STRUCTURAL CONTROLLABILITY OF DRIFTLESS BILINEAR CONTROL SYSTEMS

BY

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THESIS

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Abstract

This thesis addresses the structural controllability of driftless bilinear systems with sparse matrices. We begin with a rigorous introduction to the controllability of nonholonomic nonlinear systems. We present the notion of structural controllability and the fact that the controllability of linear systems is a generic property. We give a detailed presentation of the structural controllability of linear systems, based on Lin (1974). Afterwards, we proceed to the analysis of the structural controllability of driftless bilinear systems. We examine two cases; in the first case the matrices of the driftless bilinear system belong to a single vector space of matrices (single pattern case); in the second case the matrices belong to more than one vector spaces (multiple pattern case). After a rigorous presentation of the preliminaries of the theory of Lie algebras, we provide a theorem which claims that in the single pattern case, the driftless bilinear systems with more than two matrices can have a realization consisting of two matrices. This important result extends the theorem of Boothby (1975) about the realization of driftless bilinear systems. We prove that the controllability of driftless bilinear systems in both single and multiple pattern cases is a generic property. We define the notion of the graph which corresponds to a vector space of matrices and we establish necessary and sufficient conditions that relate the connectivity of this graph with the structural controllability of the driftless bilinear system in both cases. For the two patterns case, we provide a theorem which states that driftless bilinear systems with more than four matrices can have a realization with four matrices and we prove that similar propositions can be stated for more than two patterns.

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Contents

Chapter 1

Introduction

1.1 Motivation

We are concerned with the problem of structural controllability of driftless bilinear control systems, where each control matrix belongs to a sparse matrix space; see Belabbas (2013). The applications of bilinear systems are various. Bilinear systems model many complex networked systems (see Liu, Slotine, and Barabasi (2011) and Ruths and Ruths (2014)) such as networked control systems, where limited number of the subsystems interact with each other, or where the controller has access only to a limited number of states.

Bilinear models arise naturally in economic models. Two macroeconomic models which give rise to dynamic bilinear systems are the growth model of a two sector economy and the simple monetary model; see Aoki (1975). Furthermore, d'Alessandro (1975) presents a set of bilinear macroeconomic models, starting with a bilinear version of the Harrod-Domar growth model. Then a completely general sensitivity theory for bilinear systems is developed in order to provide both a useful tool for the identification of this class of models, and means to improve their use.

Bilinear systems are also used to model microbial cell-growth; see Williamson (1977). The results from the controllability of bilinear systems are used in the control of systems with bilinear hysteresis. In Nagy and Shekhawat (2009) the transient and steady-state response of an oscillator with hysteretic force and sinusoidal excitation are investigated. Hysteresis is modeled by using the bilinear model of Caughey with a hybrid system formulation.

Bilinear systems appear also in vehicle control. Langson and Alleyne (1997) address the stabilization of the lateral motion dynamics of an automobile. The bilinear terms are used to model the effect of steer angles on the effective moment arm associated with brake or drive torques applied at the relevant wheel. Furthermore, bilinear systems are of central importance in the control of quantum systems. Weakly coupled systems are a class of infinite-dimensional conservative bilinear control systems with discrete spectrum. An important feature of these systems is that they can be precisely approached by finite-dimensional Galerkin approximations. This property is of particular interest for the approximation of quantum system dynamics and the control of the bilinear Schrödinger equation; see Boussaïd, Caponigro, and Chambrion (2013).

Linear systems have simple dynamics which are fully studied. On the other hand, nonlinear systems are described by complex dynamics which are much more difficult to study in the general case. The bilinear system is the simplest class of nonlinear system to study. In many cases bilinearization of strongly nonlinear systems provides a better approximation of a system around a point than the linearization. Schwartz (1988) shows how bilinear models of essentially nonlinear technical systems may be constructed by two methods. The first method starts from the nonlinear system equation and a bilinear model is computed from two linear models for two suitably chosen operating points. The second method uses system-realization theory for bilinear systems.

1.2 Previous works in the area

The notion of controllability of bilinear systems was introduced by Piechottka and Frank (1992), which found conditions for the controllability of homogeneous-in-the-state bilinear systems in state spaces of dimensions two and three. In Rink and Mohler (1968), sufficient conditions for complete controllability of systems that are bilinear in state and control are established by geometrical arguments. The global controllability for a class of bilinear systems is studied in Wei and Pearson (1978). Specifically, using fixed-point arguments, sufficient conditions are derived for global controllability of a nonhomogeneous bilinear system and a related class of nonlinear systems. The topic of bilinear control systems is extensively analyzed in the books by Elliot (2009) and Brockett (1973).

Discrete time bilinear systems are also studied. Controllability of time-invariant discrete-time bilinear systems with bounded control inputs is discussed in Tarn, Elliot, and Goka (1973). Furthermore, the controllability of a class of discrete time bilinear systems is presented in Evans and Murthy (1977). In Louati and Ouzahra (2014) it is proven that in a Banach state space of infinite dimensions discrete bilinear systems are uncontrollable, and the control of the projections of the state on finite-dimensional subspaces is examined. Then finite-dimensional results on near-controllability are generalized. In Ball, Marsden, and Slemrod (1982) we are given a detailed study on the controllability of distributed bilinear systems.

The generic properties of linear systems have been studied by Dion, Commault, and Woude (2003). It is proven that the controllability of linear systems is a generic property; see Lee and Markus (1986). The notion of structural controllability introduced by Lin (1974) is based on the generic property of controllability in linear control systems. The main result of Lin (1974) states that the pair (A,b) is structurally controllable if and only if the graph corresponding to (A,b) is spanned by a special graph named "cactus". The notion of structural controllability and observability of linear systems is also discussed in Willems (1986). The topic of strong structural controllability is introduced in Mayeda and Yamada (1979). A system is strongly structurally controllable if, whatever values (other than zero) the indeterminate parameters of the system may take, the system is controllable. The notion of minimal structural controllability is presented in Lin (1976).

The study of structural controllability of linear systems was followed by the study of controllability of special classes of bilinear systems. In Ghosh and Ruths (2014b) the necessary and sufficient conditions are provided for structural controllability of discrete-time single-input bilinear systems with an input matrix of rank one. A control configuration design for a class of structural bilinear systems is presented in Ghosh and Ruths (2014a). Late advances on the structural controllability of sparse bilinear control systems are given in Belabbas and Gharesifard (2016). Important contributions to the mathematical theory of controllability and structural controllability of bilinear systems are presented in Kuranishi (1951), Boothby (1975) and Wilson (1979). Finally, the notion of structural controllability is used in the study of multi-agent systems, see Zamani and Lin (2009), and of hybrid systems, see Liu, Lin, and Chena (2013).

1.3 The structure of the thesis

The thesis consists of six chapters. In chapter 1, we present the motivation behind the contributions of the thesis and the previous works in this research area. We also give a detailed description of the contributions of our work. In chapter 2, we provide necessary background on the controllability of linear systems and nonholonomic nonlinear systems with special emphasis on the controllability of driftless bilinear systems. In chapter 3, we introduce the notion of structural controllability and the notion of generic rank. In chapter 4, we present in detail the results on the structural controllability of linear systems which were initially introduced by Lin (1974). In chapter 5, we state the problem of the structural controllability of driftless bilinear systems. Based on algebraic and graph theoretic results, we provide necessary and sufficient conditions which characterize a driftless bilinear system as controllable or not. We examine two cases: in the first case the matrices of the driftless bilinear system belong to a single vector space of matrices (single pattern case); in the second case the matrices belong to more than one vector space (multiple pattern case). Chapter 6 contains the conclusion as well as a description of future research directions.

1.4 Contribution

We focus on the study of the structural controllability of driftless bilinear systems with sparse matrices. We examine two cases: in the first case the matrices of the driftless bilinear system belong to a single vector space of matrices (single pattern case); in the second case the matrices belong to more than one vector spaces (multiple pattern case). Furthermore, we define the notion of structural controllability of bilinear systems in the single pattern case as well in the multiple pattern case. The theory of Lie algebras is of central importance in the study of the controllability of bilinear systems; see Boothby and Wilson (1979), Boothby (1975) and Brockett (1973). In section 5.2, we present the necessary background of the theory of Lie algebras and we give the definition of the transitive Lie algebra. In our work, we focus on two particular types of transitive Lie algebras: the $\mathfrak{sl}(n)$ and the $\mathfrak{so}(n) + aI$, $a \in$. The proof that $\mathfrak{sl}(n)$ and the $\mathfrak{so}(n) + aI$, $a \in$ are transitive Lie algebras is included in the thesis.

In section 5.3, we prove the proposition that the Lie algebra of a vector space of matrices is of dimension two. Based on this result, we can deduce the theorem of Kuranishi (1951) as a corollary. Furthermore, we can extend the theorem of Boothby (1975) to theorem 5.3.3, which states that in the single case a controllable bilinear system of more than two matrices can have a realization of two matrices only. In the end of section 5.3, we prove that the controllability of bilinear systems is a generic property in the single pattern and in the multiple patterns case.

In section 5.4, we define the graph which corresponds to a vector space of matrices and we provide results which relate the connectivity of this graph with the structural controllability of driftless bilinear systems with matrices in this vector space. Specifically, if the vector space is a subset of $\mathfrak{sl}(n)$, the bilinear system is structurally controllable if and only if the corresponding graph is strongly connected. If the vector space is a subset of $\mathfrak{so}(n) + aI$ and contains all aI, $a \in$, the bilinear system is structurally controllable if and only if the corresponding graph is connected.

In section 5.5 we extend our results for the single pattern case to the multiple pattern case. We prove that in the case of two patterns a system with more than four matrices with at least two matrices in each pattern can have a realization with four matrices and exactly two matrices in each pattern. We show that our results can be extended for more than two patterns.

Chapter 2

Controllability of linear and nonholonomic nonlinear systems

In this chapter, we provide a short analysis of the controllability of linear and nonholonomic control systems. Special emphasis is put on the bilinear control systems. In each section, we describe all the necessary mathematical tools. An extended analysis of linear systems can be found in (Hespanha 2009) and (Belabbas 2016). For a detailed description on the controllability of nonholonomic systems, see Belabbas (2016) and Brockett (1973).

2.1 Controllability of linear systems

In this section we give the theory of the controllability of linear time-varying systems. The time varying case captures also the time invariant case as a special case. So, we focus on the linear time-varying controlled system.

$$\dot{x} = A(t)x(t) + B(t)u(t) \tag{2.1}$$

Definition 2.1.1. *(Controllability).*
We say that the system (2.1) is controllable over the time interval $[t_0, t_1]$ *if for any* x_0, $x_1 \in \mathbb{R}^n$, *there exists an integrable control input u(t) defined over the time interval* $[t_0, t_1]$ *such that* $x(t_1) = x_1$ *if* $x(t_0) = x_0$ *in (2.1).*

The solution of differential equation (2.1) for an initial condition x_0 is:

$$x(t_1) = \Phi(t_1, t_0)x_0 + \int_{t_0}^{t_1} \Phi(t_1, s)B(s)u(s)ds \tag{2.2}$$

where $\Phi(t_1, t_0)$ is the transition matrix of the differential equation (2.1). Using the above equation, we define an affine operator from the space of integrable functions to $\mathbb{R}^n$; more specifically, the operator maps $u(t)$ to $x(t_1)$. The system (2.1) is controllable if the operator

$$L(u) = \int_{t_0}^{t_1} \Phi(t_1, s)B(s)u(s)ds \tag{2.3}$$

is onto $\mathbb{R}^n$. We have to identify the range space of operator L defined in (2.3). This is done by showing that the range space of L is equivalent to the range space of the following operator on $\mathbb{R}^n$:

$$W(t_0, t_1) = \int_{t_0}^{t_1} \Phi(t_0, s)B(s)B^T(s)\Phi(t_0, s)^{-1}ds \tag{2.4}$$

The operator $W(t_0, t_1)$ is called the controllability gramian of the linear system (2.1).

Lemma 2.1.2. *Let P(s) be a continuous function over the interval $[t_0, t_1]$, let*

$$L : C([t_0, t_1]) \mapsto \mathbb{R} : u \mapsto \int_{t_0}^{t_1} P(s)u(s)ds \tag{2.5}$$

and

$$Q = \int_{t_0}^{t_1} P^T(s)P(s)ds \tag{2.6}$$

Then the range space of $\mathbb{Q}$ and the range space of $\mathbb{L}$ are the same.

We finish this section with the following useful theorem:

Theorem 2.1.3. *There exists a u(s) that drives system (2.1) from x_0 at t_0 to x_1 at t_1 if and only if $\Phi(t_0, t_1)x(t_1) - x_0$ is in the range space of*

$$W(t_0, t_1) = \int_{t_0}^{t_1} \Phi(t_0, s)B(s)B^T(s)\Phi^T(t_0, s)ds \tag{2.7}$$

Moreover, if $y_1 \in \mathbb{R}^n$ is such that $\Phi(t_0, t_1)x(t_1) - x_0 = W(t_0, t_1)y_0$, then $u(t) = -B^T(t)\Phi^T(t_0, t)y_0$ is a control that achieves the desired transfer.

2.2 Controllability of nonholonomic nonlinear systems

In this section, we provide the fundamental results for the controllability of nonholonomic systems. So, we focus on systems of the form:

$$\dot{x} = \sum_{i=1}^{p} u_i g_i(x) \tag{2.8}$$

for g_i differentiable vector fields on a manifold M, where the u_i are the control inputs. At this point we give the definition of Lie bracket and Lie algebra which will be widely used in the thesis.

Definition 2.2.1. *(Lie bracket).*
Let $f(x)$ and $g(x)$ be differentiable vector fields in $\mathbb{R}^n$. We call the Lie bracket of f and g, denoted by [f,g](x), the vector field:

$$[f,g] = \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g \tag{2.9}$$

Definition 2.2.2. *(Lie algebra).*
A Lie algebra is a vector space $\mathfrak{g}$ over some field F together with a binary operator $[.,.] : \mathfrak{g}\times\mathfrak{g}\rightarrow\mathfrak{g}$ called the Lie bracket that satisfies the following axioms:

Bilinearity *[ax+by,z]=a[x,z]+b[y,z], [z,ax+by]=a[z,x]+b[z,y] for all scalars a, b in F and all elements x, y, z in $\mathfrak{g}$.*

Alternativity *[x,x]=0, for all x in $\mathfrak{g}$.*

The Jacobi identity *[x,[y,z]]+[z,[x,y]]+[y,[z,x]]=0, for all x, y, z in $\mathfrak{g}$.*

Using bilinearity to expand the Lie bracket [x+y,x+y] and using alternativity [x+y,x+y]=0, we get [x,y]+[y,x]=0 for all elements x, y in $\mathfrak{g}$, showing that bilinearity and alternativity together imply:

Anticommutativity *[x,y]=-[y,x] for all elements x, y in $\mathfrak{g}$. Anticommutativity only implies the alternating property if the field's characteristic is not 2.*

Definition 2.2.3. *(Derived distributions).*
Given a set of smooth vector fields $g_i(x)$ on M, we set
$D_0(x) = span\{g_1(x), ..., g_p(x)\}$. Generally, we define the kth derived distribution as:

$$D_k(x) = D_{k-1}(x) \bigoplus span\{[\tilde{g}_i(x), \tilde{g}_j(x)] \quad \tilde{g}_i, \tilde{g}_j \in D_{k-1}(x)\} \tag{2.10}$$

The dimension of $D_k(x)$ is clearly upper bounded by n, where n is the dimension of the tangent space T_xM. Thus, there exists a smallest integer j such that D_j is involutive. Now, we provide a theorem which determines the reachable space of (2.8):

Theorem 2.2.4. *For the system (2.8) with initial condition x_0, the reachable space is the integral manifold of the distribution D_∞ (or D_k such that $D_k = D_{k+1}$).*

We conclude this section with the well-known Rashevsky-Chow theorem.

