K-THEORETIC SCHUBERT CALCULUS AND APPLICATIONS

BY

OLIVER PECHENIK

DISSERTATION

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Mathematics
in the Graduate College of the
University of Illinois at Urbana-Champaign, 2016

Urbana, Illinois

Doctoral Committee:

Professor Hal Schenck, Chair
Associate Professor Alexander Yong, Director of Research
Professor Philippe Di Francesco
Professor Rinat Kedem

Abstract

A central result in algebraic combinatorics is the Littlewood-Richardson rule that governs products in the cohomology of Grassmannians. A major theme of the modern Schubert calculus is to extend this rule and its associated combinatorics to richer cohomology theories.

This thesis focuses on K-theoretic Schubert calculus. We prove the first Littlewood-Richardson rule in torus-equivariant K-theory. We thereby deduce the conjectural rule of H. Thomas and A. Yong, as well as a mild correction to the conjectural rule of A. Knutson and R. Vakil. Our rule manifests the positivity established geometrically by D. Anderson, S. Griffeth and E. Miller, and moreover in a stronger 'squarefree' form that resolves an issue raised by A. Knutson. Our work is based on the combinatorics of genomic tableaux, which we introduce, and a generalization of M.-P. Schützenberger's jeu de taquin. We further apply genomic tableaux to obtain new rules in non-equivariant K-theory for Grassmannians and maximal orthogonal Grassmannians, as well as to make various conjectures relating to Lagrangian Grassmannians. This is joint work with Alexander Yong.

Our theory of genomic tableaux is a semistandard analogue of the increasing tableau theory initiated by H. Thomas and A. Yong. These increasing tableaux carry a natural lift of M.-P. Schützenberger's promotion operator. We study the orbit structure of this action, generalizing a result of D. White by establishing an instance of the cyclic sieving phenomenon of V. Reiner, D. Stanton and D. White. In joint work with J. Bloom and D. Saracino, we prove a homomesy conjecture of J. Propp and T. Roby for promotion on standard tableaux, which partially generalizes to increasing tableaux. In joint work with K. Dilks and J. Striker, we relate the action of K-promotion on increasing tableaux to the rowmotion operator on plane partitions, yielding progress on a conjecture of P. Cameron and D. Fon-der-Flaass. Building on this relation between increasing tableaux and plane partitions, we apply the K-theoretic jeu de taquin of H. Thomas and A. Yong to give, in joint work with Z. Hamaker, R. Patrias and N. Williams, the first bijective proof of a 1983 theorem of R. Proctor, namely that that plane partitions of height k in a rectangle are equinumerous with plane partitions of height k in a trapezoid.

Acknowledgments

First thanks go to my advisor, Alexander Yong, for years of incredibly dedicated support and guidance. He has been an exceptional mentor and I have benefited immensely from our innumerable conversations. I am very grateful for the amount of time and energy he has contributed toward my success.

I would also like the thank the other members of my thesis committee: Philippe Di Francesco, Rinat Kedem and Hal Schenck. In addition to their service on my committee, all of them have been valuable to me through their teaching and through many less formal conversations. Other faculty whose teaching at Illinois I have particularly benefited from include Jayadev Athreya, Maarten Bergvelt, Eugene Lerman, Tom Nevins, Charles Rezk, and Doug West. Outside of Illinois, I had valuable interactions with more mathematicians than I can possibly name. Among the Illinois students, I would like to single out for gratitude my mathematical siblings Cara Monical, Dominic Searles, and Anna Weigandt.

I am especially grateful to my coauthors Jonathan Bloom, Kevin Dilks, Zachary Hamaker, Rebecca Patrias, Dan Saracino, Jessica Striker, Nathan Williams, and Alexander Yong for their collaboration on various projects described in this thesis. I have learned a lot from working with all of them.

Finally I want to my family, especially my wife Ardea Thurston-Shaine.

This research was supported by an Illinois Distinguished Fellowship from the University of Illinois at Urbana-Champaign, a National Science Foundation Graduate Research Fellowship, and National Science Foundation Grant DMS 0838434. I also benefited from travel funding provided by National Science Foundation Grants DMS 1201595 and DMS 1500691.

Table of Contents

Chapter 1

Introduction

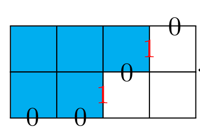
This thesis studies the combinatorics of Young tableaux and their relations to the Schubert calculus of the Grassmannian and related spaces. In this introductory chapter, we recall relevant known work from the literature. Our main references for this chapter are the textbooks [Fu97, Sta99, Man01, Mac95, Mac98].

1.1 The Grassmannian

Let $X = \mathrm{Gr}_k(\mathbb{C}^n)$ be the **Grassmannian** of k-dimensional linear subspaces of $\mathbb{C}^n$. The defining action of $\mathsf{GL}_n(\mathbb{C})$ on $\mathbb{C}^n$ passes to an action on X. We may restrict this action to the Borel subgroup B of invertible upper triangular matrices or further to the maximal torus T of invertible diagonal matrices.

It is easy to see that a k-dimensional linear subspace of $\mathbb{C}^n$ is T-stable exactly when it is a coordinate hyperplane. Hence the action of T on X has exactly $\binom{n}{k}$ fixed points. Letting $\{e_1, \ldots, e_n\}$ denote the standard basis of $\mathbb{C}^n$, we may usefully index a T-stable hyperplane H and its corresponding T-fixed point $p \in X$ by the binary sequence recording which e_i are in H. For example, if $n = 6$ and $H = \langle e_2, e_4 \rangle$, then we label H as 010100.

Alternatively, we may substitute these binary sequences by partitions inside a $k \times (n-k)$ rectangle in the following way. Starting at the upper right corner of the $k \times (n-k)$ rectangle, construct a lattice path from the binary sequence by reading each 0 as the segment $(-1, 0)$ and each 1 as the segment $(0, -1)$. Since the number of 0's is k and the number of 1's is $n-k$, this lattice path necessarily ends at the lower left corner of the rectangle. The region of the rectangle northwest of this path is naturally (the Young diagram of) a partition. Continuing the example above, we obtain the partition $(3, 2)$ as the index for H:



.

Leaving our focus on T-stable sets, a general k-dimensional linear subspace of $\mathbb{C}^n$ can be specified by giving k vectors that span it, say as a $k \times n$ matrix of rank k. This assignment of a matrix A to a linear subspace H is not unique, as we may replace A by any other matrix of the same shape whose rows span H. The set of such potential replacements is precisely the set of matrices related to A by row operations. Thus, a canonical matrix representative for H is given by choosing $A = A_H$ to be in reduced row-echelon form. For example, if $\hat{H} = \langle (2,1,0,4),(2,1,1,5)\rangle$, then

$$A_{\hat{H}} = \begin{bmatrix} 1 & \frac{1}{2} & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

as may be found by performing row reductions on

$$\begin{bmatrix} 2 & 1 & 0 & 4 \\ 2 & 1 & 1 & 5 \end{bmatrix}.$$

The matrix A_H then has k pivot columns, each containing a single 1 and $k-1$ 0's (for $A_{\hat{H}}$, columns 1 and 3). To the left of each pivot 1, the remaining entries of each row are 0's. Deleting the pivot columns of A_H, these forced 0's naturally form the (mirror image of the) shape of a partition λ_H. (In $A_{\hat{H}}$, only the 0 in position $(2,2)$ is left of a pivot 1 and not in a pivot column; hence $\lambda_{\hat{H}} = \square$.) All other entries of A_H are arbitrary complex numbers, which specify the subspace H. Since T acts on H by scaling the columns of A_H, H is T-stable if and only if these arbitrary complex numbers are in fact 0. In this case, λ_H is the partition previously associated to the stable subspace H.

Fixing a set of k columns to be the pivots, we obtain the set of all H such that A_H has those pivots. By the above discussion, this set is an affine cell in the Grassmannian. Each such cell contains a single T-fixed point. Indexing the cell by the corresponding partition λ, we call it the **Schubert cell** X_λ°. The dimension of X_λ° is the number of free entries in A_H, i.e. $kn - k^2 - |\lambda|$, where $|\lambda|$ is the size of the partition λ. The cell containing $\hat{H}$, for example, is $X_{\lambda_{\hat{H}}}^\circ$, given by fixing columns 1 and 3 to be pivots, and hence is a 3-dimensional affine cell consisting of all H for which A_H is of the form

$$\begin{bmatrix} 1 & \star & 0 & \star \\ 0 & 0 & 1 & \star \end{bmatrix},$$

where the $\star$'s denote arbitrary independently-chosen complex numbers. The unique T-fixed point in $X_{\lambda_{\hat{H}}}^\circ$ is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

For $\lambda = (0)$, X_λ° is a dense open subset of X, so $\dim X = kn - k^2 - 0 = k(n-k)$. Thus in general the codimension of X_λ° inside X is $|\lambda|$. Note that the Schubert cells X_λ° are exactly the orbits of B on X, acting on matrix representatives by rightward column operations.

The closure of X_λ° in X is the **Schubert variety** X_λ. It is not hard to see that as a set

$$X_\lambda = \bigsqcup_{\lambda \subset \mu} X_\mu,$$

where $\lambda \subset \mu$ denotes containment of Young diagrams. A key feature of the Schubert varieties is that, since the Schubert cells give a cell decomposition of X, their Poincaré duals—the **Schubert classes** $\{\sigma_\lambda\}_{\lambda \subseteq k \times (n-k)}$—are a $\mathbb{Z}$-linear basis for the cohomology ring $H^\star(X)$ with σ_λ in degree $|\lambda|$.

Thus understanding the structure of $H^\star(X)$ as a graded $\mathbb{Z}$-module reduces to understanding the set of partitions contained in the rectangle $k \times (n-k)$. Let $[n]_q$ be the **q-integer** $[n]_q = 1 + q + q^2 + \cdots + q^{n-1}$. Observe that setting $q \mapsto 1$ recovers the ordinary integer n. Further define $[n]!_q := [n]_q[n-1]_q \dots [1]_q$ and $\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]!_q}{[k]!_q[n-k]!_q}$ by analogy with the standard formulas for factorials and binomial coefficients. Then it is an easy induction that the generating function for partitions in $k \times (n-k)$ by size (and hence the Poincaré polynomial of X) is $\begin{bmatrix} n \\ k \end{bmatrix}_q$. (Taking $q \mapsto 1$, this recovers that the total number of such partitions is $\binom{n}{k}$.)

Having made completely explicit the $\mathbb{Z}$-module structure of $H^\star(X)$, a natural next step in elucidating X is to determine the multiplication on $H^\star(X)$. Since we have a basis $\{\sigma_\lambda\}$, it suffices to determine the structure coefficients $c_{\lambda,\mu}^\nu$ defined by

$$\sigma_\lambda \smile \sigma_\mu = \sum_\nu c_{\lambda,\mu}^\nu \sigma_\nu.$$

These numbers $c_{\lambda,\mu}^\nu$ are the celebrated **Littlewood-Richardson coefficients**. Not only are they the Schubert structure coefficients of $H^\star(X)$, but they are also multiplicities in tensor products of $\mathsf{GL}_n(\mathbb{C})$-representations and in induction products of $\mathfrak{S}_n$-representations, in addition to governing exact sequences of abelian p-groups and eigenvalues of sums of Hermitian matrices.

Thus we would like a way to determine $c_{\lambda,\mu}^\nu$. From the geometric and representation-theoretic interpretations of Littlewood-Richardson coefficients, it is immediate that $c_{\lambda,\mu}^\nu \in \mathbb{Z}_{\geq 0}$. Hence it would be desirable if our algorithm for computing $c_{\lambda,\mu}^\nu$ manifested this nonnegative integrality: Instead of a rule requiring addition and subtraction of rational numbers, we would prefer a rule of the form:

> "The structure coefficient $c_{\lambda,\mu}^\nu$ is the cardinality of an explicit set of combinatorial objects."

One advantage of such a rule is that it is thereby possible to show $c_{\lambda,\mu}^\nu > 0$ without determining the exact value of $c_{\lambda,\mu}^\nu$. These counting formulas are known as **Littlewood-Richardson rules**. The development of these rules has lead to significant new combinatorics of more general impact, and has deepened our understanding of the common combinatorial laws that govern diverse situations. In the next section, we review several Littlewood-Richardson rules whose extensions will be the main subject of this thesis.

1.2 Cohomological Schubert calculus

In this section, we describe three combinatorial rules for computing $c_{\lambda,\mu}^{\nu}$.

1.2.1 Rule H.1: Ballot semistandard Young tableaux

The following is essentially the original Littlewood-Richardson rule stated in 1934 by D.E. Littlewood and A.R. Richardson [LiRi34], though not rigorously proven until work of M.-P. Schützenberger in the 1970's. D.E. Littlewood and A.R. Richardson considered $c_{\lambda,\mu}^{\nu}$ in its representation-theoretic avatar. The connection to $H^{\star}(X)$ is due to L. Lesieur [Le47].

For Young diagrams $\lambda \subseteq \nu$, the **skew partition** ν/λ is the set-theoretic difference $\nu \backslash \lambda$. A **semistandard Young tableau** T of shape ν/λ is a filling of the boxes of ν/λ by positive integers such that the labels of each row weakly increase from left to right and the labels of each column strictly increase from top to bottom. If $\lambda = \emptyset$, we say T is of **straight shape**; otherwise T is a **skew tableau**.

Example 1.1. For $\nu = \ydiagram{5,4,2,1}$ and $\lambda = \ydiagram{3,1,1}$, one semistandard Young tableau of shape ν/λ is

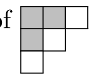
$$U = \begin{ytableau} *(gray) & *(gray) & *(gray) & 1 & 1 \\ *(gray) & 1 & 2 & 2 \\ *(gray) & 6 \\ 3 \end{ytableau}. \qquad \diamondsuit$$

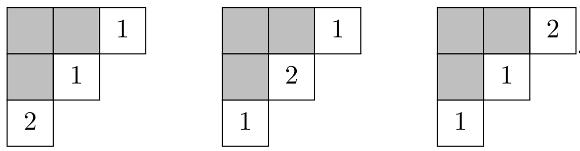
The **reading word** of T is the word read(T) formed by reading down the columns of T from right to left. The **content** of a word w is the vector content$(w) = (a_1, a_2, \dots)$, where a_i records the number of entries i in w. We say a semistandard tableau T is **ballot** (sometimes called *lattice* or *Yamanouchi*) if the content of every initial segment of read(T) is a partition, i.e. if as we read read(T), we have always read at least as many i's as $(i+1)$'s. The **content** of a semistandard tableau T is content$(T) :=$ content(read(T)).

Example 1.2. Continuing Example 1.1 above, content(U) is $(3, 2, 1, 0, 0, 1, 0, 0, \dots)$ and its reading word is read$(U) = 1122163$. The tableau U is not ballot, since read(U) contains a 6 but no 5. Replacing the 6 by either 2 or 3 yields a ballot semistandard tableau. $\diamondsuit$

Theorem 1.1 (Littlewood-Richardson rule (H.1))**.**

$$c^{\nu}_{\lambda,\mu} = \#\{\textit{Ballot semistandard tableaux of shape } \nu/\lambda \textit{ and content } \mu\} \qquad \square$$

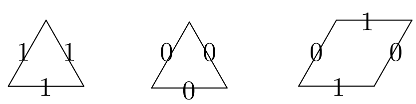
Example 1.3. To calculate $c^{(3,2,1)}_{(2,1),(2,1)}$, we find all semistandard fillings of  with content $(2,1)$:



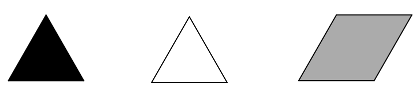
Of these, only the first two are ballot. Hence $c^{(3,2,1)}_{(2,1),(2,1)} = 2$. $\diamondsuit$

1.2.2 Rule H.2: Puzzles

For this rule, we use the translation of Section 1.1 to work with binary strings of length n with k 1's in place of partitions inside $k \times (n-k)$. Consider the n-length equilateral triangle oriented as Δ. A **puzzle** is a filling of Δ with the following **puzzle pieces**:



A **filling** requires that the common edges of adjacent puzzle pieces share the same label. The pieces may be rotated but not reflected. To avoid complicating the picture with too many edge labels, we will illustrate these three puzzle pieces as



respectively.

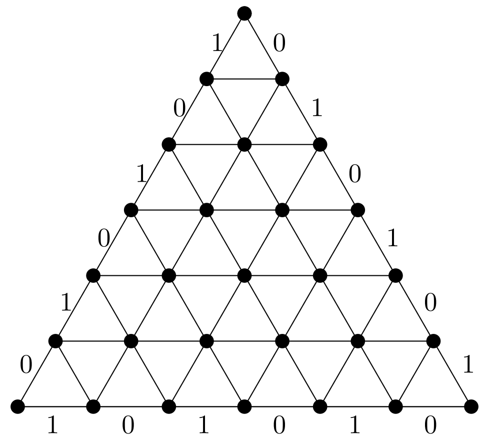
Let $\Delta_{\lambda,\mu,\nu}$ be Δ with the boundary given by

- λ as read $\nearrow$ along the left side;
- μ as read $\searrow$ along the right side; and
- ν as read $\rightarrow$ along the bottom side.

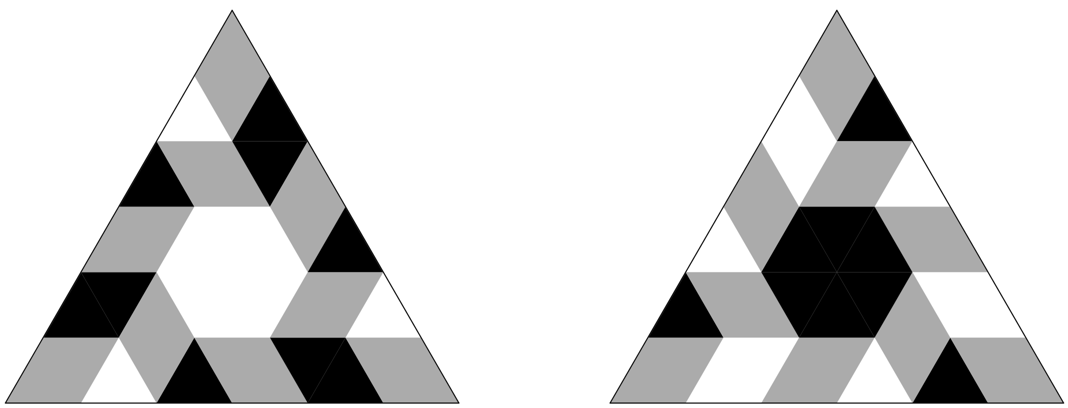
Theorem 1.2 (Littlewood-Richardson rule (H.2), A. Knutson–T. Tao–C. Woodward [KnTaWo04])**.**

$$c^{\nu}_{\lambda,\mu} = \#\{\textit{Puzzles with boundary } \Delta_{\lambda,\mu,\nu}\}. \qquad \square$$

Example 1.4. As in Example 1.3, we calculate $c^{(3,2,1)}_{(2,1),(2,1)}$. We may assume $n = 6$ and $k = 3$. Then the large triangle $\Delta_{\lambda,\mu,\nu}$ is



There are only two ways to tile $\Delta_{\lambda,\mu,\nu}$ with the puzzle pieces, subject to the boundary conditions:



Thus we recover the fact that $c^{(3,2,1)}_{(2,1),(2,1)} = 2$. $\diamondsuit$

1.2.3 Rule H.3: Rectification of standard Young tableaux

In this section, we return to treating λ, μ, ν as partitions and recall another Littlewood-Richardson rule in terms of tableaux. A **standard Young tableau** of shape ν/λ is a semistandard tableau of shape ν/λ in which each of the numbers $1, 2, \ldots, |\nu/\lambda|$ appears exactly once.

For b a box of the shape ν/λ, we write $\mathsf{b}^{\rightarrow}$ for the box immediately east of b, $\mathsf{b}^{\downarrow}$ for the box immediately south of b, etc. An **inner corner** of the shape ν/λ is a box $\mathsf{b} \in \lambda$ such that $\mathsf{b}^{\rightarrow} \notin \lambda$ and $\mathsf{b}^{\downarrow} \notin \lambda$. An **outer corner** of ν/λ is a box $\mathsf{b} \in \nu/\lambda$ such that $\mathsf{b}^{\rightarrow} \notin \nu/\lambda$ and $\mathsf{b}^{\downarrow} \notin \nu/\lambda$.

Given a standard Young tableau T of skew shape ν/λ, M.-P. Schützenberger's **jeu de taquin** is an algorithm to produce a standard Young tableau of some straight shape θ with $|\theta| = |\nu/\lambda|$. Choose any inner corner c of ν/λ and fill it with $\bullet$. Say the **SE-neighbors** of a label in box b are the entries of the boxes $\mathsf{b}^{\rightarrow}$ and $\mathsf{b}^{\downarrow}$, if they exist. Repeatedly switch the positions of $\bullet$ and its smallest SE-neighbor until no such SE-neighbor exists, that is until $\bullet$ is in an outer corner of ν/λ. Now delete $\bullet$. It is not hard to see that the result is a new standard Young tableau $\tilde{T}$ of some shape $\tilde{\nu}/\tilde{\lambda}$ where $|\tilde{\nu}| = |\nu| - 1$ and $|\tilde{\lambda}| = |\lambda| - 1$. Choose an inner corner of $\tilde{T}$ to fill with $\bullet$ and repeat this process. After $|\lambda|$ iterations, the tableau produced will be a standard Young tableau of some straight shape θ with $|\theta| = |\nu/\lambda|$. We call this tableau $\mathrm{rect}(T)$, the **rectification** of T.

Example 1.5. Let

$$T = \begin{ytableau} *(gray) & *(gray) & 1 \\ *(gray) & 2 \\ 3 & 4 \end{ytableau}$$

and choose the upper inner corner. Then we obtain

$$\begin{ytableau} *(gray) & \bullet & 1 \\ *(gray) & 2 \\ 3 & 4 \end{ytableau} \longrightarrow \begin{ytableau} *(gray) & 1 \\ *(gray) & 2 \\ 3 & 4 \end{ytableau},$$

leaving only one choice of new inner corner:

$$\begin{ytableau} *(gray) & 1 \\ \bullet & 2 \\ 3 & 4 \end{ytableau} \longrightarrow \begin{ytableau} *(gray) & 1 \\ 2 & \bullet \\ 3 & 4 \end{ytableau} \longrightarrow \begin{ytableau} *(gray) & 1 \\ 2 & 4 \\ 3 \end{ytableau}.$$

Finally we have

$$\begin{ytableau} \bullet & 1 \\ 2 & 4 \\ 3 \end{ytableau} \longrightarrow \begin{ytableau} 1 & \bullet \\ 2 & 4 \\ 3 \end{ytableau} \longrightarrow \begin{ytableau} 1 & 4 \\ 2 \\ 3 \end{ytableau} = \mathrm{rect}(T).$$

$\diamondsuit$

Theorem 1.3 (Confluence (cf. [Man01, Corollary 1.5.19]))**.** *The tableau* $\mathrm{rect}(T)$ *does not depend on the choices of inner corners.* $\square$

Example 1.6. If we rectify T from Example 1.5, first choosing the lower inner corner, we obtain

$$\begin{ytableau} *(gray) & *(gray) & 1 \\ \bullet & 2 \\ 3 & 4 \end{ytableau} \longrightarrow \begin{ytableau} *(gray) & *(gray) & 1 \\ 2 & \bullet \\ 3 & 4 \end{ytableau} \longrightarrow \begin{ytableau} *(gray) & *(gray) & 1 \\ 2 & 4 \\ 3 \end{ytableau},$$

leaving only one choice of inner corner:

$$\begin{ytableau} *(gray) & \bullet & 1 \\ 2 & 4 \\ 3 \end{ytableau} \longrightarrow \begin{ytableau} *(gray) & 1 \\ 2 & 4 \\ 3 \end{ytableau}.$$

Finally we have

$$\begin{ytableau} \bullet & 1 \\ 2 & 4 \\ 3 \end{ytableau} \longrightarrow \begin{ytableau} 1 & \bullet \\ 2 & 4 \\ 3 \end{ytableau} \longrightarrow \begin{ytableau} 1 & 4 \\ 2 \\ 3 \end{ytableau} = \mathrm{rect}(T),$$

as in Example 1.5 and in agreement with Theorem 1.3. ♢

Theorem 1.4 (Littlewood-Richardson rule (H.3))**.** *Let M be any fixed standard Young tableau of shape μ. Then*

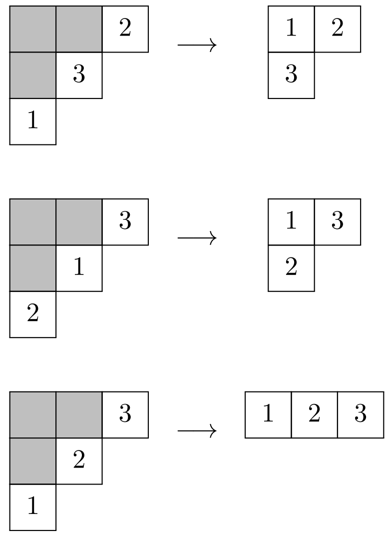
$$c^{\nu}_{\lambda,\mu} = \#\{\textit{standard Young tableaux } T : \mathrm{rect}(T) = M\}. \qquad \square$$

Example 1.7. As in Examples 1.3 and 1.4, we calculate $c^{(3,2,1)}_{(2,1),(2,1)}$. There are 6 standard Young tableaux of shape $(3,2,1)/(2,1)$, which rectify as follows:

$$\begin{ytableau} *(gray) & *(gray) & 1 \\ *(gray) & 2 \\ 3 \end{ytableau} \longrightarrow \begin{ytableau} 1 \\ 2 \\ 3 \end{ytableau}$$

$$\begin{ytableau} *(gray) & *(gray) & 1 \\ *(gray) & 3 \\ 2 \end{ytableau} \longrightarrow \begin{ytableau} 1 & 3 \\ 2 \end{ytableau}$$

$$\begin{ytableau} *(gray) & *(gray) & 2 \\ *(gray) & 1 \\ 3 \end{ytableau} \longrightarrow \begin{ytableau} 1 & 2 \\ 3 \end{ytableau}$$



Letting either $M = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array}$ or $M = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array}$ be the fixed standard Young tableau of shape $(2, 1)$, we count that two of these six tableaux rectify to M. Thus $c^{(3,2,1)}_{(2,1),(2,1)} = 2$. $\diamondsuit$

For later use, we note that there is also a notion of **reverse jeu de taquin**, where we start with a $\bullet$ in an outer corner and repeatedly switch it with its larger NW-neighbor. If a (skew) tableau T can be reached from a (skew) tableau T' by a sequence of forward and reverse jeu de taquin slides, we say that T and T' are **jeu de taquin equivalent**. By Theorem 1.3, each tableau is jeu de taquin equivalent to a unique tableau of straight shape.

1.2.4 Bijections between the Littlewood-Richardson rules

Fix partitions λ, μ, ν and let M be a standard Young tableau of shape μ. Since the cardinality of each set is $c^{\nu}_{\lambda,\mu}$, we have

$$\begin{aligned}\#\{\text{puzzles with boundary } \Delta_{\lambda,\mu,\nu}\} &= \#\{\text{ballot semistandard tableaux of shape } \nu/\lambda \text{ and content } \mu\} \\ &= \#\{\text{standard Young tableaux } T : \mathrm{rect}(T) = M\}.\end{aligned}$$

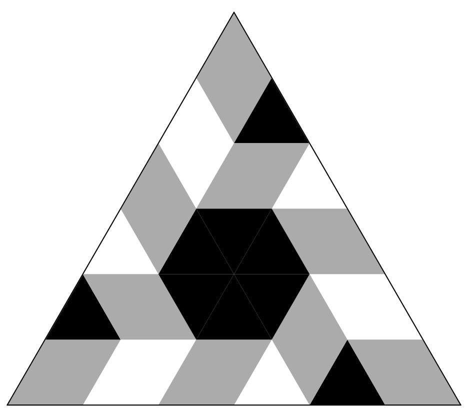
In this section, we show how to biject these three sets, that is how to biject the tableaux of rule (H.1) with the tableaux of rule (H.3) and the puzzles of rule (H.2). We omit here the proofs that these maps are bijections or even well-defined, as both these facts will follow from more general arguments in later chapters.

Puzzles to Ballot Tableaux

This bijection perhaps appeared first in [Pu08], though it is based on a bijection discovered by T. Tao and given in [Va06, Figure 11]. Consider a puzzle $P \in \{$puzzles with boundary $\Delta_{\lambda,\mu,\nu}\}$ and look at the 1's along the bottom edge (the ν-edge). We will produce disjoint trails of puzzles pieces, one for each of these 1's. We will then read these trails to construct $T_P \in \{$ballot semistandard tableaux of shape ν/λ and content $\mu\}$.

Think of P as the floorplan of a palace, where the puzzle pieces are the (triangular and rhombic) rooms and the 1 labeled edges are doorways. We enter the palace through one of the doors on the ν-edge. Whenever we enter a room, we will leave it by a different door than we came in by. Thus we traverse ▰ from bottom to top and ◥◣ from left to right. When we enter the base of ▲, we choose to exit through the door on our right, while when we enter the lower left door of ▼, we choose to exit through the door on our left. Since we cannot get stuck on this walk, and move always northeast, we will eventually exit the palace through a door on the μ-side. Record the shapes and orientations of the rooms visited on this trip; this is the **track** associated to the initial door on the ν-side of P.

Example 1.8. For P the puzzle



there are three doors on the ν-side of P. The track for the leftmost door is ▰ ▲ ◥◣ ▼▲▼ ▰ ▲. Entering instead through the middle door, we get the track ▰ ▲▼▲◥◣ , while the rightmost door gives the short track ▲◥◣ . ♢

We now convert P into a ballot semistandard tableau T_P of shape ν/λ with content μ. The track for the ith door from the left tells how to label labels i in ν/λ. Each ▲ in the track is preceded by a (possibly empty) sequence of consecutive ▰ 's. Replace the track by the integer vector $(a_1^i, a_2^i, \ldots, a_k^i)$ whose entries are the lengths of these sequences. Now we produce the tableau T_P by placing a_j^i instances of i in the $(i+j)$th row.

It is perhaps not clear that this process can always be carried out or that the result will be a tableau with the desired properties; we will later prove these facts in greater generality. For now, we satisfy ourselves with an example.

Example 1.9. Continuing Example 1.8, we convert the three tracks into the vectors $(1, 0, 1)$, $(1, 0)$, and (0). Hence we place one 1 in each of the first and third rows, and one 2 in the second row, obtaining the tableau $\begin{array}{|c|c|c|}\hline \cellcolor{lightgray} & \cellcolor{lightgray} & 1 \\ \hline \cellcolor{lightgray} & 2 \\ \cline{1-2} 1 \\ \cline{1-1}\end{array}$. Note that this is one of the ballot tableaux of Example 1.3. The reader may check that this process also converts the other puzzle of Example 1.4 to the other ballot tableau of Example 1.3. $\diamondsuit$

Ballot Tableaux to Rectifying Standard Tableaux

For a straight shape μ, the **superstandard** tableau T_μ of shape μ is the filling of μ along rows by consecutive positive integers.

Example 1.10. For $\mu = (4, 2, 1)$, $T_\mu = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 \\ \cline{1-2} 7 \\ \cline{1-1}\end{array}$. $\diamondsuit$

Let $S \in \{$ballot semistandard tableaux of shape ν/λ and content $\mu\}$. We will produce from S a standard Young tableau of shape ν/λ that rectifies to T_μ. Although we will not prove this yet, the map that this process defines is a bijection.

