EXTREMAL PROBLEMS ON CYCLE STRUCTURE AND COLORINGS OF GRAPHS

BY

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DISSERTATION

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Abstract

In this Thesis, we consider two main themes: conditions that guarantee diverse cycle structure within a graph, and the existence of strong edge-colorings for a specific family of graphs.

In Chapter 2 we consider a question closely related to the Matthews-Sumner conjecture, which states that every 4-connected claw-free graph is Hamiltonian. Since there exists an infinite family of 4-connected claw-free graphs that are not pancyclic, Gould posed the problem of characterizing the pairs of graphs, $\{X, Y\}$, such that every 4-connected $\{X, Y\}$-free graph is pancyclic. In this chapter we describe a family of pairs of graphs such that if every 4-connected $\{X, Y\}$-free graph is pancyclic, then $\{X, Y\}$ is in this family. Furthermore, we show that every 4-connected $\{K_{1,3}, N(4, 1, 1)\}$-free graph is pancyclic. This result, together with several others, completes a characterization of the family of subgraphs, $\mathcal{F}$ such that for all $H \in \mathcal{F}$, every 4-connected $\{K_{1,3}, H\}$-free graph is pancyclic.

In Chapters 3 and 4 we consider refinements of results on cycles and chorded cycles. In 1963, Corrádi and Hajnal proved a conjecture of Erdős, showing that every graph G on at least $3k$ vertices with minimum degree at least $2k$ contains k disjoint cycles. This result was extended by Enomoto and Wang, who independently proved that graphs on at least $3k$ vertices with minimum degree-sum at least $4k - 1$ also contain k disjoint cycles. Both results are best possible, and recently, Kierstead, Kostochka, Molla, and Yeager characterized their sharpness examples. A chorded cycle analogue to the result of Corrádi and Hajnal was proved by Finkel, and a similar analogue to the result of Enomoto and Wang was proved by Chiba, Fujita, Gao, and Li. In Chapter 3 we characterize the sharpness examples to these statements, which provides a chorded cycle analogue to the characterization of Kierstead et al.

In Chapter 4 we consider another result of Chiba et al., which states that for all integers r and s with $r + s \geq 1$, every graph G on at least $3r + 4s$ vertices with $\delta(G) \geq 2r + 3s$ contains r disjoint cycles and s disjoint chorded cycles. We provide a characterization of the sharpness examples to this result, which yields a transition between the characterization of Kierstead et al. and the main result of Chapter 3.

In Chapter 5 we move to the topic of edge-colorings, considering a variation known as strong edge-coloring. In 1990, Faudree, Gyárfás, Schelp, and Tuza posed several conjectures regarding strong edge-colorings of

subcubic graphs. In particular, they conjectured that every subcubic planar graph has a strong edge-coloring using at most nine colors. We prove a slightly stronger form of this conjecture, showing that it holds for all subcubic planar loopless multigraphs.

This is dedicated to my beautiful wife Sarah,

and our wonderful children, Jessica, Stephanie, Aria, and Jonathan.

Coming home has never been better.

Acknowledgments

This has been an amazing journey for me, and I honestly can't believe that this has happened. When I was a kid, I used to think about how great it would be if I could get paid to do math. Now, here I am. It really is humbling to think about how blessed I am and have been my whole life, especially when there are so many people in this world who will never get the chance to do what they love.

I have made so many friends and been impacted by so many people along the way. It would be impossible to successfully name them all, so I will only highlight a few. I owe a lot to Brooks Reid and Andre Kündgen. Their advice, mentorship, and friendship has meant so much to me over the years. Thanks to Victor Rocha and Keith Trujillo for believing in me and pushing me towards pursuing a doctoral degree. Thanks to all my colleagues (i.e., friends) over the years. I wouldn't want to laugh, struggle, argue, and succeed with anyone else.

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To the family into which I was born, my friends at CCESCO, those who became my family in Belize, the faculty and friends I made at Palomar, CSUSM, and UIUC, and to my wonderful family of Sarah, Jessica, Stephanie, Aria, Jonathan, along with any others that we may add to our family over the years, all I can

say is thank you. I mean that from the bottom of my heart.

Lastly, but most importantly, I am here by none other than the grace of God, so thank You for everything. I was 14 when I first read the following verses, and while I haven't always kept my end of the bargain, it has been amazing to see how You have directed my path.

"Trust in the Lord with all your heart and lean not on your own understanding;

In all your ways acknowledge Him, and He shall direct your path."

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List of Symbols

In the list below, t is a positive integer, as are $k_1, \ldots, k_t$. G and H are graphs, S is a collection of vertices, and u and v are individual vertices. Additional symbols and notation will be introduced as needed.

$V(G)$	The set of vertices of G
$E(G)$	The set of edges of G
$\|G\|$	Number of vertices in G
$\overline{G}$	Complement of G
$d_G(v)$	Degree of v in G
$N_G(v)$	Neighborhood of v in G
$\delta(G)$	Minimum degree of G
$\Delta(G)$	Maximum degree of G
$\alpha(G)$	Independence number of G
$\sigma_2(G)$	$\min\{d_G(u) + d_G(v) : uv \notin E(G)\}$
$\chi(G)$	Chromatic number of G
$\chi'(G)$	Chromatic index of G
$\chi'_s(G)$	Strong chromatic index of G
$G[S]$	The subgraph of G induced by the vertices in S
$L(G)$	The line graph of G
G^2	The square of G
K_t	The complete graph on t vertices
$K_{k_1,\ldots,k_t}$	The complete t-partite graph with parts of size $k_1, \ldots, k_t$
P_t	The path on t vertices
C_t	The cycle on t vertices
$K_{1,3}$	The claw
$\tilde{N}$	The net
$N(i, j, k)$	The generalized net

$G + H$	The disjoint union of G and H
$G \vee H$	The join of G and H
$G \square H$	The Cartesian product of G and H

Chapter 1

Overview

In this Chapter we give an overview of the contents of this Thesis, beginning with several necessary definitions. A large portion of the definitions and notation used are taken directly from or strongly influenced by those used in [56].

1.1 Definitions and Notation

All graphs are assumed to be *simple* (i.e., have no loops or multiple edges) unless they are explicitly referred to as multigraphs. In the following, G and H are graphs, and $\mathcal{F}$ is a nonempty family of graphs.

Given a drawing of a multigraph, a *crossing* occurs when two edges contain a common internal point. A multigraph that can be drawn in the plane without crossings is said to be *planar*, and a particular planar drawing of a multigraph is a *plane multigraph.*

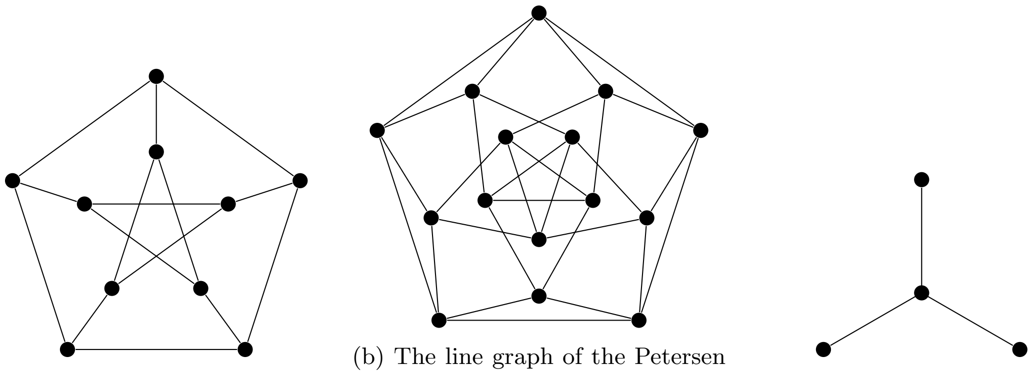
Given a vertex $v \in V(G)$, its *degree* is the number of edges incident to it, denoted by $d_G(v)$. The *maximum degree of* G is the largest degree among all the vertices of G; it is denoted by $\Delta(G)$. The *minimum degree of* G, denoted by $\delta(G)$, is defined similarly. G is *subcubic* if $\Delta(G) \leq 3$.

The *minimum degree-sum of* G, also referred to as the *Ore-degree of* G, is given by $\sigma_2(G) = \min\{d_G(u) + d_G(v) : u, v \in V(G), uv \notin E(G)\}$.

Given a graph G, its *square* is the graph G^2 with $V(G^2) = V(G)$ and $uv \in E(G^2)$ if either $uv \in E(G)$ or u and v have a common neighbor in G. The *line graph of* G, denoted by $L(G)$, is the graph with $V(L(G)) = E(G)$ and $ee' \in E(L(G))$ if e and e' are incident edges in G. See Figures 1.1a and 1.1b.

For $S \subseteq V(G)$, $G[S]$ denotes the *graph induced by* S, meaning $V(G[S]) = S$ and for $u, v \in S$, $uv \in E(G[S])$ if and only if $uv \in E(G)$. Suppose H is a subgraph of G, and let H' denote a particular copy of H in G. We say that H' is *induced* in G if $G[V(H')] \cong H$.

A graph is H*-free* if it does not contain H as an induced subgraph, and for a family $\mathcal{F}$, a graph is said to be $\mathcal{F}$*-free* if it is H-free for all $H \in \mathcal{F}$. The *claw* is the graph $K_{1,3}$ (see Figure 1.1c), and we often say that a graph is *claw-free* rather than $K_{1,3}$-free.



(a) The Petersen graph (b) The line graph of the Petersen graph (c) The claw $K_{1,3}$

Figure 1.1: Various graphs

A graph is *connected* if for any pair of vertices u and v, there exists a path from u to v. For a positive integer k, G is said to be *k-connected* if for any $S \subseteq V(G)$ with $|S| < k$, the graph $G[V(G) \setminus S]$ is still connected.

The *girth* of a graph is the length of a shortest cycle in the graph. A *Hamiltonian cycle* (resp. *Hamiltonian path*) in a graph is a cycle (resp. path) that contains all the vertices of the graph, and a graph is said to be *Hamiltonian* if it contains a Hamiltonian cycle. A graph G is *Hamiltonian-connected* if for every pair of vertices u and v, there exists a Hamiltonian path in G whose endpoints are precisely u and v. For $n \geq 3$, an n-vertex graph is *pancyclic* if it contains cycles of lengths $3, 4, \ldots, n$.

Let H be a subgraph of G, and suppose H has a cycle which spans the vertices of H (i.e., H is Hamiltonian). If $G[V(H)]$ is not an induced cycle, then H contains edges not in its Hamiltonian cycle. We call these edges *chords*, and we say H is a *chorded cycle*. If H is a chorded cycle with exactly one chord, then H is a *singly chorded cycle*, and if H has at least two chords, then H is a *doubly chorded cycle*.

Additional definitions and notation will be introduced as needed.

1.2 Forbidden Subgraphs and Pancyclicity (Chapter 2)

The following well-known conjecture of Matthews and Sumner [40] has provided the impetus for a great deal of research into the Hamiltonicity of claw-free graphs.

Conjecture 1.1 (The Matthews–Sumner Conjecture)**.** *If G is a 4-connected claw-free graph, then G is Hamiltonian.*

In [48], Ryjáček demonstrated that this is equivalent to a conjecture of Thomassen [54] that every 4-connected line graph is Hamiltonian. Also in [48], Ryjáček showed that every 7-connected claw-free graph is Hamiltonian. Kaiser and Vrána [32] then showed that every 5-connected claw-free graph with minimum degree at least 6 is Hamiltonian, which currently represents the best general progress towards affirming Conjecture 1.1. In [51], Conjecture 1.1 was also shown to be equivalent to the statement that every 4-connected claw-free graph is Hamiltonian-connected.

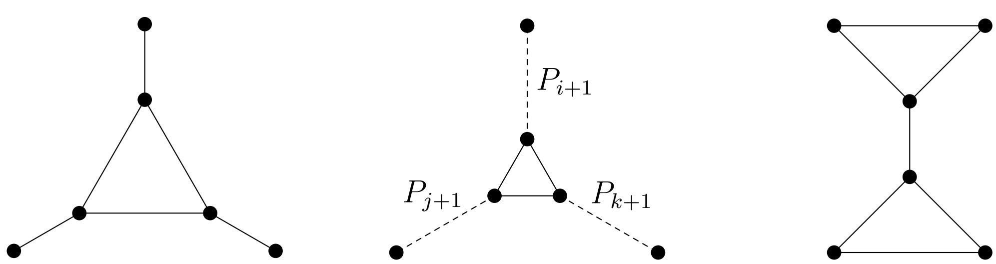
The Matthews–Sumner Conjecture has also fostered a large body of research into other properties of the cycle structure of claw-free graphs. In this chapter, we are specifically interested in the pancyclicity of highly connected claw-free graphs. Significantly fewer results of this type can be found in the literature, in part because it has been shown in many cases [49, 50] that closure techniques such as those in [48] do not apply to pancyclicity. Furthermore, for $k \geq 2$, Brandt, Favaron, and Ryjáček [4] provided an infinite number of k-connected claw-free graphs that are not pancyclic.

As forbidding the claw from a highly-connected graph is insufficient to guarantee pancyclicity, this leads to the natural question of attempting to forbid pairs of subgraphs in order to assure pancyclicity. In this Thesis we call such pairs, *pairs of forbidden subgraphs.*

In [15], Faudree and Gould characterized the pairs of forbidden subgraphs that imply pancyclicity in 2-connected graphs. Here $N(i,j,k)$ is the *generalized net* obtained by identifying an endpoint of each of the paths P_{i+1}, P_{j+1}, and P_{k+1} with distinct vertices of a triangle. See Figures 1.2a and 1.2b for the standard net and generalized net.

Theorem 1.2 (Faudree–Gould [15])**.** *Let X and Y be connected graphs with $P_3 \notin \{X,Y\}$, and let G be a 2-connected n-vertex graph with $n \geq 10$ and $G \not\cong C_n$. The property of being $\{X,Y\}$-free implies pancyclicity if and only if $X = K_{1,3}$ and Y is an induced subgraph of either P_6 or $N(2,0,0)$.*

Gould, Łuczak, and Pfender [24] obtained the following characterization of pairs of forbidden subgraphs that imply pancyclicity in 3-connected graphs. Here Ł will be used to denote two disjoint triangles joined by a single edge as in Figure 1.2c.



(a) The net $\tilde{N}$ (b) The generalized net $N(i,j,k)$ (c) Ł

Figure 1.2: Nets and Ł

Theorem 1.3 (Gould–Łuczak–Pfender [24]). *Let X and Y be connected graphs on at least three vertices with $P_3 \notin \{X, Y\}$. For 3-connected graphs, the property of being $\{X, Y\}$-free implies pancyclicity if and only if $X = K_{1,3}$ and Y is an induced subgraph of one of the graphs in the family $\{P_7, L, N(i, j, k) : i, j, k \geq 0, i + j + k = 4\}$.*

Motivated by the Matthews–Sumner Conjecture and the aforementioned results, Gould posed the following problem at the 2010 SIAM Discrete Math meeting in Austin, TX.

Problem 1. *Characterize the pairs of forbidden subgraphs that imply that a 4-connected graph is pancyclic.*

The first progress towards this problem appears in [18].

Theorem 1.4 (Ferrara–Morris–Wenger [18]). *If G is a 4-connected $\{K_{1,3}, P_{10}\}$-free graph, then either G is pancyclic or G is the line graph of the Petersen graph. Consequently, every 4-connected $\{K_{1,3}, P_9\}$-free graph is pancyclic.*

The line graph of the Petersen graph is 4-connected claw-free and contains no cycle of length 4 (see Figure 1.1b). Observing that in Theorem 1.3, all generalized nets of the form $N(i, j, 0)$ with $i + j = 4$ are in the family $\mathcal{F}$, Ferrara, Gould, Gehrke, Magnant, and Powell [17] showed the following.

Theorem 1.5 (Ferrara–Gould–Gehrke–Magnant–Powell [17]). *Every 4-connected $\{K_{1,3}, N(i, j, 0)\}$-free graph with $i + j = 6$ is pancyclic. This result is best possible, since the line graph of the Petersen graph is $N(i, j, 0)$-free for all $i, j \geq 0$ with $i + j = 7$.*

Recently, Carraher, Ferrara, Morris, and Santana [6] proved the following similar result.

Theorem 1.6 (Carraher–Ferrara–Morris–Santana [6]). *Every 4-connected $\{K_{1,3}, N(i, j, k)\}$-free graph with $i, j, k \geq 1$ and $i + j + k = 6$ is pancyclic. This result is best possible, since the line graph of the Petersen graph is $N(i, j, k)$-free for all $i, j, k \geq 1$ with $i + j + k = 7$.*

In addition, they characterized the graphs Y such that every 4-connected $\{K_{1,3}, Y\}$-free graph is pancyclic. In particular, they proved the following two theorems.

Theorem 1.7 (Carraher et al. [6]). *Let X and Y be connected graphs with at least three edges such that every 4-connected $\{X, Y\}$-free graph is pancyclic. Without loss of generality, $X \in \{K_{1,3}, K_{1,4}\}$ and Y is an induced subgraph of one of the graphs in the family $\{P_9, L, N(i, j, k) : i, j, k \geq 0, i + j + k = 6\}$.*

Theorem 1.8 (Carraher et al. [6]). *Let Y be a connected graph with at least three edges. Every 4-connected $\{K_{1,3}, Y\}$-free graph is pancyclic if and only if Y is an induced subgraph of one of the graphs in the family $\{P_9, L, N(i, j, k) : i, j, k \geq 0, i + j + k = 6\}$.*

The main goal of Chapter 2 is prove Theorem 1.7 as well as the following case of Theorem 1.6.

Theorem 1.9 (Main Result of Chapter 2). *Every 4-connected $\{K_{1,3}, N(4,1,1)\}$-free graph is pancyclic.*

1.3 Disjoint Cycles and Chorded Cycles (Chapters 3 and 4)

In these chapters, the word *disjoint* is always taken to mean vertex-disjoint.

In 1963, Corrádi and Hajnal verified a conjecture of Erdős, by proving the following.

Theorem 1.10 (Corrádi–Hajnal, [8]). *Every graph G with at least $3k$ vertices having $\delta(G) \geq 2k$ contains k disjoint cycles.*

This result of Corrádi and Hajnal has been generalized in various ways. One such generalization is a strengthening by Enomoto and by Wang, who independently proved the following.

Theorem 1.11 (Enomoto [12] , Wang [55]). *Every graph G with at least $3k$ vertices having $\sigma_2(G) \geq 4k-1$ contains k disjoint cycles.*

Theorems 1.10 and 1.11 are both sharp, leading to the following natural question of Dirac.

Question 1 (Dirac [10]). *Which $(2k-1)$-connected multigraphs do not contain k disjoint cycles?*

Question 1 was answered in the case of simple graphs in [34], and then for multigraphs in [35]. Indeed, [34] together with [36] answer a more general question for simple graphs, describing graphs with Ore-degree at least $4k-3$ without k disjoint cycles.

Theorem 1.12 ([34], [36]). *Given $k \in \mathbb{N}$ with $k \geq 4$, a graph G on at least $3k$ vertices having $\sigma_2(G) \geq 4k-3$. Then G contains k disjoint cycles if and only if none of the following hold:*

1. $\alpha(G) \geq |G| - 2k + 1$.
2. $G \cong (K_c + K_{2k-c}) \vee \overline{K_k}$ *for some odd c*
3. $G \cong (K_1 + K_{2k}) \vee \overline{K_{k-1}}$
4. $|G| = 3k$ *and $\overline{G}$ is not k-colorable*

In 2008, Finkel proved the following analogue of Theorem 1.10 in terms of chorded cycles.

Theorem 1.13 (Finkel [19]). *Every graph G on at least $4k$ vertices having $\delta(G) \geq 3k$ contains k disjoint chorded cycles.*

A stronger vertion of Theorem 1.13 was conjectured by Bialostocki, Finkel, and Gyárfás in [2], and it was proved by Chiba, Fujita, Gao, and Li in [7].

Theorem 1.14 (Chiba–Fujita–Gao–Li, [7]). *Let r and s be integers with $r + s \geq 1$. Every graph G on at least $3r + 4s$ vertices having $\sigma_2(G) \geq 4r + 6s - 1$ contains a collection of r disjoint cycles and s disjoint chorded cycles.*

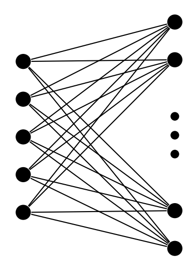
In Chapters 3 and 4, we characterize the sharpness examples to two corollaries of this result of Chiba et al.

1.3.1 Disjoint Chorded Cycles (Chapter 3)

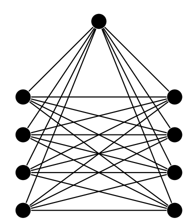
In Chapter 3, we consider the following corollary to Theorem 1.14.

Corollary 1.15 (Chiba–Fujita–Gao–Li, [7]). *Every graph G on at least $4k$ vertices having $\sigma_2(G) \geq 6k - 1$ contains a collection of k disjoint chorded cycles.*

All hypotheses in Theorem 1.13 and Corollary 1.15 are sharp. First, since any chorded cycle contains at least four vertices, if $|G| < 4k$ then G does not contain k disjoint chorded cycles. Also, the conditions $\delta(G) \geq 3k$ and $\sigma_2(G) \geq 6k - 1$ are best possible, as demonstrated by the two graphs below.



(a) $K_{3k-1,n-3k+1}$, shown for $k = 2$



(b) $K_{1,3k-2,3k-2}$, shown for $k = 2$

Figure 1.3: Sharpness examples to Corollary 1.15

This leads us to ask a question similar to Question 1: which graphs G with $\sigma_2(G) \geq 6k-2$ do *not* contain k disjoint chorded cycles? This question was answered by Molla, Santana, and Yeager [41]; the theorem is the main result of Chapter 3.

Theorem 1.16 (Main result of Chapter 3). *For $k \geq 2$ and $n \geq 4k$, let G be an n-vertex graph having $\sigma_2(G) \geq 6k - 2$. The graph G does not contain k disjoint chorded cycles if and only if G is isomorphic to either:*

1. *$K_{3k-1,n-3k+1}$, with $n \geq 6k - 2$, or*

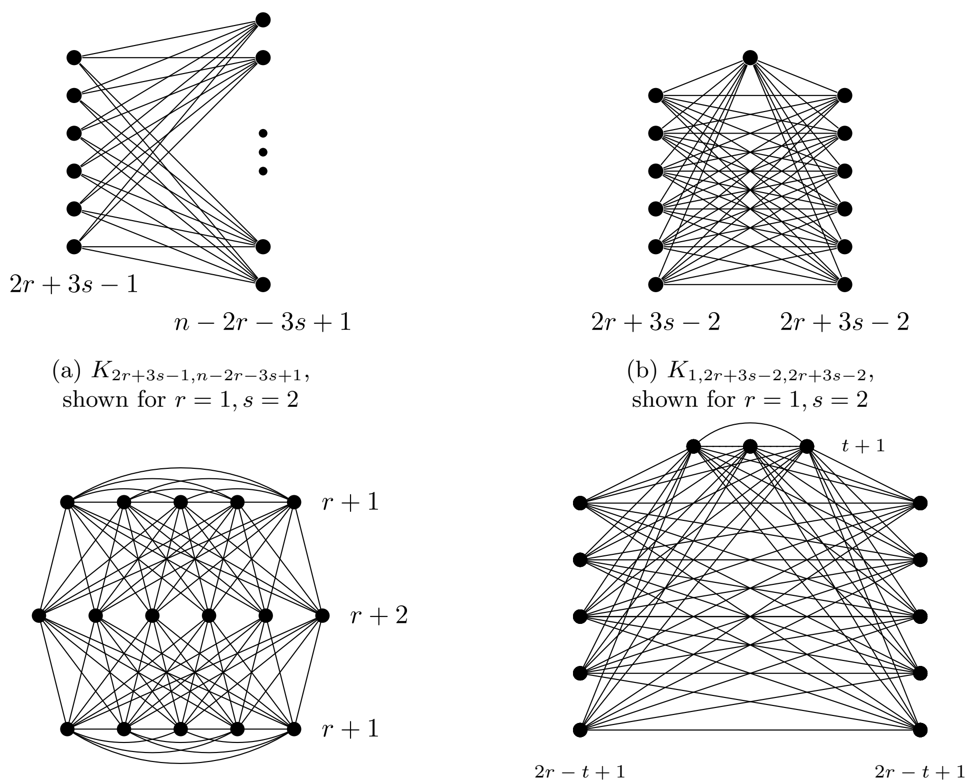
2. $K_{1,3k-2,3k-2}$.

1.3.2 Mixed Cycles (Chapter 4)

In Chapter 4, we consider the following corollary to Theorem 1.14 in terms of the minimum degree.

Corollary 1.17 (Chiba–Fujita–Gao–Li [7])**.** *Let r and s be integers with $r + s \geq 1$. Every graph G on at least $3r + 4s$ vertices having $\delta(G) \geq 2r + 3s$ contains a collection of r disjoint cycles and s disjoint chorded cycles.*

All hypotheses in Corollary 1.17 are sharp. Since every cycle contains at least three vertices, and every chorded cycle contains at least four vertices, if $|G| < 3r + 4s$, then G does not contain r disjoint cycles and s disjoint chorded cycles. In addition, the condition $\delta(G) \geq 2r + 3s$ is best possible, as demonstrated by the graphs in Figure 1.4.



(a) $K_{2r+3s-1,n-2r-3s+1}$, shown for $r = 1, s = 2$

(b) $K_{1,2r+3s-2,2r+3s-2}$, shown for $r = 1, s = 2$

(c) $\overline{K_{r+2}} \vee (K_{r+1} + K_{r+1})$, shown for $r = 4$

(d) $K_{t+1} \vee K_{2r-t+1,2r-t+1}$, shown for $r = 3, t = 2$

Figure 1.4: Sharpness examples to Corollary 1.17

As before, this leads us to ask a question similar to Question 1: which graphs G with $\delta(G) \geq 2r+3s-1$ do not contain r disjoint cycles and s disjoint chorded cycles? This question was answered by Molla, Santana, and Yeager [42]; the theorem is the main result of Chapter 4.

Theorem 1.18 (Main Result of Chapter 4). *Let r and s be positive integers, and $n \geq 3r + 4s$. If G is an n-vertex graph having $\delta(G) \geq 2r + 3s - 1$, then G contains r disjoint cycles and s disjoint chorded cycles, unless*

1. *$G \cong K_{2r+3s-1,n-2r-3s+1}$, with $n \geq 4r + 6s - 2$, or*
2. *$G \cong K_{1,2r+3s-2,2r+3s-2}$, or*
3. *$s = 1$, r is even, and $G \cong \overline{K_{r+2}} \vee (K_{r+1} + K_{r+1})$, or*
4. *$s = 1$ and $G \cong H$, where $K_{t+1,2r-t+1,2r-t+1} \subseteq H \subseteq K_{t+1} \vee K_{2r-t+1,2r-t+1}$, for some t with $0 \leq t \leq r - 1$.*

1.4 Strong Edge-Colorings (Chapter 5)

A *strong k-edge-coloring* of a multigraph G is a function $\phi : E(G) \to [k]$ such that if $\phi(e) = \phi(e')$ for $e, e' \in E(G)$, then e and e' are not incident and are not incident to a common edge. That is, each color class of ϕ forms an induced matching.

The *strong chromatic index* of G, denoted by $\chi'_s(G)$, is the minimum k for which G has a strong k-edge-coloring. Observe that $\chi'_s(G) = \chi((L(G))^2)$.

The notion of strong edge-coloring was introduced by Fouquet and Jolivet [20, 21] and was used to solve a problem involving radio networks and their frequencies. More details on this application can be found in [45, 47].

For a multigraph G with $\Delta(G) = \Delta$, the greedy algorithm provides an upper bound on $\chi'_s(G)$ of $2(\Delta - 1)^2 + 2(\Delta - 1) + 1$. At a seminar in Prague in 1985, Erdős and Nešetřil conjectured that a stronger upper bound holds, which if true, is best possible.

Conjecture 1.19 (Erdős–Nešetřil [13, 14]). *If G is a graph with maximum degree Δ, then*

$$\chi'_s(G) \leq \begin{cases} \frac{5}{4}\Delta^2, & \text{if } \Delta \text{ is even,} \\ \frac{5}{4}\Delta^2 - \frac{1}{2}\Delta + \frac{1}{4}, & \text{if } \Delta \text{ is odd} \end{cases}$$

For subcubic graphs, the conjecture was verified by Horák, Qing, and Trotter [29], and independetly by Andersen [1], who extended the result to subcubic loopless multigraphs.

In general, this problem remains open. However, an upper bound of $1.998\Delta^2$ was proved by Molloy and Reed [43]. Also, Bruhn and Joos [5] improved this bound to $1.93\Delta^2$. Both results are for Δ sufficiently large.

For subcubic graphs, Faudree, Gyárfás, Schelp, and Tuza [16] posed the following list of conjectures.

Conjecture 1.20 (Faudree–Gyárfás–Schelp–Tuza [16]). *Let G be a subcubic graph.*

1. $\chi'_s(G) \leq 10$.
2. *If G is bipartite, then* $\chi'_s(G) \leq 9$.
3. *If G is planar, then* $\chi'_s(G) \leq 9$.
4. *If G is bipartite and the sum of the degrees of any two adjacent vertices is at most 5, then* $\chi'_s(G) \leq 6$.
5. *If G is bipartite with girth at least 6, then* $\chi'_s(G) \leq 7$.
6. *If G is bipartite with large girth, then* $\chi'_s(G) \leq 5$.

As mentioned, Andersen [1] and Horák, Qing, and Trotter [29] independently proved Conjecture 1.20.1. Conjecture 1.20.2 was verified by Steger and Yu [53]. Conjecture 1.20.4 was confirmed by Wu and Lin [57] and was generalized by Nakprasit and Nakprasit [44]. The only known results in regards to Conjecture 1.20.6 are restricted to planar bipartite graphs. In particular, a result of Borodin and Ivanova [3] showed that $\chi'_s(G) \leq 5$ for any subcubic planar bipartite graph G with girth at least 42. This was recently improved by DeOrsey et al. [9] who reduced the girth requirement to 30. We know of no results that pertain to Conjecture 1.20.5.

Conjecture 1.20.3 was proved recently by Kostochka, Li, Ruksasakchai, Santana, Wang, and Yu [37]. They proved a slightly stronger statement, which is the main result of Chapter 5.

Theorem 1.21 (Main result of Chapter 5). *For every subcubic planar multigraph G with no loops,* $\chi'_s(G) \leq 9$.

This result is best possible, as shown by $K_3 \square P_2$ in Figure 1.5.

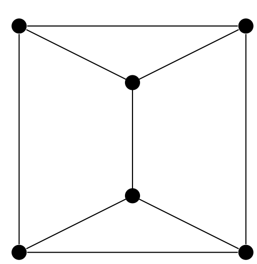


Figure 1.5: $K_3 \square P_2$

Chapter 2

Pancyclicity

The following results are joint work with James Carraher, Michael Ferrara, and Timothy Morris, appearing in [6].

2.1 Introduction

As mentioned in Section 1.2, the main purpose of this chapter is to prove the following two statements.

Theorem 2.1. *Let X and Y be connected graphs with at least three edges such that every 4-connected $\{X, Y\}$-free graph is pancyclic. Without loss of generality, $X \in \{K_{1,3}, K_{1,4}\}$ and Y is an induced subgraph of one of the graphs in the family $\{P_9, L, N(i, j, k) : i, j, k \geq 0, i + j + k = 0\}$.*

Theorem 2.2. *Every 4-connected $\{K_{1,3}, N(4, 1, 1)\}$-free graph is pancyclic.*

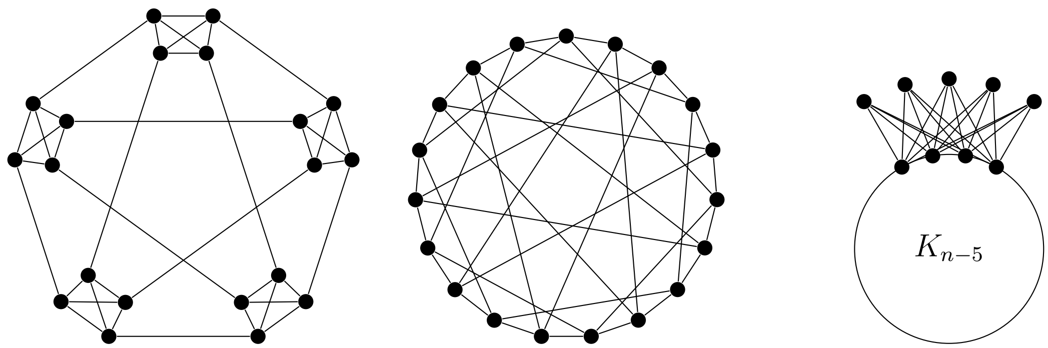
When proving Theorem 2.2, we will often consider subgraphs isomorphic to $K_{1,3}$ and $N(4, 1, 1)$. To better describe these subgraphs, we let $\langle a; a_1, a_2, a_3\rangle$ denote a copy $K_{1,3}$ in G with center vertex a and pendant edges aa_1, aa_2, and aa_3, and let $N(abc; a_1 \ldots a_i, b_1 \ldots b_j, c_1 \ldots c_k)$ denote a copy of $N(i, j, k)$ with central triangle abc and pendant paths $aa_1 \ldots a_i$, $bb_1 \ldots b_j$, and $cc_1 \ldots c_k$.

Furthermore, for a subgraph H of G and a set $S \subseteq E(K_{|G|})$, we write $H \to S$ if either H is an induced subgraph of G or $S \cap E(G) \neq \emptyset$. The purpose of this notation is the following. In our proof of Theorem 2.2, we proceed by contradiction, and consider $H \in \{K_{1,3}, N(4, 1, 1)\}$, where G is $\{K_{1,3}, N(4, 1, 1)\}$-free. Thus, if we can show $S \cap E(G) = \emptyset$, then H is induced, which is a contradiction. Oftentimes we will omit edges from S that are shown to not exist in G by lemmas that we will prove.

The proofs of Theorems 2.1 and 2.2 are given as follows. In Section 2.2, we present the proof of Theorem 2.1, while the remaining sections will focus on proving Theorem 2.2. In particular, in Section 2.3 we show that every 4-connected $\{K_{1,3}, N(4, 1, 1)\}$-free graph has cycles of length 3, 4, and 5. In Section 2.4, we prove several technical lemmas that we will use in Section 2.5 to prove that if a 4-connected $\{K_{1,3}, N(4, 1, 1)\}$-free graph has a cycle of length s, where $5 \leq s \leq n-1$, then it contains a cycle of length $s+1$. This will complete the proof of Theorem 2.2. We end this chapter with some questions for further research.

2.2 Proof of Theorem 2.1

In this section we prove Theorem 2.1 in a manner similar to that used in [24]. Let $\tilde{P}$ denote the Petersen graph. Observe that $L(\tilde{P})$ (see Figure 1.1b), $L(S(K_5))$, G_1, G_2 (see Figure 2.1), and $K_{4,4}$ are each 4-connected. In addition, they are not pancyclic, as they do not contain C_4, C_5, C_4, C_n, and C_3, respectively. Also, $L(\tilde{P})$ is $\{K_{1,3}, K_{1,4}\}$-free.



(a) $L(S(K_5))$ (b) G_1 (c) G_2

Figure 2.1: Some 4-connected graphs that are not pancyclic

Lemma 2.3. *Let X and Y be connected graphs with at least three edges. If each 4-connected $\{X, Y\}$-free graph is pancyclic, then $\{X, Y\} \cap \{K_{1,3}, K_{1,4}\} \neq \emptyset$.*

Proof. Suppose on the contrary that $X, Y \notin \{K_{1,3}, K_{1,4}\}$. As $K_{4,4}$ is not pancyclic, we may conclude without loss of generality that X is an induced subgraph of $K_{4,4}$. As $X \notin \{P_3, K_{1,3}, K_{1,4}\}$, X must contain an induced copy of C_4.

As G_1 does not contain C_4, it must contain Y as an induced subgraph. Therefore, Y must have girth at least 5 and maximum degree 4. Furthermore, G_2 is C_4-free, so Y must also be an induced subgraph of G_2. However, the only induced subgraphs of G_2 with girth at least 5 and maximum degree 4 are $K_{1,3}$ and $K_{1,4}$. So, Y must contain an induced $K_{1,3}$ or $K_{1,4}$.

Lastly, $L(\tilde{P})$ is also C_4-free so that Y must be an induced subgraph of $L(\tilde{P})$. However, neither $K_{1,3}$ nor $K_{1,4}$ is an induced subgraph of $L(\tilde{P})$. Hence Y cannot be chosen to complete the pair unless $X \in \{K_{1,3}, K_{1,4}\}$. □

In the remainder of this section, we will assume that X and Y are connected graphs with at least three edges such that $X \in \{K_{1,3}, K_{1,4}\}$ and every 4-connected $\{X, Y\}$-free graph is pancyclic. To complete the proof of Theorem 2.1, we must characterize the possibilities for Y. In doing so, we will make use of the following family of graphs developed by Lubotsky, Phillips, and Sarnak [38].

Theorem 2.4. *For infinitely many d and all n, there exists a connected, d-regular, vertex-transitive graph on at least n vertices that has arbitrarily large girth. In addition, these graphs exist for $d = p + 1$, where p is an odd prime.*

These graphs, often called *Ramanujan graphs*, were used by Brandt, Favaron, and Ryjáček [4] to show that for each $k \geq 2$, there exists a k-connected claw-free graph that is not pancyclic. We use a very similar approach to prove the following lemma, which with Lemma 2.3 immediately implies Theorem 2.1.

Lemma 2.5. *There exists a 4-connected, claw-free, non-pancyclic graph G such that if Y is an induced subgraph of each of $L(\tilde{P})$, $L(S(K_5))$, and G, then Y is an induced subgraph of one of P_9, L, or $N(i, j, k)$, with $i + j + k = 6$.*

Proof. Let H be a connected, 4-regular, vertex-transitive graph with girth $g \geq 9$, as guaranteed by [38]. By a result of Mader [39], a connected, vertex-transitive, d-regular graph must also be d-edge-connected, implying that H is also 4-edge connected. It follows that $L(H)$ is a 6-regular, 4-connected, claw-free graph. Note that each vertex v of H is represented by a graph $G_v \cong K_4$ in $L(H)$, where $xy \in E(H)$ corresponds to a vertex $z \in L(H)$ in exactly two copies of K_4.

Let H' be obtained from $L(H)$ by performing a 4-split on each vertex as follows. Let $v \in V(L(H))$ with neighbors $\{x_1, x_2, x_3, y_1, y_2, y_3\}$, where the x_i's and y_i's are in distinct copies of K_4. Delete v and replace it with adjacent vertices x, y such that $N(x) = \{y, x_1, x_2, x_3\}$ and $N(y) = \{x, y_1, y_2, y_3\}$. It is well known that if a graph F is 4-connected and F' is obtained from F by performing 4-splits, then F' also is 4-connected. Thus, H' is 4-connected, and it is easy to verify that H' is claw-free, as for every three neighbors of a vertex v, two must be in one copy of K_4. Note that H' contains 3-cycles and 4-cycles but does not contain cycles of length t, for $5 \leq t < 2g$ (recall $g \geq 9$).

For a given G_v in H', subdivide each edge of G_v exactly twice. For clarity, color these new vertices white, color the original vertices of H' black, and then add edges so that the 12 new white vertices induce a clique. Let $\tilde{G}_v$ be this new subgraph of order 16, and repeat this for each G_v in H' to obtain the graph G (see Figure 2.2).

We claim first that G is claw-free. Indeed, if a black vertex is the center of a claw, then at least two of the other vertices in the claw must be white vertices lying in a common $\tilde{G}_v$. A similar argument shows that no white vertex is the center of a claw.

To establish that G is 4-connected, consider a set S of at most three vertices in G. If S has any white vertices, then it must contain three white vertices, as removing at most two white vertices will not disconnect any $\tilde{G}_v$, let alone G. However, deleting three white vertices from a single $\tilde{G}_v$ cannot disconnect G, as in

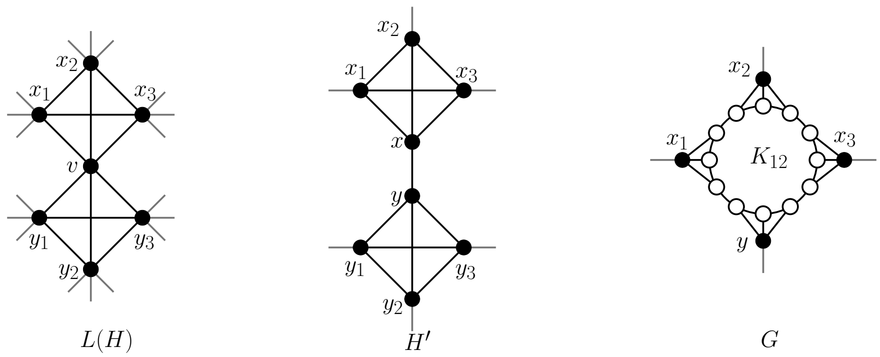


Figure 2.2: Forming the graph G

the worst case these three vertices would have a common black neighbor $v' \in \tilde{G}_v$. If $G - S$ is disconnected, then separating a vertex x from G_v is similar to disconnecting H' by deleting only x. As H' is 4-connected, this is not possible. So, we may assume S contains only black vertices. This directly corresponds to deleting vertices in H', which is 4-connected. Thus, in all cases, $G - S$ is connected.

We also claim that G is not pancyclic. Indeed, G contains cycles of length $3, \ldots, 16$. However, any cycle of length 17 must contain vertices from distinct modified copies of K_4 in G. If we ignore all white vertices of our cycle, this corresponds to a cycle in H' using distinct vertices from distinct copies of K_4. As the smallest cycles in H' are of lengths 3,4 and $2g$, where $g \geq 9$, our corresponding cycle must have length at least $2g \geq 18$ in H', and thus has length at least 14 in G. Consequently, G has no cycle of length 17, and so, it is not pancyclic.

Lastly, let Y be an induced subgraph of each of $L(\tilde{P})$, $L(S(K_5))$, and G. It remains to show that Y is an induced subgraph of P_9, Ł, or $N(i,j,k)$ with $i + j + k = 6$.

To begin, we claim that Y is either a tree or has girth 3. Suppose that Y is not a tree and has girth at least 4. Since $L(\tilde{P})$ and $L(S(K_5))$ are C_4-free and C_5-free, respectively, Y must have girth at least 6. In addition, $L(S(K_5))$ implies that Y has girth at most 10, else it contains a 3-cycle. However, any induced subgraph of G with girth less than 18 contains 3-cycle, a contradiction.

Suppose now that Y is not a tree. If Y has two distinct cycles, then by the above argument, we may assume that Y has at least two distinct 3-cycles. Considering $L(\tilde{P})$, no two 3-cycles can share two vertices. Considering $L(S(K_5))$, no two 3-cycles can share exactly one vertex. So, they must be joined by a nontrivial path. By considering $L(\tilde{P})$, it is clear that if two 3-cycles are joined by a nontrivial path, they are joined by a single edge. That is, Ł is an induced subgraph of Y. While there are many induced subgraphs of Ł in G, it is easy to see that if $Y \neq$ Ł, then Y must contain a 4-cycle, a contradiction to $Y \subseteq L(\tilde{P})$. So, unless $Y =$ Ł, Y cannot contain two distinct cycles.

Thus, if Y has a cycle, it must be a 3-cycle, and Y must be unicyclic. That is, Y is a generalized net. As noted in [17], $L(\tilde{P})$ is $N(i,j,k)$-free when $k=0$ and $i+j=7$. It is also easy to note that $L(\tilde{P})$ is $N(i,j,k)$-free when $i,j,k \geq 1$ and $i+j+k=7$. Thus, Y must be an induced subgraph of $N(i,j,k)$ where $i+j+k=6$.

Lastly, if Y is a tree, then since $L(\tilde{P})$ is $K_{1,3}$-free, Y must be a path, and by [18], Y must be an induced subgraph of P_9. This completes the proof. □

2.3 Short Cycles

In the remaining sections we prove Theorem 2.2. In this particular section we prove that a 4-connected, $\{K_{1,3}, N(4,1,1)\}$-free graph contains cycles of length 3, 4, and 5. We will use the following proposition throughout this section.

Proposition 2.6. *If G is 4-connected, claw-free, and does not contain C_4, then G is 4-regular and for all $v \in V(G)$, $N(v)$ induces $2K_2$.*

Proof. Let G be a 4-connected claw-free graph that does not contain C_4, and let $v \in V(G)$. Let $N(v) = \{x_1, x_2, \ldots, x_t\}$, where $t \geq 4$ as G is 4-connected. By considering the claw $\langle v; x_1, x_2, x_3\rangle$, we assume without loss of generality that $x_1x_2 \in E(G)$. Observe that $x_ix_j \notin E(G)$ for all $i \in \{1,2\}$ and all $j \in \{3,4,\ldots,t\}$, otherwise G contains the C_4 $vx_{3-i}x_ix_jv$.

Therefore, $\langle v; x_1, x_3, x_4\rangle \to \{x_3x_4\}$. If $t=4$, then we are done. If $t \geq 5$, then $\langle v; x_1, x_3, x_5\rangle \to \{x_3x_5\}$, which forms the 4-cycle $vx5x_3x_4v$, a contradiction. □

We are now ready to prove the main result of this section.

Lemma 2.7. *If G is a 4-connected $\{K_{1,3}, N(4,1,1)\}$-free graph, then G contains cycles of length 3, 4, and 5.*

Proof. Let G be a 4-connected $\{K_{1,3}, N(4,1,1)\}$-free graph. Note that since G is claw-free and has minimum degree at least 4, G necessarily contains a triangle.

By Theorem 1.5, if G is $\{K_{1,3}, N(5,1,0)\}$-free, then G is pancyclic. Therefore, G must contain an induced copy of $N(5,1,0)$, which we denote by $N_1 = N(a_0b_0c_0; a_1a_2a_3a_4a_5, b_1)$. Since G has minimum degree at least 4 and N_1 is induced, c_0 is adjacent to two vertices u_1 and u_2 that lie outside of N_1. Let N_{u_i} be the copy of $N(4,1,1)$ given by $N(a_0b_0c_0; a_1a_2a_3a_4, b_1, u_i)$ for $i \in \{1,2\}$.

Suppose first that G does not contain a 4-cycle, so that by Proposition 2.6 u_1 and u_2 are adjacent. Now, since G contains no C_4, the vertices u_1 and u_2 can have no common neighbor aside from c_0, and further if

u_1 and u_2 are adjacent to distinct vertices x and y, respectively, then $xy \notin E(G)$. This is a contradiction, since $N_{u_i} \to \{a_2u_i, a_3u_i, a_4u_i\}$ for $i \in \{1,2\}$, as all other possible edges immediately result in a C_4. If a_2u_i is an edge, then the claw $\langle a_2; a_1, a_3, u_i\rangle \to \{a_3u_i\}$. Thus u_1 and u_2 must have either a common neighbor or adjacent neighbors amongst $\{a_2, a_3, a_4\}$, implying $C_4 \subseteq G$.

Suppose then that G does not contain C_5. This implies that u_i has no neighbor in $\{a_1, a_2, b_1\}$, and that (for $i \in \{1,2\}$) if u_i is adjacent to b_0, then u_{3-i} is not adjacent to a_0. Assume first that neither u_1 nor u_2 is adjacent to either of a_0 and b_0. As G is $N(4,1,1)$-free, N_{u_i} is not an induced subgraph, so u_i must have some neighbor $a_j \in \{a_3, a_4\}$. The claw $\langle a_j; u_i, a_{j-1}, a_{j+1}\rangle$ then requires that each u_i is adjacent to two adjacent vertices in $\{a_3, a_4, a_5\}$. This implies $C_5 \subseteq V(G)$.

Thus, we may assume that u_1 is adjacent to one of a_0 or b_0. Since N is not induced and G contains no 5-cycle, the appropriate choice of $\langle a_0; u_1, a_1, b_0\rangle$ or $\langle b_0; u_1, b_1, a_0\rangle$ implies that a_0 and b_0 are both adjacent to u_1. Since either u_2b_0 or u_2a_0 would create a 5-cycle, $N_{u_2} \to \{u_2a_3, u_2a_4\}$. Suppose first that $u_2a_3 \in E(G)$. Since u_2 is not adjacent to a_2, the claw $\langle a_3; u_2, a_2, a_3\rangle$ requires that $u_2a_4 \in E(G)$, so we may assume $u_2a_4 \in E(G)$. This then implies that u_1 has no neighbor in $\{a_3, a_4, a_5\}$, as any of these possible edges would complete a 5-cycle in G. If u_1u_2 is an edge of G, then $u_1u_2c_0a_0b_0u_1$ is a 5-cycle, so we conclude that u_1 must have some neighbor v that lies outside of $V(N_1) \cup \{u_2\}$. Since G has no 5-cycle, $N(a_0b_0u_1; a_1a_2a_3a_4, b_1, v) \to \{va_3\}$. However, $\langle a_3; a_2, a_4, v\rangle \to \{a_2v, a_4v\}$, which implies that G contains C_5, a contradiction. $\square$

2.4 Technical Lemmas

In this section, we present notation and prove several technical lemmas that will simplify the case structure of our proof of Theorem 2.2.

2.4.1 Setup

Throughout this chapter, we will assume that all cycles C have an inherent clockwise orientation. For some vertex v on C we will denote the first, second, and i^{th} predecessor of v as v^-, v^{--}, and v^{-i} respectively. Similarly we denote the first, second, and i^{th} successor of v as v^+, v^{++}, and v^{+i} respectively. We let xCy denote the path $xx^+ \dots y$ and xC^-y denote the path $xx^- \dots y$. Also, $xCyx$ denotes the cycle formed by adding an edge to the endpoints of the path xCy. Further, let $[u,v]_C$ denote the set of vertices on uCv, and let $(u,v)_C$ denote the set of vertices on u^+Cv^-. The intervals $(u,v]_C$ and $[u,v)_C$ are defined similarly.

Let G be a 4-connected claw-free graph, and let C be a cycle in G of length s, where $5 \le s < |V(G)|$. Assume that G contains no $(s+1)$-cycle. Since G is 4-connected, for each vertex $v \in V(G) \setminus V(C)$ there exist

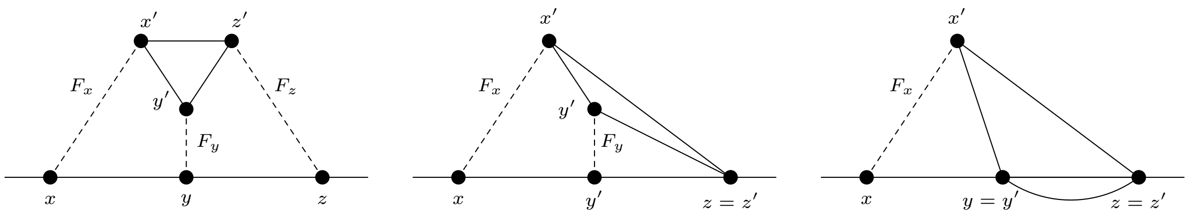


Figure 2.3: Possible structure of F

four internally disjoint v, C-paths, containing distinct vertices in C. Let $w, x, y, z \in V(C)$ be these vertices, and let P_α denote the path containing α, for $\alpha \in \{w, z, y, z\}$. Assume that amongst all choices of v, w, x, y, and z, $|P_x| + |P_y| + |P_z|$ is minimum, and subject to this assumption, we assume $|P_w|$ is minimum. Recall that $|P|$ denotes $|V(P)|$.

As the claw centered at v with one vertex from each of P_x, P_y, and P_z is not induced, v lies on a triangle T. For $a \in \{w, x, y, z\}$ let F_a denote the (unique) a, T-path that is a subpath of P_a, and let a' be the endpoint of F_a in T, where $w' = v$. It is possible that $a' = a$ if v is adjacent to a, and it is also possible that F_a is a trivial path of one vertex. However, since v is in $V(G) - V(C)$ and v is in $\{x', y', z'\}$, at most two of x', y' or z' lie on C. Finally, let $F = T \cup \left(\bigcup_{a \in \{x,y,z\}} F_a\right)$, and note that the minimality of $|P_x| + |P_y| + |P_z|$ implies that $F - \{x, y, z\}$ is induced.

Let $xx_1 \ldots x_{p+1}$, $yy_1 \ldots y_{q+1}$, and $zz_1 \ldots z_{t+1}$ denote the vertices on F_x, F_y and F_z, respectively, where x_{p+1}, y_{q+1}, and z_{t+1} denote x', y', and z', respectively. Similarly, let $ww_1 \ldots w_{r+1}$ denote the vertices on P_w, where w_{r+1} denotes w', which is v. Also, let $I_x = x_1 \ldots x_p$, $I_y = y_1 \ldots y_q$, $I_z = z_1 \ldots z_t$, and $I_w = w_1 \ldots w_r$. These are the interior subpaths of F_x, F_y, F_z, and P_w, respectively, and note that I_x, I_y, I_z, or I_w may be empty. In this case, a_1 is taken to be the neighbor of a on P_a, where $a \in \{w, x, y, z\}$.

Up to relabeling and reversing the orientation of C, assume $|I_x| \geq |I_y| \geq |I_z|$ and also that x, y and z appear on C in this order when traversing C in the clockwise direction. As a result, if v is adjacent to exactly one vertex on C, then this neighbor is $z' = z$, and if v is adjacent to exactly two vertices on C they are $y' = y$ and $z' = z$. See Figure 2.3.

The assumption that G contains no $(s+1)$-cycle also yields that $a^- a^+ \in E(G)$ for $a \in \{w, x, y, z\}$, as $\langle a; a_1, a^-, a^+ \rangle$ is not induced.

For the remainder of this section, when convenient we will let a denote an arbitrary element of $\{w, x, y, z\}$ and we will use a in a flexible manner that allows us to introduce notation relating to all of the vertices in $\{w, x, y, z\}$ without the need for tedious repetition. For instance, given the notation defined above, when unambiguous we will refer to P_a, F_a, I_a and so on.

2.4.2 Lemmas

Our first lemma follows routinely from the minimality of $|P_x| + |P_y| + |P_z|$ and the assumption that G contains no $(s+1)$-cycle.

Lemma 2.8. *If $\{x', y', z'\} \cap V(C) = \emptyset$, then there are no edges joining $V(F) \setminus \{x, y, z\}$ and $V(C)$ except for xx_1, yy_1 and zz_1. If $\{x', y', z'\} \cap V(C) = \{z\}$, i.e. $z = z'$, then there are no edges joining $V(F) \setminus \{x, y, z\}$ and $V(C)$ except xx_1, yy_1, $x'z$, and $y'z$. If $\{x', y', z'\} \cap V(C) = \{y, z\}$ and $|I_x| \geq 1$, then there are no edges joining $V(F) \setminus \{x, y, z\}$ and $V(C)$ except vy, vz, x_1x, and possibly x_1u for at most one $u \in V(C) \setminus \{x\}$.*

Our next lemmas provide useful structural information about various intervals of vertices on C.

Lemma 2.9. *If p and q are vertices on C such that $[p, q]_C \subseteq N(a) \cup \{a\}$, then $[p, q]_C$ induces a clique in G.*

Proof. Suppose $b, c \in [p, q]_C$ such that $bc \notin E(G)$. Since $[p, q]_C \subseteq N(a) \cup \{a\}$, $a \notin \{b, c\}$. Observe that $\langle a; a_1, b, c\rangle \to \{a_1b, a_1c\}$. Without loss of generality, assume $a_1b \in E(G)$. This implies $b \notin \{a^-, a^+\}$. Up to reversing the orientation of C, we assume $c \in (b, q]_C$. This implies that $b^+ \in [p, q]_C$ so that $b^+a \in E(G)$. However, $a^+Cba_1ab^+C^+a^-a^+$ is an $(s+1)$-cycle in G, a contradiction. □

For $a \in \{w, x, y, z\}$, let $Q_C(a) = [a_\ell, a_r]_C$ be the largest interval of C such that $a \in [a_\ell, a_r]_C$ and $[a_\ell, a_r]_C \subseteq N(a) \cup \{a\}$. When the context is clear, we will simply write $Q(a)$. By Lemma 2.9, $Q(a)$ induces a clique in G. Note that $Q(a)$ contains, at a minimum, the vertices a, a^- and a^+. Also, if $G[V(C)] \cong K_s$ we have $Q(a) = V(C)$ for all choices of a.

If $G[V(C)] \ncong K_s$, then Lemma 2.9 and the maximality of $Q(a)$ imply that a is adjacent to neither a_ℓ^- nor a_r^+. Additionally, as $Q(a)$ is a clique, no pair of vertices in $Q(a)$ can have a common neighbor in $V(G) - V(C)$, lest G contain a cycle of length $s+1$.

We will often use the following two lemmas without reference.

Lemma 2.10. *If $V(C)$ does not induce a complete graph, then a_ℓ and a_r are only adjacent to vertices in $V(C)$. In particular, neither a_ℓ nor a_r is in $\{w, x, y, z\}$.*

Proof. Suppose a_ℓ is adjacent to some vertex u not on C. Since $G[V(C)] \ncong K_s$, $a_\ell^- \notin Q(a)$ so that $a_\ell^- a \notin E(G)$. Further, as $a_\ell, a \in Q(a)$, a_ℓ and a cannot have a common neighbor off of C. In particular, $au \notin E(G)$.