Theorem 2.2.5. *(Rashevsky-Chow).*
Let N be a connected submanifold of M and let D be a distribution such that $D_\infty(x) = T_xN$ for all $x \in N$. Then for any pair of x_1, $x_2 \in$ N, there exists a curve $c(t)$ joining x_1 to x_2 such that $\dot{c}(t) \in$D.

2.2.1 Controllability of driftless bilinear systems

A special case of nonholonomic systems are the driftless bilinear systems:

$$\dot{x}(t) = \sum_{i=1}^{p} A_i x(t) u_i(t) \tag{2.11}$$

where the A_i are $n \times n$ matrices and n is the dimension of the state.

From theorem 2.2.2 the reachable space of 2.11 is the integral manifold of the distribution D_∞ which is equal to the Lie algebra of the matrices A_1, A_2, ..., A_p, which are denoted by $\{A_1, A_2, ..., A_p\}_{LA}$. We observe that if $\{A_1, A_2, ..., A_p\}_{LA} = \mathbb{R}^{n\times n}$ then the system 2.11 is controllable. However, this sufficient condition is not necessary for any arbitrary bilinear system. In general, the system 2.11 is controllable if and only if D_∞ is a transitive Lie algebra.

Chapter 3

Introduction to structural controllability

A control system operates properly if and only if it is controllable and observable; see Chen (1995). This fundamental result has been further strengthened by the fact that controllability and observability are robust properties in linear control systems; see Lee and Markus (1986). That is, the set of all controllable pairs (A,B) is open and dense in the space of all such pairs. This result was used by Lin (1974) to introduce the concept of structural controllability, which states that all uncontrollable systems structurally equivalent to a structurally controllable system are atypical. This result dramatically decreased the computational effort needed to decide the controllability of a system, especially for systems of high dimensions; see Siljak (1991).

3.1 Mathematical preliminaries in structural controllability

In this section, we provide the basic theorems that led to the formulation of structural controllability. The most important theorem is that the controllable pairs (A,B) are open and dense in the space of all pairs; see Lee and Markus (1986). In this section, we provide the statement and the proof of this theorem.

Theorem 3.1.1. *Consider an autonomous linear system in* $\mathbb{R}^n$ *with the control input* $u \in R^m$*:*

$$\dot{x} = A_0 x + B_0 u \tag{3.1}$$

If (3.1) is controllable, then there exists an $\epsilon_1 > 0$ such that every autonomous linear process

$$\dot{x} = Ax + Bu \tag{3.2}$$

with $\|A - A_0\| < \epsilon_1$ and $\|B - B_0\| < \epsilon_1$ is also controllable.
If (3.1) is not controllable, then for each $\epsilon > 0$, there exists a controllable system:

$$\dot{x} = A_1 x + B_1 u \tag{3.3}$$

with $\|A_1 - A_0\| < \epsilon$ $\|B_1 - B_0\| < \epsilon$.
That is, the set of all controllable systems is open and dense in the metric space of all autonomous linear systems in $\mathbb{R}^n$, the distance from (3.3) to (3.1) being $\|A_1 - A_0\| + \|B_1 - B_0\|$.

Proof. If (3.1) is controllable in $\mathbb{R}^n$, the rows of $[B_0, A_0B_0, \ldots, A_0^{n-1}B_0]$ describe n linearly independent vectors in $\mathbb{R}^{nm}$. If $\|A - A_0\| < \epsilon_1$ and $\|B - B_0\| < \epsilon_1$ for a sufficiently small $\epsilon_1 > 0$, then the rows of $[B, AB, \ldots, A^{n-1}B]$ must approximate these n vectors of $\mathbb{R}^{nm}$ and hence must also be linearly independent. In this case (3.2) is also controllable.
On the other hand assume that (3.1) is not controllable. For a given $\epsilon > 0$ choose matrices A_1 and B_1 with $\|A_1 - A_0\| < \epsilon$, $\|B_1 - B_0\| < \epsilon$ such that all entries of A_1 and B_1 are algebraically independent over the rational numbers (that is, no nontrivial rational polynomial relations hold between the entries of A_1 and B_1 - the existence of such A_1 and B_1 is a standard property of the arithmetic of real numbers). Then

$$rank[B_1, A_1B_1, \ldots, A_1^{n-1}B_1] = n \tag{3.4}$$

since no $n \times n$ subdeterminant can be zero because each such determinant is a polynomial in the entries of A_1 and B_1. Thus, (3.3) is controllable. □

Theorem (3.1.1) assures us that, in the typical or generic case, an autonomous linear system (3.2) is controllable. If (3.2) is to represent an actual physical system that involves parameters only approximately determined, then we can always assume that (3.2) is controllable.

3.2 Definition of structural controllability

In this section, we give the basic definitions of structural controllability. An analysis of structural controllability is presented in Siljak (1991).

Definition 3.2.1. *An $n \times m$ matrix $\tilde{M} = (\tilde{m}_{ij})$ is said to be a structured matrix if its elements $\tilde{m}_{ij}$ are either fixed zeros or independent free parameters.*

A 2×2 structured matrix is:

$$\tilde{M} = \begin{bmatrix} 0 & * \\ * & 0 \end{bmatrix} \tag{3.5}$$

To relate a numerical $n \times m$ matrix $M = (m_{ij})$ to a structured matrix $\tilde{M}$ we define $n = \{1, 2, \ldots, n\}$, $m = \{1, 2, \ldots, m\}$ and state the following:

Definition 3.2.2. *A numerical matrix M is said to be admissible with respect to a structured matrix $\tilde{M}$, that is, $M \in \tilde{M}$, if and only if $\tilde{m}_{ij} = 0$ implies $m_{ij} = 0$ for all $i \in n$ and $j \in m$.*

The matrix

$$M = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \tag{3.6}$$

is admissible with respect to $\tilde{M}$ of (3.5).

To state the definition of structural controllability, let us associate with system (3.2) a structured system $(\tilde{A}, \tilde{B})$ so that $(A, B) \in (\tilde{A}, \tilde{B})$. Structural controllability of the system $\tilde{S}$ is defined via the pair $(\tilde{A}, \tilde{B})$ as in Lin (1974):

Definition 3.2.3. *A pair of matrices $(\tilde{A}, \tilde{B})$ is said to be structurally controllable if there exists a controllable pair (A, B) such that $(A, B) \in (\tilde{A}, \tilde{B})$.*

3.3 The notion of generic rank

In order to state the necessary and sufficient conditions for structural controllability of $\tilde{S}$, we need the notion of *generic rank* (or term rank) of a structured matrix $\tilde{M}$, which we denote by $\tilde{\rho}(\tilde{M})$. Simply, the generic rank of $\tilde{M}$ is the maximal rank that $\tilde{M}$ can achieve by choosing numerical values for indeterminate elements of $\tilde{M}$. Therefore, a matrix $\tilde{M}$ has full generic rank if and only if there exists a matrix M of full rank such that $M \in \tilde{M}$.

To make the idea of the generic rank precise, we need certain notions from algebraic geometry; see Wonham (1985). Let us assume that a matrix $\tilde{M}$ has elements in $\mathbb{R}$, there are v indeterminate entries, and the rest of the entries are fixed zeros. Then, with $\tilde{M}$ we can associate a parameter space $\mathbb{R}^v$ such that every data point p $\in \mathbb{R}^v$ defines a matrix M $\in \tilde{M}(p)$, which is obtained by replacing the arbitrary entries $\tilde{m}_{ij}$ of $\tilde{M}$ by the corresponding elements of $p = (p_1, p_2, \ldots, p_v)^T$.

In a physical problem, it is important to know that if a matrix M has a certain rank at a nominal parameter vector p^o, it has the same rank at a vector p close to p^o, which corresponds to small deviations of the parameters from their nominal values. Most often, it turns out that the rank holds true for all $p \in \mathbb{R}^v$ except at the points p that lie on an algebraic surface in $\mathbb{R}^v$ and which are, therefore, atypical. An arbitrarily small perturbation of such points restores the rank of M.

Let us denote by $\phi_k(p_1, p_2, ..., p_v)$, $k \in \boldsymbol{K}$, $\boldsymbol{K} = \{1, 2, ..., K\}$, a set of K polynomials generated by all the r-th order minors of M, and consider a variety $V \subset \mathbb{R}$ which is the locus of common zeros of the polynomials ϕ_k:

$$\boldsymbol{V = \{p \in \mathbb{R}^v : \phi_k(p_1, p_2, ..., p_v) = 0, k \in K\}} \tag{3.7}$$

The variety V is proper if $V \neq \mathbb{R}^v$ and nontrivial if $V \neq 0$. We say that the rank r of the matrix M holds generically relative to V when the parameter values, which make the rank of M smaller than r, all lie on a proper variety V in $\mathbb{R}^v$. In other words, the variety V is either the whole space $\mathbb{R}^v$ in which case $\tilde{\rho}(\tilde{M})$, or the complement V^c of V is open and dense in $\mathbb{R}^v$ and therefore generic. What this means is that, if the rank condition fails at $p^o \in \mathbb{R}^v$, then the condition can be restored by an arbitrarily small perturbation of the parameter vector p.

Chapter 4

Structural controllability of linear systems

An introduction to the concept of structural controllability and the notion of generic rank was given in chapter 3. In this chapter, we provide the theory of structural controllability of linear time-invariant control systems, described by a pair (A,b) where A $\in M^{n\times n}(\mathbb{R})$ and b is column vector in $\mathbb{R}^n$; see Lin (1974). The graph of a pair (A,b) is also defined and it gives another way of describing the structure of this pair. The property of structural controllability is reduced to a property of the graph of the pair (A,b). The concepts of the graph "cactus" and the graph "precactus" are introduced. The main result in Lin (1974) states that the pair (A,b) is structurally controllable if and only if the graph of (A,b) is spanned by a cactus. The result is also expressed in terms of linear algebraic properties of the pair (A,b).

4.1 Structural controllability of linear systems

We begin this section with the following useful definition:

Definition 4.1.1. *The pair (A,b) has the same structure as another pair* $(\tilde{A}, \tilde{b})$ *of the same dimensions, if for every fixed (zero) entry of the matrix/column pair (A,b), the corresponding entry of the matrix/column pair* $(\tilde{A}, \tilde{b})$ *is fixed (zero) and, at the same time, for every fixed (zero) entry of* $(\tilde{A}, \tilde{b})$*, the corresponding entry of (A,b) is also fixed (zero). Then one defines the pair* (A_0, b_0) *to be structurally controllable if and only if there exists a controllable pair (A,b) which has the same structure as* (A_0, b_0).

We extend theorem 3.1.1 that the set of all controllable linear systems is open and dense, for the case of structural controllability, by the following proposition:

Proposition 4.1.2. *The pair* (A_0, b_0) *is structurally controllable if and only if* $\forall\ \epsilon > 0$*, there exists a controllable pair* (A_1, b_1) *of the same structure as* (A_0, b_0) *such that* $\|A_1 - A_0\| < \epsilon$ *and* $\|b_1 - b_0\| < \epsilon$*.*

Proof. Let (A_0, b_0) be a pair. If $\forall\ \epsilon > 0$, there exists a controllable pair (A_1, b_1) of the same structure as (A_0, b_0) such that $\|A_1 - A_0\| < \epsilon$ and $\|b_1 - b_0\| < \epsilon$, then the pair (A_0, b_0) is obviously structurally controllable because (A_1, b_1) is controllable and has the same structure as (A_0, b_0).

Conversely, assume that the pair (A_0, b_0) is structurally controllable; then by definition there exists a controllable pair (A_2, b_2) of the same structure as (A_0, b_0). Consider now the pairs, $A(\lambda) = (1 - \lambda)A_0 + \lambda A_2$; $b(\lambda) = (1 - \lambda)b_0 + \lambda b_2$, where $\lambda \in [0, 1]$. Then $\psi(\lambda) = det(b(\lambda), A(\lambda)b(\lambda), \ldots, A(\lambda)^{n-1}b(\lambda))$ is a polynomial in λ, and this polynomial is not identically zero (since it is different from zero for λ=1). Given an arbitrary $\epsilon > 0$, one can find $\lambda_0 \in [0, 1]$ such that $\|A(\lambda) - A_0\| < \epsilon$ and $\|b(\lambda) - b_0\| < \epsilon$, $\forall \lambda \in][0, \lambda_0]$. Further one can find $\lambda_1 \in [0, \lambda_0]$ such that $\psi(\lambda_1) \neq 0$ since each polynomial has a finite number of zeros. As a result, the pair $(A(\lambda_1), b(\lambda_1))$ is controllable. □

If no entry of the pair (A,b) is fixed, then the pair (A,b) is structurally controllable. However, if some entries of (A,b) are fixed, the pair may not be structurally controllable. We consider two specific forms of the pair (A,b) that lead to systems which are not structurally controllable; see Lin (1974) and Siljak (1991).

Form I: We consider the pair (A,b) of the form

$$A = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ b_2 \end{bmatrix} \tag{4.1}$$

where $A_{11} \in \mathbb{R}^{k \times k}$, $A_{21} \in \mathbb{R}^{(n-k) \times k}$, $A_{22} \in \mathbb{R}^{(n-k) \times (n-k)}$, $b_2 \in \mathbb{R}^{(n-k)}$, $1 \leq k \leq n$ and by 0, one denotes a matrix or a vector containing only fixed (zero) entries.

We see that $rank(b, Ab, \ldots, A^{n-1}b)$ is less than n, independently of the values of A_{11}, A_{21}, A_{22}, and b_2. Thus, the pair (4.1) is not structurally controllable.