For each i, there is at most one i in any column of S, since S is semistandard. The **standardization** $\Phi(S)$ of S is the tableau formed by replacing the 1's of S with the integers $1, 2, \ldots, \mu_1$ from left to right, replacing the 2's of S with the integers $\mu_1 + 1, \mu_1 + 2, \ldots \mu_1 + \mu + 2$, etc. It is not hard to see that $\Phi(S)$ is a standard Young tableau of the desired shape and content. We will later show that $\mathrm{Rect}(\Psi(S)) = T_\mu$.

Example 1.11. For $S = \begin{array}{|c|c|c|}\hline \cellcolor{lightgray} & \cellcolor{lightgray} & 1 \\ \hline \cellcolor{lightgray} & 2 \\ \cline{1-2} 1 \\ \cline{1-1}\end{array}$, we have $\Phi(S) = \begin{array}{|c|c|c|}\hline \cellcolor{lightgray} & \cellcolor{lightgray} & 1 \\ \hline \cellcolor{lightgray} & 3 \\ \cline{1-2} 2 \\ \cline{1-1}\end{array}$. We confirmed in Example 1.7 that $\Phi(S)$ rectifies to $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} = T_{(2,1)}$. $\diamondsuit$

1.3 Symmetric functions and the Schur polynomials

Consider the polynomial ring $R = \mathbb{Z}[x_1, \ldots, x_n]$. The symmetric group $\mathfrak{S}_n$ naturally acts on R by permuting variables. For example, the transposition (12) acts on the polynomial $x_1^2 x_3 + x_1 x_2 x_4$ to produce $x_2^2 x_3 + x_1 x_2 x_4$. Let $\Lambda_n := R^{\mathfrak{S}_n}$, the $\mathfrak{S}_n$ invariants. It is easy to see that Λ_n is a subring of R; we call it the **ring of**

symmetric polynomials in n variables. Λ_n is moreover a graded ring, inheriting the grading by degree from R. The homogeneous pieces are denoted $\Lambda_n^{(m)}$.

For $m \leq n$, we can map Λ_n onto Λ_m by setting the last $n-m$ variables equal to 0. The inverse limit of the $\{\Lambda_n\}$ with respect to these restriction maps is called the **ring of symmetric functions** Λ, although its elements are not functions, but rather formal power series in infinitely many variables. The distinction between Λ_n and Λ will be of little consequence for us, and we will use whichever is more convenient. Note that $f \in \Lambda$ is a finite sum of homogeneous elements.

For a composition α, we define a monomial $x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \dots$. For a partition λ, let

$$m_\lambda := \sum_\alpha x^\alpha,$$

where the sum is over all distinct compositions that can be obtained by rearranging the parts of α. If x^α is a monomial of the symmetric function f, then necessarily x^β is also a monomial of f for every β that can be obtained by rearranging the parts of α. Thus f can be written uniquely as a finite positive sum of the **monomial symmetric functions** m_λ. Therefore $\{m_\lambda\}$ is a $\mathbb{Z}$-linear basis of Λ and the dimension of $\Lambda^{(m)}$ is the number of partitions of m.

We may consider a second action of $\mathfrak{S}_n$ on R where a permutation acts by permuting variables and then multiplying by the sign of the permutation. The invariants of this action are the **alternating polynomials in n variables**, $v_n\Lambda_n$. These are precisely the polynomials where setting any two variables equal yields 0. The sum of two alternating polynomials is alternating, but the product is not. Hence $v_n\Lambda_n$ is not a subring of R, though it is a module over Λ. As with Λ_n, $v_n\Lambda_n$ is graded by degree, though for technical reasons one might prefer to shift the degree by $\binom{n}{2}$.

Let $v_n := \prod_{1 \leq i < j \leq n}(x_i - x_j)$ be the **Vandermonde determinant**. This is an alternating polynomial and moreover divides every other alternating polynomial. The quotients are necessarily symmetric. Hence every alternating polynomial can be written as v_n times a symmetric polynomial. (This fact justifies the notation $v_n\Lambda_n$ for the module of alternating polynomials.)

For a weak composition α (i.e., a finite sequence of nonnegative integers), define

$$\tilde{a}_\alpha := \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) x^{\sigma(\alpha)}.$$

Note that if x^α is a term of the alternating polynomial f, then so is every other term of $\tilde{a}_\alpha$. Moreover if α has any repeated parts, then clearly $\tilde{a}_\alpha = 0$. Hence $v_n\Lambda_n$ has a natural basis of polynomials $\tilde{a}_\theta$, for θ ranging over **strict partitions**, that is partitions with distinct parts.

Every strict partition may be written uniquely as $\delta+\lambda$, where $\delta = (n-1, n-2, \ldots, 0)$ and λ is a partition. We write $a_\lambda := \tilde{a}_{\delta+\lambda}$, to obtain a basis of $v_n\Lambda$ indexed by partitions. That is, the dimension of the space of alternating polynomials of degree $m\binom{n}{2}$ equals the dimension of the space of symmetric polynomials of degree m. Indeed, we can even identify the isomorphism; it is just multiplication by $v_n = \tilde{a}_\delta = a_{(0)}$. If we shifted the grading of $v_n\Lambda_n$ as suggested above (so that v_n is in degree 0), then multiplication by v_n is an isomorphism $\Lambda_n \to v_n\Lambda_n$ of *graded* Λ-modules.

The basis of Λ that we get by pulling back the a_λ basis of $v_n\Lambda_n$ is not the basis of monomial symmetric functions, but rather something more interesting. We call these polynomials

$$s_\lambda := \frac{a_\lambda}{v_n},$$

the **Schur polynomials**.

The Schur basis is our favorite basis of Λ, so we will study it in some more detail. The most basic question one might wish to resolve about s_λ is the following: What are the terms of the polynomial s_λ?

Theorem 1.5. $s_\lambda = \sum_{T \in \mathrm{SSYT}(\lambda)} x^{\mathrm{content}(T)}$. $\square$

Proving the above formula requires some combinatorial machinery that we will not otherwise need, so we omit the proof here. What will be useful to us instead is merely to prove combinatorially that the polynomial of the right side of Theorem 1.5 is in fact symmetric.

First we need to extend the jeu de taquin of Section 1.2.3 to semistandard tableaux. To do so, it suffices to decide how to break ties of the form $\begin{array}{|c|c|}\hline \bullet & i \\ \hline i \\ \cline{1-1}\end{array}$. We declare that in this case, we treat the left i as smaller:

$$\begin{array}{|c|c|}\hline \bullet & i \\ \hline i \\ \cline{1-1}\end{array} \mapsto \begin{array}{|c|c|}\hline i & i \\ \hline \bullet \\ \cline{1-1}\end{array}.$$

We can now define **infusion** for semistandard tableaux. Let $T \in \mathrm{SSYT}(\alpha)$ where α is possibly a skew shape. Let $U \in \mathrm{SSYT}(\beta/\alpha)$ be a semistandard tableau on a disjoint alphabet from T. We will here write the labels of U as circled numbers to distinguish them from the uncircled labels of T. Consider the layered tableau (T, U) that is the union of T and U. Then

$$\mathsf{inf}(T, U) = (U^\star, T^\star)$$

is obtained as follows: Consider the largest number M that appears in T. The rightmost box of T that contains M is an inner corner I for U. Replace this M by $\bullet$ and apply jeu de taquin to U at this inner corner, until the $\bullet$ reaches an outer corner of U. Place M in this outer corner. Now consider the largest number M' that appears in T', the remainder of T. The rightmost box of T' that contains M' is an inner corner I' for U', the modified U. Now apply jeu de taquin to U' at I'. We continue in this manner as many times as there are boxes of T. The "inner" tableau of circled numbers is $U^\star$ and the "outer" tableau of uncircled numbers is $T^\star$. If α is a straight shape, then $U^\star = \mathrm{rect}(U)$. Furthermore:

Proposition 1.1. *Infusion is an involution:* $\mathsf{inf}(U^\star, T^\star) = (T, U)$. □

Example 1.12. Let (T, U) be the layered tableau

$$\begin{array}{|c|c|c|}\hline 1 & 1 & \textcircled{3} \\ \hline 2 & \textcircled{1} \\ \cline{1-2} \textcircled{1} & \textcircled{2} \\ \cline{1-2}\end{array}.$$

Then to produce $\mathsf{inf}(T, U)$, we first perform jeu de taquin on U at the 2 in T, then place 2 in the vacated outer corner of U:

$$\begin{array}{|c|c|c|}\hline 1 & 1 & \textcircled{3} \\ \hline 2 & \textcircled{1} \\ \cline{1-2} \textcircled{1} & \textcircled{2} \\ \cline{1-2}\end{array} \mapsto \begin{array}{|c|c|c|}\hline 1 & 1 & \textcircled{3} \\ \hline \textcircled{1} & \textcircled{1} \\ \cline{1-2} \textcircled{2} & 2 \\ \cline{1-2}\end{array}.$$

Next we perform jeu de taquin at the right 1 and then at the left 1, producing in turn

$$\begin{array}{|c|c|c|}\hline 1 & \textcircled{1} & \textcircled{3} \\ \hline \textcircled{1} & 1 \\ \cline{1-2} \textcircled{2} & 2 \\ \cline{1-2}\end{array} \text{ and then } \begin{array}{|c|c|c|}\hline \textcircled{1} & \textcircled{1} & \textcircled{3} \\ \hline \textcircled{2} & 1 \\ \cline{1-2} 1 & 2 \\ \cline{1-2}\end{array} =: \mathsf{inf}(T, U).$$

The reader may check that performing infusion on the layered tableau $\mathsf{inf}(T, U)$ recovers (T, U). ♢

Next we define the **Bender-Knuth involutions** originally introduced in [BeKn72]. Given a semistandard tableau Ω, consider the subtableau T consisting of those boxes containing i and the subtableau U consisting of those boxes containing $i+1$. Circle the labels of U. Now define $\mathrm{BK}_i(\Omega)$ to be obtained from Ω by replacing the subtableau (T, U) with $(U^\star, T^\star)$, switching the labels i and $i+1$, and removing circlings.

Example 1.13. Let $\Omega = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 2 \\ \hline 2 & 3 \\ \cline{1-2}\end{array}$. Then $\mathrm{BK}_1(\Omega) = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 2 \\ \hline 2 & 3 \\ \cline{1-2}\end{array}$, $\mathrm{BK}_2(\Omega) = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 3 \\ \hline 2 & 3 \\ \cline{1-2}\end{array}$, and $\mathrm{BK}_3(\Omega) = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 2 \\ \hline 2 & 4 \\ \cline{1-2}\end{array}$. ♢

Proposition 1.2. BK_i *is an involution. Moreover,* BK_i *is a bijection from* $\{\Omega \in \mathrm{SSYT}(\nu/\lambda) : \mathrm{content}(\Omega) = (\gamma_1, \ldots, \gamma_i, \gamma_{i+1}, \ldots)\}$ *to* $\{\Omega \in \mathrm{SSYT}(\nu/\lambda) : \mathrm{content}(\Omega) = (\gamma_1, \ldots, \gamma_{i+1}, \gamma_i, \ldots)\}$.

Proof. The first sentence is immediate from Proposition 1.1. The second sentence follows from the definition of BK_i and the first sentence. □

Corollary 1.1. $\sum_{T \in \mathrm{SSYT}(\lambda)} x^{\mathrm{content}(T)}$ *is a symmetric function.*

Proof. Immediate from the above proposition since the symmetric group is generated by the adjacent transpositions $(i(i+1))$. □

The last fact that we need about Schur functions is that multiplying Schur functions is the same as computing in the cohomology of Grassmannians:

Theorem 1.6. *For partitions λ, μ, ν, we have*

$$s_\lambda \cdot s_\mu = \sum_\nu c^\nu_{\lambda,\mu} s_\nu.$$ □

1.4 The Robinson-Schensted-Knuth correspondence

The **Robinson-Schensted-Knuth (RSK) correspondence** is an algorithm for converting a string w of positive integers into a pair of tableaux $(P(w), Q(w))$ of the same shape, of which the first is semistandard and the second standard. (In fact the full RSK algorithm is somewhat more general, but we will only need the case defined here and some extensions of that to be described later.) RSK is usually described as a "bumping" algorithm, but we can define half of it through jeu de taquin. We will start with the latter description, since we have already developed some ideas about jeu de taquin.

Let w be a word of n positive integers. Arrange the letters of w along the antidiagonal from southwest to northeast. More precisely, we fill the skew shape δ_{n+1}/δ_n to produce the semistandard skew tableau T_w. Now $P(w) := \mathrm{rect}(T_w)$.

To find the standard tableau $Q(w)$, we will have to construct $P(w)$ instead by bumping. Let T be a semistandard tableau and i a positive integer. The **bump** of i into T is the semistandard tableau $T \Lleftarrow i$ constructed as follows. If i is at least as large as every label of the first row of T, place i at the end of the first row; the result is $T \Lleftarrow i$. Otherwise find the leftmost label j of the first row that is larger than i and replace it with i. If j is at least as large as every label of the second row, place j at the end of the second row; the result is $T \Lleftarrow i$. Otherwise find the leftmost label k of the second row that is larger than j and replace it with j. Continue by inserting k into the third row, etc. The algorithm is guaranteed to terminate, since there are only finitely many rows in T.

Example 1.14. Let

$$T = \begin{ytableau} 1 & 1 & 1 & 2 & 3 & 5 \\ 2 & 3 & 4 \\ 4 & 4 \end{ytableau}.$$

We will construct $T \Lleftarrow 1$. First we insert 1 into the first row of T, bumping out 2 and obtaining

$$\begin{ytableau} 1 & 1 & 1 & 1 & 3 & 5 \\ 2 & 3 & 4 \\ 4 & 4 \end{ytableau}.$$

Then we insert 2 into the second row, bumping out 3 to obtain

$$\begin{ytableau} 1 & 1 & 1 & 1 & 3 & 5 \\ 2 & 2 & 4 \\ 4 & 4 \end{ytableau}.$$

Next we insert 3 into the third row, bumping out the left 4 to obtain

$$\begin{ytableau} 1 & 1 & 1 & 1 & 3 & 5 \\ 2 & 2 & 4 \\ 3 & 4 \end{ytableau}.$$

Finally we insert 4 into the fourth row. Since the fourth row is empty, 4 is at least as large as every label in the fourth row. Hence we place 4 in the fourth row to find the tableau

$$\begin{ytableau} 1 & 1 & 1 & 1 & 3 & 5 \\ 2 & 2 & 4 \\ 3 & 4 \\ 4 \end{ytableau},$$

which is $T \Lleftarrow 1$.

In contrast, $T \Lleftarrow 5$ is just obtained by appending a 5 to the end of the first row of T. ♢

Let T_0 be the empty tableau of shape $\emptyset$. One can show that

$$P(w) = T_0 \Lleftarrow w := (\cdots((T_0 \Lleftarrow w_1) \Lleftarrow w_2) \Lleftarrow \dots) \Lleftarrow w_n,$$

where $w = w_1 w_2 \dots w_n$. Observe that the shape of T is always contained in the shape of $T \Lleftarrow i$ and that

$T \Leftarrow i$ always has exactly one more box than T. We define $Q(w)$ to be the standard tableau that has k in the box

$$\operatorname{shape}(T_0 \Leftarrow w^{(k)})/\operatorname{shape}(T_0 \Leftarrow w^{(k-1)}),$$

where $w^{(k)}$ denotes the first k letters of w.

Example 1.15. Let $w = 21143$. We have

$$(P(w^{(1)}), Q(w^{(1)})) = \left(\begin{ytableau} 2 \end{ytableau}, \begin{ytableau} 1 \end{ytableau} \right),$$

$$(P(w^{(2)}), Q(w^{(2)})) = \left(\begin{ytableau} 1 \\ 2 \end{ytableau}, \begin{ytableau} 1 \\ 2 \end{ytableau} \right),$$

$$(P(w^{(3)}), Q(w^{(3)})) = \left(\begin{ytableau} 1 & 1 \\ 2 \end{ytableau}, \begin{ytableau} 1 & 3 \\ 2 \end{ytableau} \right),$$

$$(P(w^{(4)}), Q(w^{(4)})) = \left(\begin{ytableau} 1 & 1 & 4 \\ 2 \end{ytableau}, \begin{ytableau} 1 & 3 & 4 \\ 2 \end{ytableau} \right),$$

$$(P(w), Q(w)) = (P(w^{(5)}), Q(w^{(5)})) = \left(\begin{ytableau} 1 & 1 & 3 \\ 2 & 4 \end{ytableau}, \begin{ytableau} 1 & 3 & 4 \\ 2 & 5 \end{ytableau} \right).$$

Observe that $Q(w)$ is standard, while $P(w)$ is semistandard. $\diamondsuit$

Given a pair of tableau (P, Q) of this form, it is not hard to see that one can reconstruct the word w by reversing the bumping algorithm on P according to the order dictated by Q. Hence RSK is a bijection between words and pairs of such tableaux.

1.4.1 Knuth equivalence

Since RSK is a bijection from words to pairs of tableaux (P, Q), if we follow RSK by forgetting the Q tableau and remembering only the P tableau, the map is many-to-one. In fact, for a fixed P of shape λ, the size of the fiber over P is f^{λ}, the number of standard Young tableaux of shape λ. We say two words are **Knuth equivalent** if they correspond to the same P tableau.

In [Kn70], D. Knuth studied this notion in detail. We think of words as elements of the free monoid $(\mathbb{Z}_{>0})^{\star}$ on the alphabet of positive integers. D. Knuth identified cubic relations such that two words are equivalent in the quotient of $(\mathbb{Z}_{>0})^{\star}$ by these relations if and only if the two words are Knuth equivalent.

Definition 1.1. The **plactic monoid** is the quotient

$$(\mathbb{Z}_{>0})^{\star}/\langle yzx = yxz, acb = cab\rangle_{\substack{x<y\leq z\\ a\leq b<c}}$$

Theorem 1.7. *Two words are Knuth equivalent if and only if they represent the same element of the plactic monoid. Two skew tableaux are jeu de taquin equivalent if and only if their reading words are Knuth equivalent.* □

A consequence of Theorem 1.7 is that every Knuth equivalence class contains the reading word of a unique tableau of straight shape; hence one may choose to identify the elements of the plactic monoid with the semistandard tableaux of straight shape. Theorem 1.7 can be used to great effect in some instances to determine rectifications of skew tableaux without performing jeu de taquin. We will use this trick, for example, in Section 7.5. In Chapter 9, we will do something similar with a related monoid corresponding to another type of tableau.

1.5 Organization

The remainder of this thesis is organized as follows.

In Chapter 2 (joint work with J. Bloom and D. Saracino [BlPeSa16]), we prove a conjecture of J. Propp–T. Roby [PrRo13b] on *homomesy* of semistandard tableaux under *promotion*, a composition of Bender-Knuth involutions. Our solution uses much of the tableau combinatorics described in this introduction, but little to none of the geometry.

In Chapter 3 (joint work with A. Yong [PeYo16]), we lift the geometric discussion of this introduction to the *K-theory* of Grassmannians. We develop there the analogous combinatorics of tableaux and puzzles governing K-theoretic Schubert calculus. We describe known rules that extend the rules (H.2) and (H.3) (described in Sections 1.2.2 and 1.2.3), and give the first combinatorial rule extending (H.1) to K-theory.

In Chapter 4 (joint work with A. Yong [PeYo15b]), we turn to *torus-equivariant K-theory* of Grassmannians, where we prove the first Littlewood-Richardson rule. Our rule extends the rule (H.1) for the cohomological coefficients and transparently exhibits the positivity established geometrically in [AnGrMi11]. In Section 4.13, we prove a conjectural rule of H. Thomas–A. Yong [ThYo13, Conjecture 4.7] for the same coefficients (generalizing the (H.3) rule).

In Chapter 5 (joint work with A. Yong [PeYo15c]), we exhibit a counterexample to a conjectural puzzle rule for these coefficients due to A. Knutson–R. Vakil [CoVa05], and prove a mild correction to it. The resulting puzzle rule generalizes the rule (H.2).

In Chapter 6 (joint work with A. Yong [PeYo16]), we extend the results of Chapter 3 to the K-theory of *maximal orthogonal Grassmannians* and provide some conjectures about the K-theoretic Schubert calculus of *Lagrangian Grassmannians*. Our rule for maximal orthogonal Grassmannians extends a cohomological rule given in another context by J. Stembridge [Ste89].

Chapter 7 (primarily derived from [Pe14]) uses the K-theoretic combinatorics developed in Chapter 3 to introduce and study a K-theoretic analogue of promotion, for which we prove an instance of the *cyclic sieving phenomenon* of V. Reiner–D. Stanton–D. White [ReStWh04]. Our main theorems generalize results of J. Stembridge [Ste95] and D. White [Wh07]. In Section 7.7 (joint work with J. Bloom and D. Saracino [BlPeSa16]), we obtain a related instance of homomesy, partially generalizing a result from Chapter 2.

Chapter 8 (joint work with K. Dilks and J. Striker [DiPeSt15]) relates K-promotion to the operation of *rowmotion* on *plane partitions*, and uses this connection to prove new results about both actions. In particular, we prove a new case of a plane partition conjecture of P. Cameron and D. Fon-der-Flaass [CaFo95].

Finally, in Chapter 9 (joint work with Z. Hamaker, R. Patrias, and N. Williams [HPPW16]), we give the first bijective proof of a 1983 theorem of R. Proctor on plane partitions by continuing to exploit the connection to K-theoretic combinatorics initiated in Chapter 8. Our proof relies heavily on the results described in Section 3.1.4.

Chapter 2

Homomesy in promotion of semistandard tableaux

This chapter describes joint work with Jonathan Bloom and Dan Saracino, previously published in [BlPeSa16].

2.1 Introduction

Let G be a group acting on a set X of combinatorial objects, with finite orbits, and $\xi : X \to \mathbb{C}$ any complex-valued function. The triple (X, G, ξ) is **homomesic** if for any orbits $\mathcal{O}_1, \mathcal{O}_2$, the average value of the statistic ξ is the same, that is

$$\frac{1}{|\mathcal{O}_1|} \sum_{x \in \mathcal{O}_1} \xi(x) = \frac{1}{|\mathcal{O}_2|} \sum_{y \in \mathcal{O}_2} \xi(y).$$

If X is finite, this implies that the average value of ξ on any orbit is the average value of ξ on X. The concept of homomesy was first isolated by J. Propp and T. Roby [PrRo13a, PrRo15], although instances of homomesy were previously conjectured in [Pa09] and proved in [ArStTh13].

Let $\mathrm{SSYT}_k(m \times n)$ denote the set of semistandard Young tableaux of shape $m \times n$, i.e., m rows and n columns, with entries bounded above by k. There is a **promotion operator** $\mathcal{P}$ on $\mathrm{SSYT}_k(m \times n)$, defined as follows: Delete all 1's, rectify, decrement all labels by 1, and fill all empty boxes with k. By Theorem 1.3, the operator is well-defined.

Example 2.1. Let $k = 6$ and $T = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 3 \\ \hline 3 & 3 & 4 & 4 \\ \hline 5 & 5 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array}$. Then $\mathcal{P}(T) = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 2 & 3 \\ \hline 2 & 3 & 6 & 6 \\ \hline 4 & 4 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array}$. ♢

For $T \in \mathrm{SSYT}_k(m \times n)$ and S a set of boxes in the $m \times n$ rectangle, define $\sigma_S(T)$ to be the sum of the entries of T in the boxes of S. Further let $\langle \mathcal{P} \rangle$ be the cyclic group generated by $\mathcal{P}$. With this language the main result of this chapter, which proves a conjecture of J. Propp and T. Roby [PrRo13b], is the following.

Theorem 2.1. *If S is fixed under 180° rotation, then $(\mathrm{SSYT}_k(m \times n), \langle \mathcal{P} \rangle, \sigma_S)$ is homomesic.*

This result looks remarkably similar to certain homomesies discovered by J. Propp and T. Roby [PrRo15]. These latter results relate to *rowmotion* (a.k.a. *Fon-der-Flaass map*, *Panyushev complementation*, etc.) and *promotion* of order ideals in rectangular posets $\mathbf{m} \times \mathbf{n}$. We do not know any concrete relation between Theorem 2.1 and any of the results of [PrRo15]. Note that 'promotion' in this order ideal context is quite different from the promotion we use for tableaux; the coincidence of terminology reflects the relation worked out in [StWi12] between tableau promotion for 2-row rectangles and order ideal promotion in the type A positive root poset. D. Einstein and J. Propp [EiPr14] have shown that tableau promotion on $\mathrm{SSYT}_k(m \times n)$ is naturally conjugate to a piecewise-linear lift of order ideal promotion to the rational points with denominator dividing n in the order polytope of $\mathbf{m} \times (\mathbf{k} - \mathbf{m})$. We will study poset promotion and rowmotion in Chapter 8, uncovering further relations to tableau promotion.

This chapter is structured as follows. In Section 2.2, we define promotion and evacuation for semistandard Young tableaux and prove various important properties. Although most of these results for *standard* Young tableaux may be readily found in the literature (see e.g. [Sta09]), analogous statements and proofs for semistandard Young tableaux were previously hard to find, if not completely missing from the literature. In Section 2.3, we prove Theorem 2.1 via growth diagrams. In Section 2.4, we describe cominuscule posets and prove an extension of Theorem 2.1 for them.

2.2 Basic facts about promotion and evacuation

Both promotion and evacuation have been extensively studied in the context of standard Young tableaux (SYT). See [Sta09] for a comprehensive survey. It has been widely believed that most results about promotion and evacuation generalize to the semistandard setting; however, explicit statements and proofs have been mostly lacking from the literature. The purpose of this section is to provide explicit definitions of promotion and evacuation for semistandard Young tableaux, and to prove some of their most important combinatorial properties.

For partitions $\mu \subset \lambda$, we denote by $\mathrm{SSYT}_k(\lambda/\mu)$ the set of all semistandard Young tableaux of skew shape λ/μ with **ceiling** k, i.e., all entries are $\leq k$. If $\mu = \emptyset$, we write simply $\mathrm{SSYT}_k(\lambda)$ and refer to λ as a **straight-shape**. If λ is an $n \times m$ rectangle, we write $\mathrm{SSYT}_k(n \times m)$ for $\mathrm{SSYT}_k(\lambda)$.

2.2.1 Promotion

For the remainder of this section we will fix an arbitrary partition λ. We will need an alternative definition of promotion based on Bender-Knuth involutions.

Theorem 2.2. *For any $T \in \mathrm{SSYT}_k(\lambda)$, we have $\mathcal{P}(T) = \mathrm{BK}_{k-1} \circ \mathrm{BK}_{k-2} \circ \cdots \circ \mathrm{BK}_1(T)$.*

Example 2.2. Using the tableau T from Example 2.1 we obtain

$$T = \begin{ytableau} 1 & 1 & 2 & 3 \\ 3 & 3 & 4 & 4 \\ 5 & 5 \end{ytableau} \xrightarrow{\mathrm{BK}_1} \begin{ytableau} 1 & 2 & 2 & 3 \\ 3 & 3 & 4 & 4 \\ 5 & 5 \end{ytableau} \xrightarrow{\mathrm{BK}_2} \begin{ytableau} 1 & 2 & 2 & 3 \\ 2 & 3 & 4 & 4 \\ 5 & 5 \end{ytableau} \xrightarrow{\mathrm{BK}_3} \begin{ytableau} 1 & 2 & 2 & 3 \\ 2 & 3 & 4 & 4 \\ 5 & 5 \end{ytableau}$$

$$\xrightarrow{\mathrm{BK}_4} \begin{ytableau} 1 & 2 & 2 & 3 \\ 2 & 3 & 5 & 5 \\ 4 & 4 \end{ytableau} \xrightarrow{\mathrm{BK}_5} \begin{ytableau} 1 & 2 & 2 & 3 \\ 2 & 3 & 6 & 6 \\ 4 & 4 \end{ytableau},$$

which we see is the same as $\mathcal{P}(T)$. $\diamondsuit$

2.2.2 Evacuation

We now define evacuation for semistandard Young tableaux.

Definition 2.1. For $T \in \mathrm{SSYT}_k(\lambda)$, define a sequence $\epsilon_1(T), \epsilon_2(T), \ldots, \epsilon_k(T)$ as follows. Let $\epsilon_1(T) = \mathcal{P}(T)$. For $j \geq 2$, obtain $\epsilon_j(T)$ by freezing the entries $k, \ldots, k-(j-2)$ in $\epsilon_{j-1}(T)$ and promoting the remaining portion. We define the **evacuation** $\mathcal{E}(T)$ of T to be $\epsilon_k(T)$.

Using the characterization of promotion by Bender-Knuth involutions, we see that evacuation has the following alternative description:

$$\mathcal{E} = \mathrm{BK}_1 \cdot (\mathrm{BK}_2\mathrm{BK}_1) \cdot \cdots \cdot (\mathrm{BK}_{k-3} \cdots \mathrm{BK}_1) \cdot (\mathrm{BK}_{k-2} \cdots \mathrm{BK}_1) \cdot (\mathrm{BK}_{k-1} \cdots \mathrm{BK}_1).$$

For rectangular $T \in \mathrm{SSYT}_k(m \times n)$, let $\texttt{rot}(T)$ denote the element of $\mathrm{SSYT}_k(m \times n)$ obtained by rotating T by $180°$ and then replacing each entry i by $k+1-i$.

We will also need the **dual evacuation** of T, which we denote by $\mathcal{E}'(T)$. This is defined analogously to evacuation except that here we use the inverse of promotion and freeze elements from smallest to largest. It is easy to see that in the context of rectangular semistandard Young tableaux,

$$\mathcal{E}'(T) = \texttt{rot}(\mathcal{E}(\texttt{rot}(T))). \tag{2.1}$$

Dual evacuation also has a characterization in terms of Bender-Knuth involutions:

$$\mathcal{E}' = \mathrm{BK}_{k-1} \cdot (\mathrm{BK}_{k-2}\mathrm{BK}_{k-1}) \cdot \cdots \cdot (\mathrm{BK}_3 \cdots \mathrm{BK}_{k-2}\mathrm{BK}_{k-1}) \cdot (\mathrm{BK}_2 \cdots \mathrm{BK}_{k-2}\mathrm{BK}_{k-1}) \cdot (\mathrm{BK}_1 \cdots \mathrm{BK}_{k-2}\mathrm{BK}_{k-1}).$$

Example 2.3. Using the T from Example 2.1, we illustrate each step in the definition of evacuation below.