Thus, $\langle a_\ell; a, a_\ell^-, u\rangle \to \{a_\ell^- u\}$. However, $a_\ell^- u a_\ell C a_\ell^-$ is an $(s+1)$-cycle. The case where a_r has some neighbor off of C is symmetric. □

Lemma 2.11. *Let $a, b \in \{w, x, y, z\}$. If $b \in Q(a)$, then $Q(a) = Q(b)$.*

Proof. We assume $G[V(C)] \ncong K_s$, otherwise we are done. Let $P = aI_a a'b'I_b b$. Since $b \in Q(a)$, a and b cannot have a common neighbor off of C.

Suppose $a_r^+ \in Q(b)$ so that $a_r \in Q(b)$. Observe that $\langle b; b_1, a, a_r^+ \rangle \to \{b_1 a_r^+\}$. However, $bb_1a_r^+Cb^-b^+Ca_rb$ is an $(s+1)$-cycle. Thus, $a_r^+ \notin Q(b)$, and by a similar argument $b_r^+ \notin Q(a)$. This implies that $a_r = b_r$, and by symmetry, $a_\ell = b_\ell$. This proves the lemma. □

Let $\mathcal{O}$ denote the set of vertices in $V(C)$ that have a neighbor off of C. By Lemma 2.10, if $u \in \mathcal{O}$, then $u \notin \{a_\ell, a_r\}$, and furthermore, u^-u^+ is an edge in G for any such u. Additionally, suppose that $u_1, \dots, u_m$, $m \geq 2$ are vertices in $\mathcal{O}$ that appear consecutively on C in that order. It is not difficult to prove by induction and considering various claws that since G contains no $(s+1)$-cycle, $u_1^- u_m^+$ is an edge in G. Thus, for any $S \subseteq \mathcal{O}$, we can naturally define a cycle C_S from C in which $V(C_S) = V(C) \setminus S$, every vertex of C_S appears in the same order as in C, and for each $a \in \{w, x, y, z\} \setminus S$, $Q_{C_S}(a)$ has the same endpoints as $Q_C(a)$, namely a_ℓ and a_r.

The following lemma by Gould, Łuczak and Pfender [24] will be useful in proving the final lemmas of this section.

Lemma 2.12. *Let H be a claw-free graph with minimum degree $\delta(H) \geq 3$, and let $\hat{C}$ be a cycle of length t with no hops, for some $t \geq 5$. Set*

$$S = \{u \in V(\hat{C}) | \text{there is no chord incident to } u\},$$

and suppose for some chord pq of $\hat{C}$ we have $|S \cap [p,q]_{\hat{C}}| \leq 2$. Then H contains cycles C' and C'' of lengths $t-1$ and $t-2$, respectively.

Lemma 2.13. *Let $a, b \in \{w, x, y, z\}$, and let P be an $a-b$ path of length λ with no internal vertices on C. If $2 \leq \lambda \leq 5$, then $|(a_r, b_\ell)_C| \geq \lambda - 1$ and $|(b_r, a_\ell)_C| \geq \lambda - 1$.*

Proof. Throughout the proof, let c and d be distinct vertices such that $\{a, b, c, d\} = \{w, x, y, z\}$. First, if $G[V(C)] \cong K_s$, then as $s \geq 5$, we obtain an $(s+1)$-cycle for $2 \leq \lambda \leq 5$. Thus, going forward we will assume that $G[V(C)] \ncong K_s$.

Suppose $a \in Q(b)$ so that $Q(a) = Q(b)$ by Lemma 2.11. We may assume that C is arranged so that a and b are consecutive along C. For $\lambda = 2, 3, 4$, let $\tilde{C} = C, C_{\{c\}}, C_{\{c,d\}}$, respectively. Then $aPb\tilde{C}a$ is an $(s+1)$-cycle.

For $\lambda = 5$, we may assume that $\mathcal{O} = \{w, x, y, z\}$. If not, then we can choose $S \subseteq \mathcal{O} \setminus \{a, b\}$ with $|S| = 3$, so that $aPbC_Sa$ is an $(s+1)$-cycle. As a result, let $\hat{C}$ denote the $(s+2)$-cycle $aPbC_{\{c,d\}}a$. Since we have no $(s+1)$-cycle, $\hat{C}$ contains no hops. We aim to apply Lemma 2.12 to $\hat{C}$.

Since $\lambda = 5$, we deduce that $s \geq 8$ so that $|V(\hat{C})| \geq s+2 \geq 10$. As a result, $(b,a)_{\hat{C}} \setminus \{c_\ell, c_r\} \neq \emptyset$. We claim that for each $u \in (b,a)_{\hat{C}} \setminus \{c_\ell, c_r\}$, u is incident to a chord in $\hat{C}$. If not, then as G is 4-connected, $d(u) \geq 4$. Since $u \notin \mathcal{O}$, u must be adjacent to both c and d. Thus, $\langle c; c_1, u, c_\ell \rangle \to \{uc_\ell\}$ and $\langle c; c_1, u, c_r \rangle \to \{uc_r\}$. However, if u is not incident to a chord in $\hat{C}$, then u must appear directly between c_ℓ and c_r in $\hat{C}$. This implies that $u \in Q_C(c)$, which in turn creates a hop in $\hat{C}$, a contradiction.

Thus, every vertex in $(b,a)_{\hat{C}} \setminus \{c_\ell, c_r\}$ is incident to a chord in $\hat{C}$. Furthermore, ab is a chord in $\hat{C}$ and setting $p = b$ and $q = a$ satisfies Lemma 2.12, which leads to a contradiction..

So we may assume $a \notin Q(b)$, and by symmetry, $b \notin Q(a)$. Suppose $a_r \in [b_\ell, b)$. Observe that $b_\ell \in (a, a_r]_C$. If $\lambda = 2$, then $aPbC^- b_\ell b^+ Ca^- b_\ell^- C^- a$ is an $(s+1)$-cycle. For $\lambda = 3, 4, 5$, let $\tilde{C} = C, C_{\{c\}}, C_{\{c,d\}}$, respectively. Then $aPb\tilde{C}^- a_r^+ b^+ \tilde{C} a^- a_r^- \tilde{C}^- a$ is an $(s+1)$-cycle.

Thus, $a_r \notin Q(b)$, and by symmetry, $a_\ell \notin Q(b), b_\ell \notin Q(a)$, and $b_r \notin Q(a)$. Without loss of generality, we may assume $|(a_r, b_\ell)| < \lambda - 1$. Suppose $a_r = b_\ell^-$. If $\lambda = 2$, then $aPbC^- b_\ell b^+ Ca^- a_r C^- a$ is an $(s+1)$-cycle. For $\lambda = 3, 4, 5$ let $\tilde{C} = C, C_{\{c\}}, C_{\{c,d\}}$, respectively. Then $aPb\tilde{C}^- b_\ell b^+ \tilde{C} a^- a_r^- \tilde{C}^- a$ is an $(s+1)$-cycle.

Thus, $|(a_r, b_\ell)_C| \geq 1$. If $|(a_r, b_\ell)_C| = \lambda - 2$, $aPbC^- b_\ell b^+ Ca^- a_r C^- a$ is an $(s+1)$-cycle. If $|(a_r, b_\ell)_C| = \lambda - 3$, then $aPbC^- b_\ell b^+ Ca^- a_r^- C^- a$ is an $(s+1)$-cycle. If $|(a_r, b_\ell)_C| = \lambda - 4$, then $aPbC^- b_\ell^+ b^+ Ca^- a_r^- C^- a$ is an $(s+1)$-cycle.

This completes all cases and proves the lemma. $\square$

Lemma 2.14. *Let $a, b \in \{w, x, y, z\}$, and let P be an $a - b$ path of length λ, $2 \leq \lambda \leq 5$, with no internal vertices on C. If there is an edge between $\{a_\ell^-, a_\ell, a\}$ and $\{b_\ell^-, b_\ell, b\}$ or between $\{a, a_r, a_r^+\}$ and $\{b, b_r, b_r^+\}$, then it is ab.*

Proof. Without loss of generality, suppose there exists an edge between $\{a_\ell^-, a_\ell, a\}$ and $\{b_\ell^-, b_\ell, b\}$. Throughout the proof, let c and d be such that $\{a, b, c, d\} = \{w, x, y, z\}$. By Lemma 2.13, $|(a_r, b_\ell)_C|, |(b_r, a_\ell)_C| \geq \lambda - 1 \geq 1$, and as a result, $s \geq 8$.

Let $\tilde{b}$ be the neighbor of b on P. We claim that $a_\ell^- \tilde{b} \notin E(G)$, so suppose on the contrary that $a_\ell^- \tilde{b} \in E(G)$. Without loss of generality, suppose $a_\ell^- \neq d$. If $\lambda = 2$, then $a_\ell^- \tilde{b} a C^- a_\ell a^+ C a_\ell^-$ is an $(s+1)$-cycle. For $\lambda = 3, 4, 5$, let $\tilde{C} = C, C_{\{b\}}, C_{\{b,d\}}$, respectively. Then $a_\ell^- \tilde{b} P a \tilde{C}^- a_\ell^+ a^+ \tilde{C} a_\ell^-$ is an $(s+1)$-cycle. Thus, $a_\ell^- \tilde{b} \notin E(G)$ as claimed.

Observe that if $a_\ell^- b \in E(G)$, then $\langle b; \tilde{b}, b_\ell, a_\ell^- \rangle \to \{a_\ell^- \tilde{b}, a_\ell^- b_\ell\}$, and if $a_\ell b \in E(G)$, then $\langle b; \tilde{b}, b_\ell, a_\ell \rangle \to \{a_\ell, b_\ell\}$. So up to symmetry, it suffices to consider the edges $a_\ell^- b_\ell^-, a_\ell^- b_\ell$, and $a_\ell b_\ell$.

We now consider the cases when $\lambda \in \{2, 3, 4\}$ and deal with the case $\lambda = 5$ at the end. We present this in the following table, where we assume without loss of generality that $a_\ell^- \neq d$.

$\lambda = 2$		$\lambda = 3$	
Edge	Cycle	Edge	Cycle
$a_\ell^- b_\ell^-$	$a_\ell^- b_\ell^- C a^+ a_\ell C a P b C^- b_\ell b^+ C a_\ell^-$	$a_\ell^- b_\ell^-$	$a_\ell^- b_\ell^- C^- a^+ a_\ell C a P b C^- b_\ell^+ b^+ C a_\ell^-$
$a_\ell^- b_\ell$	$a_\ell^- b_\ell C a^+ a_\ell C a P b C^- b_\ell^+ b^+ C a_\ell^-$	$a_\ell^- b_\ell$	$a_\ell^- b_\ell C^- a^+ a_\ell^+ C a P b C^- b_\ell^+ b^+ C a_\ell^-$
$a_\ell b_\ell$	$a_\ell b_\ell C^- a^+ a_\ell^+ C a P b C^- b_\ell^+ b^+ C a_\ell$	$a_\ell b_\ell$	$a_\ell b_\ell C_{\{d\}}^- a^+ a_\ell^+ C_{\{d\}} a P b C_{\{d\}}^- b_\ell^+ b^+ C_{\{d\}} a_\ell$

$\lambda = 4$	
Edge	Cycle
$a_\ell^- b_\ell^-$	$a_\ell^- b_\ell^- C^- a^+ a_\ell^+ C a P b C^- b_\ell^+ b^+ C a_\ell^-$
$a_\ell^- b_\ell$	$a_\ell^- b_\ell C_{\{d\}}^- a^+ a_\ell^+ C_{\{d\}} a P b C_{\{d\}}^- b_\ell^+ b^+ C_{\{d\}} a_\ell^-$
$a_\ell b_\ell$	$a_\ell b_\ell C_{\{c,d\}}^- a^+ a_\ell^+ C_{\{c,d\}} a P b C_{\{c,d\}}^- b_\ell^+ b^+ C_{\{c,d\}} a_\ell$

So we may assume $\lambda = 5$. Suppose first that $a_\ell b_\ell \in E(G)$. Then $\mathcal{O} = \{w, x, y, z\}$, otherwise we can choose $S \subseteq \mathcal{O} \setminus \{a, b\}$ with $|S| = 3$ so that $a_\ell b_\ell C_S^- a^+ a_\ell^+ C_S a P b C_S^- b_\ell^+ b^+ C_S a_\ell$ is an $(s+1)$-cycle.

Since we can easily rearrange vertices within $Q_C(a)$ and $Q_C(b)$, we may assume that $a^- = a_\ell$ and $b^- = b_\ell$. As a result, let $\hat{C}$ denote the $(s+2)$-cycle $a_\ell b_\ell C_{\{c,d\}}^- a P b C_{\{c,d\}} a_\ell$ (see Figure 2.4). Since we have no $(s+1)$-cycle, $\hat{C}$ contains no hops. We aim to apply Lemma 2.12 to $\hat{C}$.

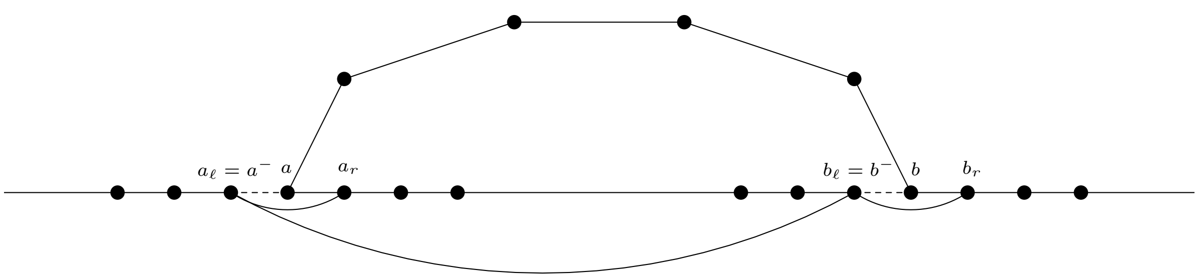


Figure 2.4: Construction of $\hat{C}$

By Lemma 2.13, $|(a_r, b_\ell)_C|, |(b_r, a_\ell)_C| \geq \lambda - 1 = 4$, and as a result, $s \geq 16$. Observe that $a_\ell a_r$ is a chord in $\hat{C}$. We claim that for each $u \in (a_\ell, a_r)_{\hat{C}} \setminus \{c_\ell, c_r\}$, u is incident to a chord in $\hat{C}$. If not, then as G is 4-connected, $d(u) \geq 4$. Since $u \notin \mathcal{O}$, u must be adjacent to both c and d. Thus, $\langle c; c_1, u, c_\ell \rangle \to \{uc_\ell\}$ and $\langle c; c_1, u, c_r \rangle \to \{uc_r\}$. However, if u is not incident to a chord in $\hat{C}$, u must appear directly between c_ℓ and c_r in $\hat{C}$. This implies that $u \in Q_C(c)$, which in turn creates a hop in $\hat{C}$, a contradiction.

Thus, every vertex in $(a_\ell, a_r)_{\hat{C}} \setminus \{c_\ell, c_r\}$ is incident to a chord in $\hat{C}$. Since $a_\ell a_r$ is a chord in $\hat{C}$, setting $p = a_\ell$ and $q = a_r$ satisfies Lemma 2.12.

So it remains to consider when either $a_\ell^- b_\ell^-$ or $a_\ell^- b_\ell$ is an edge. If $a_\ell^- \notin \mathcal{O}$, then without loss of generality assume $b_\ell^- \neq d$. If $a_\ell^- b_\ell^- \in E(G)$, then $a_\ell^- b_\ell^- C_{\{c\}}^- a^+ a_\ell^+ C_{\{c\}} a P b C_{\{c\}}^- b_\ell^+ b^+ C_{\{c\}} a_\ell^-$ is an $(s+1)$-cycle. If

$a_\ell^- b_\ell \in E(G)$, then $a_\ell^- b_\ell C_{\{c,d\}}^- a^+ a_\ell^+ C_{\{c,d\}} a P b C_{\{c,d\}}^- b_\ell^+ b^+ C_{\{c,d\}} a_\ell^-$ is an $(s+1)$-cycle.

So we may assume $a_\ell^- \in \mathcal{O}$. Now $a_\ell^- \in \{c,d\}$, otherwise the previous two cycles produce $(s+1)$-cycles. So without loss of generality, assume $a_\ell^- = c$. Now if $a_\ell^- b_\ell = cb_\ell \in E(G)$, then $\langle c; c_1, a_\ell, b_\ell \rangle \to \{a_\ell b_\ell\}$. However, we have already dealt with the case when $a_\ell b_\ell \in E(G)$. Thus, we may assume $a_\ell^- b_\ell^- \in E(G)$, and by symmetry, $b_\ell^- = d$. Lastly, $\mathcal{O} = \{w,x,y,z\}$, as otherwise we can replace $C_{\{c\}}$ with $C_{\{e\}}$ in the previous paragraph, where $e \in \mathcal{O} \setminus \{w,x,y,z\}$.

Since we can easily rearrange the vertices within $Q_C(a)$ and $Q_C(b)$, we may assume that $a^- = a_\ell$ and $b^- = b_\ell$. Let $\hat{C}$ denote the $(s+2)$-cycle $a_\ell^- b_\ell^- C^- a P b C a_\ell^-$ (see Figure 2.5). Since we have no $(s+1)$-cycle, $\hat{C}$ contains no hops. As before, we aim to use Lemma 2.12.

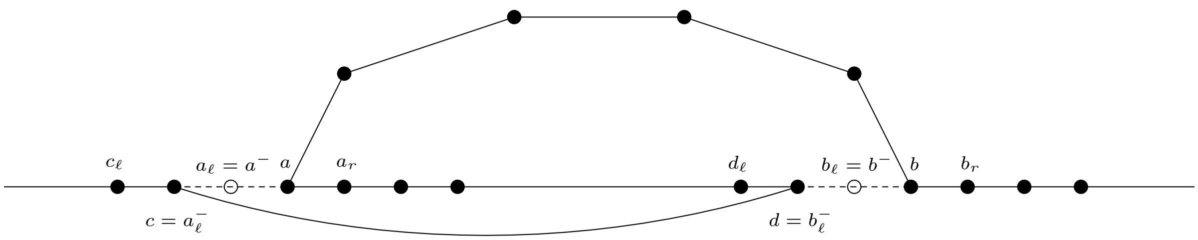


Figure 2.5: Construction of $\hat{C}$

Recall that $|(a_r, b_\ell)_C| \geq 4$ by Lemma 2.13. Thus, $(a_r, d_\ell)_C$ is nonempty, and we claim that every vertex $u \in [a_r, d_\ell)_C$ is incident to a chord in $\hat{C}$. If not, then as G is 4-connected, $d(u) \geq 4$. Since $u \notin \mathcal{O}$, u must be adjacent to both $a_\ell = c_r$ and $b_\ell = d_r$. Thus $\langle b_\ell; b, d, u \rangle \to \{ub, ud\}$, as $d = b_\ell^-$. However, both of these are clearly chords in $\hat{C}$, a contradiction.

So every vertex in $[a_r, d_\ell)_C$ is incident to a chord in $\hat{C}$, and by symmetry, this holds for every every vertex in $[a_r, d_\ell]_C$. A similar argument shows that this also applies to every vertex in $[b_r, c_\ell]_C$. Thus, the only vertices in $\hat{C}$ that are potentially not incident to a chord are $a_\ell^- = c$, $b_\ell^- = d$, and the vertices on P. Thus, as both $[a_r, d_\ell]_C$ and $[b_r, c_\ell]_C$ are both nonempty, any chord incident to any of these vertices suffices for our choice of pq in Lemma 2.12.

This completes all cases and proves the lemma. $\square$

Lemma 2.15. *Let $a, b \in \{w,x,y,z\}$, and let $P = aI_a a' b' I_b b$ have length λ, $2 \leq \lambda \leq 5$. If there exists an edge, e, between $\{a_\ell^-, a_\ell, a\}$ and $\{b_\ell^-, b_\ell, b\}$, or between $\{a, a_r, a_r^+\}$ and $\{b, b_r, b_r^+\}$, then either*

- *$3 \leq \lambda \leq 5$, $e = xw$, and $y', z' \in V(C)$, or*

- *$\lambda = 2$, $e = ab$, and $P = avb$.*

In particular, if $a, b \in \{x, y, z\}$ *and* $3 \leq \lambda \leq 5$*, then there is no edge between* $\{a_\ell^-, a_\ell, a\}$ *and* $\{b_\ell^-, b_\ell, b\}$*, or between* $\{a, a_r, a_r^+\}$ *and* $\{b, b_r, b_r^+\}$*.*

Proof. By Lemmas 2.13 and 2.14, $|(a_r, b_\ell)_C|, |(b_r, a_\ell)_C| \geq \lambda - 1 \geq 1$, and $e = ab$, respectively. This implies that $G[V(C)] \ncong K_s$. Suppose $3 \leq \lambda \leq 5$. Consider first the case in which $a, b \in \{x, y, z\}$. Since $\lambda \geq 3$, either a' or b' is not on C. In addition, by Lemma 2.8, either $a_1 b$ or ab_1 is not an edge. Without loss of generality, suppose $a_1 b \notin E(G)$. However, $\langle a; a_1, a_\ell, b\rangle \to \{a_\ell b\}$, which contradicts Lemma 2.14.

So suppose $a = w$ and $b \in \{x, y, z\}$. As $w' = v$, v is an internal vertex on P. Suppose $v = b_1$. Then $\langle b; v, b_\ell, w\rangle \to \{vw, b_\ell w\}$. Since $\lambda \geq 3$ and $v = b_1$, $vw \in E(G)$ contradicts the choice of P_w. So $b_\ell w \in E(G)$, however this contradicts Lemma 2.14. Thus, $v \neq b_1$.

By Lemma 2.14, $\langle b; b_1, b_\ell, w\rangle \to \{b_1 w, b_\ell w\}$. By Lemma 2.14, $b_1 w \in E(G)$. Let c and d be vertices such that $\{b, c, d\} = \{x, y, z\}$. Suppose that y' is not on C so that neither x' nor y' are on C. Consider the two paths $b_1 w$ and $b_1 b$. If $b \neq z$, then add to these two paths $b_1 b_2 \ldots b' z' I_z z$. Otherwise, add the path $b_1 b_2 \ldots b' y' I_y y$. In both cases, the three paths from b_1 to C contain w, but do not contain x' or x, contradicting the minimality of $|P_x| + |P_y| + |P_z|$. Thus, y' is on C, which implies that z' is on C. Since $b_1 \neq v$, b must be x, which completes the proof of the first statement of the lemma.

Suppose $\lambda = 2$. Since Lemma 2.14 implies that $e = ab$, it suffices to show that $P = avb$. Since $P = aI_a a' b' I_b b$, if $w \in \{a, b\}$, or if $a' = a$ and $b' = b$, then we are done. Since $\lambda = 2$, it cannot be that both $a' \neq a$ and $b' \neq b$, so we may assume $a' \neq a$ and $b' = b$.

If $a' = v$, then we are done. Thus, $b \neq y$, else $z' = z$ and $a' = x' = v$. So $b = z$ and $a \in \{x, y\}$. However, $\langle b; b_\ell, a, v\rangle \to \{ab_\ell\}$, which contradicts Lemma 2.14. This completes the proof of the lemma. $\square$

Together these lemmas assist us in showing a 4-connected $\{K_{1,3}, N(4,1,1)\}$-free graph G containing an s-cycle, C, also contains an $(s+1)$-cycle. In particular, we will often use Lemmas 2.8, 2.13, and 2.15 without reference to help simplify this case analysis.

2.5 Long Cycles for $N(4,1,1)$

In this section we prove the following lemma.

Lemma 2.16. *Every* 4*-connected* $\{K_{1,3}, N(4,1,1)\}$*-free graph of order* n *containing a cycle of length* s*, where* $5 \leq s \leq n-1$*, has an* $(s+1)$*-cycle.*

Lemma 2.7 implies G has cycles of length 3, 4, and 5. This together with Lemma 2.16 completes the proof of Theorem 2.2. Throughout the remainder of the paper we adopt the terminology and structure developed in Section 2.4.

We proceed by contradiction and assume that G is 4-connected, $\{K_{1,3}, N(4,1,1)\}$-free, and contains a cycle of length s, where $5 \leq s \leq n-1$, but does not contain an $(s+1)$-cycle. We show that either G has an $(s+1)$-cycle or has an induced $N(4,1,1)$, both of which give contradictions. The proof is broken up into cases based on how many vertices of x', y' and z' are in $V(C)$. We ultimately reduce to the case in which v has at least four neighbors on C, and that $N_C(v)$ induces a clique.

2.5.1 $V(C) \cap \{x', y', z'\} = \emptyset$

Recall that by the minimality of $|P_x|+|P_y|+|P_z|$, $F - \{x, y, z\}$ is induced. Furthermore, Lemma 2.8 implies that the possible only edges between $V(F) - \{x, y, z\}$ and $V(C)$ are xx_1, yy_1, and zz_1. Thus, we will never consider the situation in which any edge exists that violates either of the previous two statements.

Furthermore, for all $a, b \in \{x, y, z\}$, the path $P = aI_a a' b' I_b b$ has length at least three. If the length of P is at most five, then by Lemma 2.15 there are no edges between $\{a_\ell^-, a_\ell, a\}$ and $\{b_\ell^-, b_\ell, b\}$, or between $\{a, a_r, a_r^+\}$ and $\{b, b_r, b_r^+\}$. We will also use this fact throughout the proof.

Case 1.1 $|I_x| \geq 4$

In any situation, $N(x'y'z'; I_x, I_y, I_z)$ contains an induced $N(4,1,1)$.

Case 1.2 $|I_x| = 3$.

When $|I_y|, |I_z| \geq 1$, $N(x'y'z'; I_x x, y_q, z_t)$ contains an induced $N(4,1,1)$. So, we may assume $|I_z| = 0$.

If $|I_y| \geq 1$, then $N(x'y'z'; I_x x, I_y, z) \to \{xz\}$. If $|I_y| = 0$, then $N(x'y'z'; I_x x, y, z) \to \{xy, xz\}$. By symmetry, we may assume in both cases that $xz \in E(G)$.

We claim that we may assume that z and x are consecutive along C in the clockwise direction. Indeed, $\langle x; x_1, x^-, z\rangle \to \{x^- z\}$, so that if z and x are not consecutive, we replace C with the cycle $x^- z x C z^- z^+ C x^-$. So we may assume this, and in addition, $Q_C(x) = Q_C(z)$ by Lemma 2.11.

If $|I_y| \geq 2$, then $N(zxz_\ell; z'y'I_y, x_1, z_\ell^-)$ contains an induced $N(4,1,1)$. If $|I_y| \leq 1$, then by Lemma 2.13, $Q_C(z)$ and $Q_C(y)$ are disjoint, and furthermore, $z_\ell^- \notin Q_C(y)$. Thus, $N(zxz_\ell; z'y'I_y y y_\ell, x_1, z_\ell^-) \to \{xy\}$. However, $\langle x; x_1, z, y\rangle$ is induced, a contradiction.

Case 1.3 $|I_x| = 2$.

In any situation, $N(x'y'z'; I_x x x_\ell, I_y, I_z)$ contains an induced $N(4,1,1)$.

Case 1.4 $|I_x| = 1$.

In any situation, $N(x'y'z'; I_x x x_\ell x_\ell^-, I_y, I_z)$ contains an induced $N(4,1,1)$.

Case 1.5 $|I_x| = 0$.

By Lemma 2.13, $|(x_r, y_\ell)_C|, |(y_r, z_\ell)_C|, |(z_r, x_\ell)_C| \geq 2$. In this case, out of all cycles on vertex set $V(C)$, assume that C is chosen to maximize the value of $|Q(x) \cup Q(y) \cup Q(z)|$. That is, for all cycles $\hat{C}$ such that $V(\hat{C}) = V(C)$, $|Q_C(x) \cup Q_C(y) \cup Q_C(z)| \geq |Q_{\hat{C}}(x) \cup Q_{\hat{C}}(y) \cup Q_{\hat{C}}(z)|$.

Claim 2.1. *There are no edges between $\{x, y, z\}$ and $V(C) \setminus (Q(x) \cup Q(y) \cup Q(z))$.*

Proof. We consider the case for x as all other cases follow similarly. Let $a, b \in \{x, y, z\}$ such that $(a_r, b_\ell)_C \cap \{x, y, z\} = \emptyset$, and suppose $x \neq b$.

Suppose on the contrary, $x\alpha \in E(G)$, where $\alpha \in (a_r, b_\ell)_C$ and $x\beta \notin E(G)$, for all $\beta \in (a_r, \alpha)_C$. First, note that $\alpha \neq b_l^-$ by Lemma 2.15. So, $\alpha \in (a_r, b_\ell^-)_C$.

Observe $\langle x; x', x^-, \alpha\rangle \to \{x^-\alpha\}$, and $\langle \alpha; \alpha^-, \alpha^+, x\rangle \to \{\alpha^-\alpha^+, x\alpha^+\}$. If $\alpha^-\alpha^+ \in E(G)$, then we obtain a new s-cycle $\hat{C} = x^-\alpha x C \alpha^-\alpha^+ C x^-$. However, $|Q_{\hat{C}}(x) \cup Q_{\hat{C}}(y) \cup Q_{\hat{C}}(z)| > |Q_C(x) \cup Q_C(y) \cup Q_C(z)|$, a contradiction to the choice of C. So, we may assume $x\alpha^+ \in E(G)$.

By Lemma 2.15, $\alpha^+ \neq b_\ell^-$ so that $\alpha^+ \in (\alpha, b_\ell^-)_C$. So $N(x\alpha\alpha^+; x'b'bb_r, \alpha^-, \alpha^{+2}) \to \{x\alpha^{+2}, \alpha^-\alpha^+, \alpha^-\alpha^{+2}, \alpha^-b, \alpha^-b_r, \alpha\alpha^{+2}, \alpha b, \alpha b_r, \alpha^+b, \alpha^+b_r, \alpha^{+2}b, \alpha^{+2}b_r\}$.

By the maximality of $|Q_C(x) \cup Q(y) \cup Q(z)|$, $\alpha^-\alpha^+, \alpha\alpha^{+2}, \alpha^-\alpha^{+2} \notin E(G)$. If $x\alpha^{+2} \in E(G)$, then $\langle x; x', \alpha, \alpha^{+2}\rangle \to \{\alpha\alpha^{+2}\}$, a contradiction. Note that if ub is an edge where $u \in V(C) \setminus Q(b)$, then $\langle b; b_1, u, b_r\rangle \to \{ub_r\}$. Thus the rest of the edges we need to consider are edges incident to b_r. The following table provides the $(s+1)$-cycles formed by these edges.

Edge	Cycle	Edge	Cycle
α^-b_r	$\alpha^-b_rCz^-z^+Cx^-\alpha Cb^-b_r^-C^-bb'x'xC\alpha^-$	αb_r	$\alpha b_rCz^-z^+Cx^-\alpha^+Cb^-b_r^-C^-bb'x'xC\alpha$
α^+b_r	$\alpha^+b_rCz^-z^+Cx^-\alpha C^-xx'b'bC^-\alpha^+$	$\alpha^{+2}b_r$	$\alpha^{+2}b_rCz^-z^+Cx^-\alpha^+C^-xx'b'bC^-\alpha^{+2}$

Thus, x is not adjacent to any vertex in $(a_r, b_\ell)_C$. This proves the claim. ☐

By Claim 2.1, $N(x'y'z'; xx_rx_r^+x_r^{+2}, y, z) \to \{x_rx_r^{+2}\}$. Similarly, $N(x'y'z'; xx_rx_r^{+2}x_r^{+3}, y, z) \to \{x_rx_r^{+3}\}$. We can inductively continue this to obtain $x_ry_\ell^- \in E(G)$.

Observe $N(xx_\ell x_r; x'z'zz_\ell, x_\ell^-, y_\ell^-) \to \{x_\ell^-x_r, x_rz_\ell\}$. However, for $\beta \in \{x_\ell^-, z_\ell\}$, $\langle x_r; x, y_\ell^-, \beta\rangle$ is induced.

This completes the case where y', z' are not in $V(C)$.

2.5.2 $V(C) \cap \{x', y', z'\} = \{z'\}$

Recall that by the minimality of $|P_x| + |P_y| + |P_z|$, $F - \{x, y, z\}$ is induced. Furthermore, Lemma 2.8 impies that the only possible edges between $V(F) - \{x, y, z\}$ and $V(C)$ are $xx_1, yy_1, x'z$, and $y'z$. Thus, we will never consider the situation in which any edge exists that violates either of the previous two statements.

Suppose first that $|I_x| \geq |I_y| \geq 1$. If $|I_x| \geq 4$, then $N(x'y'z; I_x, I_y, z_\ell)$ contains an induced $N(4,1,1)$. If $1 \leq |I_x| \leq 3$, then by Lemma 2.13, $|(x_r, z_\ell)_C| \geq 1$ and $|(z_r, x_\ell)_C| \geq 1$. Here $N(x'y'z; I_x x x_\ell x_\ell^-, I_y, z_\ell)$ contains an induced $N(4,1,1)$.

So in the following cases, we always assume $|I_y| = 0$, so that by Lemma 2.13, $|(y_r, z_\ell)_C| \geq 1$ and $|(z_r, y_\ell)_C| \geq 1$. In this case, Lemma 2.15 allows for the existence of xz or yz as edges. However, we will show that is not the case. Indeed, if say $xz \in E(G)$, which implies that $|I_x| = 0$ and $x = v$ by Lemma 2.15, then $\langle z; z_\ell, y', x\rangle$ is induced. A similar argument shows that $yz \notin E(G)$.

Case 2.1 $|I_x| \geq 4$.

Here $N(x'y'z; I_x, y, z_\ell)$ contains an induced $N(4,1,1)$.

Case 2.2 $|I_x| = 3$.

Observe that $N(x'y'z; I_x x, y, z_\ell) \to \{xy\}$. Additionally, $\langle x; x_1, x^+, y\rangle \to \{x^+y\}$. We claim that we may assume that x and y are consecutive along C in the clockwise direction. Indeed, we can replace C with the cycle $xyx^+Cy^-y^+Cx$. So we may assume this, and in addition, $Q_C(x) = Q_C(y)$ by Lemma 2.11. However, $N(yxx_\ell, y'zz_\ell z_\ell^-, x_1, x_\ell^-)$ is an induced $N(4,1,1)$.

Case 2.3 $|I_x| = 2$.

Here $N(x'y'z; I_x x x_\ell, y, z_\ell)$ is an induced $N(4,1,1)$.

Case 2.4 $|I_x| = 1$.

Here $N(x'y'z; I_x x x_\ell x_\ell^-, y, z_\ell)$ is an induced $N(4,1,1)$.

Case 2.5 $|I_x| = 0$.

By Lemma 2.13, $|(x_r, y_\ell)_C| \geq 2$ and $|(y_r, z_\ell)_C|, |(z_r, x_\ell)_C| \geq 1$. In this case, out of all cycles on the vertex set $V(C)$, assume that C is chosen to maximize the value of $|Q(x) \cup Q(y) \cup Q(z)|$.

Claim 2.2. *There are no edges between $\{x, y, z\}$ and $V(C) - (Q(x) \cup Q(y) \cup Q(z))$.*

Proof. We consider the case for z as the cases for x and y are similar to that in Claim 2.1. Let $a, b \in \{x, y, z\}$ such that $(a_r, b_\ell)_C \cap \{x, y, z\} = \emptyset$, and $b \neq z$.

Suppose on the contrary, that $z\alpha \in E(G)$ where $\alpha \in (a_r, b_\ell)_C$ and $z\beta \notin E(G)$ for all $\beta \in (a_r, \alpha)_C$. By Lemma 2.15, $\alpha \neq b_\ell^-$ so that $\alpha \in (a_r, b_\ell^-)_C$. Now $\langle z; x', z^-, \alpha\rangle \to \{\alpha z^-\}$, and $\langle \alpha; \alpha^-, \alpha^+, z\rangle \to \{\alpha^-\alpha^+, \alpha^+ z\}$. If $\alpha^-\alpha^+ \in E(G)$, then we obtain a new s-cycle $\hat{C} = z^-\alpha z C \alpha^- \alpha^+ C z^-$. However, $|Q_{\hat{C}}(x) \cup Q_{\hat{C}}(y) \cup Q_{\hat{C}}(z)| > |Q_C(x) \cup Q_C(y) \cup Q_C(z)|$, a contradiction to the choice of C. So we may assume $\alpha^+ z \in E(G)$.

By Lemma 2.15, $\alpha^+ \neq b_\ell^-$ so that $\alpha^+ \in (\alpha, b_\ell^-)_C$. So $N(z\alpha\alpha^+; b'bb_r b_r^+, \alpha^-, \alpha^{+2}) \to \{z\alpha^{+2}, \alpha^-\alpha^+, \alpha^-\alpha^{+2}, \alpha\alpha^{+2}, \gamma\delta : \gamma \in \{b, b_r, b_r^+\}, \delta \in \{\alpha^-, \alpha, \alpha^+, \alpha^{+2}\}\}$.

By the maximality of $|Q_C(x) \cup Q_C(y) \cup Q_C(z)|$, $\alpha^-\alpha^+, \alpha^-\alpha^{+2}, \alpha\alpha^{+2} \notin E(G)$. if $z\alpha^{+2} \in E(G)$, then $\langle z; x', \alpha, \alpha^{+2}\rangle \to \{\alpha\alpha^{+2}\}$, a contradiction. Note that if $ub \in E(G) - Q(b)$, then $\langle b; b', b_r, u\rangle \to \{ub_r\}$. Thus, the rest of the edges we need to consider are either incident to b_r or b_r^+. The following table provides the $(s+1)$-cycles formed by these edges.

Edge	Cycle	Edge	Cycle
$\alpha^- b_r$	$\alpha^- b_r C z^- \alpha C b^- b_r^- C^- b b' z C \alpha^-$	$\alpha^- b_r^+$	$\alpha^- b_r^+ C z^- \alpha C b^- b_r C^- b b' z C \alpha^-$
αb_r	$\alpha b_r C z^- \alpha^+ C b^- b_r^- C^- b b' z C \alpha$	αb_r^+	$\alpha b_r^+ C z^- \alpha^+ C b^- b_r C^- b b' z C \alpha$
$\alpha^+ b_r$	$\alpha^+ b_r C z^- \alpha C^- z b' b C b_r^- b^- C^- \alpha^+$	$\alpha^+ b_r^+$	$\alpha^+ b_r^+ C z^- \alpha C^- z b' b C b_r b^- C^- \alpha^+$
$\alpha^{+2} b_r$	$\alpha^{+2} b_r C z^- \alpha^+ C^- z b' b C b_r^- b^- C^- \alpha^{+2}$	$\alpha^{+2} b_r^+$	$\alpha^{+2} b_r^+ C z^- \alpha^+ C^- z b' b C b_r b^- C^- \alpha^{+2}$

Thus, z is not adjacent to any vertex in $(a_r, b_\ell)_C$, and as mentioned similar arguments hold for x and y. This proves the claim. $\square$

Recall that $|(y_r, z_\ell)_C|, |(z_r, x_\ell)_C| \geq 1$. If $|(y_r, z_\ell)_C| = 1$, then $yy'x'zC^- z_\ell z^+ C y^- y_r C^- y$ is an $(s+1)$-cycle. So, $|(y_r, z_\ell)_C| \geq 2$, and by a similar argument, $|(z_r, x_\ell)_C| \geq 2$. Our goal is to show that $z_\ell^- z_r^+, x_\ell^- x_r^+ \in E(G)$ and then apply an inductive argument to obtain $x_r z_r^+ \in E(G)$, a contradiction to Lemma 2.15. This will complete the case.

Claim 2.3. *$z_\ell^- z_r^+ \in E(G)$.*

Proof. Suppose on the contrary that $z_\ell^- z_r^+ \notin E(G)$. As a result, $z_\ell^- z_r, z_\ell z_r^+ \notin E(G)$, otherwise $\langle z_r; z, z_\ell^-, z_r^+\rangle$ and $\langle z_r; z, z_\ell, z_r^+\rangle$ are induced claws. So, $N(zz_\ell z_r; y'yy_r y_r^+, z_\ell^-, z_r^+) \to \{y_r z_\ell^-, y_r z_\ell, y_r^+ z_\ell^-, y_r^+ z_\ell\}$. Note that if $y_r z_\ell^- \in E(G)$, then $\langle y_r; y, y_r^+, z_\ell^-\rangle \to \{y_r^+ z_\ell^-\}$, and similarly, if $y_r z_\ell, y_r^+ z_\ell \in E(G)$, then we may assume $y_r^+ z_\ell^- \in E(G)$. By a similar argument, we may assume $x_r^+ z_\ell^- \in E(G)$.

Observe that $\langle z_\ell^-; z_\ell, x_r^+, y_r^+\rangle \to \{x_r^+ z_\ell, y_r^+ z_\ell\}$, so that without loss of generality, $x_r^+ z_\ell \in E(G)$. Then, $N(x_r^+ z_\ell^- z_\ell; x_r x x' y', y_r^+, z_r) \to \{x_r z_\ell^-, x_r z_\ell, y_r^+ z_\ell, z_\ell^- z_r\}$. Recall from above that $z_\ell^- z_r \notin E(G)$. If $y_r^+ z_\ell \in E(G)$, then $\langle z_\ell; z, y_r^+, x_r^+\rangle$ is an induced claw. If $x_r z_\ell^- \in E(G)$, then $\langle z_\ell^-; z_\ell, y_r^+, x_r\rangle \to \{x_r z_\ell\}$. So we may assume that $x_r z_\ell \in E(G)$, and furthermore, $\langle z_\ell; z, z_\ell^-, x_r\rangle \to \{x_r z_\ell^-\}$. Thus, both $x_r z_\ell^-$ and $x_r z_\ell$ are edges. However, $N(x_r z_\ell^- z_\ell; xx'y'y, y_r^+, z_r)$ is an induced $N(4,1,1)$, a contradiction. This proves the claim. $\square$

Claim 2.4. *$x_\ell^- x_r^+ \in E(G)$.*

Proof. Suppose on the contrary that $x_\ell^- x_r^+ \notin E(G)$. As a result, $x_\ell^- x_r, x_\ell x_r^+ \notin E(G)$, otherwise $\langle x_r; x, x_\ell^- x_r^+\rangle$ and $\langle x_\ell; x, x_\ell^-, x_r^+\rangle$ are induced claws. So, $N(xx_\ell x_r; x'y'yy_r, x_\ell^-, x_r^+) \to \{x_\ell y_r, x_\ell^- y_r\}$. If $x_\ell y_r \in E(G)$, then $\langle x_\ell; x, x_\ell^-, y_r\rangle \to \{x_\ell^- y_r\}$, so we may assume $x_\ell^- y_r \in E(G)$. Then, $N(yy_\ell y_r; y'zz_\ell z_\ell^-, y_\ell^-, x_\ell^-) \to \{y_r y_\ell^-, y_r z_\ell^-, y_r z_\ell\}$. However, if $\alpha \in \{y_\ell^-, z_\ell^-, z_\ell\}$, then $\langle y_r; y, x_\ell^-, \alpha\rangle$ is an induced claw. This proves the claim. $\square$

We now show that $x_r z_r^+ \in E(G)$. As the basis of our inductive argument, note that $x_\ell^- x_r \in E(G)$, otherwise $N(x'y'z; xx_rx_r^+x_\ell^-, y, z_r) \to \{z_r x_\ell^-\}$, and $x_\ell^- z_r C^- z^+ z_\ell C z x' x C^- x_\ell x^+ C z_\ell^- z_r^+ C x_\ell^-$ is an $(s+1)$-cycle. Let $k \geq 1$ be such that $x_r x_\ell^{-i} \in E(G)$ and $x_\ell^{-i} \neq z_r^+$, for all $i, 1 \leq i \leq k$. We claim that $x_r x_\ell^{-(k+1)} \in E(G)$.

Observe that $N(x'y'z; xx_rx_\ell^{-k}x_\ell^{-(k+1)}, y, z_\ell) \to \{x_r x_\ell^{-(k+1)}, x_r z_\ell, z_\ell x_\ell^{-k}, z_\ell x_\ell^{-(k+1)}\}$. If $x_r z_\ell \in E(G)$, then $\langle x_r; x, x_\ell^-, z_\ell\rangle$ is an induced claw. If $z_\ell x_\ell^{-(k+1)} \in E(G)$, then $z_\ell x_\ell^{-(k+1)} C^- z^+ z_\ell^+ C z x' x C x_r^- x^- C^- x_\ell^{-k} x_r C C z_\ell$ is an $(s+1)$-cycle. A similar $(s+1)$-cycle is obtained if $z_\ell x_\ell^{-k} \in E(G)$, where if $k=1$, $x_\ell^{-(k-1)}$ is taken to be x_ℓ. So, as desired, $x_r x_\ell^{-(k+1)} \in E(G)$, and by induction we obtain $x_r z_r^+ \in E(G)$, a contradiction to Lemma 2.15. This completes the proof of the case.

2.5.3 $V(C) \cap \{x', y', z'\} = \{y', z'\}$ and $|I_x| \geq 1$

In the remainder of the proof, $x' = v$ and $yz \in E(G)$. Furthermore, by Lemma 2.13, $|(y_r, z_\ell)_C| \geq 1$, and in particular, $G[V(C)] \not\cong K_s$. Recall that in this case, Lemma 2.8 allows for x_1 to be adjacent to at most one vertex in $V(C) \setminus \{x\}$.

However, if $x_1 y \in E(G)$, then $\langle y; x_1, y_r, z\rangle \to \{x_1 z\}$, which contradicts Lemma 2.8 as x_1 would be adjacent to both y and z. A similar argument shows $x_1 z \notin E(G)$. Therefore, the only possible edges between $V(F) - \{x, y, z\}$ and $V(C)$ are $vy, vz, x_1 x$ and $x_1 u$ for at most one $u \in V(C) - \{x, y, z\}$. Recall however, that because $G[V(C)] \not\cong K_s$, Lemma 2.10 implies x_1 is not adjacent to a_ℓ or a_r for any $a \in \{w, x, y, z\}$.

If $|I_x| \geq 4$, then $N(x'yz; I_x, y_\ell, z_\ell)$ contains an induced $N(4,1,1)$. Furthermore, if $2 \leq |I_x| \leq 3$, then $N(x'yz; I_x x_\ell x_\ell^-, y_\ell, z_\ell)$ contains an induced $N(4,1,1)$. Thus, $|I_x| = 1$, and $N(x'yz; I_x x_\ell x_\ell^-, y_\ell, z_\ell) \to \{x_1 x_\ell^-\}$. However, if $x_1 x_\ell^- \in E(G)$, then $x_\ell^- x_1 x C^- x_\ell x^+ C x_\ell^-$ is an $(s+1)$-cycle, a contradiction. This proves the case.

2.5.4 $V(C) \cap \{x', y', z'\} = \{y', z'\}$, $|I_x| = 0$, and $|I_w| \geq 1$

As in Section 2.5.3, $x' = v = w'$ and $yz \in E(G)$. Furthermore, by Lemma 2.13, $|(x_r, y_\ell)_C|, |(y_r, z_\ell)_C|, |(z_r, x_\ell)_C| \geq 1$. In the following proof, we will avoid the use of x as much as possible. Recall that by Lemma 2.15, if $P = wI_w vy$ has length between 3 and 5, then no edge exists between $\{w_\ell^-, w_\ell, w\}$ and $\{y_\ell^-, y_\ell, y\}$ or between $\{w, w_r, w_r^+\}$ and $\{y, y_r, y_r^+\}$. A similar statement holds when replacing y with z.

Furthermore, recall that we are assuming w to be chosen such that $|P_w|$ is minimized so that P_w is induced. Let w_k denote the neighbor of v on P_w.

Claim 2.5. *If $|I_w| \geq 2$, then for $2 \leq i < k$, w_i is not adjacent to any vertex in $V(C)$. In addition, w_k is not adjacent to any vertex in $V(C) \setminus \{x, y, z\}$, and w_1 is not adjacent to any vertex in $\{x, y, z\}$.*

Proof. If $2 \le i \le k$, then $w_i u \notin E(G)$ for all $u \in V(C) \setminus \{x, y, z\}$ otherwise we contradict the choice of P_w. If $1 \le i < k$, then $w_i a \notin E(G)$ for all $a \in \{x, y, z\}$ otherwise $\langle a; v, a_r, w_i \rangle$ is induced. □

Observe that if $w_k y \in E(G)$, then $\langle y; y_r, w_k, z \rangle \to \{w_k z\}$, and a similar argument holds if $w_k z \in E(G)$. Thus $w_k y \in E(G)$ if and only if $w_k z \in E(G)$.

Case 3.2.1 $|I_w| \ge 5$.

If $w_k y, w_k z \notin E(G)$, then $N(vyz; I_w, y_r, z_r)$ contains an induced $N(4, 1, 1)$. If $w_k y, w_k z \in E(G)$, then $N(w_k yz; I_w, y_r, z_r)$ contains an induced $N(4, 1, 1)$.

Case 3.2.2 $|I_w| = 4$.

If $w_k y, w_k z \notin E(G)$, then $N(vyz; I_w, y_r, z_r)$ contains an induced $N(4, 1, 1)$. So $w_k y, w_k z \in E(G)$. Thus, $ww_1w_2w_3w_4y$ is a path of length five so that by Lemma 2.13, $|(w_r, y_\ell)_C|, |(y_r, w_\ell)_C| \ge 4$. Similarly, $|(w_r, z_\ell)_C|, |(z_r, w_\ell)_C| \ge 4$.

However, $N(w_4yz; w_3w_2w_2w, y_r, z_r) \to \{wy, wz\}$, and both edges contradict Lemma 2.14.

Case 3.2.3 $2 \le |I_w| \le 3$.

Observe that in the remaining cases, Lemma 2.15 implies there are no edges between $\{w_\ell^-, w_\ell, w\}$ and $\{a_\ell^-, a_\ell, a\}$, or between $\{w, w_r, w_r^+\}$ and $\{a, a_r, a_r^+\}$, for $a \in \{y, z\}$.

If $w_k y, w_k z \notin E(G)$, then $N(vyz; I_w ww_r, y_r, z_r)$ contains an induced $N(4, 1, 1)$. If $w_k y, w_k z \in E(G)$, then $N(w_k yz; w_{k-1} \dots w_1 ww_r w_r^+, y_r, z_r) \to \{w_1 w_r^+\}$. However, if $w_1 w_r^+ \in E(G)$, then $ww_1 w_r^+ C w^- w_r C^- w$ is an $(s+1)$-cycle.

Case 3.2.3 $|I_w| = 1$.

Note that here $k = 1$, and as in the above cases $w_1 y \in E(G)$ if and only if $w_1 z \in E(G)$. If $w_1 y, w_1 z \notin E(G)$, then $N(vyz; w_1 ww_r w_r^+, y_r, z_r)$ is an induced $N(4, 1, 1)$.

So we assume $w_1 y, w_1 z \in E(G)$. Observe that w_1 has no other neighbors on C other than w, y, z, as this would contradict the choice of v and w, x, y, z. Furthermore, as in some previous cases, we assume that C is arranged so that $|Q_C(w) \cup Q_C(x) \cup Q_C(y) \cup Q_C(z)|$ is maximized.

Claim 2.6. *There are no edges between $\{w, x, y, z\}$ and $V(C) - \{Q(w) \cup Q(x) \cup Q(y) \cup Q(z)\}$.*

Proof. Much of what we will do in this proof is similar to what was done in Claims 2.1 and 2.2. Therefore, we will prove the statement for w, as the cases are similar. Let $a, b \in \{w, x, y, z\}$ such that no other vertex from $\{w, x, y, z\}$ appears in $(a_r, b_\ell)_C$. Suppose on the contrary that w has a neighbor in $(a_r, b_\ell)_C$, and furthermore, let $\alpha \in (a_r, b_\ell)_C$ such that $w\alpha \in E(G)$, but $w\beta \notin E(G)$ for all $\beta \in (a_r, \alpha)_C$. First note that $\alpha \ne b_\ell^-$ by Lemma 2.15 so that $\alpha \in (a_r, b_\ell^-)_C$.

Now $\langle w; w_1, w_r, \alpha\rangle \to \{w_r\alpha\}$, and $\langle \alpha; \alpha^-, \alpha^+, w\rangle \to \{\alpha^-\alpha^+, \alpha^+ w\}$. By the maximality of $|Q_C(w) \cup Q_C(x) \cup Q_C(y) \cup Q_C(z)|$, $\alpha^-\alpha^+ \notin E(G)$ so that $\alpha^+ w \in E(G)$.

By Lemma 2.15, $\alpha^+ \neq b_\ell^-$ so that $\alpha^+ \in (\alpha, b_\ell^-)_C$. Without loss of generality, we may assume that $a \neq y$. So $N(w\alpha\alpha^+; w_1 y y_r y_r^+, \alpha^-, \alpha^{+2}) \to \{w\alpha^{+2}, \alpha^-\alpha^+, \alpha^-\alpha^{+2}, \alpha\alpha^{+2}, \gamma\delta : \gamma \in \{y, y_r, y_r^+\}, \delta \in \{\alpha^-, \alpha, \alpha^+, \alpha^{+2}\}\}$.

From here, the argument is the same as in Claim 2.2. This proves the claim. $\square$

Claim 2.7. $w_\ell^- w_r^+ \in E(G)$.

Proof. Suppose on the contrary that $w_\ell^- w_r^+ \notin E(G)$. As a result, $w_\ell^- w_r, w_\ell w_r^+ \notin E(G)$, otherwise $\langle w_r; w, w_\ell^-, w_r^+\rangle$ and $\langle w_\ell; w, w_\ell^-, w_r^+\rangle$ are induced. So $N(w w_\ell w_r; w_1 y y_r y_r^+; w_\ell^-, w_r^+) \to \{w_\ell^- y_r, w_\ell^- y_r^+, w_\ell y_r, w_\ell y_r^+\}$. Observe that if $w_\ell^- y_r \in E(G)$, then $\langle y_r; y, y_r^+, w_\ell^-\rangle \to \{w_\ell^-, y_r^+\}$. A similar argument holds for $w_\ell y_r$ and $w_\ell y_r^+$ so that we may assume $w_\ell^- y_r^+ \in E(G)$. By replace y with z, we obtain $w_\ell^- z_r^+ \in E(G)$.

Observe that $\langle w_\ell^-; w_\ell, y_r^+, z_r^+\rangle \to \{w_\ell y_r^+, w_\ell z_r^+\}$ so that without loss of generality, we assume $w_\ell y_r^+ \in E(G)$. Here $N(w_\ell w_\ell^- y_r^+; w w_1 v x; z_r^+, y_r) \to \{w_\ell^- y_r, w_\ell y_r, wx, w_1 x\}$. Recall that $w_1 x \notin E(G)$ as $w_1 y, w_1 z \in E(G)$, and this would contradict the choice of v. By Lemma 2.15, $wx \in E(G)$ is allowed, however $\langle x; v, x_r, w\rangle \to \{w x_r\}$, a contradiction. So either $w_\ell^- y_r$ or $w_\ell y_r$ is an edge. We aim to show that both must be edges.

If $w_\ell y_r \in E(G)$, then $\langle w_\ell; w, w_\ell^-, y_r\rangle \to \{w_\ell^- y_r\}$. If $w_\ell^- y_r \in E(G)$, then $\langle w_\ell^-; z_r^+, w_\ell, y_r\rangle \to \{w_\ell z_r^+, w_\ell y_r\}$. If $w_\ell z_r^+ \in E(G)$, then $\langle w_\ell; w, y_r^+, z_r^+\rangle$ is induced. So we may assume both $w_\ell^- y_r, w_\ell y_r \in E(G)$.

Then $N(y_r w_\ell w_\ell^-; y v x x_r, w_r, z_r^+) \to \{w_\ell^- x_r, w_\ell^- w_r, w_\ell x_r\}$. Recall that as $w_\ell^- w_r^+ \notin E(G)$, then $w_\ell^- w_r \notin E(G)$. If $w_\ell^- x_r \in E(G)$, then $\langle w_\ell^-; x_r, y_r, z_r^+\rangle$ is induced, and if $w_\ell x_r \in E(G)$, then $\langle w_\ell; w_r, x_r, y_r\rangle$ is induced. This completes the proof of the claim. $\square$

Observe that $N(w_1 y z; w w_\ell w_\ell^- w_r^+, y_\ell, z_\ell) \to \{w_r^+ y_\ell, w_r^+ z_\ell\}$. Without loss of generality, suppose $w_r^+ z_l \in E(G)$. Then $N(z z_\ell z_r; v x x_r x_r^+, w_r^+, z_r^+) \to \{x_r z_\ell, x_r^+ z_\ell, z_\ell z_r^+\}$. However, if $z_\ell u \in E(G)$ for $u \in \{x_r, x_r^+, z_r^+\}$, then $\langle z_\ell; z, u, w_r^+\rangle$ is induced. A similar contradiction is derived when $w_r^+ y_\ell \in E(G)$.

2.5.5 $|N_C(v)| \geq 4$ and $N_C(v)$ does not induce a clique.