Form II: We consider the pair (A,b) in which the $n{\times}(n+1)$ matrix $[A|b]$ can be written as:

$$[A|b] = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \tag{4.2}$$

where P_2 is an $(n-k){\times}(n+1)$ matrix, and P_1 is a $k{\times}(n+1)$ matrix ($k{\geq}1$) with no more than k-1 non-zero columns (all the other columns of P_1 having only fixed (zero) entries). Then one obtains directly that $rank([A|b]) < n$, independently of the non-zero entries in (4.2). From the fact that $rank([A|b]) < n$ together with the Hautus-Rosenbrock test, we get that the pair (A,b) is not controllable. Thus, the pair (4.2) is not structurally controllable.

In this chapter, we find the necessary and sufficient conditions of structural controllability. We will prove that every pair (A,b) which is not structurally controllable can be brought (after a suitable permutation of the coordinates) to one of the forms (4.1) and (4.2).

4.2 The graph of a pair (A,b)

Given a pair (A,b), where $A{\in}M^{n\times n}(\mathbb{R})$, $b{\in}\mathbb{R}^n$, one defines its graph C, as the graph which contains exactly n + 1 nodes, v_1, v_2, ..., v_n, all of whose edges are obtained as follows: For every non-fixed entry c_{ij} of the $n{\times}(n+1)$ matrix $[A|b]$, the graph contains the oriented edge (v_j, v_i) (an arrow going from v_j to v_i). The node v_{n+1}, which corresponds to the (n+1)-th column of $[A|b]$, will be called the "origin" of G. From the definition it follows that no arrow can point towards v_{n+1}. For every oriented edge (v_j, v_i) in G, the node v_j will be called the "origin" of this edge; a node v in the vertex set of G will be called the "final" node, if v is not the origin of any oriented edge in G; see Lin (1974).

We give two characteristic types of graphs corresponding to a pair (A,b): the stem and the bud. For the pair (A_1, b_1) of the form

$$A_1 = \begin{bmatrix} 0 & a_1 & 0 & \dots & 0 \\ 0 & 0 & a_2 & \dots & 0 \\ . & . & . & \dots & . \\ 0 & 0 & 0 & \dots & a_{n-1} \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} b_1 = \begin{bmatrix} 0 \\ 0 \\ . \\ 0 \\ a_n \end{bmatrix} \tag{4.3}$$

the corresponding graph is shown in Fig. 4.1.

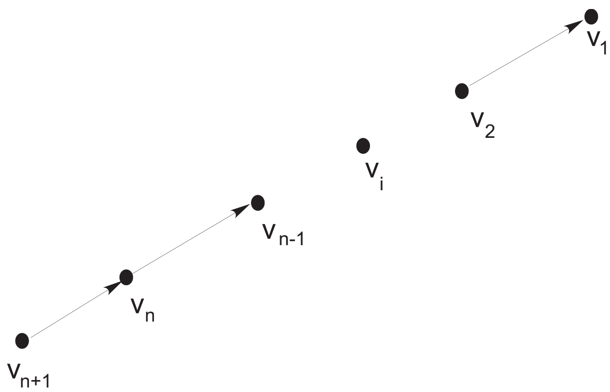

FIGURE 4.1: stem

A graph of this form will be called a stem. For the pair (A_2, b_2) of the form

$$A_1 = \begin{bmatrix} 0 & a_1 & 0 & \dots & 0 \\ 0 & 0 & a_2 & \dots & 0 \\ . & . & . & \dots & . \\ 0 & 0 & 0 & \dots & a_{n-1} \\ a_n & 0 & 0 & \dots & 0 \end{bmatrix} b_1 = \begin{bmatrix} 0 \\ 0 \\ . \\ 0 \\ a_{n+1} \end{bmatrix} \tag{4.4}$$

the corresponding graph is shown in Fig. 4.2.

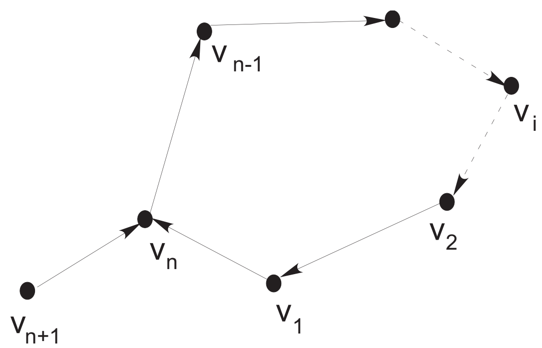

FIGURE 4.2: bud

A graph of this form will be called a bud. The node v_{n+1} is called the "origin" of the bud and the edge (v_{n+1}, v_n) is called the "distinguished edge" of the bud. Both of the pairs (4.3) and (4.4) are easily seen to be structurally controllable.

Now, we give two examples which correspond to (4.1) and to (4.2) respectively. Consider, the following pair of the form (4.1):

$$A_3 = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} b_3 = \begin{bmatrix} 0 \\ 0 \\ a_{31} \end{bmatrix} \tag{4.5}$$

The graph of this pair is depicted in Fig. 4.3, where the nodes v_l and v_2 are said to be non-accessible.

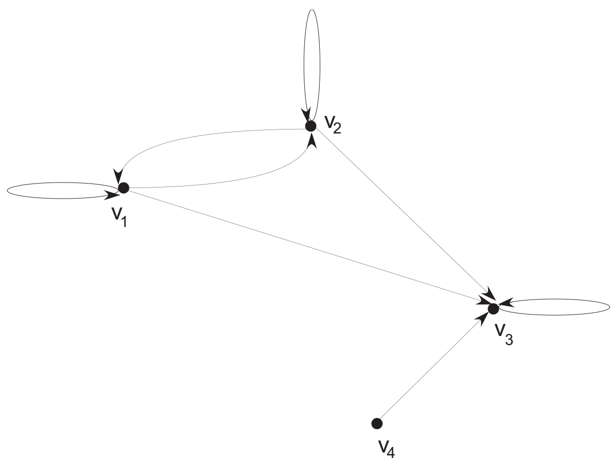

FIGURE 4.3: pair (4.5)

In general, a node v_i (other than the origin) in the graph of a pair (A,b) is called non-accessible if and only if there is no possibility of reaching the node v_i starting from the origin v_{n+l} and going to v_i only in the direction of the arrows, along a path in the graph of the pair (A,b). It is easy to see that in general, the graph of a pair (A,b) of the form (4.1) contains at least one non-accessible node. Moreover, the converse is also true: If the graph of a pair (A,b) is such that there exists at least one non-accessible node, then (after a permutation of the coordinates) (A,b) can be brought to the form (4.1) and therefore (A,b) is not structurally controllable.

Now, we consider a pair of the form (4.2)

$$A_4 = \begin{bmatrix} 0 & a_{12} & 0 \\ 0 & a_{22} & 0 \\ 0 & a_{32} & 0 \end{bmatrix} b_4 = \begin{bmatrix} a_{14} \\ a_{24} \\ a_{34} \end{bmatrix} \tag{4.6}$$

The graph of this pair is shown in Fig. 4.4.

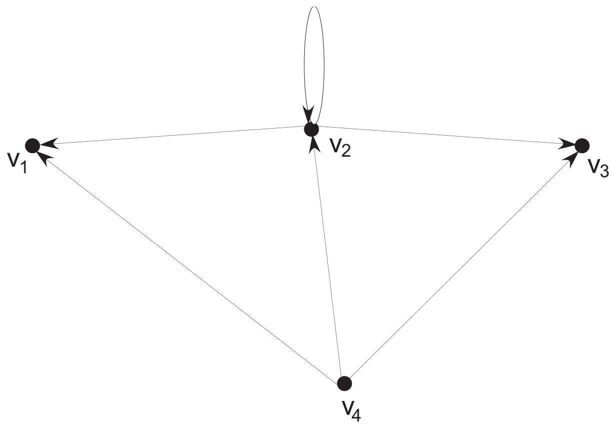

FIGURE 4.4: pair (4.6)

Consider here the set S formed by the nodes v_1, v_2, and v_3 ($S = \{v_1, v_2, v_3\}$). Determine the set $T(S)$ containing all the nodes v_i with the property that there is an oriented edge going from v_j to a node in S. Clearly, $T(S) = \{v_2, v_4\}$. Here, the T(S) contains two elements while S contains three elements. One says that the graph contains a dilation. More generally, the graph of a pair (A,b) contains a dilation if and only if there is a set S of k nodes in the vertex set of the graph (not containing the origin v_{n+l}) such that there are no more than $k-1$ nodes v_j in T(S). One denotes by T(S) the set of all the nodes v_j with the property that there exists an oriented edge from v_j to a node in S. Note that the origin v_{n+1} is not allowed to belong to S, but may belong to T(S).

One can easily see that if the pair (A,b) has the form (4.2), then its graph contains n dilation. Conversely, if the graph of a pair (A,b) contains a dilation, then (after a permutation of the coordinates) the matrix $[A|b]$ can be brought to the form of (4.2), and, therefore, the pair (A,b) is not structurally controllable.

4.2.1 The graph "Cacti"

If the graph of a pair is a stem or a bud, then the pair is structurally controllable. We extend the conclusion to some special combinations of stems and buds which we call "cacti". The lemma and the propositions presented in this section were initially presented and proven in Lin (1974); the presentation of the proofs is beyond the scope of the thesis.

Lemma 4.2.1. *Suppose that G is a graph of a structurally controllable pair. Let B be a bud with the origin e, and suppose e is the only node which belongs at the same time to the vertex set of G and to the vertex set of B. Then $G \cup B$ is the graph of a structurally controllable pair.*

The concept of a cactus initially introduced by Lin (1974), can be defined in the following descriptive form: The graph P of a pair (A,b) is a cactus if and only if one can write $P = S \cup B_1 \cup B_2 \cup \ldots \cup B_p$ where S is a stem and B_i are buds and, for every $i = 1, 2, \ldots, p$, the origin e_i of B_i is also the origin of an oriented edge in the graph $S \cup B_1 \cup \ldots \cup B_{i-1}$. Moreover, e_i is the only node which belongs at the same time to the vertex set B_i and to the vertex set of $S \cup B_1 \cup B_2 \cup \ldots \cup B_{i-1}$; see Fig. 4.5.

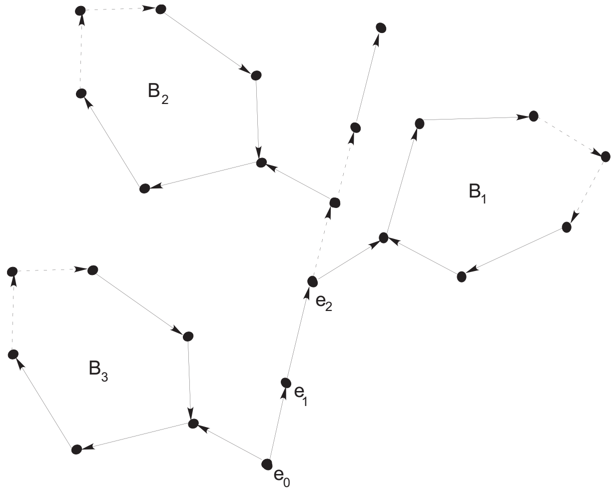

FIGURE 4.5: cactus

From the above definition and from Lemma 1, one obtains the following propositions:

Proposition 4.2.2. *If the graph of a pair (A,b) is a cactus, then the pair is structurally controllable.*

We say that the graph of a pair (A,b) is spanned by a cactus if it becomes a cactus after removing some or none of the edges from the graph. Then, we have the following proposition:

Proposition 4.2.3. *If the graph of a pair (A,b) is spanned by a cactus, then the pair (A,b) is structurally controllable.*

4.2.2 A class of graphs which are cacti

We focus on a class of graphs which are cacti. In the following lemmas we assume that G is the graph of a pair (A,b) and has the following properties:
1) There is no non-accessible node in the vertex set of G.
2) There is no dilation.
3) G is minimal (after deleting any edge of the graph, one of the properties (1) and (2) is violated).

Lemma 4.2.4. *Every node in G is accessible from the origin along one and only one simple path.*

We introduce the following notation: If M is any set of nodes, denote by N(M) the number of distinct nodes in M. Furthermore, we denote by V_i the set of all the nodes which can be reached from the origin of G by passing through the edge e_i. For every V_i, we denote by G_i the subgraphs of G whose set of nodes is exactly $V_i \cup \{e\}$ (e is the origin) and whose edges are all the edges from G of the form (a, b) with $a \in V_i \cup \{e\}$ and $b \in V_i$. The subgraphs G_i, defined as above, will be called "bunches". Clearly, $G = G_1 \cup G_2 \cup \ldots \cup G_r$, and G_i are edge disjoint (Lemma 2). A subgraph G_i, defined as above, is called a "terminal bunch" if there exists a subset $S \subset V_i$, such that $N(T(S)) = N(S)$ and $T(S)$ contains the origin of G. Now, we are ready to provide a sequence of important lemmas.

Lemma 4.2.5. *If G_i is not a terminal bunch, then for every set $S \subset V_i$ such that T(S) contains the origin, one has $N(T(S)) - N(S) \geq 1$.*

Lemma 4.2.6. *There exists at most one terminal bunch in G.*

A graph H is a precactus if and only if one can write
$H = B_1 \cup B_2 \cup \ldots \cup B_p$, where B_i are the buds such that for every $i = 2, 3, \ldots p$ the origin e_i of B_i is also the origin of one oriented edge in the graph of $B_1 \cup B_2 \cup \ldots \cup B_{i-1}$. Moreover, e_i is the only node which belongs at the same time to the vertex set of B_i and to the vertex set of $B_1 \cup B_2 \cup \ldots \cup B_{i-1}$.

Lemma 4.2.7. *Every precactus becomes a cactus after eliminating one or more suitable edges.*

Lemma 4.2.8. *Any non-terminal bunch becomes a precactus, possibly after eliminating some edges of the bunch. In other words, any non-terminal bunch is "spanned" by a precactus.*

Lemma 4.2.9. *There always exists a terminal bunch in G. Furthermore, any terminal bunch G_1 is spanned by a cactus.*

Proposition 4.2.10. *If the graph G of a pair (A,b) satisfies the properties:*

1 *There is no non-accessible node in the vertex set of G*

2 *There is no dilation*

3 *G is minimal (after deleting any edge of the graph, one of the properties 1 and 2 is violated)*

then G is a cactus.