The shading at each step denotes the boxes that are frozen.

$$T = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 3 \\ \hline 3 & 3 & 4 & 4 \\ \hline 5 & 5 \\ \cline{1-2}\end{array} \longrightarrow \begin{array}{|c|c|c|c|}\hline 1 & 2 & 2 & 3 \\ \hline 2 & 3 & \cellcolor{gray}6 & \cellcolor{gray}6 \\ \hline 4 & 4 \\ \cline{1-2}\end{array} \longrightarrow \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 2 \\ \hline 2 & 3 & \cellcolor{gray}6 & \cellcolor{gray}6 \\ \hline 3 & \cellcolor{gray}5 \\ \cline{1-2}\end{array} \longrightarrow \begin{array}{|c|c|c|c|}\hline 1 & 1 & \cellcolor{gray}4 & \cellcolor{gray}4 \\ \hline 2 & 2 & \cellcolor{gray}6 & \cellcolor{gray}6 \\ \hline \cellcolor{gray}4 & \cellcolor{gray}5 \\ \cline{1-2}\end{array} \longrightarrow \begin{array}{|c|c|c|c|}\hline 1 & 1 & \cellcolor{gray}4 & \cellcolor{gray}4 \\ \hline 3 & 3 & \cellcolor{gray}6 & \cellcolor{gray}6 \\ \hline \cellcolor{gray}4 & \cellcolor{gray}5 \\ \cline{1-2}\end{array}$$

$$\longrightarrow \begin{array}{|c|c|c|c|}\hline 1 & 1 & \cellcolor{gray}4 & \cellcolor{gray}4 \\ \hline \cellcolor{gray}3 & \cellcolor{gray}3 & \cellcolor{gray}6 & \cellcolor{gray}6 \\ \hline \cellcolor{gray}4 & \cellcolor{gray}5 \\ \cline{1-2}\end{array} \longrightarrow \begin{array}{|c|c|c|c|}\hline \cellcolor{gray}2 & \cellcolor{gray}2 & \cellcolor{gray}4 & \cellcolor{gray}4 \\ \hline \cellcolor{gray}3 & \cellcolor{gray}3 & \cellcolor{gray}6 & \cellcolor{gray}6 \\ \hline \cellcolor{gray}4 & \cellcolor{gray}5 \\ \cline{1-2}\end{array},$$

So

$$\mathcal{E}(T) = \begin{array}{|c|c|c|c|}\hline 2 & 2 & 4 & 4 \\ \hline 3 & 3 & 6 & 6 \\ \hline 4 & 5 \\ \cline{1-2}\end{array}. \qquad \diamondsuit$$

2.2.3 A fundamental theorem on promotion and evacuation

The following theorem contains the results we will need about promotion and evacuation. For the special case of standard tableaux, proofs of parts (a), (b), and (c) are readily available in the literature (see, e.g., [Sta09, Theorem 2.1]) and are essentially due to M.-P. Schützenberger.

Theorem 2.3. *Let $T \in \mathrm{SSYT}_k(\lambda)$. Then*

(a) $\mathcal{E}^2(T) = T$,

(b) $\mathcal{E} \circ \mathcal{P}(T) = \mathcal{P}^{-1} \circ \mathcal{E}(T)$,

(c) if λ is rectangular, $\mathcal{P}^k(T) = T$,

(d) if λ is rectangular, $\mathcal{E}(T) = \mathtt{rot}(T)$.

Proof of parts (a)-(c). We take part (d) as given. (Part (d) is proved below without reference to (a)–(c).) We imitate the proof of [Sta09, Theorem 2.1] (based on an idea of Haiman [Ha92]), using the formulation of promotion in terms of Bender-Knuth involutions.

Let G be the quotient of the free group with generators $x_1, \ldots, x_{k-1}$ by the relations

$$\begin{aligned} x_i^2 &= 1, \quad 1 \le i \le k-1 \\ x_i x_j &= x_j x_i, \quad \text{if } |i-j| > 1. \end{aligned} \tag{2.2}$$

Let

$$y = x_{k-1}x_{k-2}\cdots x_1$$
$$z = x_1 \cdot (x_2x_1) \cdot \cdots \cdot (x_{k-3}\cdots x_1) \cdot (x_{k-2}\cdots x_1) \cdot (x_{k-1}\cdots x_1)$$
$$z' = x_{k-1} \cdot (x_{k-2}x_{k-1}) \cdot \cdots \cdot (x_3\cdots x_{k-2}x_{k-1}) \cdot (x_2\cdots x_{k-2}x_{k-1}) \cdot (x_1\cdots x_{k-2}x_{k-1}).$$

Since the Bender-Knuth involutions $\mathrm{BK}_1, \ldots, \mathrm{BK}_{k-1}$ satisfy the defining relations (2.2) of G, letting x_i act as BK_i defines a permutation representation of G on $\mathrm{SSYT}_k(\lambda)$. Under this representation y acts as $\mathcal{P}$, z acts as $\mathcal{E}$ and z' acts as $\mathcal{E}'$. By [Sta09, Lemma 2.2], the following hold in G:

$$\begin{aligned} z^2 &= (z')^2 = 1 \\ y^k &= z'z \\ zy &= y^{-1}z. \end{aligned} \tag{2.3}$$

This proves (a) and (b).

Now assume $T \in \mathrm{SSYT}_k(m \times n)$ is rectangular. By (2.3), $\mathcal{P}^k = \mathcal{E}' \circ \mathcal{E}$. Additionally,

$$\mathcal{E}' \circ \mathcal{E}(T) = \mathcal{E}'(\texttt{rot}(T)) = \texttt{rot}(\mathcal{E}(T)) = T,$$

where the first and third equalities follow from (d) and the second equality from (2.1). This proves (c). □

For the proof of (d) we will need the Robinson-Schensted-Knuth (RSK) correspondence, as described in Section 1.4. The main ingredient for our proof of (d) is the following standard fact, which is a special case of part 4 of the Duality Theorem of [Fu97, p. 184].

Fact 2.1. Fix $k > 0$ and let $w = w_1 \cdots w_n$ be a word in the letters $\{1, \ldots, k\}$ and $\texttt{rot}(w) = (k+1-w_n)(k+1-w_{n-1})\ldots(k+1-w_1)$. Then $P(\texttt{rot}(w)) = \mathcal{E}(P(w))$.

For any tableau P we let $\mathrm{Rread}(P)$ denote its **row reading word**, the word formed by reading the rows from right to left and from top to bottom.

Proof of (d). As T is rectangular, we have $\mathrm{Rread}(\texttt{rot}(T)) = \texttt{rot}(\mathrm{Rread}(T))$. Hence Fact 2.1 yields

$$\texttt{rot}(T) = P(\mathrm{Rread}(\texttt{rot}(T))) = P(\texttt{rot}(\mathrm{Rread}(T))) = \mathcal{E}(P(\mathrm{Rread}(T))) = \mathcal{E}(T),$$

where the first and last equalities are the standard fact that the insertion tableau of a row reading word is just the underlying tableau. □

2.3 Proof of Theorem 2.1

Definition 2.2. Let $T \in \mathrm{SSYT}_k(m \times n)$. For a box B in T, we define $\mathrm{Dist}(T, B)$ to be the multiset

$$\mathrm{Dist}(T, B) = \{\sigma_{\{B\}}(\mathcal{P}^i(T)) \,:\, 0 \le i \le k-1\}.$$

Lemma 2.1. *If $T \in \mathrm{SSYT}_k(m \times n)$ and B is a box in T, then* $\mathrm{Dist}(T, B) = \mathrm{Dist}(\mathcal{E}(T), B)$.

We delay the proof of Lemma 2.1, first showing how Theorem 2.1 follows immediately.

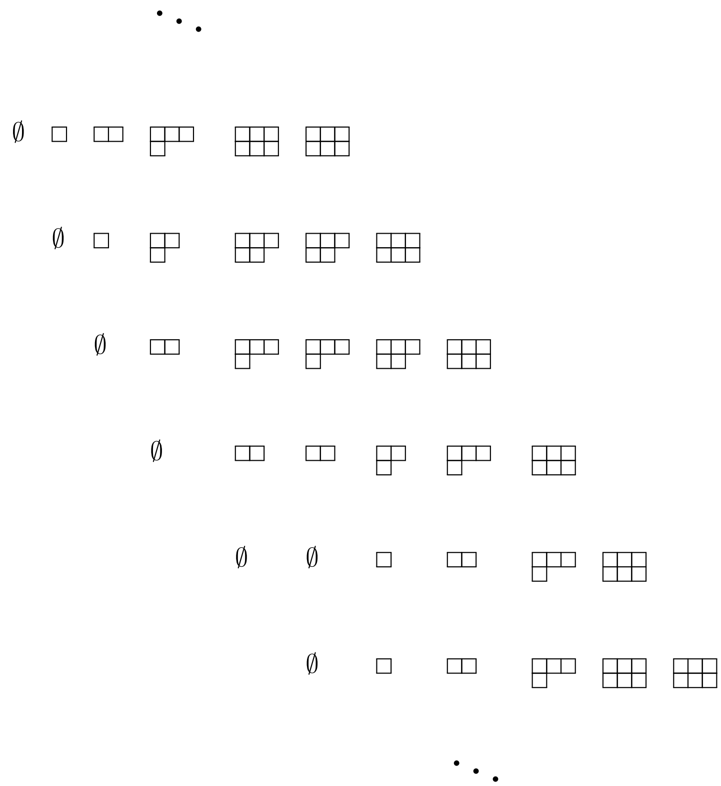
Proof of Theorem 2.1. If $T \in \mathrm{SSYT}_k(m \times n)$, then it follows from Theorem 2.3(b, c), that the orbits of T and $\mathcal{E}(T)$ under promotion are of the same size ℓ, and that $\ell | k$. By Lemma 2.1 and Theorem 2.3(b) we have the following multiset equalities

$$\{\sigma_{\{B\}}(\mathcal{P}^i(T)) \,:\, 0 \le i < \ell\} = \{\sigma_{\{B\}}(\mathcal{P}^i \circ \mathcal{E}(T)) \,:\, 0 \le i < \ell\} = \{\sigma_{\{B\}}(\mathcal{E} \circ \mathcal{P}^i(T)) \,:\, 0 \le i < \ell\}.$$

Theorem 2.3(d) now implies that $\mathrm{Dist}(T, B) = \{k+1-i \,:\, i \in \mathrm{Dist}(T, B')\}$, where B' is the box corresponding to B under $180°$ rotation. This last statement immediately implies Theorem 2.1. Specifically, the average value of σ_S on any orbit is $\frac{(k+1)|S|}{2}$. □

The remainder of this section is devoted to a proof of Lemma 2.1, using the growth diagrams of S. Fomin. (For additional information on growth diagrams, cf. [Sta99, Appendix 1] or [Sta09, §5].) For $T \in \mathrm{SSYT}_k(\lambda)$, the **growth diagram** of T is built as follows. Let $T_{\le j}$ denote the Ferrers diagram consisting of those boxes of T with entry $i \le j$. Identify T with a particular multichain in the Young lattice, explicitly with the sequence of Ferrers diagrams $(T_{\le j})_{0 \le j \le k}$. Note that this sequence uniquely encodes T. Now write this sequence of Ferrers diagrams horizontally from left to right. Below this sequence, draw, in successive rows, the sequences of Ferrers diagrams associated to $\mathcal{P}^i(T)$ for $i \ge 1$. Above this sequence, draw, in successive rows, the sequences of Ferrers diagrams associated to $\mathcal{P}^i(T)$ for $i \le -1$. This gives a doubly infinite array. Now offset each row one position to the right of the row immediately above it. Example 2.4 shows an example of this construction. The **rank** of a partition in the growth diagram is the number of partitions appearing strictly left of it in its row, or equivalently the number of partitions appearing strictly below it in its column.

Example 2.4. Let $T \in \mathrm{SSYT}_5(2 \times 3)$ be the semistandard Young tableau $\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 3 & 4 & 4 \\ \hline\end{array}$. Then the growth diagram of T is



where the top displayed row corresponds to T and the bottom displayed row to $\mathcal{P}^5(T) = T$. Each row encodes a chain of length 5, since we consider $T \in \mathrm{SSYT}_5(2 \times 3)$. $\diamondsuit$

Proof of Lemma 2.1. Let T and B be as in the statement of the lemma, and consider the growth diagram of T. We darken all shapes in the growth diagram that contain the box B (as in Example 2.5). Consider any row and the tableau R it encodes. Now look at the column containing the rightmost Ferrers diagram in this row. It is well known that, for standard T, this column is the multichain of shapes that encodes $\mathcal{E}(R)$. (See [Sta99, p. 427].) In fact the same is true for semistandard T. To verify this, we observe that, for $1 \leq j \leq k$, the shape with rank $k-j$ in the indicated column is $\mathcal{P}^j(R)_{\leq k-j}$, and we only need to verify that this is $\mathcal{E}(R)_{\leq k-j}$, i.e., that $\mathcal{P}^j(R)_{\leq k-j} = \epsilon_j(R)_{\leq k-j}$. But in fact more than this is true. The placements of the integers $1, \ldots, k-j$ in $\mathcal{P}^j(R)_{\leq k-j}$ are exactly the same as the placements of these integers in $\epsilon_j(R)$. This follows from the fact that for any V in $\mathrm{SSYT}_k(\lambda)$, and every positive integer $m \leq k$, the placements of $1, \ldots, m+1$ in V determine the placements of $1, \ldots, m$ in $\mathcal{P}(V)$.

It now follows from Theorem 2.3(b) that if a column of the growth diagram encodes a tableau V, then the column to the left of this column encodes $\mathcal{P}(V)$.

Returning to the growth diagram of T, note that if we fix any set $\{R_i : i \in I\}$ of k consecutive rows, then as multisets $\mathrm{Dist}(T, B) = \{\mathrm{rank}(D_i) : i \in I\}$, where D_i is the leftmost darkened shape in R_i. Similarly if we fix any set $\{C_j : j \in J\}$ of k consecutive columns, then $\mathrm{Dist}(\mathcal{E}(T), B) = \{\mathrm{rank}(D_j) : j \in J\}$, where D_j is the bottommost darkened shape in C_j. We call a darkened shape **row-minimal** if it is the leftmost darkened shape in some row, and **column-minimal** if it is the bottommost darkened shape in some column. We call a darkened shape **minimal** if it is either row-minimal or column-minimal.

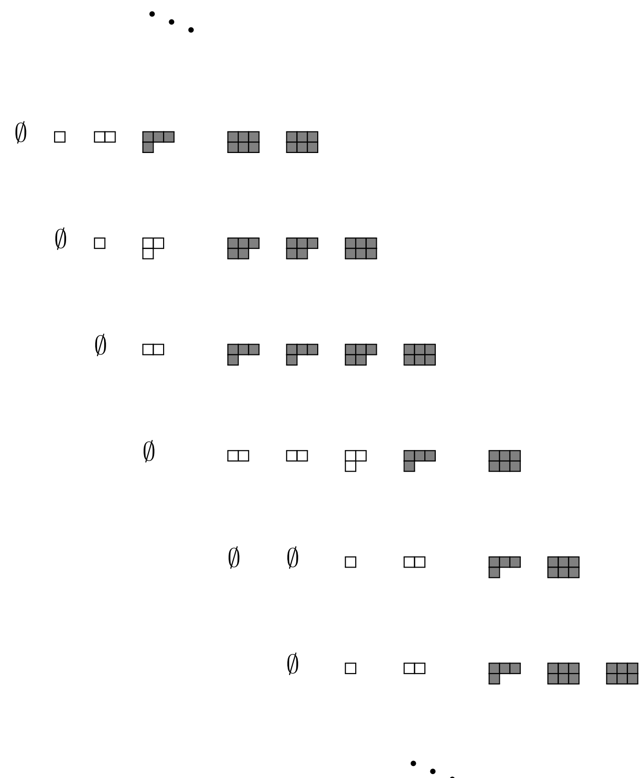
To see that $\mathrm{Dist}(T, B) = \mathrm{Dist}(\mathcal{E}(T), B)$, let $R_0, \ldots, R_k$ be any set of $k+1$ consecutive rows of the growth diagram in descending order. Let D_0 and D_k be the row-minimal shapes in rows R_0 and R_k, respectively, and note that the column containing D_k is k columns to the right of the column containing D_0. Now list all the minimal shapes in row R_0 from left to right, followed by all the minimal shapes in row R_1, and so on, concluding with just the single minimal shape D_k from row R_k. Consider all these shapes to be vertices. Note that two successive vertices D_j, D_i in this list may have the same rank, r, if D_j is column-minimal and D_i is row-minimal (in the next row). Whenever this occurs we insert a new vertex of rank $r+1$ to the right of D_j and above D_i. If the elements of the augmented list of vertices are $v_0, v_1, \ldots$, we define a path P in the first quadrant of the xy-plane by replacing each v_i by the point $(i, \mathrm{rank}(v_i))$, and connecting successive points with up-steps $(1, 1)$ and down-steps $(1, -1)$.

By the preceding paragraph $\mathrm{Dist}(T, B)$ is the multiset of ranks of row-minimal shapes in rows R_1 through R_k. By the construction of P this is the multiset M_1 of heights of right endpoints of down-steps in P. Since P starts and ends at the same height, M_1 equals the multiset M_2 of heights of left endpoints of up steps in P. By the construction of P, M_2 is the multiset of ranks of column-minimal shapes in rows R_0 through R_{k-1}, i.e., M_2 is $\mathrm{Dist}(\mathcal{E}(T), B)$. This concludes the proof. $\square$

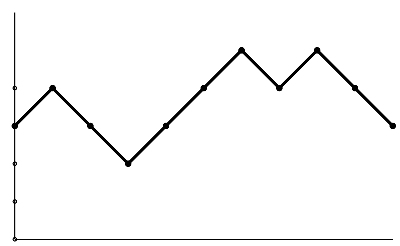
Example 2.5. As in Example 2.4, let

$$T = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 3 & 4 & 4 \\ \hline \end{array},$$

where we have shaded the box B. If we now shade all the Ferrers diagrams containing B, we obtain the following shaded growth diagram:



We have $\mathrm{Dist}(T, B) = \{2, 3, 3, 4, 4\} = \mathrm{Dist}(\mathcal{E}(T), B)$, and we obtain the path

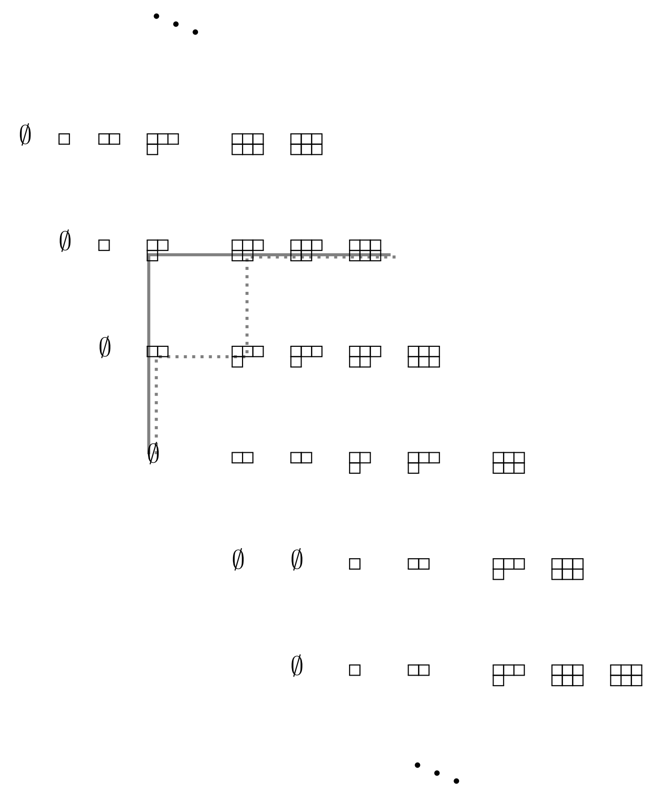


.

$\diamondsuit$

Remark 2.1. Note that the same proof shows that Lemma 2.1 remains true for $T \in \mathrm{SSYT}_k(\lambda)$ even when λ is not rectangular.

Remark 2.2. Growth diagrams are closely related to the Bender-Knuth involutions of Section 2.2. We illustrate with an example. Consider a path through the below growth diagram that starts at the left side and reaches the right by a sequence of 'hops', either one Ferrers diagram up or one Ferrers diagram to the right. This path encodes a semistandard tableau in an obvious way. In this example, the solid line encodes the tableau $A = \begin{array}{|c|c|c|}\hline 1&1&3\\ \hline 2&3&5\\ \hline\end{array}$, while the dotted line encodes $B = \begin{array}{|c|c|c|}\hline 1&1&2\\ \hline 2&3&5\\ \hline\end{array}$. It follows easily from the definitions, that 'bending' the path at a corner (or at either end) corresponds to applying a single Bender-Knuth involution, BK_i. In this example, $B = \mathrm{BK}_2(A)$ and $A = \mathrm{BK}_2(B)$. Note that this observation gives an alternative way of seeing that the central column encodes the evacuation of the top row.



$\square$

2.4 Linear extensions of cominuscule posets

In this section, all posets are assumed finite. A **linear extension** of a poset P is an order-preserving bijection onto a chain $\mathbf{d} := 1 < 2 < \cdots < d$, where $d = |P|$. Observe that standard Young tableaux of shape λ may be identified with linear extensions of λ, where we think of λ as a poset in which each box is covered by those immediately below it and to its right. We write $\mathrm{SYT}(P)$ for the set of linear extensions of a poset P. (Note that we do not generally have a notion corresponding to *semistandard* tableaux.) There are analogous definitions of promotion and evacuation in this setting (cf. [Sta09]), which we denote by $\mathcal{P}$ and $\mathcal{E}$ respectively. If S is a set of elements in the poset P and $T : P \to \mathbf{d}$ is a linear extension, we define similarly to before:

$$\sigma_S(T) := \sum_{s \in S} T(s).$$

We now prove a generalization of Theorem 2.1 to the larger class of cominuscule posets. Although we define this class algebraically, it may also be described purely combinatorially. In defining this class of posets, we mostly follow the notation and exposition of [ThYo09a]. We recommend [BiLa00] and [ThYo09a] for further details and references regarding these well-studied posets and associated geometry.

Let G be a complex connected reductive Lie group with maximal torus T. Let W denote the Weyl group $N(T)/T$. Let $\Phi = \Phi^+ \sqcup \Phi^-$ denote the root system of G, as partitioned into positive and negative roots, with Δ denoting the choice of simple roots. The set Φ^+ of positive roots has a poset structure $(\Phi^+, <)$ defined as the transitive closure of the covering relation $\alpha \lessdot \beta$ if and only if $\beta - \alpha \in \Delta$.

We say a simple root μ is **cominuscule** if for every $\alpha \in \Phi^+$, μ appears with multiplicity at most 1 in the simple root expansion of α. For μ cominuscule, let $\Lambda_\mu \subseteq (\Phi^+, <)$ be the subposet of positive roots for which μ appears in the simple root expansion. We call such a poset **cominuscule**. These posets govern much of the geometry of the so-called *cominuscule varieties*, which "next to $\mathbb{P}^n$, may be considered as the simplest examples of projective varieties" [BiLa00, §9]. In the case $G = GL_n(\mathbb{C})$, every simple root is cominuscule, the corresponding cominuscule varieties are complex Grassmannians, and the corresponding cominuscule posets are rectangles. We will study the geometry of (co)minuscule varieties in Chapter 6.

The cominuscule posets are completely classified: there are three infinite families (rectangles, shifted staircases, propellers) and two exceptional examples. These are all illustrated in Figure 2.1. We will prove additional results on cominuscule posets in Chapter 9.

The parabolic subgroups of W are in canonical bijection with the subsets of Δ. For μ cominuscule, let w_μ denote the longest element of the parabolic subgroup $W_\mu \leq W$ corresponding to the subset $\Delta \setminus \{\mu\}$. It is not hard to show that w_μ acts as an involution on Λ_μ. Following [ThYo09a, §2.2], we denote this action on Λ_μ by `rotate`. For rectangles, propellers, and the Cayley poset, this action is exactly 180° rotation. For shifted staircases and the Freudenthal poset, it is reflection across the antidiagonal.

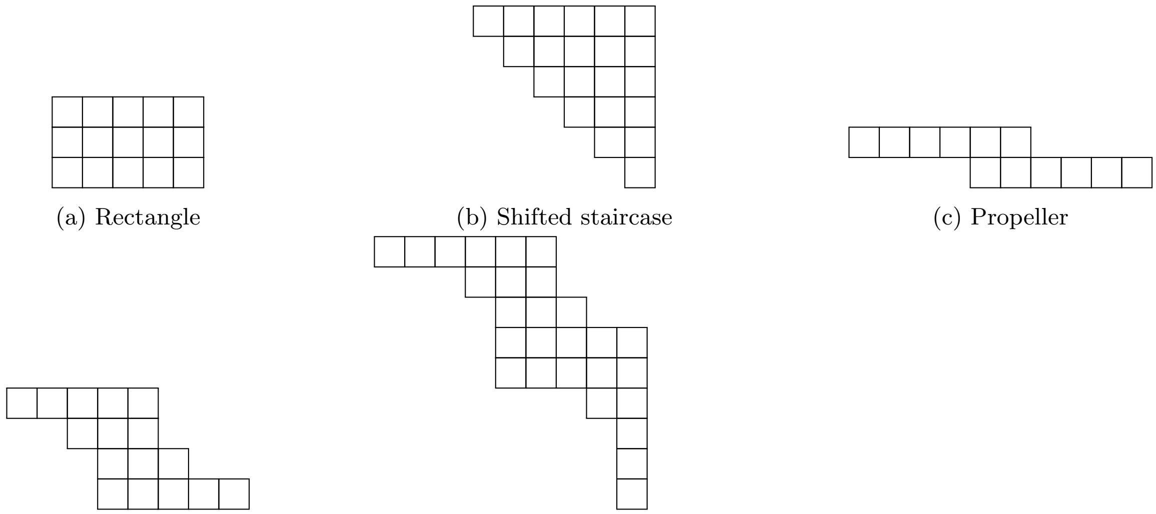
The following theorem generalizes Theorem 2.1 to include nonrectangular cominuscule posets.

Theorem 2.4. *Let P be a cominuscule poset, $S \subseteq P$ a set of elements fixed under* `rotate`*, and $\mathcal{C} = \langle c \rangle$, the cyclic group with c acting on* $\mathrm{SYT}(P)$ *by promotion. Then*

$$(\mathrm{SYT}(P), \mathcal{C}, \sigma_S)$$

is homomesic.

Proof. Let $T \in \mathrm{SYT}(P)$ with P cominuscule. By [ThYo15, Lemma 5.2], $\mathcal{E}(T)$ may be formed by applying `rotate` and reversing the alphabet (so i becomes $|P| + 1 - i$).



(a) Rectangle (b) Shifted staircase (c) Propeller

(d) Cayley poset (e) Freudenthal poset

Figure 2.1: The five families of cominuscule posets. The boxes are the elements of the poset, and each box is covered by any box immediately below it or immediately to its right. Rectangles may have arbitrary height and width. Shifted staircases have arbitrary width, and height equal to their width; hence a shifted staircase of width n contains $\binom{n+1}{2}$ elements. Propellers consist of two rows of arbitrary but equal length, overlapping by two boxes in the center. The Cayley and Freudenthal posets are unique, containing 16 and 27 elements, respectively.

The theorem then follows from a poset analogue of Lemma 2.1. For this the growth diagram proof of Lemma 2.1 may be copied nearly verbatim, using the cardinality of the promotion orbit in place of k. $\square$

Chapter 3

K-theoretic Schubert calculus

This chapter derives from joint work with Alexander Yong from [PeYo16].

3.1 Introduction

Recall $X = \mathrm{Gr}_k(\mathbb{C}^n)$ denotes the **Grassmannian** of k-dimensional linear subspaces of $\mathbb{C}^n$. Textbook discussion of Schubert calculus revolves around classes of X_λ in the cohomology ring $H^\star(X, \mathbb{Z})$; see, e.g., [Fu97]. As discussed in Chapter 1, these classes form a $\mathbb{Z}$-linear basis of $H^\star(X, \mathbb{Z})$. Their structure constants

$$\sigma \smile \sigma_\mu = \sum_\nu c^\nu_{\lambda,\mu} \sigma_\nu$$

with respect to the cup product are given by the classical Littlewood-Richardson rules (described in Section 1.2) that govern the multiplication of Schur functions.

There has been significant attention on K-theoretic Schubert calculus, which provides a richer setting for study; see, e.g., [Br05, Bu05, Va06, Kn14] and the references therein. The **Grothendieck ring** $K^0(X)$ is the free abelian group generated by isomorphism classes $[V]$ of algebraic vector bundles over X under the relation

$$[V] = [U] + [W]$$

whenever there is a short exact sequence

$$0 \to U \to V \to W \to 0.$$

The product structure on $K^0(X)$ is given by the tensor product of vector bundles, i.e.,

$$[U] \cdot [V] = [U \otimes V].$$

Since X is a smooth projective variety, the structure sheaf $\mathcal{O}_{X_\lambda}$ has a resolution

$$0 \to V_N \to V_{N-1} \to \cdots \to V_1 \to V_0 \to \mathcal{O}_{X_\lambda} \to 0$$

by locally free sheaves. Therefore it makes sense to define the class $[\mathcal{O}_{X_\lambda}]$ by

$$[\mathcal{O}_{X_\lambda}] := \sum_{j=0}^{N} (-1)^j [V_j] \in K^0(X).$$

Now, $\{[\mathcal{O}_{X_\lambda}]\}$ forms a $\mathbb{Z}$-linear basis of $K^0(X)$. Thus, define structure constants by

$$[\mathcal{O}_{X_\lambda}] \cdot [\mathcal{O}_{X_\mu}] = \sum_{\nu} a^{\nu}_{\lambda,\mu} [\mathcal{O}_{X_\nu}]. \tag{3.1}$$

A. Buch [Bu02] gave a combinatorial rule for $a^{\nu}_{\lambda,\mu}$, thereby establishing

$$(-1)^{|\nu|-|\lambda|-|\mu|} a^{\nu}_{\lambda,\mu} \geq 0.$$

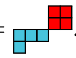
A number of other rules have been discovered since, see, e.g., [Va06, BKSTY08, ThYo09b] and the references therein, as well as the references above. In this chapter, we will prove yet another combinatorial rule and develop some related combinatorics. Our work here will pay dividends in later chapters, when we apply these ideas to other situations in which no combinatorial rule was known.

3.1.1 A. Buch's combinatorial rule for $a^{\nu}_{\lambda,\mu}$

We briefly recall the original combinatorial rule for $a^{\nu}_{\lambda,\mu}$ found by A. Buch [Bu02, Theorem 5.4]. Another proof of this original rule has recently been given in [IkSh14]. One proof of our new rule will be by bijection with this rule.

We first recall some definitions from [Bu02]. A **set-valued tableau** T of (skew) shape ν/λ is a filling of the boxes of ν/λ with non-empty finite subsets of $\mathbb{N}$ with the property that any tableau obtained by choosing exactly one label from each box is a (classical) semistandard tableau. The **column reading word** of T, denoted $\texttt{colword}(T)$ is obtained by reading the entries of T from bottom to top along columns and from left to right. The entries in a non-singleton box are read in increasing order. Such a word $(w_1, w_2, \ldots, w_N)$ is a **reverse lattice word** if the content of $(w_L, w_{L+1}, \ldots, w_N)$ is a partition for every $1 \leq L \leq N$, that is to say if its reverse is ballot. Finally, the shape $\mu \star \lambda$ is the skew shape obtained by placing μ and λ in southwest to northeast orientation with μ's northeast corner incident to λ's southwest corner. In other words

$$\mu \star \lambda = (\mu_1 + \lambda_1, \ldots, \mu_1 + \lambda_{\ell(\lambda)}, \mu_1, \mu_2, \ldots, \mu_{\ell(\mu)})/(\mu_1^{\ell(\lambda)}).$$

Example 3.1. If $\lambda =$  and $\mu =$ , then $\mu \star \lambda =$ . $\diamondsuit$

Theorem 3.1 (A. Buch [Bu02, Theorem 5.4])**.** $(-1)^{|\lambda|+|\mu|-|\nu|} a^{\nu}_{\lambda,\mu}$ *equals the number of set-valued tableaux* T *of shape* $\mu \star \lambda$ *and content* ν *such that* $\texttt{colword}(T)$ *is reverse lattice.*

Let $\texttt{Buch}_\nu(\mu \star \lambda)$ be the set of tableaux from Theorem 3.1.