In the remainder of this proof we assume that v has four neighbors on C labelled w, x, y, z such that $yz \in E(G)$. Furthermore, for all $a, b \in N_C(v)$, Lemmas 2.13 and 2.15 imply that $|(a_r, b_\ell)_C| \geq 1$, and the only possible edge between $\{a_\ell^-, a_\ell, a\}$ and $\{b_\ell^-, b_\ell, b\}$, or between $\{a, a_r, a_r^+\}$ and $\{b, b_r, b_r^+\}$ is ab. As a consequence, $va_r^+, va_\ell \notin E(G)$.

Let $a, b, c \in N_C(v)$ such that $bc \in E(G)$. If $ab \in E(G)$, then $\langle b; b_r, a, c\rangle \to \{ac\}$. In particular, for all $a \in N_C(v) \setminus \{y, z\}$, $ay \in E(G)$ if and only if $az \in E(G)$. Thus, since $N_C(v)$ does not induce a clique, we can find $a \in N_C(v) \setminus \{y, z\}$ such that $ay, az \notin E(G)$.

Claim 2.8. *$a_\ell^- a_r^+ \in E(G)$.*

Proof. Suppose on the contrary that $a_\ell^- a_r^+ \notin E(G)$. Thus, $a_\ell^- a_r, a_\ell a_r^+ \notin E(G)$, as otherwise $\langle a_r; a, a_r^+, a_\ell^- \rangle$ and $\langle a_\ell; a, a_\ell^-, a_r^+ \rangle$ are induced.

Observe that $N(aa_\ell a_r; vyy_r y_r^+, a_\ell^-, a_r^+) \to \{a_\ell^- y_r, a_\ell^- y_r^+, a_\ell y_r, a_\ell y_r^+\}$. If $a_\ell y_r \in E(G)$, then $\langle a_\ell; a_\ell^-, ay_r \rangle \to \{a_\ell^- y_r\}$, and if $a_\ell^- y_r \in E(G)$, then $\langle y_r; y, y_r^+, a_\ell^- \rangle \to \{a_\ell^- y_r^+\}$. A similar argument holds if $a_\ell y_r^+ \in E(G)$ so that we may assume $a_\ell^- y_r^+ \in E(G)$. By symmetry, $a_\ell^- z_r^+ \in E(G)$.

Let $b \in N_C(v) \setminus \{a, y, z\}$. If $ab \notin E(G)$, then we can repeat the above argument to obtain $a_\ell^- b_r^+ \in E(G)$. However, $\langle a_\ell^-; b_r^+, y_r^+, z_r^+ \rangle$ is induced. So $ab \in E(G)$. Furthermore, $by, bz \notin E(G)$, as otherwise, if say $by \in E(G)$, then $\langle b; a, y, b_r \rangle$ is induced.

Observe that $\langle a_\ell^-; a_\ell, y_r^+, z_r^+ \rangle \to \{a_\ell y_r^+, a_\ell z_r^+\}$. Without loss of generality, suppose $a_\ell y_r^+ \in E(G)$. Then $N(y_r^+ a_\ell^- a_\ell; y_r yvb, z_r^+, a_r) \to \{a_\ell^- a_r, a_\ell^- y_r, a_\ell y_r, a_\ell z_r^+\}$. Recall that $a_\ell^- a_r \notin E(G)$ as we are assuming $a_\ell^- a_r^+ \notin E(G)$. If $a_\ell z_r^+ \in E(G)$, then $\langle a_\ell; a_r, y_r^+, z_r^+ \rangle$ is induced. So either $a_\ell^- y_r$ or $a_\ell y_r$ is an edge. We claim that in fact both are edges of G.

Indeed, if $a_\ell^- y_r \in E(G)$, then $\langle a_\ell^-; a_\ell, y_r, z_r^+ \rangle \to \{a_\ell y_r, a_\ell z_r^+\}$, however $a_\ell z_r^+ \notin E(G)$ as shown above. Similarly, if $a_\ell y_r \in E(G)$, then $\langle a_\ell; a_\ell^-, y_r, a_r \rangle \to \{a_\ell^- y_r, a_\ell^- a_r\}$, but again $a_\ell^- a_r \notin E(G)$ as shown above.

So $a_\ell^- y_r, a_\ell y_r \in E(G)$. Then $N(y_r a_\ell^- a_\ell; yvbb_r, z_r^+, a_r) \to \{a_\ell^- a_r, a_\ell^- b_r, a_\ell b_r, a_\ell z_r^+\}$. In the above, we showed that $a_\ell^- a_r, a_\ell z_r^+ \notin E(G)$. If $a_\ell^- b_r \in E(G)$, then $\langle a_\ell^-; b_r, y_r, z_r^+ \rangle$ is induced, and if $a_\ell b_r \in E(G)$, then $\langle a_\ell; a_r, b_r, y_r \rangle$ is induced. This completes the proof of the claim. $\square$

Observe that $N(vyz; aa_\ell a_\ell^- a_r^+, y_\ell, z_\ell) \to \{a_r^+ y_\ell, a_r^+ z_\ell\}$. Without loss of generality, suppose $a_r^+ z_\ell \in E(G)$. Then $N(zz_\ell z_r; vyy_r y_r^+, a_r^+, z_r^+) \to \{y_r z_\ell, y_r z_\ell^+, z_\ell z_r^+\}$. However, if $z_\ell u \in E(G)$ for$u \in \{y_r, y_r^+, z_r^+\}$, then $\langle z_\ell; z, u, a_r^+ \rangle$ is induced.

2.5.6 $|N_C(v)| \geq 4$ and $N_C(v)$ induces a clique.

As above, for all $a, b \in N_C(v)$, Lemmas 2.13 and 2.15 imply that $|(a_r, b_\ell)_C| \geq 1$, and the only possible edge between $\{a_\ell^-, a_\ell, a\}$ and $\{b_\ell^-, b_\ell, b\}$, or between $\{a, a_r, a_r^+\}$ and $\{b, b_r, b_r^+\}$ is ab. As a consequence, $va_r^+, va_\ell \notin E(G)$.

We will now complete our proof of Theorem 2.2 by proving several claims that will lead to a contradiction in our final case.

Claim 2.9. *Let $a, b \in N_C(v)$. Then $a_r b_\ell, a_\ell b_r \notin E(G)$.*

Proof. Suppose on the contrary, $a_r b_\ell \in E(G)$. Let $c, d \in N_C(v)$ such that $|\{a, b, c, d\}| = 4$ and b, d appear consecutively on C such that $(b, d)_C \cap N_C(v) = \emptyset$. Then, $N(acd; a_r b_\ell b_r b_r^+, c_r, d_r) \to \{b_\ell c_r, b_\ell d_r, b_\ell b_r^+\}$. If $b_\ell c_r \in E(G)$, then the claw $\langle b_\ell; b, a_r, c_r\rangle$ is induced. Similarly, if $b_\ell d_r \in E(G)$. So, we must have $b_\ell b_r^+ \in E(G)$. By Lemma 2.15, $b_r^+ \neq d_\ell^-$.

Let $k \geq 1$ be such that $b_\ell b_r^{+i} \in E(G)$, $b_r^{+i} \neq d_\ell^-$ for all $i, 1 \leq i \leq k$. Then, $N(acd; a_r b_\ell b_r^{+k} b_r^{+(k+1)}, c_r, d_r) \to \{b_\ell b_r^{+(k+1)}, b_\ell c_r, b_\ell d_r, u b_r^{+k}, u b_r^{+(k+1)} : u \in \{a, a_r, c, c_r, d, d_r\}\}$. If $b_\ell c_r \in E(G)$, then $\langle b_\ell; b_r, a_r, c_r\rangle$ is induced, and similarly if $b_\ell d_r \in E(G)$. For $\alpha \in \{a, c, d\}$ and $\beta \in \{b_r^{+k}, b_r^{+(k+1)}\}$, if $\alpha\beta \in E(G)$, then $\langle \alpha; v, \alpha_r, \beta\rangle \to \{\alpha_r \beta\}$

So if $\alpha_r b_r^{+(k+1)}$, for $\alpha \in \{a, c, d\}$, then $b_r^{+(k+1)} \alpha_r C b^- b^+ C b_r^{+k} b v \alpha C \alpha_r^- \alpha^- C^- b_r^{+(k+1)}$ is an $(s+1)$-cycle. If $\alpha_r b_r^{+k} \in E(G)$, then we obtain an $(s+1)$-cycle by replacing $b_r^{+(k+1)}$ and b_r^{+k} in the previous cycle with b_r^{+k} and $b_r^{+(k-1)}$, respectively, where if $k = 1$, then b_r^0 is taken to be b_r.

So by induction, $b_\ell d_\ell^- \in E(G)$, a contradiction to Lemma 2.15. So, $a_r b_\ell \notin E(G)$ and similarly for $a_\ell b_r$. $\square$

Up to relabeling, let $w, x, y, z \in N_C(v)$ such that they appear along C in this order with $[w, z]_C \cap N_C(v) = \{w, x, y, z\}$. Consider the path on C formed by $[w_r, x_\ell]_C$, and let $t_1 \in [w_r, x_\ell]_C$ such that $wt_1 \in E(G)$, but $wu \notin E(G)$ for all $u \in (t_1, x_\ell]_C$. Observe that such a t_1 exists as it is possible that $t_1 = w_r$. Furthermore, $\langle w; v, t_1, w_\ell\rangle \to \{t_1 w_\ell\}$ so that $t_1 \in [w_r, x_\ell^-)_C$. That is, $(t_1, x_\ell)_C$ is nonempty.

Claim 2.10. *$t_1 u \notin E(G)$, where $u \in \{x, x_r, x_r^+, y, y_r, y_r^+, z, z_r, z_r^+\}$.*

Proof. Suppose on the contrary that $t_1 u \in E(G)$ for some u listed above. By Lemma 2.15 and Claim 2.9, $t_1 \notin \{w_r, x_\ell\}$.

If $\alpha \in \{x, y, z\}$, then $\langle \alpha; v, \alpha_r, t_1\rangle \to \{t_1 \alpha_r\}$. So it suffices to consider when $t_1 \alpha_r$ or $t_1 u_r^+$ is an edge, for $\alpha \in \{x, y, z\}$. Suppose $t_1 \alpha_r \in E(G)$, then $\langle t_1; w, t_1^+, \alpha_r\rangle \to \{t_1^+ \alpha_r\}$.

Since $wt_1 \in E(G)$, $\langle w; v, w_\ell, t_1\rangle \to \{t_1 w_\ell\}$. However, $\alpha_r t_1^+ C \alpha^- \alpha^+ C \alpha_r^- \alpha v w C^- w_\ell^+ w^+ C t_1 w_\ell C^- \alpha_r$ is an $(s+1)$-cycle.

If $t_1 \alpha_r^+ \in E(G)$, then as above we deduce that $t_1^+ \alpha_r^+, t_1 w_\ell \in E(G)$, and furthermore, we obtain a similar $(s+1)$-cycle by replacing α_r and α_r^- with α_r^+ and α_r, respectively. $\square$

Claim 2.11. *$t_1 u \notin E(G)$, where $u \in \{x_\ell, y_\ell, z_\ell\}$.*

Proof. Suppose on the contrary that $t_1 u \in E(G)$ for some u listed above. Without loss of generality, suppose $t_1 x_\ell \in E(G)$. By Claim 2.9, $t_1 \neq w_r$ so that $t_1 \in (w_r, x_\ell^-)_C$. Thus, $N(wyz; t_1 x_\ell x_r x_r^+, y_r, z_r) \to \{x_\ell x_r^+\}$, where here, and in the remainder of this paper, we omit the edges that contradict Claims 2.9 and 2.10.

By Lemma 2.15, $x_r^+ \neq y_\ell^-$. So let $k \geq 1$ be such that $x_\ell x_r^{+i} \in E(G)$, $x_\ell x_r^{+i} \in E(G)$, and $x_r^{+i} \neq y_\ell^-$, for all $i, 1 \leq i \leq k$. We claim $x_\ell x_r^{+(k+1)} \in E(G)$.

Observe $N(wyz; t_1 x_\ell x_r^{+k} x_r^{+(k+1)}, y_r, z_r) \to \{x_\ell x_r^{+(k+1)}, \gamma\delta : \gamma \in \{w, t_1, y, y_r, z, z_r\}, \delta \in \{x_r^{+k}, x_r^{+(k+1)}\}\}$. If $\alpha\beta \in E(G)$, where $\alpha \in \{w, y, z\}$ and $\beta \in \{x_r^{+k}, x_r^{+(k+1)}\}$, then $\langle \alpha; v, \alpha_r, \beta \rangle \to \{\alpha_r \beta\}$.

So if $\alpha_r x_r^{+(k+1)} \in E(G)$, for $\alpha \in \{w, y, z\}$, then $x_r^{+(k+1)} \alpha_r C x^- x^+ C x_r^{+k} x v \alpha C \alpha_r^- \alpha^- C^- x_r^{+(k+1)}$ is an $(s+1)$-cycle. If $\alpha_r x_r^{+k} \in E(G)$, then we obtain an $(s+1)$-cycle by replacing $x_r^{+(k+1)}$ and x_r^{+k} in the previous cycle with x_r^{+k} and $x_r^{+(k-1)}$, respectively, where if $k = 1$, then x_r^0 is taken to be x_r.

If $t_1 x_r^{+(k+1)} \in E(G)$, then $\langle t_1; w, t_1^+, x_r^{+(k+1)} \rangle \to \{t_1^+ x_r^{+(k+1)}\}$, as we have already considered the case when $w x_r^{+(k+1)} \in E(G)$. Recall that $t_1 \in (w_r, x_\ell^-)_C$ so that $t_1^+ \in (t_1, x_\ell)_C$. Recall also that $\langle w; v, w_\ell, t_1 \rangle \to \{t_1 w_\ell\}$. However, $x_r^{+(k+1)} t_1^+ C x^- x^+ C x_r^{+k} x v w C^- w_\ell^+ w^+ C t_1 w_\ell C^- x_r^{+(k+1)}$ is an $(s+1)$-cycle. As above, we obtain a similar $(s+1)$-cycle if $t_1 x_r^{+k} \in E(G)$.

Thus, $x_\ell x_r^{+(k+1)} \in E(G)$, and by induction $x_\ell y_\ell^- \in E(G)$, a contradiction to Lemma 2.15. This completes the proof. $\square$

Define $t_2 \in (t_1, x_\ell)_C$ such that $t_1 t_2 \in E(G)$, but $t_1 u \notin E(G)$ for all $u \in (t_2, x_\ell)_C$. Observe that t_2 exists as $t_1 t_1^+ \in E(G)$ and $t_1^+ \neq x_\ell$ by Claim 2.11. Recall that by the choice of t_1, $w t_2 \notin E(G)$.

Claim 2.12. *$t_2 u \notin E(G)$, where $u \in \{x, x_r, y, y_r, z, z_r\}$.*

Proof. Suppose on the contrary, $t_2 u \in E(G)$ for some u listed above. Without loss of generality, assume either $t_2 x$ or $t_2 x_r$ is an edge. If $t_2 x \in E(G)$, then $\langle x; v, x_r, t_2 \rangle \to \{t_2 x_4\}$. So we may assume $t_2 x_r \in E(G)$.

Observe that $N(wyz; t_1 t_2 x_r x_r^+, y_r, z_r) \to \{t_2 x_r^+, t_2 y, t_2 y_r, t_2 z, t_2 z_r\}$. If $t_2 y \in E(G)$, then $\langle y; v, y_r, t_2 \rangle \to \{t_2 y_r\}$. However, $\langle t_2; t_1, x_r, y_r \rangle$ is induced. Similar arguments hold for $t_2 z$ and $t_2 z_r$. Hence, $t_2 x_r^+ \in E(G)$.

Observe that $N(wyz; t_1 t_2 x_r x_\ell, y_r, z_r) \to \{t_2 x_\ell, t_2 y, t_2 y_r, t_2 z, t_2 z_r\}$, where as in previous proofs, we omit the edges that contradict Claims 2.9, 2.10, and 2.11. Since we have already shown in the above argument that $t_2 y, t_2 y_r, t_2 z, t_2 z_r \notin E(G)$, we deduce that $t_2 x_\ell \in E(G)$.

Since $t_2 x_r^+ \in E(G)$, $\langle t_2; t_1, x_\ell, x_r^+ \rangle \to \{x_\ell x_r^+\}$. Thus, $t_2 x_r^+, x_\ell x_r^+ \in E(G)$, and by Lemma 2.15, $x_r^+ \neq y_\ell^-$. Let $k \geq 1$ such that $t_2 x_r^{+i}, x_\ell x_r^{+i} \in E(G)$ with $x_r^{+i} \neq y_\ell^-$, for all $i, 1 \leq i \leq k$. We aim to show that $t_2 x_r^{+(k+1)}, x_\ell x_r^{+(k+1)} \in E(G)$.

Observe that $N(wyz; t_1 t_2 x_r^{+k} x_r^{+(k+1)}; y_r, z_r) \to \{t_2 x_r^{+(k+1)}, t_2 y, t_2 y_r, t_2 z, t_2 z_r, \gamma\delta : \gamma \in \{w, t_1, y, y_r, z, z_r\}, \delta \in \{x_r^{+k}, x_r^{+(k+1)}\}\}$. Recall that we have already shown $t_2 y, t_2 y_r, t_2 z, t_2 z_r \notin E(G)$. Let $\alpha \in \{w, y, z\}$ and $\beta \in \{x_r^{+k}, x_r^{+(k+1)}\}$. If $\alpha\beta \in E(G)$, then $\langle \alpha; v, \alpha_r, \beta \rangle \to \{\alpha_r \beta\}$ so that is suffices to consider $\alpha_r \beta \in E(G)$.

If $\alpha_r x_r^{+(k+1)} \in E(G)$, then $x_r^{+(k+1)} \alpha_r C x_\ell x_r^{+k} C^- x^+ x_\ell^+ C x v \alpha C \alpha_r^- \alpha^- C^- x_r^{+(k+1)}$ is an $(s+1)$-cycle. If $\alpha_r x_r^{+k} \in E(G)$, then we obtain a similar $(s+1)$-cycle by replacing $x_r^{+(k+1)}$ and x_r^{+k} in the previous cycle with x_r^{+k} and $x_r^{+(k-1)}$, respectively, where if $k=1$, then x_r^0 is taken to be x_r.

So it remains to consider the edges $t_2x_r^{+(k+1)}, t_1x_r^{+k}$, and $t_1x_r^{+(k+1)}$. Recall that we wish to show $t_2x_r^{+(k+1)} \in E(G)$. Recall also that t_1^+ exists in $(t_1, x_\ell)_C$, where possibly $t_1^+ = t_2$. So for $\beta \in \{x_r^{+k}, x_r^{+(k+1)}$, if $t_1\beta \in E(G)$, then $\langle t_1; w, t_1^+, \beta\rangle \to \{t_1^+\beta\}$, as we have already shown that $w\beta \notin E(G)$.

Thus, if $t_1^+x_r^{+(k+1)} \in E(G)$, then $\langle w; v, w^-, t_1\rangle \to \{t_1w^-\}$, however, $x_r^{+(k+1)}t_1^+Cx_\ell x_r^{+(k+1)}C^-x^+x_\ell^+Cxv$ $wCt_1w^-C^-x_r^{+(k+1)}$ is an $(s+1)$-cycle. We obtain a similar $(s+1)$-cycle if $t_1^+x_r^{+k} \in E(G)$ by replacing $x_r^{+(k+1)}$ and x_r^{+k} as above.

So $t_2x_r^{+(k+1)} \in E(G)$, and $\langle t_2; t_1, x_\ell, x_r^{+(k+1)}\rangle \to \{x_\ell x_r^{+(k+1)}\rangle$, as we have already shown that $t_1x_r^{+(k+1)} \notin E(G)$. Thus, by induction $t_2y_\ell^-, x_\ell y_\ell^- \in E(G)$, however this contradicts Lemma 2.15. $\square$

Claim 2.13. *$t_2u \notin E(G)$, where $u \in \{x_\ell, y_\ell, z_\ell\}$.*

Proof. Suppose on the contrary that $t_2x_\ell \in E(G)$. Then, $N(wyz; t_1t_2x_\ell x_r, y_r, z_r)$ is induced by Lemma 2.15 and Claims 2.9, 2.10, 2.11, and 2.12, a contradiction. Similar arguments hold if t_2y_ℓ or $t_2z_e \ll$ are edges. $\square$

Define $t_3 \in (t_2, y_\ell)_C$ such that $t_2t_3 \in E(G)$, but $t_2u \notin E(G)$ for all $u \in (t_3, x_\ell)_C$. Observe that t_3 exists since $t_2t_2^+ \in E(G)$ and $t_2^+ \neq x_\ell$ by Claim 2.13. By the choice of t_1 and t_2, $xt_3, t_1t_3 \notin E(G)$.

Claim 2.14. *$t_3u \notin E(G)$, where $u \in \{x, x_r, y, y_r, z, z_r\}$.*

Proof. Suppose that $t_3\alpha \in E(G)$ for $\alpha \in \{x, y, z\}$. Then $\langle \alpha; v, \alpha_r, t_3\rangle \to \{t_3\alpha_r\}$ so that it suffices to consider $t_3\alpha_r \in E(G)$. Without loss of generality, suppose $t_3x_r \in E(G)$.

Observe that $N(wyz; t_1t_2t_3x_r, y_r, z_r) \to \{t_3y_r, t_3z_r\}$. However, $\langle t_3; t_2, x_r, y_r\rangle$ and $\langle t_3; t_2, x_r, z_r\rangle$ are induced, respectively. Similar arguments hold if t_3y_r or t_3z_r are edges. $\square$

Claim 2.15. *$t_3x_\ell \notin E(G)$.*

Proof. If $t_3x_\ell \in E(G)$, then $N(wyz; t_1t_2t_3x_\ell, y_r, z_r)$ is induced by Lemma 2.15 and Claims 2.9, 2.10, 2.11, 2.12, 2.13, and 2.14, a contradiction. $\square$

We now complete the proof of this case. Define $t_4 \in (t_3, x_\ell)_C$ such that $t_3t_4 \in E(G)$, but $t_3u \notin E(G)$ for all $u \in (t_3, x_\ell)_C$. Observe that t_4 exists since $t_3t_3^+ \in E(G)$ and $t_3^+ \neq x_\ell$ by Claim 2.15. By the choice of t_1, t_2, t_3, we cannot have $xt_2, xt_3, xt_4, t_1t_3, t_1t_4$ or t_2t_4 in $E(G)$. By Lemma 2.15 and Claims 2.9, 2.10, 2.11, 2.12, 2.13,2.14, and 2.15, $N(wyz; t_1t_2t_3t_4, y_r, z_r) \to \{t_4y_r, t_4z_r\}$. Without loss of generality, assume $t_4y_r \in E(G)$.

Then $N(wxz; t_1t_2t_3t_4, x_r, z_r) \to \{t_4x_r, t_4z_r\}$. However, since $t_4y_r \in E(G)$, the claws $\langle t_4; t_3, y_r, x_r\rangle$ and $\langle t_4; t_3, y_r, z_r\rangle$ are induced. This completes the proof of Lemma 2.16.

2.6 Future Questions

Outside of the conjecture by Matthews and Sumner, the major problem in this area is the question of Gould (see Problem 1), which asks to characterize the pairs of forbidden subgraphs that imply pancyclicity in 4-connected graphs.

Theorem 2.1 yields a partial answer to this problem by providing a list of potential pairs of forbidden subgraphs. However, this list includes $K_{1,4}$ as a possible member of pairs of forbidden subgraphs. It would be desirable to show that in fact, $K_{1,4}$ cannot appear in any pair of forbidden subgraphs. This would answer the question of Gould and yield a characterization similar to those given by Theorems 1.2 and 1.3.

If on the other hand, $K_{1,4}$ can appear in a pair of forbidden subgraphs, then it remains to characterize the subgraphs whose removal implies pancyclicity in 4-connected $K_{1,4}$-free graphs. It would be interesting to see whether or not this family of subgraphs is the same as the family of subgraphs given in Theorem 1.8.

Chapter 3

Chorded Cycles

The following results are joint work with Theodore Molla and Elyse Yeager, appearing in [41].

3.1 Introduction

As stated in Section 1.3.1, the main purpose of this chapter is to prove the following statement

Theorem 3.1 (Molla–Santana–Yeager [41])**.** *For $k \geq 2$ and $n \geq 4k$, let G be an n-vertex graph having $\sigma_2(G) \geq 6k - 2$. The graph G does not contain k disjoint chorded cycles if and only if G is isomorphic to either:*

1. *$K_{3k-1,n-3k+1}$, with $n \geq 6k - 2$, or*

2. *$K_{1,3k-2,3k-2}$.*

Observe that if $n \geq 6k - 2$, then $|K_{3k-1,n-3k+1}| = n \geq 4k$. Also $\delta(K_{3k-1,n-3k+1}) = 3k - 1$ and $\sigma_2(K_{3k-1,n-3k+1}) = 6k - 2$. Since each chorded cycle in $K_{3k-1,n-3k+1}$ uses at least three vertices from each part, $K_{3k-1,n-3k+1}$ does not contain k disjoint chorded cycles.

Similarly, for $k \geq 2$, we have $|K_{1,3k-2,3k-2}| = 6k - 3 \geq 4k$. Also $\delta(K_{1,3k-2,3k-2}) = 3k - 1$, and $\sigma_2(K_{1,3k-2,3k-2}) = 6k - 2$. Each chorded cycle in $K_{1,3k-2,3k-2}$ either uses three vertices from each of the big parts, or uses the dominating vertex and at least two vertices from a big part. Thus, $K_{1,3k-2,3k-2}$ does not contain k chorded cycles. This shows that the graphs in Theorem 3.1 and Figure 1.3 are sharpness examples for Corollary 1.15.

The condition $k \geq 2$ in Theorem 3.1 is necessary, as subividing every edge of a graph results in a new graph with no chorded cycles. Thus, for $k = 1$, we obtain the following characterization, which is analogous to the characterization of acyclic graphs as the graphs for which there exists at most one path between every pair of vertices.

Proposition 3.2. *A graph G has no chorded cycle if and only if for all $uv \in E(G)$, $G - uv$ has at most one path between u and v.*

Every graph G with $\delta(G) \geq 3k-1$ also satisfies $\sigma_2(G) \geq 6k-2$. Therefore, Theorem 3.1 is a refinement of both Theorem 1.13 and Corollary 1.15. Two other immediate corollaries of Theorem 3.1 are listed here.

Corollary 3.3. *For $k \geq 2$, every graph G on at least $4k$ vertices having $\sigma_2(G) \geq 6k-2$ and $\alpha(G) \leq n-3k$, contains k disjoint chorded cycles.*

Every graph G with $\sigma_2(G) \geq 6k-2$ also satisfies $\alpha(G) \leq n-3k+1$. So, requiring $\alpha(G) \leq n-3k$ in Corollary 3.3 is equivalent to requiring the seemingly weaker condition $\alpha(G) \neq n-3k+1$.

Corollary 3.4. *For $k \geq 2$, if G is a graph satisfying $4k \leq |G| \leq 6k-4$ and $\sigma_2(G) \geq 6k-2$, then G contains k disjoint chorded cycles.*

We have already discussed one direction in the proof of Theorem 3.1 by showing that $K_{3k-1,n-3k+1}$ and $K_{1,3k-2,3k-2}$ do not contain k disjoint chorded cycles. In the remaining sections we prove the other direction. In Section 3.2, we detail the setup of our proof and present several important lemmas that will be used throughout this chapter. In particular, we find and choose an 'optimal' collection of $k-1$ disjoint cycles, and we use R to denote the subgraph induced by the vertices outside our collection. In Section 3.3, we consider the case when R does not have a spanning path, and in Section 3.4, we consider the case when R has a spanning path. We end this chapter with some questions for further research.

3.2 Setup and Preliminaries

3.2.1 Notation

Let G be a graph. For $A, B \subseteq V(G)$, not necessarily disjoint, we define $\|A, B\| = \sum_{a \in A} |N_G(a) \cap B|$. When $A = \{a\}$ or A is the vertex set of some subgraph $\mathcal{A}$, we will often replace A in the above notation with a or $\mathcal{A}$, respectively. Also, if $\mathcal{L}$ is a collection of graphs, then $\|A, \mathcal{L}\| = \|A, \bigcup_{L \in \mathcal{L}} V(L)\|$. If A is the vertex set of some subgraph $\mathcal{A}$, we will write $G[\mathcal{A}]$ for $G[A]$, the subgraph of G induced by the vertices of $\mathcal{A}$. Furthermore, if $\mathcal{B}$ is a subgraph of G with vertex set B, we will use $\mathcal{A} \setminus \mathcal{B}$ to denote $G[A \setminus B]$, and if $B = \{b_1, \ldots, b_k\}$ and k is small, we will also use $\mathcal{A} - b_1 - \cdots - b_k$. For a vertex $v \in V(G) - A$, we additionally write $\mathcal{A} + v$ for $G[A \cup \{v\}]$.

If $P = v_1 \ldots v_m$ is a path, then for $1 \leq i \leq j \leq m$, $v_i P v_j$ is the path $v_i \ldots v_j$. An n-cycle is a cycle with n vertices.

3.2.2 Setup

We let $k \geq 2$ and consider a graph G' on n vertices such that $n \geq 4k$ and $\sigma_2(G') = 6k - 2$, where G' does not contain k disjoint chorded cycles. We then let G be a graph with vertex set $V(G')$ such that $E(G') \subseteq E(G)$ and G is "edge-maximal" in the sense that, for any $e \in E(\overline{G})$, $G + e$ does contain k disjoint chorded cycles. We then prove that G is $K_{3k-1,n-3k+1}$ or $K_{1,3k-2,3k-2}$, which implies that $G = G'$, because any proper spanning subgraph of $K_{3k-1,n-3k+1}$ or $K_{1,3k-2,3k-2}$ has Ore-degree less than $6k-2$. Since we have already observed that $K_{3k-1,n-3k+1}$ and $K_{1,3k-2,3k-2}$ do not contain k disjoint chorded cycles, this will prove Theorem 3.1.

Note that $G \not\cong K_n$, else G contains k disjoint chorded cycles. So there exists $e \in E(\overline{G})$, and by our edge-maximality condition, G contains $k-1$ disjoint chorded cycles. Over all possible collections of $k-1$ disjoint chorded cycles in G, let $\mathcal{C}$ be such a collection which satisfies the following conditions, where R denotes the subgraph of G induced by the vertices not in $\mathcal{C}$:

(O1) the number of vertices in $\mathcal{C}$ is minimum,

(O2) subject to (O1), the total number of chords in the cycles of $\mathcal{C}$ is maximum, and

(O3) subject to (O1) and (O2), the length of the longest path in R is maximum.

We use the convention that P is a longest path in R. Since $G[P]$ may have several paths spanning $V(P)$ and the endpoints of such paths will have similar properties, we let

$$\mathcal{P} = \{v \in V(P) : v \text{ is an endpoint of a path spanning } V(P)\}.$$

3.2.3 Preliminary Results

We begin with a number of observations about G that follow directly from our setup. For the sake of brevity, the observations in this paragraph will often be used in the text without citation. Since G does not contain k disjoint chorded cycles, R does not contain any chorded cycle, and for any $C \in \mathcal{C}$, $G[V(R) \cup V(C)]$ does not contain two disjoint chorded cycles. If p is an endpoint of P and has a neighbor in $R \setminus P$, then we can extend P. Thus, $\|p, R\| = \|p, P\|$. If $\|p, P\| \geq 3$, then $G[P]$ contains a chorded cycle, so $\|p, R\| \leq 2$. Similarly, to avoid a chorded cycle in R, $\|q, P\| \leq 3$ and for any $v \in P$, $\|v, P\| \leq 4$. If p has two neighbors in P, then $G[P]$ contains two distinct spanning paths.

An immediate consequence of (O1) is that, for any chorded cycle $C \in \mathcal{C}$, no vertex of C is incident to two chords; otherwise, we could replace C with a chorded cycle on fewer vertices. We will use this fact in the proof of the following lemma.

Lemma 3.5. *Let $v \in R$ and $C \in \mathcal{C}$.*

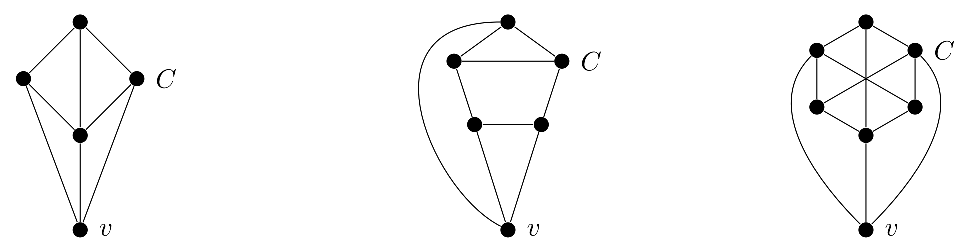
1. *If $\|v, C\| \geq 4$, then $\|v, C\| = 4 = |C|$, and $G[C] \cong K_4$.*

2. *If $\|v, C\| = 3$, then $|C| \in \{4, 5, 6\}$. Moreover:*

   (a) if $|C| = 4$, then C has a chord incident to the non-neighbor of v (see Figure 3.1a);

   (b) if $|C| = 5$, then C is singly chorded, and the endpoints of the chord are disjoint from the neighbors of v (see Figure 3.1b); and

   (c) if $|C| = 6$, then C has three chords, with $G[C] \cong K_{3,3}$ and $G[C + v] \cong K_{3,4}$ (see Figure 3.1c).



(a) $|C| = 4$, $\|v, C\| = 3$ (b) $|C| = 5$, $\|v, C\| = 3$ (c) $|C| = 6$, $\|v, C\| = 3$

Figure 3.1: Lemma 3.5.2

Proof. If there exist vertices $c_1, c_2 \in C$ that are adjacent along the cycle of C such that $\|v, C - c_1 - c_2\| \geq 3$, then $(C - c_1 - c_2) + v$ contains a chorded cycle with strictly fewer vertices than C, contradicting (O1). This proves that if $\|v, C\| = 3$, then $|C| \leq 6$. Similarly, if $\|v, C\| \geq 4$, then $|C| = 4$ and $\|v, C\| = 4$. If $\|v, C\| = 4$ and $|C| = 4$, then v together with a triangle in C give a doubly chorded 4-cycle, so by (O2), $G[C] \cong K_4$.

Suppose $\|v, C\| = 3$. If $|C| = 4$, then let $c \in C$ be the non-neighbor of v in C. If c is not incident to a chord, then $(C - c) + v$ gives a doubly chorded 4-cycle, preferable to C by (O2). This proves (2a).

So $|C| \in \{5, 6\}$. Since the vertices in $V(C) \setminus N_G(v)$ cannot be adjacent along the cycle C, $C - c + v$ contains a chorded cycle C' of the same length as C, for any $c \in V(C) \setminus N_G(v)$, If c is not incident to a chord, then C' has strictly more chords than C, violating (O2). So every vertex in $V(C) \setminus N_G(v)$ is incident to a chord.

If $|C| = 6$, then v is adjacent to every other vertex along the cycle, and every $c \in V(C) \setminus N_G(v)$ is incident to a chord. Since no vertex in C is incident to two chords, (O1) implies (2c). If $|C| = 5$, then (O1) implies that the only possible chord has the two non-neighbors of v as its endpoints, which proves (2b). □

Lemma 3.6. *Let Q be a path in R such that $|Q| \geq 4$ and let $C \in \mathcal{C}$. If $F \subseteq V(Q)$ such that $|F| = 4$, then $\|F, C\| \leq 12$. Furthermore, if $G[C] \cong K_4$ and there exists an endpoint v of Q such that $\|v, C\| \geq 3$, then $\|Q, C\| \leq 12$ with $\|Q, C\| = 12$ only if $\|v, C\| = 4$.*

Proof. Assume $\|F, C\| \geq 13$ for some $F \subseteq V(Q)$, $|F| = 4$, and let u_1, u_2, u_3, u_4 be the vertices of F in the order they appear on the path Q. By Lemma 3.5, $G[C] \cong K_4$, so there exists $c \in C$ such that $\|c, F\| \geq 4$. Since $\|\{u_1, u_4\}, C\| \geq 5$, there exists $i \in \{1, 4\}$ such that $\|u_i, C\| \geq 3$. So $Q - u_i + c$ and $C - c + u_i$ both contain chorded cycles, a contradiction.

To prove the second statement, suppose $G[C] \cong K_4$ and let v be an endpoint of Q such that $\|v, C\| \geq 3$. Note that for every $c \in C$, $C - c + v$ and $Q - v + c$ both contain chorded cycles if $\|c, Q - v\| \geq 3$. Thus, $\|Q, C\| \leq 12$, and furthermore, if $\|Q, C\| = 12$, then $\|c, Q\| = 3$ and $\|c, v\| = 1$ for every $c \in C$. $\square$

Lemma 3.7. *If $C \in \mathcal{C}$ and $\|v_1, C\|, \|v_2, C\| \geq 3$ for distinct $v_1, v_2 \in R$, then $|C| \in \{4, 6\}$.*

Proof. If $C \notin \{4, 6\}$, then $|C| = 5$ and $N_C(v_1) = N_C(v_2)$, by Lemma 3.5. Furthemore, Lemma 3.5, implies that there are two adjacent vertices $c, c' \in N_C(v_1) = N_C(v_2)$, but then $v_1 c v_2 c' v_1$ is a chorded cycle contradicting (O1). $\square$

In the following sections, we will often show that every C in $\mathcal{C}$ is a 6-cycle. Furthermore, it will often be the case that there exists some $u \in R$ such that $\|u, C\| = 3$ for every $C \in \mathcal{C}$. The following lemma will be useful in considering the neighbors of u in R and their adjacencies in C.

Lemma 3.8. *Let $C \in \mathcal{C}$ with $|C| = 6$, and let $u, v \in R$ such that $uv \in E(G)$. If $\|u, C\| = 3$ and $\|v, C\| \geq 1$, then $N_C(u) \cap N_C(v) = \emptyset$.*

Proof. By Lemma 3.5, we may assume that $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$ are the partite sets of $G[C] \cong K_{3,3}$ with $N_C(u) = A$. Suppose on the contrary that $va_1 \in E(G)$. Then $ua_2b_1a_1vu$ is a 5-cycle with chord ua_1. This contradicts (O1). $\square$

Lemma 3.9. *Suppose H is a graph with no chorded cycle. Let U and W be two disjoint paths in H and let u_1 and u_2 be the endpoints of U. Then $\|\{u_1, u_2\}, W\| \leq 3$. If equality holds, then $u_1 \neq u_2$ and for some $i \in [2]$, $\|u_i, W\| = 2$ and $\|u_{3-i}, W\| = 1$, with the neighbor of u_{3-i} strictly between the neighbors of u_i on W; in addition, $\|U, W\| = 3$.*

Proof. Let $W = w_1 w_2 \ldots w_t$ for some $t \geq 1$. $\|u_1, W\| \leq 2$ and $\|u_2, W\| \leq 2$, as H does not contain a chorded cycle. Thus, if $\|\{u_1, u_2\}, W\| \geq 3$, we may assume that $u_1 \neq u_2$, and, without loss of generality, that $\|u_1, W\| = 2$ and $\|u_2, W\| \geq 1$. Suppose $u_1 w_i, u_1 w_j \in E(H)$ such that $i < j$, and let $u_2 w_\ell \in E(H)$ for some ℓ.

If $\ell \leq i$, then $w_\ell W w_j u_1 U u_2 w_\ell$ is a cycle with chord $u_1 w_i$. If $\ell \geq j$, then $w_i W w_\ell u_2 U u_1 w_i$ is a cycle with chord $u_1 w_j$. Thus, the neighbors of u_2 in W are internal vertices of the path $w_i W w_j$. If $\|u_2, W\| = 2$, then suppose ℓ is the largest index such that $u_2w_\ell \in E(H)$. However, $w_iWw_\ell u_2Uu_1w_i$ is a cycle containing a chord incident to u_2. So $\|u_2, W\| = 1$.

Now if v is an internal vertex on U such that $vw_m \in E(H)$, then by replacing u_2 with v, we deduce that $i \leq m \leq j$. If $m \leq \ell$, then $w_iWw_\ell u_2Uu_1w_i$ is a cycle with chord vw_m, and if $m > \ell$, then $w_\ell Ww_ju_1Uu_2w_j$ is a cycle with chord vw_m. This proves the lemma. $\square$

3.3 Suppose $V(R) \neq V(P)$.

In this section, we make the assumption that $V(R) \neq V(P)$. That is, there exists some vertex $v \in R \setminus P$. In addition, we will use the convention that p and p' are the endpoints of P, and q (resp. q') is the neighbor of p (resp. p') on P. By the maximality of P, $vp \notin E(G)$ so that $d_G(v) + d_G(p) \geq 6k - 2$. Similarly for v and p'.

Our aim is to show that $G = K_{3k-1,n-3k+1}$, which is a complete bipartite graph. To aid us, we define a set of vertices $T = \{v \in R : d_R(v) = 2\}$. We will show that T is contained in one of the partite sets of $K_{3k-1,n-3k+1}$.

Lemma 3.10. *If $v \in R \setminus P$, then $\|\{v, p\}, C\| \leq 6$ for every $C \in \mathcal{C}$, with equality only if*

(i) $|C| \in \{4, 6\}$ *and* $N_C(v) = N_C(p)$, *or*

(ii) $\|p, C\| = |C| = 4$.

Proof. Suppose $v \in R \setminus P$ and $\|\{v, p\}, C\| \geq 6$ for some $C \in \mathcal{C}$. If $\|\{v, p\}, C\| \geq 7$, then either $\|v, C\| = 4$ or $\|p, C\| = 4$, so that $G[C] \cong K_4$ by Lemma 3.5. If $\|v, C\| = 4$, then $\|p, C\| = 0$, lest we extend P by adding a neighbor of p in C, and replace said neighbor in C with v, violating (O3). If $\|p, C\| = 4$, then $\|v, C\| \leq 2$, else there exists $c \in C$ such that $C - c + v \cong K_4$, and we can extend P by adding c, violating (O3). So, $\|\{v, p\}, C\| \leq 6$, and if equality holds, then either (ii) occurs, or $\|v, C\| = \|p, C\| = 3$. We may assume $\|v, C\| = \|p, C\| = 3$, so that $|C| \in \{4, 5, 6\}$ by Lemma 3.5.

By Lemma 3.7, $|C| \in \{4, 6\}$. Suppose $|C| = 4$ and $\|v, C\| = \|p, C\| = 3$. Note that $G[N_C(v) \cup \{v\}]$ forms a chorded 4-cycle with at least the same number of chords as C. If p is adjacent to the vertex in $V(C) \setminus N_G(v)$, we use that vertex to extend P, violating (O3). So (i) holds.

Finally, suppose $|C| = 6$. By Lemma 3.5, if v and p do not have the same neighborhood, they are adjacent to disjoint sets of vertices, and $C + p$ and $C + v$ both contain $K_{3,4}$. In this case, we extend P using any $c \in N_C(p)$, and replace C with a chorded cycle in $C - c + v$. This violates (O3), so (i) holds. $\square$

Lemma 3.11. *For any $v \in R \setminus P$, $\|\{v, p\}, R\| \geq 4$, so that $\|v, R\| \geq 2$. Moreover, $|P| \geq 3$.*

Proof. Let $v \in R \setminus P$. By the maximality of P, $pv \notin E(G)$. Thus, by Lemma 3.10,

$$2(3k-1) \le d_G(v) + d_G(p) = \|\{v,p\},\mathcal{C}\| + \|\{v,p\},R\| \le 6(k-1) + \|\{v,p\},R\|,$$

so $\|\{v,p\},R\| \ge 4$. Since $\|p,R\| \le 2$, it follows that $\|v,R\| \ge 2$. Then v and two of its neighbors form a path of length three in R, hence $|P| \ge 3$. □

Lemma 3.12. *For any maximal path P' in $R \setminus P$, label the (not necessarily distinct) endpoints v_1 and v_2 so that $\|v_1,P\| \le \|v_2,P\|$. Then:*

1. $\|v_2,P\| \le 2$, *and if* $v_1 \ne v_2$ *then* $\|v_1,P\| \le 1$,
2. $d_R(v_1) = 2$ *(this implies $v_1 \in T \setminus V(P)$ so that $T \setminus V(P) \ne \emptyset$), and*
3. *if* $\|v_2,P\| = 2$ *and* $\|v_1,P\| = 1$, *then* $\|P' - v_1 - v_2, P\| = 0$.

Proof. Since R contains no chorded cycle, no vertex in $R \setminus P$ has three neighbors in P, so $\|v_2,P\| \le 2$. Lemma 3.9 then gives (a) and (c).

It remains to show (b). If $\|v_1,P\| = 0$, then using Lemma 3.11 and the maximality of P', $d_R(v_1) = \|v_1,P'\| = 2$. If $v_1 = v_2$, then $\|v_1,R\| = \|v_2,P\| = 2$. So suppose $\|v_1,P\| = 1$ and $v_1 \ne v_2$. Since $\|v_2,P\| \ge \|v_1,P\| = 1$, there exist $a_1,a_2 \in P$ (perhaps $a_1 = a_2$) such that $v_1a_1, v_2a_2 \in E(G)$. Then $v_1P'v_2a_2Pa_1v_1$ is a cycle. Since it has no chord, $\|v_1,P'\| = 1$, so $\|v_1,R\| = 2$ and $v_1 \in T$. □

Lemma 3.13. $d_R(p) = d_R(p') = 2$. *Additionally, for every $v \in T \setminus V(P)$ and every $C \in \mathcal{C}$:*

1. $|C| \in \{4,6\}$,
2. $\|p,C\| = 3$, *and*
3. $N_{\mathcal{C}}(v) = N_{\mathcal{C}}(p)$.

Proof. By Lemma 3.12, $v \in T \setminus V(P)$ exists so that $d_R(v) = 2$. Lemma 3.11 implies $\|\{v,p\},R\| \ge 4$, and hence, $d_R(p) = 2$ and $\|\{v,p\},R\| = 4$. Since $vp \notin E(G)$, $\|\{v,p\},\mathcal{C}\| \ge (6k-2) - 4 = 6(k-1)$. By Lemma 3.10, $\|\{v,p\},C\| = 6$ for all $C \in \mathcal{C}$. If we can show that $\|p,C\| = 3$ for all $C \in \mathcal{C}$, then we are done by Lemma 3.10.

If not, then there exists $C \in \mathcal{C}$ such that $\|p,C\| > 3$, so $\|p,C\| = 4$ and $G[C] \cong K_4$ by Lemma 3.5. Thus, $\|v,C\| = 2$, and by Lemma 3.10, there exists $u \in N_C(p')$. Since $\|p,P\| = 2$, $P + u$ forms a chorded cycle, so since $C - u + v$ also forms chorded cycles, we have a contradiction. Thus, $\|p,C\| = 3$ as desired. □

From Lemma 3.13 we immediately obtain the following.

Corollary 3.14. *$d_G(p) = d_G(p') = 3k-1$, and consequently, $d_G(v) \geq 3k-1$ for all $v \in R \setminus P$.*

Recall that $\mathcal{P}$ is the set of vertices in P that are the endpoint of a path spanning $V(P)$. Note Lemmas 3.10, 3.11, 3.12, and 3.13 apply to each $p^* \in \mathcal{P}$. Thus, $\mathcal{P} \subseteq T$, and furthermore, for all $p_1^*, p_2^* \in \mathcal{P}$, $N_{\mathcal{C}}(p_1^*) = N_{\mathcal{C}}(p_2^*)$.

Lemma 3.15. *For every $C \in \mathcal{C}$, $G[C] \cong K_{3,3}$.*

Proof. If not, by Lemma 3.5 and Lemma 3.13, we may assume that there exists $C \in \mathcal{C}$ with $|C| = 4$. Suppose $V(C) = \{c_1, c_2, c_3, c_4\}$. Let $v \in T \setminus V(P)$, which we know exists by Lemma 3.12. By Lemmas 3.5 and 3.13, we may assume that $N_C(p) = N_C(p') = N_C(v) = \{c_1, c_2, c_3\}$ and $c_2c_4 \in E(G)$. Since $\|p, P\| = 2$ by Lemma 3.13, $P + c_1$ and $C - c_1 + v$ contain chorded cycles, a contradiction. □

For the remainder of this section, we will use the fact that for each $C \in \mathcal{C}$, $G[C] \cong K_{3,3}$ and, that there exist $A \subseteq C$ such that A is a partite set of C and such that, for every $p^* \in \mathcal{P}$, $N_C(p^*) = A$, without mentioning Lemmas 3.13, and 3.15.

Lemma 3.16. *For every $C \in \mathcal{C}$, if $v \in R \setminus P$ has a neighbor in C, then $N_C(v) \subseteq N_C(p)$, unless $|N_C(v)| = 1$.*

Proof. Fix $C \in \mathcal{C}$, and let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$ be the partite sets of C such that $N_C(p) = N_C(p') = A$. Suppose on the contrary, there exists $v \in R \setminus P$ with $|N_C(v)| \geq 2$ such that, say $vb_3 \in E(G)$.

By Lemma 3.13, $\|p, P\| = 2$ so that $P + a_i$ contains a chorded cycle for each $i \in [3]$. If $vb_2 \in E(G)$, then $vb_3a_3b_1a_2b_2v$ is a cycle with chord a_2b_3. However, $P + a_1$ also contains a chorded cycle, a contradiction.

So we may assume that $va_3 \in E(G)$. However, $vb_3a_2b_2a_3v$ is a 5-cycle with chord a_3b_3 contradicting (O1). Thus, $N_C(v) \subseteq A = N_C(p)$, as desired. □

Lemma 3.17. *$R \setminus P$ is an independent set, and $V(R \setminus P) \subseteq T$.*

Proof. Suppose $R \setminus P$ is not an independent set. Then there exists a maximal path P' in $R \setminus P$ with distinct endpoints v_1 and v_2, labeled as in Lemma 3.12. Thus,$\|v_2, P\| \leq 2$, and, hence, $d_R(v_2) \leq 4$. Since $pv_2 \notin E(G)$, Lemma 3.14 implies that $d_G(v_2) \geq 3k-1 > 4$, which implies that there exists $C \in \mathcal{C}$ such that v_2 has a neighbor $c \in C$.

Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$ be the partite sets of C such that $N_C(p) = N_C(p') = A$. By Lemmas 3.12 and 3.13, $v_1 \in T \setminus V(P)$ and $N_C(v_1) = A$. We can assume $a_1 \neq c$, so that there exists a path W in $C - a_1$ that contains a_2 and a_3 for which c is an endpoint. Since $\|v_1, W\| \geq 2$ and v_2 is adjacent to an endpoint of W, $\|\{v_1, v_2\}, W\| \geq 3$ and Lemma 3.9 implies there is a chorded cycle in $G[V(P') \cup V(C - a_1)]$. However, as $\|p, P\| = 2$, $P + a_1$ also contains a chorded cycle, a contradiction. □

Let $\mathcal{S} = N_{\mathcal{C}}(p)$, and let $\mathcal{T} = ((\bigcup_{C\in\mathcal{C}} V(C)) \setminus \mathcal{S}) \cup T$.

Proposition 3.18. *$G[\mathcal{S} \cup \mathcal{T}] \cong K_{3k-3,|\mathcal{T}|}$, and no vertex in G has neighbors in both $\mathcal{S}$ and $\mathcal{T}$.*

Proof. By Lemma 3.15, $\mathcal{C}$ consists of $k-1$ copies of $K_{3,3}$. Lemmas 3.13 and 3.17 tell us that, for every $v \in R \setminus P$, $N_{\mathcal{C}}(v) = \mathcal{S}$. Given $C \in \mathcal{C}$, $a \in V(C) \cap \mathcal{T}$, and $v \in R \setminus P$, we can create a chorded cycle C' by swapping a and v in C. Note $G[C'] \cong K_{3,3}$, and we have not changed any vertices in P. Then replacing C with C' in $\mathcal{C}$ results in a collection of $k-1$ chorded cycles satisfying (O1) through (O3). Thus all the previous lemmas apply, and, in particular, Lemma 3.12 and Lemma 3.17 imply that $a \in T$. So by Lemma 3.13, and the fact that $N_C(a) = V(C) \cap \mathcal{S}$, we conclude $N_{\mathcal{C}}(a) = \mathcal{S}$. Hence, every vertex in $\mathcal{T}$ is adjacent to every vertex in $\mathcal{S}$, and $G[\mathcal{S} \cup \mathcal{T}]$ contains a copy of $K_{|\mathcal{S}|,|\mathcal{T}|}$.

We claim $G[\mathcal{S} \cup \mathcal{T}]$ has no additional edges. Note $|\mathcal{T}| > 3(k-1)$ and $|\mathcal{S}| = 3(k-1)$. If there exists any edge with both endpoints in $\mathcal{T}$, or both endpoints in $\mathcal{S}$, then we find a set of $k-1$ chorded cycles, $k-2$ of which are 6-cycles, and one of which is a 4-cycle, violating (O1). So $G[\mathcal{S} \cup \mathcal{T}] \cong K_{|\mathcal{S}|,|\mathcal{T}|} \cong K_{3k-3,|\mathcal{T}|}$.

If any vertex of $V(G) \setminus (\mathcal{S} \cup \mathcal{T})$ has neighbors in both $\mathcal{S}$ and $\mathcal{T}$, then in a similar manner, we find $k-1$ disjoint chorded cycles, one of which is a 5-cycle and the rest of which are 6-cycles, again violating (O1). $\square$

Recall that q and q' were defined as the neighbors of p and p', respectively, on P. Since $\|p, P\| = 2$ by Lemma 3.13, there exists $w \in N_R(p) \setminus \{q\}$. As a consequence of Proposition 3.18, $w \neq p'$. Now the neighbor of w on pPw is the endpoint of a path that spans $V(P)$. Thus, $|\mathcal{P}| \geq 3$.

Lemma 3.19. $|\mathcal{P}| = 3$

Proof. Suppose $|\mathcal{P}| \geq 4$, with p_1, p_2, p_3, p_4 the first four members of $\mathcal{P}$ along P. In particular, $p_1 = p$. Fix $C \in \mathcal{C}$, and let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$ be the partite sets of C such that $N_C(p_i) = A$ for each $i \in [4]$.

By Lemma 3.8, $N_C(q) \subseteq B$. So in particular, $q \neq p_2$. If q has a neighbor in C, say b_1, then $qb_1a_1b_2a_2p_1q$ is a 6-cycle with chord p_1a_1 and $p_2Pp_4a_3p_2$ is a cycle with chord p_3a_3, a contradiction.

So we may assume that for every $C \in \mathcal{C}$, $N_C(q) = \emptyset$. That is, $\|q, R\| = d_G(q)$. Since $\|p_3, P\| = 2$ by Lemma 3.13, q is not adjacent to p_3. Then since $d_G(p_3) = 3k-1$ by Corollary 3.14, $d_G(q) \geq 3k-1 \geq 5$. Since $\|q, P\| \leq 3$, q must be adjacent to two vertices $v_1, v_2 \in R \setminus P$. By Lemma 3.13, $N_C(v_1) = N_C(v_2) = A$. However, this yields the cycles $v_1qv_2a_2b_1a_1v_1$ and $p_2Pp_4a_3p_2$ with chords v_1a_2 and p_3a_3, respectively, a contradiction. $\square$

Lemma 3.20. $G[P] \cong K_{2,3}$

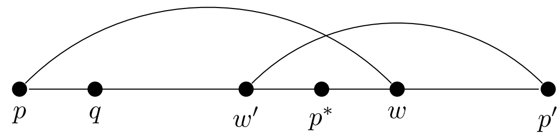


Figure 3.2: Setup for Lemma 3.20

Proof. By Lemma 3.19, we may assume that $\mathcal{P} = \{p, p', p^*\}$. Recall that $\mathcal{P} \subseteq \mathcal{T}$, so Lemma 3.18 implies that $\mathcal{P}$ is an independent set. Lemma 3.13 implies that $\|p, P\| = \|p', P\| = \|p^*, P\| = 2$, so there exist w and w' on P such that $w \neq q$ and $w' \neq q'$ and $N_P(p) = \{q, w\}$ and $N_P(p') = \{q', w'\}$. Furthermore, since $|\mathcal{P}| = 3$, and both the neighbor of w on pPw and the neighbor of w' on $w'Pp'$ are in $\mathcal{P}$, we can conclude that $w \neq w'$ and $N_P(p^*) = \{w, w'\}$, i.e. wPw' is the path on three vertices wp^*w'.

Since $G[P]$ does not contain a chorded cycle, $qq' \notin E$, so if $w = q'$ and $w' = q$, then $G \cong K_{2,3}$. So if $G \ncong K_{2,3}$, then without loss of generality we can assume that $q \neq w'$ as in Figure 3.2. Thus, $qp', pw' \notin E(G)$ so, by Corollary 3.14, $d_G(q), d_G(w') \geq 3k - 1$.

Fix $C \in \mathcal{C}$ with partite sets $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$ such that $N_C(p) = N_C(p') = N_C(p^*) = A$. By Lemma 3.8, $N_C(q) \subseteq B$ and $N_C(w') \subseteq B$.