We can summarize the results of this section by the following theorem:

Theorem 4.2.11. *The following properties are equivalent:*

1 *The pair (A,b) is structurally controllable.*

2 *There is no permutation of coordinates, bringing the pair (A,b) to one of the forms (4.1) and (4.2).*

3 *The graph of (A,b) contains no non-accessible node and no dilation.*

4 *The graph of (A,b) is spanned by a cactus.*

Chapter 5

Structural controllability of driftless bilinear systems

5.1 Structural controllability of driftless bilinear systems

5.1.1 Problem Statement

Let Σ_β be the vector space of matrices in $\mathbb{R}^{n\times n}$, where $\beta \subset \{1,\ldots,n\}\times\{1,\ldots,n\}$ and all entries not in β are forced to be zero. It is common to represent a matrix in Σ_β with a star symbol "*" in the non-zero entries and with a 0 in the entries forced to be zero. We assume that the matrix vector space Σ_β contains sparse matrices. We usually refer to Σ_β as a sparse matrix space (SMS). Consider now the bilinear control system

$$\dot{x}(t) = (A_\ell + \sum_{\ell=1}^{m} u_\ell(t)B_\ell)x(t) \tag{5.1}$$

where $A_\ell \in \Sigma_\alpha$, $B_\ell \in \Sigma_\beta$ (β possibly different than α), and $u_\ell : \mathbb{R}_{>0} \to \mathbb{R}$ (control input) is a continuous function, for all $\ell \in \{1,\ldots,m\}$, and $m \in \mathbb{Z}_{\geq 1}$. We refer to this system as an m-bilinear sparse control system. The controllability problem for bilinear control systems with drift is to a large extent open; see Elliot (2009). For this reason, we focus our attention on the class of bilinear control systems without drift:

$$\dot{x}(t) = (\sum_{\ell=1}^{m} B_\ell u_\ell(t))x(t) \tag{5.2}$$

where $x(t) \in \mathbb{R}^n$ for all $t \geq 0$, and $B_\ell \in \mathbb{R}^{n\times n}$. We would like to make clear that for the driftless bilinear control system the space of admissible control inputs that we consider is the space of all piecewise constant functions on $\mathbb{R}$. At this point, we introduce the notion of structural controllability of bilinear control systems, extending the definition of structural controllability of linear systems initially presented in Lin (1974).

Definition 5.1.1. *The system* (5.2) *with* $B_\ell \in \Sigma_\beta$, $\ell \in \{1,\ldots,m\}$, *is structurally controllable if and only if there exist* $B_\ell \in \Sigma_\beta$, $\ell \in \{1,\ldots,m\}$ *such that the system* (5.2) *is controllable.*

By definition 5.1.1 it is evident that if a system is controllable then it is also structurally controllable. We are interested in the problem of structural controllability of (5.2). We examine the case where all $B_\ell \in \Sigma_\beta$ for all $\ell \in \{1,\ldots,m\}$, which is the single pattern case. Then, we extend our results for $B_\ell \in \Sigma_{\beta_1} \cup \Sigma_{\beta_2} \cup \ldots \cup \Sigma_{\beta_k}$, where $k \leq m$ and $\ell \in \{1,\ldots,m\}$, with at least one matrix in each Σ_{β_i}, $1 \leq i \leq k$ which is the multiple patterns case. First, we provide the necessary background on Lie algebras.

5.2 Preliminaries

5.2.1 The theory of Lie algebras

We begin with the definition of the Lie algebra of a matrix group G, see Tapp (2005).

Definition 5.2.1. *Let G be a matrix group. The Lie algebra of G is the tangent space to G at I, which is denoted by*

$$T_I G := \{\gamma'(0) | \gamma : (-\epsilon, \epsilon) \to G, \text{ is differentiable with } \gamma(0) = I\} \tag{5.3}$$

Proposition 5.2.2. *The Lie algebra* $\mathfrak{g}$ *of a matrix group* $G \subseteq GL_n(\mathbb{R})$ *is not only a vector subspace but also a real subspace of* $M_n(\mathbb{R})$.

Proof. Since $\mathfrak{g}$ is the tangent space of matrix group G, which has elements of $n \times n$ matrices, then, trivially, the elements of $\mathfrak{g}$ are $n \times n$ matrices. Now, we must show that $\mathfrak{g}$ is closed under scalar multiplication and addition. Let λ be a real number, and let A be an element of $\mathfrak{g}$. Then, by definition, $A = \gamma'(0)$ for some differentiable path γ in G, and $\gamma(0) = I$. Consider the path $\sigma : (-\frac{\epsilon}{\lambda}, \frac{\epsilon}{\lambda}) \to G$, given by $\sigma(t) = \gamma(\lambda t)$. This is in G and passes through the identity matrix I. It has initial velocity vector $\sigma'(0) = \lambda\gamma'(0) = \lambda A$, which proves that λA belongs to $\mathfrak{g}$. Next, let A, B be in $\mathfrak{g}$. This means that $A = \alpha'(0)$, $B = \beta'(0)$ for some differentiable paths α, β in G with $\alpha(0) = \beta(0) = I$. We construct the product path $\delta(t){:=}\alpha(t)\beta(t)$, which lies in G, because G is closed under multiplication. This new path satisfies $\delta(0) = I$ and $\delta'(0) = \alpha'(0)\beta(0) + \alpha(0)\beta'(0) = A{\cdot}I + I{\cdot}B = A + B$, by the product rule. This shows that $A + B$ is also in $\mathfrak{g}$. □

We recall some of the preliminaries required for Lie algebras that are crucial for the study of bilinear systems; see Boothby and Wilson (1979) and Boothby (1975). Throughout, we assume that $\mathfrak{g}$ is a subalgebra of $\mathfrak{gl}(n, \mathbb{R})$, where $\mathfrak{gl}(n, \mathbb{R})$ is the Lie algebra of $GL(n, \mathbb{R})$, the set of all invertible n by n matrices on reals. We denote by $\mathfrak{g}$ the Lie algebra with the Lie product $[\cdot, \cdot]$, and the unique Lie subgroup of $GL(n, \mathbb{R})$ associated to it by G. The center of $\mathfrak{g}$ is defined to be

$$Z(\mathfrak{g}) = \{Z \in \mathfrak{g} \mid [A, Z] = 0, \text{ for all } A \in \mathfrak{g}\}$$

The centralizer of a subset $\mathfrak{s}$ of $\mathfrak{g}$ is

$$\zeta_{\mathfrak{g}}(\mathfrak{s}) = \{A \in \mathfrak{g} \mid [A, S] = 0, \text{ for all } S \in \mathfrak{s}\}$$

and the normalizer of $\mathfrak{s}$ is

$$N_{\mathfrak{g}}(\mathfrak{s}) = \{A \in \mathfrak{g} \mid [A, S] \in \mathfrak{s}, \text{ for all } S \in \mathfrak{s}\}$$

We introduce the definition of a Lie algebra associated to a vector space of matrices Σ_β.

Definition 5.2.3. *Let Σ_β be a vector space of matrices. The Lie algebra associated to Σ_β is denoted by $\{\Sigma_\beta\}_{LA}$ and is defined as the intersection of all Lie algebras that contain Σ_β.*

The Lie algebras $\{\Sigma_\beta\}_{LA}$ for which the system (5.2) is controllable have been classified in Boothby (1975) and are called transitive Lie algebras. In the following definition we recall the notion of transitive matrix group and of transitive Lie algebra which are of central importance in our work.

Definition 5.2.4. *The matrix group G is called transitive if*

$$span\{Bx|B{\in}G\} = \mathbb{R}^n \backslash \{0\}, \forall x{\in}\mathbb{R}^n \backslash \{0\}$$

The Lie algebra $\mathfrak{g}$ corresponding to a transitive matrix group G is called a transitive Lie algebra.

The following theorem of Boothby (1975) is a consequence of the fact that the dimension of the center of a Lie algebra $\mathfrak{g}$ is less than or equal to 2 if G is transitive on $\mathbb{R}^n \backslash \{0\}$ and of the theorem of Kuranishi (1951), which is recalled in Theorem 5.2.5.

Theorem 5.2.5 (Kuranishi 1951). *Let $\mathfrak{g}_0$ be a real semisimple Lie subalgebra of $\mathfrak{gl}(n, \mathbb{R})$. Then there exist two elements A_1 and A_2 of $\mathfrak{g}_0$ which generate $\mathfrak{g}_0$.*

Theorem 5.2.6 (Boothby 1975). *Suppose system (5.2) is controllable on $\mathbb{R}^n \backslash \{0\}$. Then there exists $B_1, B_2 \in \mathbb{R}^{n \times n}$ such that the control system (5.2) is equivalent to*

$$\dot{x}(t) = (uB_1 + vB_2)x(t)$$

where u and v are piecewise constant functions on $\mathbb{R}$.

Based on definition 5.2.4 it is evident that $\mathfrak{gl}(n, \mathbb{R})$, the Lie algebra of $GL(n, \mathbb{R})$, is a transitive Lie algebra. We focus our attention on two commonly used Lie algebras in the realization of bilinear systems: the $\mathfrak{sl}(n, \mathbb{R})$ which is the Lie algebra of traceless square matrices and the $\mathfrak{so}(n, \mathbb{R}) + aI$ which is the Lie algebra of skew symmetric square matrices augmented by aI. The diagonal entries of a matrix in $\mathfrak{sl}(n)$ are dependent because they sum to zero. So, $dim(\mathfrak{sl}(n)) = n^2 - 1$. For a matrix $A \in \mathfrak{so}(n) + aI$ we see that for the entries $A(i, j)$ and $A(j, i)$ it holds that $A(i, j) = -A(j, i)$ and all the diagonal elements are equal. So, $dim(\mathfrak{so}(n) + aI) = \frac{(n-1)n}{2} + 1$. Results similar to those presented in this thesis can be derived for the rest of the transitive Lie algebras classified in Boothby (1975) following the same procedure. At this point, we provide two important lemmas about the transitivity of the Lie algebras $\mathfrak{sl}(n, \mathbb{R})$ corresponding to $SL(n, \mathbb{R})$ and $\mathfrak{so}(n, \mathbb{R}) + aI$ corresponding to $SO(n, \mathbb{R}) \cdot a$, $a \in \mathbb{R}$.

Lemma 5.2.7. *$\mathfrak{sl}$(n) is a transitive Lie algebra.*

Proof. Based on definition 3.2, we prove that for any x $\in \mathbb{R}^n \backslash \{0\}$ and y $\in \mathbb{R}^n \backslash \{0\}$ there exists a matrix A $\in SL(n)$ such that $y = Ax$. First, we prove an easier case, namely that for any u $\in \mathbb{R}^n \backslash \{0\}$ there exists a matrix B $\in SL(n)$ such that $u = Be_1$, where $e_1 = [1, 0, \ldots, 0]$, $e_1 \in \mathbb{R}^n$.
1) If $u_1 \neq 0$ then B is a lower triangular matrix and the first column of

B is the vector u^T. The diagonal elements $b_{(i,i)} = 1$ for $2 \leq i \leq n-1$, and $b_{(n,n)} = \frac{1}{u_1}$. All the other entries of B are equal to zero.
2) If $u_1 = 0$, we know that at least one of the entries of u must be non-zero, let $u_i \neq 0$ for $i \neq 1$. Then we swap u_1 with u_i and we get $u' = [u_i, \ldots, u_1, \ldots]$. Based on 1), we consider a matrix C such that $u' = Ce_1$ and $det(C) = 1$. We multiply the i-th column of C by -1 and we get D. It still holds that $u' = De_1$. Finally, we interchange the first row of D with the i-th row of D and we get a new matrix E. It is clear that $u = Ee_1$ and $det(E) = -det(D) = -(-det(C)) = det(C) = 1$.

So, in both cases there exists a matrix M $\in SL(n)$ such that $u = Me_1$. Now, we can prove that given x $\in \mathbb{R}^n \setminus \{0\}$ and y $\in \mathbb{R}^n \setminus \{0\}$ there exists $A \in SL(n)$ such that $y = Ax$. We have proven that there exist A_1 and A_2 such that $y = A_1 e_1$ and $x = A_2 e_1 \Rightarrow e_1 = A_2^{-1} x \Rightarrow y = A_1 A_2^{-1} x$ and $det(A_1 A_2^{-1}) = det(A_1) det(A_2^{-1}) = 1$. □

The Lie algebra $\mathfrak{so}(n)$ is not transitive. However, we establish the following theorem which was originally presented in Boothby (1975):

Lemma 5.2.8. *$\mathfrak{so}(n) + aI$, $a \in \mathbb{R}$ is a transitive Lie algebra.*

Proof. The $so(n)$ is not a transitive Lie algebra. A matrix A $\in SO(n)$ represents rotation around the origin. So, given a vector $x \in \mathbb{R}^n \setminus \{0\}$, there exists no matrix $B \in SO(n)$ which gives us $2x$. This issue is resolved for the group $SO(n) \cdot a$, where $a \in \mathbb{R}$ performs the scaling. The Lie algebra which corresponds to the transitive matrix group $SO(n) \cdot a$ is the transitive Lie algebra: $\mathfrak{so}(n) + aI$, where $a \in \mathbb{R}$; see Boothby (1975). □

Now, we prove that transitivity implies controllability. First, we provide the following definitions.

Definition 5.2.9. *We denote by $e^{tf} x_0$ the solution of the Cauchy problem $\dot{x} = f(x)$, $x(0) = x_0$ at time t.*

Definition 5.2.10. *Let $\{B_1, B_2, \ldots, B_n\}$, $n \in \mathbb{N}$ be a finite set of matrices. The Lie algebra associated to the set $\{B_1, B_2, \ldots, B_n\}$ is denoted by $\{B_1, B_2, \ldots, B_n\}_{LA}$ and is equal to the Lie algebra associated to the vector space $span\{B_1, B_2, \ldots, B_n\}$;*
i.e. $\{B_1, B_2, \ldots, B_n\}_{LA} = \{span\{B_1, B_2, \ldots, B_n\}\}_{LA}$.

Based on definition 5.2.10, we easily deduce the following corollary.

Corollary 5.2.11. *If* $span\{B_1, \ldots, B_n\} = span\{C_1, \ldots, C_m\}$ *then* $\{span\{B_1, \ldots, B_n\}\}_{LA} = \{span\{C_1, \ldots, C_m\}\}_{LA}$.