Example 3.2. Let $\lambda = (2,1), \mu = (1,1)$ and $\nu = (3,2,1)$. Then $\texttt{Buch}_\nu(\mu \star \lambda)$ consists of the two tableaux

$$B_1 = \begin{array}{|c|c|c|}\hline & 1 & 1 \\ \hline & 2 & \\ \hline 1,2 & & \\ \hline 3 & & \\ \hline \end{array} \quad \text{and } B_2 = \begin{array}{|c|c|c|}\hline & 1 & 1 \\ \hline & 2 & \\ \hline 1 & & \\ \hline 2,3 & & \\ \hline \end{array}.$$

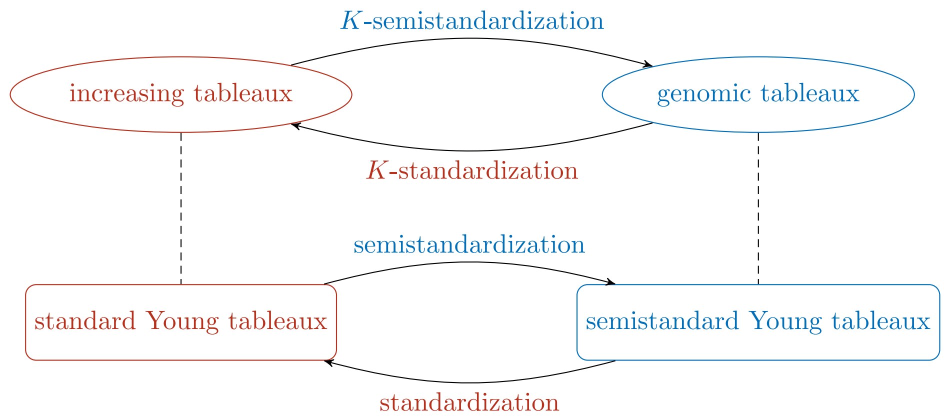
Hence $a^{\nu}_{\lambda,\mu} = -2$. $\diamondsuit$

3.1.2 History of K-theoretic combinatorics

There is interest in finding K-analogues of elements of the classical Young tableau theory; see, e.g., [Le00, Bu02, BKSTY08, ThYo09b, BuSa13, GMPPRST16, PaPy14, IkSh14, HKPWZZ15, LiMoSh16]. Although these ideas were originally studied for geometric reasons, the combinatorics has been part of a broader conversation in algebraic and enumerative combinatorics, e.g., *Hopf algebras* [LaPy07, PaPy16, Pa15], *cyclic sieving* [Pe14, Rh15, PrStVi14], *Demazure characters* [RoYo15, Mo16+], *homomesy* [BlPeSa16], longest increasing subsequences of random words [ThYo11], *plane partitions* [DiPeSt15, HPPW16], and poset edge densities [ReTeYo16].

In [ThYo09b], H. Thomas and A. Yong introduced a jeu de taquin theory for **increasing tableaux**. These tableaux are fillings of Young diagrams ν/λ with $[\ell] := 1, 2, \ldots, \ell$ where $\ell \leq |\nu/\lambda|$ and the entries increase in rows and columns (labels may be repeated). If $\ell = |\nu/\lambda|$, these are standard Young tableaux and increasing tableau results closely parallel those for standard Young tableaux. An outcome was a new Littlewood-Richardson rule for $a^{\nu}_{\lambda,\mu}$ (after [Bu02]) and its *minuscule* extension (see [ThYo09a, BuRa12, ClThYo14, BuSa13]). We recall this jeu de taquin for increasing tableau in Section 3.1.4.

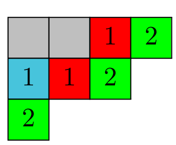
In [ThYo13], H. Thomas and A. Yong conjectured a *jeu de taquin*-based Littlewood-Richardson rule for torus-equivariant K-theory of Grassmannians. In [PeYo15b], A. Yong and the author proved this conjecture by defining *genomic tableaux* as a semistandard analogue of increasing tableaux. These ideas will be the main topic of Chapter 4.



Our goal is a theory of genomic tableaux parallel to that of [ThYo09b] for increasing tableaux. The Schubert calculus application in [PeYo15b, PeYo15c] used *edge-labeled* genomic tableaux. In this chapter, we give a logically independent development of genomic tableau combinatorics in the basic (i.e., non-edge labeled) case. Our first application is to give a new Littlewood-Richardson-type rule for (ordinary) K-theory of Grassmannians. Recently M. Gillespie and J. Levinson have applied genomic tableaux to the real geometry of Schubert curves [GiLe16], while in forthcoming work R. Kaliszewski and J. Morse relate genomic tableaux to the theory of Macdonald polynomials [KaMo16+].

3.1.3 Genomic tableau results

Let S be a semistandard Young tableau of a shape ν/λ. Place a total order on those boxes with entry i using left to right order. A **gene** $\mathcal{G}$ (of family i) is a collection of consecutive boxes in this order, where no two lie in the same row; we write $\mathtt{family}(\mathcal{G}) = i$. A **genomic tableau** T is a semistandard tableau together with a partition of its boxes into genes. We indicate the partition by color-coding the boxes. The **content** of T is the number of genes of each family. Note, a semistandard tableau T is a genomic tableau where each gene is a single box. Moreover, the content of T agrees with the usual notion for semistandard tableaux.

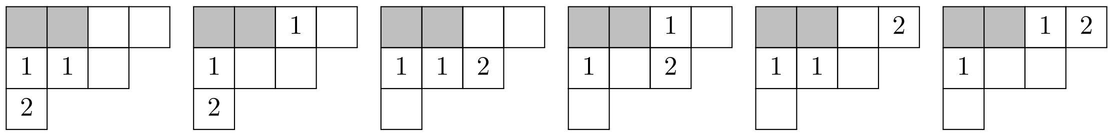
Example 3.3. $T =$ 

has content $(2, 1)$ since there are two genes of family 1 and one of family 2. $\diamondsuit$

A **genotype** G of a genomic tableau T is a choice of a single box from each gene.[1] We depict G by erasing the entries in all unchosen boxes of T.

[1] The genomic analogy is that boxes of a gene are *alleles* and the other genes of the same family are *paralogs*.

Example 3.4. Continuing Example 3.3,



are the six genotypes of T. $\diamondsuit$

Suppose U is any filling of a subset of boxes of a shape. The **sequence** $\texttt{seq}(U)$ of U is the reading word obtained by reading its entries along rows from right to left and from top to bottom (ignoring empty boxes). Now, $\texttt{seq}(U)$ is a **ballot sequence** if the number of i's that appear is always weakly greater than the number of $(i+1)$'s that appear, at any point in the sequence. A genomic tableau T is **ballot** if $\texttt{seq}(G)$ is a ballot sequence for every genotype G of T. Notice if each gene of T is a single box, there is a unique genotype (namely, the underlying semistandard tableau of T) and the concept of a ballot tableau coincides with the same notion for semistandard tableaux.

Example 3.5. The genotypes of Example 3.4 respectively have sequences: 112, 112, 211, 121, 211, and 211. Since 211 is not a ballot sequence, T is not ballot. $\diamondsuit$

Our combinatorial results are:

1. A K-analogue of the *(semi)standardization maps* between standard and semistandard tableaux. This relates genomic tableaux to increasing tableaux.

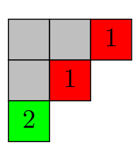
2. Using (1), we acquire genomic analogues of *Knuth equivalence*, *jeu de taquin*, *infusion* and *Bender-Knuth involutions.*

3. Using (2), we describe a new basis $\{U_\lambda\}$ of Λ where each U_λ is a generating series over genomic tableaux of shape λ. This is a deformation of the Schur basis.

Using the above results, we prove the following.

Theorem 3.2 (Genomic Littlewood-Richardson rule)**.** $a^{\nu}_{\lambda,\mu} = (-1)^{|\nu|-|\lambda|-|\mu|}$ *times the number of ballot genomic tableaux of shape* ν/λ *and content* μ*.*

Actually, in the case $|\nu| = |\lambda| + |\mu|$, $a^{\nu}_{\lambda,\mu} = c^{\nu}_{\lambda,\mu}$ and Theorem 3.2 recovers the original rule (H.1) of D.E. Littlewood-A.R. Richardson for multiplication of Schur functions [LiRi34], as described in Section 1.2.1.

Example 3.6. The tableau  is the unique witness of $a^{(3,2,1)}_{(2,1),(1,1)} = -1$. $\diamondsuit$

Using the tableau results outlined in Section 3.1.3, the proof of Theorem 3.2 is derived from the corresponding Littlewood-Richardson rule in [ThYo09b] by extending the discussion from Section 1.2.4.

3.1.4 The Thomas-Yong rule: Jeu de taquin for increasing tableaux

Let T be a filling of ν/λ with letters from the alphabet $\mathcal{A}$. For $a \in \mathcal{A}$, let T_a be the set of boxes of T that share an edge with a distinct box containing a. For two letters $a, b \in \mathcal{A}$, we may obtain a new filling of ν/λ by switching the letters to obtain $\mathsf{swap}_{a,b}(T)$; the entry of box $\mathsf{b} \in \nu/\lambda$ of $\mathsf{swap}_{a,b}(T)$ is determined as follows:

$$\mathsf{swap}_{a,b}(T)(\mathsf{b}) := \begin{cases} a, & \text{if } T(\mathsf{b}) = b \text{ and } \mathsf{b} \in T_a; \\ b, & \text{if } T(\mathsf{b}) = a \text{ and } \mathsf{b} \in T_b; \\ T(\mathsf{b}), & \text{otherwise.} \end{cases}$$

Suppose T is an increasing tableau of shape ν/λ and I a set of inner corners. Let Bullet_I be the operator that adds $\bullet$'s to the boxes in I. The **slide** of T into I is

$$\mathsf{jdt}_I(T) := \mathsf{DelBullets} \circ \mathsf{swap}_{M,\bullet} \circ \cdots \circ \mathsf{swap}_{2,\bullet} \circ \mathsf{swap}_{1,\bullet} \circ \mathsf{AddBullets}_I(T),$$

where M is the largest label of T and $\mathsf{DelBullets}$ deletes all $\bullet$'s together with their boxes.

Example 3.7.

$$\begin{array}{|c|c|c|c|} \hline & & \bullet & 1 \\ \hline & & 1 & 2 \\ \hline \bullet & 3 \\ \cline{1-2} \end{array} \xrightarrow{\mathsf{swap}_{1,\bullet}} \begin{array}{|c|c|c|c|} \hline & & 1 & \bullet \\ \hline & & \bullet & 2 \\ \hline \bullet & 3 \\ \cline{1-2} \end{array} \xrightarrow{\mathsf{swap}_{2,\bullet}} \begin{array}{|c|c|c|c|} \hline & & 1 & 2 \\ \hline & & 2 & \bullet \\ \hline \bullet & 3 \\ \cline{1-2} \end{array} \xrightarrow{\mathsf{swap}_{3,\bullet}} \begin{array}{|c|c|c|c|} \hline & & 1 & 2 \\ \hline & & 2 & \bullet \\ \hline 3 & \bullet \\ \cline{1-2} \end{array} \xrightarrow{\mathsf{DelBullets}} \begin{array}{|c|c|c|c|} \hline & & 1 & 2 \\ \hline & & 2 \\ \cline{1-3} 3 \\ \cline{1-1} \end{array}$$

$\diamondsuit$

When T is standard and $|I| = 1$, this process recovers M.-P. Schützenberger's jeu de taquin, as described in Section 1.2.3. If O is a set of outer corners of T, we also have a notion of **reverse slide** of T into O:

$$\mathsf{jdt}_O(T) := \mathsf{DelBullets} \circ \mathsf{swap}_{1,\bullet} \circ \cdots \circ \mathsf{swap}_{M-1,\bullet} \circ \mathsf{swap}_{M,\bullet} \circ \mathsf{AddBullets}_O(T),$$

Two increasing tableaux T and T' are called **jeu de taquin equivalent** if they are related by a sequence of (forward or reverse) slides.

If T is an increasing tableau of skew shape, a **rectification** of T is a increasing tableau T' that can be reached from T by a sequence of forward slides for some choices of sets of inner corners. Interestingly this rectification is not generally unique, i.e. the analogue of Theorem 1.3 does not hold for increasing tableaux.

A **unique rectification target (URT)** is an increasing tableau of straight shape that is the not jeu de taquin equivalent to any other increasing tableau of straight shape. In particular, if U is a URT and U is a rectification of T, then U is the only rectification of T; in this case only, we may safely write $\mathrm{Rect}(T) = U$.

Let T_λ be the **superstandard tableau** of shape λ, that is the standard filling of λ by row with consecutive numbers.

Example 3.8. For $\lambda = (4, 2, 1)$, $T_\lambda = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 \\ \cline{1-2} 7 \\ \cline{1-1} \end{array}$. ♢

Theorem 3.3 ([ThYo09b, Theorem 1.2])**.** *For any λ, T_λ is a unique rectification target.* □

Let $\mathrm{Inc}(\nu/\lambda)$ denote the set of increasing tableaux of shape ν/λ.

Theorem 3.4 ([ThYo09b, Theorem 1.4] and [BuSa13, Corollary 3.19])**.** *Let U be any unique rectification target of shape μ (for example T_μ). Then*

$$a^{\nu}_{\lambda,\mu} = (-1)^{|\nu|-|\lambda|-|\mu|} \#\{T \in \mathrm{Inc}(\nu/\lambda) : \mathrm{Rect}(T) = U\}. \qquad \square$$

As described for semistandard tableaux in Section 1.3, there is an **infusion** operator on layered pairs of increasing tableaux of disjoint alphabets. Let $T \in \mathrm{Inc}(\alpha)$ where α is possibly a skew shape. Let $U \in \mathrm{Inc}(\beta/\alpha)$ be an increasing tableau on a disjoint alphabet from T. We here write the labels of U as circled numbers to distinguish them from the uncircled labels of T. Consider the layered tableau (T, U) that is the union of T and U. Then

$$\mathsf{inf}(T, U) = (U^\star, T^\star)$$

is obtained as follows: Consider the largest number M that appears in T. The set of boxes of T containing M is a set I of inner corners for U. Replace these M's by $\bullet$'s and apply jeu de taquin to U at I, until the $\bullet$'s reach outer corners of U. Place M in these outer corners. Now consider the largest number M' that appears in T', the remainder of T. The boxes of T' containing M' are inner corners I' for U', the modified U. Now apply jeu de taquin to U' at I'. We continue in this manner as many times as there are distinct labels in T. The "inner" tableau of circled genes is $U^\star$ and the "outer" tableau of uncircled genes is $T^\star$. Just as for semistandard tableaux:

Proposition 3.1 ([ThYo09b, Theorem 3.1])**.** *Infusion is an involution on layered pairs of increasing tableaux:*

$$\mathsf{inf}(U^\star, T^\star) = (T, U). \qquad \square$$

3.2 K-(semi)standardization maps

Let

$$\mathtt{Gen}_\mu(\nu/\lambda) = \{\text{genomic tableaux of shape } \nu/\lambda \text{ with content } \mu = (\mu_1, \mu_2 \ldots, \mu_{\ell(\mu)})\},$$

and

$$\mathrm{Inc}\nu/\lambda = \{\text{increasing tableaux of shape } \nu/\lambda\}.$$

Define an order on the genes of $T \in \mathtt{Gen}_\mu(\nu/\lambda)$ by $\mathcal{G}_1 < \mathcal{G}_2$ if $\mathtt{family}(\mathcal{G}_1) < \mathtt{family}(\mathcal{G}_2)$ or if $\mathtt{family}(\mathcal{G}_1) = \mathtt{family}(\mathcal{G}_2)$ with all boxes of $\mathcal{G}_1$ west of all boxes of $\mathcal{G}_2$.

Lemma 3.1. *The order $<$ on genes of T is a total order.*

Proof. When showing two genes $\mathcal{G}_1$ and $\mathcal{G}_2$ are comparable in the order $<$, the only concern is if $\mathtt{family}(\mathcal{G}_1) = \mathtt{family}(\mathcal{G}_2) = k$. By definition, a gene of family k consists of boxes of entry k that are consecutive in the left to right order on such boxes. Hence either all boxes of $\mathcal{G}_1$ are west of the boxes of $\mathcal{G}_2$ or *vice versa*. □

The K**-standardization map**,

$$\Phi : \mathtt{Gen}_\mu(\nu/\lambda) \to \mathrm{Inc}\nu/\lambda,$$

is defined by filling the kth gene in the $<$-order with the entry k. Since any $T \in \mathtt{Gen}_\mu(\nu/\lambda)$ is also a semistandard tableau (by forgetting the gene structure) and since no two boxes of the same gene can be in the same row, it follows that $\Phi(T) \in \mathrm{Inc}\nu/\lambda$. Note that when each gene is a single box, Φ is the usual standardization map.

Example 3.9. If T is the genomic tableau $\begin{array}{|c|c|c|c|} \hline & & 1 & 2 \\ \hline 1 & 1 & 2 & \multicolumn{1}{c}{} \\ \cline{1-3} 2 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}$ then $\Phi(T) = \begin{array}{|c|c|c|c|} \hline & & 2 & 3 \\ \hline 1 & 2 & 3 & \multicolumn{1}{c}{} \\ \cline{1-3} 3 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}$. ♢

A **horizontal strip** is a skew shape with no two boxes in the same column. Following [ThYo09b], a **Pieri filling** is an increasing tableau of horizontal strip shape where, in addition, labels weakly increase from southwest to northeast.

Let

$$\mathcal{P}_k(\mu) := \left\{ 1 + \sum_{i<k} \mu_i, 2 + \sum_{i<k} \mu_i, \ldots, \sum_{j \le k} \mu_j \right\}.$$

That is,

$$\mathcal{P}_1(\mu) = \{1, 2, \ldots, \mu_1\},\ \mathcal{P}_2(\mu) = \{\mu_1 + 1, \ldots, \mu_1 + \mu_2\},\ \text{etc.}$$

We say $S \in \mathrm{Inc}\nu/\lambda$ is μ**-Pieri-filled** if for each $k \le \ell(\mu)$, the entries of S in $\mathcal{P}_k(\mu)$ form a Pieri filling of a horizontal strip.

Example 3.10. The increasing tableau $\begin{array}{cccc} & & \boxed{2} & \boxed{3} \\ & \boxed{1} & \boxed{4} & \\ \boxed{1} & \boxed{3} & & \end{array}$ is not $(2,2)$-Pieri-filled, as the entries 3 and 4 do not form a Pieri filling. However, it is $(2,1,1)$-Pieri-filled. $\diamondsuit$

Let

$$\mathtt{PF}_\mu(\nu/\lambda) = \{S \in \mathrm{Inc}\nu/\lambda \text{ that are } \mu\text{-Pieri-filled}\}.$$

Theorem 3.5. $\Phi : \mathtt{Gen}_\mu(\nu/\lambda) \to \mathtt{PF}_\mu(\nu/\lambda)$ *is a bijection.*

Proof. We begin by defining the K**-semistandardization map**

$$\Psi : \mathtt{PF}_\mu(\nu/\lambda) \to \mathtt{Gen}_\mu(\nu/\lambda).$$

This extends the classical semistandardization map from standard Young tableaux to semistandard Young tableaux. Suppose $S \in \mathtt{PF}_\mu(\nu/\lambda)$. Construct a filling T of ν/λ by placing into each box the unique positive integer k such that $i \in \mathcal{P}_k(\mu)$, where i is the entry of the corresponding box of S. Clearly, T is a semistandard tableau.

Declare boxes of T to be in the same gene if and only if the corresponding boxes of S contain the same value. Since S is an increasing tableau, each gene of T has at most one box in any row. Since the entries of S in $\mathcal{P}_k(\mu)$ form a Pieri filling, given any two genes $\mathcal{G}_1$, $\mathcal{G}_2$ of family k in T, every box $\mathcal{G}_1$ appears west of every box of $\mathcal{G}_2$ (or *vice versa*). Hence $T \in \mathtt{Gen}_\mu(\nu/\lambda)$.

We now show that Φ is well-defined, i.e.,

$$\operatorname{im}\Phi \subseteq \mathtt{PF}_\mu(\nu/\lambda).$$

Fix

$$T \in \mathtt{Gen}_\mu(\nu/\lambda)$$

and set

$$S := \Phi(T) \in \mathrm{Inc}\nu/\lambda.$$

Let $\gamma \subseteq \nu/\lambda$ be the set of boxes that contain k in T. By (column) semistandardness of T, γ is a horizontal strip. Since Φ puts the labels of $\mathcal{P}_k(\mu)$ into γ (in S) so as to increase southwest to northeast, the resulting filling is μ-Pieri-filled.

By construction we have that

$$\Phi \circ \Psi = \mathrm{id}_{\mathtt{PF}_\mu(\nu/\lambda)} \text{ and } \Psi \circ \Phi = \mathrm{id}_{\mathtt{Gen}_\mu(\nu/\lambda)}.$$

It is straightforward from the definitions that Ψ are Φ are injective maps. Therefore, we conclude that Φ and Ψ are mutually inverse bijections. $\square$

3.3 Genomic words and Knuth equivalence

A **genomic word** is a word s of colored positive integers such that all i's of a fixed color are consecutive among the set of all i's. A **genotype** of s is a subword that selects one letter of each color. Say s is **ballot** if every genotype of s is ballot.

Example 3.11. 212112 is a genomic word, whereas 212112 is not because the subword of 1's is 111 and the 1's are not consecutive. $\diamondsuit$

Let $\mathtt{genomicseq}(T)$ be the colored row reading word (taken in right to left and top to bottom order) of a genomic tableau T.

Lemma 3.2. *For a genomic tableau T, $\mathtt{genomicseq}(T)$ is a genomic word.*

Proof. The follows from the semistandardness of T together with the condition that the boxes of each gene of family i are consecutive in the left-to-right order on i's in T. $\square$

We extend the K-standardization map Φ to genomic words by $\Phi(s) := \mathtt{seq}(\Phi(\widehat{T}(s)))$ where $\widehat{T}(s)$ is the antidiagonal of disconnected boxes filled from northeast to southwest by the given genomic word.

Lemma 3.3.

(I) Every genomic word s is $\mathtt{genomicseq}(T)$ for some genomic tableau T.

(II) T is ballot if and only if $\mathtt{genomicseq}(T)$ is ballot.

(III) If $\mathtt{genomicseq}(T) = s$, then $\Phi(s) = \mathtt{seq}(\Phi(T))$.

Proof. For (I), in particular, one can take $T = \widehat{T}(s)$. (II) is clear. (III) is straightforward. $\square$

Example 3.12. If T is the genomic tableau $\begin{array}{|c|c|c|c|}\hline & & 1 & 2 \\ \hline 1 & 1 & 2 \\ \cline{1-3} 2 \\ \cline{1-1}\end{array}$, then $\mathtt{genomicseq}(T) = 212112$. By selecting one green letter, one red letter, and one blue letter from 212112 we arrive at three possible genotypes of $\mathtt{genomicseq}(T)$: 211, 121 and 112. Thus $\mathtt{genomicseq}(T)$ is not ballot. $\diamondsuit$

Genomic Knuth equivalence is the equivalence relation $\equiv_G$ on genomic words obtained as the transitive closure of

$$\mathbf{u}ii\mathbf{v} \equiv_G \mathbf{u}i\mathbf{v}, \tag{G.1}$$

$$\mathbf{u}iji\mathbf{v} \equiv_G \mathbf{u}jij\mathbf{v}, \tag{G.2}$$

$$\mathbf{u}jik\mathbf{v} \equiv_G \mathbf{u}jki\mathbf{v}, \tag{G.3}$$

$$\mathbf{u}pqj\mathbf{v} \equiv_G \mathbf{u}qpj\mathbf{v}, \tag{G.4}$$

where $i \leq j < k$, $p < j \leq q$, and red, blue, green are distinct colors. This equivalence relation is a genomic version of the *K-Knuth equivalence* introduced by A. Buch–M. Samuel [BuSa13, §5]. It furthermore generalizes Knuth equivalence [Kn70] in the sense that it agrees with this older notion on words where each letter is of a distinct color, obviating (G.1) and (G.2).

Theorem 3.6. *If $\mathbf{x} \equiv_G \mathbf{y}$, then $\mathbf{x}$ is ballot if and only if $\mathbf{y}$ is ballot.*

Proof. Let $\mathbf{x}$ be a genomic word. It suffices to show that (G.1)–(G.4) do not change the ballotness of $\mathbf{x}$.

(G.1) and (G.2) preserve the set of genotypes and therefore ballotness.

(G.3) clearly preserves ballotness unless $k = i + 1$. In this case, since $i \leq j < k$, this means $i = j$. Suppose therefore

$$\mathbf{x} = \mathbf{u}jjk\mathbf{v}, \text{ and that } \mathbf{y} = \mathbf{u}jkj\mathbf{v}.$$

Clearly if $\mathbf{x}$ is not ballot, then $\mathbf{y}$ is not ballot. Conversely, assume $\mathbf{x}$ is ballot. It is enough to show that $\mathbf{u}jk$ is ballot. Since $\mathbf{x}$ is ballot, the initial segment $\mathbf{u}j$ is ballot. Now, deleting the last letter of a ballot word leaves a ballot word. Since the last letter in the case at hand is j it follows that the subsequence of $\mathbf{u}j$ formed by deleting every j is ballot. In particular, every genotype of $\mathbf{u}j$ has strictly more j's than k's. Thus $\mathbf{u}jk$ (and hence $\mathbf{y}$) is ballot.

(G.4) is only a concern if $q = p + 1$. In this case, since $p < j \leq q$, we must also have $j = q$. Thus,

$$\mathbf{x} = \mathbf{u}pqq\mathbf{v} \text{ and } \mathbf{y} = \mathbf{u}qpq\mathbf{v}.$$

If $\mathbf{y}$ is ballot, then $\mathbf{x}$ is ballot. Conversely, assume $\mathbf{x}$ is ballot. It suffices to show that $\mathbf{u}qp$ is ballot. Since it is an initial segment of $\mathbf{x}$, $\mathbf{u}pqq$ is ballot. Given any two genes of family q in any genomic word, one appears entirely right of the other. Thus q does not appear in $\mathbf{u}$, and hence every genotype of $\mathbf{u}pq$ has strictly more p's than q's. Thus $\mathbf{u}qp$ is ballot. ∎

3.4 Genomic jeu de taquin

If $T \in \mathtt{Gen}_\mu(\nu/\lambda)$, an **inner corner** of T is a maximally southeast box of λ. Let I be any set of inner corners of T. We obtain a genomic tableau $\mathsf{jdt}(T)_I$ as follows: Place a $\bullet$ in each box of I; let $T^\bullet$ denote the result. Two boxes of a tableau are **neighbors** if they share a horizontal or vertical edge. For each gene $\mathcal{G}$, define the operator $\mathsf{swap}_{\mathcal{G},\bullet}$ as follows. Every box of $\mathcal{G}$ with a neighbor containing a $\bullet$ becomes a box containing a $\bullet$, while simultaneously every box with a $\bullet$ and a $\mathcal{G}$ neighbor becomes a box of $\mathcal{G}$. The remaining boxes are unchanged by $\mathsf{swap}_{\mathcal{G},\bullet}$.

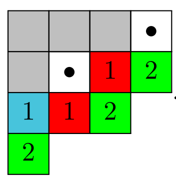
Index the genes of T as

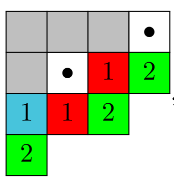
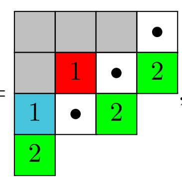
$$\mathcal{G}_1 < \mathcal{G}_2 < \cdots < \mathcal{G}_{|\mu|}$$

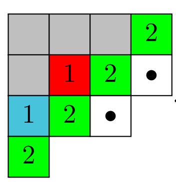
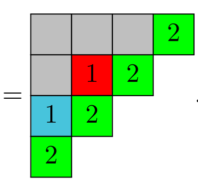
according to the total order on genes from Lemma 3.1. Then

$$\mathsf{jdt}(T)_I := \mathsf{DelBullets} \circ \mathsf{swap}_{\mathcal{G}_{|\mu|},\bullet} \circ \cdots \circ \mathsf{swap}_{\mathcal{G}_2,\bullet} \circ \mathsf{swap}_{\mathcal{G}_1,\bullet}(T^\bullet).$$

(This algorithm reduces to M.-P. Schützenberger's jeu de taquin for semistandard tableaux in the case each gene contains only a single box.)

Example 3.13. Suppose $T^\bullet$ is the genomic tableau . Then

$\mathtt{switch}_{1}^{\bullet}(T^\bullet) =$ , $\mathtt{switch}_{1}^{\bullet} \circ \mathtt{switch}_{1}^{\bullet}(T^\bullet) =$ , and

$\mathtt{switch}_{2}^{\bullet} \circ \mathtt{switch}_{1}^{\bullet} \circ \mathtt{switch}_{1}^{\bullet}(T^\bullet) =$ . So $\mathsf{jdt}(T)_I =$ . $\diamondsuit$

Define **jeu de taquin equivalence** $\sim_G$ on genomic tableaux as the symmetric, transitive closure of the relation $T \sim_G \mathsf{jdt}(T)_I$. We now state the genomic analogue of [BuSa13, Theorem 6.2] (restated as Theorem 3.8 below):

Theorem 3.7. *Let T, U be genomic tableaux. Then $T \sim_G U$ if and only if $\mathtt{genomicseq}(T) \equiv_G \mathtt{genomicseq}(U)$.*

Proof. K**-Knuth equivalence** [BuSa13, §5] is the the symmetric, transitive closure of the following K-

Knuth relations (our conventions are reversed from those of [BuSa13]; this has no effect on the applicability of their results): For words $\mathbf{u}, \mathbf{v}$ and integers $0 < i < j < k$,

$$\mathbf{u}ii\mathbf{v} \equiv_K \mathbf{u}i\mathbf{v}, \tag{K.1}$$

$$\mathbf{u}iji\mathbf{v} \equiv_K \mathbf{u}jij\mathbf{v}, \tag{K.2}$$

$$\mathbf{u}jik\mathbf{v} \equiv_K \mathbf{u}jki\mathbf{v}, \tag{K.3}$$

$$\mathbf{u}ikj\mathbf{v} \equiv_K \mathbf{u}kij\mathbf{v}. \tag{K.4}$$

Define jdt**-equivalence** ($\sim_K$) on increasing tableaux as the symmetric, transitive closure of the relation $T \sim_K \mathsf{jdt}_I(T)$. The key relationship between these two equivalence relations is:

Theorem 3.8. *[BuSa13, Theorem 6.2] $T \sim_K U$ if and only if $\mathtt{seq}(T) \equiv_K \mathtt{seq}(U)$.*

Let $\mathtt{Gen}(\nu/\lambda)$ be the set of all genomic tableaux of shape ν/λ.