Since $d_G(q) \geq 3k-1$, $\|q, C \cup R\| \geq (3k-1) - 3(k-2) = 5$. Also, $\|q, P\| \leq 3$. This holds for w' as well. Thus, both q and w' have two neighbors in $B \cup (R \setminus P)$. Let v_1 and v_2 be distinct vertices in $B \cup (R \setminus P)$ such that $v_1q, v_2w' \in E(G)$. We may assume that $v_2 \neq b_3$. Observe that $N_C(v_1) = N_C(v_2) = A$. Then the cycle $pqv_1a_1b_3a_3p$ has chord pa_1, and the cycle $w'v_2a_2p'Pw'$ has chord a_2p^*, a contradiction. $\square$

Lemma 3.21. $G \cong K_{3k-1,n-3k+1}$

Proof. By Lemma 3.20, let $\{p_1, p_2, p_3\}$ and $\{q_1, q_2\}$ denote the partite sets of $G[P]$. Recall that $\mathcal{P} \subseteq T$ so that $G[\mathcal{S} \cup \mathcal{T}]$ contains every vertex of G except for q_1 and q_2.

By Lemmas 3.13 and 3.17 and Corollary 3.14, $\|v, P\| = 2$ for all $v \in R \setminus P$, and by Proposition 3.18, $N_R(v) = \{q_1, q_2\}$. Since $\mathcal{T}$ is an independent set in G, for each $u \in \mathcal{T} \setminus T$, $\|u, R\| \geq (3k-1) - 3(k-1) = 2$. Thus, $uq_1, uq_2 \in E(G)$, and so $N_G(q_i) \supseteq \mathcal{T}$ for $i \in [2]$. That is, $G \supseteq K_{|\mathcal{S}|+2,|\mathcal{T}|} = K_{3k-1,|G|-3k+1} = K_{3k-1,n-3k+1}$. Since adding any edge to $K_{3k-1,n-3k+1}$ results in a graph with k disjoint chorded cycles, we conclude $G \cong K_{3k-1,n-3k+1}$. $\square$

3.4 Suppose $V(R) = V(P)$

In this section, we assume $V(P) = V(R)$. Since adding any edge to G results in k chorded cycles, by (O1) $|P| \geq 4$. If $|P| \geq 6$, we label $P = p_1q_1r_1 \cdots r_2q_2p_2$. Note that, since $G[R]$ has no chorded cycles, for every

$v \in R$, $\|r, R\| \leq 4$. When $|P| = 5$, we let $P = p_1q_1rq_2p_2$, and when $|P| = 4$, we let $P = p_1q_1q_2p_2$. We call an edge in $E(G[P]) \setminus E(P)$ a *hop*. If $Q = v_1 \cdots v_{|R|}$ is a spanning path of R, then we call an edge v_iv_j a *hop (on Q)* if $|i - j| > 1$.

Lemma 3.22. *If $Q = v_1 \cdots v_{|R|}$ is a spanning path of R and v_iv_j is a hop with $i < j$, then v_{i+1} and v_{i+2} cannot both be incident to hops, and similarly, v_{j-1} and v_{j-2} cannot both be incident to hops.*

Proof. Suppose that, on the contrary, $v_{i+1}v_k$ and $v_{i+2}v_{k'}$ are both hops. Note that, if we consider only the hop v_iv_j and the hop $v_{i+1}v_k$, $v_jv_iv_{i+1}Qv_j$ is a chorded cycle if $i + 3 \leq k \leq j$, and $v_kQv_iv_jQv_{i+1}v_k$ is a chorded cycle if $k \leq i - 1$, so $k > j$. Repeating this argument but now only considering the hops $v_{i+1}v_k$ and $v_{i+2}v_{k'}$ gives us that $k' > k$, but then $v_iv_{i+1}v_{i+2}v_{k'}Qv_jv_i$ is a cycle with chord $v_{i+1}v_k$, a contradiction. By symmetry, the lemma holds. $\square$

Lemma 3.23. *For any $p \in \mathcal{P}$, $d_R(p) = 2$ unless R is a path.*

Proof. Let $v_1 \cdots v_{|R|}$ be a spanning path in R, and let $p = v_1$. Assume $d_R(p) = 1$, and that R is not a path. Since R is not a path, hops exist. Let v_iv_j, $i < j$, be a hop such that for all $k, j < k \leq |R|$, v_k is not incident to a hop. Note, because $d_R(p) = 1$, that $i \neq 1$.

Let D be the cycle $v_jv_iv_{i+1} \cdots v_{j-1}v_j$. Since R contains no chorded cycles, v_j is incident to exactly one hop and v_{j-1} is incident to at most one hop. If v_{j-1} is not incident to a hop let $x = v_{j-1}$ and $y = v_j$, and if v_{j-1} is incident to exactly one hop, let $x = v_{j-2}$ and $y = v_{j-1}$. By Lemma 3.22, when v_{j-1} is incident to a hop, v_{j-2} is not incident to a hop, so in either case, $xy \in E(D)$, $d_R(x) + d_R(y) \leq 5$, and $px, py \notin E(G)$. Therefore,

$$2\|p, \mathcal{C}\| + \|\{x, y\}, \mathcal{C}\| \geq 2(6k - 2) - (2\|p, R\| + \|\{x, y\}, R\|) > 12(k - 1).$$

So there exists $C \in \mathcal{C}$ such that $2\|p, C\| + \|\{x, y\}, C\| \geq 13$. Thus, $\|v, C\| = 4$ for some $v \in \{p, x, y\}$, and by Lemma 3.5, $G[C] \cong K_4$. Further, $\|\{x, y\}, C\| \geq 5$ so that there exists $c \in C$ such that $xc, yc \in E(G)$ and $D + c$ contains a chorded cycle. Also $2\|p, C\| \geq 5$, which implies $\|p, C - c\| \geq 2$ so that $C - c + p$ contains a chorded cycle, a contradiction. $\square$

Lemma 3.24. *If $|R| \geq 6$, then there exists $F^+ \subseteq V(R)$ such that $|F^+| = 6$ and such that for every $C \in \mathcal{C}$ and every pair of distinct vertices $u, u' \in F^+$, $\|\{u, u'\}, C\| \geq 1$.*

Proof. First we find $F^+ \subseteq V(R)$ such that $\|F^+, R\| \leq 15$. If R is a path, this is trivial, so we assume R has at least one hop. By Lemmas 3.22 and 3.23, p_i is incident to a hop so that q_i and r_i cannot both be incident to hops. If $d_R(r_i) \leq 3$ for some $i \in [2]$, then since $d_R(q_i) \leq 3$ and $d_R(p_i) = 2$ by Lemma 3.23,

$\|\{p_i, q_i, r_i\}, R\| \leq 7$. If $d_R(r_i) = 4$, then $d_R(p_i) = d_R(q_i) = 2$, so that $\|\{p_i, q_i, r_i\}, R\| \leq 8$. Therefore, $F^+ = \{p_1, q_1, r_1, r_2, q_2, p_2\}$ suffices when either $d_R(r_1) \leq 3$ or $d_R(r_2) \leq 3$. In this case, we let $r_1^* = r_1$.

When $d_R(r_1) = d_R(r_2) = 4$, $|R| \geq 7$, since R has no chorded cycles, and there exists a vertex u following r_1 on P with $d_R(u) \leq 3$. Here, we let $F^+ = \{p_1, q_1, u, r_2, q_2, p_2\}$. and let $r_1^* = u$. Thus, in both cases, $F^+ = \{p_1, q_1, r_1^*, r_2, q_2, p_2\}$.

We claim that we can partition F^+ into three sets so that each set will consist of two nonadjacent vertices. Define $F_1 = \{p_1, q_1, r_1^*\}$ and $F_2 = \{p_2, q_2, r_2\}$, and let H be the subgraph of G on the vertex set F^+ containing precisely those edges of G with one endpoint in F_1 and the other in F_2. Because R contains no chorded cycle, every vertex in F_2 has at most two neighbors in F_1, and vice-versa. That is, $H \subseteq 3K_2$. Therefore we can label $F_1 = \{f_1, f_2, f_3\}$ so that f_1p_2, f_2q_2, and f_3r_2 are all nonedges.

Therefore, $\|F^+, \mathcal{C}\| \geq 3(6k-2) - 15 = 18(k-1) - 3$. Suppose there exists $C \in \mathcal{C}$ for which $\|F^+, C\| \leq 14$ so that there exists $C' \in \mathcal{C}$ such that $\|F^+, C'\| \geq 19$. If we can find $v_1, v_2 \in F^+$ such that $\|\{v_1, v_2\}, C'\| \leq 6$, then $\|F' - v_1 - v_2, C'\| \geq 13$, contradicting Lemma 3.6. So for $F^+ = \{v_1, v_2, \ldots, v_6\}$, $\|\{v_i, v_{i+1}\}, C'\| \geq 7$ for $i \in \{1, 3, 5\}$. However this implies $\|\{v_1, v_2, v_3, v_4\}, C'\| \geq 14$, a contradiction to Lemma 3.6.

Thus, $\|F^+, C\| \geq 15$ for every $C \in \mathcal{C}$. If there exists a pair of distinct vertices $u, u' \in F^+$ such that $\|\{u, u'\}, C\| = 0$, then $\|F^+ - u - u', C\| \geq 15$, again a violation of Lemma 3.6. $\square$

Lemma 3.25. *There exists $F \subseteq V(R)$ such that $p_1, p_2 \in F$, $|F| = 4$ and*

1. $\|F, \mathcal{C}\| \geq 12(k-1) - 2$ *if* $R \cong K_{2,3}$, $\|F, \mathcal{C}\| \geq 12(k-1) + 2$ *if R is a path, and* $\|F, \mathcal{C}\| \geq 12(k-1)$ *otherwise, and*

2. *if R is not a path, then for every $u \in F$, there exists a path Q in $R - u$ such that $F - u \subseteq V(Q)$.*

Proof. If R is a path or $R \cong K_{2,3}$, let $F = \{p_1, q_1, q_2, p_2\}$. When R is a path, $\|F, R\| = 6$, and $p_1q_2, p_2q_1 \notin E(G)$; when $R \cong K_{2,3}$, $\|F, R\| = 10$, and $p_1p_2, q_1q_2 \notin E(G)$. In both cases, 1 and 2 hold.

So we assume $R \ncong K_{2,3}$ and R is not a path. By Lemma 3.23, for $i \in [2]$, $\|p_i, P\| = 2$. Thus, p_i has a neighbor $w_i \in P - q_i$. Let t_i denote the neighbor of w_i on w_iPp_i. Observe that $t_i \in \mathcal{P}$, so by Lemma 3.23, $\|t_i, P\| = 2$. Suppose $t_1 \neq t_2$, and, in this case, let $F = \{p_1, t_1, t_2, p_2\}$. Then $F \subseteq \mathcal{P}$, so 2 holds and $\|F, R\| \leq 8$. If either $p_1t_1, p_2t_2 \notin E(G)$ or $p_1t_2, p_2t_1 \notin E(G)$, then 1 holds. Suppose (say) $p_1t_1 \in E(G)$. Then $t_1 = q_1$, and $t_1p_2 \notin E(G)$. Then $w_2 \notin \{p_1, t_1\}$, hence $t_2 \notin \{t_1, w_1\} = N_R(p_1)$, so also $p_1t_2 \notin E(G)$. So in this case also, 1 holds.

So assume $t_1 = t_2$, which implies $\|u, P\| = 2$ for all $u \in V(P) - w_1 - w_2$, as otherwise R contains a chorded cycle. Also, when $t_1 = t_2$, we may assume that $q_1 \neq w_2$ since R is not isomorphic to $K_{2,3}$. In this case, let $F := \{p_1, q_1, t_1, p_2\}$ and note that $p_1t_1, q_1p_2 \notin E(G)$. Since $d_R(u) = 2$ for all $u \in F$, 1 holds. Since $t_1 = t_2$, $p_1w_1t_1w_2p_2$ is a path in $R - q_1$ containing $F - q_1$ and $F - q_1 \subseteq \mathcal{P}$, 2 holds. $\square$

Corollary 3.26. *R is not a path.*

Proof. Let $F \subseteq V(R)$ be as guaranteed in Lemma 5.3. If R is a path, then $\|F, \mathcal{C}\| \geq 12(k-1)+2$, so that there exists $C \in \mathcal{C}$ such that $\|F, C\| \geq 13$, which violates Lemma 3.6. So R is not path. $\square$

Lemma 3.27. *Let $F \subseteq V(R)$ be as guaranteed in Lemma 5.3. If $\|F, C\| = 12$ for any $C \in \mathcal{C}$, then $G[C] \cong K_{3,3}$.*

Proof. Let $F \subseteq V(R)$ be as guaranteed in Lemma 5.3 and let $C \in \mathcal{C}$. Suppose that $\|F, C\| = 12$. By Lemmas 3.6 and 5.3, this is true for all $C \in \mathcal{C}$, unless $R \cong K_{2,3}$. By Lemmas 3.5 and 3.7, $C \cong K_{3,3}$ unless $|C| = 4$, so assume $|C| = 4$. Note that for any $u \in F$ and $c \in C$, if $C - c + u$ is a chorded cycle, then $\|c, F - u\| \leq 2$, because there exists a path Q in R such that $F - u \subseteq V(Q)$ and $G[Q + c]$ cannot contain a chorded cycle.

First assume that C is singly chorded, so we can label $V(C) = \{c_1, c_2, c_3, c_4\}$ such that $c_1c_2c_3c_4$ is a cycle and c_2c_4 is the chord. By Lemma 3.5, $\|u, C\| = 3$ for every $u \in F$, and $\|c_i, F\| = 4$, for $i \in \{1, 3\}$. Recall that $p_1, p_2 \in F$ so that $C - c_1 + p_1$ and $P - p_1 + c_1$ both contain chorded cycles, a contradiction.

So for the remainder of the proof, we assume $G[C] \cong K_4$, with $V(C) = \{c_1, c_2, c_3, c_4\}$. Fix $u \in F$, and by Lemma 5.3, let Q be a path in $R - u$ such that $F - u \subseteq V(Q)$. Suppose $\|u, C\| = 3$, so $\|F - u, C\| = 9$, and there exists $c \in C$ such that c is adjacent to all three vertices in $F - u$. This implies $Q + c$ and $C - c + u$ both contain chorded cycles, a contradiction.

Now suppose $\|u, C\| = 2$ and $N_C(u) = \{c_1, c_2\}$. Then $\|F - u, C\| = 10$, and there exist two vertices in C adjacent to all three vertices in $F - u$. If c' is one of these two vertices and $c' \notin \{c_1, c_2\}$, then $Q + c'$ and $C - c' + u$ both contain chorded cycles, a contradiction. Therefore, every vertex in F is adjacent to both c_1 and c_2. Since $\|F, C\| = 12$ and $\|u, C\| = 2$, there exists $v \in F - u$ such that $\|v, C\| = 4$. By Lemma 5.3, there exists a path Q' in $R - v$ such that $F - v \subseteq V(Q')$, so that $C - c_1 + v$ and $Q' + c_1$ both contain chorded cycles, a contradiction.

So $\|u, C\| \in \{0, 1, 4\}$, for every $u \in F$. Since $\|F, C\| = 12$, there exists $u' \in F$ such that $\|u', C\| = 0$ and $\|u, C\| = 4$ for every $u \in F - u'$. By Lemma 5.3, $p_1, p_2 \in F$, so we may assume $\|p_1, C\| = 4$. Thus, for all $c \in C$, $C - c + p_1$ is a chorded cycle, and further $\|c, P - p_1\| \leq 2$, else $P - p_1 + c$ contains a chorded cycle. Therefore, if $\|R \setminus F, C\| > 0$, we can pick c such that $\|c, P - p_1\| \geq 3$ so that $P - p_1 + c$ has a chorded cycle, a contradiction.

Thus $\|R \setminus F, C\| = 0$. By Lemma 3.24, $|R| \leq 5$, as otherwise we can find $F^+ \subseteq V(R)$ with $|F^+| = 6$ so that for distinct $v, v' \in F^+ \setminus F$, $\|\{v, v'\}, C\| \geq 1$, a contradiction. If $|R| = 4$, then u' has a neighbor $v \in F - u'$. Since R is not a path, by Lemma 3.23 $R \cong C_4$, so replacing C with $C' = C - c + v$ in $\mathcal{C}$ gives a collection of $k-1$ chorded cycles that satisfies (O1) - (O3), but $R' = R - v + c$ has a path P' such that $|P'| = |R'|$ and such that u' is an endpoint and such that $\|u', R'\| = 1$. This is a contradiction to Lemma 3.23.

So assume $|R| = 5$ so that $P = p_1q_1rq_2p_2$. By Lemma 3.23, either $p_1r, p_2r \in E(G)$, or $R \in \{C_5, K_{2,3}\}$. In each of these cases, we can assume that $F = \{p_1, q_1, q_2, p_2\}$, by the proof Lemma 5.3. Recall that $\|p_1, C\| = 4$ and $\|u', C\| = 0$ for some $u' \in F$. Furthermore, since $\|R \setminus F, C\| = 0$, $\|r, C\| = 0$.

Suppose $R \in \{C_5, K_{2,3}\}$. Let $F' = \{q_1, r, q_2, p_2\}$, so that $u' \in F'$, $\|F', C\| \leq 8$ and $\|F', R\| \leq 10$. Since q_1q_2, $rp_2 \notin E(G)$, $\|F', \mathcal{C} - C\| \geq 12(k-2) + 2$ so that $k \geq 3$ and $\|F', C'\| \geq 13$ for some $C' \in \mathcal{C} - C$, a contradiction to Lemma 3.6.

Thus $p_1r, p_2r \in E(G)$. Since three of the five vertices in R send four edges to C, there exists $i \in [2]$, such that at least two vertices in $\{r, q_i, p_i\}$ have four neighbors in C, and so have a common neighbor $c \in C$. This implies that $G[\{r, q_i, p_i, c\}]$ contains a chorded cycle. Furthermore, there exists $v \in \{p_{3-i}, q_{3-i}\}$ such that v has four neighbors in C, and so $C - c + v$ contains a chorded cycle, a contradiction.

Thus, $|C| \neq 4$ and $G[C] \cong K_{3,3}$, as desired. $\square$

Lemma 3.28. *If $R \ncong K_{2,3}$, then $G[C] \cong K_{3,3}$ for all $C \in \mathcal{C}$. If $R \cong K_{2,3}$, then $G[C] \cong K_{3,3}$ for all but at most one $C \in \mathcal{C}$, and for any such C, $G[C] \cong K_{1,1,2}$ and $G[V(R) \cup V(C)] \cong K_{1,4,4}$.*

Proof. Let $F \subseteq V(R)$ be as guaranteed by Lemma 5.3. If R is not isomorphic to $K_{2,3}$, then $\|F, \mathcal{C}\| \geq 12(k-1)$. By Lemma 3.6, $\|F, C\| \leq 12$ for all $C \in \mathcal{C}$ so that in fact, equality holds for all $C \in \mathcal{C}$. Thus, by Lemma 5.4, $G[C] \cong K_{3,3}$ for all $C \in \mathcal{C}$.

So assume $R \cong K_{2,3}$ with partite sets $A = \{p_1, p_2, p_3\}$ and $B = \{q_1, q_2\}$ with $|A| = 3$ and $|B| = 2$. Since A and B are independent, we have $\|B, \mathcal{C}\| \geq 6k - 8$ and

$$2\|A, \mathcal{C}\| = \sum_{a \in A} 2\|a, \mathcal{C}\| \geq 3(6k-2) - 12 = 18k - 18,$$

so $\|A, \mathcal{C}\| \geq 9(k-1)$ and $\|R, \mathcal{C}\| \geq 15k - 17 = 15(k-1) - 2$. If $\|R, C\| \geq 16$ for some $C \in \mathcal{C}$, then there exists some $u \in R$ such that $\|u, C\| = 4$. By Lemma 3.6, $\|R - u, C\| \leq 12$ so that there exists $u' \in R - u$ such that $\|u', C\| \leq 3$. However, $\|R - u', C\| \geq 13$, a contradiction to Lemma 3.6.

We therefore have that, for ever $C \in \mathcal{C}$, $13 \leq \|R, C\| \leq 15$. Fix $C \in \mathcal{C}$. At least two vertices in R have three neighbors each in C so that by Lemmas 3.5 and 3.7, $|C| = 4$ or $G[C] \cong K_{3,3}$. We claim that

$G[C] \ncong K_4$.

Suppose on the contrary, $G[C] \cong K_4$. If $\|p_i, C\| \geq 3$ for some $i \in [3]$, Lemma 3.6 implies that $\|R, C\| \leq 12$, a contradiction. So $\|p_i, C\| \leq 2$ for all $i \in [3]$. Hence $\|B, C\| \geq 7$ so that for all $c \in C$ and $j \in [2]$, $C - c + q_j$ is a chorded cycle. As $\|R, C\| \geq 13$, there exists $c \in C$ such that $\|c, R\| \geq 4$. Without loss of generality, $N_R(c) \supseteq \{p_1, p_2, q_1\}$. However, $C - c + q_2$ and $p_1 c p_2 q_1 p_1$ each contain chorded cycles, a contradiction.

So for all $C \in \mathcal{C}$, either $|C| = 4$ and C is singly chorded or $G[C] \cong K_{3,3}$. By Lemma 3.5, $\|u, C\| \leq 3$ for all $u \in A$ and $C \in \mathcal{C}$. Since $\|A, \mathcal{C}\| \geq 9(k-1)$, we deduce that $\|A, C\| = 9$ and so $\|u, C\| = 3$ for all $u \in A$ and $C \in \mathcal{C}$.

Suppose $|C| = 4$ and C is singly chorded. We can label $V(C) = \{c_1, c_2, c_3, c_4\}$ such that $c_1c_2c_3c_4$ is a cycle and c_2c_4 is the chord. By Lemma 3.5, $uc_1, uc_3 \in E(G)$ for all $u \in A$. Since, $C - c_i + u$ is a chorded cycle for $i \in \{1, 3\}$, $R - u + c_i$ cannot contain a chorded cycle, which implies that $N_R(c_i) = A$. Hence, for every $v \in B$, $N_C(v) \subseteq \{c_2, c_4\}$, and since $\|R, C\| \geq 13$, equality holds and $N_C(v) = \{c_2, c_4\}$ for every $v \in B$.

Fix $u \in A$. Without loss of generality, assume $N_C(u) = \{c_1, c_3, c_4\}$. Then $C - c_2 + u$ is a chorded cycle. If $u' \in A - u$ has $c_2 \in N_C(u)$, then $R - u + c_2$ contains a chorded cycle, a contradiction. Thus, for all $w \in A$, $N_C(w) = \{c_1, c_3, c_4\}$ so that $N_R(c_4) = V(R)$ and $G[R \cup C] \cong K_{4,4,1}$.

Recall that $\|R, \mathcal{C}\| \geq 15(k-1) - 2$ and $\|R, C'\| \leq 15$ for all $C' \in \mathcal{C}$. Further, $\|u, C'\| \leq 3$ for all $u \in R$ and $C' \in \mathcal{C}$. Since $\|R, C\| = 13$, $\|R, C''\| = 15$ for every $C'' \in \mathcal{C} - C$. However, for any $u \in A$, $\|u, C'\| \leq 3$ so that $F = R - u$ satisfies $\|F, C''\| \geq 12$. Furthermore, F satisfies all the hypotheses of Lemmas 5.3 and 5.4, so that $G[C''] \cong K_{3,3}$ for all $C'' \in \mathcal{C} - C$.

This completes the proof of the lemma. $\square$

Lemma 3.29. *For every $u \in R$ and $C \in \mathcal{C}$, $\|u, C\| \leq 3$. If P' is path that spans R, p is an endpoint of P' and q is adjacent to p on P', then $d_G(p) = 3k - 1$ and $d_G(q) \geq 3k - 1$. In particular, for every $C \in \mathcal{C}$ $\|p, C\| = 3$ and $\|q, C\| \geq 2$.*

Proof. Let p and p' be the two endpoints of P', and let q and q' be the neighbors of p and p', respectively, on P'. By Lemmas 3.5 and 3.28, $\|u, C\| \leq 3$ for all $u \in R$ and $C \in \mathcal{C}$. Therefore, if $d_R(u) = 2$, then $d_G(u) \leq 3k-1$, so in particular, $d_G(p) \leq 3k-1$ and $d_G(p') \leq 3k-1$. If $pp' \notin E$, then $d_G(p') = d_G(p) = 3k-1$. Otherwise, $pp' \in E$ and p is not adjacent to q'. In this case, $d_R(q') = 2$ so that $d_G(p) = 3k - 1$. Since $\|u, C\| \leq 3$ for all $u \in R$ and $C \in \mathcal{C}$, it follows that $\|p, C\| = 3$. By symmetry, this holds for p' as well.

Since $\|q, R\| \leq 3$, if we can show that $d_G(q) \geq 3k - 1$, it follows that $\|q, C\| \geq 2$ for all $C \in \mathcal{C}$. So assume $d_G(q) \leq 3k - 2$. Now, $qp' \in E(G)$, as otherwise $d_G(q) \geq 3k - 1$. If $|R| = 4$, then by Lemma 3.23, R contains a chorded cycle. So $|R| > 4$, and as a result $qq' \notin E(G)$. Since $d_G(q) \leq 3k - 2$, we get $d_G(q') \geq 3k$, and furthermore, since $d_R(q') \leq 3$ and $\|q', C\| \leq 3$ for all $C \in \mathcal{C}$, we deduce that $\|q', C\| = 3$ and $d_R(q') = 3$. This implies $pq' \in E(G)$, as otherwise we get a chorded cycle in R. Furthermore, $d_G(q) = 3k-2$ and $\|q, R\| \leq 3$ so that $\|q, C\| \geq 1$ for all $C \in \mathcal{C}$.

Since $|R| \geq 5$, there exists $r' \notin \{p, p'\}$ a neighbor of q' on P'. Note that $r' \in \mathcal{P}$ so that by the above, $d_G(r') = 3k-1$ and $\|r', C\| = 3$ for all $C \in \mathcal{C}$. If $|R| \geq 6$, then $r'q \notin E(G)$ and $d_G(q) \geq 3k-1$, a contradiction. Hence, $|R| = 5$, and, furthermore, $R \cong K_{2,3}$ with partite sets $\{q, q'\}$ and $\{p, p', r'\}$. Observe that for all $u \in \{p, r', q', p'\}$ and $C \in \mathcal{C}$, $\|u, C\| = 3$.

If know fix $C \in \mathcal{C}$, such that $\|q, C\| \leq 2$, which must exist because $d(q) = 3k-2$ and $d_R(q) = 3$. By Lemma 3.28, $G[C] \in \{K_{3,3}, K_{1,1,2}\}$. Furthermore, if $G[C] \cong K_{1,1,2}$, then $G[C \cup R] = K_{1,4,4}$, but this contradicts the fact that $\|q', C \cup R\| = 6$. Hence, $C \cong K_{3,3}$ and let A and B denote its partite sets. By Lemmas 3.5 and 3.8, we may assume $N_C(p) = N_C(r') = N_C(p') = A$, $N_C(q') = B$, and $N_C(q) \subseteq B$. Since $\|q, C\| \leq 2$, there exists $b \in B \setminus N_C(q)$. We can replace C with $C - b + p'$ and replace P' with $bq'P'p$. Our new collection and path satisfy (O1)-(O3). However, b is an endpoint of our new path and by the above, $d_G(b) = 3k-1$. Since $bq \notin E(G)$, $d_G(q) \geq 3k-1$, a contradiction. $\square$

Lemma 3.30. *R is either isomorphic to $K_{2,3}$ or $K_{2,2}$.*

Proof. If $|R| = 4$, then Lemmas 3.23 implies that $R \cong K_{2,2}$, so assume $|R| \geq 5$ and R is not isomorphic to $K_{2,3}$. Let $P = u_1, \ldots, u_{|R|}$, $p = u_1$, $q = u_2$, $q' = u_{|R|-1}$ and $p' = u_{|R|}$. Let $C \in \mathcal{C}$. By Lemma 3.28, $G[C] \cong K_{3,3}$, so we let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$ be its partite sets. Recall that by Lemma 3.5, if $\|u, C\| = 3$ for any $u \in R$, then $N_C(u) \in \{A, B\}$.

First assume that R is Hamiltonian (that is, R contains a cycle of size $|R|$). Since every vertex in R is the endpoint of a path spanning R, by Lemma 3.29, $\|u, C\| = 3$ for every $C \in \mathcal{C}$ and $u \in R$. By Lemma 3.8, we can assume that $N_C(u_i) = A$ if i is odd and $N(u_i) = B$ is i is even. Therefore, Lemma 3.8 implies that $|R|$ is even, which further implies that $|R| \geq 6$. Then for any $a \in A$ and $b \in B$, $G[\{u_1, \ldots, u_4, a, b\}]$ and $C - a - b + u_5 + u_6$ contain chorded cycles, a contradiction.

So we can assume R is not Hamiltonian. Let pw be a hop on P so that $w \neq p'$. First assume $w \neq q'$. Without loss generality assume that $N_C(p') = A$. By Lemmas 3.8 and 3.29, $N_C(p) \cap N_C(q) = \emptyset$, and so there exists $cc' \in E(C)$ such that $pcc'qPwp$ is a cycle with chord pq. By Lemmas 3.8 and 3.29, $|N_C(p') - c - c'| \geq 2$ and $|N_C(q') - c - c'| \geq 1$, so $C - c - c' + p' + q'$ contains a chorded cycle, a contradiction.

Now we can assume that both pq' and qp' are edges. Since $R \neq K_{2,3}$, we have that $|R| \geq 6$. Let $r \neq p$ and $r' \neq p'$ be the neighbors of q and q', respectively, on P. Note that r and r' are endpoints of paths spanning R so that $\|r, C\| = \|r', C\| = 3$. By Lemmas 3.8 and 3.29, and because $pq', qp' \in E(G)$, we may assume that $N_C(p) = N_C(r) = N_C(r') = N_C(p') = A$ and $N_C(q) \cup N_C(q') \subseteq B$. In particular, we may assume $qb_1 \in E(G)$ so that $pa_1b_2a_2b_1qp$ is a cycle with chord pa_2, and $rPp'a_3r$ is a cycle with chord a_3r', a contradiction.

So $|R| = 5$ and $R \cong K_{2,3}$, as desired. $\square$

Lemma 3.31. *If $G[C] \cong K_{3,3}$ for every $C \in \mathcal{C}$, then $G \cong K_{3k-1,n-3k+1}$.*

Proof. By Lemma 3.30, $R \in \{K_{2,2}, K_{2,3}\}$. So let $U_1, U_2 \subseteq V(R)$ be the partite sets of R such that $|U_1| \geq |U_2| = 2$, and let $u_1 \in U_1$, $V_2 = N_G(u_1)$, and $V_1 = V(G) \setminus V_2$. Since u_1 is the end of spanning path of R, Lemma 3.29 implies that $|V_2| = 3k-1$. Since $|G| \leq 6(k-1)+5$, $|V_1| \leq 3k$. We aim to show that $N_G(v) = V_2$ for all $v \in V_1$. This will imply that $G \cong K_{3k-1,n-3k+1}$.

Fix $v \in V_1 - u_1$. Since $u_1v \notin E(G)$, Lemma 3.29 implies that $d_G(v) \geq 3k-1$. If $v \in U_1$, then v is the end of a spanning path of R, and by Lemmas 3.5, 3.8 and 3.29, $N_G(v) = N_G(u_1) = V_2$. So we may assume $v \in V_1 \setminus U_1$, and in particular, $v \in C$ for some $C \in \mathcal{C}$.

Define $V_1' = \{u \in V_1 : \|u, U_2\| \geq 1\}$, and suppose $v \in V_1' \setminus U_1$. Recall that we are assuming $G[C] \cong K_{3,3}$ for all $C \in \mathcal{C}$ so that by Lemma 3.5, $G[C - v + u_1] \cong K_{3,3}$. Furthermore, v is an end of a path of length $|R|$ in $R' = R - u_1 + v$. This new collection and path satisfy (O1)-(O3), so by Lemma 3.30, $R' \cong R$ and $N_G(v) = N_G(u_1) = V_2$.

Now suppose $v \in V_1 \setminus V_1'$. Since $d_G(v) \geq 3k-1$ and v has at most $3(k-1)$ neighbors in V_2, v must have two neighbors in V_1. By Lemmas 3.8 and 3.29, for every $u_2 \in U_2$, $d_G(u_2) \geq 3k-1$ and $N_G(u_2) \subseteq V_1$, so that $|V_1'| \geq 3k-1$. Since $|V_1| \leq 3k$, v has a neighbor, say v', in V_1'. However, by the above, $N_G(v') = V_2$, which contradicts the fact that vv' is an edge. Therefore, $V_1' = V_1$ which finishes the proof of the lemma. $\square$

Lemma 3.32. *Suppose there exists $C \in \mathcal{C}$ with $|C| = 4$. Then $G \cong K_{1,3k-2,3k-2}$.*

Proof. By Lemmas 3.28 and 3.30, we can assume $R \cong K_{2,3}$, $G[C] \cong K_{1,1,2}$, and $G[R \cup C] \cong K_{1,4,4}$. Let A' and B' be the two partite sets of size four and $\{c\}$ be the partite set of size one in $G[R \cup C]$. By symmetry, we can assume that any $v \in A' \cup B'$ is an end of a spanning path in R or the end of a spanning path of $G[V(G) \setminus V(\mathcal{C}')]$ for some collection $\mathcal{C}'$ of $k-1$ vertex disjoint cycles that satisfies (O1)-(O3), so, by Lemma 3.29, $d_G(v) = 3k-1$ and $\|v, \mathcal{C} - C\| = 3(k-2)$. By Lemma 3.28, for all $D \in \mathcal{C} - C$, $G[D] \cong K_{3,3}$, and, with Lemma 3.8, we deduce that $\|v, D\| = 3$ and that we can label the partite sets of D as A_D and B_D so that for every $p \in A'$, $N_D(p) = B_D$ and for every $q \in B'$, $N_D(q) = A_D$. Therefore, there exists a partition $\{A, B, \{c\}\}$ of $V(G)$ such that for every $p \in A'$, $N_G(p) = B + c$, for every $q \in B'$, $N_G(q) = A + c$, and $|A| = |B| = 3k-2$.

If $u \in V(G) \setminus (A' \cup B')$, then there exists $D \in \mathcal{C} - C$, such that $u \in D$. Let $p \in A' \cap V(R)$, and $q \in B' \cap V(R)$ and label $\{w, w'\} = \{p, q\}$ so that $uw \notin E(G)$ and $uw' \in E(G)$. We have that $G[D - u + w] \cong K_{3,3}$ and

$G[R-w+u] \cong K_{3,2}$, so there exists a collection $\mathcal{C}'$ of $k-1$ vertex disjoint cycles containing C that satisfies (O1)-(O3), and there exists a spanning path of of $G[V(G) \setminus V(\mathcal{C}')]$ such that u is an endpoint or u is the neighbor of an endpoint. Therefore, by Lemma 3.29, $d_G(u) \geq 3k-1$, so, with Lemma 3.28, we have that $N_C(u) = (V(C) \setminus N_C(w')) + c$ and, for any $D' \in \mathcal{C}' - C$, by Lemma 3.8, $N_{D'}(u) = D' \setminus N_{D'}(w')$. Therefore, either $N_G(u) \supseteq B + c$ if $u \in A$ or $N_G(u) \supseteq A + c$ if $u \in B$. Hence, G contains $K_{1,3k-2,3k-2}$ as a spanning subgraph. As $K_{1,3k-2,3k-2}$ is edge-maximal with respect to not containing k disjoint chorded cycles, $G \cong K_{1,3k-2,3k-2}$. □

Using Lemmas 3.28, 3.30, 3.31, and 3.32, we conclude $G \in \{K_{3k-1,n-3k+1}, K_{1,3k-2,3k-2}\}$.

3.5 Future Questions

There are many extensions of Theorem 3.1 outside of those mentioned in Chapter 1. In particular, one can consider different conditions, such as neighborhood union conditions or bounding the difference between the number of vertices of 'large' degree and the number of vertices of 'small' degree. In addition, results on doubly chorded cycles, or chorded cycles with a specified number of chords, would also be of interest.

Chapter 4

Mixed Cycles

The following results are joint work with Theodore Molla and Elyse Yeager, appearing in [42].

4.1 Introduction

As mentioned in Section 1.3.2, the main purpose of this chapter is to prove the following statement.

Theorem 4.1 (Molla–Santana–Yeager [42]). *Let r and s be positive integers, and $n \geq 3r + 4s$. If G is an n-vertex graph having $\delta(G) \geq 2r + 3s - 1$, then G contains r disjoint cycles and s disjoint chorded cycles, unless*

1. *$G \cong K_{2r+3s-1,n-2r-3s+1}$, with $n \geq 4r + 6s - 2$, or*
2. *$G \cong K_{1,2r+3s-2,2r+3s-2}$, or*
3. *$s = 1$, r is even, and $G \cong \overline{K_{r+2}} \vee (K_{r+1} + K_{r+1})$, or*
4. *$s = 1$ and $G \cong H$, where $K_{t+1,2r-t+1,2r-t+1} \subseteq H \subseteq K_{t+1} \vee K_{2r-t+1,2r-t+1}$, for some t with $0 \leq t \leq r - 1$,*

For $n \geq 4r + 6s - 2$, $|K_{2r+3s-1,n-2r-3s+1}| = n \geq 3r + 4s$ and $\delta(K_{2r+3s-1,n-2r-3s+1}) = 2r + 3s - 1$. Since each cycle and chorded cycles uses at least two and three vertices, respectively, from each part, $K_{2r+3s-1,n-2r-3s+1}$ does not contain r disjoint cycles and s disjoint chorded cycles s of which are chorded.

For $r, s \geq 1$, $|K_{1,2r+3s-2,2r+3s-2}| = 4r + 6s - 3 \geq 3r + 4s$ and $\delta(K_{1,2r+3s-2,2r+3s-2}) = 2r + 3s - 1$. Each cycle in $K_{1,2r+3s-2,2r+3s-2}$ either uses two vertices from each of the big parts or uses the dominating vertex and at least one vertex from each big part. Each chorded cycle in $K_{1,2r+3s-2,2r+3s-2}$ either uses three vertices from each of the big parts or uses the dominating vertex and at least two vertices from a big part. Thus, $K_{1,2r+3s-2,2r+3s-2}$ does not contain r disjoint cycles and s disjoint chorded cycles.

For $s = 1$ and r even, $|\overline{K_{r+2}} \vee (K_{r+1} + K_{r+1})| = 3r + 4 = 3r + 4s$ and $\delta(\overline{K_{r+2}} \vee (K_{r+1} + K_{r+1})) = 2r + 2 = 2r + 3s - 1$. In order to find r disjoint cycles and one chorded cycle, we must obtain r copies of K_3

and one chorded 4-cycle; in addition, every vertex must be contained in exactly one of these $r+1$ disjoint objects. In particular, every vertex in $\overline{K_{r+2}}$ must be contained in these cycles and chorded cycle. This is only possible if r of these vertices are in r disjoint cycles and two of these vertices are in the chorded 4-cycle. Thus, every copy of K_3 uses two vertices from one of the K_{r+1}'s. Since $r+1$ is odd, this leaves exactly one vertex from each copy of K_{r+1} to form a chorded 4-cycle with two vertices from $\overline{K_{r+2}}$, however this cannot happen. Thus, $\overline{K_{r+2}} \vee (K_{r+1} + K_{r+1})$ does not contain r disjoint cycles and one chorded cycle.

For $s = 1$ and $0 \leq t \leq r-1$, let H be a graph such that $K_{t+1,2r-t+1,2r-t+1} \subseteq H \subseteq K_{t+1} \vee K_{2r-t+1,2r-t+1}$. Observe that $|H| = 4r - t + 3 \geq 3r + 4 = 3r + 4s$ and $\delta(H) = 2r + 2 = 2r + 3s - 1$. In order to show that H does not have r disjoint cycles and one chorded cycle, it suffices to consider $H \cong K_{t+1} \vee K_{2r-t+1,2r-t+1}$. As every triangle in H uses at least one vertex from the copy of K_{t+1}, H contains at most $t+1$ disjoint triangles. Since $|H| = 4r - t + 3 = 3(t+1) + 4(r-t)$, in order to obtain r disjoint cycles and one chorded cycle we must use $t+1$ triangles, each of which contains a single vertex from K_{t+1} and each of the partite sets of $K_{2r-t+1,2r-t+1}$. However, this leaves $K_{2r-2t,2r-2t}$, which does not contain $r-t-1$ disjoint cycles and one chorded cycle. Thus, H does not contain r disjoint cycles and one chorded cycle.

We leave it to the reader to check that $K_{2r+3s-1,n-2r-3s+1}$ for $n \geq 4r + 6s - 2$, $K_{1,2r+3s-2,2r+3s-2}$, $\overline{K_{r+2}} \vee (K_{r+1} + K_{r+1})$ for $s = 1$ and r even, and $K_{t+1} \vee K_{2r-t+1,2r-t+1}$ for $s = 1$ and $0 \leq t \leq r-1$, are each edge-maximal with respect to not having r disjoint cycles and s disjoint chorded cycles.

4.1.1 Setup and Outline

In order to prove Theorem 4.1, we consider a graph G on at least $3r + 4s$ vertices having $\delta(G) \geq 2r + 3s - 1$ and is edge-maximal with respect to not having a collection of r disjoint cycles and s chorded cycles. Since $G \ncong K_n$, there exists $e \notin E(G)$ such that $G + e$ has r disjoint cycles and s chorded cycles. Thus, we can consider two general cases in G; when G has a collection of $r-1$ disjoint cycles and s disjoint chorded cycles, and when G has a collection of r disjoint cycles and $s-1$ disjoint chorded cycles. In both cases we choose an 'optimal' collection $\mathcal{U}$ of $r+s-1$ disjoint objects, let R denote the subgraph of G induced by the vertices outside of $\mathcal{U}$, and let P be a longest path in R.

This chapter is structured as follows. In Sections 4.2 - 4.4, we assume that G has a collection of $r-1$ disjoint cycles and s disjoint chorded cycles. We choose $\mathcal{U}$ to be such a collection, optimized by certain constraints given in Section 4.2. Also in this section, we present several lemmas that we will use throughout our proof. In Sections 4.3 and 4.4, we consider the cases when $V(R) \neq V(P)$ and $V(R) = V(P)$, respectively. We complete our proof in Section 4.5 by assuming that G has no collection of $r-1$ disjoint cycles and s disjoint chorded cycles; that is, we choose $\mathcal{U}$ to be an optimal collection of r disjoint cycles and $s-1$ chorded

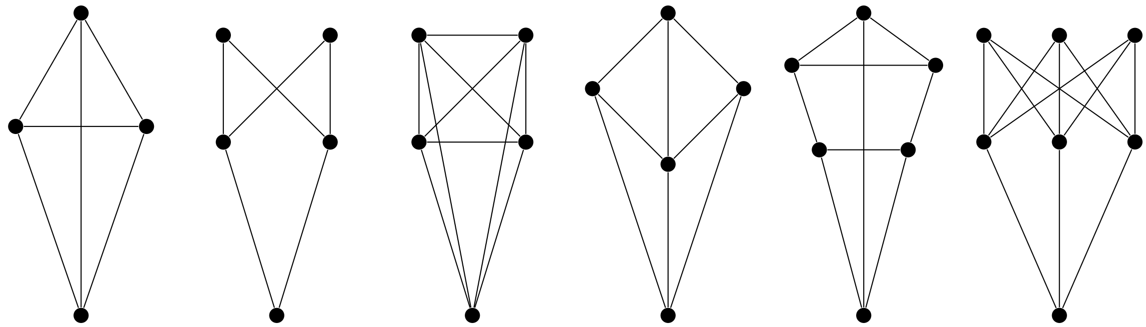


Figure 4.1: Figures for Lemma 4.2

cycles. We end this chapter with some questions for further research.

4.2 Prelimary Lemmas

In this section as well as in Sections 4.3 and 4.4, we assume G has a collection of $r-1$ disjoint cycles and s disjoint chorded cycles. Let $\mathcal{U} = \mathcal{C} \cup \mathcal{D}$ be such a collection in G, in which $\mathcal{D}$ contains the s disjoint chorded cycles, and $\mathcal{C}$ contains the $r-1$ disjoint cycles. Furthermore, we choose $\mathcal{U} = \mathcal{C} \cup \mathcal{D}$ to be a collection which satisfies the following conditions when $R = G \setminus \mathcal{U}$:

(O1) the number of vertices in $\mathcal{U}$ is minimum,

(O2) subject to (O1), the total number of chords in the cycles of $\mathcal{D}$ is maximum, and

(O3) subject to (O1) and (O2), the length of the longest path in R is maximum.

We use the convention that P is a longest path in R. By (O1), every cycle in $\mathcal{C}$ is an induced cycle in G; that is, C has no chords. In addition, for any chorded cycle $D \in \mathcal{D}$, no vertex of D is incident to two chords.

Lemma 4.2. *Let $v \in R$, $C \in \mathcal{C}$ and $D \in \mathcal{D}$.*

1. *If $\|v, C\| \geq 3$, then $\|v, C\| = 3$ and $G[C] \cong K_3$.*

2. *If $\|v, C\| = 2$, then $|C| \in \{3, 4\}$. Moreover, if $|C| = 4$, then $G[C] \cong K_{2,2}$ and $G[C + v] \cong K_{2,3}$.*

3. *If $\|v, D\| \geq 4$, then $\|v, D\| = 4$ and $G[D] \cong K_4$.*

4. *If $\|v, D\| = 3$, then $|D| \in \{4, 5, 6\}$. Moreover:*

   (a) if $|D| = 4$, then D has a chord incident to the non-neighbor of v;

   (b) if $|D| = 5$, then D is singly-chorded, and the endpoints of the chord are disjoint from the neighbors of v;

(c) if $|D| = 6$, then $G[D] \cong K_{3,3}$, and $G[D+v] \cong K_{3,4}$.

Proof. We first prove 4.2.1. Suppose $\|v, C\| \geq 3$ with $c_1, c_2, c_3 \in N_C(v)$ appearing in this order on C. If there exists $c \in V(C) \setminus \{c_1, c_2, c_3\}$, then without loss of generality, assume c appears on C after c_2 and before c_3. Let $\tilde{P}$ denote the path on C between c_1 and c_2 that does not contain c. Then $vc_1\tilde{P}c_2v$ is a cycle strictly smaller than C, contradicting (O1). Thus, $N_C(v) = \{c_1, c_2, c_3\} = V(C)$, which proves 4.2.1.

Suppose $\|v, C\| = 2$ with $N_C(v) = \{c_1, c_2\}$. Let P_1 and P_2 be the two paths between c_1 and c_2 on C. Suppose without loss of generality there exists internal vertices c and c' on $c_1P_1c_2$. Then $vc_1P_2c_2v$ is a cycle with fewer vertices than C, contradicting (O1). Thus, $|C| \in \{3, 4\}$, and furthermore, if $|C| = 4$, then $G[C+v] \cong K_{2,3}$. This proves 4.2.2.

We now consider $D \in \mathcal{D}$ and prove 4.2.3. If there exist vertices $d_1, d_2 \in D$ that are adjacent along the cycle of D such that $\|v, D - d_1 - d_2\| \geq 3$, then $(D - d_1 - d_2) + v$ contains a chorded cycle with strictly fewer vertices than D, contradicting (O1). This proves that if $\|v, D\| = 3$, then $|D| \leq 6$. Similarly, if $\|v, D\| \geq 4$, then $|D| = 4$ and $\|v, D\| = 4$. If $\|v, D\| = 4$ and $|D| = 4$, then v together with a triangle in D yields K_4. So by (O2), $G[D] \cong K_4$. This proves 4.2.3.

Suppose $\|v, D\| = 3$. If $|D| = 4$, then let $d \in D$ be the non-neighbor of v in D. If d is not incident to a chord, then $(D - d) + v \cong K_4$, preferable to D by (O2).

So $|D| \in \{5, 6\}$. Since the vertices in $V(D) \setminus N_G(v)$ cannot be adjacent along the cycle D, for any $d \in V(D) \setminus N_G(v)$, $D - d + v$ contains a chorded cycle D' of the same length as D. If d is not incident to a chord, then D' has strictly more chords than D, violating (O2). So every vertex in $V(D) \setminus N_G(v)$ is incident to a chord.

If $|D| = 6$, then v is adjacent to every other vertex along the cycle, and every $d \in V(D) \setminus N_G(v)$ is incident to a chord. Since no vertex in D is incident to two chords, (O1) implies that $G[D] \cong K_{3,3}$ and $G[D+v] \cong K_{3,4}$.

If $|D| = 5$, then (O1) implies that the only possible chord has the two non-neighbors of v as its endpoints, which completes the proof of 4.2.4. □

Lemma 4.3. *Let $C \in \mathcal{C}$ with $|C| = 4$, $D \in \mathcal{D}$ with $|D| = 6$, and let $u, v \in R$ such that $uv \in E(G)$. If $\|u, C\| = 2$ and $\|v, C\| \geq 1$, then $N_C(u) \cap N_C(v) = \emptyset$. Similarly, if $\|u, D\| = 3$ and $\|v, D\| \geq 1$, then $N_D(u) \cap N_D(v) = \emptyset$.*

Proof. By Lemma 4.2, $G[C] \cong K_{2,2}$ with partite sets $A = \{a_1, a_2\}$ and $B = \{b_1, b_2\}$, where $N_C(u) = A$. Suppose on the contrary, $va_1 \in E(G)$. Then we can replace C with the smaller cycle uva_1u to obtain a new collection $\mathcal{U}'$ that contradicts (O1).

Similarly, we may assume $G[D] \cong K_{3,3}$ with partite sets $A' = \{a'_1, a'_2, a'_3\}$ and $B' = \{b'_1, b'_2, b'_3\}$, where $N_D(u) = A'$. If $va'_1 \in E(G)$, then we can replace D with the smaller chorded cycle $ua'_2b'_1ba'_1vu$, contradicting (O1). □

Lemma 4.4. *If $D \in \mathcal{D}$ and $\|v_1, D\|, \|v_2, D\| \geq 3$ for distinct $v_1, v_2 \in R$, then $|D| \in \{4, 6\}$.*

Proof. If $|D| \notin \{4, 6\}$, then $|D| = 5$ and $N_D(v_1) = N_D(v_2)$ by Lemma 4.2. Furthermore, there are two adjacent vertices $d, d' \in N_D(v_1) = N_D(v_2)$, but then $v_1dv_2d'v_1$ is a chorded cycle contradicting (O1). □

Lemma 4.5. *Let p be an endpoint of P and let v be a vertex in $R \setminus P$, if it exists. Let $F = \{p, v\}$. Then $\|F, C\| \leq 4$ and $\|F, D\| \leq 6$ for all $C \in \mathcal{C}$ and $D \in \mathcal{D}$. Furthermore,*

(1) if $\|F, C\| = 4$, then either $\|p, C\| = |C| = 3$, or $|C| \in \{3, 4\}$ and $N_C(p) = N_C(v)$;

(2) if $\|F, D\| = 6$, then either $\|p, D\| = |D| = 4$, or $|D| \in \{4, 6\}$ and $N_D(p) = N_D(v)$.

Proof. Suppose $\|F, C\| \geq 4$. Assume $|C| = 3$ with $V(C) = \{c_1, c_2, c_3\}$. If we can partition $V(C)$ such that without loss of generality $pc_1, vc_2, vc_3 \in E(G)$, then we can replace C with vc_2c_3v and P with $P + c_1$, contradicting (O3). This implies that $\|F, C\| = 4$, and furthermore, if $\|p, C\| \leq 2$, then $\|p, C\| = 2 = \|v, C\|$ with $N_C(p) = N_C(v)$.

Assume $|C| = 4$ with $C = c_1c_2c_3c_4c_1$. By Lemma 4.2, we may assume $N_C(p) = \{c_1, c_3\}$ and $N_C(v) = \{c_2, c_4\}$. However, we can replace C with $vc_2c_3c_4v$ and P with $P + c_1$, contradicting (O3). This completes the proof of (1).

Consider $D \in \mathcal{D}$ and suppose $\|F, D\| \geq 6$. Assuem $G[D] \cong K_4$ with $V(D) = \{d_1, d_2, d_3, d_4\}$. If we can partition $V(D)$ such that, without loss of generality, $d_1, d_2, d_3 \in N_G(v)$ and $d_4 \in N_G(p)$, then we can replace D and P with $D - d_4 + v$ and $P + d_4$, respectively, which violates (O3). Thus, if $G[D] \cong K_4$, then either $\|p, D\| = |D| = 4$ or $N_D(p) = N_D(v)$.

So we may assume $G[D] \ncong K_4$. As a result, $\|\{v, p\}, D\| = 6$ and $\|v, D\| = \|p, D\| = 3$, so that $|D| \in \{4, 6\}$ by Lemmas 4.2 and 4.4. Suppose $|D| = 4$ so that $G[D] \cong K_{1,1,2}$. Let $d \in N_D(p)$ such that d is not incident to a chord in D. Then we can replace D and P with $D - d + v$ and $P + d$, respectively, which violates (O3).

Finally, suppose $|D| = 6$. By Lemma 4.2, if v and p do not have the same neighborhood, they are adjacent to disjoint sets of vertices, and $D + p$ and $D + v$ both contain $K_{3,4}$. In this case, we extend P using any $d \in N_D(p)$, and replace D with a chorded cycle in $D - d + v$. This violates (O3), and completes the proof. □

Lemma 4.6. *Let u_1, v_1, u_2, v_2 be distinct vertices, and let P_1 and P_2 be vertex-disjoint paths in R from u_1 to v_1 and u_2 to v_2, respectively. Let $F = \{u_1, v_1, u_2, v_2\}$. Then $\|F, C\| \leq 7$ and $\|F, D\| \leq 11$ for all $C \in \mathcal{C}$ and $D \in \mathcal{D}$.*

Proof. Fix $C \in \mathcal{C}$ and suppose $\|F, C\| \geq 8$. By Lemma 4.2, $|C| \in \{3, 4\}$. Suppose first that $|C| = 3$ with $V(C) = \{c_1, c_2, c_3\}$. Assume $\|u_1, C\| = 3$. If u_2, v_2 have a common neighbor in C, say c_1, then we replace C with $c_1u_2P_2v_2c_1$ and $u_1c_2c_3u_1$, a contradiction. So $\|\{u_2, v_2\}, C\| \leq 3$ and $\|v_1, C\| \geq 2$.

If $\|v_1, C\| = 3$, then $\|\{u_2, v_2\}, C\| \geq 2$, and we may assume $c_1, c_2 \in N_C(u_2) \cup N_C(v_2)$ so that $G[P_2 + c_1 + c_2]$ contains a cycle. However, $c_3u_1P_1v_2c_3$ is another cycle, a contradiction.

So $\|v_1, C\| = 2$ and $\|\{u_2, v_2\}, C\| = 3$. Without loss of generality, suppose $N_C(v_1) = \{c_1, c_2\}$. If u_2 is adjacent to both c_2 and c_3, then we obtain cycles $u_2c_2c_3u_2$ and $u_1c_1v_1P_1u_1$, a contradiction. A similar contradiction holds if u_2 is adjacent to both c_1 and c_3, and the same holds for v_2 in place of u_2.

Since $\|\{u_2, v_2\}, C\| = 3$ and u_2 and v_2 don't have a common neighbor in C, we may assume $u_2c_1, u_2c_2, v_2c_3 \in E(G)$. However, this yields $u_2c_2c_3v_2P_2u_2$ and $u_1c_1v_1P_1u_1$, a contradiction.

So by symmetry, $\|w, C\| = 2$ for all $w \in F$. Without loss of generality, assume $N_C(u_1) = \{c_1, c_2\}$ and $c_1 \in N_C(v_1)$. Let c be the common neighbor of u_2 and v_2 in C. If $c \neq c_2$, then obtain $cu_2P_2v_2c$ and $c_1u_1P_1v_1c_1$ a contradiction. So we may assume $N_C(u_2) = \{c_1, c_2\}$ and $N_C(v_2) = \{c_1, c_3\}$. However, this yields $c_2u_2P_2v_2c_3c_2$ and $c_1u_1P_1v_1c_1$, a contradiction.

Thus we may assume that $|C| = 4$, and by Lemma 4.2, $\|u_i, C\| = \|v_i, C\| = 2$ for $i \in \{1, 2\}$. Let $C = c_1c_2c_3c_4c_1$, so that without loss of generality $N_C(u_1) = \{c_1, c_3\}$ by Lemma 4.2. Suppose $N_C(v_1) = \{c_2, c_4\}$. If u_2 and v_2 have a common neighbor in C, say c_1, then we obtain cycles $c_1u_2P_2v_2c_1$ and $c_2c_3u_1P_1v_1c_2$, a contradiction. So without loss of generality, $N_C(u_2) = N_C(u_1)$ and $N_C(v_2) = N_C(v_1)$. However, this yields $c_1c_2v_1P_1u_1c_1$ and $c_3c_4v_2P_2u_2c_3$, a contradiction.

Thus, $N_C(u_1) = N_C(v_1)$ and by symmetry, $N_C(u_2) = N_C(v_2)$. In particular, there exists vertices $c, c' \in V(C)$ such that c is a common neighbor of u_1 and v_1, and c' is a common neighbor of u_2 and v_2. However, this yields $u_1cv_1P_1u_1$ and $u_2c'v_2P_2u_2$, a contradiction. Thus, $\|F, C\| \leq 7$ for all $C \in \mathcal{C}$, as desired.