Theorem 5.2.12. *The system* (5.2) *is controllable if and only if the Lie algebra* $\{B_1, \ldots, B_m\}_{LA}$ *is transitive.*

Proof. We consider the system (5.2) where $\{B_1, \ldots, B_m\} \in \mathfrak{g}$ and $\mathfrak{g}$ is transitive Lie algebra. The control inputs are piecewise constant. Based on definition 5.2.9 the solution of the differential equation (5.2) for the initial condition x_0 is

$$x(t) = e^{t_1(u_1B_1+\ldots+u_mB_m)} \cdot \ldots \cdot e^{t_k(u_1B_1+\ldots+u_mB_m)} x_0 \tag{5.4}$$

where $t = t_1 + \ldots + t_k$. Since $\mathfrak{g}$ is a transitive Lie algebra which corresponds to the transitive matrix group G, it holds for $1 \le i \le k$, we have $e^{t_i(u_1B_1+\ldots+u_mB_m)} = G_i$, where $G_i \in G$. So, $x(t) = G_1 \ldots G_k x_0$ and $G_1 \ldots G_k = H \in G$. Thus, $x(t) = Hx_0$. By the definition of transitive matrix, we have for any $x_0, x_T \in \mathbb{R}^n \setminus \{0\}$ there exists a matrix $H \in G$ such that $x_T = Hx_0$. The system (5.2) can be driven from x_0 to x_T, in finite time T. This means that the system (5.2) is controllable. Conversely, if the system (5.2) is controllable, it can be driven from x_0 to x_T, in finite time T. Thus, there exists a matrix $H \in G$ such that $x_T = Hx_0$, for any $x_0, x_T \in \mathbb{R}^n \setminus \{0\}$. So, G is transitive. □

We conclude this section with the definition of the set of matrices which can be written as a Lie bracket of arbitrary length and contain as entry at least one matrix $A \in \Sigma_{\beta_1}$ and at least one matrix $B \in \Sigma_{\beta_2}$, where Σ_{β_1} and Σ_{β_2} are matrix vector spaces.

Definition 5.2.13. *Let* Σ_{β_1} *and* Σ_{β_2} *be two vector spaces, subspaces of the Lie algebra* $\{\Sigma_{\beta_1} \bigoplus \Sigma_{\beta_2}\}_{LA}$. *The set* $L^{(n)}(\Sigma_{\beta_1}, \Sigma_{\beta_2})$ *contains all the Lie brackets of* $n+1$ *matrices each, where at least one matrix* $A_1 \in \Sigma_{\beta_1}$, *at least one matrix* $A_2 \in \Sigma_{\beta_2}$ *and all the other matrices* $A_3, \ldots, A_{n+1}$ *belong to* $\Sigma_{\beta_1} \cup \Sigma_{\beta_2}$.

$L^{(1)}(\Sigma_{\beta_1}, \Sigma_{\beta_2}) = \{[A_1, A_2], [A_2, A_1]\}$, *where* $A_1 \in \Sigma_{\beta_1}$, $A_2 \in \Sigma_{\beta_2}$.
$L^{(2)}(\Sigma_{\beta_1}, \Sigma_{\beta_2}) = \{[[A_1, A_2], A_3], [[A_2, A_3], A_1], [[A_3, A_1], A_2], [A_1, [A_3, A_2]], \ldots\}$, *where* $A_1 \in \Sigma_{\beta_1}$, $A_2 \in \Sigma_{\beta_2}$, $A_3 \in \Sigma_{\beta_1} \cup \Sigma_{\beta_2}$.
$L^{(3)}(\Sigma_{\beta_1}, \Sigma_{\beta_2}) = \{[A_1, [A_2, [A_3, A_4]]], [A_2, [A_3, [A_4, A_1]]], [A_3, [A_4, [A_1, A_2]]], \ldots, [[A_1, A_3], [A_2, A_4]], \ldots\}$, *where* $A_1 \in \Sigma_{\beta_1}$, $A_2 \in \Sigma_{\beta_2}$, $A_3, A_4 \in \Sigma_{\beta_1} \cup \Sigma_{\beta_2}$.
$\vdots$

$L^{(i)}(\Sigma_{\beta_1}, \Sigma_{\beta_2}) = \{[A_1, [A_2, \ldots [A_i, A_{i+1}]]], [A_2, [A_3, \ldots [A_i, [A_{i+1}, A_1]]]],$
$[A_3, [A_4, \ldots [A_i, [A_{i+1}, [A_1, A_2]]]]], \ldots\}$, *where* $A_1 \in \Sigma_{\beta_1}$, $A_2 \in \Sigma_{\beta_2}$, $A_3, \ldots,$
$A_{i+1} \in \Sigma_{\beta_1} \cup \Sigma_{\beta_2}$.
$\vdots$
Finally, we define $L(\Sigma_{\beta_1}, \Sigma_{\beta_2}) = \bigcup\limits_{i=1}^{\infty} L^{(i)}(\Sigma_{\beta_1}, \Sigma_{\beta_2})$.

Definition 5.2.14. *Let* Σ_{β_1} *and* Σ_{β_2} *be two SMS. Let* $B_1 \in \Sigma_1$ *and* $B_2 \in \Sigma_2$. *We denote by D the set of iterated Lie brackets of* B_1 *and* B_2 *defined as follows:*
$D^{(0)} = \{B_1, B_2\}$,
$D^{(1)} = \{[B_1, B_2], [B_2, B_1]\}$
$D^{(2)} = \{[[B_1, B_2], B_1], [[B_1, B_2], B_2], [[B_2, B_1], B_1], [[B_2, B_1], B_2], [B_1, B_2],$
$[B_2, B_1]\}$
$\vdots$
$D^{(i)} = \{[H_k, H_j]\}$ *for all* $H_k, H_j \in D^{(0)} \cup \ldots \cup D^{(i-1)}$.
$\vdots$
So, $D(B_1, B_2) = \bigcup\limits_{j=0}^{\infty} D^{(j)}$.

Proposition 5.2.15. *If* $D^{(k)} \subseteq span\{D^{(0)} \cup \ldots \cup D^{(k-1)}\}$ *then* $D^{(m)} \subseteq span\{D^{(0)} \cup \ldots \cup D^{(k-1)}\}$, *where* $m \geq k$, $k, m \in \mathbb{N}$.

Proof. By definition, $D^{(k+1)} = \{[H_k, H_j]\}$ for all $H_k, H_j \in D^{(0)} \cup \ldots \cup D^{(k)}$. The assumption $D^{(k)} \subseteq span\{D^{(0)} \cup \ldots \cup D^{(k-1)}\}$ and the bilinearity of the Lie bracket implies that $span\{D^{(k+1)}\} = span\{D^{(k)}\}$. Thus, $D^{(k+1)} \subseteq span\{D^{(k+1)}\} = span\{D^{(k)}\} \subseteq span\{D^{(0)} \cup \ldots \cup D^{(k-1)}\}$. Inductively, we get that $D^{(m)} \subseteq span\{D^{(0)} \cup \ldots \cup D^{(k-1)}\}$. □

Lemma 5.2.16. *Let* Σ_{β_1} *and* Σ_{β_2} *be two SMS. Let* $B_1 \in \Sigma_1$ *and* $B_2 \in \Sigma_2$. *The set* $D^{(0)} \cup \ldots \cup D^{(n^2-1)}$ *contains a basis of* $D(B_1, B_2)$.

Proof. We assume B_1, B_2 linearly independent. Otherwise, $D(B_1, B_2) = \{B_1\}$. Thus, $dim\{D^{(0)}\} = 2$. If $D^{(i)} \not\subseteq span\{D^{(0)} \cup \ldots \cup D^{(i-1)}\}$, for $0 \leq i \leq n^2 - 1$, then $dim\{span\{D^{(0)} \cup \ldots \cup D^{(i)}\}\} \geq dim\{span\{D^{(0)} \cup \ldots \cup D^{(i-1)}\}\} + 1$. So, for $D^{(n^2-1)} \not\subseteq span\{D^{(0)} \cup \ldots \cup D^{(n^2-2)}\}$, we get $dim\{span\{D^{(0)} \cup \ldots \cup D^{(n^2)}\}\} = n^2$ which is the maximum number of linearly independent $n \times n$ matrices and the statement is proven. If for some $i \leq n^2 - 1$, $D^{(i)} \subseteq span\{D^{(0)} \cup \ldots \cup D^{(i-1)}\}$ then from proposition 5.2.15, $D^{(0)} \cup \ldots \cup D^{(i-1)}$ contains a basis of $D(B_1, B_2)$ and $D^{(0)} \cup \ldots \cup D^{(i-1)} \subseteq D^{(0)} \cup \ldots \cup D^{(n^2-1)}$.

□

5.3 Structural controllability of driftless bilinear sparse control systems

In this section, we wish to establish the basic theorems for the study of structural controllability of bilinear systems. First, we provide the following proposition which will be used in the proof of the following theorems:

Proposition 5.3.1. *Let $\Sigma_\beta \subset \mathfrak{g}$ be a subspace of dimension $n < \dim \mathfrak{g}$ and such that $\{\Sigma_\beta\}_{LA} = \mathfrak{g}$. Then there exists $B_1, B_2 \in \Sigma_\beta$ such that $\{B_1, B_2\}_{LA} = \mathfrak{g}$.*

Proof. We construct a sequence $\Sigma_\beta \supset \Sigma_\beta^1 \supset \Sigma_\beta^2 \supset \cdots \supset \Sigma_\beta^{n-2}$, where the codimension of Σ_β^i in Σ_β is i, such that $\{\Sigma_\beta^i\}_{LA} = \mathfrak{g}$. We construct this sequence by induction on i and using a contradiction argument. To this end, let us assume that there is no subspace $\Sigma_\beta^1 \subset \Sigma_\beta$ of codimension one such that $\{\Sigma_\beta^1\}_{LA} = \mathfrak{g}$. Using this assumption and since $\{\Sigma_\beta\}_{LA} = \mathfrak{g}$, it is evident that all subspaces of codimension one of Σ_β are subalgebras of $\mathfrak{g}$.

Furthermore, since the intersection of subalgebras is again a subalgebra, and because all codimension two subspaces of Σ_β can be described as the intersection of two codimension one subspaces of Σ_β, we have that all codimension two subspaces of Σ_β are subalgebras. Using this argument iteratively, we conclude that all two-dimensional subspaces of Σ_β are subalgebras.

Now let $e_1, \ldots, e_n$ be a basis of Σ_β. Because $\{\Sigma_\beta\}_{LA} = \mathfrak{g}$ but $\Sigma_\beta \neq \mathfrak{g}$, there exist two elements $z, w \in \Sigma_\beta$ so that $[z, w] \notin \Sigma_\beta$. Let us write $z = \sum_{i=1}^n \alpha_i e_i$ and $w = \sum_{i=1}^n \beta_i e_i$ for $\alpha_i, \beta_i \in \mathbb{R}$. Then

$$[z, w] = \sum_{j>i} (\alpha_i \beta_j - \alpha_j \beta_i)[e_i, e_j]$$

But we have shown that if no Σ_β^1 of codimension one in Σ_β is such that $\{\Sigma_\beta^1\}_{LA} = \mathfrak{g}$, then all dimension two subspaces of Σ_β are subalgebras. Hence for all pairs e_i, e_j, there exists $\gamma_{ij}, \delta_{ij} \in F$ such that

$$[e_i, e_j] = \gamma_{ij} e_i + \delta_{ij} e_j$$

Plugging this last relation in system 5.2, we obtain that $[z, w] \in \Sigma_\beta$, which is a contradiction.

We thus have that there exists $\Sigma_\beta^1 \subset \mathfrak{g}$ of dimension $n-1$ so that $\{\Sigma_\beta^1\}_{LA} = \mathfrak{g}$. Using the argument above inductively, we can show the existence of the Σ_β^i's described above. □

The following lemma can be easily deduced from the conclusion of proposition 5.3.1.

Lemma 5.3.2. *Let $\mathfrak{g}$ be a semi-simple Lie algebra. Then there exists a codimension one subspace $\Sigma_\beta \subset g$ with $\{\Sigma_\beta\}_{LA} = \mathfrak{g}$.*

At this point we have to note that 5.2.5 can be considered as a corollary of proposition 5.3.1. The result is proven by induction on the codimension of a subspace $\Sigma_\beta \subset \mathfrak{g}$ that is generating. The inductive step is obtained using Prop. 5.3.1 inductively and the base case using lemma 5.3.2. From 5.2.5 we deduce 5.2.6. Now, we provide the theorem about the minimum number of matrices needed for the realization of the driftless bilinear systems.

Theorem 5.3.3. *The system 5.2 with $B_\ell \in \Sigma_\beta$, $\ell \in \{1, \ldots, m\}$ is structurally controllable if and only if the system*

$$\dot{x}(t) = (uB_1 + vB_2)x(t) \tag{5.5}$$

is structurally controllable, where $B_1, B_2 \in \Sigma_\beta$.

Proof. If system (5.5) is structurally controllable then there exists $B_1, B_2 \in \Sigma_\beta$ so that $\{B_1, B_2\}_{LA} = \mathfrak{g}$ and $\{B_1, B_2, B_3, \ldots, B_m\}_{LA} \supseteq \mathfrak{g}$ for arbitrary $B_3, \ldots B_m$ which implies that system (5.2) is structurally controllable. If system (5.2) is structurally controllable then there exist $B_1, B_2, \ldots, B_m \in \Sigma_\beta$ such that $\{B_1, B_2, \ldots, B_m\}_{LA} = \mathfrak{g}$. Set $\Sigma = \operatorname{span}\{B_1, B_2, \ldots, B_m\}$. Then by assumption, $\{\Sigma\}_{LA} = \mathfrak{g}$. Then by proposition 5.3.1, there exists $B_1, B_2 \in \Sigma \subseteq \Sigma_\beta$ so that $\{B_1, B_2\}_{LA} = \mathfrak{g}$ which implies that system (5.5) is structurally controllable. □

We will prove that the controllability of the bilinear system 5.5 with $B_1 \in \Sigma_{\beta_1}$, $B_2 \in \Sigma_{\beta_2}$ is a generic property. This means that if a bilinear system is structurally controllable then for almost all pairs of matrices (B_1, B_2) in $\Sigma_{\beta_1} \times \Sigma_{\beta_2}$ the system (5.5) is controllable. With the term for almost all pairs (B_1, B_2) we mean all pairs (B_1, B_2) except from a zero measure set of pairs in $\Sigma_{\beta_1} \times \Sigma_{\beta_2}$.

Theorem 5.3.4. *If system (5.5) is structurally controllable then for almost all pairs of matrices $B_1 \in \Sigma_{\beta_1}$, $B_2 \in \Sigma_{\beta_2}$ the system (5.5) is controllable.*

Proof. Since system (5.5) is structurally controllable there exist $B_1 \in \Sigma_{\beta_1}$, $B_2 \in \Sigma_{\beta_2}$ such that system (5.5) is controllable. This implies that the set of iterated Lie brackets 5.2.14, of B_1 and B_2 denoted by $D(B_1, B_2)$, has rank $m = dim(\{B_1, B_2\}_{LA})$. As a result, there exist m linearly independent matrices Q_1, Q_2, ..., $Q_m \in D(B_1, B_2)$. For two arbitrary matrices $X \in \Sigma_{\beta_1}$, $Y \in \Sigma_{\beta_2}$, we consider the set of iterated Lie brackets 5.2.14, of X, Y $D(X, Y) = \{G_1(X, Y), \ldots, G_i(X, Y), \ldots\}$, and by $G_i(X, Y)$ we denote the matrices of the set of the iterated Lie brackets $D(X, Y)$. By construction, there exist indexes i_1, i_2, ..., i_m such that $G_{i_1}(B_1, B_2) = Q_1$, $G_{i_2}(B_1, B_2) = Q_2$, ..., $G_{i_m}(B_1, B_2) = Q_m$.