Lemma 3.4. *For $T \in \mathtt{Gen}(\nu/\lambda)$ and set of inner corners I, $\Phi(\mathsf{jdt}(T)_I) = \mathsf{jdt}_I(\Phi(T))$.*

Proof. From the definitions, this is an easy induction on the number of genes of T. □

Lemma 3.5. *For any genomic words $\mathbf{u}$ and $\mathbf{v}$, we have $\mathbf{u} \equiv_G \mathbf{v}$ if and only if $\Phi(\mathbf{u}) \equiv_K \Phi(\mathbf{v})$.*

Proof. Immediate from the definitions of $\equiv_K$ and $\equiv_G$. □

By Lemma 3.4, $T \sim_G U$ if and only if $\Phi(T) \sim_K \Phi(U)$. By Theorem 3.8, the latter relation is equivalent to

$$\mathtt{seq}(\Phi(T)) \equiv_K \mathtt{seq}(\Phi(U)).$$

By Lemma 3.5 and Lemma 3.3(III), we see that

$$\mathtt{seq}(\Phi(T)) \equiv_K \mathtt{seq}(\Phi(U))$$

is equivalent to

$$\mathtt{genomicseq}(T) \equiv_G \mathtt{genomicseq}(U).$$

□

Corollary 3.1. *If $T \sim_G U$, then T is ballot if and only if U is ballot.*

Proof. By Theorem 3.7 and Theorem 3.6. □

Let T_μ be the **highest weight tableau** of shape μ, i.e., the semistandard tableau whose i-th row uses only the label i. Note T_μ may be also regarded as a genomic tableau in a unique manner. Let $S_\mu := \Phi(T_\mu)$ be the **row superstandard tableau** of shape μ (this is the tableau whose first row has entries $1, 2, 3, \ldots, \mu_1$, and whose second row has entries $\mu_1 + 1, \mu_2 + 2, \ldots, \mu_1 + \mu_2$ etc.).

Corollary 3.2 (of Lemma 3.4)**.** *For $T \in \texttt{Gen}(\nu/\lambda)$, $T \sim_G T_\mu$ if and only if $\Phi(T) \sim_K S_\mu$.*

Proof. This is immediate from Lemma 3.4 because $S_\mu = \Phi(T_\mu)$. □

3.5 Three proofs of the Genomic Littlewood-Richardson rule (Theorem 3.2)

3.5.1 Proof 1: Bijection with increasing tableaux

Our first proof uses the results of Sections 3.2–3.4 to prove Theorem 3.2. Let

$$\texttt{Ballot}_\mu(\nu/\lambda) := \{T \in \texttt{Gen}_\mu(\nu/\lambda) : T \text{ is ballot}\}.$$

Also, let

$$\texttt{IncRect}_\mu(\nu/\lambda) := \{T \in \mathrm{Inc}(\nu/\lambda) : \mathrm{Rect}(T) = S_\mu\}.$$

Lemma 3.6. *Let $T \in \texttt{Gen}_\mu(\nu/\lambda)$. Then $T \in \texttt{Ballot}_\mu(\nu/\lambda)$ if and only if $\Phi(T) \in \texttt{IncRect}_\mu(\nu/\lambda)$.*

Proof. Suppose T is ballot. By iterating application of $\texttt{jdt}_I$ (under arbitrary choices of nonempty sets I of inner corners) starting with T, we have that $T \sim_G R$ for some straight-shaped tableau R (*a priori*, R might depend on the choices of I). By Corollary 3.1, R is ballot. Since genomic jeu de taquin preserves tableau content, $R = T_\mu$. Hence, by Lemma 3.4, $\Phi(T)$ rectifies to S_μ.

Conversely, suppose $\Phi(T)$ rectifies to S_μ. Then by Lemma 3.4, T rectifies to T_μ. But T_μ is a ballot genomic tableau. Hence by Corollary 3.1, T is also ballot. □

Lemma 3.7. $\texttt{IncRect}_\mu(\nu/\lambda) \subseteq \texttt{PF}_\mu(\nu/\lambda)$.

Proof. This is part of [ThYo09b, Proof of Theorem 1.2]. □

In view of Lemmas 3.6 and 3.7, we may define

$$\phi : \texttt{Ballot}_\mu(\nu/\lambda) \to \texttt{IncRect}_\mu(\nu/\lambda)$$

as the restriction

$$\Phi|_{\mathtt{Ballot}_\mu(\nu/\lambda)}$$

and define

$$\psi : \mathtt{IncRect}_\mu(\nu/\lambda) \to \mathtt{Ballot}_\mu(\nu/\lambda)$$

as the restriction

$$\Psi|_{\mathtt{IncRect}_\mu(\nu/\lambda)}.$$

Now Φ and Ψ are mutually inverse bijections (cf. Theorem 3.5). Thus ϕ and ψ are mutually inverse bijections between $\mathtt{IncRect}_\mu(\nu/\lambda)$ and $\mathtt{Ballot}_\mu(\nu/\lambda)$. Hence the theorem follows from the $\mathtt{Kjdt}$ rule of [ThYo09b] for $a^\nu_{\lambda,\mu}$. □

3.5.2 Proof 2: Bijection with set-valued tableaux

In our next proof, we relate genomic tableaux to the original rule for $a^\nu_{\lambda,\mu}$ found by A. Buch [Bu02, Theorem 5.4] and described in Section 3.1.1.

We define a map

$$\Xi : \mathtt{Buch}_\nu(\mu \star \lambda) \to \mathtt{Ballot}_\mu(\nu/\lambda)$$

as follows. Let $T \in \mathtt{Buch}_\nu(\mu \star \lambda)$. Start with an empty shape λ. Read the columns of the μ portion of T from top to bottom and right to left. Suppose a set

$$S = \{s_1 < \ldots < s_t\}$$

gives the entries of a box in row i of the μ shape. Then place a new gene of family i in the rows $s_1, \ldots, s_t$ (as far left as possible in each case). Then Ξ clearly has a (putative) inverse

$$\Theta : \mathtt{Ballot}_\mu(\nu/\lambda) \to \mathtt{Buch}_\nu(\mu \star \lambda)$$

that records in row i and column 1 of the μ shape the rows that the leftmost gene of family i sits in. Similarly, in row i and column 2 we record the rows that the second leftmost gene of family i sits in, etc.

Theorem 3.9. $\Xi : \mathtt{Buch}_\nu(\mu \star \lambda) \to \mathtt{Ballot}_\mu(\nu/\lambda)$ *and* $\Theta : \mathtt{Ballot}_\mu(\nu/\lambda) \to \mathtt{Buch}_\nu(\mu \star \lambda)$ *are well-defined and mutually inverse bijections.*

Example 3.14. Continuing Example 3.2, we have

$$\Xi(B_1) = \begin{array}{|c|c|c|} \hline & & 1 \\ \hline & 1 & \multicolumn{1}{c}{} \\ \cline{1-2} 2 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} \text{ and } \Xi(B_2) = \begin{array}{|c|c|c|} \hline & & 1 \\ \hline & 2 & \multicolumn{1}{c}{} \\ \cline{1-2} 2 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}.$$

The reader can check that these are the unique two elements of $\mathtt{Ballot}_\mu(\nu/\lambda)$. $\diamondsuit$

Proof of Theorem 3.9. Let $T \in \mathtt{Buch}_\nu(\mu \star \lambda)$ and set $U := \Xi(T)$.

(Ξ is well-defined): By definition, the number of genes of family i is μ_i. Hence the content of U is μ, as required. Next, observe that since in each row of T the entries increase weakly from left to right, no two genes of the same family interweave. Also note that no two labels of the same gene are in the same column since otherwise we would obtain that $\mathtt{colword}(T)$ is not reverse lattice, since labels in the same box are read in increasing order, a contradiction.

The hypothesis that $\mathtt{colword}(T)$ is reverse lattice precisely guarantees that when adding the boxes in the rows of S one takes a Young diagram to a larger Young diagram. Thus U is a tableau of (skew) Young diagram shape. Note that since S is a set, no row of U contains two boxes of the same gene.

We next verify the semistandardness conditions. Suppose U violates the horizontal semistandardness requirement. That is, there is a box x directly left and adjacent to a box y in U such that $\mathtt{lab}_U(\mathsf{x}) > \mathtt{lab}_U(\mathsf{y})$. Let x' and y' be the boxes in T that added x and y during the execution of Ξ. Since $\mathtt{lab}_U(\mathsf{x}) > \mathtt{lab}_U(\mathsf{y})$, by Ξ's definition, the row of x' is strictly south of the row of y'. Moreover, since x is left of y we know that x' is read before y' in $\mathtt{colword}(T)$. Therefore, y' is strictly north and strictly west of x'. However, since T is a (set-valued) semistandard tableau, the labels of y' in T are all strictly smaller than those of x'. This implies that y is in a row strictly north of that of x, a contradiction. The argument that U satisfies the vertical semistandardness requirement is similar.

It remains to check that U is ballot. To do this, make an arbitrary but fixed choice of genotype G_U of U. The labels of family i and $i+1$ may be blamed on labels in rows i and $i+1$ of T. Suppose the sets of labels in those rows are

$$Q_1, Q_2, \ldots, Q_t, Q_{t+1}, \ldots, Q_{t+s} \quad (\text{row } i) \text{ and } \quad R_1, R_2, \ldots, R_t \quad (\text{row } i+1)$$

where $s \geq 0$. Since we know U is semistandard, the labels associated to rows i and $i+1$ separately form a Pieri strip. Here Q_1 is associated to the rightmost gene of family i (in U) and Q_{t+s} is associated to the leftmost gene of family i (in U). Similarly, R_1 is associated to the rightmost gene of family $i+1$ (in U) and

R_t is associated to the leftmost gene of family $i+1$ (in U). By the vertical semistandardness of T, we have

$$\max Q_i < \min R_i \text{ for } 1 \leq i \leq t.$$

This clearly implies that the mth rightmost label of family $i+1$ in G_U is strictly south and weakly west of the mth rightmost label of family i in G_U, for $1 \leq m \leq t$. Since this is true for each i, G_U is ballot.

(Ξ is injective): Clear.

(Θ is well-defined:) This is proved with the same arguments (said in reverse) as those given in the well-definedness of Ξ.

(Θ is injective): Clear.

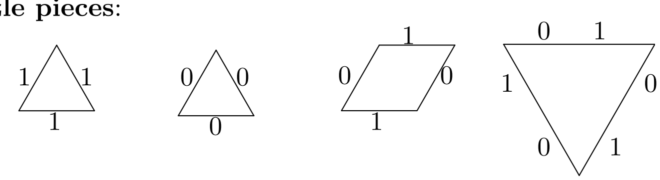
The theorem follows since Ξ and Θ are mutually inverse injections. $\square$

Composing Theorem 3.9 with the bijection of Section 3.5.1 permits one to biject the above rule of A. Buch with the K-theoretic jeu de taquin rule of [ThYo09b].

3.5.3 Proof 3: Bijection with puzzles

A third proof of Theorem 3.2 is by extending the bijection given in Section 1.2.4 between the tableaux of (H.1) and the puzzles of (H.2). In Chapter 5, we describe and prove a bijection between more general genomic tableaux and (a slight modification of) the Knutson-Vakil puzzles of [CoVa05, §5]. It is not hard to see that this restricts to a bijection between the tableaux of Theorem 3.2 and the ordinary K-theory puzzles discovered by T. Tao [Va06, §3.3]. Since the latter are known to calculate $a_{\lambda,\mu}^{\nu}$, Theorem 3.2 follows. We describe the rule here but omit the details of the bijection, in favor of the full argument in Chapter 5.

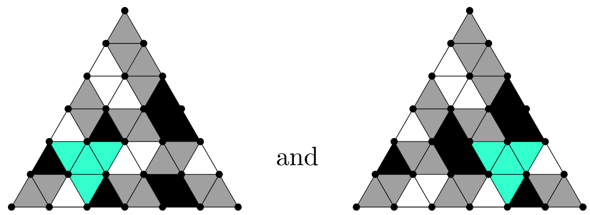
Consider the n-length equilateral triangle oriented as Δ. Impose boundary conditions according to λ, μ, ν (thought of as binary sequences) to produce $\Delta_{\lambda,\mu,\nu}$ as in Section 1.2.2. A **K-puzzle** is a filling of $\Delta_{\lambda,\mu,\nu}$ with the following **puzzle pieces**:



Henceforth, we color code these pieces as black, white, gray, and blue respectively, dropping the numerical labels. A **filling** requires that the common edges of adjacent puzzle pieces share the same label. The first three may be rotated but the fourth (**K-piece**) may not. A **K-puzzle** is a puzzle filling of Δ.

Theorem 3.10 (T. Tao [Va06, §3.3]). $(-1)^{|\nu|-|\lambda|-|\mu|}a_{\lambda,\mu}^{\nu} = \#\{K\text{-}puzzles\ of\ \Delta_{\lambda,\mu,\nu}\}$. $\square$

Example 3.15. Continuing Example 3.14 and assuming the Grassmannian in question is $\mathrm{Gr}_3(\mathbb{C}^6)$, the bijection of Chapter 5 matches $\Xi(B_1)$ and $\Xi(B_2)$ to the puzzles



respectively, where we use the color-coding of puzzle pieces described above. It is straightforward to check that these are the only K-puzzles in the sense of [Va06, §3.3] for this structure constant. ♢

3.6 Infusion, Bender-Knuth involutions, and the genomic Schur function

We first define **genomic infusion**. Let $T \in \mathtt{Gen}(\alpha)$ and $U \in \mathtt{Gen}(\beta/\alpha)$ where α is possibly a skew shape. We think of a layered tableau (T, U) that is the union of T and U. For convenience, the labels of T will be circled. Then

$$\mathtt{geninf}(T, U) = (U^\star, T^\star)$$

is obtained by the following procedure. Consider the largest gene $\mathcal{G}$ (under the $<$ order) that appears in T. The boxes of this gene are inner corners I with respect to U. Now apply $\mathtt{jdt}_I(U)$, leaving some outer corners of β. Place into these outer corners $\mathcal{G}$. Now consider the second largest gene $\mathcal{G}'$ that appears in T. These will form inner corners I' with respect to $U' := \mathtt{jdt}_I(U)$. Now apply $\mathtt{jdt}_{I'}(U')$ again leaving some outer corners of which we will fill with $\mathcal{G}'$. We continue in this manner until we have exhausted all genes of T. The "inner" tableau of uncircled genes is $U^\star$ and the "outer" tableau of circled genes is $T^\star$. Clearly, if α is a straight shape, then $U^\star$ is a genomic rectification of U where the order of rectification is imposed by T. Furthermore:

Proposition 3.2. *Genomic infusion is an involution, i.e.,*

$$\mathtt{geninf}(U^\star, T^\star) = (T, U).$$

Proof. This follows from the fact that *K-infusion* as defined in [ThYo09b, §3.1] is an involution [ThYo09b, Theorem 3.1], combined with Lemma 3.4. □

Next we define **genomic Bender-Knuth** involutions. Given a genomic tableau V consider the genomic subtableau T consisting of genes of family i and consider the genomic subtableau U consisting of genes of family $i+1$. Now define $\mathtt{genBK}_i(V)$ to be obtained by replacing inside V the subtableau (T, U) with $(U^\star, T^\star)$, switching the labels i and $i+1$, keeping all other boxes of V the same (and removing any circlings).

Proposition 3.3. *$\mathtt{genBK}_i$ is an involution. Moreover, $\mathtt{genBK}_i$ defines a bijection from the set of genomic tableaux of a shape ν/λ of content $\gamma = (\gamma_1, \ldots, \gamma_i, \gamma_{i+1}, \ldots)$ to the set of genomic tableaux of shape ν/λ of content $\gamma = (\gamma_1, \ldots, \gamma_{i+1}, \gamma_i, \ldots)$.*

Proof. The first sentence is immediate from Proposition 3.2. The second sentence follows from the definition of $\mathtt{genBK}_i$ and the first sentence. □

From these genomic Bender-Knuth involutions, one can define genomic versions of M.-P. Schützenberger's *promotion* and *evacuation* operators. (The classical theory was described in Section 2.2.) We do not analyze these notions further in this thesis.

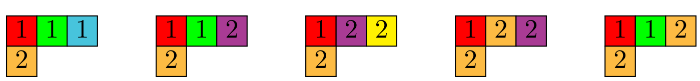
We explore the **genomic Schur function**, which we define as

$$U_{\nu/\lambda} := \sum_{T \in \mathrm{Gen}(\nu/\lambda)} \mathbf{x}^T$$

where

$$\mathbf{x}^T := \prod_i x_i^{\#\text{ genes of family } i \text{ in } T}.$$

Example 3.16. The polynomial $U_{31}(x_1, x_2)$ is computed by the tableaux



Hence $U_{31}(x_1, x_2) = x_1^3 x_2 + x_1^2 x_2^2 + x_1 x_2^3 + x_1 x_2^2 + x_1^2 x_2 = s_{31}(x_1, x_2) + s_{21}(x_1, x_2)$. ♢

Theorem 3.11. $U_{\nu/\lambda} \in \Lambda$.

Proof. The argument is an extension of the combinatorial proof of symmetry of Schur functions, as given in Corollary 1.1. Here we use Proposition 3.3 in place of Proposition 1.2. □

Since

$$U_{\nu/\lambda} = s_{\nu/\lambda} + \text{lower degree terms},$$

$\lambda\backslash\mu$	1	2	11	3	21	111	31	22	211	32	311	221	33	321	222	331	322	332	333
1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
21	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
111	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
31	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0
22	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0
211	0	0	0	0	0	2	0	0	1	0	0	0	0	0	0	0	0	0	0
32	0	0	0	0	0	1	0	1	1	1	0	0	0	0	0	0	0	0	0
311	0	0	0	0	0	1	0	0	2	0	1	0	0	0	0	0	0	0	0
221	0	0	0	0	0	1	0	0	1	0	0	1	0	0	0	0	0	0	0
33	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0
321	0	0	0	0	0	1	0	0	2	0	1	2	0	1	0	0	0	0	0
222	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
331	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	1	0	0	0
322	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	0	1	0	0
332	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	1	0
333	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

Table 3.1: Transition matrix between the $\{U_\lambda(x_1, x_2, x_3)\}$ to $\{s_\mu(x_1, x_2, x_3)\}$ bases.

by Theorem 3.11 we have that $\{U_\lambda\}$, where λ ranges over all (straight) partitions, is a basis of Λ.

While in small examples $U_{\nu/\lambda}$ is Schur-positive (cf. Table 3.1), this is not true in general:

Example 3.17. One may check that 38 tableaux contribute to $U_{333}(x_1, x_2, x_3, x_4)$. Expanding this polynomial in the Schur basis yields

$$\begin{aligned} U_{333}(x_1,x_2,x_3,x_4) &= s_{333}(x_1,x_2,x_3,x_4) + s_{3221}(x_1,x_2,x_3,x_4) \\ &\qquad + s_{2221}(x_1,x_2,x_3,x_4) - s_{2222}(x_1,x_2,x_3,x_4). \qquad \diamondsuit \end{aligned}$$

Also, the structure coefficients for the U-basis do not possess any positivity or alternating positivity properties:

Example 3.18. Using Table 3.1, one can check that $U_{22} \cdot U_1 = U_{32} + U_{221} - U_{22} - U_{111}$. $\diamondsuit$

At present, we are unaware of any geometric significance of these polynomials.

Chapter 4

A Littlewood-Richardson rule in torus-equivariant K-theory

This chapter derives from joint work with A. Yong [PeYo15b].

4.1 Introduction

4.1.1 Overview

Recall $X = \mathrm{Gr}_k(\mathbb{C}^n)$ denotes the Grassmannian of k-dimensional subspaces of $\mathbb{C}^n$ and that the (classical) **Schubert structure constants** $c_{\lambda,\mu}^{\nu}$ are defined by $\sigma_\lambda \cdot \sigma_\mu = \sum_\nu c_{\lambda,\mu}^{\nu} \sigma_\nu$, where $\{\sigma_\theta\}$ are the cohomological Schubert classes. Combinatorially, $c_{\lambda,\mu}^{\nu}$ is computed, in a manifestly nonnegative manner, by Littlewood-Richardson rules as described in Section 1.2.

In the modern Schubert calculus, there is significant attention on the problem of generalizing the above work to richer cohomology theories. The structure coefficients for the multiplication of the Schubert structure sheaves in K-theory were studied in Chapter 3, where we gave several combinatorial rules that are positive after accounting for a predictable alternation of sign. In [KnTa03], A. Knutson-T. Tao introduced certain other puzzles to give a rule for torus-equivariant Schubert calculus that is positive in the sense of [Gr01].

In this chapter, we turn to a unification of these problems. Let $K_{\mathsf{T}}(X)$ denote the Grothendieck ring of T-equivariant vector bundles over X. This ring has a natural $K_{\mathsf{T}}(\mathrm{pt})$-module structure and an additive basis given by the classes of Schubert structure sheaves; for background, we refer the reader to, e.g., [KoKu90, AnGrMi11] and the references therein. The analogues of Littlewood-Richardson coefficients are the Laurent polynomials $K_{\lambda,\mu}^{\nu} \in \mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}] \cong K_{\mathsf{T}}(\mathrm{pt})$ defined by

$$[\mathcal{O}_{X_\lambda}] \cdot [\mathcal{O}_{X_\mu}] = \sum_{\nu \subseteq k \times (n-k)} K_{\lambda,\mu}^{\nu} [\mathcal{O}_{X_\nu}],$$

where $[\mathcal{O}_{X_\lambda}]$ is the class of the structure sheaf of X_λ. These coefficients may be algebraically computed using *double Grothendieck polynomials*; see [LaSc82, FuLa94]. The problem addressed by this paper is to prove a combinatorial rule for $K_{\lambda,\mu}^{\nu}$.

We summarize past contributions to the problem: A. Knutson-R. Vakil conjectured a formula for $K_{\lambda,\mu}^{\nu}$ in terms of puzzles (reported in [CoVa05, §5]). V. Kreiman [Kre05] proved a rule for the case $\lambda = \nu$, corresponding to a certain localization (cf. Section 4.4). C. Lenart-A. Postnikov [LePo07] gave a rule for the case $\lambda = (1)$ (in a broader context applicable to any generalized flag variety); we use this result. Later, W. Graham-S. Kumar [GrKu08] determined the coefficients in the case $X = \mathbb{P}^{n-1}$. "Positivity" of $K_{\lambda,\mu}^{\nu}$ (in a more general context) was geometrically established by D. Anderson-S. Griffeth-E. Miller [AnGrMi11]. More recently, A. Knutson [Kn10] obtained a puzzle rule in $K_{\mathsf{T}}(X)$ for the different problem of multiplying the class of a Schubert structure sheaf by that of an *opposite* Schubert structure sheaf. Finally, H. Thomas and A. Yong conjectured the first Young tableau rule for $K_{\lambda,\mu}^{\nu}$ [ThYo13, Conjecture 4.7]; they showed their conjectural rule is [AnGrMi11]-positive; see [ThYo13, §4.1]. No combinatorial rule for structure coefficients of $K_{\mathsf{T}}(X)$ with respect to any fixed basis has earlier been proved.

This paper introduces and proves an [AnGrMi11]-positive rule for the structure coefficients $K_{\lambda,\mu}^{\nu}$ (Theorem 4.1); in fact, our rule exhibits a further property of the coefficients which seems at present not to have a geometric explanation. The rule allows us to deduce the aforementioned conjecture of [ThYo13]. Indeed, we complete the strategy set out in *loc. cit.* and our Theorem 4.1 is a generalization of the rule of [ThYo13] for T-equivariant cohomology. The first step of our proof is to relate our combinatorial rule to a K-theoretic generalization of a recurrence proven by A. Molev-B. Sagan [MoSa99] and A. Knutson-T. Tao [KnTa03] (who also credit A. Okounkov). A similar step was employed by A. Buch [Bu15] who gave a rule for the equivariant quantum cohomology of Grassmannians, cf. [BuMi11]. (The case of non-equivariant quantum cohomology had been previously handled geometrically by [Co09] and combinatorially by [BKPT14], cf. [BuKrTa03].)

In Chapter 5, we use our new rule to also resolve the 2005 puzzle conjecture of A. Knutson-R. Vakil. More precisely, we first show that their conjecture is false by explicit counterexample. On the other hand, our rule suggests a mild correction of their conjecture, which we then prove.

The main innovation of this paper is *genomic tableaux* and a generalization of M.-P. Schützenberger's *jeu de taquin* [Sc77]. We anticipate additional applications of these ideas. C. Monical has reported use of genomic tableaux in the study of *Lascoux polynomials* (see, e.g., [RoYo15] and references therein) and K-theoretic analogues of *Demazure atoms*, extending results of [HLMvW11]. These tableaux also give a new rule for (non-equivariant) K-theory of Grassmannians; the announcement [PeYo15a] outlines applications to analogous problems when X is replaced by Lagrangian or maximal orthogonal Grassmannians. A full development of these results within a theory of genomic tableaux will be found in our forthcoming work [PeYo16]. Moreover, closely related to the equivariant Schubert calculus of X, the combinatorial rule of A. Molev-B. Sagan [MoSa99] solves a *triple Schubert calculus* problem in $H^{\star}(\mathsf{GL}_n(\mathbb{C})/\mathsf{B} \times X \times \mathsf{GL}_n(\mathbb{C})/\mathsf{B})$

(see [KnTa03, §6]). Our methods should extend to give a K-theoretic analogue, cf. [KnTa03, §6.2]. Finally, we remark that A. Buch (private communication) has shown us a short argument that turns our rule into an [AnGrMi11]-positive rule for the structure coefficients with respect the basis of $K_{\mathsf{T}}(X)$ dual to $\{[\mathcal{O}_\lambda]\}$.

4.1.2 Genomic tableaux

A **genomic** tableau is a Young diagram filled with (subscripted) labels i_j where $i \in \mathbb{Z}_{>0}$ and the j's that appear for each i form an initial segment of $\mathbb{Z}_{>0}$. It is **edge-labeled** of shape ν/λ if each horizontal edge of a box weakly below the southern border of λ (viewed as a lattice path from $(0,0)$ to $(k, n-k)$) is filled with a subset of $\{i_j\}$.

Let $\mathsf{x}^{\rightarrow}$ be the box immediately east of x, $\mathsf{x}^{\uparrow}$ the box immediately north of x, etc. For a box x, let $\overline{\mathsf{x}}$ denote the upper horizontal edge of x and $\underline{\mathsf{x}}$ denote the lower horizontal edge. We write $\mathtt{family}(i_j) = i$. We distinguish two orders on subscripted labels. Say $i_j < k_\ell$ if $i < k$. Write $i_j \prec k_\ell$ if $i < k$ or $i = k$ with $j < \ell$. Note that $\prec$ is a total order, while $<$ is not.

A genomic tableau T is **semistandard** if the following four conditions hold:

(S.1) $\mathtt{label}(\mathsf{x}) \prec \mathtt{label}(\mathsf{x}^{\rightarrow})$;

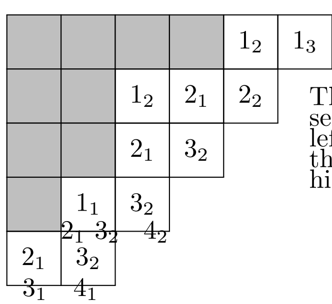
(S.2) every label is $<$-strictly smaller than any label South[1] in its column;

(S.3) if i_j, k_ℓ appear on the same edge then $i \neq k$;

(S.4) if i_j is West of i_k, then $j \leq k$.

Refer to the multiset $\{i_j\}$ (for fixed i and j) collectively as a **gene**. The **content** of T is $(c_1, c_2, c_3, \ldots)$ where c_i is the number of genes of family i. Suppose x is in row r. A label i_j is **too high** if $i \geq r$ and $i_j \in \overline{\mathsf{x}}$, or alternatively if $i > r$ and $i_j \in \mathsf{x}$ or $i_j \in \underline{\mathsf{x}}$.

Example 4.1. For $\lambda = (4,2,2,1)$ and $\nu = (6,5,4,3,2)$ consider the genomic tableau T:



The content of T is $(3,2,2,2)$. The tableau T is not semistandard, since the second column from the left fails (S.2). If we deleted the 3_2 from the edge, the result would be semistandard. No label is too high.

$\diamondsuit$

[1]Throughout, we write "West", "west" and "NorthWest" to mean "strictly west", "weakly west" and "strictly north and strictly west" respectively, etc.

4.1.3 The ballot property

A **genotype** G of T is a choice of one label from each gene of T. Let $\texttt{word}(G)$ be obtained by reading G down columns from right to left. (If there are multiple labels on an edge, read them from smallest to largest in $\prec$-order.) Then G is **ballot** if in every initial segment of $\texttt{word}(G)$, there are at least as many labels of family i as of family $i+1$, for each $i \geq 1$. We say T is **ballot** if all of its genotypes are ballot. Let $\texttt{BallotGen}(\nu/\lambda)$ be the set of ballot, semistandard, edge-labeled genomic tableaux of shape ν/λ where no label is too high.

Example 4.2. Let $T = \begin{array}{|c|c|}\hline & 1_2 \\ \hline 1_1 & 2_1 \\ \hline\end{array}$ and $U = \begin{array}{|c|c|}\hline & 1_1 \\ \hline 1_1 & 2_1 \\ \hline\end{array}$. Then T is ballot: the one genotype (itself) has reading word is $1_2 2_1 1_1$, which is a ballot sequence. U is not ballot: it has two genotypes $\begin{array}{|c|c|}\hline & \\ \hline 1_1 & 2_1 \\ \hline\end{array}$ and $\begin{array}{|c|c|}\hline & 1_1 \\ \hline & 2_1 \\ \hline\end{array}$ and the word for the former is $2_1 1_1$, which is not ballot. ♢

4.1.4 Tableau weights and the main theorem

Let $T \in \texttt{BallotGen}(\nu/\lambda)$. For a box x, $\mathsf{Man}(\mathsf{x})$ is the "Manhattan distance" from the southwest corner (point) of $k \times (n-k)$ to the northwest corner (point) of x (the length of any north-east lattice path between the corners).

For a gene $\mathcal{G}$, let $N_{\mathcal{G}}$ be the number of genes $\mathcal{G}'$ with $\texttt{family}(\mathcal{G}') = \texttt{family}(\mathcal{G})$ and $\mathcal{G}' \succ \mathcal{G}$. For instance, in Example 4.1, $N_{1_1} = 2$ since the genes 1_2 and 1_3 are of the same family as 1_1 (namely family 1) but $1_1 \prec 1_2, 1_3$.

If $\ell = i_j \in \underline{\mathsf{x}}$ and x is in row r, then

$$\texttt{edgefactor}(\ell) := \texttt{edgefactor}_{\underline{\mathsf{x}}}(i_j) := 1 - \frac{t_{\mathsf{Man}(\mathsf{x})}}{t_{r-i+N_{i_j}+1+\mathsf{Man}(\mathsf{x})}}. \tag{4.1}$$

The **edge weight** $\texttt{edgewt}(T)$ is $\prod_\ell \texttt{edgefactor}(\ell)$; the product is over edge labels of T.

A nonempty box x in row r is **productive** if $\texttt{label}(\mathsf{x}) < \texttt{label}(\mathsf{x}^{\rightarrow})$. If $i_j \in \mathsf{x}$, set

$$\texttt{boxfactor}(\mathsf{x}) := \frac{t_{\mathsf{Man}(\mathsf{x})+1}}{t_{r-i+N_{i_j}+1+\mathsf{Man}(\mathsf{x})}}. \tag{4.2}$$

The **box weight** of a tableau T is $\texttt{boxwt}(T) := \prod_{\mathsf{x}} \texttt{boxfactor}(\mathsf{x})$, where the product is over all productive boxes of T. The **weight** of T is $\texttt{wt}\, T := (-1)^{d(T)} \times \texttt{boxwt}(T) \times \texttt{edgewt}(T)$. Here $d(T) = \sum_{\mathcal{G}}(|\mathcal{G}| - 1)$, where the sum is over all genes $\mathcal{G}$ and $|\mathcal{G}|$ is the (multiset) cardinality of $\mathcal{G}$. Set

$$L^{\nu}_{\lambda,\mu} := \sum_T \texttt{wt}\, T,$$

where the sum is over all $T \in \texttt{BallotGen}(\nu/\lambda)$ that have content μ.