Consider $D \in \mathcal{D}$ and suppose $\|F, D\| \geq 12$. We first consider the case where $G[D] \cong K_4$. Suppose $\|u_1, D\| = 4$, so that u_2 and v_2 do not have a common neighbor d in D, otherwise we obtain $u_2dv_2P_2u_2$ and $D - d + u_1$. Hence, $\|\{u_2, v_2\}, D\| \leq 4$, which implies $\|v_1, D\| = \|\{u_2, v_2\}, D\| = 4$. Since $\|\{u_1, v_1\}, D\| = 8 > 4$, a symmetric argument shows that $1 \leq \|u_2, D\|, \|v_2, D\| \leq 3$. As u_2 and v_2 do not have a common neighbor, we may assume u_2 and v_2 have distinct neighbors in D, say d and d', respectively. However this yields $u_2dd'v_2P_2u_2$ and $D - d - d' + u_1 + v_1$, a contradiction.

Thus, we may assume that $\|u_i, D\| = \|v_i, D\| = 3$ for $i \in \{1, 2\}$. In particular, u_1 and v_1 have a common neighbor d, which yields $u_1dv_1P_1u_1$ and $D-d+u_2$, a contradiction.

So $G[D] \not\cong K_4$, and by Lemma 4.2, $\|u_i, D\| = \|v_i, D\| = 3$ for $i \in \{1,2\}$. By Lemma 4.4, $|D| \in \{4,6\}$. If $|D| = 4$, then let $d_1d_2d_3d_4d_1$ be a cycle of D with chord d_2d_4. By Lemma 4.2, u_1 and v_1 are both adjacent to d_1. Thus we obtain $u_1d_1v_1P_1u_1$ and $D-d_1+u_2$, a contradiction.

So $|D| = 6$ and $G[D] \cong K_{3,3}$. Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$ be the partite sets of D with $N_D(u_1) = A$. Suppose $N_D(v_1) = B$. If u_2 and v_2 have a common neighbor, say a_1, then we obtain $u_2a_1v_2P_2u_2$ and $D-a_1+v_1$, a contradiction. So without loss of generality, $N_D(u_2) = A$ and $N_D(v_2) = B$. This yields $a_3b_3v_2P_2u_2a_3$ and $D-a_3-b_3+u_1+v_1$, a contradiction.

So $N_D(u_1) = N_D(v_1) = A$, and by symmetry, $N_D(u_2) = N_D(v_2)$. Without loss of generality, u_2 and v_2 have a common neighbor $d \notin \{a_1, a_2, b_1\}$. However, this yields $u_2dv_2P_2u_2$ and $u_1a_1b_1a_2v_1P_1u_1$, a contradiction. This completes the proof of the lemma. □

Lemma 4.7. *Let $\tilde{P}$ be a path contained in R, let u_1, u_2, u_3, u_4 be distinct vertices on $\tilde{P}$ appearing in this order (not necessarily consecutive), and let $F = \{u_1, u_2, u_3, u_4\}$. Then $\|F, C\| \le 7$ and $\|F, D\| \le 9$ for all $C \in \mathcal{C}$ and $D \in \mathcal{D}$.*

Proof. By Lemma 4.6, $\|F, C\| \le 7$. So let $D \in \mathcal{D}$ and suppose $\|F, D\| \ge 10$. We first consider when $G[D] \cong K_4$ with $V(D) = \{d_1, d_2, d_3, d_4\}$. Suppose $\|u_1, D\| \ge 3$. If u_i and u_j have a common neighbor d for distinct $i, j \in \{2,3,4\}$, then we obtain $D-d+u_1$ and $u_idu_j\tilde{P}u_i$, a contradiction. Thus, $\|\{u_2, u_3, u_4\}, D\| \le 4$, a contradiction as $\|F, D\| \ge 10$.

So $\|u_1, D\| \le 2$ and by symmetry $\|u_4, D\| \le 2$. Without loss of generality, assume $\|u_2, D\| \ge 3$. If u_3 and u_4 have a common neighbor d, then $u_3du_4\tilde{P}u_3$ and $D-d+u_2$, a contradiction. So $\|\{u_3, u_4\}, D\| \le 4$. Thus, $\|\{u_1, u_2\}, D\| \ge 6$, which implies $\|u_1, D\| = 2$, $\|u_2, D\| = 4$, and $\|u_3, D\| \ge 2$. We can find $i \in \{3,4\}$ such that without loss of generality, $d_1, d_2 \in N_D(u_i)$ and $N_D(u_1) \ne \{d_1, d_2\}$. Hence we may assume $d_3 \in N_D(u_1)$. However, this yields $u_id_1d_2u_i$ and $u_2d_4d_3u_1\tilde{P}u_2$, a contradiction.

Thus, $G[D] \not\cong K_4$ and by Lemma 4.2, $\|u_i, D\| \le 3$ for $i \in \{1,2,3,4\}$. Since $\|F, D\| \ge 10$, there exists $i, j \in \{1,2,3,4\}$ such that $\|u_i, D\| = \|u_j, D\| = 3$ so that by Lemmas 4.2 and 4.4, $|D| \in \{4,6\}$. Suppose $|D| = 4$ with cycle $d_1d_2d_3d_4d_1$ and chord d_2d_4. Assume $\|u_1, D\| = 3$ with $N_D(u_1) = \{d_1, d_2, d_3\}$. Thus, $\|\{u_2, u_3, u_4\}, D\| \ge 7$, which implies that u_i, u_j for $i, j \in \{2,3,4\}$ have a common neighbor $d \ne d_2$. However, this yields $u_idu_j\tilde{P}u_i$ and $D-d+u_1$, a contradiction.

So $\|u_1, D\| \le 2$ and by symmetry $\|u_4, D\| \le 2$. Thus, $\|u_2, D\| = \|u_3, D\| = 3$ with $\|u_1, D\| = \|u_4, D\| = 2$. Without loss of generality, $N_D(u_2) = \{d_1, d_2, d_3\}$. If u_4 is adjacent to d_1, then we obtain $u_3d_1u_4\tilde{P}u_3$ and $D-d_1+u_2$, a contradiction. A similar argument shows $u_4d_3 \notin E(G)$ so that $N_D(u_4) = \{d_2, d_4\}$. If

$u_3d_4 \in E(G)$, then $u_3d_4u_4\tilde{P}u_3$ and $D-d_4+u_2$, a contradiction. Thus, $N_D(u_3) = N_D(u_2)$, and by symmetry, $N_D(u_1) = N_D(u_4)$. However, this yields $u_2d_1u_3\tilde{P}u_2$ and $u_1d_2u_4d_4u_1$, a contradiction.

Thus $|D| = 6$ and by Lemma 4.2, $G[D] \cong K_{3,3}$. Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$ denote the partite sets of D. Suppose $\|u_1, D\| = 3$ with $N_D(u_1) = A$. Thus, $\|\{u_2, u_3, u_4\}, D\| \geq 7$ so that there exists u_i and u_j, $i, j \in \{2, 3, 4\}$ with a common neighbor d in D. If $d \in B$, then we obtain $u_idu_j\tilde{P}u_i$ and $D - d + u_1$, a contradiction. If $d \in A$, say $d = a_1$, then we obtain $u_1a_1u_j\tilde{P}u_1$ (assuming $i < j$) and $a_2b_2a_3b_3a_2$, a contradiction.

So $\|u_1, D\| \leq 2$ and by symmetry, $\|u_4, D\| \leq 2$. Thus, $\|u_2, D\| = \|u_3, D\| = 3$ with $\|u_1, D\| = \|u_4, D\| = 2$. Without loss of generaltiy, $N_D(u_2) = A$. Suppose $N_D(u_3) = A$ as well. If $N_D(u_1) \not\subset A$, then without loss of generality, suppose $a_3 \notin N_D(u_1)$. However we obtain the cycle $u_2a_3u_3\tilde{P}u_2$, and $D - a_3 + u_1$ contains a chorded cycle. So by symmetry, $N_D(u_1) \cup N_D(u_4) \subseteq A$. Without loss of generality, $a_1 \in N_D(u_1)$ and $a_3 \in N_D(u_4)$. Yet, this yields $u_3a_3u_4\tilde{P}u_3$ and $u_1a_1b_1a_2u_2\tilde{P}u_1$, a contradiction.

Thus, $N_D(u_2) = A$ and $N_D(u_3) = B$. If u_1 is adjacent any $a_i \in A$, then we obtain $u_1a_iu_2\tilde{P}u_1$ and $D - a_i + u_3$, a contradiction. So without loss of generality $N_D(u_1) = \{b_1, b_2\}$, and by symmetry $N_D(u_4) = \{a_1, a_2\}$. However, this yields $u_1b_1a_1u_2\tilde{P}u_1$ and $u_3b_2a_3b_3a_2u_4\tilde{P}u_3$, a contradiction. This completes the proof of the lemma. □

Lemma 4.8. *Let $\tilde{P}$ be a path contained in R, let u_1, u_2, u_3 be distinct vertices on $\tilde{P}$ appearing in this order (not necessarily consecutive), and let $F = \{u_1, u_2, u_3\}$. Then $\|F, C\| \leq 7$ and $\|F, D\| \leq 9$ for all $C \in \mathcal{C}$ and $D \in \mathcal{D}$. Furthermore,*

(1) if $\|F, C\| = 7$, then $|C| = 3$, $N_C(u_2) = V(C)$, $N_C(u_1) = N_C(u_3)$, and no other vertex on $\tilde{P}$ has a neighbor in C;

(2) if $\|F, D\| = 9$, then $D \cong K_{3,3}$, $N_D(u_1) = N_D(u_3)$, and no other vertex on $\tilde{P}$ has a neighbor in D.

Proof. By Lemma 4.7, $\|F, C\| \leq 7$ and $\|F, D\| \leq 9$ for all $C \in \mathcal{C}$ and $D \in \mathcal{D}$. So fix $C \in \mathcal{C}$ and suppose $\|F, C\| = 7$. For some $i \in \{1, 2, 3\}$, $\|u_i, C\| = 3$ so that by Lemma 4.2, $G[C] \cong K_3$. Suppose $\|u_1, C\| = 3$. Then u_2 and u_3 have a common neighbor c in C, and we obtain $u_2cu_3\tilde{P}u_2$ and $C - c + u_1$, a contradiction.

So by symmetry, $\|u_1, C\| = \|u_3, C\| = 2$ and $\|u_2, C\| = 3$. Let $V(C) = \{c_1, c_2, c_3\}$ and suppose without loss of generality that $N_C(u_1) = \{c_1, c_2\}$. If $N_C(u_3) \neq N_C(u_1)$, then say $N_C(u_3) = \{c_2, c_3\}$. However this yields $u_3c_2c_3u_3$ and $u_1c_1u_2\tilde{P}u_1$, a contradiction. So $N_C(u_1) = N_C(u_3) = \{c_1, c_2\}$.

Suppose there exists some vertex $u \notin \{u_1, u_2, u_3\}$ on $\tilde{P}$ such that $uc_i \in E(G)$ for some $i \in \{1, 2, 3\}$. Without loss of generality assume u is on $u_1\tilde{P}u_2$. If $uc_1 \in E(G)$, then we obtain $uc_1u_1\tilde{P}u$ and $u_2c_3c_2u_3\tilde{P}u_2$, a contradiction. A similar argument holds if $uc_2 \in E(G)$. If $uc_3 \in E(G)$, then we obtain $uc_3u_2\tilde{P}u$ and $u_1c_1u_3c_2u_1$, a contradiction. This proves (1).

Fix $D \in \mathcal{D}$ and suppose $\|F, D\| = 9$. We first consider when $G[D] \cong K_4$. Without loss of generality, suppose $\|u_1, D\| \geq 3$. Since $\|F, D\| = 9$, u_2 and u_3 have a common neighbor d in D. However this yields $u_2du_3\tilde{P}u_2$ and $D - D + u_1$, a contradiction.

So $G[D] \ncong K_4$ and by Lemma 4.2, $\|u_i, D\| = 3$ for $i \in \{1, 2, 3\}$. By Lemma 4.4, $|D| \in \{4, 6\}$. If $|D| = 4$, let $d_1d_2d_3d_4d_1$ be a cycle of D with chord d_2d_4. However, this yields $u_1d_1u_2\tilde{P}u_1$ and $D - d_1 + u_3$, a contradiction.

Thus, $|D| = 6$ and by Lemma 4.2, $G[D] \cong K_{3,3}$. Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$ be the partite sets of D with $N_D(u_1) = A$. If $N_D(u_3) = B$, then without loss of generality suppose $N_D(u_2) = A$. However this yields $u_1a_1u_2\tilde{P}u_1$ and $D - a_1 + u_3$, a contradiction.

Suppose there exists some vertex $u \notin \{u_1, u_2, u_3\}$ on $\tilde{P}$ such that $ud \in E(G)$ for some d in D. Without loss of generality, assume u is on $u_1\tilde{P}u_2$. Suppose $d = a_1$. If $N_D(u_2) = B$, then we obtain $u_1a_1u\tilde{P}u_1$ and $D - a_1 + u_2$, a contradiction. If $N_D(u_2) = A$, then we obtain $u_1a_1u_2\tilde{P}u_1$ and $a_2b_2a_3b_3a_2$, a contradiction.

So we may assume $d = b_1$. Then we obtain the cycle $u_1a_1b_1u\tilde{P}u_1$, and regardless of $N_D(u_2) \in \{A, B\}$, $D - a_1 - b_1 + u_2\tilde{P}u_3$ contains a chorded cycle, a contradiction. This completes the proof of the lemma. $\square$

In Sections 4.3 and 4.4, we will quickly show that $|P| = 3$. Thus, if $\|P, D\| = 9$ for some $D \in \mathcal{D}$, then Lemmas 4.2, 4.3, and 4.8 imply that $G[P + D] \cong K_{4,5}$.

Lemma 4.9. *Let u_1, u_2, u_3, v be distinct vertices, and let P_1, P_2, P_3 be pairwise internally disjoint paths from u_1, u_2, u_3, respectively, to v, with each path contained in R. Let $F = \{u_1, u_2, u_3\}$. Then $\|F, C\| \leq 6$ and $\|F, D\| \leq 9$ for all $C \in \mathcal{C}$ and $D \in \mathcal{D}$. Furthermore,*

(1) if $\|F, C\| = 6$, then no other vertex on any P_i, except possibly v, has a neighbor in C, and either

(a) $N_C(u_1) = N_C(u_2) = N_C(u_3)$, or

(b) $|C| = 3$ and there exists $j, k \in \{1, 2, 3\}$ such that $N_C(u_j) = N_C(u_k) = V(C)$;

(2) if $\|F, D\| = 9$, then $D \cong K_{3,3}$, $N_D(u_1) = N_D(u_2) = N_D(u_3)$, and no other vertex on any P_i, except possibly v, has a neighbor in D.

Proof. Fix $C \in \mathcal{C}$ and suppose $\|F, C\| \geq 7$. By Lemma 4.2, $G[C] \cong K_3$ and without loss of generality, $\|u_1, C\| = 3$. Since $\|\{u_2, u_3\}, C\| \geq 4$, u_2 and u_3 have a common neighbor c in C. Thus we obtain $u_2cu_3P_3vP_2u_2$ and $C - c + u_1$, a contradiction. Thus, in the remainder of this proof we assume $\|F, C\| = 6$.

We first consider when $G[C] \cong K_3$. Suppose $\|u_1, C\| = 3$. As above, u_2 and u_3 cannot have a common neighbor. Thus, $\|\{u_2, u_3\}, C\| = 3$. Suppose $1 \leq \|u_2, C\|, \|u_3, C\| \leq 2$. Then without loss of generality, assume $N_C(u_2) = \{c_1, c_2\}$ and $N_C(u_3) = \{c_3\}$. However this yields $u_2c_1c_2u_2$ and $u_1c_3u_3P_3vP_1u_1$, a contradiction. So without loss of generality, $\|u_2, C\| = 3$ and $\|u_3, C\| = 0$.

If there exists $u \notin \{u_1, u_2, u_3, v\}$ on some P_i such that $uc \in E(G)$ for some $c \in C$, then u is not on P_3, as otherwise for $\tilde{F} = \{u_1, u_2, u\}$, $\|\tilde{F}, C\| \geq 7$, a contradiction to the above. So without loss of generality, suppose such a u exists on P_1. However this yields $u_1cuP_1u_1$ and $C - c + u_2$, a contradiction. This completes the proof of *(1b)*.

So suppose $G[C] \cong K_3$ with $\|u_i, C\| \leq 2$ for all $i \in \{1, 2, 3\}$. Since $\|F, C\| = 6$, we have $\|u_i, C\| = 2$ for all i. Let $V(C) = \{c_1, c_2, c_3\}$ and without loss of generality, $N_C(u_1) = \{c_1, c_2\}$. If u_2 and u_3 are both adjacent to c_3 then we obtain $u_2c_3u_3P_3vP_2u_2$ and $u_1c_1c_2u_1$, a contradiction. So without loss of generality, $N_C(u_3) = N_C(u_1)$ and $N_C(u_2) = \{c_2, c_3\}$. However, this yields $u_1c_1u_3P_3vP_1u_1$ and $u_2c_2c_3u_2$, a contradiction. Thus, $N_C(u_1) = N_C(u_2) = N_C(u_3) = \{c_1, c_2\}$.

Suppose there exists $u \notin \{u_1, u_2, u_3, v\}$ on some P_i such that $uc \in E(G)$ for some $c \in C$. Without loss of generality assume u is on P_1. If $uc_1 \in E(G)$, then we obtain $u_1c_1uP_1u_1$ and $u_2c_2u_3P_3vP_2u_2$, a contradiction. A similar argument holds if $uc_2 \in E(G)$. However, if $uc_3 \in E(G)$, then we obtain $uc_3c_1u_1P_1u$ and $u_2c_2u_3P_3vP_2u_2$, a contradiction. This completes the proof of *(1a)* in the case that $G[C] \cong K_3$.

So suppose $G[C] \cong K_{2,2}$ with partite sets $A = \{a_1, a_2\}$ and $B = \{b_1, b_2\}$. By Lemma 4.2, $\|u_i, C\| = 2$ for $i \in \{1, 2, 3\}$, and we may assume that $N_C(u_1) = A$. If $N_C(u_2) = B$, then without loss of generality suppose $N_C(u_3) = A$. This yields $u_1a_1u_3P_3vP_1u_1$ and $C - a_1 + u_2$, a contradiction. Thus, $N_C(u_1) = N_C(u_2) = N_C(u_3)$ by symmetry.

Suppose there exists $u \notin \{u_1, u_2, u_3, v\}$ on some P_i such that $uc \in E(G)$ for some $c \in C$. Without loss of generality, assume u is on P_1. If $u = a$, then we obtain $ua_1u_1P_1u$ and $u_2a_2u_3P_3vP_2u_2$, a contradiction. If $u = b_1$, then we obtain $ub_1a_1u_1P_1u$ and $u_2a_2u_3P_3vP_2u_2$, a contradiciton. This completes the proof of 4.9.1a and proves 4.9.1.

Fix $D \in \mathcal{D}$ and suppose $\|F, D\| \geq 9$. Suppose $G[D] \cong K_4$. Without loss of generality we may assume that $\|u_1, D\| \geq 3$. This implies that $\|\{u_2, u_3\}, D\| \geq 5$ so that u_2 and u_3 have a common neighbor d in D. However this yields $D - d + u_1$ and $u_2du_3P_3vP_2u_2$, a contradiction.

Thus, $G[D] \ncong K_4$ and by Lemma 4.2, $\|u_i, D\| = 3$ for $i \in \{1, 2, 3\}$. By Lemma 4.4, $|D| \in \{4, 6\}$. Suppose $|D| = 4$ with cycle $d_1d_2d_3d_4d_1$ and chord d_2d_4. Then we obtain $u_1d_1u_2P_2vP_1u_1$ and $D - d_1 + u_3$, a contradiction.

So $|D| = 6$ and by Lemma 4.2, $G[D] \cong K_{3,3}$. Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$ be the partite sets of D with $N_D(u_1) = A$. If say $N_D(u_2) = B$, then assume without loss of generality that $N_D(u_3) = A$. However this yields $u_1a_1u_3P_3vP_1u_1$ and $D - a_1 + u_2$, a contradiction. Thus by symmetry, $N_D(u_1) = N_D(u_2) = N_D(u_3) = A$.

Suppose there exists some vertex $u \notin \{u_1, u_2, u_3, v\}$ on say P_1 such that $ud \in E(G)$ for some d in D. If $d = a_1$, then we obtain $u_1a_2b_1a_1uP_1a_1$ and $u_2a_3u_3P_3vP_2u_2$, a contradiction. If $d = b_1$, then we obtain $u_1a_2b_2a_1b_1uP_1u_1$ and $u_2a_3u_3P_3vP_2u_2$, again a contradiction. This proves the lemma. □

4.3 $V(R) \neq V(P)$

Lemma 4.10. *$R \setminus P$ is an independent set.*

Proof. Suppose on the contrary that there exists a nontrivial component in $R \setminus P$, and let $\tilde{P}$ be a maximal path in that component with endpoints t_1 and t_2.

Claim 4.10.1. $d_R(t_1) = d_R(t_2) = 1$.

Proof. Suppose without loss of generality, $d_R(t_1) \geq 2$. As R is acyclic and both P and $\tilde{P}$ are maximal, t_1 has exactly one neighbor on P, and it is not p_1 or p_2. Furthermore, $d_R(t_2) = d_R(p_1) = d_R(p_2) = 1$ and $d_R(t_1) = 2$. Let $F = \{p_1, p_2, t_2\}$. By Lemma 4.9, $\|F, \mathcal{C}\| \leq 6(r-1)$ and $\|F, \mathcal{D}\| \leq 9s$ so that $\|F, \mathcal{U}\| \leq 6r + 9s - 6$. Since $\|F, R\| = 3$, $\|F, \mathcal{U}\| \geq 3(2r + 3s - 1) - 3 = 6r + 9s - 6$. Therefore, $\|F, \mathcal{C}\| = 6(r-1)$ and $\|F, \mathcal{D}\| = 9s$. This implies $\|F, C\| = 6$ and $\|F, D\| = 9$ for all $C \in \mathcal{C}$ and $D \in \mathcal{D}$. By Lemma 4.9, $\|t_1, \mathcal{U}\| = 0$. However, $\|t_1, \mathcal{U}\| \geq 2r + 3s - 1 - 2 \geq 2$, a contradiction. □

Let $\tilde{F} = \{p_1, p_2, t_1, t_2\}$. By the above claim and the maximality of P, $d_R(p_1) = d_R(p_2) = d_R(t_1) = d_R(t_2) = 1$. Thus $\|\tilde{F}, \mathcal{U}\| \geq 4(2r + 3s - 1) - 4 = 8r + 12s - 8$. By Lemma 4.6, $\|\tilde{F}, \mathcal{C}\| \leq 7(r-1)$ and $\|\tilde{F}, \mathcal{D}\| \leq 11s$, so that $\|\tilde{F}, \mathcal{U}\| \leq 7r + 11s - 7$. This implies $8r + 12s - 8 \leq 7r + 11s - 7$ so that $r + s \leq 1$, a contradiction as $r, s \geq 1$. □

Lemma 4.11. $|P| \geq 3$.

Proof. Let p be an endpoint of P, $v \in R \setminus P$, and let $\tilde{F} = \{v, p\}$. By Lemma 4.5, $\|\tilde{F}, C\| \leq 4$ and $\|\tilde{F}, D\| \leq 6$ for all $C \in \mathcal{C}$ and $D \in \mathcal{D}$. Thus, $\|\tilde{F}, \mathcal{U}\| \leq 4r + 6s - 4$. If $\|\tilde{F}, R\| < 2$, then $\|\tilde{F}, \mathcal{U}\| > 2(2r + 3s - 1) - 2 = 4r + 6s - 4$. However, this implies that $4r + 6s - 4 < 4r + 6s - 4$, a contradiction.

So $\|\tilde{F}, R\| \geq 2$. Since P is a longest path, $d_R(p) \leq 1$. By Lemma 4.47, the only neighbors of v in R are internal vertices of P. Thus, P must have an internal vertex so that $|P| \geq 3$. □

Lemma 4.12. $|P| = 3$. *That is, $G[P] \cong K_{1,2}$.*

Proof. Suppose on the contrary, $|P| \geq 4$.

Claim 4.12.1. *At most one vertex on P has a neighbor in $R \setminus P$.*

Proof. Suppose on the contrary that there exists u_1 and u_2 on P that has neighbors t_1 and t_2, respectively, in $R \setminus P$. As R is acyclic, $t_1 \neq t_2$, and by the maximality of P, neither u_1 or u_2 is an endpoint of P. Without loss of generality, suppose p_1, u_1, u_2, p_2 appear on P in this order.

Let P_i be the path $p_i P u_i t_i$ for $i \in [2]$. Note that P_1 and P_2 are two vertex-disjoint paths. Let $\tilde{F} = \{p_1, p_2, t_1, t_2\}$, so that by Lemma 4.6, $\|F, \mathcal{C}\| \leq 7$ and $\|F, \mathcal{D}\| \leq 11$ for all $C \in \mathcal{C}$ and $D \in \mathcal{D}$. Thus, $\|F, \mathcal{U}\| \leq 7r + 11s - 7$. By Lemma 4.47, the maximality of P, and R being acyclic, $d_R(p_1) = d_R(p_2) = d_R(t_1) = d_R(t_2) = 1$.

Thus, $\|F, \mathcal{U}\| \geq 4(2r + 3s - 1) - 4 = 8r + 12s - 8$. This implies $8r + 12s - 8 \leq 7r + 11s - 7$ so that $r + s \leq 1$, a contradiction as $r, s \geq 1$. This proves the claim. $\square$

Claim 4.12.2. *No vertex on P has a neighbor in $R \setminus P$.*

Proof. Suppose on the contrary, there exists a vertex u on P has a neighbor $t \in R \setminus P$. By the maximality of P, u must be an internal vertex of P. Since $|P| \geq 4$, P has another internal vertex, call it v with $d_R(v) = 2$ by the above claim.

Let $\tilde{F} = \{p_1, p_2, t\}$. By Lemma 4.9, $\|F, C\| \leq 6$ and $\|F, D\| \leq 9$ for all $C \in \mathcal{C}$ and $\mathcal{D}$. Thus, $\|F, \mathcal{U}\| \leq 6r + 9s - 6$. As R is acyclic, P is maximal, and by Lemma 4.47, $\|F, R\| = 3$ so that $\|F, \mathcal{U}\| \geq 3(2r + 3s - 1) - 3 = 6r + 9s - 6$. This implies that $\|F, C\| = 6$ and $\|F, D\| = 9$ for all $C \in \mathcal{C}$ and $D \in \mathcal{D}$, and by Lemma 4.9 $\|v, \mathcal{U}\| = 0$. However, $\|v, \mathcal{U}\| \geq 2r + 3s - 1 - 2 \geq 2$, a contradiction. $\square$

Thus, no vertex on P has a neighbor in $R \setminus P$. Since $|P| \geq 4$, let u, v be internal vertices of P, and let $\tilde{F} = \{p_1, u, v, p_2\}$. By Lemma 4.7, $\|\tilde{F}, \mathcal{C}\| \leq 7(r - 1)$ and $\|\tilde{F}, \mathcal{D}\| \leq 9s$. Since $d_R(p_1) = d_R(p_2) = 1$ and $d_R(u) = d_R(v) = 2$, $\|\tilde{F}, \mathcal{U}\| \geq 4(2r + 3s - 1) - 6 = 8r + 12s - 10$. This implies that $8r + 12s - 10 \leq 7r + 9s - 7$ so that $r + 3s \leq 3$, a contradiction as $r, s \geq 1$. $\square$

By Lemma 4.12, we can label the vertices of P so that $P = p_1 q p_2$.

Lemma 4.13. $R \cong K_{1,|R|-1}$.

Proof. We claim that for all $v \in R \setminus P$, $vq \in E(G)$. Suppose not. Then $d_R(v) = 0$ by Lemma 4.47 and the maximality of P. Let $\tilde{F} = \{v, p_1\}$ so that by Lemma 4.5, $\|\tilde{F}, C\| \leq 4$ and $\|\tilde{F}, D\| \leq 6$ for all $C \in \mathcal{C}$ and $D \in \mathcal{D}$. Thus, $\|\tilde{F}, \mathcal{U}\| \leq 4r + 6s - 4$.

Since $d_R(p_1) = 1$, $\|\tilde{F}, \mathcal{U}\| \geq 2(2r+3s-1)-1 = 4r+6s-3$. However, this implies $4r+6s-3 \leq 4r+6s-4$, a contradiction.

Thus, $vq \in E(G)$ so that q is a dominating vertex in R. Since R is acyclic, $R \cong K_{1,|R|-1}$, as desired. $\square$

Lemma 4.14. *For all $u, v \in R \setminus \{q\}$, $N_G(u) = N_G(v)$. Furthermore, $\|v, C\| = 2$ and $\|v, D\| = 3$ for all $C \in \mathcal{C}$ and $D \in \mathcal{D}$.*

Proof. Let $v \in R \setminus P$, and let $\tilde{F} = \{v, p_1\}$. By Lemma 4.13, every vertex in $R \setminus \{q\}$ is the endpoint of a longest path in R, thus by symmetry, it suffices to show that $N_{\mathcal{U}}(p_1) = N_{\mathcal{U}}(v)$.

First, since $d_R(p_1) = d_R(v) = 1$, we deduce $\|\tilde{F}, \mathcal{U}\| \geq 2(2r+3s-1)-2 = 4r+6s-4$. By Lemma 4.5, $\|\tilde{F}, C\| \leq 4$ and $\|\tilde{F}, D\| \leq 6$ for all $C \in \mathcal{C}$ and $D \in \mathcal{D}$. Thus, $\|\tilde{F}, \mathcal{U}\| \leq 4r+6s-4$. This implies that $\|\tilde{F}, C\| = 4$ and $\|\tilde{F}, D\| = 6$ for all $C \in \mathcal{C}$ and $D \in \mathcal{D}$.

Fix $C \in \mathcal{C}$. We first show that $N_C(p_1) = N_C(v)$. By Lemma 4.5, either $\|p_1, C\| = |C| = 3$ or $|C| \in \{3, 4\}$ and $N_C(p_1) = N_C(v)$. Suppose $\|p_1, C\| = |C| = 3$ so that $\|v, C\| = 1$. Let $V(C) = \{c_1, c_2, c_3\}$ such that $vc_3 \in E(G)$. We can replace C and P with $p_1c_1c_2p_1$ and p_2qvc_3, respectively, a contradiction to $(O3)$.

A similar proof show that $N_D(p_1) = N_D(v)$ for all $D \in \mathcal{D}$. This proves the lemma. $\square$

Lemma 4.15. *$C \cong K_{2,2}$ and $D \cong K_{3,3}$ for all $C \in \mathcal{C}$ and $D \in \mathcal{D}$.*

Proof. Let $v \in R \setminus P$, and fix $D \in \mathcal{D}$. By Lemmas 4.5 and 4.24, we may assume that D has cycle $d_1d_2d_3d_4d_1$ with chord d_2d_4, and $N_D(p_1) = N_D(p_2) = N_D(v) = \{d_1, d_2, d_3\}$. However, $vd_3d_4d_2v$ is a cycle with chord d_2d_3, and $p_1d_1p_2qp_1$ is a cycle, a contradiction. Thus, $|D| = 6$ and by Lemma 4.2, $D \cong K_{3,3}$ and $D + p_1 \cong D + p_2 \cong D + v \cong K_{3,4}$.

Now fix $C \in \mathcal{C}$. By Lemmas 4.5 and 4.24, we may assume that $V(C) = \{c_1, c_2, c_3\}$ and $N_C(p_1) = N_C(p_2) = N_C(v) = \{c_1, c_2\}$. Since $s \geq 1$, there exists $D \in \mathcal{D}$, $D \cong K_{3,3}$. Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$ be the partite sets of D such that $N_D(p_1) = N_D(p_2) = A$.

However, $vc_1c_3c_2v$ is a cycle with chord c_1c_2, and $p_1qp_2a_1p_1$ and $a_2b_2a_3b_3a_2$ are two disjoint cycles, a contradiction. $\square$

Lemma 4.16. *$G \cong K_{2r+3s-1, n-2r-3s+1}$.*

Proof. Let $v \in R \setminus P$, and fix $C \in \mathcal{C}$. By Lemma 4.15, we may assume $A = \{a_1, a_2\}$ and $B = \{b_1, b_2\}$ are the partite sets of C with $N_C(p_1) = N_C(p_2) = N_C(v) = A$. We can replace C with va_1p_2qv so that this new collection of $r+s-1$ satisfies $(O1)$ and $(O2)$. Furthermore, we can replace P with either $b_1a_2p_1$ or $b_2a_2p_1$ satisfying $(O3)$. Thus, all the previous lemmas apply to this new collection, and in particular, Lemma 4.24 implies that $N_{\mathcal{U}}(b_1) = N_{\mathcal{U}}(b_2) = N_{\mathcal{U}}(p_1)$, as b_1 and b_2 play the same role as p_2.

Now, fix $D \in \mathcal{D}$. By Lemma 4.15, we may assume $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$ are the partite sets of D with $N_D(p_1) = N_D(p_2) = N_D(v) = A$. We can replace D with $D - b_i + v \cong K_{3,3}$. This yields a new collection that satisfies $(O1) - (O3)$ so that all the previous lemmas apply. In particular, b_i must be adjacent to q, and furthermore, Lemma 4.24 implies that $N_{\mathcal{U}}(b_i) = N_{\mathcal{U}}(p_1)$. Thus, for all $v \in V(G) \setminus N_G(p_1)$, $N_{\mathcal{U}}(v) = N_{\mathcal{U}}(p_1)$.

By Lemma 4.15, $d_G(p_1) = 2(r-1) + 3s + 1 = 2r + 3s - 1$, and G contains a complete bipartite subgraph with a partite set of size $2r + 3s - 1$. That is, G contains $K_{2r+3s-1,n-2r-3s+1}$ as a subgraph. However, as it is edge-maximal with respect to not having $r + s$ cycles, s of which are chorded, the lemma holds. □

4.4 $V(R) = V(P)$

Lemma 4.17. $R \cong K_{1,2}$.

Proof. Since R is acyclic and contains a spanning path, $R \cong P_{|R|}$. Suppose $|R| \geq 4$. By Lemma 4.7, $\|R, \mathcal{U}\| \leq 7(r-1) + 9s = 7r + 9s - 7$. However, $\|R, R\| = 6$ so that $\|R, \mathcal{U}\| \geq 4(2r + 3s - 1) - 6 = 8r + 12s - 10$. This implies that $r + 3s \leq 3$, a contradiction as $r, s \geq 1$. So $|R| = 3$ and $R \cong K_{1,2}$. □

As a result of this lemma, we label the vertices of P so that $P = p_1 q p_2$. In addition, we will often use the fact that $\|P, R\| = \|P, P\| = 4$, without reference.

4.4.1 $\|P, C\| \leq 6$ for all $C \in \mathcal{C}$

Here we will assume that $\|P, C\| \leq 6$ for all $C \in \mathcal{C}$.

Lemma 4.18. *Either*

1. $\|P, C\| = 6$ *for all* $C \in \mathcal{C}$ *and* $\|P, D\| = 9$ *for all but at most one* $D \in \mathcal{D}$, *or*

2. $\|P, D\| = 9$ *for all* $D \in \mathcal{D}$ *and* $\|P, C\| = 6$ *for all but at most one* $C \in \mathcal{C}$.

Furthermore if $\|P, D\| < 9$ *for some* $D \in \mathcal{D}$, *then* $\|P, D\| = 8$, *and if* $\|P, C\| < 6$ *for some* $C \in \mathcal{C}$, *then* $\|P, C\| = 5$.

Proof. By Lemma 4.8, $\|P, D\| \leq 9$ for all $D \in Dd$. Since $\|P, R\| = 4$ and $\|P, C\| \leq 6$ for all $C \in \mathcal{C}$, we have $\|P, \mathcal{D}\| \geq 3(2r + 3s - 1) - 4 - 6(r - 1) = 9s - 1$. Thus, $\|P, D\| < 9$ for at most one $D \in \mathcal{D}$, and if such a D exists, then $\|P, D\| = 8$ and $\|P, C\| = 6$ for all $C \in \mathcal{C}$. If there exists $C \in \mathcal{C}$ such that $\|P, C\| < 6$, then a similar inequality implies that $\|P, \mathcal{D}\| \geq 9s$. Thus, $\|P, D\| = 9$ for all $D \in \mathcal{D}$, and C is the only cycle in $\mathcal{C}$ for which $\|P, C\| < 6$. In particular, $\|P, C\| = 5$, as desired. □

Lemma 4.19. *If there exists $D \in \mathcal{D}$ such that $G[D] \cong K_{3,3}$, then $\|u, C\| \leq 1$ for all $u \in P$ and $C \in \mathcal{C}$ where $G[C] \cong K_3$. In particular, $\|P, C\| \leq 3$ for all $C \in \mathcal{C}$ such that $G[C] \cong K_3$.*

Proof. Suppose on the contrary, there exists $u \in P$ such that $\|u, C\| \geq 2$ where $G[C] \cong K_3$. We then replace C with a 4-cycle from D and replace D with $C + u$. This contradicts (O1). □

Lemma 4.20. *$\|P, C\| = 6$ for all $C \in \mathcal{C}$.*

Proof. By Lemma 4.18, if the statement does not hold, there exists a unique $\hat{C} \in \mathcal{C}$ such that $\|P, \hat{C}\| = 5$. Furthermore, $\|P, C\| = 6$ and $\|P, D\| = 9$ for all $C \in \mathcal{C} - \hat{C}$ and $D \in \mathcal{D}$. Since $s \geq 1$, Lemma 4.8 implies there exists $D \in \mathcal{D}$ such that $G[D] \cong K_{3,3}$. Thus, by Lemma 4.19, $G[C] \ncong K_3$ for all $C \in \mathcal{C}$. In particular, $\|u, C\| \leq 2$ for all $u \in P$ and $C \in \mathcal{C}$.

By Lemma 4.8, $\|p_i, D\| = 3$ for all $D \in \mathcal{D}$ and $i \in [2]$. Thus, $\|p_i, \mathcal{C}\| \geq 2r + 3s - 1 - 1 - 3s = 2(r-1)$. So $\|p_i, C\| = 2$ for all $C \in \mathcal{C}$. In particular, $\|\{p_1, p_2\}, \hat{C}\| = 4$ so that $\|q, \hat{C}\| = 1$. By Lemma 4.2 and 4.3, we let $A = \{a_1, a_2\}$ and $B = \{b_1, b_2\}$ be the partite sets of $\hat{C}$ so that $N_{\hat{C}}(p_1) = N_{\hat{C}}(p_2) = A$ and $N_{\hat{C}}(q) = \{b_1\}$. In particular, $qb_2 \notin E(G)$.

We replace $\hat{C}$ and P with $C' = p_1qb_1a_1p_1$ and $p_2a_2b_2$, respectively, to obtain a new collection $\tilde{\mathcal{U}} = \tilde{\mathcal{C}} \cup \tilde{\mathcal{D}}$ satisfying (O1)-(O3). Note that $\tilde{\mathcal{U}}$ consists of C_4's and $K_{3,3}$'s just as $\mathcal{U}$. Thus, $\|b_2, \tilde{C}\| \leq 2$ and $\|b_2, \tilde{D}\| \leq 3$ for every $\tilde{C} \in \tilde{\mathcal{C}}$ and $\tilde{D} \in \tilde{\mathcal{D}}$, which implies $\|b_2, \tilde{\mathcal{C}}\| \geq 2r + 3s - 1 - 1 - 3s = 2(r-1)$. Hence, $\|b_2, \tilde{C}\| = 2$ for every $\tilde{C} \in \tilde{\mathcal{C}}$.

In particular, $\|b_2, C'\| = 2$. However, $N_{C'}(p_2) = \{q, a_1\}$ and $N_{C'}(q) = \{p_1, b_1\}$. Thus, by Lemmas 4.2 and 4.3, $N_{C'}(b_2) = N_{C'}(p_2) = \{q, a_1\}$. However, recall that $qb_2 \notin E(G)$, a contradiction. □

Lemma 4.21. *If $\|P, C\| = 6$ for all $C \in \mathcal{C}$ and $\|P, D\| = 9$ for all $D \in \mathcal{D}$, then $G \cong K_{2r+3s-1, n-2r-3s+1}$.*

Proof. Since $s \geq 1$, Lemma 4.8 implies there exists $D \in \mathcal{D}$ such that $G[D] \cong K_{3,3}$. By Lemma 4.19, $G[C] \ncong K_3$ as $\|P, C\| = 6$ for all $C \in \mathcal{C}$. Thus, by Lemmas 4.2 and 4.3, $\|u, C\| = 2$ for all $u \in P$, and $G[P + C] \cong K_{3,4}$ for all $C \in \mathcal{C}$. Furthermore, since $\|P, D\| = 9$ for all $D \in \mathcal{D}$, we obtain $G[P + D] \cong K_{4,5}$ for all $D \in \mathcal{D}$. In particular, $N_{\mathcal{U}}(p_1) = N_{\mathcal{U}}(p_2)$.

Let $V_1 = N_G(p_1)$ and let $V_2 = V(G) \setminus V_1$. Note that $|V_1| = d_G(p_1) = 2(r-1) + 3s + 1 = 2r + 3s - 1$. We claim that for all $u \in V_2$, $N_G(u) = N_G(p_1) = V_1$. Note that if $u = p_2$, then we are done. So suppose $u \in \mathcal{U} \cap V_2$.

Assume that $u \in V(C) \cap V_2$ for some $C \in \mathcal{C}$. By the above, we may assume that $A = \{a_1, a_2\}$ and $B = \{b_1, b_2\}$ are the partite sets of C so that $N_C(p_1) = N_C(p_2) = A$, $N_C(q) = B$, and $u \in B$. Without loss of generality, suppose $u = b_1$. Then we can replace C with $p_2a_1b_2a_2$ and obtain a new collection satisfying (O1)-(O3), where p_1qu replaces P.

If $u \in V(D) \cap V_2$ for some $D \in \mathcal{D}$. By the above, we may assume $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$ are the partite sets of D so that $N_D(p_1) = N_D(p_2) = A$, $N_D(q) = B$, and $u \in B$. Without loss of generality, suppose $u = b_1$. Then we can replace D with $G[D - u + p_2] \cong K_{3,3}$ and obtain a new colleciton satisfying (O1)-(O3), where p_1qu replace P.

In either case, this new collection, $\tilde{\mathcal{U}}$, has every cycle is isomorphic to C_4 and every chorded cycle is isomorphic to $K_{3,3}$, just as in $\mathcal{U}$. Thus, by Lemma 4.3, $N_{\tilde{\mathcal{U}}}(u) \cap N_{\tilde{\mathcal{U}}}(q) = \emptyset$, and in particular, $N_{\tilde{\mathcal{U}}}(u) \subseteq N_{\tilde{\mathcal{U}}}(p_1)$. So $N_G(u) \subseteq N_G(p_1) = V_1$. However, $|V_1| = 2r+3s-1$ and $d_G(u) \geq 2r+3s-1$. Thus, $N_G(u) = V_1$, as desired.

So G contains $K_{|V_1|,|V_2|} = K_{2r+3s-1,n-2r-3s+1}$ as a spanning subgraph. As $K_{2r+3s-1,n-2r-3s+1}$ is edge-maximal with respect to not having r disjoint cycles and s chorded cycles, $G \cong K_{2r+3s-1,n-2r-3s+1}$, as desired. □

As a result of this lemma, for the remainder of this section we will let $\hat{D} \in \mathcal{D}$ such that $\|P, \hat{D}\| = 8$. Thus, for all $D \in \mathcal{D} - \hat{D}$, $\|P, D\| = 9$ and $G[P+D] \cong K_{4,5}$ by Lemmas 4.3 and 4.8. In particular, $\|u, D\| = 3$ for all $u \in P$.

Lemma 4.22. $G[\hat{D}] \cong K_{1,1,2}$

Proof. By Lemma 4.2, $|\hat{D}| \in \{4, 5, 6\}$.

Claim 4.22.1. $G[\hat{D}] \ncong K_4$.

Proof. Suppose on the contrary that $G[\hat{D}] \cong K_4$ with $V(\hat{D}) = \{d_1, d_2, d_3, d_4\}$. We first show that $\|u, \hat{D}\| \geq 2$ for all $u \in P$.

If $\|u, \hat{D}\| \leq 1$ for some $u \in P$, then $\hat{D} + P - u$ contains two disjoint triangles. Since $\|u, D\| = 3$ for all $D \in \mathcal{D} - \hat{D}$ and $\|u, R\| \leq 2$, we obtain $\|u, \mathcal{C}\| \geq 2r + 3s - 1 - 2 - 3(s-1) - 1 = 2(r-1) + 1$. This implies that $r \geq 2$, and furthermore, there exists $C \in \mathcal{C}$ such that $\|u, C\| = 3$. By Lemma 4.2, $G[C] \cong K_3$. However, we can replace $\hat{D}$ and C with the two disjoint triangles from $\hat{D} + P - u$ and the chorded cycle $C + u$.

So $\|u, \hat{D}\| \geq 2$ for all $u \in P$. Suppose $\|p_1, \hat{D}\| = 2$ with $N_{\hat{D}}(p_1) = \{d_1, d_2\}$. Then $\|\{q, p_2\}, \hat{D}\| = 6$ so that q and p_2 have two common neighbors $d, d' \in \hat{D}$. If say $d \notin \{d_1, d_2\}$, then we obtain qdp_2q and $\hat{D} - d + p_1$, a contradiction. Thus, $p_1, p_2, q \in N_G(d_1)$. However, this yields $P + d_1$ and $\hat{D} - d_1$, a contradiction.

Thus, $\|p_1, \hat{D}\| \geq 3$, which by symmetry implies $\|p_1, \hat{D}\| = \|p_2, \hat{D}\| = 3$ and $\|q, \hat{D}\| = 2$. However, q and p_2 have a common neighbor $d \in \hat{D}$, which yields qdp_2q and $D - d + p_1$, a contradiction. □

By Lemma 4.2, $\|u, \hat{D}\| \leq 3$ for all $u \in P$. In addition, Lemmas 4.2 and 4.4 imply $G[\hat{D}] \cong K_{3,3}$ with partite sets $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$. Without loss of generality, suppose $\|p_1, \hat{D}\| = 3$ so that

$N_{\hat{D}}(p_1) = A$. By Lemma 4.3, $N_{\hat{D}}(q) \subseteq B$ and $N_{\hat{D}}(p_2) \subseteq A$. So without loss of generality, $b_1, b_2 \in N_{\hat{D}}(q)$ and $a_1, a_2 \in N_{\hat{D}}(p_2)$.

If $\|p_2, \hat{D}\| = 2$, then as $\|p_2, D\| = 3$ for all $D \in \mathcal{D} - \hat{D}$, we obtain $\|p_2, \mathcal{C}\| \geq 2r + 3s - 1 - 1 - 2 - 3(s-1) = 2r - 1$. This implies $r \geq 2$, and furthermore, there exists $C \in \mathcal{C}$ such that $\|p_2, C\| = 3$. By Lemma 4.2, $G[C] \cong K_3$. However, we can replace C and D with the cycles $p_1a_1b_3a_2p_1$ and $qb_1a_3b_2q$ and the chorded cycle $C + p_2$, a contradiction.

So $N_{\hat{D}}(p_2) = N_{\hat{D}}(p_1) = A$ and $N_{\hat{D}}(q) = \{b_1, b_2\}$. In particular, $qb_3 \notin E(G)$.

Claim 4.22.2. *$G[C] \cong C_4$ for all $C \in \mathcal{C}$.*

Proof. Since $\|P, C\| = 6$ for all $C \in \mathcal{C}$, if there exists $C \in \mathcal{C}$ such that $G[C] \ncong C_4$, then $G[C] \cong K_3$. Thus, without loss of generaliy, $\|p_2, C\| \geq 2$. However, we can replace C and $\hat{D}$ with the cycles $p_1a_1b_3a_2p_1$ and $qb_1a_3b_2q$ and the chorded cycle $C + p_2$. □

We can replace $\hat{D}$ and P with $D' = \hat{D} - a_1 - b_3 + q + p_2$ and $p_1a_1b_3$, respectively. Observe that $D' \cong K_{3,3} \cong \hat{D}$ so that we obtain new collection $\tilde{\mathcal{U}} = \tilde{\mathcal{C}} \cup \tilde{\mathcal{D}}$ satisfying (O1)-(O3). In particular, $\mathcal{C} = \tilde{\mathcal{C}}$ and every chorded cycle in $\tilde{\mathcal{D}}$ is isomorphic to $K_{3,3}$ just as in $\mathcal{D}$. Thus, $\|b_3, \tilde{D}\| \leq 3$ for all $\tilde{D} \in \tilde{\mathcal{D}} - D'$.

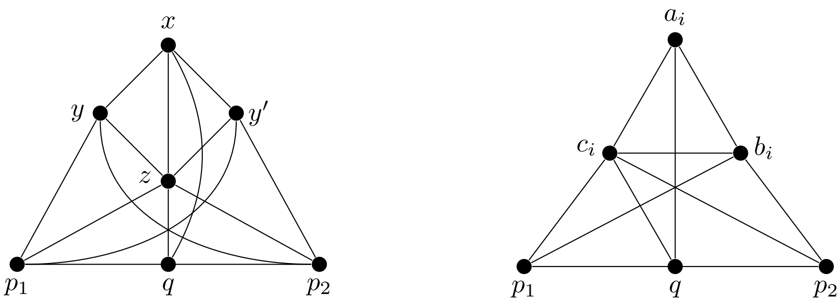
If there exists $\tilde{C} \in \tilde{\mathcal{C}}$ such that $\|b_3, \tilde{C}\| = 3$, then $G[\tilde{D}] \cong K_3$ by Lemma 4.2, and we can replace $\tilde{C}$ and D' with the cycles $p_1qb_1a_2p_1$ and $a_1b_2a_3p_2a_1$ and the chorded cycle $\hat{C} + p_2$. So $\|b_3, \tilde{C}\| \leq 2$ for all $\tilde{C} \in \tilde{\mathcal{C}}$. However, this implies that $\|b_3, D'\| \geq 2r + 3s - 1 - 2(r-1) - 1 - 3(s-1) = 3$. By Lemma 4.2, $N_{D'}(b_3) = \{q, a_2, a_3\}$. Yet recall that $qb_3 \notin E(G)$, a contradiction. □

In the following two lemmas we will consider an arbitrary collection satisfying (O1)-(O3), and furthermore, we will not require that $\|P, C\| = 6$ for all $C \in \mathcal{C}$.

Lemma 4.23. *Let $\tilde{\mathcal{U}} = \tilde{\mathcal{C}} \cup \tilde{\mathcal{D}}$ be a collection satisfying (O1)-(O3) with longest path $\tilde{P} = prp'$. If there exists $\tilde{D} \in \tilde{\mathcal{U}}$ such that $\|\tilde{P}, \tilde{D}\| = 8$ and $G[\tilde{D}] \cong K_{1,1,2}$, then $G[\tilde{D} + \tilde{P}] \cong K_{1,3,3}$.*

Proof. $\|u, \tilde{C}\| \leq 2$ for all $u \in \tilde{P}$ and $\tilde{C} \in \tilde{\mathcal{C}}$, else we can replace $\tilde{C}$ and $\tilde{D}$ with a 3-cycle from $K_{1,1,2}$ and $G[\tilde{C} + u] \cong K_4$, which contradicts (O2). Thus $\|p, \tilde{\mathcal{D}}\| \geq 2r + 3s - 1 - 1 - 2(r-1) = 3s$, and the same holds for $\|p, \tilde{\mathcal{D}}\|$. Since Lemmas 4.8 and 4.18 imply that $G[D'] \cong K_{3,3}$, for all $D' \in \tilde{\mathcal{D}} - \tilde{D}$, we deduce that $\|p_i, D'\| = 3$ so that $\|p_i, \tilde{D}\| = 3$. Thus, $\|q, \tilde{D}\| = 2$.

Let $d_1d_2d_3d_4d_1$ be the cycle of $\tilde{D}$ with chord d_2d_4. By Lemma 4.2, $d_1, d_3 \in N_{\tilde{D}}(p_1) \cap N_{\tilde{D}}(p_2)$. If $d_i \in N_{\tilde{D}}(q)$ for $i \in \{1, 3\}$, then we replace $\tilde{D}$ with the cycle $p_1d_iqp_1$ and chorded cycle $\tilde{D} - d_i + p_2$. Thus, $N_{\tilde{D}}(q) = \{d_2, d_4\}$. Without loss of generality, assume $N_{\tilde{D}}(p_1) = \{d_1, d_2, d_3\}$. If $d_4 \in N_{\tilde{D}}(p_2)$, then we replace



(a) $G[P + \hat{D}] \cong K_3 \vee \overline{K_3}$ (b) $G[P + C_i] \cong K_{1,2,3}$

Figure 4.2: P with special triangles and $\hat{D}$

$\tilde{D}$ with the cycle $p_1 d_2 q p_1$ and chorded cycle $\tilde{D} - d_2 + p_2$, a contradiction. Thus, $G[P + \tilde{D}] \cong K_{1,3,3}$, as desired. $\square$

Lemma 4.24. *Let $\tilde{\mathcal{U}} = \tilde{\mathcal{C}} \cup \tilde{\mathcal{D}}$ be a collection satisfying (O1)-(O3) with longest path $\tilde{P} = prp'$. If there exists $\tilde{D} \in \tilde{\mathcal{U}}$ such that $G[\tilde{D}] \cong K_{1,1,2}$, then $N_G(p) = N_G(p')$. Furthermore, $G[\tilde{P} + \tilde{C}] \in \{K_{1,2,3}, K_{3,4}\}$ and $G[\tilde{P} + \tilde{D}] \cong K_{1,3,3}$.*

Proof. If $\|u, \tilde{C}\| = 3$ for some $\tilde{C} \in \tilde{\mathcal{C}}$, then we can replace $\tilde{C}$ and $\tilde{D}$ with a 3-cycle from $\tilde{D}$ and $G[\tilde{C} + u] \cong K_4$, respectively, to obtain a collection that contradicts (O2). Thus, $\|u, \tilde{C}\| \leq 2$ for all $u \in \tilde{P}$, and $\|\tilde{P}, \tilde{C}\| \leq 6$ for all $\tilde{C} \in \tilde{\mathcal{C}}$. By Lemma 4.8, $\|\tilde{P}, D'\| \leq 9$ for all $D' \in \tilde{\mathcal{D}} - \tilde{D}$ and $\|\tilde{P}, \tilde{D}\| \leq 8$. Therefore, $\|\tilde{P}, \tilde{\mathcal{C}}\| \geq 3(2r + 3s - 1) - 4 - (9s - 1) = 6(r - 1)$ and $\|\tilde{P}, \tilde{C}\| = 6$ for all $\tilde{C} \in \tilde{\mathcal{C}}$.

Thus, by Lemma 4.18, $\|\tilde{P}, D'\| = 9$ for all $D' \in \tilde{\mathcal{D}} - \tilde{D}$ and $\|\tilde{P}, \tilde{D}\| = 8$. So by Lemmas 4.2, 4.3, and 4.23, $G[\tilde{P} + D'] \cong K_{4,5}$ and $G[\tilde{P} + \tilde{D}] \cong K_{1,3,3}$. In particular, $N_{\tilde{\mathcal{D}}}(p) = N_{\tilde{\mathcal{D}}}(p')$.

Since $\|u, \tilde{C}\| \leq 2$ and $\|\tilde{P}, \tilde{C}\| = 6$, equality holds for all $\tilde{C} \in \tilde{\mathcal{C}}$ and $u \in \tilde{P}$. If $G[\tilde{C}] \cong C_4$, then by Lemmas 4.2 and 4.3, $G[\tilde{P} + \tilde{C}] \cong K_{3,4}$, and in particular $N_{\tilde{C}}(p) = N_{\tilde{C}}(p')$. So we may assume $G[\tilde{C}] \cong K_3$ with $\tilde{C} = c_1 c_2 c_3 c_1$.

If u and v are consective vertices on $\tilde{P}$ and $N_{\tilde{C}}(u) = N_{\tilde{C}}(v) = \{c_i, c_j\}$, then we can replace $\tilde{D}$ with $G[\{c_i, c_j, u, v\}] \cong K_4$, a contradiction to (O2). So without loss of generality, $N_{\tilde{C}}(p) = \{c_1, c_2\}$ and $N_{\tilde{C}}(r) = \{c_1, c_3\}$. If $p'c_3 \in E(G)$, then since r and p' are consecutive vertices on $\tilde{P}$, $p'c_2 \in E(G)$. We then replace $\tilde{C}$ with the cycles prc_1p and $p'c_2c_3p'$, a contradiction. So $G[\tilde{C}] \cong K_{1,2,3}$, and in particular, $N_{\tilde{C}}(p) = N_{\tilde{C}}(p')$.

Thus, $N_{\tilde{\mathcal{C}}}(p) = N_{\tilde{\mathcal{C}}}(p')$ and $N_G(p) = N_G(p')$, as desired. $\square$

We now return to considering $\mathcal{U} = \mathcal{C} \cup \mathcal{D}$ with $\|P, C\| = 6$ for all $C \in \mathcal{C}$.