We establish a procedure which transforms each matrix $G_{i_1}(X, Y)$, ..., $G_{i_m}(X, Y)$ to $m \times 1$ vectors $v_1(X, Y)$, ..., $v_m(X, Y)$, respectively. If $\Sigma_{\beta_1}, \Sigma_{\beta_2} \subseteq \mathfrak{sl}(n)$ then $\{B_1, B_2\}_{LA} = \mathfrak{sl}(n)$; so the $n \times n$ matrix $G_{i_1}(X, Y)$ is turned to a $(n^2 - 1) \times 1$ column vector by placing the columns of $G_{i_1}(X, Y)$, one below the other in order, and the entry $G_{i_1}(X, Y)_{(n,n)}$ is omitted because it depends on the other diagonal entries; $G_{i_1}(X, Y)_{(n,n)} = -\sum_{l=1}^{n-1} G_{i_1}(X, Y)_{(l,l)}$. We follow the same process for the rest of the matrices $G_{i_2}(X, Y)$, ..., $G_{i_m}(X, Y)$.

If $aI \subseteq \Sigma_{\beta_1}, \Sigma_{\beta_2} \subseteq \mathfrak{so}(n) + aI$ then $\{B_1, B_2\}_{LA} = \mathfrak{so}(n) + aI$ and the $n \times n$ matrix $G_{i_1}(X, Y)$ is turned to a $(\frac{n(n-1)}{2} + 1) \times 1$ column vector in the following way. The first column of $G_{i_1}(X, Y)$ is preserved complete, while for the rest of the columns we keep only the entries (i, j) for which $i > j$, while the others are omitted. This is because $G_{i_1}(X, Y)_{(i,j)} = -G_{i_1}(X, Y)_{(j,i)}$ and all the diagonal elements are equal. Finally, we place the remaining parts of the columns, one below the other in order, and we form a $(\frac{n(n-1)}{2} + 1) \times 1$ column vector. We follow the same process for the rest of the matrices $G_{i_2}(X, Y)$, ..., $G_{i_m}(X, Y)$.

We define the polynomial $p : \Sigma_{\beta_1} \times \Sigma_{\beta_2} \to \mathbb{R}$, $p(X, Y) = det[v_1(X, Y), \ldots, v_m(X, Y)]$. By construction, $p(X, Y)$ is not permanently equal to a positive (or negative) constant. A polynomial is either identically zero or non-zero almost everywhere; see Federer (1969). As we mentioned, since (5.5) is structurally controllable, there exist $B_1 \in \Sigma_1$, $B_2 \in \Sigma_2$ such that $p(B_1, B_2) \neq 0$. Thus, $p(X, Y)$ is non-zero almost everywhere in $\Sigma_{\beta_1} \times \Sigma_{\beta_2}$. From the definition of p, it is clear that the system (5.5) is controllable for almost all pairs $B_1 \in \Sigma_{\beta_1}$, $B_2 \in \Sigma_{\beta_2}$. □

Remark 5.3.5. *There is an alternative way to define the polynomial* $p(X,Y) : \Sigma_{\beta_1}\times\Sigma_{\beta_2}\rightarrow\mathbb{R}$. *From lemma 5.2.16, we know that* $D^{(0)}\cup\ldots\cup D^{(n^2-1)}$ *contains a basis of* $D(X,Y)$. *We transform the matrices in* $D^{(0)}\cup\ldots\cup D^{(n^2-1)}$ *into vectors as described in theorem 5.3.4 and we create a matrix* M *whose columns are the vectors. It holds that* $rank\{M\} = rank\{MM^T\}$ *and* MM^T *is a square* $m\times m$ *matrix, where* $m = dim(\{\Sigma_{\beta_1}\bigoplus\Sigma_{\beta_2}\}_{LA})$. *Thus, we can define the polynomial as* $p(X,Y) = det(MM^T)$.

For $\Sigma_{\beta_1} = \Sigma_{\beta_2} = \Sigma_{\beta}$, we get the single pattern case. Theorem 5.3.4 can be easily extended for bilinear control systems with more than two matrices and more than two SMS Σ_{β_i}.

Definition 5.3.6. *We consider system* (5.2) *with* $B_\ell \in \Sigma_{\beta_1}\cup\Sigma_{\beta_2}\cup\ldots\cup\Sigma_{\beta_k}$, *where* $k\leq m$ *and* $\ell \in \{1,\ldots,m\}$, *with at least one matrix in each* Σ_{β_i}, $1\leq i\leq m$. *The system* (5.2) *is structurally controllable if and only if there exist* $B_\ell \in \Sigma_{\beta_1}\cup\Sigma_{\beta_2}\cup\ldots\cup\Sigma_{\beta_k}$, *where* $k\leq m$ *and* $\ell \in \{1,\ldots,m\}$ *with at least one matrix in each* Σ_{β_i}, $1\leq i\leq m$, *such that the system* (5.2) *is controllable.*

Theorem 5.3.7. *We consider the SMS* Σ_{β_1}, ..., Σ_{β_k}. *If system* (5.2) *with* $B_1\in\Sigma_{\beta_{i_1}}$, ..., $B_m\in\Sigma_{\beta_{i_m}}$, $i_1,\ldots,i_m\in\{1,\ldots,k\}$, $k\leq m$ *with at least one matrix in each* Σ_{β_i}, $1\leq i\leq k$, *is controllable, then system* (5.2) *is controllable for almost all tuples* $(B_1,\ldots,B_m)$ *in* $\Sigma_{\beta_{i_1}}\times\ldots\times\Sigma_{\beta_{i_m}}$.

In the following two sections, we investigate the structural controllability of bilinear control systems for the single pattern case as well as for the multiple pattern case. We introduce the notion of the graph corresponding to one vector space or more than one matrix vector spaces, for the single pattern and the multiple patterns case respectively. We provide theorems that relate the structural controllability of the bilinear system with the connectivity of the graph. For undirected graphs we talk about simple connectivity, for directed graphs we talk about strong connectivity, and for colored edges graphs we talk about connectivity through paths with at least one alternation in color. If the matrices of bilinear systems are dense, the connectivity is almost always guaranteed. The study of sparse matrix spaces is of particular interest because we need to find the necessary and sufficient conditions that guarantee the connectivity of the corresponding graph.

5.4 The single pattern case

In this section, we investigate the structural controllability of bilinear control systems for the single pattern case. First, we provide the following useful definitions.

Definition 5.4.1. *The graph G corresponding to the vector space Σ_β of $n{\times}n$ matrices is defined as the directed graph of n vertices $v_1, v_2, \ldots, v_n$ for which the edge $\vec{v_i v_j} \in E(G)$ if and only if there exists a matrix $A \in \Sigma_\beta$ such that for the entry $A(i,j)$ it holds that $A(i,j){\neq}0$.*

Definition 5.4.2. *Let Σ_β be an SMS. We define ${\Sigma_\beta}^{(0)}{:=}\Sigma_\beta$ and ${\Sigma_\beta}^{(i+1)}{:=}$ ${\Sigma_\beta}^{(i)}\bigoplus[{\Sigma_\beta}^{(i)}, {\Sigma_\beta}^{(i)}]$, where $[{\Sigma_\beta}^{(i)}, {\Sigma_\beta}^{(i)}] = \{[A,B] | A, B {\in} {\Sigma_\beta}^{(i)}\}$.*

Definition 5.4.3. *Let Σ_β be an SMS. Let G the digraph corresponding to Σ_β. We define $G^{(0)}{:=}G$ and $G^{(i+1)}$ is the graph taken from $G^{(i)}$ after one-step transitive closure, see Fig. 5.1.*

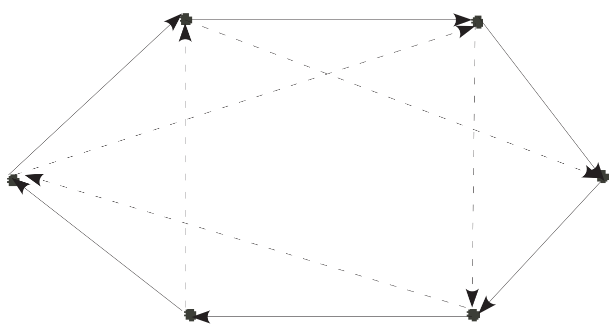

FIGURE 5.1: The graph obtained by the plain and dashed edges is the one-step transitive closure of the graph which consists of the plain edges.

At this point, we would like to make clear that the matrix $E_{(i,j)}$ denotes the matrix with zero in all entries, except from the entry (i,j) which is equal to 1. The (i,j) entry of a matrix A is denoted by $A(i,j)$. Now, we establish the correspondence between graph $G^{(i)}$ and the matrix vector space ${\Sigma_\beta}^{(i)}$.

Proposition 5.4.4. *Given an SMS $\Sigma_\beta{\subseteq}\mathfrak{sl}(n)$, the corresponding SMS of $G^{(i)}$ is ${\Sigma_\beta}^{(i)}$.*

Proof. We proceed by induction on i.
Induction Basis (i=0): We have ${\Sigma_\beta}^{(0)}{:=}\Sigma_\beta$ and let $G^{(0)}$ the digraph corresponding to ${\Sigma_\beta}^{(0)}$.

Let ${\Sigma_\beta}^{(1)}{:=}{\Sigma_\beta}^{(0)}\bigoplus[{\Sigma_\beta}^{(0)}, {\Sigma_\beta}^{(0)}]$ and let $G^{(1)}$ be the digraph resulting from the one step closure of $G^{(0)}$. We prove that $G^{(1)}$ is the corresponding graph of the SMS ${\Sigma_\beta}^{(1)}$.

First, we prove that for every edge $v_i\vec{v}_j$ $i{\neq}j$ there exists a matrix A $\in {\Sigma_\beta}^{(1)}$ such that $A(i,j){\neq}0$. For an edge $v_i\vec{v}_j \in E(G^{(1)})$ and $v_i\vec{v}_j \in E(G^{(0)})$ then $E_{(i,j)} \in{\Sigma_\beta}^{(0)} \Rightarrow E_{(i,j)} \in{\Sigma_\beta}^{(1)}$. So, let us assume that for v_i, $v_j \in V(G^{(0)})$ the edge $v_i\vec{v}_j \in E(G^{(1)})$ and $v_i\vec{v}_j \notin E(G^{(0)})$. Then there exist $v_k \in V(G^{(0)})$ such that $v_i\vec{v}_k \in E(G^{(0)})$ and $v_k\vec{v}_j \in E(G^{(0)})$. So, for the matrices $E_{(i,k)}$, $E_{(k,j)} \in {\Sigma_\beta}^{(0)} \Rightarrow [E_{(i,k)}, E_{(k,j)}] = E_{(i,j)}{\in}{\Sigma_\beta}^{(1)}$. So, in both cases, namely $v_i\vec{v}_j \notin E(G^{(0)})$ and $v_i\vec{v}_j \in E(G^{(0)})$, we have that $E_{(i,j)}{\in}\Sigma_\beta^{(1)}$, where ${\Sigma_\beta}^{(1)} = {\Sigma_\beta}^{(0)}\bigoplus[{\Sigma_\beta}^0, {\Sigma_\beta}^0]$.

Now, we examine the special case of the self-loop edge. If the self-loop $v_i\vec{v}_i \in E(G^{(0)})$, then by definition there exist matrix $A{\in}{\Sigma_\beta}^{(0)}{\Rightarrow}$ $A{\in}{\Sigma_\beta}^{(1)}$, such that $A(i,i){\neq}0$. If the self-loop $v_i\vec{v}_i \notin E(G^{(0)})$ and $v_i\vec{v}_i \in E(G^{(1)})$ then there exist edges $v_i\vec{v}_j$ and $v_j\vec{v}_i$ in $E(G^{(0)})$. So, $E_{(i,j)}$, $E_{(j,i)}$ $\in {\Sigma_\beta}^{(0)} \Rightarrow [E_{(i,j)}, E_{(j,i)}] = E_{(i,i)} - E_{(j,j)}{\in}{\Sigma_\beta}^{(1)}$.

Second, we prove that if $E_{(i,j)}{\in}{\Sigma_\beta}^{(1)} = {\Sigma_\beta}^{(0)}\bigoplus[{\Sigma_\beta}^0, {\Sigma_\beta}^0]$ then the edge $v_i\vec{v}_j \in E(G^{(1)})$. If $E_{(i,j)}{\in}\ \Sigma_\beta^{(0)}$ then by definition the edge $v_i\vec{v}_j \in E(G^{(0)})$ and since $G^{(1)}$ is the one step transitive closure of $G^{(0)}$, we have that $v_i\vec{v}_j \in E(G^{(1)})$. If $E_{(i,j)}{\notin}\ {\Sigma_\beta}^{(0)} \Rightarrow E_{(i,j)}{\in}[{\Sigma_\beta}^{(0)}, {\Sigma_\beta}^{(0)}] \Rightarrow \exists$ $E_{(i,k)}$, $E_{(k,j)} \in {\Sigma_\beta}^{(0)}$ such that $E_{(i,j)} = [E_{(i,k)}, E_{(k,j)}]$. Since $E_{(i,k)}, E_{(k,j)}{\in}$ ${\Sigma_\beta}^{(0)}$ then $v_i\vec{v}_k$ and by definition $v_k\vec{v}_j \in E(G^{(0)}) \Rightarrow$ so from one step transitive closure $v_i\vec{v}_j \in E(G^{(1)})$.

Induction step: Let us assume that the argument holds for $i = m$; we will prove that it also holds for $i = m+1$. By definition, ${\Sigma_\beta}^{(m+1)} = {\Sigma_\beta}^{(m)}\bigoplus[{\Sigma_\beta}^{(m)}, {\Sigma_\beta}^{(m)}]$, and $G^{(m+1)}$ is the one step transitive closure of $G^{(m)}$. So, if we consider $G^{(m)}$ as a new $G^{(0)}$ and ${\Sigma_\beta}^{(m)}$ as a new ${\Sigma_\beta}^{(0)}$ we can follow the procedure described in the basis step and we can prove that $G^{(m+1)}$ is graph that corresponds to ${\Sigma_\beta}^{(m+1)}$. This concludes the induction. □

If the one step transitive closure does not increase the number of edges, we get the final graph $\bar{G}_{\Sigma_\beta}$. By definition, the graph $\bar{G}_{\Sigma_\beta}$ corresponds to $\{\Sigma_\beta\}_{LA}$. Based on proposition 5.4.4, we are ready to establish the following corollary.