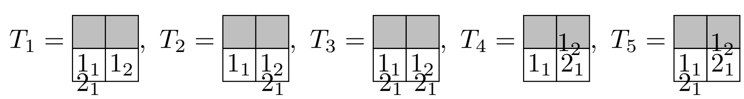
Theorem 4.1 (Main Theorem)**.** $K^{\nu}_{\lambda,\mu} = L^{\nu}_{\lambda,\mu}$.

This provides the first proved rule for $K^{\nu}_{\lambda,\mu}$ that is manifestly [AnGrMi11]-positive. That is, let $z_i := \frac{t_i}{t_{i+1}} - 1$. For $j > i$, we have

$$\frac{t_i}{t_j} = \prod_{k=i}^{j-1}(z_k + 1) \quad \text{and} \quad 1 - \frac{t_i}{t_j} = -\left(\prod_{k=i}^{j-1}(z_k + 1) - 1\right). \tag{4.3}$$

Therefore, $(-1)^{\#\text{edge labels}} \times \texttt{boxwt}(T) \times \texttt{edgewt}(T)$ is z-positive. Since clearly $d(T) = |\nu| - |\lambda| - |\mu| + \#\text{edge labels}$, we have that $(-1)^{|\nu|-|\lambda|-|\mu|} L^{\nu}_{\lambda,\mu} = \sum_T (-1)^{|\nu|-|\lambda|-|\mu|}\, \texttt{wt}\, T$ is z-positive. This positivity is the same as that of [AnGrMi11, Corollary 5.3] after the substitution $z_i \mapsto e^{\alpha_i} - 1$ where α_i is the i-th simple root for the root system A_{n-1}.

Example 4.3. To compute $K^{(2,2)}_{(2),(2,1)}$ for $\mathrm{Gr}_2(\mathbb{C}^4)$, the required tableaux are



Then

- $\texttt{edgewt}(T_1) = 1 - \frac{t_1}{t_2}$, $\texttt{boxwt}(T_1) = \frac{t_3}{t_4}$ and $d(T_1) = 0$;
- $\texttt{edgewt}(T_2) = 1 - \frac{t_2}{t_3}$, $\texttt{boxwt}(T_2) = \frac{t_3}{t_4}$ and $d(T_2) = 0$;
- $\texttt{edgewt}(T_3) = (1 - \frac{t_1}{t_2})(1 - \frac{t_2}{t_3})$, $\texttt{boxwt}(T_3) = \frac{t_3}{t_4}$ and $d(T_3) = 1$;
- $\texttt{edgewt}(T_4) = (1 - \frac{t_3}{t_4})$, $\texttt{boxwt}(T_4) = \frac{t_2}{t_4}$ and $d(T_4) = 0$; and
- $\texttt{edgewt}(T_5) = (1 - \frac{t_1}{t_2})(1 - \frac{t_3}{t_4})$, $\texttt{boxwt}(T_5) = \frac{t_2}{t_4}$ and $d(T_5) = 1$.

Hence

$$K^{(2,2)}_{(2),(2,1)} = \left(1 - \frac{t_1}{t_2}\right)\frac{t_3}{t_4} + \left(1 - \frac{t_2}{t_3}\right)\frac{t_3}{t_4} - \left(1 - \frac{t_1}{t_2}\right)\left(1 - \frac{t_2}{t_3}\right)\frac{t_3}{t_4} + \left(1 - \frac{t_3}{t_4}\right)\frac{t_2}{t_4} - \left(1 - \frac{t_1}{t_2}\right)\left(1 - \frac{t_3}{t_4}\right)\frac{t_2}{t_4}.$$

Observe that, after rewriting using (4.3), each term is z-negative, in agreement with the discussion above;

that is,

$$\begin{aligned}(-1)^{|(2,2)|-|(2)|-|(2,1)|}K_{(2),(2,1)}^{(2,2)} &= -(-z_1)(z_3+1)-(-z_2)(z_3+1)+(-z_1)(-z_2)(z_3+1)\\ &\quad -(-z_3)(z_2+1)(z_3+1)+(-z_1)(-z_3)(z_2+1)(z_3+1)\\ &= z_1(z_3+1)+z_2(z_3+1)+z_1z_2(z_3+1)\\ &\quad +z_3(z_2+1)(z_3+1)+z_1z_3(z_2+1)(z_3+1)\end{aligned}$$

is z-positive (without any cancellation needed). $\diamondsuit$

There is a stronger positivity property exhibited by the rule of Theorem 4.1. The work of [AnGrMi11] generalizes the positivity of W. Graham [Gr01]: the equivariant Schubert structure coefficients are polynomials with nonnegative integer coefficients in the simple roots α_i. In [Kn10], A. Knutson observes W. Graham's geometric argument further implies the coefficients can be expressed as polynomials with nonnegative integer coefficients in the positive roots *such that each monomial is square-free.* Moreover, A. Knutson raises the issue of finding a "proper analogue" in equivariant K-theory for this square-free property. For X, we have:

Corollary 4.1 (Strengthened [AnGrMi11]-positivity)**.** *Let* $z_{ij} := \frac{t_i}{t_j}-1$*. Then* $(-1)^{|\nu|-|\lambda|-|\mu|}K_{\lambda,\mu}^{\nu}$ *is expressible as a polynomial with nonnegative integer coefficients in the* z_{ij}*'s such that each monomial is square-free.*

Proof. The nonnegativity of the coefficients is immediate from each z_{ij} being positive in the z_i's. It remains to show each monomial in our expression $L_{\lambda,\mu}^{\nu}$ is square-free.

Consider a $T \in \mathtt{BallotGen}(\nu/\lambda)$. Every $\mathtt{edgefactor}(\ell)$ is of the form $-z_{ij}$, while every $\mathtt{boxfactor}(\mathsf{x})$ is of the form $z_{ij}+1$. Define an (i,j)-label to be either an edge label with $\mathtt{edgefactor}(\ell) = -z_{ij}$ or a label ℓ in a productive box x with $\mathtt{boxfactor}(\mathsf{x}) = z_{ij}+1$.

Suppose ℓ, ℓ' are (i,j)-labels of T. Say $\ell \in \mathsf{x}$ or $\overline{\mathsf{x}}$ and $\ell' \in \mathsf{y}$ or $\overline{\mathsf{y}}$. Since both are $(i,-)$-labels, $\mathsf{Man}(\mathsf{x}) = \mathsf{Man}(\mathsf{y})$. Hence x and y are boxes of the same diagonal. We may assume x northwest of y. Let ℓ be an instance of m_n and ℓ' and instance of p_q. Since both are $(-,j)$-labels, $\mathrm{row}(\mathsf{x}) - m + N_{m_n} = \mathrm{row}(\mathsf{y}) - p + N_{p_q}$. By (S.1) and (S.2), $m + r(\mathsf{y}) - r(\mathsf{x}) \leq p$, so $N_{m_n} = r(\mathsf{y}) - r(\mathsf{x}) + m - p + N_{p_q} \leq p - p + N_{p_q} = N_{p_q}$. Hence by ballotness of T, $\mathsf{x} = \mathsf{y}$ and moreover $m = p$. Therefore by (S.2) and (S.3), $\ell = \ell'$, and thus T contains at most one (i,j)-label and each monomial in our expression is square-free. $\square$

We do not know a geometric explanation for Corollary 4.1. However, based on this result, one speculates that for any G/P, if for each positive root α we set $z_\alpha := e^\alpha - 1$, then the corresponding Schubert structure coefficients for $K_T(G/P)$ may be expressed in a square-free manner with nonnegative coefficients in the z_α's.

4.1.5 Organization

The first key to the proof is to reformulate Theorem 4.1 in terms of the more technical *bundled* tableaux that are appropriate for the inductive argument; this is presented in Section 4.2. In Section 4.3, we outline this inductive argument that the rule of Theorem 4.1 satisfies the key recurrence alluded to above. The base case is in Section 4.4. Both the plan of induction and the base case may be considered routine.

The core of the argument lies in Sections 4.5–4.12. The central innovation of this paper is a genomic generalization of M.-P. Schützenberger's *jeu de taquin*. This permits us to establish a combinatorial map of formal sums of tableaux. This part of the argument is developed as a sequence of four main ideas:

(1) To show well-definedness of the map, we identify and characterize the class of *good tableaux* that arise via genomic *jeu de taquin* (Sections 4.5, 4.6 and 4.7).

(2) To establish surjectivity, we develop reverse genomic *jeu de taquin* (Sections 4.8 and 4.9).

(3) To prove that the map respects the coefficients of the key recurrence, we define and prove properties of a *reversal tree* (Sections 4.10 and 4.11).

(4) The map is weight-preserving. However, a significant subtlety is that it is not generally weight-preserving on *individual* tableaux. To establish this property of the map, we need involutions that pair tableaux (Section 4.12).

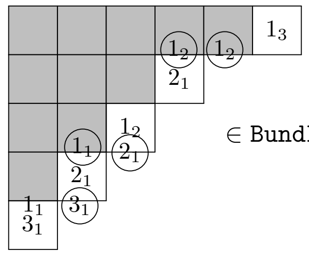
In Section 4.13, we recall the conjecture of [ThYo13] and prove it from Theorem 4.1; this argument is essentially independent of the rest of the chapter.

4.2 Bundled tableaux and a reformulation of Theorem 4.1

A tableau $T \in \mathtt{BallotGen}(\nu/\lambda)$ is **bundled** if every edge label is the westmost label of its gene. For example, in Example 4.3, only T_3 is *not* bundled (the eastmost 2_1 is to blame). We denote the set of bundled tableaux of shape ν/λ by $\mathtt{Bundled}(\nu/\lambda)$.

Define a surjection $\mathtt{Bun} : \mathtt{BallotGen}(\nu/\lambda) \twoheadrightarrow \mathtt{Bundled}(\nu/\lambda)$. This sends T to $\mathtt{Bun}(T)$ by deleting each edge label of T that is not maximally west in its gene. If $B \in \mathtt{Bundled}(\nu/\lambda)$, then any $T \in \mathtt{Bun}^{-1}(B)$ differs from B by having (possibly 0) additional edge labels. Let E_{i_j} be the edges where i_j appears in some $T \in \mathtt{Bun}^{-1}(B)$ but not in B, i.e., the set of edges of B where adding an i_j would yield an element of $\mathtt{BallotGen}(\nu/\lambda)$. We say B has a **virtual label** i_j on each edge of E_{i_j}. We denote a virtual label i_j by i_j.

Example 4.4. All virtual labels are depicted below:



$\in$ $\texttt{Bundled}((6,4,3,2,1)/(5,3,2,1))$

$\diamondsuit$

For $B \in \texttt{Bundled}(\nu/\lambda)$, let

$$\texttt{wt}(B) = \sum_{T \in \texttt{Bun}^{-1}(B)} \texttt{wt}(T). \tag{4.4}$$

Let $B^{\nu}_{\lambda,\mu}$ denote the set of tableaux in $\texttt{Bundled}(\nu/\lambda)$ with content μ.

Proposition 4.1. $L^{\nu}_{\lambda,\mu} = \sum_{B \in B^{\nu}_{\lambda,\mu}} \texttt{wt}(B)$.

Proof. Immediate from (4.4) and the definition of $L^{\nu}_{\lambda,\mu}$. $\square$

Compute $\widetilde{\texttt{wt}}(B)$ as a product: an edge label ℓ contributes a factor of $\texttt{edgefactor}(\ell)$ and each productive box x contributes a factor of $\texttt{boxfactor}(\mathsf{x})$. Each virtual label $\textcircled{\ell} \in \underline{\mathsf{x}}$ contributes $1 - \texttt{edgefactor}_{\underline{\mathsf{x}}}(\ell)$ (where the latter is calculated as if $\textcircled{\ell}$ were instead ℓ). Multiply by $(-1)^{d(T)}$ where $d(T) = \sum_{\mathcal{G}}(|\mathcal{G}| - 1)$ and here $|\mathcal{G}|$ is interpreted to be the multiset cardinality of non-virtual $\mathcal{G}$ in T.

Example 4.5. For B from Example 4.4, $\widetilde{\texttt{wt}}(B) = (-1)^1 \cdot \left(1 - \frac{t_2}{t_8}\right) \cdot \frac{t_2}{t_4}\frac{t_4}{t_6}\frac{t_6}{t_9}\frac{t_8}{t_8}\frac{t_{11}}{t_{11}} \cdot \frac{t_3}{t_5}\frac{t_4}{t_9}\frac{t_5}{t_7}\frac{t_8}{t_{10}}\frac{t_9}{t_{11}}$. $\diamondsuit$

Lemma 4.1. $\texttt{wt}(B) = \widetilde{\texttt{wt}}(B)$.

Proof. Let m be the number of virtual labels in B and a_i be the non-virtual weight of the i-th virtual label (listed in some given order). By the weights' definitions, the lemma follows from the "inclusion-exclusion" identity $\prod_{i \in [m]} a_i = \sum_{S \subseteq [m]} (-1)^{|S|} \prod_{i \in S}(1 - a_i)$. $\square$

4.3 Structure of the proof of Theorem 4.1

Let

$$\lambda^+ := \{\rho \supsetneq \lambda : \rho/\lambda \text{ has no two boxes in the same row or column}\}$$

and

$$\nu^- := \{\delta \subsetneq \nu : \nu/\delta \text{ has no two boxes in the same row or column}\}.$$

For a set D of boxes, let $\texttt{wt}\, D := \prod_{\mathsf{x} \in D} \frac{t_{\mathsf{Man}(\mathsf{x})}}{t_{\mathsf{Man}(\mathsf{x})+1}}$.

Proposition 4.2 (Key recurrence)**.**

$$\sum_{\rho \in \lambda^+} (-1)^{|\rho/\lambda|+1} K^{\nu}_{\rho,\mu} = K^{\nu}_{\lambda,\mu}(1 - \mathtt{wt}\,\nu/\lambda) + \sum_{\delta \in \nu^-} (-1)^{|\nu/\delta|+1} K^{\delta}_{\lambda,\mu}\,\mathtt{wt}\,\delta/\lambda. \tag{4.5}$$

Proof. The Chevalley formula in equivariant K-theory [LePo07, Corollary 8.2] implies:

$$[\mathcal{O}_{X_\lambda}][\mathcal{O}_{X_\square}] = [\mathcal{O}_{X_\lambda}](1 - \mathtt{wt}\,\lambda) + \sum_{\rho \in \lambda^+} (-1)^{|\rho/\lambda|+1} [\mathcal{O}_{X_\rho}]\,\mathtt{wt}\,\lambda.$$

Thus, the coefficient of $[\mathcal{O}_{X_\nu}]$ in $\left([\mathcal{O}_{X_\lambda}][\mathcal{O}_{X_\square}]\right)[\mathcal{O}_{X_\mu}]$ is

$$K^{\nu}_{\lambda,\mu}(1 - \mathtt{wt}\,\lambda) + \sum_{\rho \in \lambda^+} (-1)^{|\rho/\lambda|+1} K^{\nu}_{\rho,\mu}\,\mathtt{wt}\,\lambda.$$

On the other hand, the coefficient of $[\mathcal{O}_{X_\nu}]$ in $\left([\mathcal{O}_{X_\lambda}][\mathcal{O}_{X_\mu}]\right)[\mathcal{O}_{X_\square}]$ is

$$K^{\nu}_{\lambda,\mu}(1 - \mathtt{wt}\,\nu) + \sum_{\delta \in \nu^-} (-1)^{|\nu/\delta|+1} K^{\delta}_{\lambda,\mu}\,\mathtt{wt}\,\delta.$$

The proposition then follows from associativity and commutativity:

$$\left([\mathcal{O}_{X_\lambda}][\mathcal{O}_{X_\square}]\right)[\mathcal{O}_{X_\mu}] = \left([\mathcal{O}_{X_\lambda}][\mathcal{O}_{X_\mu}]\right)[\mathcal{O}_{X_\square}].$$ □

To prove $K^{\nu}_{\lambda,\mu} = L^{\nu}_{\lambda,\mu}$, we induct on $|\nu/\lambda|$. Proposition 4.3 is the base case: $K^{\lambda}_{\lambda,\mu} = L^{\lambda}_{\lambda,\mu}$; this is proved using the description of $L^{\lambda}_{\lambda,\mu}$ from Section 4.1.

The remaining cases use the description of $L^{\nu}_{\lambda,\mu}$ from Proposition 4.1. Assume $K^{\tau}_{\theta,\mu} = L^{\tau}_{\theta,\mu}$ when $|\tau/\theta| \leq h$. Suppose we are given λ, ν with $|\nu/\lambda| = h + 1$. We show that $L^{\nu}_{\lambda,\mu}$ satisfies (4.5). Since Proposition 4.2 asserts $K^{\nu}_{\lambda,\mu}$ also satisfies (4.5) we will be done by induction.

Fix λ, μ, ν with $\lambda \subsetneq \nu$. Define the formal sum

$$\Lambda^+ := \sum_{\rho \in \lambda^+} (-1)^{|\rho/\lambda|+1} \sum_{T \in B^{\nu}_{\rho,\mu}} T.$$

Similarly define

$$\Lambda := (1 - \mathtt{wt}\,\nu/\lambda) \sum_{T \in B^{\nu}_{\lambda,\mu}} T \quad \text{and} \quad \Lambda^- := \sum_{\delta \in \nu^-} (-1)^{|\nu/\delta|+1} (\mathtt{wt}\,\delta/\lambda) \sum_{T \in B^{\delta}_{\lambda,\mu}} T.$$

In Section 4.7.2, we define an operation $\texttt{slide}_{\rho/\lambda}$ that takes as input $T \in \Lambda^+$ and returns a formal sum of tableaux with coefficients from $\mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$. The construction of $\texttt{slide}_{\rho/\lambda}$ and proof of its correctness are found in Sections 4.5–4.7. Specifically, Corollary 4.2 shows the tableaux in the formal sum are from $B^{\nu}_{\lambda,\mu} \cup \left(\bigcup_{\delta \in \nu^-} B^{\delta}_{\lambda,\mu}\right)$.

In Section 4.11 we prove that

$$\texttt{slide}(\Lambda^+) := \sum_{\rho \in \lambda^+} (-1)^{|\rho/\lambda|+1} \sum_{T \in B^{\nu}_{\rho,\mu}} \texttt{slide}_{\rho/\lambda}(T) = \Lambda + \Lambda^-;$$

see Proposition 4.13 for the precise statement. Finally Proposition 4.14 shows that $\texttt{wt}\,\Lambda^+ = \texttt{wt}\,\texttt{slide}(\Lambda^+)$, so $\sum_{\rho \in \lambda^+} (-1)^{|\rho/\lambda|+1} L^{\nu}_{\rho,\mu} = L^{\nu}_{\lambda,\mu}(1 - \texttt{wt}\,\nu/\lambda) + \sum_{\delta \in \nu^-} (-1)^{|\nu/\delta|+1} L^{\delta}_{\lambda,\mu}\,\texttt{wt}\,\delta/\lambda$. This completes the proof that the Laurent polynomials $L^{\nu}_{\lambda,\mu}$ defined by the rule of Proposition 4.1 equal $K^{\nu}_{\lambda,\mu}$. Hence we have completed our proof of Theorem 4.1. □

4.4 The base case of the recurrence

A different rule for the case $K^{\lambda}_{\lambda,\mu}$ was given by V. Kreiman [Kre05]. We give an independent proof of the following:

Proposition 4.3 (Base case of the recurrence). $K^{\lambda}_{\lambda,\mu} = L^{\lambda}_{\lambda,\mu}$.

Proof. We use the original (unbundled) definition of $L^{\lambda}_{\lambda,\mu}$ from Section 4.1.

One says that $\pi \in S_n$ is a **Grassmannian permutation** if there is at most one k such that $\pi(k) > \pi(k+1)$. The Grassmannian permutation for $\lambda \subseteq k \times (n-k)$ is the (unique) Grassmannian permutation $\pi_\lambda \in S_n$ defined by $\pi_\lambda(i) = i + \lambda_{k-i+1}$ for $1 \leq i \leq k$ and $\pi(i) < \pi(i+1)$ for $i \neq k$.

Let $w', v' \in S_n$ be the Grassmannian permutations for the conjugate diagrams $\lambda', \mu' \subseteq (n-k) \times k$. The following identity relates $K^{\lambda}_{\lambda,\mu}$ to the localization of the class $[\mathcal{O}_{X_\lambda}]$ at the T-fixed point e_μ, expressed as a specialization of a double Grothendieck polynomial:

Lemma 4.2. $K^{\lambda}_{\lambda,\mu} = \overline{\mathfrak{G}_{v'}(t_{w'(1)}, \ldots, t_{w'(n)}; t_1, \ldots, t_n)}$, *where* $\overline{f(t_1, \ldots, t_n)}$ *is obtained by applying the substitution* $t_j \mapsto t_{n-j+1}$ *to* $f(t_1, \ldots, t_n)$.

Proof. This lemma is known to experts, but for completeness we give details and references. Suppose X_w is a Schubert variety in $\mathsf{GL}_n(\mathbb{C})/\mathsf{B}$. We have in $K_{\mathsf{T}}(\mathsf{GL}_n(\mathbb{C})/\mathsf{B})$,

$$[\mathcal{O}_{X_v}][\mathcal{O}_{X_w}] = K^{w}_{v,w}[\mathcal{O}_{X_w}] + \sum_{\theta \neq w} K^{\theta}_{v,w}[\mathcal{O}_{X_\theta}]. \tag{4.6}$$

It is known that $K^{\theta}_{v,w} = 0$ unless $v \leq \theta$ in Bruhat order; this follows for instance from the equivariant K-theory localization formula of M. Willems [Wi06] together with the *mutatis mutandis* modification of the proof of [KnTa03, Proposition 1].

Now, let $[\mathcal{O}_{X_v}]|_{e_w}$ denote the localization of the class $[\mathcal{O}_{X_v}]$ at the T-fixed point $e_w := w\mathsf{B}/\mathsf{B}$. Localization is a $\mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$-module homomorphism from $K_{\mathsf{T}}(\mathsf{GL}_n(\mathbb{C})/\mathsf{B})$ to $K_{\mathsf{T}}(e_w) \cong \mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$. Applying the localization map to (4.6) gives

$$[\mathcal{O}_{X_v}]|_{e_w}[\mathcal{O}_{X_w}]|_{e_w} = K^w_{v,w}[\mathcal{O}_{X_w}]|_{e_w}.$$

All terms in the summation vanish because $[\mathcal{O}_{X_\pi}]|_{e_\rho} = 0$ unless $\rho \leq \pi$ in Bruhat order. This vanishing condition appears in [Wi06] for generalized flag varieties; it also follows in the case at hand from, e.g., from the later work [WoYo12, Theorem 4.5] (see specifically the proof). For similar reasons, $[\mathcal{O}_{X_w}]|_{e_w} \neq 0$. Hence dividing by this shows $K^w_{v,w} = [\mathcal{O}_{X_v}]|_{e_w}$.

Consider the natural projection $\mathsf{GL}_n(\mathbb{C})/\mathsf{B} \twoheadrightarrow X$. The pullback of of each Schubert variety in X is a distinct Schubert variety in $\mathsf{GL}_n(\mathbb{C})/\mathsf{B}$ (see, e.g., [Br05, Example 1.2.3(6)]). Thus the Schubert basis of X is sent into the Schubert basis of $\mathsf{GL}_n(\mathbb{C})/\mathsf{B}$. Hence we obtain an injection $K_{\mathsf{T}}(X) \hookrightarrow K_{\mathsf{T}}(\mathsf{GL}_n(\mathbb{C})/\mathsf{B})$. Thus, if $\lambda, \mu \subseteq k \times (n-k)$ and $w, v \in S_n$ are respectively their Grassmannian permutations, then $K^{\lambda}_{\lambda,\mu} = K^w_{w,v}$. The lemma now follows from [WoYo12, Theorem 4.5] (after chasing conventions). $\square$

Since v' is Grassmannian, by [KnMiYo09, Theorem 5.8] $\mathfrak{G}_{v'}(X;Y) = \sum_T \texttt{SVSSYTwt}(T)$, where the sum is over all set-valued semistandard Young tableaux T of shape μ' with entries bounded above by $n-k$. Here $\texttt{SVSSYTwt}(T) = (-1)^{|L(T)|-|\mu'|}\prod_{\ell \in L(T)}(1 - \frac{x_\ell}{y_{\ell+\mathrm{col}(\mathsf{x})-\mathrm{row}(\mathsf{x})}})$, where $L(T)$ is the set of labels in T and x is the box containing ℓ.

Let $\texttt{SVSSYTeqwt}(T)$ be the result of the substitution $x_j \mapsto t_{w'(j)}, y_j \mapsto t_j$. Define $\mathcal{A}$ to be the set of $T \in \texttt{BallotGen}(\lambda/\lambda)$ that have content μ. Define $\mathcal{B}$ to be the set of set-valued semistandard tableaux U of shape μ' where $\texttt{SVSSYTeqwt}(U) \neq 0$.

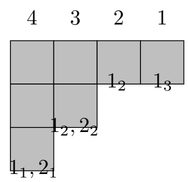
Lemma 4.3. *There is a bijection $\xi : \mathcal{A} \to \mathcal{B}$, with $\texttt{wt}(T) = \overline{\texttt{SVSSYTeqwt}(\xi(T))}$ for all $T \in \mathcal{A}$.*

Proof. Index columns of $k \times (n-k)$ by $1, 2, \ldots, n-k$ from *right to left.* To construct $\xi(T)$, begin with a Young diagram of shape μ'. For each label in T, we add a label to $\xi(T)$ as follows: If i_j appears in column c in T, place a label c in position $(\mu_i + 1 - j, i)$ in $\xi(T)$.

We have a candidate inverse map $\xi^{-1} : \mathcal{B} \to \mathcal{A}$: For each label c in (matrix) position (r, i) in $U \in \mathcal{B}$, we place an i_{μ_i+1-r} at the bottom of column c of λ/λ.

Example 4.6. Let $n = 7$, $k = 3$, $\lambda = (4, 2, 1)$ and $\mu = (3, 2, 0)$. Then T, together with the column labels

$1, 2, 3, 4$, and $\xi(T)$ are depicted below:

$T =$ 

$\mapsto \xi(T) = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2,3 & 4 \\ \hline 4 & \\ \cline{1-1} \end{array}.$

We compute that $\mathtt{wt}(T) = (-1)^1 \left(1 - \frac{t_1}{t_6}\right)\left(1 - \frac{t_3}{t_6}\right)\left(1 - \frac{t_5}{t_7}\right)\left(1 - \frac{t_6}{t_7}\right)\left(1 - \frac{t_1}{t_4}\right)\left(1 - \frac{t_3}{t_4}\right)$, where the first four factors correspond to the labels 1_j of T from left to right and the last two factors correspond to the labels 2_j of T from left to right. Now,

$$\mathtt{SVSSYTwt}(\xi(T)) = (-1)^1 \left(1 - \frac{x_4}{y_2}\right)\left(1 - \frac{x_3}{y_2}\right)\left(1 - \frac{x_2}{y_1}\right)\left(1 - \frac{x_1}{y_1}\right)\left(1 - \frac{x_4}{y_4}\right)\left(1 - \frac{x_3}{y_4}\right),$$

where the factors correspond to the entries of $\xi(T)$ as read up columns from left to right (i.e., consistent with the order of factors of $\mathtt{wt}(T)$ above).

Since $\lambda' = (3, 2, 1, 1)$ we have $w' = 2357146$ (one-line notation). So substituting, we get

$$\mathtt{SVSSYTeqwt}(\xi(T)) = (-1)^1 \left(1 - \frac{t_7}{t_2}\right)\left(1 - \frac{t_5}{t_2}\right)\left(1 - \frac{t_3}{t_1}\right)\left(1 - \frac{t_2}{t_1}\right)\left(1 - \frac{t_7}{t_4}\right)\left(1 - \frac{t_5}{t_4}\right).$$

The reader can check $\overline{\mathtt{SVSSYTeqwt}(\xi(T))} = \mathtt{wt}(T)$, in agreement with the lemma. $\diamondsuit$

(ξ^{-1} is well-defined and is weight-preserving): Let $U \in \mathcal{B}$. That $\xi^{-1}(U)$ is an edge-labeled genomic tableau is immediate from the column strictness of U. Ballotness follows from the row increasingness of U.

We now check that no label of $\xi^{-1}(U)$ is too high. Suppose c is a **bad** label in U in (matrix) position (r, i), i.e., one such that the label i_{μ_i+1-r} placed in column c of λ/λ is too high. Observe that every label c' North of c and in the same column of U is also bad: this is since c' corresponds to placing another label of family i in the weakly shorter column c' East of column c (since $c' < c$). Thus we may assume c is in the northmost row of U, i.e., $r = 1$. Now if $i = 1$, then since c is bad, it must be that $\lambda'_{n-k-c+1} = 0$, so $w'(c) = c + 0$. Now c contributes a factor of $1 - \frac{x_c}{y_c}$ to $\mathtt{SVSSYTwt}(U)$ and hence a factor of $1 - \frac{t_{c+0}}{t_c} = 0$ to $\mathtt{SVSSYTeqwt}(U)$. That is, $\mathtt{SVSSYTeqwt}(U) = 0$, so $U \notin \mathcal{B}$, a contradiction. Otherwise, we may also assume $i > 1$ is smallest such that a label in $(r = 1, i)$ is bad. Since no label c' in $(r = 1, i-1)$ of U is bad, it must be that c is "barely" bad, i.e.,

$$\lambda'_{n-k-c+1} = i - 1 \tag{4.7}$$

(column c is one box too short). However, c contributes a factor of $1 - \frac{x_c}{y_{c+i-1}}$ to $\mathtt{SVSSYTwt}(U)$ and hence a factor of $1 - \frac{t_{c+\lambda'_{n-k-c+1}}}{t_{c+i-1}}$ to $\mathtt{SVSSYTeqwt}(U)$. This latter factor is 0 precisely by (4.7). Hence again $U \notin \mathcal{B}$, a contradiction. Thus U has no bad labels and thus no label of $\xi^{-1}(U)$ is too high, as desired.

The sign appearing in $\mathtt{wt}\,\xi^{-1}(U)$ records the difference between $|\mu|$ and the number of labels in $\xi^{-1}(U)$, while the sign in $\overline{\mathtt{SVSSYTeqwt}(U)}$ records the difference between $|\mu|$ and number of labels in U. Since the number of labels in U is clearly the same as the number of labels in $\xi^{-1}(U)$, these signs are equal.

We check that the weight assigned to a label c of U in position (r,i) is the same as the $\mathtt{edgefactor}$ assigned to the corresponding label i_{μ_i+1-r} at the bottom of column c in $\xi^{-1}(U)$. The label c is assigned the weight

$$\mathtt{SSYTeqfactor}_{(r,i)}(c) := 1 - \frac{x_c}{y_{c+i-r}} = 1 - \frac{t_{c+\lambda'_{n-k+1-c}}}{t_{c+i-r}}.$$

Hence we must show the equality of these two quantities:

$$\begin{aligned}\overline{\mathtt{SSYTeqfactor}_{(r,i)}(c)} &= 1 - \frac{t_{n+1-c-\lambda'_{n-k+1-c}}}{t_{n+1-c+r-i}} \text{ and}\\ \mathtt{edgefactor}_{\underline{\mathsf{x}}}(i_{\mu_i+1-r}) &= 1 - \frac{t_{\mathtt{Man}(\mathsf{x})}}{t_{\lambda'_{n-k+1-c}-i+r+\mathtt{Man}(\mathsf{x})}},\end{aligned}$$

where $\underline{\mathsf{x}}$ is the southern edge of λ in column c.