Lemma 4.25. *Either $G \cong K_{1,2r+3s-2,2r+3s-2}$, or $G \cong K_{t+1} \vee K_{2r-t+1,2r-t+1}$ for some $t, 0 \leq t \leq r - 1$.*

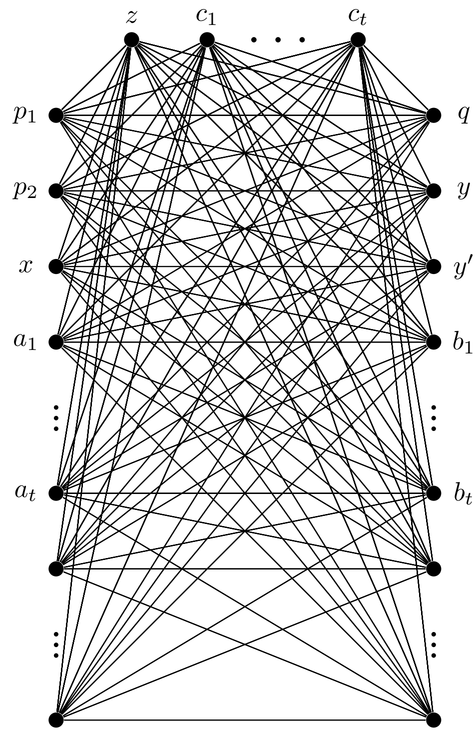


Figure 4.3: $K_{t+1,2r+t-1,2r+t-1}$

Proof. Let $C_1, \dots, C_t$ be exactly the cycles in $\mathcal{C}$ such that $G[C_i] \cong K_3$ for all C_i. Note that $\{C_1, \dots, C_t\} = \emptyset$ if $t = 0$.

By Lemma 4.18, $\|P, D'\| = 9$ for all $D' \in \mathcal{D} - \hat{D}$ and $\|P, C\| = 6$ for all $C \in \mathcal{C}$. In particular, by Lemmas 4.2, 4.3, 4.23, and 4.24, $G[P + D'] \cong K_{4,5}$, $G[P + \hat{D}] \cong K_{1,3,3}$, and $G[P + C] \in \{K_{1,2,3}, K_{3,4}\}$. Let x, y, y', z be the vertices of $\hat{D}$ such that y and y' are in the same partite set, and z is the dominating vertex in $G[P + \hat{D}]$ (see Figure 4.2a). Since $G[P + C_i] \cong K_{1,2,3}$, let $C_i = a_i b_i c_i a_i$ such that $qb_i \notin E(G)$ and $p_1 a_i \notin E(G)$. So c_i is adjacent to every vertex on P (see Figure 4.2b).

Let $V_3 = \{z, c_1, \dots, c_t\}$, $V_1 = N_G(p_1) \setminus V_3$, and $V_2 = V(G) \setminus (V_1 \cup V_3)$.

Claim 4.25.1. *Every vertex in V_2 is adjacent to every vertex in $V_1 \cup V_3$.*

Proof. Let $v \in V_2$. If $v = p_2$, then by Lemma 4.24, $N_G(p_1) = N_G(p_2) = V_1 \cup V_3$.

Suppose $v \in D$, where $D \in \mathcal{D} - \hat{D}$. By Lemmas 4.2 and 4.3, let $U = \{u_1, u_2, u_3\}$ and $W = \{w_1, w_2, w_3\}$ such that $N_D(p_1) = N_D(p_2) = U$ and $N_D(q) = W$. Without loss of generality suppose $v = w_1$. Then we replace D and P with $G[D - w_1 + p_2] \cong K_{3,3}$ and $p_1 q v$, respectively, to obtain a new collection that satisfies (O1)-(O3). By Lemma 4.24, $N_G(v) = N_G(p_1) = V_1 \cup V_3$, as desired.

If $v \in \hat{D}$, then $v = x$. We replace $\hat{D}$ and P with $G[\hat{D} - x + p_2] \cong K_{1,1,2}$ and $p_1 q x$, respectively, to obtain a new collection that satisfied (O1)-(O3). By Lemma 4.24, $N_G(v) = N_G(p_1) = V_1 \cup V_3$, as desired.

Suppose $v \in C$ for some $C \in \mathcal{C}$. If $G[C] \cong C_4$, then by Lemmas 4.2 and 4.3, let $U = \{u_1, u_2\}$ and $W = \{w_1, w_2\}$ be the partite sets of C such that $N_C(p_1) = N_C(p_2) = U$ and $N_C(q) = W$. Without loss of generality suppose $v = w_1$. Then we can replace C and P with $p_2u_1w_2u_2p_2$ and p_1qv, respectively, to obtain a new collection that satisfies (O1)-(O3). By Lemma 4.24, $N_G(v) = N_G(p_1) = V_1 \cup V_3$, as desired.

If $v \in C$ with $G[C] \cong K_3$, then $C = C_i$ for some $i, 1 \leq i \leq t$. Further, as $N_C(p_1) = \{b_i, c_i\}$, $v = a_i$. We then replace C and P with the cycle $p_2b_ic_ip_2$ and p_1qa_i, respectively, to obtain a new collection that satisfies (O1)-(O3). By Lemma 4.24, $N_G(v) = N_G(p_1) = V_1 \cup V_3$, as desired.

Thus, every vertex in V_2 is adjacent to every vertex in $V_1 \cup V_3$. That is, G contains $K_{|V_2|, n-|V_2|}$ as a spanning subgraph. $\square$

We now aim to show that every vertex in V_1 is adjacent to every vertex in V_3. That is, G contains $K_{|V_1|,|V_2|,|V_3|}$ as a spanning subgraph (see Figure 4.3).

Claim 4.25.2. *If $t = 0$, then $G \cong K_{1,2r+3s-2,2r+3s-2}$.*

Proof. Since $t = 0$, $V_3 = \{z\}$ and Lemma 4.24 implies $G[C] \cong C_4$ and $G[P + C] \cong K_{3,4}$ for all $C \in \mathcal{C}$. We aim to show that every vertex in V_1 is adjacent to z.

Let $v \in V_1$. If $v = q$, then $vz \in E(G)$. So assume $v \in C$ for some $C \in \mathcal{C}$ or $v \in D$ for some $D \in \mathcal{D}$. If $v \in C$, then we replace C and P with $C - v + q$ and p_1vp_2, respectively. If $v \in D$, we replace D and P with $D - v + q$ and p_1vp_2, respectively. In either case, we obtain a new collection satisfying (O1)-(O3), and by Lemma 4.24, $vz \in E(G)$.

Thus, G contains $K_{|\{z\}|,|V_1|,|V_2|}$ as a spanning subgraph. Since we are assuming $t = 0$, then $|V_1| = |N_G(p_1)| - 1 = (2(r-1)+3s+1) - 1 = 2r+3s-2$. Further, since $|G| = 4(r-1)+6(s-1)+4+3 = 4r+6s-3$, $|V_2| = |G| - |V_1| - 1 = 2r + 3s - 2$. Thus, G contains $K_{1,2r+3s-2,2r+3s-2}$ a spanning subgraph, and as it is edge-maximal with respect to not having r disjoint cycles and s chorded cycles, $G \cong K_{1,2r+3s-2,2r+3s-2}$. $\square$

In the remainder of this proof, we may assume $t \geq 1$. Since $\|P, C_1\| = 6$ and $\|P, D\| = 9$ for all $D \in \mathcal{D} - \hat{D}$, Lemma 4.19 implies that $s = 1$. Our goal is to show that every vertex in V_1 is adjacent to every vertex in V_3. To do so, we will use the following claim.

Claim 4.25.3. *For all $v \in V_1 \setminus \{q\}$,*

- *$z \in N_G(v)$,*
- *$y, y' \notin N_G(v)$, and*
- *for each $i, 1 \leq i \leq t$, $|N_G(v) \cap \{b_i, c_i\}| = 1$.*

Proof. Suppose $v \in C$ for some $C \in \mathcal{C}$. We then replace C and P with $C-v+q$ and $\tilde{P} = P-q+v$, respectively, to obtain a new collection, $\tilde{\mathcal{U}}$ that satisfies (O1)-(O3) that contains $G[\hat{D}] \cong K_{1,1,2}$. By Lemma 4.24, $G[\tilde{P}+\hat{D}] \cong K_{1,3,3}$ so that $z \in N_G(v)$ and $y, y' \notin N_G(v)$, as desired. Furthermore, $G[\tilde{P}+\tilde{C}] \in \{K_{1,2,3}, K_{3,4}\}$ for each $\tilde{C} \in \tilde{\mathcal{U}}$. Thus, either $vb_i \in E(G)$ and $vc_i \notin E(G)$, or $vb_i \notin E(G)$ and $vc_i \in E(G)$.

Suppose $v \in \hat{D}$ so that $v \in \{y, y'\}$. Clearly, $z \in N_G(v)$ and $y, y' \notin N_G(v)$. We then replace $\hat{D}$ and P with $\hat{D} - v + q$ and $P - q + v$, respectively to obtain a new collection that satisfies (O1)-(O3). Since $G[\hat{D} - v + q] \cong K_{1,1,2}$, by a similar argument, either $vb_i \in E(G)$ and $vc_i \notin E(G)$, or $vb_i \notin E(G)$ and $vc_i \in E(G)$, as desired. This proves our claim. □

If $yb_i \in E(G)$ for some $i, 1 \leq i \leq t$, then replace C_i and $\hat{D}$ with cycles yb_ip_2y and qc_ia_iq and chorded cycle $\hat{D}-y+p_1$, a contradiction. A similar argument holds if $y'b_i \in E(G)$ so that the claim implies $yc_i, y'c_i \in E(G)$ for each $i, 1 \leq i \leq t$.

We now show that for all $v \in V_1 \setminus \{q\}$, $N_G(v) = N_G(q) = V_2 \cup V_3$. We have just shown that this holds for $v \in \{y, y'\}$. Thus, suppose $v \in C$ for some $C \in \mathcal{C}$. If $N_G(v) \neq N_G(q)$, then by the claim $vb_i \in E(G)$ and $vc_i \notin E(G)$, for some $i, 1 \leq i \leq t$. This implies that $v \notin C_i$ so that $C \neq C_i$.

However, we can replace $\hat{D}, C_i$ and P with $G[\{v, b_i, p_1, z\}] \cong K_4$, yc_ia_iy, $xy'p_2$, respectively. If $C = C_j$ for some $j \neq i$, then we replace C with qc_ja_jq, otherwise, we replace C with $C - v + q$. In either case, we obtain a new collection that satisfies (O1), but contradicts (O2), as we have replaced $G[\hat{D}] \cong K_{1,1,2}$ with K_4.

Thus, $N_G(v) = N_G(q) = V_2 \cup \{z, c_1, \ldots, c_t\}$ for all $v \in V_1 \setminus \{q\}$. Hence, G contains $K_{t+1,|V_1|,|V_2|}$ as a spanning subgraph. Observe that $|V_1| = |V_2| = 3 + t - 1 + 2(r - t - 1) = 2r - t + 1$. So G contains $K_{t+1,2r-t+1,2r-t+1}$ as a spanning subgraph (see Figure 4.3).

Suppose G contains an edge uv such that u are in the same partite set of size $2r-t+1$. We then use two vertices from the other partite set of size $2r - t + 1$ to form the chorded cycle $K_{1,1,2}$. The resulting graph contains $K_{t+1,2r-t-1,2r-t-1}$ which contains $t+1$ triangles and $r - t - 1$ C_4's, a contradiction. Thus, G only differs from $K_{t+1,2r+t-1,2r+t-1}$ by edges in the partite set of size $t + 1$. As G is edge-maximal with respect to not having r disjoint cycles and one chorded cycle, and $K_{t+1} \vee K_{2r-t+1,2r-t+1}$ also has this property, $G \cong K_{t+1} \vee K_{2r-t+1,2r-t+1}$, as desired. □

4.4.2 $\|P, C\| = 7$ for some $C \in \mathcal{C}$

By Lemma 4.8, if there exists $C \in \mathcal{C}$ such that $\|P, C\| = 7$, then $G[P + C] \cong K_3 \vee \overline{K_3}$. We will call such $C \in \mathcal{C}$ a *special triangle*. In this section we will assume the existence of a special triangle and let C^* denote one such triangle.

Lemma 4.26. *$G[D] \cong K_4$ for all $D \in \mathcal{D}$.*

Proof. Suppose D is a cycle of length t, $c_1c_2\dots c_tc_1$ with chord c_ic_j, $i<j$. If $t \leq 5$, then without loss of generality, $c_1c_2\dots c_ic_j\dots c_tc_1$ is a cycle of length at most $t-2$. However, we can replace C^* and D with $c_1c_2\dots_i c_j \dots_t c_1$ and $G[C^*+q] \cong K_4$, respectively, to obtain a new collection that contradicts (O1). So $|D| = t = 4$. If $G[D] \ncong K_4$, then we replace C^* and D with a 3-cycle from D and $G[C^*+q] \cong K_4$, respectively to obtain a new collection that contradicts (O2). This proves the lemma. □

Lemma 4.27. *For $i \in \{1,2\}$ and $D \in \mathcal{D}$, p_i and q do not have a common neighbor in D. In particular, $\|\{p_i,q\},D\| \leq 4$ for all $D \in \mathcal{D}$.*

Proof. Suppose on the contrary, there exists $D \in \mathcal{D}$ such that p_1 and q have a common neighbor, say d. However, we can replace C^* and D with two cycles p_1qdp_1 and $D-d$ and a chorded cycle, C^*+p_2, a contradiction. As a consequence, $\|\{p_i,q\},D\| \leq 4$ for all $D \in \mathcal{D}$. □

Lemma 4.28. *Fix $C \in \mathcal{C}$ and $D \in \mathcal{D}$. If $\|u,C\| \geq 3$ for some $u \in D$, then $G[C] \cong K_3$. Further, if $\|u,C\| = 2$, then $|C| \in \{3,4\}$, and if $|C| = 4$, then $G[C+u] \cong K_{2,3}$.*

Proof. Suppose $\|u,C\| \geq 3$. If $|C| \geq 4$, then for some $v \in V(C) - N_C(u)$, $G[C-v+u]$ contains a chorded cycle. We then replace C and D with a 3-cycle from D and the chorded cycle from $G[C-v+u]$, respectively, to obtain a collection contradicting (O1).

Suppose $\|u,C\| = 2$ with $C = v_1v_2\dots v_tv_1$. Let v_i and v_j, $i<j$, be the neighbors of u. If $|C| \geq 5$, or if $|C| = 4$ and u is adjacent to consecutive vertices on C, then we may assume that $uv_iv_{i+1}\dots v_ju$ is a cycle of length at most $t-2$, where $|C| = t$. Since $|C| \geq 4$, $C \neq C^*$. Thus, we can replace C, D, and C^* with two cycles $uv_iv_{i+1}\dots v_ju$ and $D-u$ and chorded cycle $G[C^*+q] \cong K_4$. This produces a collection, again contradicting (O1). □

Lemma 4.29. *We can always find a collection $\tilde{\mathcal{U}} = \tilde{\mathcal{C}} \cup \tilde{\mathcal{D}}$ and path $\tilde{P}$ satisfying (O1)-(O3) such that $\|\tilde{p},\tilde{D}\| \geq 2$ for some $\tilde{D} \in \tilde{\mathcal{D}}$, where $\tilde{p}$ is an endpoint of $\tilde{P}$*

Proof. If not, then for all $D \in \mathcal{D}$ and $i \in \{1,2\}$, $\|p_i,D\| \leq 1$. Fix $\hat{D} \in \mathcal{D}$. By Lemma 4.27, we can always find $u \in \hat{D}$ such that $\|u,P\| \leq 1$. Let $F = \{p_1,q,p_2,u\}$ so that $\|F,P\| \leq 5$. Since $\|p_i,D\| \leq 1$ for all $D \in \mathcal{D}$ and $i \in \{1,2\}$, Lemma 4.27 implies that $\|P,D\| \leq 5$ for all $D \in \mathcal{D}$. Thus, $\|F,\hat{D}\| \leq 8$, and $\|F,D\| \leq 9$ for all $D \in \mathcal{D} - \hat{D}$.

So $\|F,\mathcal{C}\| \geq 4(2r+3s-1)-5-8-9(s-1) > 8(r-1)$. That is, there exists $C \in \mathcal{C}$ such that $\|F,C\| \geq 9$. By Lemmas 4.2 and 4.28, $G[C] \cong K_3$ and let $V(C) = \{c_1,c_2,c_3\}$. Suppose $\|u,C\| = 3$. If $\|p_1,C\| \geq 2$, then we can replace C and D with $D-u$ and $\tilde{D} = C+u$, respectively, to obtain a collection satifsying

(O1)-(O3), in which $\|p_1, \tilde{D}\| \geq 2$. Thus, $\|p_1, C\| \leq 1$ and by symmetry, $\|p_2, C\| \leq 2$. However, this implies that $\|P, C\| \leq 5$, a contradiction as $\|F, C\| \geq 9$.

We may assume $\|u, C\| \leq 2$ so that $\|P, C\| \geq 7$. Thus, by Lemma 4.8, C is a special triangle and $\|u, C\| = 2$. Recall that for $i \in \{1, 2\}$ and $D \in \mathcal{D}$, $\|p_i, D\| \leq 1$. Thus, $\|\{p_1, p_2\}, \mathcal{C}\| \geq 2(2r+3s-1)-2-2s > 4(r-1)$. That is, there exists $\tilde{C} \in \mathcal{C}$ such that $\|\{p_1, p_2\}, \tilde{C}\| \geq 5$, and without loss of generality, $\|p_2, \tilde{C}\| = 3$ so that $\|p_1, \tilde{C}\| \geq 2$. Observe that $\tilde{C} \in \mathcal{C} - C$ as C is a special triangle.

Since $\|u, C\| = 2$, we may assume $N_C(p_1) = N_C(p_2) = \{c_1, c_2\}$ such that $c_1 \in N_C(u)$. We then replace $C, \tilde{C}, D$, and P with $D - u$, qc_3c_2q, $G[\tilde{C} + p_2] \cong K_4$, and uc_1p_1, respectively, to obtain a new collection satisfying (O1)-(O3), in which p_1 has two neighbors in $G[\tilde{C} + p_2]$ as $\|p_1, \tilde{C}\| \geq 2$.

Thus, in any case we can obtain our desired collection. □

Lemma 4.30. *$s = 1$, and furthermore, $G[P + D] \cong \overline{K_3} \vee (K_1 + K_3)$*

Proof. By Lemma 4.29, we may assume there exists $\hat{D} \in D$ such that $\|p_2, \hat{D}\| \geq 2$. Let $u, v \in \hat{D}$ such that $\|p_2, \hat{D} - u - v\| = 2$. That is, $G[\hat{D} - u - v + p_2] \cong K_3$ and for $w \in \{u, v\}$, $G[\hat{D} - w + p_2]$ contains a chorded cycle. Let $F = \{p_1, q, u, v\}$. Observe that by Lemma 4.27, $\|F, \hat{D}\| \leq 10$. We will use this fact in proving the following claims.

Claim 4.30.1. *$\|F, C\| \leq 8$ for all $C \in \mathcal{C}$.*

Proof. Suppose on the contrary, $\|F, C\| \geq 9$ for some $C \in \mathcal{C}$. By Lemmas 4.2 and 4.28, $G[C] \cong K_3$. Let $V(C) = \{c_1, c_2, c_3\}$, and suppose $\|u, C\| = 3$. If p_1 and q have a common neighbor, say c_1, then we replace C and $\hat{D}$ with cycles $p_1c_1qp_1$ and uc_2c_3u and the chorded cycle $G[\hat{D} - u + p_2]$, a contradiction. Thus, $\|\{p_1, q\}, C\| \leq 3$, which implies that there exists a special triangle $C^* \in \mathcal{C} - C$. Additionally, $\|v, C\| = 3$, and $\|w, C\| \geq 2$ for some $w \in \{p_1, q\}$. Suppose $c_2, c_3 \in N_C(w)$. We then replace C, C^*, and $\hat{D}$ with cycles $G[\hat{D} - u - v + p_2], uvc_1u, wc_2c_3w$, and the chorded cycle $G[C^* + p_1 + q - w]$, a contradiction.

Thus, we may assume $\|u, C\| \leq 2$ and $\|v, C\| \leq 2$ so that $\|\{p_1, q\}, C\| \geq 5$. Suppose $\|p_1, C\| = \|q, C\| = 3$. Without loss of generality, $\|u, C\| \geq 2$ and $c_1, c_2 \in N_C(u)$. Then we replace C and $\hat{D}$ with cycles $p_1qc_3p_1$, uc_1c_2u, and the chorded cycle $G[\hat{D} - u + p_2]$, a contradiction.

So $\|\{p_1, q\}, C\| = 5$ and $\|u, C\| = \|v, C\| = 2$. Thus, u and v have a common neighbor, say c_1, and $\|\{p_1, q\}, C - c_1\| \geq 3$. That is, $G[\{c_2, c_3, p_1, q\}]$ contains a chorded cycle. We then replace C and $\hat{D}$ with cycles $uvc_1u, G[\hat{D}-u-v+p_2]$, and chorded cycle $G[\{c_2, c_3, p_1, q\}]$, a contradiction. This proves the claim. □

Claim 4.30.2. *$\|p_2, \hat{D}\| = 3$.*

Proof. Suppose $\|p_2, \tilde{D}\| = 2$ so that by the choice of u and v, $\|\{u,v\}, p_2\| = 0$. By Lemma 4.27, $\|\{p_1, q\}, \{u, v\}\| \leq 2$. Thus, $\|F, P\| \leq 5$, and recall that $\|F, \hat{D}\| \leq 10$. Together with the above claim, $\|F, \mathcal{D} - \hat{D}\| \geq 4(2r + 3s - 1) - 5 - 10 - 8(r - 1) > 12(s - 1)$. That is, there exists $D \in \mathcal{D} - \hat{D}$ such that $\|F, D\| \geq 13$, a contradiction as $\|\{p_1, q\}, D\| \leq 4$ and $\|\{u, v\}, D\| \leq 8$.

Suppose $\|p_2, \hat{D}\| = 4$. By Lemma 4.27, $\|q, \hat{D}\| = 0$. If $\|p_1, \hat{D}\| \geq 2$, let $V(\hat{D}) = \{d_1, d_2, d_3, d_4\}$ so that $d_1, d_2 \in N_{\hat{D}}(p_1)$. Then as there exists a special triangle C^*, we can replace C^* and $\hat{D}$ with cycles $p_1 d_1 d_2 p_1$, $p_2 d_3 d_4 p_2$, and chorded cycle $G[C+q]$, a contradiction. Thus, $\|p_1, \hat{D}\| \leq 1$ so that $\|F, \hat{D}\| \leq 7$. As $\|p_2, \hat{D}\| = 4$, we deduce that $\|F, P\| \leq 6$. Then together withthe above claim, $\|F, \mathcal{D} - \hat{D}\| \geq 4(2r+3s-1)-6-7-8(r-1) > 12(s - 1)$. That is, there exists $D \in \mathcal{D} - \hat{D}$ such that $\|F, D\| \geq 13$, a contradiction as $\|\{p_1, q\}, D\| \leq 4$ and $\|\{u, v\}, D\| \leq 8$. This proves the claim. □

By this claim and the choice of u and v, $\|p_2, \{u, v\}\| = 1$. Furthermore, Lemma 4.27 implies that $\|\{p_1, q\}, \{u, v\}\| \leq 2$ so that $\|F, P\| \leq 6$.

Claim 4.30.3. $s = 1$.

Proof. Recall that $\|F, \hat{D}\| \leq 10$. Thus, $\|F, \mathcal{D}\| \geq 4(2r + 3s - 1) - 6 - 10 - 8(r - 1) = 12(s - 1)$. By Lemma 4.27, $\|F, D\| \leq 12$ for all $D \in \mathcal{D} - \hat{D}$, and since $s \geq 2$, equality holds. In particular, $\|\{p_1, q\}, D\| = \|u, D\| = \|v, D\| = 4$ for all $D \in \mathcal{D} - \hat{D}$.

Recall that there exists a special triangle $C^* \in \mathcal{C}$, and observe that both $C^* + p_1$ and $C^* + q$ are chorded cycles. Fix $D \in \mathcal{D} - \hat{D}$, and let $\{x, \bar{x}\} = \{p_1, q\}$ so that say $\|x, D\| \geq 2$. Further, let $w \in D$ such that $\|x, D - w\| \geq 2$. Then replace C^*, D, and $\hat{D}$ with cycles $wuvw$, $\hat{D} - u - v + p_2$, and chorded cycles $D - w + x$ and $C^* + \bar{x}$. This proves the claim. □

Thus, it remains to show that $G[P + \hat{D}] \cong H$. Since $\|F, P\| \leq 6$, we deduce that $\|F, \hat{D}\| \geq 4(2r+3-1) - 6 - 8(r - 1) = 10$. As $\|\{u, v\}, \hat{D}\| = 6$, we have $\|\{p_1, q\}, \hat{D}\| \geq 4$. By Lemma 4.27, equality holds for both inequalities. Further, since $\|p_2, \hat{D}\| = 3$, we deduce that $\|q, \hat{D}\| \leq 1$ and $\|p_1, \hat{D}\| \geq 3$. If $N_{\hat{D}}(p_1) \neq N_{\hat{D}}(p_2)$, let $V(\hat{D}) = \{d_1, d_2, d_3, d_4\}$ with $N_{\hat{D}}(p_1) = \{d_1, d_2, d_3\}$ and say, $d_3, d_4 \in N_{\hat{D}}(p_2)$. Since there exists a special triangle C^*, we replace C^* and $\hat{D}$ with cycles $p_1 d_1 d_2 p_1, p_2 d_3 d_4 p_2$, and chorded cycle $C^* + q$, a contradiction.

Thus, $\|q, \hat{D}\| = 1$ and $N_{\hat{D}}(p_1) = N_{\hat{D}}(p_2)$. By Lemma 4.27, $G[P + \hat{D}] \cong H$, as desired. □

By Lemma 4.30, $\delta(G) \geq 2r + 2$. In addition, let $\mathcal{D} = \{D\}$, and let $V(D) = \{\tilde{p}, x_1, x_2, x_3\}$ such that $N_D(p_1) = N_D(p_2) = \{x_1, x_2, x_3\} = X$ and $q\tilde{p} \in E(G)$. Furthermore, let $\overline{P} = \{p_1, p_2, \tilde{p}\}$. For a special triangle C^*, let $V(C^*) = \{y, y', z\}$ such that $N_{C^*}(p_1) = N_{C^*}(p_2) = \{y, y'\}$, and let $qz \in E(G)$. See Figures 4.4a and 4.4b.

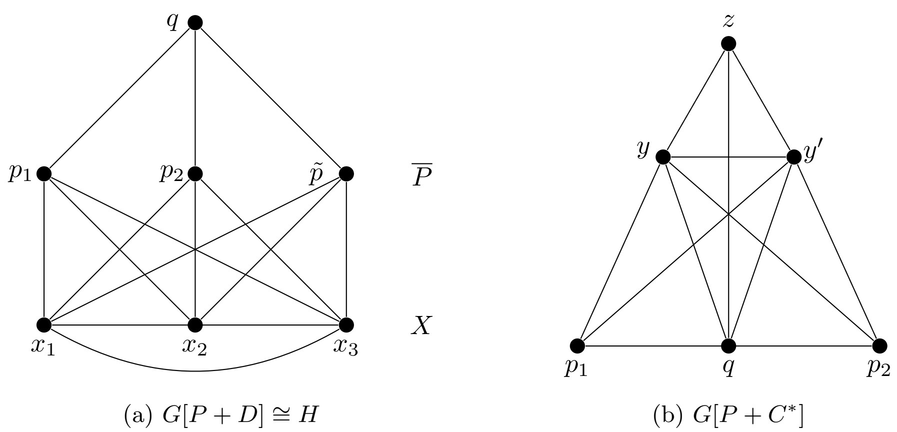


Figure 4.4: P together with D and C^*

Lemma 4.31. *If $r = 2$, then $G \cong \overline{K_4} \vee (K_3 + K_3)$*

Proof. If $r = 2$, then $\mathcal{C} = \{C^*\}$, where C^* is a special triangle. Observe that in order to show that $\overline{K_4} \vee (K_3 + K_3)$ is a spanning subgraph of G, we only need to show that $zx_i, \tilde{p}y, \tilde{p}y' \in E(G)$ for $i \in \{1, 2, 3\}$.

We can replace D with $D - \tilde{p} + p_2 \cong K_4$ and P with $\tilde{P} = p_1 q \tilde{p}$ to obtain a new collection $\tilde{\mathcal{U}} = \tilde{\mathcal{C}} \cup \tilde{\mathcal{D}}$ that satisfies (O1)-(O3). Note that $\tilde{\mathcal{C}} = \mathcal{C} = \{C^*\}$. If $\|\tilde{P}, C^*\| \leq 6$ for all $\tilde{C} \in \tilde{\mathcal{C}}$, then we can proceed as in Section 4.4.1. Therefore, we may assume $\|\tilde{P}, C^*\| = 7$, so that C^* is a special triangle in this collection as well. In particular, $\tilde{p}y, \tilde{p}y' \in E(G)$.

Similarly, we can replace C^* with $\tilde{C} = C^* - z + p_2$ and P with $\tilde{P} = p_1 q z$. This yields a new collection $\tilde{\mathcal{U}} = \tilde{\mathcal{C}} \cup \tilde{\mathcal{D}}$ that satisfies (O1)-(O3). Observe that $\tilde{\mathcal{D}} = \mathcal{D} = \{D\}$, and $\tilde{C}$ is a special triangle in this collection as well. Thus, we can apply Lemma 4.30 to obtain $G[\tilde{P} + D] \cong H$. In particular, $zx_i \in E(G)$ for all $i \in \{1, 2, 3\}$.

Thus, $\overline{K_4} \vee (K_3 + K_3)$ is a spanning subgraph of G, and as it is edge-maximal with respect to not having r disjoint cycles and s disjoint chorded cycles, $G \cong \overline{K_4} \vee (K_3 + K_3)$, as desired. $\square$

Thus, in the remainder of this section we will assume $r \geq 3$.

Lemma 4.32. *If $C_1, C_2 \in \mathcal{C}$ with $|C_1| \geq |C_2|$ and $|C_1| \geq 4$, then $\|C_1, C_2\| \leq \max\{2|C_1|, |C_1| + 4\} = 2|C_1|$.*

Proof. If $\|c, C_2\| \leq 2$ for every $c \in C_1$, the result follows. So, suppose there exists $c \in C_1$ with $\|c, C_2\| \geq 3$.

Suppose $\|c, C_2\| \geq 4$. If there exists $a \in C_2$ with $\|a, C_1\| \geq 3$, then a has neighbors $c', c'' \in C_1 - c$. Note that C_1 contains a path, call it P, with endpoints c' and c'' that also avoid c. However, this yields a cycle $ac'Pc''a$, and $C_2 - a + c$ contains a cycle with strictly fewer vertices than C_2. This contradicts (O1). Thus, $\|a, C_1\| \leq 2$ for all $a \in C_2$. Hence, $\|C_1, C_2\| \leq 2|C_2| \leq 2|C_1|$.

Observe that the above argument does not rely on $|C_1| \geq |C_2$. Therefore, $\|a, C_1\| \leq 3$ for all $a \in C_2$.

Since $\|c, C_2\| \geq 3$, we may assume equality holds. We claim that no vertex $a \in C_2 - N_C(c)$ has more than one neighbor in C_1. If so, then a has two neighbors $c', c'' \in C_1 - c$. We then repeat the same argument as above to arrive at a contradiction.

If $\|a, C_1\| \leq 2$ for all $a \in N_{C_2}(c)$, then $\|C_1, C_2\| \leq |C_2| - 3 + 6 = |C_2| + 3 \leq |C_1| + 4$, and we are done. So suppose $\|a, C_1\| \geq 3$ for some $a \in N_{C_2}(c)$, and as noted above, equality holds. Since $C_1 - c + a$ and $C_2 - a + c$ both contain cycles, we deduce that $N_{C_1}(a) - c = N_{C_1}(c)$, as otherwise these new cycles contradict (O1). Similarly, $N_{C_2}(c) - a = N_{C_2}(a)$.

If $\|a', C_1\| \leq 2$ for $a' \in N_{C_2}(c) - a$, then $\|C_1, C_2\| \leq |C_2| - 3 + 3 + 4 \leq |C_1| + 4$. So we may assume $\|a', C_1\| \geq 3$ for some $a' \in N_{C_2}(c) - a$, and as observed above, equality holds. By repeating the same argument for a' as we did for a, we deduce that $N_{C_1}(a') - c = N_{C_1}(c)$ and $N_{C_2}(c) - a' = N_{C_2}(a')$. That is, $|C_2| = 3$.

Let $N_{C_2}(c) = \{a, a', a''\}$, and let $N_{C_1}(a) = N_{C_1}(a') = \{c, c', c''\}$. If $\|a'', C_1\| \leq 1$, then $\|C_1, C_2\| \leq |C_2| - 3 + 7 \leq |C_1| + 4$. So we may assume $\|a'', C_1\| \geq 2$, and without loss of generality, $c'' \in N_{C_1}(a'')$. However, since $|C_1| \geq 4$, we obtain the two cycles $a'c'ca'$ and $a''ac''a''$, contradicting (O1). This proves the lemma. □

Lemma 4.33. *Every $C \in \mathcal{C}$ is a triangle.*

Proof. Let C_1 be a longest cycle in $\mathcal{C}$, and suppose that $|C_1| \geq 4$. By (O1), every cycle in $\mathcal{C}$ is induced so that $\|C_1, C_1\| = 2|C_1|$. By Lemma 4.32 and the assumption of C_1 being longest, $\|C_1, C\| \leq 2|C_1|$ for all $C \in \mathcal{C} - C_1$. Therefore, $\|C_1, \mathcal{C}\| \leq 2|C_1|(r-1)$, and so $\|C_1, H\| \geq |C_1|(2r+2) - 2|C_1|(r-1) = 4|C_1|$.

Suppose $\|d, C_1\| \geq 3$ for some $d \in D$. Then since $|C_1| \geq 4$, $C_1 + d$ contains a chorded cycle $\tilde{D}$ that does not contain all the vertices of C_1. We then replace C_1 and D with $D - d$ and $\tilde{D}$, respectively, which contradicts (O1). Thus, $\|d, C_1\| \leq 2$ for all $d \in D$, and $\|C_1, D\| \leq 8$ as a result.

Since $|C_1| \geq 4$, we also conclude that $\|u, C_1\| \leq 2$ for all $u \in P$, as otherwise we could replace C_1 with a shorter cycle. That is, $\|C_1, P\| \leq 6$.

Thus, $4|C_1| \leq \|C_1, H\| \leq 6 + 8 = 14 < 4|C_1|$, a contradiction. □

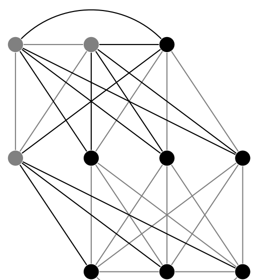
Lemma 4.34. *Let $\mathcal{C}$ be partitioned into $\mathcal{C} = \bigcup_{i=0}^{3} Q_i$ so that $C \in Q_i$ if and only if $\|q, C\| = i$. Then:*

1. *Given $C \in Q_0$, $\|H, C\| \leq 15$.*

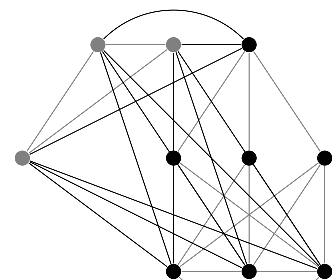
2. *Given $C \in Q_1$, $\|H, C\| \leq 16$. Further, if equality holds, then $N_C(x) = V(C)$ for all $x \in X$, and if $c \in C$ is the neighbor of q, then both vertices of $C - c$ are adjacent to every vertex of $\overline{P}$ (Figure 4.5e).*

3. *Given $C \in Q_2$, $\|H, C\| \leq 14$. Further, if equality holds, then either there exists $c \in C$ such that $\|c, H\| = 0$ and both vertices of $C - c$ are adjacent to every vertex of H (Figure 4.5c), or $N_H(c) = \overline{P}$, $N_H(c') = X \cup \{q\}$, and $N_H(c'') = V(H)$ (Figure 4.5d).*

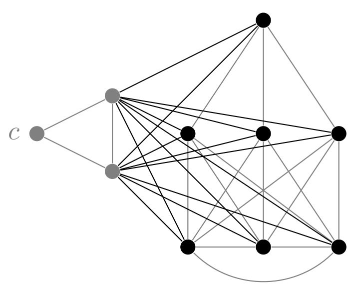
4. *Given $C \in Q_3$, $\|H, C\| \leq 12$. Further, if equality holds, then either C is special (Figure 4.5a), or $\|\overline{P}, C\| = 0$ and every vertex of C is adjacent to every vertex of X (Figure 4.5b).*



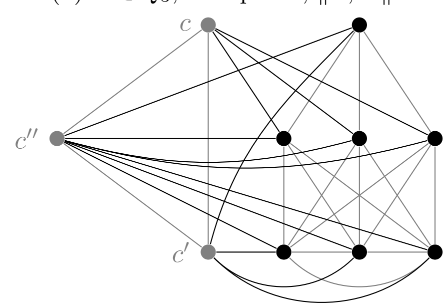
(a) $C \in Q_3$, special, $\|C, H\| = 12$



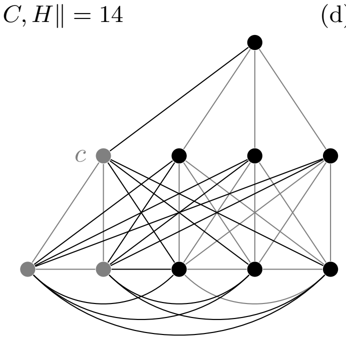
(b) $C \in Q_3$, not special, $\|C, H\| = 12$



(c) $C \in Q_2$, $\|c, H\| = 0$, $\|C, H\| = 14$



(d) $C \in Q_2$, $\|c, H\| \neq 0$, $\|C, H\| = 14$



(e) $C \in Q_1$, $\|C, H\| = 16$

Figure 4.5: Cycles in Q_i, shown in gray, with their optimal configurations of edges to H.

Proof. Fix $\hat{C} \in \mathcal{C}$ with $V(\hat{C}) = \{c, c', c''\}$. Let $\overline{P} = \{p, p', p''\}$.

Claim 4.34.1. *$\|p, \hat{C}\| \leq 2$, and by symmetry, $\|\overline{P}, \hat{C}\| \leq 6$.*

Proof. Suppose $\|p, \hat{C}\| = 3$. We can replace $\hat{C}, D$ and P with $\tilde{C} = x_1x_2x_3x_1$, $\tilde{D} = \hat{C} + p \cong K_4$, and $\tilde{P} = p'qp''$, respectively. This yields a new collection $\tilde{\mathcal{U}} = \tilde{\mathcal{C}} = \cup\tilde{\mathcal{D}}$ that satisfies (O1)-(O3). If $\|\tilde{P}, C\| \leq 6$ for all $C \in \tilde{\mathcal{C}}$, then we proceed as in Section 4.4.1. Hence $\|\tilde{P}, C\| = 7$ for some $C \in \tilde{\mathcal{C}}$ so that we apply the previous lemmas. In particular, Lemma 4.30 implies that $\|q, \tilde{D}| = 1$.

However, we can also replace $\hat{C}, D$, and P with $\tilde{C} = p'x_1x_2p'$, $\tilde{D} = \hat{D} + p \cong K_4$, and $\tilde{P} = qp''x_3$, respectively. A similar argument shows that $\|q, \tilde{D}\| = 3$. Yet $\tilde{D}$ is the same in both cases, a contradiction. □

Case 1. $\hat{C} \in Q_0 \cup Q_1$.

Since $\|q, \hat{C}\| \leq 1$, $\|\overline{P}, \hat{C}\| \leq 6$, and $\hat{C}$ is a triangle, we deduce that $\|H, \hat{C}\| \leq 1 + 6 + 9 = 16$, with equality exactly when $\hat{C} \in Q_1$, $\|x_i, \hat{C}\| = 3$ for all $i \in \{1, 2, 3\}$, and $\|p, \hat{C}\| = 2$ for all $p \in \overline{P}$.

Suppose $\hat{C} \in Q_1$ and $\|H, \hat{C}\| = 16$. Let $qc \in E(G)$. If $pc \in E(G)$ for some $p \in \overline{P}$, then we can replace $\hat{C}$ and D with cycles p_1cqp_1, $c'c''x_1c'$, and chorded cycle $x_2p'x_3p''x_2$, a contradiction. Thus, $N_{\hat{C}}(p) = \{c', c''\}$ for all $p \in \overline{P}$. This yields Figure 4.5e.

Case 2. $\hat{C} \in Q_2$.

Suppose $\|H, \hat{C}\| \geq 14$. Let $N_{\hat{C}}(q) = \{c', c''\}$ with $\|H, c''\| \geq \|H, c'\|$. Since $qc'c''q$ is a triangle, c does not have a neighbor in both $\overline{P}$ and X. Since $cq \notin E(G)$, this implies $\|H, c\| \leq 3$. Since $\|H, \hat{C}\| \geq 14$, then $\|H, \{c', c''\}\| \geq 11$, so we may assume $\|H, c''\| \geq 6$.

Suppose a vertex $u \in \overline{P} \cup X$ is adjacent to both c and c'. Since $\|H, c''\| \geq 6$, we may assume $u \neq p$ and $pc'' \in E(G)$. However, this yields cycles $ucc'u, pc''qp$, and chorded cycle $H - q - u - p$. Therefore, c and c' have no common neighbors in H, and consequently, $\|u, \hat{C}\| \leq 2$ for all $u \in H$. That is, $\|H, \hat{C}\| \leq 14$. Thus equality holds, and every vertex of H is adjacent to precisely one vertex of $\{c, c'\}$. In addition, every vertex of H is adjacent to c''.

If $\|c, H\| = 0$, then we obtain the configuration in Figure 4.5c. So we may assume $\|c, H\| \geq 1$. Suppose $cx_i \in E(G)$ for $x_i \in X$. If $pc' \in E(G)$, then we obtain cycles $pc'qp, x_icc''x_i$, and chorded cycle $D - x_i - p$. So by symmetry, every vertex in $\overline{P}$ is adjacent to both c and c''. However, this yields cycles $pcx_ip, qc'c''q$, and chorded cycle $D - x_i - p$. Thus, $cx_i \notin E(G)$ for all $i \in \{1, 2, 3\}$. That is, $N_{\hat{C}}(x_i) = \{c', c''\}$.

Since $\|c, H\| \geq 1$, we may assume $pc \in E(G)$. If $p'c' \in E(G)$, then we obtain cycles $qc'p'q, pcc''p$, and chorded cycle $D - p - p'$, a contradiction. Thus, $N_{\hat{C}}(u) = \{c, c''\}$ for all $u \in \overline{P}$, which yields the configuration in Figure 4.5d.

Case 3. $\hat{C} \in Q_3$.

Observe that for any $c \in \hat{C}$, $u \in \overline{P}$, and $x_i \in X$, $C - c + q$ is a cycle and $H - u - x_i$ is a chorded cycle. Thus, u, x_i, and c cannot form a cycle. As a result, $\|c, H\| \leq 4$ so that $\|H, \hat{C}\| \leq 12$. If equality holds, then by the previous argument, $N_H(c)$ is either $\overline{P} \cup \{q\}$ or $X \cup \{q\}$, for all $c \in \hat{C}$.

By Lemma 4.8, at most two of the vertices on $\hat{C}$ have their neighbors in $\overline{P} \cup \{q\}$. If none have their neighbors in $\overline{P} \cup \{q\}$ so that $N_H(c) = X \cup \{q\}$ for all $c \in \hat{C}$, then we arrive at the configuration in Figure 4.5b.

If only $c \in \hat{C}$ is adjacent to $\overline{P}$, then we obtain cycles $cqpc$ and $C - c + x_1$, and the chorded cycle $x_2p'x_3p''x_2$. Thus, exactly two vertices on $\hat{C}$ have their neighbors in $\overline{P} \cup \{q\}$, which implies that $\hat{C}$ is a special triangle and yields the configuration in Figure 4.5a ☐

Lemma 4.35. *$Q_0 = \emptyset$ and $|Q_1| = |Q_3| - 1$. Furthermore, $\|H, C\| = 16$ for all $C \in Q_1$, $\|H, C\| = 14$ for all $C \in Q_2$, and $\|H, C\| = 12$ for all $C \in Q_3$.*

Proof. Observe that $Q_0 \cup Q_1 \cup Q_2 \cup Q_3$ is a partition of $\mathcal{C}$. Since $\|q, H\| = 3$ and $|\mathcal{C}| = r - 1$, we have $\|q, \mathcal{C}\| \geq 2r + 2 - 3 = 2|Q_3| + 2|Q_2| + 2|Q_1| + 2|Q_0| + 1$. This yields,

$$3|Q_3| + 2|Q_2| + |Q_1| = \|q, \mathcal{C}\| \geq= 2|Q_3| + 2|Q_2| + 2|Q_1| + 2|Q_0| + 1. \tag{4.1}$$

Since $\|H, H\| = 30$ and $|\mathcal{C}| = r - 1$, we have $\|H, \mathcal{C}\| \geq 7(2r+2) - 30 = 14|Q_3| + 14|Q_2| + 14|Q_1| + 14|Q_0| - 2$. Therefore, Lemma 4.34 yields,

$$12|Q_3| + 14|Q_2| + 16|Q_1| + 15|Q_0| \geq \|H, \mathcal{C}\| \geq 14|Q_3| + 14|Q_2| + 14|Q_1| + 14|Q_0| - 2. \tag{4.2}$$

Observe that 4.1 simplifies to $|Q_3| \geq |Q_1| + 2|Q_0| + 1$, and 4.2 simplifies to $2|Q_1| + |Q_0| \geq 2|Q_3| - 2$. Using these two, we obtain $2|Q_1| + |Q_0| \geq 2|Q_1| + 4|Q_0| + 2 - 2$, from which we deduce that $|Q_0| = 0$.

Plugging $|Q_0| = 0$ into 4.1 and 4.2, they reduce to $|Q_3| \geq |Q_1| + 1$ and $|Q_1| \geq |Q_3| - 1$, respectively. That is, $|Q_3| = |Q_1| + 1$.

Since $Q_0 = \emptyset$, $r - 1 = |\mathcal{C}| = |Q_3| + |Q_2| + |Q_1|$, and $\|H, H\| = 30$, we have

$$
\begin{aligned}
\|H, Q_1\| &\geq 7(2r+2) - 30 - 14|Q_2| - 12|Q_3| \\
&= 14r - 16 - 14|Q_2| - 12|Q_1| - 12 \\
&= 14(|Q_1| + |Q_2| + |Q_3| + 1) - 28 - 14|Q_2| - 12|Q_1| \\
&= 2|Q_1| + 14|Q_3| - 14 \\
&= 16|Q_1|
\end{aligned}
$$

By Lemma 4.34, $\|H, C\| = 16$ for all $C \in Q_1$. Similar inequalities show that $\|H, C\| = 14$ for all $C \in Q_2$, and $\|H, C\| = 12$ for all $C \in Q_3$. □

Lemma 4.36. *Every cycle in Q_3 is a special triangle.*

Proof. Observe that by Lemmas 4.34 and 4.35, $\|\overline{P}, C\| = 0$ where $C \in Q_3$ and C is not a special triangle, and $\|\overline{P}, C\| = 6$ for all other $C \in \mathcal{C}$. Since $\|\overline{P}, \mathcal{C}\| \geq 3(2r+2) - \|\overline{P}, H\| = 6r + 6 - 12 = 6|\mathcal{C}|$, we deduce that every triangle in Q_3 is a special triangle. □

Lemma 4.37. $Q_2 = \emptyset$.

Proof. Suppose there exists $C \in Q_2$. Let $C = cc'c''c$, with $\|c, H\| \leq \|c', H\| \leq \|c'', H\|$.

By Lemma 4.34, $\|c, H\| \leq 3$. Then $\|c, \mathcal{C}\| \geq 2r + 2 - \|c, H\| \geq 2r + 2 - 3 = 2(r-1) + 1$. So there exists a triangle $C' \in \mathcal{C} - C$ with $\|c, C'\| = 3$. Then $C' + c$ is a chorded cycle, so that $H + c' + c''$ does not contain three disjoint triangles. Given the two possible configurations from Lemma 4.34, this implies that $\|c, H\| \neq 0$. So c'' is adjacent to every vertex in H, $N_H(c') = \{q, x_1, x_2, x_3\}$, and $N_H(c) = \overline{P}$.

If we can show that $C' \notin Q_1 \cup Q_2 \cup Q_3$, this leads to a contradiction by Lemma 4.35 so that $Q_2 = \emptyset$.

Suppose $C' \in Q_1$. Label $C' = tuvt$ where $tq \in E(G)$. However, we can replace C, C', and D with $c\tilde{p}uc$, x_1tvx_1, and $qc'c''q$, which are triangles, and $x_2p_1x_3p_2x_2$, which is a chorded cycle . Therefore, $C' \notin Q_1$.

Suppose $C' \in Q_2$. By the above, we can label $C' = tuvt$ so that $N_H(t) = \overline{P}$ and $x_1, x_2, x_3 \in N_H(u) \cap N_H(v)$. However, we can replace C, C', and D with uvx_1u, $ct\tilde{p}c$, and $qc'c''q$,which are triangles, and $x_2p_1x_3p_2x_2$, which is a chorded cycle. So, $C' \notin Q_2$.

Let $C' \in Q_3$. By Lemma 4.36, C' is a special triangle. Label $C' = tuvt$ where $N_H(t) = \{q, x_1, x_2, x_3\}$. However, we can replace C, C', and D with $cu\tilde{p}c$, $qtvq$, and $x_1c'c''x_1$, which are triangles, and $x_2p_1x_3p_2x_2$, which is a chorded cycle. So $C' \notin Q_3$. □

Lemma 4.38. *r is even, and $G \cong \overline{K_{r+2}} \vee (K_{r+1} + K_{r+1})$.*

Proof. By Lemmas and 4.35 and 4.37, $r-1 = |\mathcal{C}| = |Q_1| + |Q_3| = (|Q_3|-1) + |Q_3|$, so that $r = 2|Q_3|$. Thus, r is even.

Because all cycles of $\mathcal{C}$ are either special, or in Q_1, we can partition $V(G)$ into three sets $\mathcal{Q} \cup \mathcal{W} \cup \mathcal{P}$.

Let $\mathcal{P}$ consist of all the non-neighbors of p_1, including p_1 itself; let $\mathcal{Q} = (N_G(q) - \mathcal{P}) \cup \{q\}$; let $\mathcal{W} = V(G) - (\mathcal{P} \cup \mathcal{Q})$. We aim to show that $G[\mathcal{P}] \cong \overline{K_{r+2}}$, $G[\mathcal{Q}] \cong G[\mathcal{Q}] \cong K_{r+1}$.

Observe that $V(H) \cap \mathcal{Q} = \{q\}, V(H) \cap \mathcal{P} = \overline{P}$, and $\mathcal{W} \cap H = X$. Since $\mathcal{C} = Q_1 \cup Q_3$ and every cycle in Q_3 is special, exactly one vertex from every $C \in \mathcal{C}$ is in $\mathcal{P}$, namely the one adjacent to x_1, x_2, x_3. If $C \in Q_1$, then the remaining two vertices are in $\mathcal{W}$, and if $C \in Q_3$, then the remaining two vertices are in $\mathcal{Q}$.

So $|\mathcal{Q}| = 2|Q_3| + 1$, $|\mathcal{W}| = 2|Q_1| + 3$, and $|\mathcal{P}| = r - 1 + 3 = r + 2$. Since $r = 2|Q_3|$ and $|Q_1| = |Q_3| - 1$, we deduce that $|\mathcal{Q}| = |\mathcal{W}| = r + 1$.

We claim that for all $p \in \mathcal{P} - p_1$, $N_G(p) = N_G(p_1)$. If $p \in \{p_2, \tilde{p}\}$, this is clear by Lemmas 4.34 and 4.35. So suppose $p \in C$ for some $C \in \mathcal{C}$. We replace C and P with $\tilde{C} = C - p + p_2$ and $\tilde{P} = p_1 q p$, respectively. This yields a new collection $\tilde{\mathcal{U}} = \tilde{\mathcal{C}} \cup \tilde{\mathcal{D}}$ satisfying (O1)-(O3) in which the only difference is that C was replaced by $\tilde{C}$. Furthermore, $\tilde{H} = H - p_2 + p \cong H$, and q has the same number of neighbors in $\tilde{\mathcal{C}}$ as in $\mathcal{C}$. Thus, all the previous lemmas in this section apply to $\tilde{\mathcal{U}}$. In particular, Lemmas 4.34 and 4.35 so that $N_G(p) = N_G(p_1)$, as desired.

Since $N_G(p_1) = \mathcal{Q} \cup \mathcal{W}$ by definition, $G[\mathcal{P} + \mathcal{Q}]$ and $G[\mathcal{P} + \mathcal{W}]$ contain $K_{|\mathcal{P}|,|\mathcal{Q}|}$ and $K_{|\mathcal{P}|,|\mathcal{W}|}$, respectively, as spanning subgraphs. That is, G contains $K_{|\mathcal{P}|,|\mathcal{Q} \cup \mathcal{W}|}$ as a spanning subgraph.

We now claim that $\mathcal{Q}$ and $\mathcal{W}$ induce cliques. Suppose on the contrary that $\mathcal{Q}$ does not induce a clique so that there exists $u \in \mathcal{Q}$ that is not adjacent to all other vertices in $\mathcal{Q}$. Since $|\mathcal{Q} - \{u\}| + |\mathcal{P}| = 2r + 2$, u must have a neighbor $v \in \mathcal{W}$. By Lemmas 4.34 and 4.35, q is adjacent to every vertex in $\mathcal{Q}$ other than itself, and for each $i \in \{1, 2, 3\}$, x_i is adjacent to every vertex in $\mathcal{W}$ other than itself. Therefore, we may assume that $u \in C$ and $v \in C'$ for some $C \in Q_3$ and $C' \in Q_1$. However, we can replace C, C', and D with cycles $u\tilde{p}vu$, $C - u + q$, $C' - v + x_1$, and the chorded cycle $x_2 p_1 x_3 p_2 x_2$, a contradiction.

Thus, G contains $\overline{K_{|\mathcal{P}|}} \vee (K_{|\mathcal{Q}|} + K_{|\mathcal{W}|}) = \overline{K_{r+2}} \vee (K_{r+1} + K_{r+1})$ as a spanning subgraph. As both G and $\overline{K_{r+2}} \vee (K_{r+1} + K_{r+1})$ are edge-maximal with respect to not having r disjoint cycles and s chorded cycles, $G \cong \overline{K_{r+2}} \vee (K_{r+1} + K_{r+1})$, as desired. $\square$

4.5 $\mathcal{U}$ contains r cycles and $s-1$ chorded cycles.

In Sections 4.2, 4.3, and 4.4, we considered the case when we can find a collection of $r-1$ disjoint cycles and s disjoint chorded cycles. Therefore, in this section we will assume that we can find no such collection. That is, we have a collection $\mathcal{U} = \mathcal{C} \cup \mathcal{D}$ in which $\mathcal{D}$ contains $s-1$ disjoint chorded cycles, and $\mathcal{C}$ contains the r disjoint cycles. Our goal in this section is to show that we always arrive at a contradiction.

To do so, we proceed with a similar setup to that used in Section 4.2. In particular, let $\mathcal{U} = \mathcal{C} \cup \mathcal{D}$ be a collection of r disjoint cycles and $s-1$ disjoint chorded cycles. Furthermore, we choose $\mathcal{U} = \mathcal{C} \cup \mathcal{D}$ subject to the same conditions (O1)-(O3) as in Section 4.2, where $R = G \setminus \mathcal{U}$ and P is a longest path in R, as before. Just as in Section 4.2, every cycle in $\mathcal{C}$ is an induced cycle in G, and for any chorded cycle $D \in \mathcal{D}$, no vertex of D is incident to two chords.

We now prove a short sequence of lemmas similar to those in Section 4.2. We will then show that R is an independent set, and from here deduce our final contradiction.

4.5.1 Preliminaries

By the edge-maximality of G and the fact that $|G| \geq 3r + 4s$, we deduce that $|R| \geq 4$. Since G does not contain a collection of $r-1$ disjoint cycles and s disjoint chorded cycles, the folliwng lemma is immediate.

Lemma 4.39. *For all $C \in \mathcal{C}$, $G[C+R]$ is chorded-cycle free.*

From this lemma, we deduce the following.

Lemma 4.40. *Let Q be a path completely contained in R. Then $\|Q, C\| \leq 2$ for all $C \in \mathcal{C}$.*

We conclude this portion with several lemmas whose statements and proofs are similar to those in Section 4.2.