Corollary 5.4.5. *The complete directed graph with self-loop at each vertex is denoted by $K_{d,n}$. $\bar{G}_{\Sigma_\beta} = K_{d,n}$ if and only if $\{\Sigma_\beta\}_{LA} = \mathfrak{sl}(n)$.*

Proof. Let us assume that $\bar{G}_{\Sigma_\beta} = K_{d,n}$. We can see that since ${\Sigma_\beta}^{(0)}$ has trace zero then every ${\Sigma_\beta}^{(i)}$ has trace zero, because the Lie bracket of traceless matrices is a traceless matrix. Given $\Sigma_\beta \subseteq \mathfrak{sl}(n)$, based on proposition 5.4.4, we can see that the graph $\bar{G}_{\Sigma_\beta} = K_{d,n}$ corresponds to $\mathfrak{sl}(n)$. This is because, for every edge $v_i\vec{v}_j \in E(\bar{G}_{\Sigma_\beta})$ with $i \neq j$, there exists the matrix $E_{(i,j)} \in \mathfrak{sl}(n)$. For a self-loop $v_i\vec{v}_i \in E(\bar{G}_{\Sigma_\beta})$, there exists the matrix $E_{(i,i)} - E_{(j,j)} \in \mathfrak{sl}(n)$, $i \neq j$. Conversely, if $\{\Sigma_\beta\}_{LA} = \mathfrak{sl}(n)$, from proposition 5.4.4, we have that for the matrix $E_{(i,j)} \in \mathfrak{sl}(n)$ there exists a directed edge $v_i\vec{v}_j \in E(\bar{G}_{\Sigma_\beta})$. For the matrix $E_{(i,i)} - E_{(j,j)} \in \mathfrak{sl}(n)$ there exists a self-loop $v_i\vec{v}_i \in E(\bar{G}_{\Sigma_\beta})$. Thus, $\bar{G}_{\Sigma_\beta} = K_{d,n}$. □

Theorem 5.4.6. *$\bar{G}_{\Sigma_\beta} = K_{d,n}$ if and only if the system (5.5) with $\Sigma_\beta \subseteq \mathfrak{sl}(n)$ is controllable for almost every pair B_1, B_2 in Σ_β.*

Proof. From corollary 5.4.5, $\bar{G}_{\Sigma_\beta} = K_{d,n}$ if and only if $\{\Sigma_\beta\}_{LA} = \mathfrak{sl}(n)$. From lemma 5.2.7 we have that $\mathfrak{sl}(n)$ is a transitive Lie algebra. Thus, $\{\Sigma_\beta\}_{LA} = \mathfrak{sl}(n)$ if and only if there exist B_1, B_2 in Σ_β such that system 5.5 is controllable. From theorem 5.3.4 the controllability is a generic property for bilinear systems. Thus, we have that $\{\Sigma_\beta\}_{LA} = \mathfrak{sl}(n)$ if and only if system 5.5 is controllable for almost every pair B_1, B_2 in Σ_β. □

At this point, we have a result which relates the structure of the final graph G_{Σ_β} corresponding to $\{\Sigma_\beta\}_{LA}$ with the controllability of system (5.5). Based on graph theoretic results, we would like to relate the structure of the initial graph G corresponding to Σ_β with the controllability of system (5.5). To this end, we introduce the following lemma.

Lemma 5.4.7. *$\bar{G}_{\Sigma_\beta} = K_{d,n}$ if and only if the initial graph $G^{(0)}$ is strongly connected.*

Proof. For the necessity, if $G^{(0)}$ is strongly connected then $\forall\ v_i$, $v_j \in V(G^{(0)})$, $1 \leq i, j \leq n$, there exists a directed path from v_i to v_j. Under transitive closure the path is reduced to a directed edge connecting v_i to v_j. Thus, if the initial graph $G^{(0)}$ is strongly connected then under transitive closure we get a complete directed final graph. For the sufficiency, let us assume that $\bar{G}_{\Sigma_\beta} = K_{d,n}$. In $\bar{G}_{\Sigma_\beta}$, we consider a directed edge $v_i\vec{v}_j$ where v_i, $v_j \in V(G^{(0)})$. If the edge $v_i\vec{v}_j$ existed in the initial graph $G^{(0)}$ we have a path between v_i and v_j in $G^{(0)}$. If the edge

$v_i\vec{v}_j$ appeared in the k-th step of the transitive closure then in the (k-1)-th step there exists vertex $v_k \in G^{(k-1)}$ such that the directed edges $v_i\vec{v}_k$, $v_k\vec{v}_j \in E(G^{(k-1)})$. Moving backwards, we repeat recursively the same argument for the pair of vertices v_i, v_k and for the pair of vertices v_k, v_j. We conclude that the initial graph contains a directed path from v_i to v_j. Thus, the initial graph is strongly connected. □

Now, we are ready to provide the following theorem.

Theorem 5.4.8. *The initial graph $G^{(0)}$ corresponding to $\Sigma_\beta \subseteq \mathfrak{sl}(n)$ is strongly connected if and only if the system (5.5) is controllable for almost all pairs B_1, B_2 in Σ_β.*

Proof. Based on lemma 5.4.7, $G^{(0)}$ is strongly connected if and only if $\bar{G}_{\Sigma_\beta} = K_{d,n}$. From theorem 5.4.6 $\bar{G}_{\Sigma_\beta} = K_{d,n}$, if and only if system (5.5) is controllable for almost all pairs B_1, B_2 in $\Sigma_\beta \subseteq \mathfrak{sl}(n)$. □

5.4.1 The $\mathfrak{so}(n) + aI$ case

Slightly easier is the examination of the case of the controllability of bilinear systems where $aI \subseteq \Sigma_\beta \subseteq so(n) + aI$, $a \in \mathbb{R}$. In this case, we work on undirected graphs. Similar to definition 5.4.1, we give the following definition.

Definition 5.4.9. *Let $aI \subseteq \Sigma_\beta \subseteq so(n) + aI$, $a \in \mathbb{R}$ be an SMS. The graph G, corresponding to the vector space Σ_β of $n \times n$ matrices, is defined as the undirected graph of n vertices $v_1, v_2, \ldots, v_n$ for which the edge $v_i v_j \in E(G)$ if and only if there exists a matrix $A \in \Sigma_\beta$ for which it holds that $A(i,j) \neq 0$ and $A(j,i) \neq 0$.*

The definitions 5.4.2 and 5.4.3 remain unchanged given that we consider the definition of the one-step transitive closure for the case of undirected graphs, which means that we do not care about the direction of the edges but only about the existence. Following a similar procedure as in proposition 5.4.4, we establish the correspondence between ${\Sigma_\beta}^{(i)}$ and $G^{(i)}$.

Proposition 5.4.10. *Given an SMS $aI \subseteq \Sigma_\beta \subseteq so(n) + aI$, $a \in \mathbb{R}$, the corresponding SMS of $G^{(i)}$ is ${\Sigma_\beta}^{(i)}$.*

Proof. We proceed by induction on i.
Induction Basis (i=0): We have ${\Sigma_\beta}^{(0)} := \Sigma_\beta$ and let $G^{(0)}$ be the graph

corresponding to ${\Sigma_\beta}^{(0)}$. Let ${\Sigma_\beta}^{(1)}{:=}{\Sigma_\beta}^{(0)}\bigoplus[{\Sigma_\beta}^{(0)}, {\Sigma_\beta}^{(0)}]$ and let $G^{(1)}$ be the graph resulting from the one step closure of $G^{(0)}$. We prove that $G^{(1)}$ is the corresponding graph of the SMS ${\Sigma_\beta}^{(1)}$. First, we prove that for every edge v_iv_j $i{\neq}j$ there exists a matrix A $\in {\Sigma_\beta}^{(1)}$ such that $A(i,j){\neq}0$ and $A(j,i){\neq}0$. For an edge $v_iv_j \in E(G^{(1)})$ and $v_iv_j \in E(G^{(0)})$ then $E_{(i,j)} + E_{(j,i)} \in{\Sigma_\beta}^{(0)}{\Rightarrow}E_{(i,j)} + E_{(j,i)} \in{\Sigma_\beta}^{(1)}$. So, let us assume that for v_i, $v_j \in V(G^{(0)})$ the edge $v_iv_j \in E(G^{(1)})$ and $v_iv_j \notin E(G^{(0)})$. Then there exist $v_k \in V(G^{(0)})$ such that $v_iv_k \in E(G^{(0)})$ and $v_kv_j \in E(G^{(0)})$. This implies that the matrices $A_1 = E_{(i,k)} + E_{(k,i)}$ and $A_2 = E_{(k,j)} + E_{(j,k)}$ are in ${\Sigma_\beta}^{(0)}$. So, $[E_{(i,k)} + E_{(k,i)}, E_{(k,j)} + E_{(j,k)}] = [E_{(i,k)}, E_{(k,j)}]{+}[E_{(i,k)}, E_{(j,k)}]{+}[E_{(k,i)}, E_{(k,j)}]{+}[E_{(k,i)}, E_{(j,k)}] = E_{(i,j)}{+}E_{(j,i)}$. Thus, $E_{(i,j)} + E_{(j,i)} \in{\Sigma_\beta}^{(1)}$. So, in both cases, namely $v_iv_j \notin E(G^{(0)})$ and $v_iv_j \in E(G^{(0)})$, we have that $E_{(i,j)} + E_{(j,i)}{\in}\Sigma_{\beta_1}$, where ${\Sigma_\beta}^{(1)} = {\Sigma_\beta}^{(0)}\bigoplus[{\Sigma_\beta}^0, {\Sigma_\beta}^0]$. For the self-loop v_iv_i, we know that $aI{\in}{\Sigma_\beta}^{(0)}$ and since ${\Sigma_\beta}^{(1)}{:=}{\Sigma_\beta}^{(0)}\bigoplus[{\Sigma_\beta}^{(0)}, {\Sigma_\beta}^{(0)}]$, it holds that $aI{\in}{\Sigma_\beta}^{(1)}$.

Second, we prove that for $i{\neq}j$ if $E_{(i,j)} + E_{(j,i)} \in{\Sigma_\beta}^{(1)} = {\Sigma_\beta}^{(0)}\bigoplus [{\Sigma_\beta}^0, {\Sigma_\beta}^0]$ then the edge $v_iv_j \in E(G^{(1)})$. If $E_{(i,j)} + E_{(j,i)} \in {\Sigma_\beta}^{(0)}$ then by definition the edge $v_iv_j \in E(G^{(0)})$ and since $G^{(1)}$ is the one step transitive closure of $G^{(0)}$, we have that $v_iv_j \in E(G^{(1)})$. If $E_{(i,j)} + E_{(j,i)} \notin {\Sigma_\beta}^{(0)} \Rightarrow E_{(i,j)} + E_{(j,i)} \in[{\Sigma_\beta}^{(0)}, {\Sigma_\beta}^{(0)}] \Rightarrow \exists\, E_{(i,k)} + E_{(k,i)}, E_{(j,k)} + E_{(k,j)} \in {\Sigma_\beta}^{(0)}$ that give $E_{(i,j)} = [E_{(i,k)} + E_{(k,i)}, E_{(j,k)} + E_{(k,j)}]$. So, v_iv_k and $v_kv_j \in E(G^{(0)})$. So from one step transitive closure $v_iv_j \in E(G^{(1)})$. For the matrix $aI{\in}{\Sigma_\beta}^{(1)}$ we know that $G^{(0)}$ contains all self-loops and from one-step transitive closure $G^{(1)}$ contains all self-loops.

Induction step: Let us assume that the argument holds for $i = m$. By definition, ${\Sigma_\beta}^{(m+1)} = {\Sigma_\beta}^{(m)}\bigoplus[{\Sigma_\beta}^{(m)}, {\Sigma_\beta}^{(m)}]$, and $G^{(m+1)}$ is the one step transitive closure of $G^{(m)}$. So, if we consider $G^{(m)}$ as a new $G^{(0)}$ and ${\Sigma_\beta}^{(m)}$ as a new ${\Sigma_\beta}^{(0)}$, we can follow the procedure described in the basis step and we can prove that $G^{(m+1)}$ is the graph that corresponds to ${\Sigma_\beta}^{(m+1)}$. This concludes the induction. □

Corollary 5.4.11. *$\bar{G}_{\Sigma_\beta} = K_n$ if and only if $\mathfrak{so}(n) + aI = \{\Sigma_\beta\}_{LA}$, $a{\in}\mathbb{R}$. By K_n we denote the complete graph with self-loop at each vertex.*

Proof. Let us assume that $\bar{G}_{\Sigma_\beta} = K_n$. Given $aI{\subseteq}\Sigma_\beta{\subseteq}\mathfrak{so}(n){+}aI$, based on proposition 5.4.10, we can see that the graph $\bar{G}_{\Sigma_\beta} = K_n$ corresponds to $\mathfrak{so}(n) + aI$. This is because, for every edge $v_i\vec{v}_j{\in}E(\bar{G}_{\Sigma_\beta})$ with $i{\neq}j$, there exists the matrix $E_{(i,j)} + E_{(j,i)}{\in}\mathfrak{so}(n) + aI$. For a self-loop $v_i\vec{v}_i \in E(\bar{G}_{\Sigma_\beta})$, there exists the matrix $aI \in \mathfrak{so}(n) + aI$. Conversely, if $\{\Sigma_\beta\}_{LA} = \mathfrak{so}(n) + aI$, from proposition 5.4.10, we have that for the matrix $E_{(i,j)} + E_{(j,i)} \in \mathfrak{so}(n) + aI$ there exists a directed edge $v_i\vec{v}_j \in E(\bar{G}_{\Sigma_\beta})$. For a matrix $aI \in \mathfrak{so}(n) + aI$ there exist the self-loops $v_i\vec{v}_i \in E(\bar{G}_{\Sigma_\beta})$, $1 \leq i \leq n$. Thus, $\bar{G}_{\Sigma_\beta} = K_n$. □

Theorem 5.4.12. *$\bar{G}_{\Sigma_\beta} = K_n$ if and only if the system (5.5) with $aI \subseteq \Sigma_\beta \subseteq \mathfrak{so}(n) + aI$ is controllable for almost every pair B_1, B_2 in Σ_β.*

Proof. From corollary 5.4.11, $\bar{G}_{\Sigma_\beta} = K_n$ if and only if $\{\Sigma_\beta\}_{LA} = \mathfrak{so}(n) + aI$. From lemma 5.2.8 we have that $\mathfrak{so}(n) + aI$ is a transitive Lie algebra. Thus, $\{\Sigma_\beta\}_{LA} = \mathfrak{so}(n) + aI$ if and only if there exist B_1, B_2 in Σ_β such that system (5.5) is controllable. From theorem 5.3.4 the controllability is a generic property for bilinear systems. Thus, we have that $\{\Sigma_\beta\}_{LA} = \mathfrak{so}(n) + aI$ if and only if system (5.5) is controllable for almost every pair B_1, B_2 in Σ_β. □

Furthermore, the graph theoretic result of lemma 5.4.7 is updated to lemma 5.4.13.