Now, counting the rows and columns separating x from the southwest corner of $k \times (n-k)$, we have

$$\mathtt{Man}(\mathsf{x}) = (n-k-c) + (k - \lambda'_{n-k+1-c} + 1) = n + 1 - c - \lambda'_{n-k+1-c}.$$

Thus, the numerators of the quotients of $\overline{\mathtt{SSYTeqfactor}(c)}$ and $\mathtt{edgefactor}(c)$ are equal. To see that the denominators are also equal, observe

$$\begin{aligned}\mathtt{Man}(\mathsf{x}) + \lambda'_{n-k+1-c} - i + r &= \left(n + 1 - c - \lambda'_{n-k+1-c}\right) + \lambda'_{n-k+1-c} - i + r\\ &= n + 1 - c - i + r.\end{aligned}$$

(ξ is well-defined and weight-preserving): Let $T \in \mathcal{A}$. We must show $\xi(T)$ is strictly increasing along columns. This is clear since T satisfies (S.3) and (S.4).

Now we show that $\xi(T)$ is weakly increasing along rows. Suppose we have a in position (r,i) and b in position $(r,i+1)$. This a comes from an i_{μ_i+1-r} in column a in T, while this b comes from an $(i+1)_{\mu_{i+1}+1-r}$ in column b. By ballotness of T, each i_{μ_i+1-r} must be weakly right of each $(i+1)_{\mu_{i+1}+1-r}$. Thus $a \le b$.

Hence $\xi(T)$ is a set-valued semistandard tableau of shape μ'. The same computations showing ξ^{-1} is weight preserving shows $0 \neq \mathtt{wt}(T) = \overline{\mathtt{SSYTeqwt}(\xi(T))}$ and so the desired conclusions hold. ☐

The proposition now follows immediately from Lemmas 4.2 and 4.3. ☐

4.5 Good tableaux

In this section, we give an intrinsic description of the tableaux that will appear during our generalized *jeu de taquin* process (defined in Section 4.7). Since we will use *box* labels $\bullet_{\mathcal{G}}$, we distinguish labels i_j as **genetic labels**. As a visual aid, we mark genetic labels $\mathcal{F}$ southeast of a $\bullet_{\mathcal{G}}$ with $\mathcal{F} \prec \mathcal{G}$ as $\mathcal{F}^!$. For a gene $\mathcal{G}$, let $\mathcal{G}^+$ (respectively, $\mathcal{G}^-$) denote the successor (respectively, predecessor) of $\mathcal{G}$ in the total order $\prec$ on genes. For example, $1_1^+ = 2_1$ if $\mu_1 = 1$, and $1_1^+ = 1_2$ if $\mu_1 > 1$. Let $\mathcal{G}_{\max}$ be the maximum gene that can appear, namely $\ell(\mu)_{\mu_{\ell(\mu)}}$ where $\ell(\mu)$ is the number of nonzero rows of μ. Declare $\mathcal{G}_{\max}^+ := (\ell(\mu)+1)_1$.

A $\mathcal{G}$**-good tableau** is an edge-labeled filling T of ν/λ by genetic labels i_j (such that $i \in \mathbb{Z}_{>0}$ and the j's that appear for each i form an initial segment of $\mathbb{Z}_{>0}$) and *box* labels $\bullet_{\mathcal{G}}$, satisfying the conditions (G.1)–(G.13) below:

(G.1) no genetic label is too high;

(G.2) no $\bullet_{\mathcal{G}}$ is southeast of another (in particular, $\bullet_{\mathcal{G}}$'s are in distinct rows and columns);

(G.3) the labels $\prec$-increase along rows (ignoring any $\bullet_{\mathcal{G}}$'s), except for possibly three consecutive labels $\boxed{\mathcal{H}}\boxed{\bullet_{\mathcal{G}}}\boxed{\mathcal{F}^!}$ with $\mathcal{H} > \mathcal{F}$;

(G.4) the labels $<$-increase down columns (ignoring any $\bullet_{\mathcal{G}}$'s), except that unmarked $\mathcal{F}$ may appear adjacent and above $\mathcal{F}^!$ when both are box labels;

(G.5) if i_j, k_ℓ appear on the same edge, then $i \neq k$;

(G.6) if i_j is West of i_k, then $j \leq k$;

(G.7) each edge label is maximally west in its gene;

(G.8) each genotype G obtained by choosing one label of each gene of T is ballot in the sense defined in Section 4.1.3.

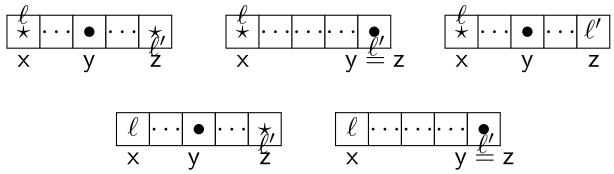
(G.9) if $\mathcal{F}$ appears northwest of $\bullet_{\mathcal{G}}$, then $\mathcal{F} \prec \mathcal{G}$;

(G.10) if $\mathcal{F}^! \in \mathsf{x}$ or $\mathcal{F}^! \in \underline{\mathsf{x}}$, then $\bullet_{\mathcal{G}}$ appears in x's row;

(G.11) $\bullet_{\mathcal{G}}$ does not appear in a column containing a marked label;

(G.12) if ℓ and ℓ' are genetic labels of the same family with ℓ NorthWest of ℓ', then there are boxes x, z in row r with x West of z, $\ell \in \mathsf{x}$ or $\overline{\mathsf{x}}$, and $\ell' \in \mathsf{z}$ or $\underline{\mathsf{z}}$; further, $\bullet_{\mathcal{G}}$ appears in some box y of r that is East of x

and west of z. Pictorially, the scenarios are:



Furthermore, if $\mathsf{y} = \mathsf{z} = \mathsf{x}^{\rightarrow}$ in the last scenario, then $\mathsf{y}^{\rightarrow}$ does not contain a marked label nor another instance of the gene of ℓ'.

We place a **virtual label** $\mathcal{H}$ on each edge $\underline{\mathsf{x}}$ where $\mathcal{H} \in \underline{\mathsf{x}}$ would

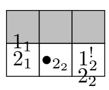
(V.1) *not* be marked (hence if $\mathcal{H}$ appears southeast of a $\bullet_{\mathcal{G}}$, then $\mathcal{H} \succeq \mathcal{G}$);

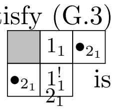
(V.2) *not* be maximally west in its gene (hence violating condition (G.7)); and

(V.3) satisfy the conditions (G.1), (G.4), (G.5), (G.6), (G.8), (G.9) and (G.12).

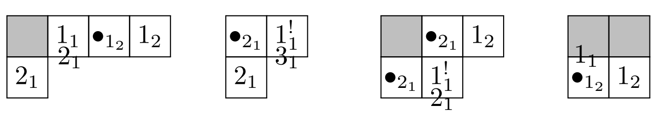
(G.13) If $\mathcal{E}^! \in \mathsf{x}$ or $\mathcal{E}^! \in \underline{\mathsf{x}}$, then there is $\mathcal{F}$ or $\mathcal{F}$ on $\underline{\mathsf{x}}$ with $N_{\mathcal{E}} = N_{\mathcal{F}}$ and $\mathtt{family}(\mathcal{F}) = \mathtt{family}(\mathcal{E}) + 1$.

A tableau is **good** if it is $\mathcal{G}$-good for some $\mathcal{G}$.

Example 4.7. The tableau  is 2_2-good. Although the labels in the second row do not increase left to right, they satisfy (G.3). Furthermore, notice the 1_1 and $1_2^!$ satisfy (G.12), as do the 2_1 and 2_2.

The tableau  is also good. Although the label 1_1 appears twice in the same column, the lower instance is marked in accordance with (G.4). $\diamondsuit$

Example 4.8. The following tableaux are *not* good:



The first fails conditions (G.1) and (G.7) because of the edge label 2_1. The second fails (G.8), as the unique genotype is not ballot. Although the marked $1_1^!$ in the third tableau has a label of family 2 on the lower edge of its box, the tableau fails (G.13) as $1 = N_{1_1} \neq N_{2_1} = 0$. It also fails (G.11) by having both a $\bullet_{2_1}$ and a marked label in the second column. The fourth tableau fails (G.12). $\diamondsuit$

Lemma 4.4. *If $T \in \mathtt{Bundled}(\nu/\lambda)$, then T is $\mathcal{G}$-good for every $\mathcal{G}$. Moreover the virtual labels of the $\mathcal{G}$-good tableau T (as defined by* (V.1)–(V.3)*) are the same as the virtual labels of the bundled tableau T (as defined in Section 4.2).*

Proof. Since T is bundled, (S.1), (S.2), (S.3) and (S.4) hold. These conditions respectively imply (G.3), (G.4), (G.5) and (G.6). (G.1), (G.7) and (G.8) are part of the definition of a bundled tableau. For (G.12), if ℓ is NorthWest of ℓ' and both are from the same family, (S.1) or (S.2) is violated. The remaining conditions are vacuous since T has no $\bullet_{\mathcal{G}}$'s. Hence T is $\mathcal{G}$-good.

The claim about virtual labels is then clear from the definitions. $\square$

Lemma 4.5 (Strong form of (G.10)). *Assume T is $\mathcal{G}$-good. Let x be a box of T in row r.*

(I) If $\mathcal{F}^! \in \underline{\mathsf{x}}$, then $\mathtt{label}(\mathsf{x})$ is marked.

(II) If $\mathcal{F}^! \in \mathsf{x}$, then there is a y West of x in r such that $\bullet_{\mathcal{G}} \in \mathsf{y}$. Every box label of r between x and y is marked.

Proof. (I): Since $\mathcal{F}^! \in \underline{\mathsf{x}}$, $\underline{\mathsf{x}}$ (and hence also x) is southeast of a $\bullet_{\mathcal{G}}$. By (G.11), $\bullet_{\mathcal{G}} \notin \mathsf{x}$. Hence some $\mathcal{E} \in \mathsf{x}$. By (G.4), $\mathcal{E} < \mathcal{F}$. Therefore the $\mathcal{E} \in \mathsf{x}$ is marked.

(II): Since $\mathcal{F}^! \in \mathsf{x}$, there is a $\bullet_{\mathcal{G}}$ northwest of x. By (G.10), there is a $\bullet_{\mathcal{G}}$ in x's row. If this latter $\bullet_{\mathcal{G}}$ is East of x, these two $\bullet_{\mathcal{G}}$'s are distinct and violate (G.2). Hence the $\bullet_{\mathcal{G}}$ in x's row is in some box y West of x. If $\mathcal{E}$ is a box label between x and y (and in the same row), it is southeast of the $\mathtt{label}(\mathsf{y}) = \bullet_{\mathcal{G}}$. By (G.3) $\mathcal{E} \prec \mathcal{F}$. Hence this $\mathcal{E}$ is also marked. $\square$

Lemma 4.6 (Strong form of (G.13)). *Let T be $\mathcal{G}$-good. Suppose $\mathcal{E}^! \in \mathsf{x}$ or $\mathcal{E}^! \in \underline{\mathsf{x}}$ with $\mathtt{family}(\mathcal{G}) - \mathtt{family}(\mathcal{E}) = k > 0$. For each $0 < h < k$, there is $\mathcal{H}^! \in \underline{\mathsf{x}}$ with $N_{\mathcal{H}} = N_{\mathcal{E}}$ and $\mathtt{family}(\mathcal{H}) = \mathtt{family}(\mathcal{E}) + h$. Also, there is a $\mathcal{G}'$ or $\mathcal{G}'$ $\in \underline{\mathsf{x}}$ with $N_{\mathcal{G}'} = N_{\mathcal{E}}$ and $\mathtt{family}(\mathcal{G}') = \mathtt{family}(\mathcal{G})$.*

Proof. This follows by repeated application of (G.13). Note that none of the $\mathcal{H}$'s of the statement can be virtual since they must be marked. $\square$

Lemma 4.7. *If $\mathcal{E} < \mathcal{F}$ are genes of a good tableau T with $N_{\mathcal{E}} = N_{\mathcal{F}}$, then no $\mathcal{F}$ or $\mathcal{F}$ is East of any $\mathcal{E}$.*

Proof. First suppose that some $\mathcal{F}$ is East of some $\mathcal{E}$. Let G be a genotype of T with $\mathcal{F} \in G$ that is East of some $\mathcal{E} \in G$. Then $\mathcal{F}$ appears before $\mathcal{E}$ in $\mathtt{word}(G)$. By (G.6), the initial segment W of $\mathtt{word}(G)$ ending at $\mathcal{F}$ contains $N_{\mathcal{F}} + 1$ labels of $\mathtt{family}(\mathcal{F})$ and at most $N_{\mathcal{E}}$ labels of $\mathtt{family}(\mathcal{E})$. Thus T's (G.8) is violated for some $\mathtt{family}(\mathcal{E}) \leq i < \mathtt{family}(\mathcal{F})$, a contradiction. Finally, if some $\mathcal{F}$ is East of some $\mathcal{E}$, then by (V.3) the tableau T' obtained by replacing that $\mathcal{F}$ by $\mathcal{F}$ satisfies (G.6) and (G.8). Now we derive the same contradiction as before, using T' in place of T. $\square$

Lemma 4.8. *If $\mathcal{E}^!$ appears in a good tableau T, then it is maximally west in its gene.*

Proof. Suppose $\mathcal{E}^! \in \mathsf{x}$ or $\mathcal{E}^! \in \underline{\mathsf{x}}$. By (G.13), there is an $\mathcal{F}$ or $\mathcal{F} \in \underline{\mathsf{x}}$ with $N_{\mathcal{E}} = N_{\mathcal{F}}$ and $\mathcal{E} < \mathcal{F}$. Thus we are done by Lemma 4.7. $\square$

Lemma 4.9. *Suppose column c of good tableau T contains labels $\mathcal{H}$ and $\mathcal{J}$ with $\mathcal{H} < \mathcal{J}$ and $N_{\mathcal{H}} = N_{\mathcal{J}}$. Then for every i such that $\mathtt{family}(\mathcal{H}) < i < \mathtt{family}(\mathcal{J})$, there is a label $\mathcal{I}$ of family i in column c such that $N_{\mathcal{H}} = N_{\mathcal{I}}$.*

Proof. Suppose not. By (G.8), there is some $\mathcal{I} \in T$ of family i such that $N_{\mathcal{H}} = N_{\mathcal{J}} = N_{\mathcal{I}}$. If this $\mathcal{I}$ is not in column c, we contradict Lemma 4.7. $\square$

Lemma 4.10. *Suppose $\mathcal{E}$ and $\mathcal{F}$ satisfy $N_{\mathcal{E}} = N_{\mathcal{F}}$ and $\mathtt{family}(\mathcal{F}) = \mathtt{family}(\mathcal{E}) + 1$. Let T be a $\mathcal{G}$-good tableau with $\mathcal{F} \in \underline{\mathsf{x}}$ and either $\mathcal{E}^! \in \mathsf{x}$ or $\mathcal{E}^! \in \underline{\mathsf{x}}$. Then $\bullet_{\mathcal{G}} \in \mathsf{x}^{\leftarrow}$ and $\mathtt{family}(\mathcal{F}) = \mathtt{family}(\mathcal{G})$.*

Proof. If $\bullet_{\mathcal{G}} \notin \mathsf{x}^{\leftarrow}$, then by Lemma 4.5, $\mathcal{D}^! \in \mathsf{x}^{\leftarrow}$. By (G.3) and (G.4), $\mathcal{D} \prec \mathcal{E}$. Also $\mathcal{E} \prec \mathcal{G}$ since $\mathcal{E}^! \in T$. Thus by (G.6) and Lemma 4.6, there is a $\widetilde{\mathcal{E}}^! \in \underline{\mathsf{x}^{\leftarrow}}$ or $\widetilde{\mathcal{E}}^! \in \mathsf{x}^{\leftarrow}$ with $\mathtt{family}(\mathcal{E}) = \mathtt{family}(\widetilde{\mathcal{E}})$ and $N_{\widetilde{\mathcal{E}}} = N_{\mathcal{D}}$. By (G.13), there is $\widetilde{\mathcal{F}}$ or $\widetilde{\mathcal{F}} \in \underline{\mathsf{x}^{\leftarrow}}$ with $\mathtt{family}(\mathcal{F}) = \mathtt{family}(\widetilde{\mathcal{F}})$ and $N_{\widetilde{\mathcal{E}}} = N_{\widetilde{\mathcal{F}}}$. Thus, by Lemma 4.7, $\mathcal{F} \neq \widetilde{\mathcal{F}}$, contradicting $\mathcal{F} \in \underline{\mathsf{x}}$. Finally, $\mathtt{family}(\mathcal{F}) = \mathtt{family}(\mathcal{G})$ by Lemma 4.6. $\square$

Lemma 4.11. *If T is $\mathcal{G}$-good, then no $\mathcal{H}$ is southEast of another.*

Proof. If some $\mathcal{H}$ is SouthEast of another $\mathcal{H}$, by (G.12) there is a $\bullet_{\mathcal{G}}$ in between the two $\mathcal{H}$'s. If two $\mathcal{H}$'s are box labels of the same row, then by (G.3) we reach the same conclusion that there is a $\bullet_{\mathcal{G}}$ in between the two $\mathcal{H}$'s. In either case, since this $\bullet_{\mathcal{G}}$ is southeast of the western $\mathcal{H}$ we have $\mathcal{H} \prec \mathcal{G}$ by (G.9). Since this $\bullet_{\mathcal{G}}$ is northwest of the eastern $\mathcal{H}$, this eastern $\mathcal{H}$ is marked. This contradicts Lemma 4.8. Finally, suppose two $\mathcal{H}$'s are edge labels on the bottom of the same row. This contradicts (G.7). $\square$

Lemma 4.12. *Let T be a $\mathcal{G}$-good tableau. Suppose $\mathtt{family}(\mathcal{F}) \leq \mathtt{family}(\mathcal{G})$, $\bullet_{\mathcal{G}} \in \mathsf{y}$ and $\mathcal{F} \in \mathsf{z}$ or $\underline{\mathsf{z}}$. Then z is not SouthEast of y.*

Proof. Suppose z is SouthEast of y. First assume $\mathcal{F} < \mathcal{G}$. Consider the box a that is in y's column and z's row. By Lemma 4.5, either a contains a marked label (contradicting (G.11)) or $\bullet_{\mathcal{G}} \in \mathsf{a}$ Southeast of y (contradicting (G.2)).

Now assume $\mathtt{family}(\mathcal{F}) = \mathtt{family}(\mathcal{G})$. (We do not assume $\mathcal{F} \preceq \mathcal{G}$.) Consider the box b of T that is in y's row and z's column. By (G.2), b contains a genetic label. By (G.4), $\mathtt{label}(\mathsf{b}) < \mathcal{F}$. Hence $\mathtt{label}(\mathsf{b})$ is marked in T. By Lemma 4.6, $\underline{\mathsf{b}}$ then contains a (possibly virtual) label of the same family as $\mathcal{F}$ and $\mathcal{G}$. This contradicts (G.4). $\square$

Lemma 4.13. *Let U be a $\mathcal{G}^{+}$-good tableau. Suppose that $\bullet_{\mathcal{G}^+} \in \mathsf{x}$ and that either $\mathcal{G} \in \mathsf{y}$ or $\mathcal{G} \in \underline{\mathsf{y}}$. Then y is not NorthWest of x.*

Proof. Suppose otherwise. Consider the box b that is in y's column and x's row. By (G.2) it contains a genetic label. By (G.4) either $\texttt{label}(\mathsf{b}) > \mathcal{G}$ or else $\mathcal{G}^! \in \mathsf{b}$. If $\mathcal{G}^! \in \mathsf{b}$, then b is southeast of a $\bullet_{\mathcal{G}^+}$ by definition. This contradicts (G.2). If $\mathcal{G} < \texttt{label}(\mathsf{b})$, we contradict (G.9). $\square$

Lemma 4.14. *Let c be a column of a $\mathcal{G}$-good tableau T. Suppose $\bullet_{\mathcal{G}} \in c$ and either $\mathcal{G} \in c$ or $\mathcal{G} \in c$. Further suppose that $\mathcal{E}^! \in \mathsf{y}$, where y is a box of column $c^{\rightarrow}$. Then $\mathcal{G} \in \underline{\mathsf{y}}$.*

Proof. Since $\mathcal{E}^!$ appears in T, $\mathcal{E} \prec \mathcal{G}$. Since $\mathcal{E}$ appears East of some $\mathcal{G}$, by (G.6) this implies $\mathcal{E} < \mathcal{G}$.

Hence by Lemma 4.6, there is either $\mathcal{G}' \in \underline{\mathsf{y}}$ or $\mathcal{G}' \in \underline{\mathsf{y}}$ with $\texttt{family}(\mathcal{G}') = \texttt{family}(\mathcal{G})$. It remains to show $\mathcal{G}' = \mathcal{G}$, for then by (G.7), $\mathcal{G} \in \underline{\mathsf{y}}$.

Suppose $\mathcal{G}' \neq \mathcal{G}$. Then by (G.4), (G.5) and (G.6), $\mathcal{G}' = \mathcal{G}^{+}$. By Lemma 4.6, $N_{\mathcal{E}} = N_{\mathcal{G}^+}$; thus $\texttt{family}(\mathcal{E}^-) = \texttt{family}(\mathcal{E})$ by (G.8). Also by (G.8), every instance of $\mathcal{E}^-$ must be read before any $\mathcal{G}$ or $\mathcal{G}$. By (G.4), $\mathcal{E}^- \notin c^{\rightarrow}$. By (G.6), $\mathcal{E}^-$ does not appear East of $c^{\rightarrow}$. But by assumption either $\mathcal{G} \in c$ or $\mathcal{G} \in c$, so $\mathcal{E}^-$ must appear in c.

Consider the box $\mathsf{y}^{\leftarrow}$. By Lemma 4.5, either $\bullet_{\mathcal{G}} \in \mathsf{y}^{\leftarrow}$ or some $\mathcal{D}^! \in \mathsf{y}^{\leftarrow}$. The latter is impossible by (G.11), since $\bullet_{\mathcal{G}} \in c$. Hence $\bullet_{\mathcal{G}} \in \mathsf{y}^{\leftarrow}$.

Now $\mathcal{E}^-$ cannot appear South of $\mathsf{y}^{\leftarrow}$ in c, for then it would be marked, in violation of (G.11). We have $\mathcal{E}^- \notin \mathsf{y}^{\leftarrow}$, since $\bullet_{\mathcal{G}} \in \mathsf{y}^{\leftarrow}$. By (G.12), $\mathcal{E}^-$ cannot appear North of $\mathsf{y}^{\leftarrow}$ in c. This contradicts that $\mathcal{E}^-$ must appear in c, and therefore the assumption $\mathcal{G}' \neq \mathcal{G}$. $\square$

4.6 Snakes of good tableaux

In this section, we give structural results about certain subsets of a good tableau; these will play a critical role in the definition of our generalized *jeu de taquin* (given in Section 4.7).

4.6.1 Snakes

Let T be a $\mathcal{G}$-good tableau. Let $\mathcal{G} = g_k$ and consider the set of *boxes* in T that contain either $\bullet_{\mathcal{G}}$ or $\mathcal{G}$. This set decomposes into edge-connected components R that we call **presnakes**. A **short ribbon** is a connected skew shape without a 2×2 subshape and where each row and column contains at most two boxes.

Lemma 4.15. *Each presnake R is a short ribbon. Any row of R with two boxes is $\begin{array}{|c|c|}\hline \bullet_{\mathcal{G}} & \mathcal{G} \\ \hline\end{array}$. Any column of R with two boxes is $\begin{array}{|c|}\hline \bullet_{\mathcal{G}} \\ \hline \mathcal{G} \\ \hline\end{array}$.*

Proof. Since T is $\mathcal{G}$-good, there is no $\mathcal{G}^!$. So any column of R has at most one $\mathcal{G}$ by (G.4) and at most one $\bullet_{\mathcal{G}}$ by (G.2). Hence any column of R has at most two boxes. By (G.9) if $\bullet_{\mathcal{G}}$ and $\mathcal{G}$ are in the same column, the $\bullet_{\mathcal{G}}$ is to the north. The description of rows of R holds by (G.2), (G.3) and (G.9). That R is a skew shape with no 2×2 subshape then follows immediately. $\square$

A **snake** S is a presnake R extended by (R.1)–(R.3):

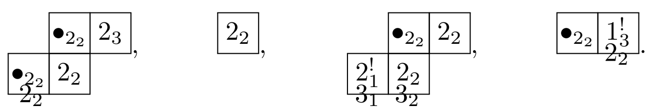
(R.1) If the box immediately right of the northmost $\bullet_{\mathcal{G}}$ in R contains $\mathcal{G}^+$ with $\texttt{family}(\mathcal{G}^+) = \texttt{family}(\mathcal{G})$, then adjoin this box to R.

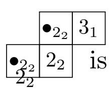
(R.2) If the box immediately left of the southmost $\mathcal{G}$ in R contains a marked label, adjoin this box to R.

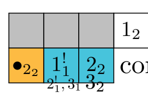
(R.3) If x in the northmost row of R contains $\bullet_{\mathcal{G}}$, $\texttt{label}(\mathsf{x}^{\rightarrow})$ is marked and either $\mathcal{G}$ or $\mathcal{G} \in \underline{\mathsf{x}^{\rightarrow}}$, then adjoin $\mathsf{x}^{\rightarrow}$ to R.

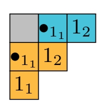
The entries of S are its box labels and labels appearing on the bottom edges of its boxes.

Example 4.9. Below are snakes for $\mathcal{G} = 2_2$:



On the other hand,  is *not* a snake, even if $3_1 = 2_2^+$ ((R.1) does not apply). $\diamondsuit$

Example 4.10 (Snakes can share a row)**.**  contains two snakes as colored. $\diamondsuit$

Example 4.11 (Snakes can share a column)**.**  has two snakes as colored. $\diamondsuit$

Lemma 4.16. *Every snake S is a short ribbon.*

Proof. S is built by adjoining boxes to a presnake R. By Lemma 4.15, R is a short ribbon. In view of Lemma 4.15, (R.1) and (R.3) only apply if the northmost row of R is a single box with $\bullet_{\mathcal{G}}$. So adjoining a box to the right maintains shortness. Similarly, (R.2) maintains shortness. $\square$

Lemma 4.17 (Disjointness and relative positioning of snakes)**.** *Suppose S, S' are snakes obtained from distinct presnakes R, R' respectively. Up to relabeling of the snakes, one of the following holds:*

(I) S is entirely SouthWest of the S' (that is, if b, b' are respectively boxes of these snakes, then b is SouthWest of b').

(II) S consists of a single box containing $\bullet_{\mathcal{G}}$ with neither $\mathcal{G}$ nor $\mathcal{G}$ on its lower edge; further, this box appears West of and in the same row as the southmost row of S', and all intervening box labels are marked; cf. Example 4.10.

(III) S involves an (R.1) extension, adjoining a $\mathcal{G}^+$ in some box w, while $S' = \{\bullet_{\mathcal{G}} \in \mathsf{w}^{\uparrow}\}$ or $S' = \{\bullet_{\mathcal{G}} \in \mathsf{w}^{\uparrow}, \mathcal{G}^+ \in \mathsf{w}^{\uparrow\rightarrow}\}$; cf. Example 4.11.

In particular, S and S' are box disjoint.

Proof. By Lemma 4.15, (G.2) and/or (G.4), R and R' share at most one row and do not share a column. Moreover, one sees that R is southWest of R' (say). By (R.1)–(R.3), S and S' share a row if and only if R and R' do.

Case 1: (R and R' share a row r): The northmost row of R and the southmost row of R' are in row r. We must show that (II) holds and that S, S' are box disjoint.

By (G.2), (G.9) and Lemma 4.11, R has in row r only a $\bullet_{\mathcal{G}} \in \mathsf{x}$ while R' has in r only $\mathcal{G} \in \mathsf{y}$. Since $S \neq S'$, $\mathsf{y} \neq \mathsf{x}^{\rightarrow}$. By (G.3), $\mathtt{label}(\mathsf{y}^{\leftarrow}) \prec \mathcal{G}$, so we have some marked label $\mathcal{F}^! \in \mathsf{y}^{\leftarrow}$. Therefore R' extends to S' by (R.2).

Claim 4.1. *No $\mathcal{G}$ or $\mathcal{G}$ appears in columns west of $\mathsf{y}^{\leftarrow}$.*

Proof. Since $\mathcal{F} \prec \mathcal{G}$, we are done by (G.4) and (G.6) if $\mathtt{family}(\mathcal{F}) = \mathtt{family}(\mathcal{G})$. Thus assume $\mathcal{F} < \mathcal{G}$. By Lemma 4.6, there is either $\mathcal{G}' \in \underline{\mathsf{y}^{\leftarrow}}$ or $\mathcal{G}' \in \underline{\mathsf{y}^{\leftarrow}}$ such that $\mathtt{family}(\mathcal{G}') = \mathtt{family}(\mathcal{G})$ and $N_{\mathcal{F}} = N_{\mathcal{G}'}$. By (G.6), $\mathcal{G}' \preceq \mathcal{G}$ because $\mathcal{G} \in \mathsf{y}$. If $\mathcal{G}' = \mathcal{G}$, then since $N_{\mathcal{F}} = N_{\mathcal{G}'(=\mathcal{G})}$, the $\mathcal{G} \in \mathsf{y}$ and $\mathcal{F}^! \in \mathsf{y}^{\leftarrow}$ combine to contradict Lemma 4.7. Thus $\mathcal{G}' \prec \mathcal{G}$ and we are done by (G.6) and (G.4). □

By Claim 4.1, $R = \{\bullet_{\mathcal{G}} \in \mathsf{x}\}$ without $\mathcal{G}$ or $\mathcal{G} \in \underline{\mathsf{x}}$. Observe that R cannot extend to S by (R.1), since (R.1) requires $\mathcal{G}^+ \in \mathsf{x}^{\rightarrow}$, which contradicts (G.3) in view of $\mathcal{G} \in \mathsf{y}$. It cannot be extended by (R.2) since $\mathcal{G} \notin \mathsf{x}$. If R were extended by (R.3), there would be a $\mathcal{G}$ or $\mathcal{G}$ in $\underline{\mathsf{x}^{\rightarrow}}$ in violation of Claim 4.1. Thus $R = S = \{\mathsf{x}\}$.

By Lemma 4.5(II), all labels strictly between x and y are marked. Hence (II) holds. Since $\mathsf{y}^{\leftarrow} \notin S$, we see by (R.1)–(R.3) that S and S' are box disjoint.

Case 2: (R and R' do *not* share a row): We may assume S and S' share a column, for if they do not, then clearly (I) and box-disjointness both hold. Since R and R' do not share a column, S and S' can only share a column if R is extended East by (R.1) or (R.3) or if R' is extended West by (R.2). Let x be the northeastmost box of R and y be the southwestmost box of R'.