Lemma 4.41. *Let $v \in R, C \in \mathcal{C}$, and $D \in \mathcal{D}$.*

1. *If $\|v, C\| \geq 2$, then $\|v, C\| = 2$, $G[C] \cong K_{2,2}$, and $G[C+v] \cong K_{2,3}$.*

2. *If $\|v, D\| \geq 4$, then $\|v, D\| = 4$ and $G[D+v] \cong K_5$.*

Proof. Suppose $\|v, C\| \geq 2$. If v is adjacent to two consecutive vertices along C, then we get a chorded cycle, which contradicts Lemma 4.39. Thus, $|C| \geq 4$. If $|C| \geq 5$, then we can replace it with a shorter cycle, contradicting (O1), just as in the proof of Lemma 4.2.2. This proves 4.41.1.

To prove 4.41.2, we proceed in the exact manner as in the proof of Lemma 4.2.3. □

Lemma 4.42. *Suppose $|P| \geq 2$, and let p and p' be the endpoints of P. Suppose there exists $u, v \in R - P$, and let $F = \{p, p', u, v\}$. Then $\|F, D\| \leq 12$ for all $D \in \mathcal{D}$.*

Proof. Suppose not. Then some vertex in F has four neighbors in D, so that $G[D] \cong K_4$ by Lemma 4.41. Let $V(D) = \{d_1, d_2, d_3, d_4\}$.

Since $\|\{p, p'\}, D\| \leq 8$, it follows that $\|\{u, v\}, D\| \geq 5$. So without loss of generaltiy, suppose $\|u, D\| \geq 3$ with $d_1, d_2, d_3 \in N_G(u)$. Then $\|\{p, p'\}, d_4\| = 0$, as otherwise we can replace D with $D - d_4 + u \cong K_4$, and replace P with $P + d_4$, which contains a path longer than P, contradicting (O3). So $\|F, d_4\| \leq 2$.

If $\|u, D\| = 4$, then a similar argument shows that $\|F, d_i\| \leq 2$ for all $i \in \{1, 2, 3, 4\}$. That is, $\|F, D\| \leq 8$, a contradiction. So $\|u, D\| = 3$, and by symmetry, $\|v, D\| \leq 3$. As a consequence, $\|\{p, p'\}, D\| \geq 7$, however this implies that $\|\{p, p'\}, d_4\| \geq 1$, a contradiction. $\square$

Lemma 4.43. *Suppose $|P| \geq 3$. Let p and p' be the endpoints of P, and let q be the neighbor of p on P. Suppose there exists $u \in R - P$, and let $F = \{p, q, p', u\}$. Then $\|F, D\| \leq 12$ for all $D \in \mathcal{D}$.*

Proof. Suppose $\|F, D\| \geq 13$ for some $D \in \mathcal{D}$. By Lemma 4.41, $G[D] \cong K_4$, and we let $V(D) = \{d_1, d_2, d_3, d_4\}$.

Suppose $\|u, D\| \geq 3$. Observe that if p, q, and p', have a common neighbor, say $d \in D$, then we obtain two chorded cycles $D - d + u$ and $pPp'dp$, a contradiction. Thus, $\|\{q, q', p'\}, D\| \leq 8$. However, this contradicts the assumption that $\|F, D\| \geq 13$. So $\|u, D\| \leq 2$, which implies that $\|\{p, q, p'\}, D\| \geq 11$.

Thus, we may assume without loss of generality that $\|p, D\| = 4$. Suppose $\|u, D\| = 2$ with $N_G(u) = \{d_1, d_2\}$. Since $\|\{p, q, p'\}, D\| \geq 11$, we have $\|q, D\| \geq 3$. In particular, we may assume $d_3 \in N_G(q)$. If $\|p', D\| = 4$, then we obtain pd_4d_3qp and $p'd_1ud_2p'$, a contradiction. Thus, $\|p', D\| = 3$ and $\|q, D\| = 4$. We may assume that $d_3 \in N_G(p')$, which yields pd_1ud_2p and $qd_4d_3p'q$, a contradiction.

Thus, $\|u, D\| = 1$, and in turn $\|w, D\| = 4$ for all $w \in F - u$. However, if $N_D(u) = \{d\}$, then we can replace D and P with $D - d + p \cong K_4$ and $P - p + d + u$, a contradiction to (O3). $\square$

Lemma 4.44. *Suppose $|P| \geq 4$. Let p and p' be the endpoints of P, and let q and q' be their neighbors on P, respectively. If $F = \{p, q, q', p'\}$, then $\|F, D\| \leq 12$ for all $D \in \mathcal{D}$.*

Proof. Suppose $\|F, D\| \geq 13$ for some $D \in \mathcal{D}$. By Lemma 4.41, $G[D] \cong K_4$, and we let $V(D) = \{d_1, d_2, d_3, d_4\}$.

Since $\|\{q, q'\}, D\| \leq 8$, we have $\|\{p, p'\}, D\| \geq 5$. So without loss of generality, assume $\|p, D\| \geq 3$. Observe that if q, q', and p' have a common neighbor, say $d \in D$, then we obtain two chorded cycles $D - d + p$ and $qPp'dq$, a contradiction. Thus, $\|\{q, q', p'\}, D\| \leq 8$. However, this contradicts the assumption that $\|F, D\| \geq 13$. $\square$

4.5.2 Determining the Size of P

Lemma 4.45. $|P| \leq 3$.

Proof. Suppose on the contrary that $|P| \geq 4$. Let p and p' be the endpoints of P, with neighbors on P, q and q', respectively. Let $F = \{p, q, q', p'\}$.

We first show that either q or q' has a neighbor in $R - P$. By Lemma 4.44, $\|F, D\| \leq 12$ for all $D \in \mathcal{D}$, and by Lemma 4.40, $\|F, C\| \leq 2$ for all $C \in \mathcal{C}$. Thus, $\|F, R\| \geq 4(2r + 3s - 1) - 2r - 12(s - 1) = 6r + 8$. Observe that $\|\{p, p'\}, R\| \leq 4$, otherwise we obtain a chorded cycle or a longer path than P in R. Hence, $\|\{q, q'\}, R\| \geq 6r + 4 \geq 10$.

Suppose without loss of generality that $\|q, R\| \geq 5$. Since $\|q, P\| \leq 3$, otherwise we obtain a chorded cycle in R, q must have a neighbor $\tilde{p} \in R - P$. Note that we can replace p with $\tilde{p}$ and obtain a new longest path $\tilde{P}$ whose endpoints are $\tilde{p}$ and p'.

By Lemma 4.40, $\|\tilde{P}, C\| \leq 2$, and in particular, $\|\{\tilde{p}, p'\}, C\| \leq 2$ for all $C \in \mathcal{C}$. Observe that $\|\{\tilde{p}, p'\}, R\| \leq 4$, otherwise we obtain a chorded cycle or a longer path than $\tilde{P}$ in R. Thus, $\|\{\tilde{p}, p'\}, \mathcal{D}\| \geq 2(2r + 3s - 1) - 2r - 4 = 6(s - 1) + 2r$.

Since $r \geq 1$, there exists some $D \in \mathcal{D}$ such that $\|\{\tilde{p}, p'\}, D\| \geq 7$. Thus, we can find $d \in D$ such that $\|\tilde{p}, D - d\| \geq 3$ and $p'd \in E(G)$. However, we can replace D with $D - d + \tilde{p} \cong K_4$, and P with $P + d$, which contains a longer path than P, contradicting (O3). □

Lemma 4.46. $|P| \leq 2$.

Proof. Suppose on the contrary that $|P| = 3$ with $P = pqp'$. Since $|R| \geq 4$, there exists some vertex in $R - P$.

Claim 4.46.1. $\|q, R\| = 2$.

Proof. Suppose on the contrary that $q\tilde{p} \in E(G)$ for some $\tilde{p} \in R - P$. Note that $\tilde{p}$ is the endpoint of a longest path in R. Let $\tilde{P} = p'q\tilde{p}$.

Observe that $\|\{p, p'\}, R\| \leq 4$, otherwise we obtain a chorded cycle or a longer path than P in R. Also recall that Lemma 4.40, $\|P, C\| \leq 2$, and in particular, $\|\{p, p'\}, C\| \leq 2$ for all $C \in \mathcal{C}$. Thus, $\|\{p, p'\}, \mathcal{D}\| \geq 2(2r + 3s - 1) - 4 - 2r = 6(s - 1) + 2r$.

Since $r \geq 1$, there exists $D \in \mathcal{D}$ such that $\|\{p, p'\}, D\| \geq 7$. Thus, we can find $d \in D$ such that $\|p, D - d\| \geq 3$ and $p'd \in E(G)$. However, we can replace D with $D - d + p \cong K_4$, and P with $\tilde{P} + d$, which contains a longer path than $\tilde{P}$, contradicting (O3). This proves the claim. □

By this claim, $\|u, R\| \leq 2$ for all $u \in R$. Otherwise, u corresponds to q in a longest path, and the above claim shows that $\|u, R\| = 2$. So let $\tilde{p} \in R - P$, and let $F = \{p, q, p', \tilde{p}\}$. As a result, $\|F, R\| \leq 8$. By Lemmas 4.40 and 4.41, $\|F, C\| \leq 4$ for all $C \in \mathcal{C}$. Thus, $\|F, \mathcal{D}\| \geq 4(2r + 3s - 1) - 8 - 4r = 12(s - 1) + 4r$.

Since $r \geq 1$, there exists some $D \in \mathcal{D}$ such that $\|F, D\| \geq 13$. However, this contradicts Lemma 4.43. □

Lemma 4.47. $|P| = 1$*, and in particular,* R *is an independent set.*

Proof. Suppose on the contrary, $|P| = 2$ with endpoints p and p'. Since $|R| \geq 4$, there exists $u, v \in R - P$. Let $F = \{p, p', u, v\}$, where possibly $uv \in E(G)$. By Lemmas 4.40 and 4.41, $\|F, C\| \leq 6$ for all $C \in \mathcal{C}$. Since $\|F, R\| \leq 4$, we obtain $\|F, \mathcal{D}\| \geq 4(2r + 3s - 1) - 4 - 6r = 12s - 8 + 2r$.

Since $r \geq 1$, there exists $D \in \mathcal{D}$ such that $\|F, D\| \geq 13$. However, this contradicts Lemma 4.42. □

Since $|R| \geq 4$, let $u_1, u_2, u_3, u_4 \in R$ and $F = \{u_1, u_2, u_3, u_4\}$. By Lemma 4.47, $\|F, R\| = 0$, and by Lemma 4.40, $\|F, C\| \leq 8$ for all $C \in \mathcal{C}$. Thus, $\|F, \mathcal{D}\| \geq 4(2r + 3s - 1) - 8r = 12s - 4$. Hence there exists $D \in \mathcal{D}$ such that $\|F, D\| \geq 13$.

By Lemma 4.41, $G[D] \cong K_4$, and we let $V(D) = \{d_1, d_2, d_3, d_4\}$. Without loss of generality, suppose $\|w_1, D\| = 4$ and $\|w_2, D\| \geq 3$ with $d_1, d_2, d_3 \in N_D(w_2)$. However, we can replace D with $D - d_4 + w_2 \cong K_4$, and P (which is a single vertex by Lemma 4.47) with $w_1 d_4$, which contradicts (O3).

This completes the section, and completes the proof of Theorem 4.1.

4.6 Future Questions

The most natural question for future research in this area is to replace the minimum degree condition in Theorem 4.1 with an Ore-condition. Such a result would yield a characterization of the sharpness examples to Theorem 1.14. While we don't know of all the possible sharpness examples to Theorem 1.14 that do not already appear in Theorem 4.1, we know that graphs of the form $(K_c + K_{2r+2-c}) \vee \overline{K_{r+2}}$ should appear.

Chapter 5

Strong Edge-Coloring

The following results are joint work with Alexandr V. Kostochka, Xiangwen Li, Watcharintorn Ruksasakchai, Tao Wang, and Gexin Yu, appearing in [37].

5.1 Introduction

As mentioned in Section 1.4, the main purpose of this chapter is the prove the following statement, which proves a conjecture of Faudre et al. in [16]. In addition, it is best possible, as shown in Figure 1.5.

Theorem 5.1 (Kostochka et al. [37])**.** *For every subcubic planar multigraph G with no loops, $\chi'_s(G) \leq 9$.*

The proof of this result yields a polynomial-time algorithm in terms of the number of vertices that will produce a strong edge-coloring of any subcubic planar loopless multigraph using at most nine colors. Theorem 5.1 implies the following corollary.

Corollary 5.2. *Every subcubic planar multigraph G with no loops contains an induced matching of size at least $|E(G)|/9$.*

This corollary extends a result of Kang, Mnich and Müller [33] to loopless multigraphs. Joos, Rautenbach and Sasse [31] later showed that the above lower bound of $|E(G)|/9$ holds for all subcubic graphs, thus proving a conjecture of Henning and Rautenbach [26].

We present our result as follows. In Section 5.2, we provide the notation that we will use along with preliminary results. The remaining sections assume the existence of a minimal counterexample. Section 5.3 contains basic properties of a minimal counterexample, including the fact that it has no cycles of length three or four. The lemmas in Section 5.4 will show that if a face has a vertex of degree 2 on its boundary, then the face has length at least 8, and additionally, if two vertices of degree 2 exist on a face, then the distance between them is at least 5 on the face. Section 5.5 contains two lemmas showing that every face of length 5 is surrounded by faces of length at least 7. Section 5.6 contains a discharging proof based on the lemmas presented in Sections 5.3, 5.4, and 5.5. We end this chapter with some questions for further research.

5.2 Preliminaries and notation

In the proof of Theorem 5.1, we will often remove vertices or edges from a minimal counterexample and obtain a strong edge-coloring of the remaining multigraph. To aid us, we introduce some notation and preliminary facts that we will use in explanations.

We will use some lower case Greek letters, such as $\alpha, \beta, \gamma, \delta$, to denote arbitrary colors, and we will use ϕ, σ, ψ to denote colorings. Also an *i-vertex* is a vertex of degree i in our multigraph, and a *j-face* is a face of length j in our plane multigraph. An *i^+-vertex* and *j^+-face* are a vertex of degree at least i and a face of length at least j, respectively.

An edge-coloring of a multigraph G is *good*, if it is a strong edge-coloring of G using at most nine colors. A *partial* coloring of a graph G is a coloring of any subset of $E(G)$, and we say it is a *good partial coloring* of G if any colored edges e_1 and e_2 incident to each other or incident to a common edge have different colors. Given edges e and e' in G, we say that e *sees* e' if e and e' are incident or some other edge e'' is incident to both. Additionally, we will also say that e *sees* a color α, if e sees an edge e' for which $\phi(e') = \alpha$, where ϕ is a partial coloring.

Let ϕ be a good partial coloring of a graph G. For $v \in V(G)$, let $\mathcal{U}_\phi(v)$ denote the set of colors used on the edges incident to v. For an uncolored edge $e \in E(G)$, let $A_\phi(e)$ denote the set of colors that can be used on e to extend ϕ to a new good partial coloring of G. For adjacent vertices u, v, let $\Upsilon_\phi(u,v) = \mathcal{U}_\phi(u) \setminus \{\phi(uv)\}$. That is, $\Upsilon_\phi(u,v)$ denotes the set of colors used on edges incident to u other than uv. As ϕ is a good partial coloring, $\Upsilon_\phi(u,v)$ and $\Upsilon_\phi(v,u)$ are disjoint. Often we will refer to only one partial coloring which will not be named. In these cases we will suppress the subscripts in the above notation.

As mentioned, we will remove vertices and edges from a multigraph G to obtain a good partial coloring, say ϕ. Often, we will consider $|A_\phi(e)|$ for every uncolored e in G, in order to apply the well known result of Hall [25] in terms of systems of distinct representatives.

Theorem (Hall [25]). *Let $A_1, \ldots, A_n$ be n subsets of a set U. A system of distinct representatives of $\{A_1, \ldots, A_n\}$ exists if and only if for every subset S of $\{1, 2, \ldots, n\}$, we have $|\bigcup_{i \in S} A_i| \geq |S|$.*

When this condition holds for the sets $A_\phi(e)$ corresponding to the uncolored edges, we obtain a coloring of the remaining uncolored edges such that for every pair of uncolored edges e_1 and e_2, they will receive distinct colors from $A_\phi(e_1)$ and $A_\phi(e_2)$, respectively. Such an extension of ϕ is a good coloring of G and yields the desired result. Thus, when left in a situation in which we can apply Hall's Theorem, we will say that we obtain a good coloring of G *by SDR*.

5.3 Basic Properties

Everywhere below we assume G to be a subcubic planar loopless multigraph contradicting Theorem 5.1. Among all such counterexamples, we assume that G has the fewest vertices, and over all such counterexamples, has the fewest edges. Note that G is connected, since otherwise we can color each component by the minimality of G and obtain a good coloring of G. As G is planar, we assume G to be a *plane* multigraph in all the following statements. That is, we consider G together with an embedding of G into the plane.

In this section, we will show several properties of G, including that G is simple, has no small cycles, and has distance at least 3 between any two 2-vertices, a fact that we will strengthen in a later section. Similar statements are proven in [27, 28, 30] while considering minimal counterexamples with different properties.

Lemma 5.3. *G has no multiple edges, i.e., G is a simple graph.*

Proof. Suppose that $e \in E(G)$ and G has another edge with the same endpoints as e. By the minimality of G, $G - e$ has a good coloring. Since e sees at most seven edges in G, we can extend this good coloring to G. □

Lemma 5.4. *G has minimum degree at least 2.*

Proof. Suppose that v is a 1-vertex and u is the neighbor of v. Then $G - v$ has a good coloring. Since uv sees at most six edges in G, we can extend this good coloring to G. □

Lemma 5.5. *G has no cut-vertex and no cut-edge.*

Proof. Since G is subcubic, the existence of a cut-vertex implies the existence of a cut-edge. Thus, we may assume that G has a cut-edge, v_1v_2. For $i \in \{1, 2\}$, let H_i be the component of v_1v_2 containing v_i. By Lemma 5.4, $|V(H_i)| \geq 2$. Define G_1 to be the graph consisting of H_1 together with v_2 and the edge v_1v_2. Similarly define G_2 to be the graph consisting of H_2 together with v_1 and the edge v_1v_2.

By the minimality of G, G_1 and G_2 have good colorings, ϕ_1 and ϕ_2, respectively. By permuting the names of colors, we may assume $\mathcal{U}_{\phi_1}(v_1) \subseteq \{1, 2, 3\}$, $\mathcal{U}_{\phi_2}(v_2) \subseteq \{1, 4, 5\}$ with $\phi_1(v_1v_2) = \phi_2(v_1v_2) = 1$. Merging these two colorings yields a good coloring of G. □

Lemma 5.6. *If $\{e_1, e_2\}$ is an edge-cut in G, then e_1 and e_2 are incident to each other.*

Proof. If not, then we have an edge-cut $\{u_1w_1, u_2w_2\}$ in G that is a matching. We may assume that u_1 and u_2 are in the same component of $G - \{u_1w_1, u_2w_2\}$. Let H_u be the component of $G - \{u_1w_1, u_2w_2\}$ containing u_1 and u_2. Let H_w be the other component. We may then let G_u be the graph consisting of H_u together with a new vertex w whose neighborhood is $\{u_1, u_2\}$. Similarly, let G_w be the graph consisting of H_w together with a new vertex u whose neighborhood is $\{w_1, w_2\}$. Observe that G_u and G_w are subcubic planar loopless multigraphs, and so by the minimality of G, G_u and G_w have good colorings ϕ_u and ϕ_w, respectively.

Now, if $|\mathcal{U}_{\phi_w}(w_1) \cup \mathcal{U}_{\phi_w}(w_2)| \leq 5$, then we may assume that $\mathcal{U}_{\phi_w}(w_1) \cup \mathcal{U}_{\phi_w}(w_2) \subseteq [5]$ with uw_i being colored i. Since $|\mathcal{U}_{\phi_u}(u_1) \cup \mathcal{U}_{\phi_u}(u_2)| \leq 6$, we may similarly assume that $\mathcal{U}_{\phi_u}(u_1) \cup \mathcal{U}_{\phi_u}(u_2) \subseteq \{1, 2, 6, 7, 8, 9\}$ with wu_i being colored i. We may then merge these two colorings to obtain a good coloring of G in which u_iw_i receives color i, for $i \in \{1, 2\}$.

So, we have $|\mathcal{U}_{\phi_w}(w_1) \cup \mathcal{U}_{\phi_w}(w_2)| = |\mathcal{U}_{\phi_u}(u_1) \cup \mathcal{U}_{\phi_u}(u_2)| = 6$. This implies $u_1u_2, w_1w_2 \notin E(G)$. Thus, we may assume that $\mathcal{U}_{\phi_u}(u_1) = \{1, 3, 4\}, \mathcal{U}_{\phi_w}(w_2) = \{2, 3, 4\}, \mathcal{U}_{\phi_u}(u_2) = \{2, 5, 6\}, \mathcal{U}_{\phi_w}(w_1) = \{1, 5, 6\}$ with uw_i, wu_i being colored i. Again, we can merge these two colorings to obtain a good coloring of G in which u_iw_i receives color i. ∎

Lemma 5.7. *G has no triangles.*

Proof. Suppose that $w_0w_1w_2w_0$ is a triangle in G. If w_0 is a 2-vertex, then as $G - w_0$ has a good coloring, and since each of w_0w_1 and w_0w_2 see at most five colored edges in G, we can extend this good coloring to G. Thus, each w_i is a 3-vertex, and we may assume $N_G(w_0) = \{u_0, w_1, w_2\}$, $N_G(w_1) = \{w_0, u_1, w_2\}$ and $N_G(w_2) = \{w_0, w_1, u_2\}$.

Now, $G - \{w_0, w_1, w_2\}$ has a good coloring, which applied to G is a good partial coloring such that $|A(w_iu_i)| \geq 3$ and $|A(w_iw_{i+1})| \geq 5$ for $i \in \{0, 1, 2\}$, taken modulo 3. If there are at least six colors available on these six uncolored edges, then we can extend to a good coloring of G by SDR. So we may assume $A(w_0w_1) = A(w_1w_2) = A(w_2w_0)$ and $|A(w_0w_1)| = 5$. Without loss of generality, we may assume $A(w_0w_1) = \{1, 2, 3, 4, 5\}$. However, this implies that for $i \in \{0, 1, 2\}$, $\mathcal{U}(u_i)$ and $\mathcal{U}(u_{i+1})$ partition $\{6, 7, 8, 9\}$, which cannot happen. ∎

Lemma 5.8. *G has no separating cycle of length 4 or 5.*

Proof. We first show that G has no 4-cycle with a 2-vertex. Suppose that $w_1w_2w_3w_4w_1$ is a 4-cycle. If w_1 is a 2-vertex, then $G - w_1$ has a good coloring, such that $|A(w_1w_2)|$, $|A(w_4w_1)| \geq 2$, and we can extend this to a good coloring of G. Thus, if G has a 4-cycle, then each vertex of the cycle is a 3-vertex. We will use this below to show that G has no separating 4-cycle or 5-cycle.

If on the contrary, G has a separating 4-cycle or 5-cycle, call it C. By Lemma 5.7, C has no chords. Since G is subcubic, each vertex of C is incident to at most one edge not on C. Since $\lfloor \frac{5}{2} \rfloor = 2$, by symmetry we may assume that there are at most two edges inside C that are incident to vertices on C (recall that G is assumed to be embedded in the plane). If there is exactly one such edge, then G has a cut-edge, contradicting Lemma 5.5. So, we have two such edges, which are in fact cut-edges. By Lemma 5.6, these edges share a common endpoint, say u, inside of C. Now, u is a 2-vertex, as otherwise it would be a cut-vertex incident to a cut-edge. However, the graph induced by u together with the vertices of C has either a triangle or a 4-cycle containing a 2-vertex, contradicting Lemma 5.7 or the above, respectively. Thus, G has no separating 4-cycle or 5-cycle. □

Lemma 5.9. *G has no 4-cycle.*

Proof. Suppose that $x_0x_1x_2x_3x_0$ is a 4-cycle in G. By Lemma 5.8, this cycle is a 4-face and, as is shown in the proof of Lemma 5.8, each x_i is a 3-vertex. Therefore, we may let y_i denote the third neighbor of x_i, which is not on this 4-cycle. By Lemmas 5.7 and 5.8, these vertices are distinct and $y_0y_2, y_1y_3 \notin E(G)$. Let G' denote the graph obtained from G by removing x_0, x_1, x_2, x_3 and adding the edge y_0y_2. Observe that G' is a subcubic planar loopless multigraph, and so by the minimality of G, G' has a good coloring. Ignoring y_0y_2, we have a good partial coloring of G that we extend by coloring x_0y_0, x_2y_2 with the same color that y_0y_2 received. This extended coloring is a good partial coloring, and we will call it ϕ. As $|A_\phi(x_1y_1)|, |A_\phi(x_3y_3)| \geq 2$, we can greedily color these two edges and obtain another good partial coloring, which we will call σ.

Note that the edges of the 4-cycle are the only uncolored edges of G under σ, and $|A_\sigma(x_ix_{i+1})| \geq 2$ for $i \in \{0,1,2,3\}$, taken modulo 4. Also $\mathcal{U}_\sigma(y_0) \cap \mathcal{U}_\sigma(y_2) = \{\sigma(x_0y_0)\}$. Therefore without loss of generality, we may assume that $\mathcal{U}_\sigma(y_0) \subseteq \{1,2,3\}$ and $\mathcal{U}_\sigma(y_2) \subseteq \{1,4,5\}$.

Suppose that $|A_\sigma(x_0x_1) \cup A_\sigma(x_2x_3)| = 2$. Without loss of generality, $A_\sigma(x_0x_1) = A_\sigma(x_2x_3) = \{8,9\}$. This implies

$$\mathcal{U}_\sigma(y_0) \cup \mathcal{U}_\sigma(y_1) \cup \{\sigma(x_3y_3)\} = \mathcal{U}_\sigma(y_2) \cup \mathcal{U}_\sigma(y_3) \cup \{\sigma(x_1y_1)\} = [7].$$

Also $\Upsilon_\sigma(y_1,x_1) = \{4,5\}, \Upsilon_\sigma(y_3,x_3) = \{2,3\}$. However, this implies $|A_\sigma(x_1x_2)|$, $|A_\sigma(x_3x_0)| \geq 4$, and we can obtain a good coloring of G by coloring the edges x_0x_1, x_2x_3, x_1x_2, x_3x_0 in this order.

Thus $|A_\sigma(x_0x_1) \cup A_\sigma(x_2x_3)| \geq 3$ and by symmetry $|A_\sigma(x_1x_2) \cup A_\sigma(x_3x_0)| \geq 3$. We may assume $|A_\sigma(x_0x_1) \cup A_\sigma(x_1x_2) \cup A_\sigma(x_2x_3) \cup A_\sigma(x_3x_0)| \leq 3$; otherwise we can obtain a good coloring of G by SDR.

Now, if $|A_\sigma(x_0x_1)| = 2$, then $\Upsilon_\sigma(y_0,x_0) = \{2,3\}$ and $2,3 \notin \mathcal{U}_\sigma(y_1) \cup \{\sigma(x_3y_3)\}$. Since $\mathcal{U}_\sigma(y_2) \subseteq \{1,4,5\}$, we have $2,3 \in A_\sigma(x_1x_2)$, but $2,3 \notin A_\sigma(x_0x_1)$. Thus, $|A_\sigma(x_0x_1) \cup A_\sigma(x_1x_2)| \geq 4$, a contradiction. So,

$|A_\sigma(x_0x_1)| = 3$, and by symmetric arguments $A_\sigma(x_0x_1) = A_\sigma(x_1x_2) = A_\sigma(x_2x_3) = A_\sigma(x_3x_0)$.

If $\Upsilon_\sigma(y_0, x_0) \subseteq \mathcal{U}_\sigma(y_1) \cup \{\sigma(x_3y_3)\}$, then $|A_\sigma(x_0x_1)| \geq 4$, a contradiction. Thus, we may assume $2 \notin \mathcal{U}_\sigma(y_1) \cup \{\sigma(x_3y_3)\}$. However, $2 \notin \mathcal{U}_\sigma(y_2)$, so $2 \in A_\sigma(x_1x_2) \setminus A_\sigma(x_0x_1)$, again a contradiction. Thus, in all cases we can extend σ and obtain a good coloring of G. □

Lemma 5.10. *The distance between any two 2-vertices is at least 3.*

Proof. Let u and v be 2-vertices in G. Suppose first that u and v are adjacent, and let w be the other neighbor of v, which may be the other neighbor of u as well. Now, $G - v$ has a good coloring, and since uv sees at most five colored edges in G and vw sees at most seven colored edges in G, we can extend this good coloring to G. Thus, u and v are at least distance 2 apart in G.

Now suppose that u and v are distance 2 apart and are both adjacent to a 3-vertex x. Let $N_G(u) = \{u', x\}$, $N_G(v) = \{v', x\}$ and $N_G(x) = \{u, v, x'\}$, where u', v', x' are not necessarily distinct. By the minimality of G, $G - \{u, v, x\}$ has a good coloring such that uu', vv', xx' each see at most six different colors, and ux, vx each see at most four different colors. Thus, we can extend this good partial coloring to G by coloring the edges uu', vv', xx', ux, vx in this order. □

5.4 Faces Without 2-Vertices

In this section, we show that if a face has a 2-vertex, then that face must have length at least 8. Additionally, if a face does have two 2-vertices on its boundary, then the distance between them along the face is at least 5.

Lemma 5.11. *Every vertex of a 5-cycle in G is a 3-vertex.*

Proof. By Lemma 5.8, it suffices to consider 5-faces. Suppose on the contrary that $x_1x_2x_3x_4x_5x_1$ is a 5-face in G and x_5 is a 2-vertex. Lemma 5.10 implies that each x_i other than x_5 has a third neighbor y_i. By Lemmas 5.7, 5.8 and 5.9, these y_i are distinct, not on our cycle and pairwise nonadjacent except for possibly y_1y_4.

Let G' denote the graph obtained from G by removing x_1, x_2, x_3, x_4, x_5 and adding the edge y_2y_4. Observe that G' is a subcubic planar loopless multigraph, and so by the minimality of G, G' has a good coloring. Ignoring y_2y_4, we have a good partial coloring of G that we can extend by coloring x_4x_5, x_2y_2 with the color of y_2y_4. Call this good partial coloring, ϕ. Note that $|A_\phi(x_3y_3)|, |A_\phi(x_4y_4)| \geq 2$, so we can color these two edges greedily to obtain a new good partial coloring σ.

Now, $|A_\sigma(x_1y_1)|, |A_\sigma(x_2x_3)|, |A_\sigma(x_3x_4)| \geq 2, |A_\sigma(x_1x_2)| \geq 3$ and $|A_\sigma(x_5x_1)| \geq 5$. If $A_\sigma(x_1y_1) \cap A_\sigma(x_3x_4) = \emptyset$, then we can extend this to a good coloring of G by SDR. So we can color x_1y_1, x_3x_4 with the same color. We can then color the remaining three uncolored edges by SDR. $\square$

Lemma 5.12. *The distance between any two 2-vertices is at least 4.*

Proof. By Lemma 5.10, we may consider a path $x_1x_2x_3x_4x_5x_6$ such that x_2, x_5 are 2-vertices. By Lemma 5.10, all other x_i are 3-vertices, and so, we let y_3, y_4 be the third neighbors of x_3, x_4, respectively. By Lemmas 5.7, 5.9, 5.8 and 5.11, y_3, y_4 are distinct, not on this path and the only possible edge among these eight vertices other than those on the path plus x_3y_3, x_4y_4, is x_1x_6. However, regardless of the existence of x_1x_6, the following argument holds.

By the minimality of G, $G - \{x_2, x_3, x_4, x_5\}$ has a good coloring such that $|A(x_1x_2)|$, $|A(x_3y_3)|$, $|A(x_4y_4)|$, $|A(x_5x_6)| \geq 3$ and $|A(x_2x_3)|, |A(x_3x_4)|, |A(x_4x_5)| \geq 5$ (when $x_1x_6 \in E(G)$, then we get $|A(x_1x_2)|, |A(x_5x_6)| \geq 4$).

If there exists $\alpha \in A(x_2x_3) \setminus A(x_4x_5)$ (or if $|A(x_4x_5)| \geq 6$), then we can color x_2x_3 with α (or color x_2x_3 first) and then color x_1x_2, x_3y_3, x_4y_4, x_3x_4, x_5x_6, x_4x_5 in this order to obtain a good coloring of G. So, we may assume that $|A(x_4x_5)| = 5$ and $A(x_2x_3) = A(x_4x_5)$.

If $A(x_1x_2) \cap A(x_2x_3) = \emptyset$, then we can color $x_5x_6, x_4x_5, x_4y_4, x_3y_3, x_3x_4$, x_2x_3, x_1x_2 in this order to obtain a good coloring of G. Thus, it remains to consider the case when $A(x_2x_3) = A(x_4x_5)$ and there exists some $\beta \in A(x_1x_2) \cap A(x_2x_3)$. In this case, we color x_1x_2 and x_4x_5 with β and then color $x_5x_6, x_4y_4, x_3y_3, x_3x_4$, x_2x_3 in this order to obtain a good coloring of G. $\square$

Lemma 5.13. *If the boundary of a face in G contains a pair of 2-vertices, then the distance on the boundary between them is at least 5.*

Proof. By Lemma 5.12, any face contradicting the statement has length at least 8 and contain a path $x_1x_2x_3x_4x_5x_6x_7$ such that x_2 and x_6 are 2-vertices. By Lemma 5.12, all other x_i are 3-vertices, and so, for $j \in \{3, 4, 5\}$ we let y_j be the neighbor of x_j other than x_{j-1}, x_{j+1}. By Lemmas 5.7, 5.8 and 5.9, we have that y_3, y_4, y_5 are distinct, pairwise nonadjacent and not on this path. By the same Lemmas, the only possible adjacencies between these ten vertices other than those on the path and x_3y_3, x_4y_4, x_5y_5, are x_1y_5, x_7y_3. However, both edges cannot exist simultaneously and their existence will not affect the following argument.

Let G' be obtained from G by removing x_2, x_3, x_4, x_5, x_6 and adding the edge y_3y_5. Observe that G' is a subcubic planar loopless multigraph, and so by the minimality of G, G' has a good coloring. Ignoring y_3y_5, we have a good partial coloring of G that we can extend by coloring x_3y_3 and x_5y_5 with the color of y_3y_5. We will refer to this coloring as ϕ. Note that $|A_\phi(x_1x_2)|$, $|A_\phi(x_4y_4)|$, $|A_\phi(x_6x_7)| \geq 2$ and $|A_\phi(x_ix_{i+1})| \geq 4$

for $i \in \{2,3,4,5\}$. From here we see that the existence of x_1y_5 does not affect coloring x_1x_2 as $\phi(x_5y_5)$ is already excluded from $A_\phi(x_1x_2)$ since x_1x_2 sees x_3y_3. Symmetrically, the existence of x_7y_3 does not affect coloring x_6x_7 as $\phi(x_3y_3)$ is already excluded from $A_\phi(x_6x_7)$ since x_6x_7 sees x_5y_5.

If there exists $\alpha \in A_\phi(x_4x_5) \setminus A_\phi(x_2x_3)$ (or if $|A_\phi(x_2x_3)| \geq 5$), then we can color x_4x_5 with α (or color x_4x_5 first) and then color x_6x_7, x_4y_4, x_5x_6, x_3x_4, x_1x_2, x_2x_3 in this order to obtain a good coloring of G. So, we may assume that $|A_\phi(x_2x_3)| = 4$ and $A_\phi(x_2x_3) = A_\phi(x_4x_5)$.

If $A_\phi(x_1x_2) \cap A_\phi(x_4x_5) = \emptyset$ (and consequently, $A_\phi(x_1x_2) \cap A_\phi(x_2x_3) = \emptyset$), then we can color x_6x_7, x_4y_4, x_5x_6, $x_4x_5, x_3x_4, x_2x_3, x_1x_2$ in this order to obtain a good coloring of G. Thus, it remains to consider the case when there exists some $\beta \in A(x_1x_2) \cap A(x_4x_5)$. In this case we color x_1x_2, x_4x_5 with β and then color x_6x_7, x_4y_4, x_5x_6, x_3x_4, x_2x_3 in this order to obtain a good coloring of G. ☐

Lemma 5.14. *Every vertex of a 6-cycle in G is a 3-vertex.*

Proof. Suppose that G has a 6-cycle C given by $x_0x_1x_2x_3x_4x_5x_0$ on which x_0 is a 2-vertex. By Lemma 5.12, x_0 is the only 2-vertex of C.

Case 1. *C is a separating 6-cycle.*

By Lemmas 5.7, 5.8 and 5.9, C has no chords. Just as in the proof of Lemma 5.8, we may assume that C has at most two edges inside C that are incident to vertices on C. If there is exactly one such edge, then G has a cut-edge, contradicting Lemma 5.5. So, we have two such edges, and by Lemma 5.6 these edges share a common endpoint, say u, inside of C. Now, u is a 2-vertex, else it is a cut-vertex with a cut-edge. However, u together with the vertices of C contains either a triangle, a 4-cycle, or a 5-cycle containing a 2-vertex, contradicting Lemmas 5.7, 5.9, 5.8, or 5.11, respectively.

Case 2. *C is not a separating 6-cycle.*

Recall that G is assumed to be embedded into the plane. Thus C must be the boundary of a 6-face. As mentioned above, each x_i, other than x_0, is a 3-vertex and so has a third neighbor y_i. We claim that these y_i's are distinct, pairwise disjoint, and not on C. Indeed, if any y_i was on C, we would create either a triangle or 4-cycle, contradicting Lemmas 5.7 and 5.9. For $i \in [4]$, if $y_i = y_{i+1}$, we have a triangle contradicting Lemma 5.7. For $i \in \{1,2,3,5\}$, taken modulo 5, if $y_i = y_{i+2}$, we have a 4-cycle contradicting Lemma 5.9. For $i \in \{1,2\}$, if $y_i = y_{i+3}$, then $y_ix_ix_{i+1}x_{i+2}x_{i+3}y_{i+3}$ is a separating 5-cycle contradicting Lemma 5.8. Thus, the y_i's are distinct. For $i \in [4]$, if $y_iy_{i+1} \in E(G)$, we have a 4-cycle contradicting Lemma 5.9. For $i \in [3]$ if $y_iy_{i+2} \in E(G)$, we have a separating 5-cycle contradicting Lemma 5.8. If $y_5y_1 \in E(G)$, then

$y_1x_1x_0x_5y_5y_1$ is a 5-cycle containing a 2-vertex contradicting Lemma 5.11. For $i \in \{1,2\}$ if $y_iy_{i+3} \in E(G)$, then $y_ix_ix_{i+1}x_{i+2}x_{i+3}y_{i+3}y_i$ is a separating 6-cycle contradicting Case 1. Thus, the y_i's are pairwise disjoint.

Now, let G' denote the plane graph obtained from G by adding a new vertex z inside the face bounded by C, deleting $x_0, \ldots, x_5$, and adding the new edges zy_1, zy_3, zy_4. Observe that G' is a subcubic planar loopless multigraph, and so by the minimality of G, it has a good coloring ϕ. Ignoring zy_1, zy_3, zy_4, this yields a good partial coloring of G that can be extended by coloring x_1y_1 and x_3x_4 with $\phi(zy_1)$. This coloring, call it σ, is indeed a good partial coloring as $\phi(zy_1)$ cannot appear in $\Upsilon_\phi(y_3, x_3) \cup \Upsilon_\phi(y_4, x_4)$ since ϕ was a partial good coloring.

Without loss of generality, suppose $\sigma(x_1y_1) = \sigma(x_3x_4) = 1$. Note that $|A_\sigma(x_iy_i)| \geq 2$ for $i \in \{2,3,4,5\}$, $|A_\sigma(x_jx_{j+1})| \geq 4$ for $j \in \{1,2,4\}$ and $|A_\sigma(x_\ell x_{\ell+1})| \geq 6$ for $\ell \in \{0,5\}$, taken modulo 6. As a result, if we can extend σ to a good partial coloring on the edges $x_2y_2, x_3y_3, x_4y_4, x_5y_5, x_2x_3, x_4x_5$, then we can extend this further by coloring x_1x_2, x_0x_1, x_0x_5 in this order to obtain a good coloring of G. Thus, it suffices to consider the edges $x_2y_2, x_3y_3, x_4y_4, x_5y_5, x_2x_3, x_4x_5$.

For $i \in \{2,3,4,5\}$, if there exists $\alpha \in A_\sigma(x_iy_i) \setminus A_\sigma(x_2x_3)$ (or $|A_\sigma(x_2x_3)| \geq 5$), then we can color x_iy_i with α (or color x_iy_i first). If $i = 2$, we color $x_3y_3, x_4y_4, x_5y_5, x_4x_4, x_2x_3$ in this order. If $i = 5$, we color $x_4y_4, x_3y_3, x_2y_2, x_4x_5, x_2x_3$ in this order. If $i \in \{2,3\}$, we color $x_{i-1}y_{i-1}, \ldots, x_2y_2, x_{i+1}y_{i+1}, \ldots, x_5y_5$, x_4x_5, x_2x_3 in this order. In all cases, we obtain our good partial coloring of G. As a consequence, $|A_\sigma(x_2x_3)|$ $= 4$ and $A_\sigma(x_iy_i) \subseteq A_\sigma(x_2x_3)$ for $i \in \{2,3,4,5\}$. By a symmetric argument, $|A_\sigma(x_4x_5)| = 4$ and $A_\sigma(x_iy_i) \subseteq$ $A_\sigma(x_4x_5)$.

Now, if there exists $\beta \in A_\sigma(x_3y_3) \setminus A_\sigma(x_2y_2)$ (or $|A_\sigma(x_2y_2)| \geq 3$), then we can color x_3y_3 with β (or color x_3y_3 first) and then color x_4y_4, x_5y_5, x_4x_5, x_2x_3, x_2y_2 in this order to obtain our good partial coloring of G. So we may assume that $|A_\sigma(x_2y_2)| = 2$ and $A_\sigma(x_2y_2) = A_\sigma(x_3y_3)$. A similar argument shows that $|A_\sigma(x_5y_5)| = 2$ and $A_\sigma(x_5y_5) = A_\sigma(x_4y_4)$.

Lastly, if there exists $\gamma \in A_\sigma(x_2y_2) \cap A_\sigma(x_4y_4)$, then we can color x_2y_2, x_4y_4 with γ and then color x_3y_3, x_5y_5, x_4x_5, x_2x_3 in this order to obtain our good partial coloring of G.

Thus, $A_\sigma(x_2y_2) = A_\sigma(x_3y_3)$ and $A_\sigma(x_4y_4) = A_\sigma(x_5y_5)$. Furthermore, $A_\sigma(x_2y_2)$ and $A_\sigma(x_4y_4)$ partition $A_\sigma(x_2x_3)$ and $A_\sigma(x_4x_5)$ so that $A_\sigma(x_2x_3) = A_\sigma(x_4x_5)$. So without loss of generality, we may assume that $A_\sigma(x_2y_2) = A_\sigma(x_3y_3) = \{2,3\}$, $A_\sigma(x_4y_4) = A_\sigma(x_5y_5) = \{4,5\}$ and $A_\sigma(x_2x_3) = A_\sigma(x_4x_5) = \{2,3,4,5\}$. We can then obtain a good partial coloring of G by coloring x_iy_i with i for $i \in \{2,3,4,5\}$, x_2x_3 with 5 and x_4x_5 with 2. As mentioned above, these good partial colorings can each be extended to obtain good colorings of G.

This completes the case that C is the boundary of a 6-face, and so proves the lemma. $\square$

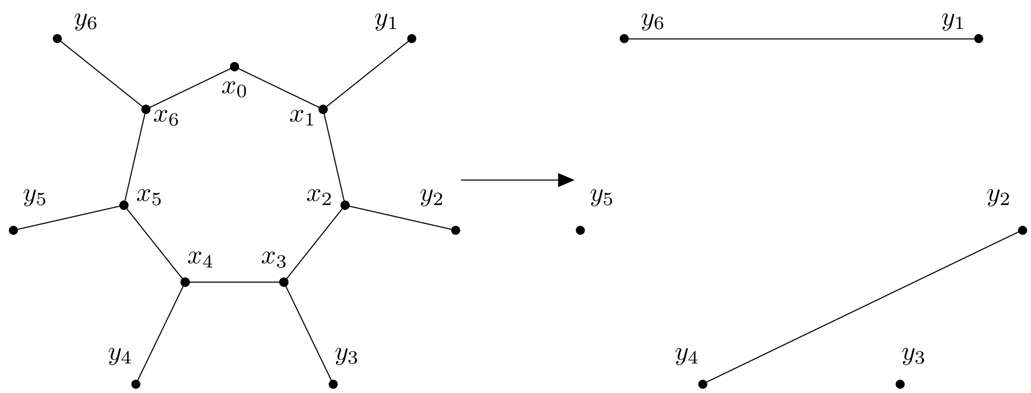


Figure 5.1: Forming G' from G in Lemma 5.15

Lemma 5.15. *Every vertex of a 7-face in G is a 3-vertex.*

Proof. Recall that G is assumed to be embedded into the plane. Suppose on the contrary that G has a 7-face with boundary $x_0x_1x_2\dots x_6x_0$ with x_0 being a 2-vertex. By Lemma 5.13, each x_i other than x_0 has a third neighbor $y_i \notin \{x_{i-1}, x_{i+1}\}$ where i is taken modulo 7. Similarly to Case 2 of Lemma 5.14, Lemmas 5.7, 5.9, 5.8, 5.11, and 5.14, imply that the y_i's are not on the 7-face, are distinct and the only possible adjacencies other than those on this face or x_iy_i, $i \in [6]$, are y_1y_4, y_2y_5, y_3y_6. Note by Lemma 5.14, $y_2y_6, y_1y_5 \notin E(G)$.

Let G' be obtained from G by removing $x_0, x_1, \dots, x_6$ and adding the edges y_1y_6, y_2y_4 (see Figure 5.1). Observe that G' is a subcubic planar loopless multigraph, and so by the minimality of G, G' has a good coloring, which ignoring y_1y_6, y_2y_4, is a good partial coloring ϕ of G.

Claim 5.15.1. $A_\phi(x_2y_2) \cap A_\phi(x_4y_4) \cap A_\phi(x_6y_6) = \emptyset$.

Proof. Without loss of generality, suppose on the contrary that $1 \in A_\phi(x_2y_2) \cap A_\phi(x_4y_4) \cap A_\phi(x_6y_6)$. We can obtain another good partial coloring of G, σ, by coloring x_2y_2, x_4y_4, x_6y_6 with 1. Recall that y_iy_{i+3}, $i \in [3]$ are possible edges of G. However, the existence of these edges will not affect the following argument as we will be sure to not color x_1y_1, x_3y_3, x_5y_5 with 1.

Note that $|A_\sigma(x_iy_i)| \geq 2$ for $i \in \{1, 3, 5\}$, $|A_\sigma(x_jx_{j+1})| \geq 4$ for $j \in [5]$ and $|A_\sigma(x_6x_0)|$, $|A_\sigma(x_0x_1)| \geq 6$. As a result, if we can somehow extend σ to a good partial coloring on the edges x_1y_1, x_3y_3, x_5y_5, x_1x_2, x_2x_3, x_3x_4, x_4x_5, then we can extend this further by coloring x_5x_6, x_6x_0, x_0x_1 in this order. Thus, it suffices to consider the edges $x_1y_1, x_3y_3, x_5y_5, x_1x_2$, x_2x_3, x_3x_4, x_4x_5.

Now, if there exists $\alpha \in A_\sigma(x_2x_3) \setminus A_\sigma(x_4x_5)$ (or $|A_\sigma(x_4x_5)| \geq 5$), we can color x_2x_3 with α (or just color x_2x_3 first) and then color $x_1y_1, x_3y_3, x_1x_2, x_3x_4, x_5y_5, x_4x_5$ in this order to obtain our good partial coloring of G. So, we may assume that $|A_\sigma(x_4x_5)| = 4$ and $A_\sigma(x_4x_5) = A_\sigma(x_2x_3)$.

If $A_\sigma(x_5y_5) \cap A_\sigma(x_2x_3) = \emptyset$ (and consequently, $A_\sigma(x_5y_5) \cap A_\sigma(x_4x_5) = \emptyset$), then we can color x_1y_1, x_3y_3, x_1x_2, $x_2x_3, x_3x_4, x_4x_5, x_5y_5$ in this order to obtain our good partial coloring of G. Thus, it remains to consider the case when there exists some $\beta \in A_\sigma(x_5y_5) \cap A_\sigma(x_2x_3)$. In this case, we color x_5y_5, x_2x_3 with β and then color $x_1y_1, x_3y_3, x_1x_2, x_3x_4, x_4x_5$ in this order to obtain a good coloring of G. This proves the claim. $\square$

Recall that we originally constructed the auxiliary graph $G - \{x_0, \dots, x_6\} + y_1y_6 + y_2y_4$ to obtain ϕ. By Claim 5.15.1, the colors placed on y_1y_6, y_2y_4 are distinct, as they are colors in $A_\phi(x_6y_6)$ and $A_\phi(x_2y_2) \cap A_\phi(x_4y_4)$, respectively. So we may assume that y_1y_6 and y_2y_4 received the colors 1 and 2, respectively.

Coloring x_1y_1, x_6y_6 with 1 and x_2y_2, x_4y_4 with 2, extends ϕ to a good partial coloring of G. Additionally, under this new partial coloring, x_5y_5 sees at most eight colored edges, including edges colored 1 and 2, so that we can extend further by coloring x_5y_5 with some α. We will refer to this new good partial coloring in which x_1y_1, x_6y_6 are colored 1, x_2y_2, x_4y_4 are colored 2 and x_5y_5 is colored α, as ψ.

Under ψ, the existence of y_1y_4, y_2y_5 will not affect our arguments as the edges x_1y_1, x_4y_4, x_2y_2, x_5y_5 are already colored in a good partial coloring. The existence of the edge y_3y_6 will not affect our arguments as we will not color x_3y_3 with 1.

Observe that $|A_\psi(x_3y_3)|, |A_\psi(x_4x_5)|, |A_\psi(x_5x_6)| \geq 2$, $|A_\psi(x_ix_{i+1})| \geq 3$ for $i \in [3]$ and $|A_\psi(x_6x_0)|$, $|A_\psi(x_0x_1)| \geq 5$. As a result, if we can somehow extend ψ to a good partial coloring on the edges $x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_6, x_3y_3$, then we can extend this further by coloring x_0x_1, x_6x_0. Thus, it suffices to consider the edges $x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_6, x_3y_3$ below.

Claim 5.15.2. *$A_\psi(x_4x_5) = A_\psi(x_5x_6)$ and $|A_\psi(x_4x_5)| = 2$.*

Proof. Suppose on the contrary that either $|A_\psi(x_5x_6)| \geq 3$ or $A_\psi(x_4x_5) \setminus A_\psi(x_5x_6) \neq \emptyset$. In either case, we color x_4x_5 first, where in the latter case we use a color from $A_\psi(x_4x_5) \setminus A_\psi(x_5x_6)$. Suppose that β is the color we can apply to x_4x_5. Note that there exists some $\gamma_1 \in A_\psi(x_3y_3) \setminus \{\beta\}$ as an available color for x_3y_3.

We aim to show that it is impossible for $A_\psi(x_2x_3) = A_\psi(x_3x_4) = \{\beta, \gamma_1, \gamma_2\}$ for some $\gamma_2 \notin \{\beta, \gamma_1\}$. If this was the case, then as $1, 2 \notin A_\psi(x_2x_3)$, we may assume that $\beta = 3, \gamma_1 = 4$ and $\gamma_2 = 5$. Additionally, as $\alpha \notin \{\beta, \gamma_1, \gamma_2\}$, we may assume that $\alpha = 6$. Thus, we have $\Upsilon_\psi(y_3, x_3) \cup \Upsilon_\psi(y_4, x_4) = \{1, 7, 8, 9\}$ and $\Upsilon_\psi(y_2, x_2) \cup \Upsilon_\psi(y_3, x_3) = \{6, 7, 8, 9\}$. This implies that $\Upsilon_\psi(y_2, x_2) \cap \Upsilon_\psi(y_4, x_4) \neq \emptyset$. However, recall that the auxiliary graph used to obtain ϕ contained y_2y_4. As a result, $\Upsilon_\psi(y_2, x_2) \cap \Upsilon_\psi(y_4, x_4) = \emptyset$, a contradiction. So we cannot have $A_\psi(x_2x_3) = A_\psi(x_3x_4) = \{\beta, \gamma_1, \gamma_2\}$, as desired.

As a result, if we color x_4x_5 with β and x_3y_3 with γ_1, we can further color x_2x_3, x_3x_4 to obtain a good partial coloring of G, which we will call τ. Let γ_2, γ_3 denote $\tau(x_2x_3), \tau(x_3x_4)$, respectively. Without loss of generality, we may assume $\gamma_1 = 7, \gamma_2 = 8, \gamma_3 = 9$. Recall that we are assuming either $|A_\psi(x_5x_6)| \geq 3$ or $\beta \in A_\psi(x_4x_5) \setminus A_\psi(x_5x_6)$. So $A_\tau(x_5x_6) \neq \emptyset$, and if $A_\tau(x_1x_2) \neq \emptyset$, we can greedily color x_1x_2, x_5x_6 to obtain a good partial coloring which we can extend to all of G as mentioned above.

Thus, we had $A_\psi(x_1x_2) = \{7, 8, 9\}$. We may also assume that $\mathcal{U}_\psi(y_1) = \{1, 3, 4\}$ and $\mathcal{U}_\psi(y_2) = \{2, 5, 6\}$. Under τ, if we could recolor x_2x_3 with either 3 or 4, then we could color x_1x_2 with 8 and color x_5x_6 last to obtain our good partial coloring of G. Thus, $3, 4 \in \Upsilon_\tau(y_3, x_3) \cup \{\beta\}$. A similar argument holds if we could recolor x_3x_4 with 1, 3, 4, 5, or 6, implying $1, 3, 4, 5, 6 \in \Upsilon_\tau(y_3, x_3) \cup \Upsilon_\tau(y_4, x_4) \cup \{\alpha, \beta\}$.

Recall that y_2y_4 was an edge of G' so that $\Upsilon_\tau(y_2, x_2) \cap \Upsilon_\tau(y_4, x_4) = \emptyset$. In particular, $5, 6 \notin \Upsilon_\tau(y_4, x_4)$. Thus, we have $5, 6 \in \Upsilon_\tau(y_3, x_3) \cup \{\alpha, \beta\}$, and consequently, $\Upsilon_\tau(y_3, x_3) \cup \{\alpha, \beta\} = \{3, 4, 5, 6\} = \Upsilon_\tau(y_1, x_1) \cup \Upsilon_\tau(y_2, x_2)$, and $1 \in \Upsilon_\tau(y_4, x_3)$.

Let us reconsider ψ. As $1 \in \Upsilon_\psi(y_4, x_3)$, we have $|A_\psi(x_4x_5)| \geq 3$. If either $|A_\psi(x_5x_6)| \geq 3$ or $|A_\psi(x_4x_5) \setminus A_\psi(x_5x_6)| \geq 2$, then instead of coloring x_4x_5 with β, we could color it with some $\beta' \neq \beta$ such that x_5x_6 would still have at least two colors available on it. By repeating an argument similar to the above, we would then conclude that $\Upsilon_\tau(y_3, x_3) \cup \{\alpha, \beta'\} = \Upsilon_\tau(y_1, x_1) \cup \Upsilon_\tau(y_2, x_2)$, a contradiction, as it would imply $\beta = \beta'$.

As a result, we have $|A_\psi(x_5x_6)| = 2$ and $|A_\psi(x_4x_5) \setminus A_\psi(x_5x_6)| = 1$. We may assume that $A_\psi(x_5x_6) = \{\delta_1, \delta_2\}$ and $A_\psi(x_4x_5) = \{\beta, \delta_1, \delta_2\}$. Recall that $\Upsilon_\psi(y_3, x_3) \cup \{\alpha, \beta\} = \{3, 4, 5, 6\}$ so that $\beta \notin \Upsilon_\psi(y_3, x_3)$, and consequently, $\beta \in A_\psi(x_3x_4)$.

If $\{\delta_1, \delta_2\} \neq \{7, 8\}$, then we can color x_4x_5 with a color in $\{\delta_1, \delta_2\} \setminus \{7, 8\}$, color x_3x_4 with β, x_3y_3 with 7, x_2x_3 with 8, x_1x_2 with 9 and color x_5x_6 last to obtain our good partial coloring of G. If $\{\delta_1, \delta_2\} = \{7, 8\}$, then we can color x_1x_2, x_4x_5 with 8 and x_3y_3, x_5x_6 with 7. This good partial coloring of G leaves at least one available color on each of x_2x_3, x_3x_4. In particular, 5 and 6 are not available on x_2x_3. If 5 or 6 is in $\Upsilon_\psi(y_3, x_3)$, then x_2x_3 has at least two available colors and we obtain our good partial coloring of G. Since we cannot have 5 or 6 in $\Upsilon_\psi(y_4, x_4)$, we must have either 5 or 6 available on x_3x_4. Thus, we can color x_3x_4, x_2x_3 and obtain our good partial coloring of G.