Lemma 5.4.13. *$G_{\Sigma_\beta} = K_n$ if and only if the initial graph $G^{(0)}$ is connected, where K_n is the complete graph with self-loop at each vertex.*

Proof. Since $aI \subseteq \Sigma_\beta \subseteq so(n) + aI$, $a \in \mathbb{R}$ every graph has a self-loop at each vertex. The remaining part of the proof is the same as in lemma 5.4.7, given that we care about a path, not a directed path, between two vertices v_i, $v_j \in V(G^{(0)})$. □

Now, we can provide the following theorem which is equivalent to theorem 5.4.8 for $aI \subseteq \Sigma_\beta \subseteq \mathfrak{so}(n) + aI$, $a \in \mathbb{R}$.

Theorem 5.4.14. *Let $aI \subseteq \Sigma_\beta \subseteq \mathfrak{so}(n) + aI$, $a \in \mathbb{R}$. The initial graph $G^{(0)}$ corresponding to Σ_β is connected if and only if the system (5.5) is controllable for almost all pairs B_1, B_2 in Σ_β.*

Proof. Based on lemma 5.4.13, $G^{(0)}$ is connected if and only if $G_{\Sigma_\beta} = K_n$. From theorem 5.4.12, $G_{\Sigma_\beta} = K_n$ if and only if system (5.5) is controllable for almost all pairs B_1, B_2 in Σ_β. □

5.5 Multiple patterns case

In this section, we investigate the case of multiple patterns, meaning that in system (5.2) we have $B_\ell \in \Sigma_{\beta_1} \cup \Sigma_{\beta_2} \cup \ldots \cup \Sigma_{\beta_k}$, where $k \leq m$ and

$\ell \in \{1, \ldots, m\}$. First, we extended proposition 5.3.1 for two patterns $\Sigma_{\beta_1}, \Sigma_{\beta_2}$.

5.5.1 On the realization of bilinear systems with multiple patterns

Proposition 5.5.1. *Let Σ_{β_1}, Σ_{β_2} be two SMS. Let $\Sigma_{\beta_1} \bigoplus \Sigma_{\beta_2} \subset \mathfrak{g}$ be a subspace of dimension $n < dim\mathfrak{g}$ and such that $\{\Sigma_{\beta_1} \bigoplus \Sigma_{\beta_2}\}_{LA} = \mathfrak{g}$. Then there exist $B_1, B_2 \in \Sigma_{\beta_1}$ and $B_3, B_4 \in \Sigma_{\beta_2}$ such that $\{B_1, B_2, B_3, B_4\}_{LA} = \mathfrak{g}$.*

Proof. According to definition 5.2.13, we have
$\{\Sigma_{\beta_1} \bigoplus \Sigma_{\beta_2}\}_{LA} = \{\Sigma_{\beta_1}\}_{LA} \bigoplus \{\Sigma_{\beta_2}\}_{LA} \bigoplus \{L(\Sigma_{\beta_1}, \Sigma_{\beta_2})\}_{LA}$. From proposition 5.3.1 there exist B_1, $B_2 \in \Sigma_{\beta_1}$ such that $\{\Sigma_{\beta_1}\}_{LA} = \{B_1, B_2\}_{LA}$ and there exist B_3, $B_4 \in \Sigma_{\beta_2}$ such that $\{\Sigma_{\beta_2}\}_{LA} = \{B_3, B_4\}_{LA}$. Furthermore, we prove that
$\{L(\Sigma_{\beta_1}, \Sigma_{\beta_2})\}_{LA} \subseteq \{B_1, B_2, B_3, B_4\}_{LA}$: Let u $\in \{L(\Sigma_{\beta_1}, \Sigma_{\beta_2})\}_{LA} \Rightarrow$
$u = \sum_{i=1}^{k} c_i A_i$, where $A_i \in L(\Sigma_{\beta_1}, \Sigma_{\beta_2})$. A_i is a Lie bracket of arbitrary length which contains elements from Σ_{β_1} and from Σ_{β_2}. In addition, $\Sigma_{\beta_1} \subseteq \{B_1, B_2\}_{LA}$ and $\Sigma_{\beta_2} \subseteq \{B_3, B_4\}_{LA} \Rightarrow A_i \in \{B_1, B_2, B_3, B_4\}_{LA}$ $\Rightarrow u \in \{B_1, B_2, B_3, B_4\}_{LA}$. So, after substitution $\{\Sigma_{\beta_1} \bigoplus \Sigma_{\beta_2}\} = \{B_1, B_2\}_{LA}$ $\bigoplus \{B_3, B_4\}_{LA} \bigoplus \{B_1, B_2, B_3, B_4\}_{LA} = \{B_1, B_2, B_3, B_4\}_{LA}$. □

We can see that there exist no matrices $A \in \Sigma_{\beta_1}$ and $B \in \Sigma_{\beta_2}$ such that $\{A, B\}_{LA} = \mathfrak{g}$. Furthermore, there exist no $A_1, A_2 \in \Sigma_{\beta_1}$ and $B_1 \in \Sigma_{\beta_2}$ such that $\{A_1, A_2, B_1\}_{LA} = \mathfrak{g}$; neither $A_1 \in \Sigma_{\beta_1}$ and $B_1, B_2 \in \Sigma_{\beta_2}$ such that $\{A_1, B_1, B_2\}_{LA} = \mathfrak{g}$. As a counterexample, we provide the following two vector spaces:

$$\Sigma_{\beta_1} = \begin{bmatrix} 0 & * & 0 & 0 & 0 & 0 \\ 0 & 0 & * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Sigma_{\beta_2} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Based on proposition 5.5.1, we have the following theorem which extends theorem 5.3.3 for the two pattern case.

Theorem 5.5.2. *The system (5.2) where $B_\ell \in \Sigma_{\beta_1} \cup \Sigma_{\beta_2}$ where $\ell \in \{1, \ldots, m\}$, $m \geq 4$, with at least two matrices in Σ_{β_1} and at least two matrices in Σ_{β_2}, is* *structurally controllable if and only if the system*

$$\dot{x} = u_1B_1x + u_2B_2x + u_3B_3x + u_4B_4x \tag{5.6}$$

is structurally controllable, with $B_1, B_2 \in \Sigma_{\beta_1}$ *and* $B_3, B_4 \in \Sigma_{\beta_2}$.

Proof. Let $\{\Sigma_{\beta_1} \bigoplus \Sigma_{\beta_2}\}$. If system (5.6) is structurally controllable then there exist $B_1, B_2 \in \Sigma_{\beta_1}$ and $B_3, B_4 \in \Sigma_{\beta_2}$ so that $\{B_1, B_2, B_3, B_4\}_{LA} = \mathfrak{g}$ and as a result $\{B_1, B_2, B_3, B_4, \ldots, B_m\} \supseteq \mathfrak{g}$ for arbitrary B_5, ..., B_m which implies that system (5.2) is structurally controllable. For the converse, if system (5.2) is structurally controllable then there exist B_1, B_2, ..., B_m in $\Sigma_{\beta_1} \cup \Sigma_{\beta_2}$ with at least two matrices in Σ_{β_1} and at least two matrices in Σ_{β_2} such that $\{B_1, B_2, \ldots, B_m\}_{LA} = \mathfrak{g}$. Set $\Sigma = \Sigma_{\beta_1} \bigoplus \Sigma_{\beta_2}$. Then by proposition 5.5.1, there exist $B_1, B_2 \in \Sigma_{\beta_1}$ and $B_3, B_4 \in \Sigma_{\beta_2}$ so that $\{B_1, B_2, B_3, B_4\}_{LA} = \mathfrak{g}$ which implies that system (5.6) is structurally controllable. □

If in system (5.2) there exist only one matrix in Σ_{β_1} and at least two matrices in Σ_{β_2} then system (5.2) is structurally controllable if and only if the system

$$\dot{x} = u_1B_1x + u_3B_3x + u_4B_4x \tag{5.7}$$

is structurally controllable with $B_1 \in \Sigma_{\beta_1}$ and $B_3, B_4 \in \Sigma_{\beta_2}$. Similarly, if in system (5.2) there exist only one matrix in Σ_{β_2} and at least two matrices in Σ_{β_1} then system (5.2) is structurally controllable if and only if the system

$$\dot{x} = u_1B_1x + u_2B_2x + u_3B_3x \tag{5.8}$$

is structurally controllable with $B_1, B_2 \in \Sigma_{\beta_1}$ and $B_3 \in \Sigma_{\beta_2}$.

We observe that theorem 5.5.2 is useful when $m \geq 5$. We can easily extend theorem 5.5.2 to cases of more than two patterns using induction on the number of patterns with basis step theorem 5.5.2; we need two matrices for each different pattern.

Theorem 5.5.3. *The system* (5.2) *where* $B_\ell \in \Sigma_{\beta_1} \cup \Sigma_{\beta_2} \cup \ldots \cup \Sigma_{\beta_k}$, *where* $k \leq m$ *and* $\ell \in \{1, \ldots, m\}$, *with at least two matrices in each* Σ_{β_i}, $1 \leq i \leq k$, *is structurally controllable if and only if the system*

$$\dot{x} = u_{\beta_1,1}B_{\beta_1,1}x + u_{\beta_1,2}B_{\beta_1,2}x + u_{\beta_2,1}B_{\beta_2,1}x + u_{\beta_2,2}B_{\beta_2,2}x + \dots + u_{\beta_k,1}B_{\beta_k,1}x + u_{\beta_k,2}B_{\beta_k,2}x \quad (5.9)$$

is structurally controllable, with $B_{\beta_i,1}, B_{\beta_i,2} \in \Sigma_{\beta_i}$, $1 \leq i \leq k$.

Proof. We proceed by induction on k. The induction basis $k = 2$ is proven in theorem 5.5.2. For the induction step let us assume that the hypothesis holds for k; we prove that it also holds for $k+1$. Let us consider a bilinear system where $B_\ell \in \Sigma_{\beta_1} \cup \Sigma_{\beta_2} \cup \dots \cup \Sigma_{\beta_{k+1}}$, where $k+1 \leq m$ and $\ell \in \{1, \dots, m\}$, with at least two matrices in each Σ_{β_i}, $1 \leq i \leq k$. Let us assume that there exists one Σ_{β_i} such that there exist at least 3 matrices that belong to it. Without loss of generality, let us assume that $i = k+1$. If such a Σ_{β_i} does not exist then the statement is true by definition. So, by proposition 5.3.1, system (5.2) is structurally controllable if and only if the system

$$\dot{x} = \sum_{l:B_l \in \Sigma_{\beta_1} \cup \dots \cup \Sigma_{\beta_k}} u_l B_l x + u_{\beta_{k+1},1}B_{\beta_{k+1},1}x + u_{\beta_{k+1},2}B_{\beta_{k+1},2}x \quad (5.10)$$

is structurally controllable. Then, by the induction hypothesis, which says that there exist $B_{\beta_i,1}, B_{\beta_i,2} \in \Sigma_{\beta_i}$, $1 \leq i \leq k$, such that $\{span(B_{\beta_i,1}, B_{\beta_i,2}, 1 \leq i \leq k)\}_{LA} = \{span(B_l \in \Sigma_{\beta_1} \cup \dots \cup \Sigma_{\beta_k})\}_{LA}$, we know that system (5.10) is structurally controllable if and only if the system

$$\dot{x} = u_{\beta_1,1}B_{\beta_1,1}x + u_{\beta_1,2}B_{\beta_1,2}x + u_{\beta_2,1}B_{\beta_2,1}x + u_{\beta_2,2}B_{\beta_2,2}x + \dots + u_{\beta_k,1}B_{\beta_k,1}x + u_{\beta_k,2}B_{\beta_k,2}x + u_{\beta_{k+1},1}B_{\beta_{k+1},1}x + u_{\beta_{k+1},2}B_{\beta_{k+1},2}x \quad (5.11)$$

is structurally controllable. This concludes the induction. □

In case we do not have at least two matrices in each pattern we establish the following theorem.

Theorem 5.5.4. *We consider system* (5.2) *with* $B_\ell \in \Sigma_{\beta_1} \cup \Sigma_{\beta_2} \cup \dots \cup \Sigma_{\beta_k}$, *where* $k \leq m$ *and* $\ell \in \{1, \dots, m\}$. *Furthermore, we assume that the summation has at least two matrices from each* Σ_{β_i}, $1 \leq i \leq r$ *and only one matrix from each* Σ_{β_j}, $r+1 \leq j \leq k$. *Then system* (5.2) *is structurally controllable if* *and only if the system*

$$\dot{x} = u_{\beta_1,1}B_{\beta_1,1}x + u_{\beta_1,2}B_{\beta_1,2}x + \ldots + u_{\beta_r,1}B_{\beta_r,1}x + u_{\beta_r,2}B_{\beta_r,2}x + \sum_{i:B_i\in\Sigma_{\beta_j}, r+1\leq j\leq k} u_iB_ix \tag{5.12}$$

is structurally controllable, with $B_{\beta_i,1}, B_{\beta_i,2} \in \Sigma_{\beta_i}$*,* $1\leq i\leq r$.

All the results developed for two patterns can be inductively extended to cases of more than two patterns.

Chapter 6

Conclusion

We have studied the structural controllability of sparse bilinear systems in the single pattern case as well as in the multiple pattern case. For both cases, we provided theorems about the realization of controllable bilinear systems which significantly extend the results of Boothby and Wilson (1979). We defined the notion of the graph which corresponds to one or more patterns and we provided a theory that relates the connectivity of the graph with the structural controllability of the bilinear system. Our results for the two patterns case can be inductively generalized for more than two patterns. We proved that the controllability of bilinear systems is a generic property in the single pattern as well as in the multiple pattern case. Given that a bilinear system is structurally controllable, the generic property permit us to conclude that the bilinear system is controllable for almost all matrices in the given vector space.

6.1 Future directions

The controllability and the structural controllability of bilinear systems with drift term is still an open issue; see Elliot (2009). For the linear systems, the structural observability is directly derived from the structural controllability results due to duality. However, in the bilinear systems the duality property between controllability and observability does not hold. Thus, the analysis of the structural observability of bilinear systems, either driftless or with drift term, is an open research topic. Finally, the theories developed from the analysis of the structural controllability of linear and bilinear systems can be used in the investigation of the properties of complex networks.

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