As mentioned above, these good partial colorings of G can be extended to good colorings of G, and this proves the claim. $\square$

Without loss of generality suppose $\alpha = 3$. As $1, 2, 3 \notin A_\psi(x_4x_5)$, we may assume that $A_\psi(x_4x_5) = A_\psi(x_5x_6) = \{8, 9\}$. Additionally, we may assume that $\Upsilon_\psi(y_6, x_6) = \{4, 5\} = \Upsilon_\psi(y_4, x_4)$ and $\Upsilon_\psi(y_5, x_5) = \{6, 7\}$. If $1 \in A_\psi(x_3x_4)$, we can color x_3x_4 with 1 and then color $x_3y_3, x_4x_5, x_5x_6, x_2x_3, x_1x_2$ in this order to obtain our good partial coloring of G. Thus, $1 \in \Upsilon_\psi(y_3, x_3)$, and so $|A_\psi(x_2x_3)| \geq 4$.

Recall that $|A_\psi(x_3x_4)| \geq 3$, and thus, x_3x_4 has an available color not in $\{8, 9\}$. As $1, 2, 3, 4, 5 \notin A_\psi(x_3x_4)$, we may assume without loss of generality that it is 6. So, we color x_3x_4 with 6 and then color $x_3y_3, x_4x_5, x_5x_6, x_2x_3$ in this order. Call this good partial coloring of G, τ. It remains only to color x_1x_2 to obtain a good partial coloring of G that we can extend to all of G.

We must have $A_\psi(x_1x_2) = \{6, \tau(x_2x_3), \tau(x_3y_3)\}$, otherwise we can color x_1x_2. Recall that our auxiliary graph G' contained the edges y_1y_6, y_2y_4 so that $\Upsilon_\psi(y_1, x_1) \cap \Upsilon_\psi(y_6, x_6) = \Upsilon_\psi(y_2, x_2) \cap \Upsilon(y_4, x_4) = \emptyset$. Since $\Upsilon_\psi(y_4, x_4) = \Upsilon(y_6, x_6) = \{4, 5\}$, we have $4, 5 \in A_\psi(x_1x_2)$, and in particular, $A_\psi(x_1x_2) = \{4, 5, 6\}$ with $\{\tau(x_2x_3), \tau(x_3y_3)\} = \{4, 5\}$.

Without loss of generality assume $\tau(x_3y_3) = 4$. We may then extend ψ by coloring x_3x_4 with 6, x_3y_3 with 4, x_1x_2 with 5 and then color x_2x_3, x_4x_5, x_5x_6 in this order to obtain our good partial coloring of G.

In all cases, we obtain a partial good coloring of G from which we can extend to a good coloring of G as mentioned above. This proves the lemma. □

5.5 Adjacent Faces

By the lemmas in Section 5.3, every face in G is a 5^+-face. In this section we show that if a face has length 5, then it can only be adjacent to 7^+-faces.

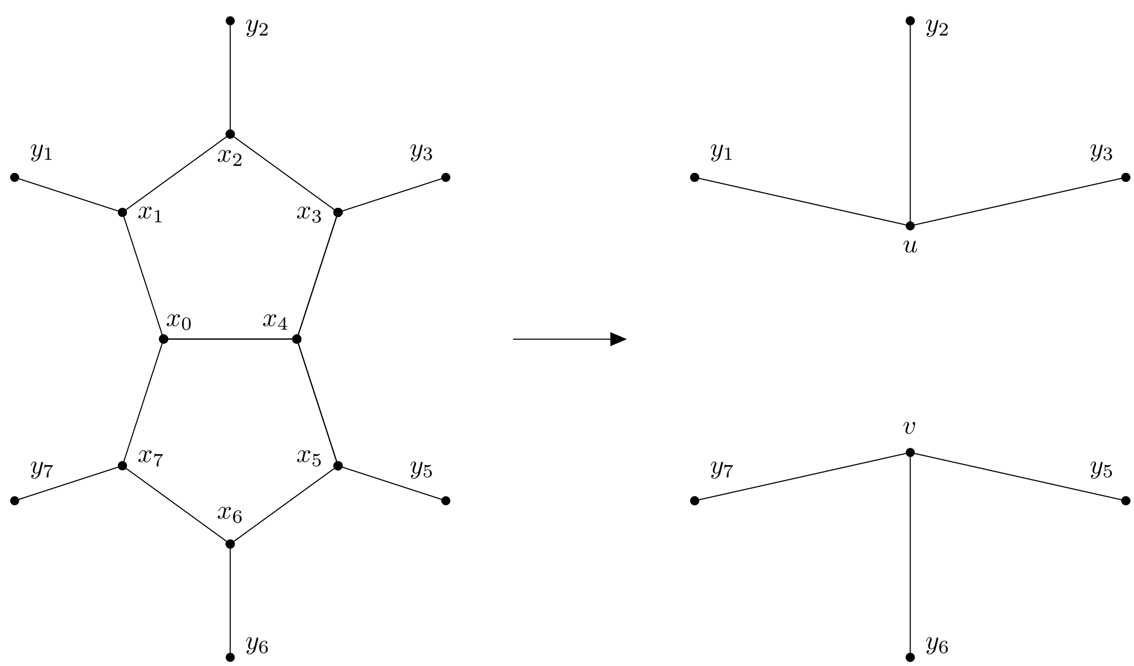


Figure 5.2: Forming G' from G in Lemma 5.16

Lemma 5.16. *No two 5-faces in G share an edge.*

Proof. Suppose the contrary. By Lemma 5.11, the boundaries of the two faces form an 8-cycle, $x_0x_1 \dots x_7x_0$

with $x_4x_0 \in E(G)$. By Lemmas 5.7, 5.9, 5.8 and 5.11, each x_i other than x_4, x_0 has a third neighbor y_i not on the 8-cycle that are distinct from each other, except possibly $y_2 = y_6$. Additionally, the only possible adjacencies between the y_i's are y_iy_j for $i \in [3]$ and $j \in \{5, 6, 7\}$.

Let G' denote the graph obtained from G by removing $x_0, \ldots, x_7$, adding two new vertices u, v and the edges $uy_1, uy_2, uy_3, vy_5, vy_6, vy_7$ (see Figure 5.2). Observe that G' is a subcubic planar loopless multigraph, and so by the minimality of G, G' has a good coloring, which ignoring $uy_1, uy_2, uy_3, vy_5, vy_6, vy_7$ gives us a good partial coloring of G that can be extended by coloring x_jy_j with the same color as uy_j, $j \in [3]$ and $x_\ell y_\ell$ with the same color as vy_ℓ, for $\ell \in \{5, 6, 7\}$. This new partial coloring of G is still a good partial coloring, and we will refer to it as ϕ.

By the construction of G', we see that $\phi(x_1y_1) \neq \phi(x_3y_3)$ and $\phi(x_5y_5) \neq \phi(x_7y_7)$. Without loss of generality, we may assume that $\phi(x_1y_1) = 1$ and $\phi(x_3y_3) = 2$. We will break the following into cases depending on $(\phi(x_5y_5), \phi(x_7y_7))$.

Case 1. $(\phi(x_5y_5), \phi(x_7y_7)) = (3, 4)$.

Observe that $|A_\phi(x_ix_{i+1})| \geq 2$ for $i \in \{1, 2, 5, 6\}$, $|A_\phi(x_jx_{j+1})| \geq 4$ for $j \in \{0, 3, 4, 7\}$, taken modulo 8 and $A_\phi(x_4x_0) = \{5, 6, 7, 8, 9\}$. By the construction of G', we can extend ϕ to another good partial coloring of G by coloring $x_3x_4, x_4x_5, x_7x_0, x_0x_1$ with $1, 4, 3, 2$, respectively. We will call this good partial coloring σ. Note that $|A_\sigma(x_ix_{i+1})| \geq 1$ for $i \in \{1, 2, 5, 6\}$ and $A_\sigma(x_4x_0) = \{5, 6, 7, 8, 9\}$.

If $|A_\sigma(x_1x_2) \cup A_\sigma(x_2x_3)|, |A_\sigma(x_5x_6) \cup A_\sigma(x_6x_7)| \geq 2$, we can color x_1x_2,x_2x_3, x_5x_6, x_6x_7, x_4x_0 in this order to obtain a good coloring of G. By symmetry, we have two subcases to consider.

Subcase 1.1. $|A_\sigma(x_1x_2) \cup A_\sigma(x_2x_3)| = |A_\sigma(x_5x_6) \cup A_\sigma(x_6x_7)| = 1$.

Let $A_\sigma(x_1x_2) = A_\sigma(x_2x_3) = \{\alpha\}$ and $A_\sigma(x_5x_6) = A_\sigma(x_6x_7) = \{\beta\}$. Since $\alpha \notin \{1, 2, 3, 4\}$, we have $3 \in \Upsilon_\sigma(y_2, x_2) \cup \Upsilon(y_3, x_3)$. However, if $3 \in \mathcal{U}_\sigma(y_2)$, then $|A_\sigma(x_1x_2)| \geq 2$, a contradiction. Thus, $\mathcal{U}_\sigma(y_3) = \{2, 3, \gamma\}$ for some $\gamma \notin [4]$, since G is a counterexample. By a similar argument, we have $4 \in \mathcal{U}_\sigma(y_1)$, and as $A_\sigma(x_1x_2) = A_\sigma(x_2x_3)$, we have $\mathcal{U}_\sigma(y_1) = \{1, 4, \gamma\}$. Symmetrically, $\mathcal{U}_\sigma(y_5) = \{2, 3, \delta\}$ and $\mathcal{U}_\sigma(y_7) = \{1, 4, \delta\}$, where $\delta \notin [4]$.

Now, as $4 \in \mathcal{U}_\sigma(y_1)$ and $|A_\sigma(x_1x_2)| = 1$, we cannot have $4 \in \mathcal{U}_\sigma(y_2)$. Thus, $4 \in A_\phi(x_2x_3)$. Similarly, $2 \in A_\phi(x_6x_7)$. Thus, we can extend ϕ by coloring x_1x_2 with α, x_2x_3 with 4, x_3x_4 with 1, x_5x_6 with β, x_6x_7 with 2, x_7x_0 with 3 and color x_4x_5, x_0x_1, x_4x_0 in this order. This gives us a good partial coloring of G and completes this subcase.

Subcase 1.2. $|A_\sigma(x_1x_2) \cup A_\sigma(x_2x_3)| \geq 2$ *and* $|A_\sigma(x_5x_6) \cup A_\sigma(x_6x_7)| = 1$.

Suppose $A_\sigma(x_5x_6) = A_\sigma(x_6x_7) = \{\beta\}$. Now $2 \notin \mathcal{U}_\phi(y_6) \cup \mathcal{U}_\phi(y_7)$, as otherwise $|A_\sigma(x_6x_7)| \geq 2$, a contradiction. Thus, $2 \in A_\phi(x_6x_7)$, and by symmetry, $1 \in A_\phi(x_5x_6)$. Now, we can alter σ to another good partial coloring by uncoloring x_0x_1 and then coloring x_5x_6 with β and x_6x_7 with 2. Call this new partial coloring ψ. Note that $|A_\psi(x_0x_1)| \geq 2$ and $|A_\psi(x_ix_{i+1})| \geq 1$ for $i \in \{1, 2\}$. Since the only change affecting the edges available on x_1x_2, x_2x_3 was the uncoloring of x_0x_1, we still have $|A_\psi(x_1x_2) \cup A_\psi(x_2x_3)| \geq 2$.

If $|A_\psi(x_0x_1) \cup A_\psi(x_1x_2) \cup A_\psi(x_2x_3)| \geq 3$, then we can obtain a good coloring of G by SDR. So we have $|A_\psi(x_0x_1)| = 2$ and $A_\psi(x_1x_2) \cup A_\psi(x_2x_3) = A_\psi(x_0x_1)$. In particular, $A_\psi(x_1x_2) \subseteq A_\psi(x_0x_1)$.

Since $|A_\psi(x_0x_1)| = 2$ and x_0x_1 sees x_7y_7 colored 4, we cannot have $4 \in \mathcal{U}_\psi(y_1) \cup \{\psi(x_2y_2)\}$. If $4 \notin \Upsilon_\psi(y_2, x_2)$, then $4 \in A_\psi(x_1x_2) \setminus A_\psi(x_0x_1)$, a contradiction to $A_\psi(x_1x_2) \subseteq A_\psi(x_0x_1)$. Thus, $4 \in \Upsilon_\psi(y_2, x_2)$, and so $|A_\psi(x_2x_3)| = 2$. Furthermore, we cannot have 4 in $\mathcal{U}_\psi(y_3) = \mathcal{U}_\sigma(y_3)$, as otherwise $|A_\psi(x_2x_3)| \geq 3$. Returning to ϕ, this implies $4 \in A_\phi(x_3x_4)$.

Recall that $1 \in A_\phi(x_5x_6)$. By a symmetric argument, $3 \in \Upsilon_\psi(y_2, x_2)$. Thus $\Upsilon_\psi(y_2, x_2) = \Upsilon_\sigma(y_2, x_2) = \{3, 4\}$. Now, we can alter σ by first uncoloring x_4x_5, then recoloring x_3x_4 with 4 and coloring x_5x_6 with 1, x_6x_7 with β. By the above, this is another good partial coloring, call it τ.

Note that $|A_\tau(x_4x_5)| \geq 1, |A_\tau(x_1x_2)|, |A_\tau(x_2x_3)| \geq 2$ and $|A_\tau(x_4x_0)| \geq 4$. We can then color $x_4x_5, x_2x_3, x_1x_2, x_4x_0$ in this order to obtain a good coloring of G.

This completes the subcase and so proves the case.

Case 2. $(\phi(x_5y_5), \phi(x_7y_7)) = (1, 3)$.

First, notice that one can recolor x_1y_1 with a color other than 1, call it α, and still maintain a good partial coloring of G. We will proceed in this case based on whether or not α is 2.

Subcase 2.1. $\alpha \neq 2$.

We can extend our good partial coloring of G by coloring x_2x_3, x_7x_0 with 1, x_4x_5 with 3 and x_0x_1 with 2. Call this new coloring σ.

Note that $|A_\sigma(x_1x_2)|, |A_\sigma(x_6x_7)| \geq 1$, $|A_\sigma(x_5x_6)| \geq 2$, $|A_\sigma(x_3x_4)| \geq 3$ and $|A_\sigma(x_4x_0)| \geq 5$. Thus, we can color $x_6x_7, x_5x_6, x_1x_2, x_3x_4, x_4x_0$ in this order to obtain a good coloring of G.

Subcase 2.2. $\alpha = 2$.

We can extend our good partial coloring of G by coloring x_2x_3, x_7x_0 with 1 and x_4x_5 with 3. Call this new coloring σ.

Note that $|A_\sigma(x_ix_{i+1})| \geq 2$ for $i \in \{1, 5, 6\}$, $|A_\sigma(x_3x_4)|, |A_\sigma(x_0x_1)| \geq 3$ and $|A_\sigma(x_4x_0)| \geq 6$. If there exists some $\beta \in A_\sigma(x_6x_7) \cap A_\sigma(x_1x_2)$, we can color x_1x_2, x_6x_7 with β and then color $x_5x_6, x_3x_4, x_1x_0, x_4x_0$

in this order to obtain a good partial coloring of G.

As a result, either $|A_\sigma(x_1x_0)| \geq 4$ or there exists some $\gamma \in (A_\sigma(x_1x_2) \cup A_\sigma(x_6x_7)) \setminus A_\sigma(x_1x_0)$. In either case, we color x_1x_2, x_6x_7 in this order (in particular, using γ on at least one edge in the latter case), then color $x_5x_6, x_3x_4, x_1x_0, x_4x_0$ in this order to obtain a good coloring of G.

This completes the subcase, and so proves the case.

Case 3. $(\phi(x_5y_5), \phi(x_7y_7)) = (1, 2)$.

As in the previous case, we can recolor x_1y_1 with a color $\alpha \neq 1$ so that we still maintain a good partial coloring of G. We proceed in subcases as above.

Subcase 3.1. $\alpha = 2$.

We can extend our good partial coloring of G by coloring x_2x_3, x_7x_0 with 1. Call this new coloring σ.

Note that $|A_\sigma(x_ix_{i+1})| \geq 2$ for $i \in \{1, 5, 6\}$, $|A_\sigma(x_jx_{j+1})| \geq 4$ for $j \in \{0, 3, 4\}$, taken modulo 8 and $|A_\sigma(x_4x_0)| \geq 7$. Now, either $|A_\sigma(x_1x_2)| \geq 4$ or there exists $\beta \in A_\sigma(x_3x_4) \setminus A_\sigma(x_1x_2)$. In either case, we color x_3x_4 first (in particular, with β in the latter case), then color $x_5x_6, x_6x_7, x_4x_5, x_0x_1, x_1x_2, x_4x_0$ to obtain our good coloring of G.

Subcase 3.2. $\alpha \neq 2$.

Just as with x_1y_1, we can recolor x_3y_3 with another color $\beta \neq 2$ and still maintain a good partial coloring of G. By the above subcase, we may assume that $\beta \neq 1$, but it is possible that $\alpha = \beta$. We can extend our good partial coloring of G by coloring x_1x_2, x_4x_5 with 2 and x_2x_3, x_7x_0 with 1. Call this new coloring σ.

Note that $|A_\sigma(x_5x_6)|, |A_\sigma(x_6x_7)| \geq 2$, $|A_\sigma(x_3x_4)|, |A_\sigma(x_0x_1)| \geq 3$ and $|A_\sigma(x_4x_0)| \geq 5$. We can then color $x_5x_6, x_6x_7, x_0x_1, x_3x_4, x_4x_0$ in this order to obtain a good partial coloring of G.

This completes the subcase and so proves the case.

Case 4. $(\phi(x_5y_5), \phi(x_7y_7)) = (2, 1)$.

Again, we recolor x_1y_1 with $\alpha \neq 1$.

Subcase 4.1. $\alpha = 2$.

This subcase is symmetric to Subcase 3.1.

Subcase 4.2. $\alpha \neq 2$.

We can extend our good partial coloring of G by coloring x_1x_2, x_4x_5 with 1 and x_7x_0 with 2. Call this new coloring σ.

Note that $|A_\sigma(x_2x_3)| \geq 1, |A_\sigma(x_5x_6)|, |A_\sigma(x_6x_7)| \geq 2, |A_\sigma(x_0x_1)| \geq 3$, $|A_\sigma(x_3x_4)| \geq 4$ and $|A_\sigma(x_4x_0)| \geq 6$. We can color $x_2x_3, x_5x_6, x_6x_7, x_0x_1, x_3x_4, x_4x_0$ in this order to obtain a good partial coloring of G. This completes the subcase and so completes the case.

Case 5. $(\phi(x_5y_5), \phi(x_7y_7)) = (3, 1)$.

Observe that $|A_\phi(x_ix_{i+1})| \geq 2$ for $i \in \{1, 2, 5, 6\}$, $|A_\phi(x_jx_{j+1})| \geq 4$ for $j \in \{3, 4\}$, $|A_\phi(x_\ell x_{\ell+1})| \geq 5$ for $\ell \in \{0, 7\}$, taken modulo 8 and $A_\phi(x_4x_0) = \{4, 5, 6, 7, 8, 9\}$. We can extend ϕ by coloring x_3x_4, x_7x_0, x_0x_1 with 1,3,2 respectively. We can further extend this new coloring by coloring x_1x_2, x_2x_3 in this order as $|A_\phi(x_1x_2) \setminus \{1, 2, 3\}| \geq 1$ and $|A_\phi(x_2x_3) \setminus \{1, 2, 3\}| \geq 2$. This is another good partial coloring of G, and we will refer to it as σ in this case. Let $\alpha = \sigma(x_2x_3)$, and since x_1x_2 sees $1, 2, 3$, we may assume that $\sigma(x_1x_2) = 4$

Note that $|A_\sigma(x_6x_7)| \geq 1$, $|A_\sigma(x_5x_6)|, |A_\sigma(x_4x_5)| \geq 2$ and $|A_\sigma(x_4x_0)| \geq 5$. We have $A_\sigma(x_6x_7) \subseteq A_\sigma(x_5x_6) = A_\sigma(x_4x_5)$ and $|A_\sigma(x_4x_5)| = 2$, otherwise we obtain a good coloring of G by SDR. So let $A_\sigma(x_5x_6) = A_\sigma(x_4x_5) = \{\beta_1, \beta_2\}$. Note that $1, 2, 3, \alpha \notin \{\beta_1, \beta_2\}$.

Since $|A_\sigma(x_4x_5)| = 2$ and x_4x_5 sees 2 and α, we cannot have $2, \alpha \in \mathcal{U}_\sigma(y_5) \cup \{\sigma(x_6y_6)\}$. As x_5x_6 must also see 2 and α, we have $\Upsilon_\sigma(y_6, x_6) = \{2, \alpha\}$. Thus, $|A_\sigma(x_6x_7)| \geq 2$, and in particular, $A_\sigma(x_6x_7) = \{\beta_1, \beta_2\}$ as $A_\sigma(x_6x_7) \subseteq A_\sigma(x_4x_5)$.

Now, we can return to ϕ and obtain a different partial coloring of G by coloring x_4x_5 with 1, x_5x_6 with β_1, x_6x_7 with β_2, x_7x_0 with 3 and x_0x_1 with 2. This partial coloring is also good, and we will denote it by ψ_1.

Note that $|A_{\psi_1}(x_1x_2)| \geq 1$ and $|A_{\psi_1}(x_2x_3)|, |A_{\psi_1}(x_3x_4)| \geq 2$. As above, we have $A_{\psi_1}(x_1x_2) \subseteq A_{\psi_1}(x_2x_3) = A_{\psi_1}(x_3x_4)$ and $|A_{\psi_1}(x_2x_3)| = 2$, otherwise we obtain a good coloring of G by SDR. As x_3x_4 sees $3, \beta_1$ and $|A_{\psi_1}(x_3x_4)| = 2$, we cannot have $3, \beta_1 \in \mathcal{U}_{\psi_1}(y_3) \cup \{\psi_1(x_2y_2)\}$. However, as $A_{\psi_1}(x_2x_3) = A_{\psi_1}(x_3x_4)$, we have $\Upsilon_{\psi_1}(y_2, x_2) = \{3, \beta_1\}$. Note that $\Upsilon_\phi(y_2, x_2) = \{3, \beta_1\}$ as a result.

Now, if we switch β_1, β_2 so that x_5x_6 is colored with β_2 and x_6x_7 is colored with β_1, we still have a good partial coloring of G, call it ψ_2. The same argument however, shows that $\Upsilon_{\psi_2}(y_2, x_2) = \{3, \beta_2\}$, so that $\Upsilon_\phi(y_2, x_2) = \{3, \beta_2\}$ and $\beta_1 = \beta_2$, a contradiction. This completes the proof of the case.

As we have exhausted all cases, the lemma holds. $\square$

Lemma 5.17. *No 5-face in G can share an edge with a 6-face.*

Proof. Suppose that a 5-face and a 6-face share an edge. By Lemmas 5.7 and 5.11, their boundaries form a 9-cycle, $u_0u_1 \dots u_8u_0$ so that $u_5u_0 \in E(G)$. By Lemmas 5.11 and 5.14, each u_i is a 3-vertex. Additionally, Lemmas 5.7, 5.8 and 5.9 imply that each u_i other than u_5, u_0 has a third neighbor v_i not on the 9-cycle.

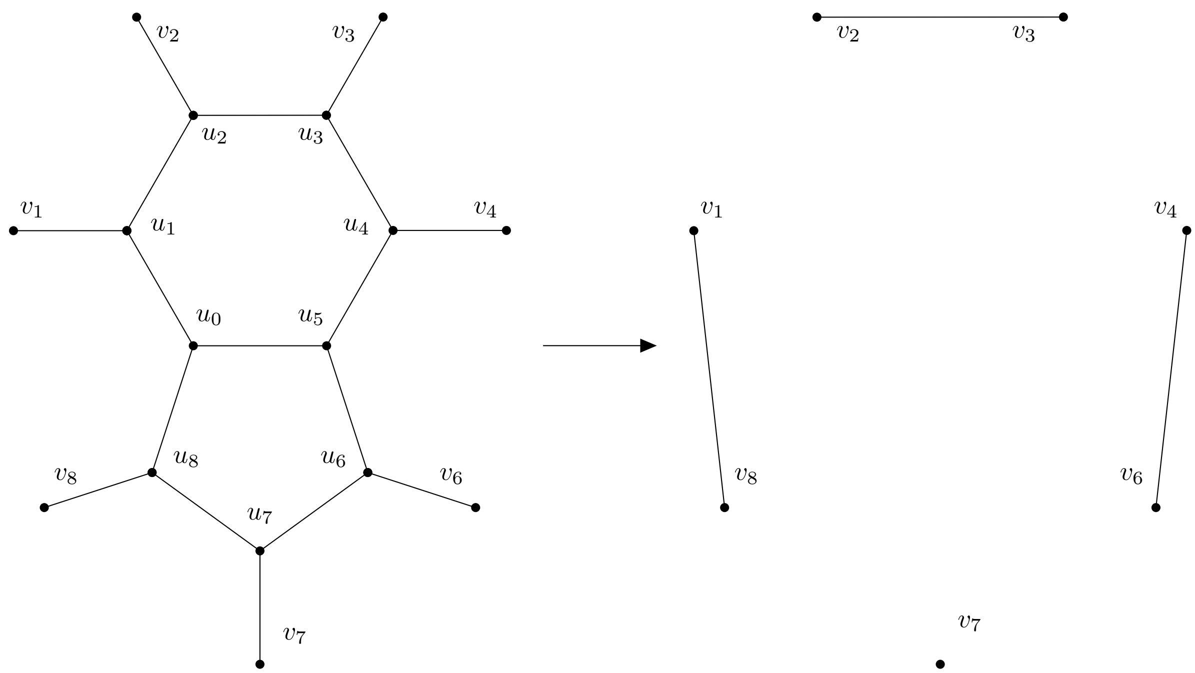


Figure 5.3: Forming G' from G in Lemma 5.17

By these same lemmas, the vertices $v_1, v_2, v_3, v_4, u_6, u_8$ are distinct from each other, as are the vertices u_4, u_1, v_6, v_7, v_8.

By Lemmas 5.8, 5.9, and 5.16, the edges v_2v_3, v_4v_6, v_8v_1 do not exist. So let G' denote the graph obtained from G by deleting $u_1, u_2, \ldots, u_0$ and adding the edges v_2v_3, v_4v_6, v_8v_1 (see Figure 5.3). Observe that G' is a subcubic planar loopless multigraph, and so by the minimality of G, G' has a good coloring. Ignoring v_2v_3, v_4v_6, v_8v_1, we have a good partial coloring of G that we can extend by coloring u_1v_1, u_8v_8 with the same color that v_8v_1 received in G' and u_4v_4, u_6v_6 with the same color that v_4v_6 received in G'. We can further extend this good partial coloring of G by coloring u_2v_2, u_3v_3 and u_7v_7. Call this extended, good partial coloring, ϕ, and let α denote $\phi(u_7v_7)$.

Case 1. $\phi(u_1v_1) \neq \phi(u_4v_4)$.

Without loss of generality, we may assume that $\phi(u_1v_1) = \phi(u_8v_8) = 2$ and $\phi(u_4v_4) = \phi(u_6v_6) = 1$.

Subcase 1.1. $1 \in \Upsilon_\phi(v_1, u_1)$ *and* $2 \in \Upsilon_\phi(v_4, u_4)$.

By the existence of v_4v_6, v_8v_1 in our auxiliary graph G', we cannot have $2 \in \mathcal{U}_\phi(v_6)$ or $1 \in \mathcal{U}_\phi(v_8)$. So, we can extend ϕ to another good partial coloring of G by coloring u_5u_6 with 2 and u_8u_0 with 1. Call this new coloring σ.

Observe $|A_\sigma(u_2u_3)| \geq 1$, $|A_\sigma(u_iu_{i+1})| \geq 2$ for $i \in \{1,3,6,7\}$, $|A_\sigma(u_4u_5)|, |A_\sigma(u_0u_1)| \geq 5$ and $|A_\sigma(u_5u_0)| \geq 7$. Thus, if we can somehow extend σ to a good partial coloring on u_1u_2, u_2u_3, u_3u_4, we can further extend this to a good coloring of G by coloring u_6u_7, u_7u_8, u_4u_5, u_0u_1, u_5u_0 in this order. Thus, it suffices to color u_1u_2, u_2u_3, u_3u_4.

If we cannot, then we have $A_\sigma(u_1u_2) = A_\sigma(u_3u_4)$ and $|A_\sigma(u_1u_2)| = 2$. As $1, 2 \notin A_\sigma(u_1u_2)$, we may assume that $A_\sigma(u_1u_2) = A_\sigma(u_3u_4) = \{8, 9\}$. Additionally, we may assume that $\mathcal{U}_\sigma(v_4) = \{1,2,3\}$, $A_\sigma(v_3) = \{4,5,6\}$ with $\sigma(u_3v_3) = 4$ and $\sigma(u_2v_2) = 7$. Since $A_\sigma(u_1u_2) = A_\sigma(u_3u_4)$, we have 5 or 6 in $\Upsilon_\sigma(v_2, u_2)$. However, v_2v_3 is an edge in our auxiliary graph G' so that $\Upsilon_\sigma(v_2, u_2) \cap \Upsilon_\sigma(v_3, u_3) = \emptyset$, a contradiction.

Subcase 1.2. $1 \in \Upsilon_\phi(v_1, u_1)$, *but* $2 \notin \Upsilon(v_4, u_4)$.

Recall that ϕ colors both u_2v_2 and u_3v_3. In this case, we may choose $\phi(u_3v_3)$ so that $\phi(u_3v_3) \neq 2$. As a result, $2 \in A_\phi(u_4u_5)$. As in Subcase 1.1, we can extend ϕ by coloring u_8u_0 with 1. Call this new, good partial coloring σ. We proceed to prove this subcase by considering whether or not 2 is in $\Upsilon_\sigma(v_3, u_3)$.

Subcase 1.2.1. $2 \notin \Upsilon_\sigma(v_3, u_3)$.

As a result, $2 \in A_\sigma(u_3u_4)$, and we can extend σ by coloring u_3u_4 with 2, and then u_2u_3, u_1u_2 in this order. Call this good partial coloring ψ. Observe that $|A_\psi(u_6u_7)|, |A_\psi(u_7u_8)| \geq 2$, $|A_\psi(u_4u_5)|, |A_\psi(u_0u_1)| \geq 3$, $|A_\psi(u_5u_6)| \geq 4$ and $|A_\psi(u_5u_0)| \geq 6$. If $|A_\psi(u_4u_5) \cup A_\psi(u_7u_8)| \geq 5$, then we obtain a good coloring of G by SDR. Otherwise, there exists some β with which we can color u_4u_5, u_7u_8 and then color $u_6u_7, u_0u_1, u_5u_6, u_5u_0$ in this order to obtain a good coloring of G.

Subcase 1.2.2. $2 \in \Upsilon_\phi(v_3, u_3)$.

Recall that $2 \in A_\sigma(u_4u_5)$. Additionally, we can recolor u_1v_1 with some $\beta \neq 2$ and still maintain a good partial coloring of G. Thus, we adjust σ by recoloring u_1v_1 with β, coloring u_1u_2, u_4u_5 with 2 and then coloring u_2u_3, u_3u_4 in this order. Call this good partial coloring ψ.

Observe that $|A_\psi(u_6u_7)|, |A_\psi(u_7u_8)| \geq 2$, $|A_\psi(u_5u_6)|, |A_\psi(u_0u_1)| \geq 3$ and $|A_\psi(u_5u_0)| \geq 5$. We then color $u_6u_7, u_7u_8, u_5u_6, u_0u_1, u_5u_0$ in this order to obtain a good coloring of G.

This completes the subcase, and by symmetry, it remains to consider the following subcase.

Subcase 1.3. $1, 2 \notin \Upsilon_\phi(v_1, u_1) \cup \Upsilon_\phi(v_4, u_4)$.

Just as in Subcase 1.2, we may assume that $\phi(u_3v_3) \neq 2$, and as a result, $2 \in A_\phi(u_4u_5)$. We proceed to prove this final subcase based on the color of $\phi(u_2v_2)$.

Subcase 1.3.1. $\phi(u_2v_2) \neq 1$.

As a result, $1 \in A_\phi(u_0u_1)$. Additionally, there exists some color in $A_\phi(u_2u_3)$. Thus, we can extend ϕ to another good partial coloring of G by coloring u_1u_0 with 1, u_4u_5 with 2 and then coloring u_2u_3 with some available color. We can further extend ϕ by coloring u_6u_7 and u_7u_8 with some β and γ, respectively. Call this good partial coloring σ.

Now, we can choose β and γ such that either $\{\alpha, \beta\} \neq \Upsilon_\sigma(v_1, u_1)$ or $\{\alpha, \gamma\} \neq \Upsilon_\sigma(v_4, u_4)$. We show the former as the latter is done by a similar argument. Since $|A_\phi(u_6u_7)|, |A_\phi(u_7u_8)| \geq 2$, if $\alpha \notin \Upsilon_\phi(v_1, u_1)$, then we are done, and if $\alpha \in \Upsilon_\phi(v_1, u_1)$, then we can choose β from $A_\phi(u_6u_7) \setminus \Upsilon_\phi(v_1, u_1)$.

Now, if $\Upsilon_\phi(v_1, u_1) \cap \Upsilon_\phi(v_4, u_4) = \emptyset$, then we can choose β and γ such that both $\Upsilon_\phi(v_1, u_1) \neq \{\alpha, \beta\}$ and $\Upsilon_\phi(v_4, u_4) \neq \{\alpha, \gamma\}$. Indeed, if $\alpha \notin \Upsilon_\phi(v_1, u_1) \cup \Upsilon_\phi(v_4, u_4)$, then we are done. So either $\alpha \in \Upsilon_\phi(v_1, u_1) \setminus \Upsilon_\phi(v_4, u_4)$ or $\alpha \in \Upsilon_\phi(v_4, u_4) \setminus \Upsilon_\phi(v_1, u_1)$. If the former holds, then we proceed as above since we are guaranteed that $\{\alpha, \gamma\} \neq \Upsilon_\phi(v_4, u_4)$, and a similar argument holds in the latter case.

In Subcase 1.3.1, we will assume that β, γ are chosen so that $\{\alpha, \gamma\} \neq \Upsilon_\phi(v_4, u_4)$. Additionally, as $\sigma(u_2v_2), \sigma(u_2u_3), \sigma(u_3v_3) \notin \{1, 2\}$ and are distinct from each other, we may assume that $\sigma(u_3v_3) = 3, \sigma(u_2v_2) = 4$ and $\sigma(u_2u_3) = 5$.

Since $A_\sigma(u_1u_2)$ and $A_\sigma(u_3u_4)$ are possibly empty, we proceed by considering whether they are empty or not.

Subcase 1.3.1.1. $A_\sigma(u_1u_2) = A_\sigma(u_3u_4) = \emptyset$.

As u_1u_2, u_3u_4 each see all nine colors and v_2v_3 was an edge of G', we may assume that $\Upsilon_\sigma(v_1, u_1) = \Upsilon_\sigma(v_3, u_3) = \{6, 7\}$ and $\Upsilon_\sigma(v_2, u_2) = \Upsilon(v_4, u_4) = \{8, 9\}$. Therefore, we can adjust σ by uncoloring u_0u_1, u_4u_5 and then coloring u_1u_2 and u_3u_4 with 1 and 2, respectively. Call this good partial coloring ψ. Since $\Upsilon_\sigma(v_1, u_1) \cap \Upsilon_\sigma(v_4, u_4) = \emptyset$, we can assume that β, γ were chosen so that $\{\alpha, \beta\} \neq \{6, 7\}$ and $\{\alpha, \gamma\} \neq \{8, 9\}$.

Note that $|A_\psi(u_iu_{i+1})| \geq 2$ for $i \in \{0, 4, 5, 8\}$, taken modulo 9 and $|A_\psi(u_5u_0)| \geq 5$. In particular, $A_\psi(u_4u_5) \subseteq \{4, 6, 7\}$ and $A_\psi(u_0u_1) \subseteq \{3, 8, 9\}$ so that $|A_\psi(u_4u_5) \cup A_\psi(u_0u_1)| \geq 4$.

Now, suppose $A_\psi(u_4u_5) = A_\psi(u_5u_6)$ and $|A_\psi(u_4u_5)| = 2$. As u_4u_5 sees edges colored 8 and 9, and $\Upsilon_\psi(v_4, u_4) \cap \Upsilon_\psi(v_6, u_6) = \emptyset$, we have $8, 9 \in \{\alpha, \beta, \gamma\}$. However, as $|A_\psi(u_4u_5)| = 2$, $\beta \notin \{8, 9\}$ so that $\{8, 9\} = \Upsilon_\psi(v_4, u_4) = \{\alpha, \gamma\}$, a contradiction. Thus, we have $|A_\psi(u_4u_5) \cup A_\psi(u_5u_6)| \geq 3$, and by a symmetric argument, $|A_\psi(u_0u_1) \cup A_\psi(u_8u_0)| \geq 3$. Thus, we obtain a good coloring of G by SDR.

Subcase 1.3.1.2. *There exists* $\delta \in A_\sigma(u_1u_2)$ *and* $A_\sigma(u_3u_4) = \emptyset$.

As u_3u_4 sees all nine colors, we may assume that $\Upsilon_\sigma(v_3, u_3) = \{6, 7\}$ and $\Upsilon_\sigma(v_4, u_4) = \{8, 9\}$. We can adjust σ by uncoloring u_4u_5 and then coloring u_3u_4 with 2 and u_1u_2 with δ. Call this good partial coloring ψ.

Observe that $|A_\psi(u_8u_0)| \geq 1$, $|A_\psi(u_4u_5)|, |A_\psi(u_5u_6)| \geq 2$ and $|A_\psi(u_5u_0)| \geq 4$. If $|A_\psi(u_4u_5) \cup A_\psi(u_5u_6)| \geq 3$, then we obtain a good coloring of G by SDR. So we have $A_\psi(u_4u_5) = A_\psi(u_5u_6)$ and $|A_\psi(u_4u_5)| = 2$. However, a similar argument to that used in Subcase 1.3.1.1 implies that $\{\alpha, \gamma\} = \Upsilon_\psi(v_4, u_4)$, a contradiction.

Subcase 1.3.1.3. *There exists $\epsilon \in A_\sigma(u_3u_4)$ and $A_\sigma(u_1u_2) = \emptyset$.*

Note that the choice of β and γ does not affect $A_\sigma(u_1u_2)$ or $A_\sigma(u_3u_4)$. Thus, we can rechoose β and γ, if necessary, so that $\{\alpha, \beta\} \neq \Upsilon_\phi(v_1, u_1)$. We then repeat a symmetric argument to the above.

Subcase 1.3.1.4. *There exist $\delta \in A_\sigma(u_1u_2)$ and $\epsilon \in A_\sigma(u_3u_4)$.*

Suppose first that $2 \notin \Upsilon_\sigma(v_3, u_3)$. We can adjust σ by uncoloring u_4u_5 and then coloring u_3u_4 with 2 and u_1u_2 with δ. From here, the argument is identical to that in Subcase 1.3.1.2. Thus, $2 \in \Upsilon_\sigma(v_3, u_3)$. By symmetry, we also have $1 \in \Upsilon_\sigma(v_2, u_2)$.

We can adjust σ by uncoloring u_2u_3 and then coloring u_3u_4, u_1u_2, u_2u_3 in this order. As each of these edges sees 1, 2, 3, and 4, we may assume that they are colored 5, 6, 7, respectively. Call this good partial coloring ψ. Observe that $|A_\psi(u_5u_6)|, |A_\psi(u_8u_0)| \geq 1$ and $|A_\psi(u_5u_0)| \geq 3$.

If $|A_\psi(u_5u_6) \cup A_\psi(u_8u_0)| \geq 2$, then we obtain a good coloring of G by SDR. So we have $A_\psi(u_5u_6) = A_\psi(u_8u_0) = \{\zeta\}$. Since u_5u_6 sees an edge colored 5, we cannot have $5 \in \{\alpha, \beta, \gamma\}$. Since $A_\psi(u_8u_0) = A_\psi(u_5u_6)$, u_8u_0 also sees 5, and so, $5 \in \Upsilon_\psi(v_8, u_8)$. Since v_8v_1 is an edge of G', we cannot have $5 \in \Upsilon_\psi(v_1, u_1)$. Similarly, as $|A_\psi(u_8u_0)| = 1$ and u_8u_0 sees 1, we cannot have $1 \in \Upsilon_\psi(v_8, u_8)$.

Thus, if we recolor u_0u_1 with 5, color u_8u_0 with 1, we can than color u_5u_6 and u_5u_0 in this order to obtain a good coloring of G.

This completes the proof of Subcase 1.3.1.

Subcase 1.3.2. $\phi(u_2v_2) = 1$.

We can extend ϕ to a good partial coloring of G, call it σ, such that u_4u_5 is colored with 2, and u_6u_7 and u_7u_8 are colored with β and γ, respectively. Just as in Subcase 1.3.1, we can choose β, γ so that $\{\alpha, \beta\} \neq \Upsilon_\sigma(v_1, u_1)$, and additionally require that $\{\alpha, \gamma\} \neq \Upsilon_\sigma(v_4, u_4)$ when $\Upsilon_\sigma(v_1, u_1) \cap \Upsilon_\sigma(v_4, u_4) = \emptyset$. Also, as $\sigma(u_3v_3) \neq 2$, we may assume that $\sigma(u_3v_3) = 3$.

Note that here, σ does not color u_2u_3. Thus, we proceed based on whether or not we can extend σ to u_1u_2, u_2u_3, u_3u_4.

Subcase 1.3.2.1. *We cannot extend σ by coloring u_1u_2, u_2u_3, u_3u_4.*

As $|A_\sigma(u_iu_{i+1})| \geq 2$ for $i \in [3]$, we may assume that $A_\sigma(u_1u_2) = A_\sigma(u_2u_3) = A_\sigma(u_3u_4) = \{4,5\}$. So without loss of generality, $\Upsilon_\sigma(v_2,u_2) = \Upsilon_\sigma(v_4,u_4) = \{8,9\}$ and $\Upsilon_\sigma(v_1,u_1) = \Upsilon_\sigma(v_3,u_3) = \{6,7\}$. Recall that just as in Subcase 1.3.1, $\Upsilon_\sigma(v_1,u_1)\cap\Upsilon_\sigma(v_4,u_4) = \emptyset$, we may assume $\{\alpha,\beta\} \neq \{6,7\}$ and $\{\alpha,\gamma\} \neq \{8,9\}$.

Now, we can adjust σ by uncoloring u_4u_5, coloring u_3u_4 with 2, and then coloring u_1u_2, u_2u_3 from $\{4,5\}$ so that u_1u_2 is not colored with β. We call this good partial coloring of G, ψ, and we may assume that $\psi(u_1u_2) = 4, \psi(u_2u_3) = 5$.

Observe that $|A_\psi(u_iu_{i+1})| \geq 2$ for $i \in \{0,4,5,8\}$, taken modulo 9 and $|A_\psi(u_5u_0)| \geq 4$. In particular, $A_\psi(u_4u_5) \subseteq \{4,6,7\}$, $A_\psi(u_0u_1) \subseteq \{3,8,9\}$ and $|A_\psi(u_4u_5) \cup A_\psi(u_0u_1)| \geq 4$. Now, as $\beta \neq 4$, we have $4 \in A_\psi(u_4u_5)$, and additionally, $4 \notin A_\psi(u_8u_0) \cup A_\psi(u_0u_1)$.

Also, $|A_\psi(u_8u_0) \cup A_\psi(u_0u_1)| \geq 3$, otherwise we can apply an argument similar to that used in Subcase 1.3.1.1 to show that $\{\alpha,\beta\} = \Upsilon_\psi(v_1,u_1)$, a contradiction. Thus, we can color u_4u_5 with 4, and then obtain a good coloring of G by SDR from the rest.

Subcase 1.3.2.2. *We can extend σ by coloring u_1u_2, u_2u_3, u_3u_4.*

Without loss of generality, we may assume that u_1u_2, u_2u_3, u_3u_4 are colored with $4,5,6$, respectively, and call this good partial coloring ψ. Observe that $|A_\psi(u_5u_6)| \geq 1$, $|A_\psi(u_8u_0)|$, $|A_\psi(u_0u_1)| \geq 2$ and $|A_\psi(u_5u_0)| \geq 3$. Additionally, $|A_\psi(u_8u_0) \cup A_\psi(u_0u_1)| \geq 3$, otherwise we can apply an argument similar to that used in Subcase 1.3.1.1 to show that $\{\alpha,\beta\} = \Upsilon_\psi(v_1,u_1)$ (observe that $|A_\psi(u_0u_1)| = 2$ implies that $|\Upsilon_\psi(v_1,u_1) \cup \{1,2,4,5,\gamma\}| = 7$).

First, $\beta,\gamma \notin \{4,6\}$, otherwise $|A_\psi(u_5u_0)| \geq 4$, and we obtain a good coloring of G by SDR.

Additionally, $1 \in \Upsilon_\psi(v_8,u_8)$, otherwise we can color u_8u_0 with 1 and then color u_5u_6, u_0u_1, u_5u_0 in this order to obtain a good coloring of G.

We claim $6 \in \Upsilon_\psi(v_1,u_1)$. If on the contrary, $6 \notin \Upsilon_\psi(v_1,u_1)$, then as $\gamma \neq 6$, we could color u_0u_1 with 6. Then we have $A_\psi(u_5u_6) = \{\delta\}$ and $A_\psi(u_8u_0) = \{6,\delta\}$, otherwise we could color u_5u_6, u_8u_0, u_5u_0 in this order to obtain a good coloring of G. However, since $|A_\psi(u_8u_0)\cup A_\psi(u_0u_1)| \geq 3$ (so that $A_\psi(u_0u_1) \neq \{6,\delta\}$), we can color u_5u_6 with δ, u_8u_0 with 6 and then color u_0u_1, u_5u_0 in this order to obtain a good coloring of G.

We may also assume that $\alpha = 6$. Observe that $6 \notin \{\beta,\gamma\}$, and as v_8v_1 is an edge of G', $6 \notin \Upsilon_\psi(v_8,u_8)$. Thus, if $\alpha \neq 6$, we can color u_8u_0 with 6 and then color u_5u_6, u_0u_1, u_5u_0 in this order to obtain a good coloring of G.

Now, we also have $4 \in \Upsilon_\psi(v_6,u_6)$. If not, then since $4 \notin \{\beta,\gamma\}$, we can color u_5u_6 with 4 and then color u_0u_1, u_8u_0, u_5u_0 in this order to obtain a good coloring of G. As v_4v_6 is an edge of G', we have $4 \notin \Upsilon_\psi(v_4,u_4)$.

Lastly, we claim that $2 \in \Upsilon_\psi(v_6, u_6)$. If not, then we can recolor u_4u_5 with 4, color u_5u_6 with 2 and then color u_8u_0, u_0u_1, u_5u_0 in this order to obtain a good coloring of G.

Now, we uncolor the edges u_6u_7, u_7u_8, and call this new coloring τ. Observe that $|A_\tau(u_iu_{i+1})| \geq 3$ for $i \in \{0, 6, 7\}$, taken modulo 9, $|A_\tau(u_8u_0)| \geq 4$ and $|A_\tau(u_5u_6)|, |A_\tau(u_5u_0)| \geq 5$. If $|A_\tau(u_6u_7) \cup A_\tau(u_0u_1)| \geq 6$, then we obtain a good coloring of G by SDR. Thus, there exists some ϵ such that we can color u_6u_7, u_0u_1 with ϵ and then color $u_7u_8, u_5u_6, u_8u_0, u_5u_0$ in this order to obtain a good coloring of G.

This completes all subcases of Case 1.

Case 2. $\phi(u_1v_1) = \phi(u_4v_4)$.

Without loss of generality, we may assume that $\phi(u_iv_i) = 1$ for $i \in \{1, 4, 6, 8\}$, $\phi(u_2v_2) = 2$ and $\phi(u_3v_3) = 3$.

Subcase 2.1. *We can extend ϕ by coloring u_1u_2, u_2u_3, u_3u_4.*

Let us extend ϕ by coloring u_1u_2, u_2u_3, u_3u_4, and then uncolor u_7v_7. Call this new good partial coloring σ. Without loss of generality, we may assume that $\sigma(u_1u_2) = 4, \sigma(u_2u_3) = 5, \sigma(u_3u_4) = 6$.

Subcase 2.1.1. *Either $6 \notin \Upsilon_\sigma(v_1, u_1)$ or $4 \notin \Upsilon_\sigma(v_4, u_4)$.*

By symmetry, we may assume that $4 \notin \Upsilon_\sigma(v_4, u_4)$. As a result, we can extend σ by coloring u_4u_5 with 4. Call this good partial coloring ψ. Note that $|A_\psi(u_7v_7)| \geq 2$, $|A_\psi(u_6u_7)|, |A_\psi(u_0u_1)| \geq 3$, $|A_\psi(u_5u_6)|$, $|A_\psi(u_7u_8)| \geq 4$, $|A_\psi(u_0u_1)| \geq 5$ and $|A_\psi(u_5u_0)| \geq 6$.

First, we show that $|A_\psi(u_7v_7) \cup A_\psi(u_0u_1)| \geq 5$. If not, then we can color u_7v_7, u_0u_1 with some β and then color $u_6u_7, u_5u_6, u_7u_8, u_8u_0, u_5u_0$ in this order to obtain a good coloring of G. In a similar manner, we show that $|A_\psi(u_6u_7) \cup A_\psi(u_0u_1)| \geq 6$ by otherwise coloring u_6u_7, u_0u_1 with some γ, and then coloring $u_7v_7, u_7u_8, u_5u_6, u_8u_0, u_5u_0$ in this order to obtain our good coloring of G.

Now, if $|A_\psi(u_7v_7) \cup A_\psi(u_5u_0)| \geq 7$, then we can obtain a good coloring of G by SDR. Otherwise, we can color u_7v_7, u_5u_0 with some δ, and then obtain a good coloring of G by SDR from the remaining edges using the above.

Subcase 2.1.2. $6 \in \Upsilon_\sigma(u_1v_1)$ *and* $4 \in \Upsilon_\sigma(u_4v_4)$.

We first note that there exists $\beta \in A_\sigma(u_7v_7) \setminus \{4\}$ and that $4 \in A_\sigma(u_5u_6)$. Thus, we can obtain another good partial coloring of G by coloring u_5u_6 with 4 and u_7v_7 with β. Call this new coloring ψ. Observe $|A_\psi(u_6u_7)|, |A_\psi(u_7u_8)| \geq 2$, $|A_\psi(u_4u_5)|, |A_\psi(u_0u_1)| \geq 3$, $|A_\psi(u_8u_0)| \geq 4$, and $|A_\psi(u_5u_0)| \geq 6$.

First, if $|A_\psi(u_6u_7) \cup A_\psi(u_0u_1)| \geq 5$, then we obtain a good coloring of G by SDR. Thus, there exists some $\gamma \in A_\psi(u_6u_7) \cap A_\psi(u_0u_1)$ so that we can color u_6u_7, u_0u_1 with γ and then color $u_7u_8, u_8u_0, u_4u_5, u_5u_0$ in this order to obtain a good coloring of G.

Subcase 2.2. *We cannot extend ϕ by coloring u_1u_2, u_2u_3, u_3u_4.*

As $|A_\phi(u_iu_{i+1})| \geq 2$ for $i \in \{1, 2, 3\}$, we may assume that $A_\phi(u_iu_{i+1}) = \{8, 9\}$ for $i \in \{1, 2, 3\}$. Thus, without loss of generality, $\Upsilon_\phi(v_2, u_2) = \Upsilon_\phi(v_4, u_4) = \{4, 5\}$ and $\Upsilon_\phi(v_1, u_1) = \Upsilon_\phi(v_3, u_3) = \{6, 7\}$. We can recolor u_4v_4 with some $\beta \neq 1$ and still maintain a good partial coloring of G.

Thus, we can obtain another good partial coloring of G by first recoloring u_4v_4 with β, color u_3u_4 with 1 and then color u_2u_3, u_1u_2 in this order. As in Subcase 2.1, we also uncolor u_7v_7, and call this new coloring σ. Note that $\{\sigma(u_1u_2), \sigma(u_2u_3)\} = \{8, 9\}$, and so without loss of generaltiy, $\sigma(u_1u_2) = 8, \sigma(u_2u_3) = 9$.

Subcase 2.2.1. *$\beta \neq 8$.*

As $8 \in A_\phi(u_3u_4)$, we cannot have $8 \in \mathcal{U}_\sigma(y_4)$. Thus, we can extend σ by coloring u_4u_5 with 8 and then proceed in the same way as in Subcase 2.1.1 replacing 8 with 4.

Subcase 2.2.2. *$\beta = 8$.*

By the existence of v_8v_1 in our auxiliary graph G, $6 \in \Upsilon_\sigma(v_1, u_1)$ implies that $6 \notin \Upsilon_\sigma(v_1, u_1)$ so that $6 \in A_\sigma(u_8u_0)$. Note that there exists some $\gamma \in A_\sigma(u_7v_7) \setminus \{6\}$.

We can then extend σ to another good coloring of G by coloring u_7v_7 with γ and u_8u_0 with 6. Call this ψ. Observe that $A_\psi(u_4u_5) = \{2, 7\}, A_\psi(u_0u_1) = \{3, 4, 5\}$, $|A_\psi(u_6u_7)|$, $|A_\psi(u_7u_8)| \geq 2$, $|A_\psi(u_5u_6)| \geq 3$ and $|A_\psi(u_5u_0)| \geq 6$. As $A_\psi(u_4u_5) \cap A_\psi(u_0u_1) = \emptyset$, coloring u_4u_5 does not affect coloring u_0u_1.

Now, if $|A_\psi(u_4u_5) \cup A_\psi(u_7u_8)| \geq 4$, we can color $u_4u_5, u_5u_6, u_6u_7, u_7u_8$ by SDR and then color u_0u_1, u_5u_0 in this order to obtain a good coloring of G. Thus, there exists some δ so that we can color u_4u_5, u_7u_8 with δ and then color $u_6u_7, u_5u_6, u_0u_1, u_5u_0$ in this order to obtain a good coloring of G.

This completes the proof of the final subcase of Case 2, and so proves the lemma. □

5.6 Proof of Theorem 5.1

We are now ready to prove Theorem 5.1 via discharging using the lemmas from Sections 5.3, 5.4 and 5.5,

Proof. By Euler's formula,

$$\sum_{v \in V(G)} (2d(v) - 6) + \sum_{f \in F(G)} (d(f) - 6) = -12.$$

Thus, if we assign to each vertex v the initial charge $2d(v)-6$ and to each face f the initial charge $d(f)-6$, then the total charge will be -12. We design appropriate discharging rules and redistribute charges among faces and vertices so that the final charge of every face and every vertex is nonnegative, a contradiction.

Discharging Rules:

(R1) Every 2-vertex receives 1 from each incident face.

(R2) Every 5-face receives $\frac{1}{5}$ from each adjacent face.

By Rule (R1), at the end of discharging, each 2-vertex will have charge $-2+1+1=0$. The charge of each 3-vertex does not change and remains 0.

By Rule (R2) and Lemmas 5.11 and 5.16, the final charge of every 5-face is $5-6+5\times\frac{1}{5}=0$.

By Lemmas 5.14 and 5.17, each 6-face gives no charge. Thus, as it starts with zero charge and does not receives any charge, the final charge is zero.

By Lemmas 5.15 and 5.16, each 7-face contains only 3-vertices and is adjacent to at most three 5-faces. Thus, the final charge is at least $7-6-3\times\frac{1}{5}=\frac{2}{5}$.

By Lemmas 5.16 and 5.13, each k-face, $k\geq 8$, is adjacent to at most $\lfloor\frac{k}{2}\rfloor$ 5-faces and contains at most $\lfloor\frac{k}{5}\rfloor$ 2-vertices on its boundary. Thus, the final charge is at least $k-6-\lfloor\frac{k}{5}\rfloor\times 1-\lfloor\frac{k}{2}\rfloor\times\frac{1}{5}$, which is positive for $k\geq 8$.

This completes the proof. $\square$

5.7 Future Questions

There are many unresolved and interesting problems in regards to strong edge-colorings aside from the original conjecture of Erdős and Nešetřil (see Conjecture 1.19) and the remaining conjectures of Faudree et al. (see Conjecture 1.20). These include list versions and on-line versions that can be considered.

However, there is still a question in regards to the sharpness of Theorem 5.1. As mentioned, $K_3\square P_2$ is a subcubic planar loopless multigraph whose strong chromatic index is 9. However, we know of no other subcubic planar loopless multigraphs that require 9 colors in a strong edge-coloring. Therefore, it is unknown as to whether or not we can improve upon Theorem 5.1 by excluding this single example or perhaps a finite family of graphs.

Lastly, it seems possible to improve upon Theorem 5.1 by relaxing the maximum degree condition. Specifically, it seems likely that if a planar loopless multigraph G satisfies $d(x)+d(y)\leq 6$ for all $xy\in E(G)$, then $\chi'_s(G)\leq 9$. This degree-sum condition is shown to be best possible by $K_{2,5}$, since $\chi'_s(K_{2,5})=10$, and it contains two adjacent vertices whose degrees sum to 7.

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