Subcase 2.1: (R is extended by (R.1)): Since $\texttt{label}(\mathsf{x}^{\rightarrow}) = \mathcal{G}^{+}$ and $\texttt{family}(\mathcal{G}^{+}) = \texttt{family}(\mathcal{G})$, by (G.6) R' cannot contain any $\mathcal{G}$'s and therefore $R' = \{\bullet_{\mathcal{G}} \in \mathsf{y}\}$. Hence (R.2) does not extend R'. By assumption, $\mathsf{x}^{\rightarrow}$ and y are in the same column. Hence by (G.4) and (G.11), $\mathsf{y} = \mathsf{x}^{\rightarrow\uparrow}$. By (G.6), R' is not extended by (R.3), since $\mathcal{G}^{+} \in \mathsf{x}^{\rightarrow}$ and (R.3) requires $\mathcal{G} \in \underline{\mathsf{y}^{\rightarrow}}$ or $\mathcal{G} \in \underline{\mathsf{y}^{\rightarrow}}$. If R' is extended by (R.1), we obtain the second scenario described by (III) (and S, S' are box disjoint). If R' is not extended by any of (R.1)–(R.3), then we have the first scenario described by (III) (and S, S' are box disjoint).

Subcase 2.2: (R is extended by (R.3)): Let c be $\mathsf{x}^{\rightarrow}$'s column. We have $\mathcal{F}^{!} \in \mathsf{x}^{\rightarrow}$ and either $\mathcal{G} \in \underline{\mathsf{x}^{\rightarrow}}$ or $\mathcal{G} \in \underline{\mathsf{x}^{\rightarrow}}$. Moreover $N_{\mathcal{F}} = N_{\mathcal{G}}$. Hence by Lemma 4.7, no $\mathcal{G}$ appears East of c. Thus $R' = \{\bullet_{\mathcal{G}} \in \mathsf{y}\}$. By (G.11), $\mathsf{y} \notin c$. Thus S and R' do not share a column. Since $\bullet_{\mathcal{G}} \in \mathsf{y}$, R' is not extended by (R.2). Thus S and S' do not share a column.

Subcase 2.3: (R' is extended by (R.2); R is not extended by either (R.1) or (R.3)): Here $\mathcal{G} \in \mathsf{y}$ and $\mathcal{F}^{!} \in \mathsf{y}^{\leftarrow}$. By Lemma 4.6, either $\texttt{family}(\mathcal{F}) = \texttt{family}(\mathcal{G})$ or else we have $\mathcal{G}' \in \underline{\mathsf{y}^{\leftarrow}}$ or $\mathcal{G}' \in \underline{\mathsf{y}^{\leftarrow}}$ such that $\texttt{family}(\mathcal{G}') = \texttt{family}(\mathcal{G})$. Hence by (G.4) and (G.11), R cannot contain a box in the column of $\mathsf{y}^{\leftarrow}$. Hence R, S' do not share a column. Hence by the assumption of the subcase, S and S' do not share a column. □

4.6.2 Snake sections

We decompose each snake S into three **snake sections** denoted $\texttt{head}(S)$, $\texttt{body}(S)$ and $\texttt{tail}(S)$ as follows:

Definition-Lemma 4.1.

(I) If a snake S has at least two rows and its southmost row has two boxes, then $\texttt{head}(S)$ is the southmost row of S, $\texttt{tail}(S)$ is the northmost row and $\texttt{body}(S)$ is the remaining rows.

(II) If a snake S has at least two rows and its southmost row has exactly one box, then $\texttt{head}(S)$ is empty, $\texttt{tail}(S)$ is the northmost row and $\texttt{body}(S)$ is the other rows.

(III) If S has exactly one row, then S is one of the following (edge labels not depicted):

$$\text{(i) } S = \boxed{\mathcal{G}} = \texttt{body}(S); \quad \text{(ii) } S = \boxed{\bullet_{\mathcal{G}}} = \texttt{head}(S); \quad \text{(iii) } S = \boxed{\bullet_{\mathcal{G}}}\boxed{\mathcal{G}} = \texttt{head}(S);$$

$$\text{(iv) } R = \boxed{\bullet_{\mathcal{G}}}\boxed{\mathcal{G}^{+}} = \texttt{head}(S); \quad \text{(v) } S = \boxed{\mathcal{F}^{!}}\boxed{\mathcal{G}} = \texttt{head}(S);$$

$$\text{(vi) } S = \boxed{\bullet_{\mathcal{G}}}\boxed{\mathcal{F}^{!}} = \texttt{tail}(S) \textit{ (with } \mathcal{G} \textit{ or } \mathcal{G} \textit{ on the lower right edge).}$$

Proof. It is only required to verify that in (III) all possible one-row snakes are shown. This is done by combining Lemma 4.15 and (R.1)–(R.3). □

Lemma 4.18 (Properties of $\texttt{head}, \texttt{body}, \texttt{tail}$)**.**

(I) If $\texttt{head}(S) = \{\mathsf{x}\}$, *then* $\bullet_{\mathcal{G}} \in \mathsf{x}$.

(II) If $\texttt{head}(S) = \{\mathsf{x}, \mathsf{x}^{\rightarrow}\}$, *then* $\texttt{head}(S) = \begin{array}{|c|c|}\hline \mathcal{F}^! & \mathcal{G} \\ \hline \end{array}$, $\begin{array}{|c|c|}\hline \bullet_{\mathcal{G}} & \mathcal{G} \\ \hline \end{array}$ *or* $\begin{array}{|c|c|}\hline \bullet_{\mathcal{G}} & \mathcal{G}^+ \\ \hline \end{array}$.

(III) $\texttt{body}(S)$ *is a short ribbon consisting only of* $\bullet_{\mathcal{G}}$*'s and* $\mathcal{G}$*'s (with no edge label* $\mathcal{G}$*'s or* $\textcircled{\mathcal{G}}$*'s).*

(IV) If $\texttt{tail}(S) = \{\mathsf{x}\}$, *then* $\texttt{tail}(S) = \begin{array}{|c|}\hline \bullet_{\mathcal{G}} \\ \hline \end{array}$ *and S has at least two rows.*

(V) If $\texttt{tail}(S) = \{\mathsf{x}, \mathsf{x}^{\rightarrow}\} = \begin{array}{|c|c|}\hline \bullet_{\mathcal{G}} & \mathcal{G} \\ \hline \end{array}$ *or* $\begin{array}{|c|c|}\hline \bullet_{\mathcal{G}} & \mathcal{G}^+ \\ \hline \end{array}$, *then S has at least two rows,* $\mathcal{G} \notin \underline{\mathsf{x}}$ *and* $\textcircled{\mathcal{G}} \notin \underline{\mathsf{x}}$.

(VI) If $\texttt{tail}(S) = \{\mathsf{x}, \mathsf{x}^{\rightarrow}\}$ *and* $\mathcal{G}$ *or* $\textcircled{\mathcal{G}} \in \underline{\mathsf{x}^{\rightarrow}}$, *then* $\texttt{tail}(S) = \begin{array}{|c|c|}\hline \bullet_{\mathcal{G}} & \underset{\mathcal{G}}{\mathcal{F}^!} \\ \hline \end{array}$ *or* $\begin{array}{|c|c|}\hline \bullet_{\mathcal{G}} & \underset{\textcircled{\mathcal{G}}}{\mathcal{F}^!} \\ \hline \end{array}$.

(VII) If S has at least two rows, then $\mathcal{G} \in \mathsf{x}^{\downarrow}$ *where* x *is the westmost box of* $\texttt{tail}(S)$.

Proof. If S has one row, then by Definition-Lemma 4.1(III) these claims are clear (or irrelevant). Thus assume S has at least two rows.

(I): Under the assumption that S has at least two rows, the claim is vacuous since by Definition-Lemma 4.1(I,II) we know $|\texttt{head}(S)| \neq 1$.

(II): Either the southmost row of S is $\begin{array}{|c|c|}\hline \mathcal{F}^! & \mathcal{G} \\ \hline \end{array}$ if (R.2) was used, or it is $\begin{array}{|c|c|}\hline \bullet & \mathcal{G} \\ \hline \end{array}$ if (R.2) was not used; cf. Lemma 4.15.

(III): That $\texttt{body}(S)$ is a short ribbon is clear, since S is a short ribbon by Lemma 4.16. Boxes of $\texttt{body}(S)$ only contain $\mathcal{G}$ or $\bullet_{\mathcal{G}}$ because (R.1)–(R.3) adjoin boxes only to the northmost or southmost row (and if the southmost row of S has two boxes, then by definition that row is not part of $\texttt{body}(S)$). By (G.12), an edge label $\mathcal{G}$ or $\textcircled{\mathcal{G}}$ can only appear in the northmost or southmost row of S. In those cases, the row is not part of $\texttt{body}(S)$ by Definition-Lemma 4.1(I,II).

(IV): $\texttt{tail}(S)$ is the northmost row of S and, since $|\texttt{tail}(S)| = 1$, it is the northmost row of the presnake of S. Thus we are done by Lemma 4.15.

(V): $\texttt{tail}(S)$ is the northmost row and by Lemma 4.15, $\mathcal{G} \in \mathsf{x}^{\downarrow}$ ($\mathsf{x}^{\downarrow}$ is in the presnake of S) so $\mathcal{G}, \textcircled{\mathcal{G}} \notin \underline{\mathsf{x}}$ by (G.4).

(VI): x is in the presnake of S and so by Lemma 4.15, $\bullet_{\mathcal{G}} \in \mathsf{x}$. By (G.2), $\bullet_{\mathcal{G}} \notin \mathsf{x}^{\rightarrow}$. By (G.4), $\texttt{label}(\mathsf{x}^{\rightarrow}) < \mathcal{G}$ and so $\texttt{label}(\mathsf{x}^{\rightarrow})$ is marked, since it is southeast of the $\bullet_{\mathcal{G}} \in \mathsf{x}$.

(VII): x and $\mathsf{x}^{\downarrow}$ are part of the presnake of S. Now apply Lemma 4.15. $\square$

4.7 Genomic jeu de taquin

4.7.1 Miniswaps

We first define **miniswaps** on snake sections of a $\mathcal{G}$-good tableau. The output is a formal sum of tableaux. Below, interpret $\bullet = \bullet_{\mathcal{G}}$ before the miniswap and $\bullet = \bullet_{\mathcal{G}^+}$ after the miniswap. We depict $\mathcal{G}$ whenever it exists. Labels and virtual labels from other genes are not depicted unless relevant to the miniswap's definition. For a box x, define

$$\beta(\mathsf{x}) := 1 - \frac{t_{\mathsf{Man}(\mathsf{x})}}{t_{\mathsf{Man}(\mathsf{x})+1}} \quad \text{and} \quad \hat{\beta}(\mathsf{x}) := 1 - \beta(\mathsf{x}) = \frac{t_{\mathsf{Man}(\mathsf{x})}}{t_{\mathsf{Man}(\mathsf{x})+1}}.$$

Note that if $\mathsf{x} = \alpha/\beta$, then $\hat{\beta}(\mathsf{x}) = \mathtt{wt}\,\alpha/\beta$, as defined in Section 4.3. If a snake section is empty, then mswap acts trivially, so below we assume otherwise.

Miniswaps on $\mathtt{head}(S)$

(Case H1: $\mathtt{head}(S) = \{\mathsf{x}\}$ and $\mathcal{G} \in \underline{\mathsf{x}}$):

$$\mathtt{head}(S) = \underset{\mathcal{G}}{\boxed{\bullet}} \mapsto \mathsf{mswap}(\mathtt{head}(S)) = \beta(\mathsf{x}) \cdot \boxed{\mathcal{G}} + \gamma \cdot \overset{\mathcal{G}}{\boxed{\bullet}}$$

Set $\gamma := 0$ if $\mathrm{row}(\mathsf{x}) = \mathtt{family}(\mathcal{G})$ (that is, if $\mathcal{G} \in \overline{\mathsf{x}}$ would be too high); otherwise set $\gamma := 1$.

(Case H2: $\mathtt{head}(S) = \{\mathsf{x}\}$ and $\mathcal{G} \in \underline{\mathsf{x}}$):

$$\mathtt{head}(S) = \underset{\mathcal{G}}{\boxed{\bullet}} \mapsto \mathsf{mswap}(\mathtt{head}(S)) = \boxed{\bullet} + \beta(\mathsf{x}) \cdot \boxed{\mathcal{G}}$$

(Case H3: $\mathtt{head}(S) = \{\mathsf{x}\}$ and Cases H1/H2 do not apply):

$$\mathtt{head}(S) = \boxed{\bullet} \mapsto \mathsf{mswap}(\mathtt{head}(S)) = \boxed{\bullet}$$

(Case H4: $\mathtt{head}(S) = \{\mathsf{x}, \mathsf{x}^{\rightarrow}\}$, $\mathcal{G} \in \mathsf{x}^{\rightarrow}$, and $\mathcal{G} \in \underline{\mathsf{x}}$):

$$\mathtt{head}(S) = \underset{\mathcal{G}}{\boxed{\bullet}}\boxed{\mathcal{G}} \mapsto \mathsf{mswap}(\mathtt{head}(S)) = 0$$

(Case H5: $\mathtt{head}(S) = \{\mathsf{x}, \mathsf{x}^{\rightarrow}\}$, $\mathcal{G} \in \mathsf{x}^{\rightarrow}$, and $\mathcal{G} \notin \underline{\mathsf{x}}$):

(Subcase H5.1: $\mathcal{H} \in \underline{\mathsf{x}^{\rightarrow}}$, $\texttt{family}(\mathcal{H}) = \texttt{family}(\mathcal{G}) + 1$ and $N_{\mathcal{H}} = N_{\mathcal{G}}$):

$$\texttt{head}(S) = \begin{array}{|c|c|}\hline \bullet & \mathcal{G}_{\mathcal{H}} \\ \hline \end{array} \mapsto \mathsf{mswap}(\texttt{head}(S)) = \begin{array}{|c|c|}\hline \bullet & \mathcal{G}^{!}_{\mathcal{H}} \\ \hline \end{array}$$

(Subcase H5.2: $\mathcal{H} \in \underline{\mathsf{x}^{\rightarrow}}$, $\texttt{family}(\mathcal{H}) = \texttt{family}(\mathcal{G}) + 1$ and $N_{\mathcal{H}} = N_{\mathcal{G}}$):

$$\texttt{head}(S) = \begin{array}{|c|c|}\hline \bullet & \mathcal{G}_{\mathcal{H}} \\ \hline \end{array} \mapsto \mathsf{mswap}(\texttt{head}(S)) = \begin{array}{|c|c|}\hline \bullet & \mathcal{G}^{!}_{\mathcal{H}} \\ \hline \end{array} + \hat{\beta}(\mathsf{x}) \cdot \begin{array}{|c|c|}\hline \mathcal{G} & \bullet \\ \hline \end{array}$$

(Subcase H5.3: Subcases H5.1/H5.2 do not apply):

$$\texttt{head}(S) = \begin{array}{|c|c|}\hline \bullet_{\mathcal{G}} & \mathcal{G} \\ \hline \end{array} \text{ or } \begin{array}{|c|c|}\hline \bullet & \mathcal{G} \\ \hline \end{array} \mapsto \mathsf{mswap}(\texttt{head}(S)) = \hat{\beta}(\mathsf{x}) \cdot \begin{array}{|c|c|}\hline \mathcal{G} & \bullet \\ \hline \end{array}$$

(Case H6: $\texttt{head}(S) = \{\mathsf{x}, \mathsf{x}^{\rightarrow}\}$, $\mathcal{G}^{+} \in \mathsf{x}^{\rightarrow}$, and $\mathcal{G} \in \underline{\mathsf{x}}$):

$$\texttt{head}(S) = \begin{array}{|c|c|}\hline \bullet_{\mathcal{G}} & \mathcal{G}^{+} \\ \hline \end{array} \mapsto \mathsf{mswap}(\texttt{head}(S)) = \beta(\mathsf{x}) \cdot \begin{array}{|c|c|}\hline \mathcal{G} & \mathcal{G}^{+} \\ \hline \end{array} + \alpha \cdot \begin{array}{|c|c|}\hline \mathcal{G} & \bullet_{\mathcal{G}^{+}} \\ \hline \end{array}$$

Set $\alpha := 0$ if the second tableau has two $\bullet_{\mathcal{G}^{+}}$'s in the same column; otherwise set $\alpha := \hat{\beta}(\mathsf{x})$.

(Case H7: $\texttt{head}(S) = \{\mathsf{x}, \mathsf{x}^{\rightarrow}\}$, $\mathcal{G}^{+} \in \mathsf{x}^{\rightarrow}$, and $\mathcal{G} \in \underline{\mathsf{x}}$):

$$\texttt{head}(S) = \begin{array}{|c|c|}\hline \bullet_{\mathcal{G}} & \mathcal{G}^{+} \\ \hline \end{array} \mapsto \mathsf{mswap}(\texttt{head}(S)) = \begin{array}{|c|c|}\hline \bullet & \mathcal{G}^{+} \\ \hline \end{array} + \beta(\mathsf{x}) \cdot \begin{array}{|c|c|}\hline \mathcal{G} & \mathcal{G}^{+} \\ \hline \end{array} + \alpha \cdot \begin{array}{|c|c|}\hline \mathcal{G} & \bullet_{\mathcal{G}^{+}} \\ \hline \end{array}$$

Set $\alpha := 0$ if the third tableau has two $\bullet_{\mathcal{G}^{+}}$'s in the same column; otherwise set $\alpha := \hat{\beta}(\mathsf{x})$.

(Case H8: $\texttt{head}(S) = \{\mathsf{x}, \mathsf{x}^{\rightarrow}\}$, $\mathcal{G}^{+} \in \mathsf{x}^{\rightarrow}$, and Cases H6 and H7 do not apply):

$$\texttt{head}(S) = \begin{array}{|c|c|}\hline \bullet & \mathcal{G}^{+} \\ \hline \end{array} \mapsto \mathsf{mswap}(\texttt{head}(S)) = \begin{array}{|c|c|}\hline \bullet & \mathcal{G}^{+} \\ \hline \end{array}$$

(Case H9: $\texttt{head}(S) = \{\mathsf{x}, \mathsf{x}^{\rightarrow}\}$, $\mathcal{F}^{!} \in \mathsf{x}$, and $\mathcal{G} \in \mathsf{x}^{\rightarrow}$):

$$\texttt{head}(S) = \begin{array}{|c|c|}\hline \mathcal{F}^{!} & \mathcal{G} \\ \hline \end{array} \mapsto \mathsf{mswap}(\texttt{head}(S)) = \begin{array}{|c|c|}\hline \mathcal{F}^{!} & \mathcal{G}^{!} \\ \hline \end{array}$$

Lemma 4.19. *Every nonempty* $\texttt{head}(S)$ *falls into exactly one of* H1–H9.

Proof. Since $\texttt{head}(S) \neq \emptyset$, $|\texttt{head}(S)| \in \{1, 2\}$ by Lemma 4.16. If $\texttt{head}(S) = \{\mathsf{x}\}$, then by Lemma 4.18(I), $\bullet_{\mathcal{G}} \in \mathsf{x}$. Then $\underline{\mathsf{x}}$ contains exactly one of $\mathcal{G}, \mathcal{G}$ or neither; these are respectively Cases H1, H2 and H3. If $\texttt{head}(S) = \{\mathsf{x}, \mathsf{x}^{\rightarrow}\}$, see Lemma 4.18(II): one possibility is $\mathcal{F}^{!} \in \mathsf{x}$ and $\mathcal{G} \in \mathsf{x}^{\rightarrow}$; this is H9. Otherwise, $\bullet_{\mathcal{G}} \in \mathsf{x}$

and $\mathsf{x}^{\rightarrow}$ contains $\mathcal{G}$ or $\mathcal{G}^+$. The cases where $\mathcal{G} \in \mathsf{x}^{\rightarrow}$ are covered by H4–H5. The cases where $\mathcal{G}^+ \in \mathsf{x}^{\rightarrow}$ are covered by H6–H8. ☐

Miniswaps on $\texttt{body}(S)$

Let $\texttt{body}_{\bullet_{\mathcal{G}}}(S) = \{\mathsf{x} \in \texttt{body}(S) \colon \bullet_{\mathcal{G}} \in \mathsf{x}\}$.

(Case B1: $\texttt{body}(S) = S$): By Definition-Lemma 4.1, $S = \boxed{\mathcal{G}}$. Define

$$\texttt{body}(S) = \boxed{\mathcal{G}} \mapsto \mathsf{mswap}(\texttt{body}(S)) = \boxed{\mathcal{G}} \quad .$$

(Case B2: The southmost row of $\texttt{body}(S)$ contains two boxes): Replace each $\mathcal{G}$ in $\texttt{body}(S)$ with $\bullet_{\mathcal{G}^+}$ and each $\bullet_{\mathcal{G}}$ with $\mathcal{G}$, emitting a weight $\prod_{\mathsf{x} \in \texttt{body}_{\bullet}(S)} \hat{\beta}(\mathsf{x})$. That is (cf. Lemma 4.18(III)),

$$\texttt{body}(S) = \begin{array}{cccc} \cline{3-4} & & \multicolumn{1}{|c|}{\bullet} & \multicolumn{1}{c|}{\mathcal{G}} \\ \cline{2-4} & \multicolumn{1}{|c|}{\bullet} & \multicolumn{1}{c|}{\mathcal{G}} & \\ \cline{1-3} \multicolumn{1}{|c|}{\bullet} & \multicolumn{1}{c|}{\mathcal{G}} & & \\ \cline{1-2} \end{array} \mapsto \mathsf{mswap}(\texttt{body}(S)) = \prod_{\mathsf{x} \in \texttt{body}_{\bullet_{\mathcal{G}}}(S)} \hat{\beta}(\mathsf{x}) \cdot \begin{array}{cccc} \cline{3-4} & & \multicolumn{1}{|c|}{\mathcal{G}} & \multicolumn{1}{c|}{\bullet} \\ \cline{2-4} & \multicolumn{1}{|c|}{\mathcal{G}} & \multicolumn{1}{c|}{\bullet} & \\ \cline{1-3} \multicolumn{1}{|c|}{\mathcal{G}} & \multicolumn{1}{c|}{\bullet} & & \\ \cline{1-2} \end{array}$$

(Case B3: Cases B1/B2 do not apply): Replace each $\mathcal{G}$ in $\texttt{body}(S)$ with $\bullet_{\mathcal{G}^+}$ and each $\bullet_{\mathcal{G}}$ with $\mathcal{G}$, emitting $-\prod_{\mathsf{x} \in \texttt{body}_{\bullet}(S)} \hat{\beta}(\mathsf{x})$. That is (cf. Lemma 4.18(III)),

$$\texttt{body}(S) = \begin{array}{ccc} \cline{2-3} & \multicolumn{1}{|c|}{\bullet} & \multicolumn{1}{c|}{\mathcal{G}} \\ \cline{1-3} \multicolumn{1}{|c|}{\bullet} & \multicolumn{1}{c|}{\mathcal{G}} & \\ \cline{1-2} \multicolumn{1}{|c|}{\mathcal{G}} & & \\ \cline{1-1} \end{array} \mapsto \mathsf{mswap}(\texttt{body}(S)) = -\prod_{\mathsf{x} \in \texttt{body}_{\bullet_{\mathcal{G}}}(S)} \hat{\beta}(\mathsf{x}) \cdot \begin{array}{ccc} \cline{2-3} & \multicolumn{1}{|c|}{\mathcal{G}} & \multicolumn{1}{c|}{\bullet} \\ \cline{1-3} \multicolumn{1}{|c|}{\mathcal{G}} & \multicolumn{1}{c|}{\bullet} & \\ \cline{1-2} \multicolumn{1}{|c|}{\bullet} & & \\ \cline{1-1} \end{array}$$

Lemma 4.20. *Every nonempty* $\texttt{body}(S)$ *falls into exactly one of* B1–B3.

Proof. If B1 applies, then by Definition-Lemma 4.1, $S = \boxed{\mathcal{G}}$. The lemma follows. ☐

Miniswaps on $\texttt{tail}(S)$

(Case T1: $\texttt{tail}(S) = \{\mathsf{x}\}$):

$$\texttt{tail}(S) = \boxed{\bullet} \mapsto \mathsf{mswap}(\texttt{tail}(S)) = -\hat{\beta}(\mathsf{x}) \cdot \boxed{\mathcal{G}}$$

(Case T2: $\texttt{tail}(S) = \{\mathsf{x}, \mathsf{x}^{\rightarrow}\}$ and $\mathcal{G} \in \mathsf{x}^{\rightarrow}$):

$$\texttt{tail}(S) = \begin{array}{|c|c|} \hline \bullet & \mathcal{G} \\ \hline \end{array} \mapsto \mathsf{mswap}(\texttt{tail}(S)) = \hat{\beta}(\mathsf{x}) \cdot \begin{array}{|c|c|} \hline \mathcal{G} & \bullet \\ \hline \end{array}$$

(Case T3: $\texttt{tail}(S) = \{\mathsf{x}, \mathsf{x}^{\rightarrow}\}$ and $\mathcal{G}^+ \in \mathsf{x}^{\rightarrow}$):

$$\texttt{tail}(S) = \begin{array}{|c|c|} \hline \bullet & \mathcal{G}^+ \\ \hline \end{array} \mapsto \mathsf{mswap}(\texttt{tail}(S)) = -\hat{\beta}(\mathsf{x}) \cdot \begin{array}{|c|c|} \hline \mathcal{G} & \mathcal{G}^+ \\ \hline \end{array} + \alpha \cdot \begin{array}{|c|c|} \hline \mathcal{G} & \begin{smallmatrix} \bullet \\ \mathcal{G}^+ \end{smallmatrix} \\ \hline \end{array}$$

Set $\alpha := 0$ if the second tableau has two $\bullet_{\mathcal{G}^+}$'s in the same column; otherwise set $\alpha := \hat{\beta}(\mathsf{x})$.

(Case T4: $\mathtt{tail}(S) = \{\mathsf{x}, \mathsf{x}^{\rightarrow}\}$, $\mathcal{G} \in \underline{\mathsf{x}^{\rightarrow}}$): Let $Z = \{\ell \in \underline{\mathsf{x}^{\rightarrow}} : \mathcal{F} \prec \ell \prec \mathcal{G}\}$.

(Subcase T4.1: $\mathcal{H} \in \underline{\mathsf{x}^{\rightarrow}}$, $\mathtt{family}(\mathcal{H}) = \mathtt{family}(\mathcal{G}) + 1$ and $N_{\mathcal{H}} = N_{\mathcal{G}}$):

$$\mathtt{tail}(S) = \begin{array}{|c|c|}\hline \bullet & \mathcal{F}^! \\ & {}_{Z,\mathcal{G},\mathcal{H}} \\ \hline\end{array} \mapsto \mathsf{mswap}(\mathtt{tail}(S)) = \begin{array}{|c|c|}\hline \bullet & \mathcal{F}^! \\ & {}_{Z,\mathcal{G}^!,\mathcal{H}} \\ \hline\end{array}$$

(Subcase T4.2: $\mathcal{H} \in \underline{\mathsf{x}^{\rightarrow}}$, $\mathtt{family}(\mathcal{H}) = \mathtt{family}(\mathcal{G}) + 1$ and $N_{\mathcal{H}} = N_{\mathcal{G}}$):

$$\mathtt{tail}(S) = \begin{array}{|c|c|}\hline \bullet & \mathcal{F}^! \\ & {}_{Z,\mathcal{G},\mathcal{H}} \\ \hline\end{array} \mapsto \mathsf{mswap}(\mathtt{tail}(S)) = \begin{array}{|c|c|}\hline \bullet & \mathcal{F}^! \\ & {}_{Z,\mathcal{G}^!,\mathcal{H}} \\ \hline\end{array} + \hat{\beta}(\mathsf{x}) \cdot \begin{array}{|c|c|} \multicolumn{1}{c}{{}^{\mathcal{F},Z}} & \multicolumn{1}{c}{} \\ \hline \mathcal{G} & \bullet \\ \hline\end{array}$$

(Subcase T4.3: Subcases T4.1/T4.2 do not apply):

$$\mathtt{tail}(S) = \begin{array}{|c|c|}\hline \bullet & \mathcal{F}^!_{Z,\mathcal{G}} \\ \hline\end{array} \mapsto \mathsf{mswap}(\mathtt{tail}(S)) = \hat{\beta}(\mathsf{x}) \cdot \begin{array}{|c|c|} \multicolumn{1}{c}{{}^{\mathcal{F},Z}} & \multicolumn{1}{c}{} \\ \hline \mathcal{G} & \bullet \\ \hline\end{array}$$

(Case T5: $\mathtt{tail}(S) = \{\mathsf{x}, \mathsf{x}^{\rightarrow}\}$, $\mathcal{G} \in \underline{\mathsf{x}^{\rightarrow}}$, $\mathcal{G} \notin \underline{\mathsf{x}}$): Let $Z = \{\ell \in \underline{\mathsf{x}^{\rightarrow}} : \mathcal{F} \prec \ell \prec \mathcal{G}\}$.

$$\mathtt{tail}(S) = \begin{array}{|c|c|}\hline \bullet & \mathcal{F}^!_{Z,\mathcal{G}} \\ \hline\end{array} \mapsto \mathsf{mswap}(\mathtt{tail}(S)) = \hat{\beta}(\mathsf{x}) \cdot \begin{array}{|c|c|} \multicolumn{1}{c}{{}^{\mathcal{F},Z}} & \multicolumn{1}{c}{} \\ \hline \mathcal{G} & \bullet \\ \hline\end{array}$$

(Case T6: $\mathtt{tail}(S) = \{\mathsf{x}, \mathsf{x}^{\rightarrow}\}$, $\mathcal{G} \in \underline{\mathsf{x}^{\rightarrow}}$, $\mathcal{G} \in \underline{\mathsf{x}}$):

$$\mathtt{tail}(S) = \begin{array}{|c|c|}\hline \bullet_{\mathcal{G}} & \mathcal{F}^!_{\mathcal{G}} \\ \hline\end{array} \mapsto \mathsf{mswap}(\mathtt{tail}(S)) = 0$$

Lemma 4.21. *Every nonempty* $\mathtt{tail}(S)$ *falls into exactly one of* T1–T6.

Proof. Since $\mathtt{tail}(S) \neq \emptyset$, $|\mathtt{tail}(S)| \in \{1, 2\}$ by Lemma 4.16. If $|\mathtt{tail}(S)| = 1$, then by Lemma 4.18(IV), $\mathtt{tail}(S) = \boxed{\bullet_{\mathcal{G}}}$; this is covered by T1. Suppose $\mathtt{tail}(S) = \{\mathsf{x}, \mathsf{x}^{\rightarrow}\}$. By Lemma 4.15, (R.1)–(R.3) and Definition–Lemma 4.1, $\mathtt{tail}(S) = \boxed{\bullet_{\mathcal{G}}}\boxed{\mathcal{G}}$ (handled by T2), $\mathtt{tail}(S) = \boxed{\bullet_{\mathcal{G}}}\boxed{\mathcal{G}^+}$ (handled by T3) or $\mathtt{tail}(S) = \boxed{\bullet_{\mathcal{G}}}\boxed{\mathcal{F}^!}$ with $\mathcal{G}$ or $\mathcal{G} \in \underline{\mathsf{x}^{\rightarrow}}$ (handled by T4, T5 or T6). ∎

4.7.2 Swaps and slides

We define $\mathsf{swap}_{\mathcal{G}}(T)$ and $\mathtt{slide}_{\{\mathsf{x}_i\}}(T)$ for a good tableau T. Define $\mathsf{swap}_{\mathcal{G}}$ on a single snake S by applying mswap simultaneously to $\mathtt{head}(S)$, $\mathtt{body}(S)$, and $\mathtt{tail}(S)$ (where the conditions on each mswap refer to the original S).

Lemma 4.22. *On the edges shared by two adjacent snake sections, the modifications to the labels given by* *the two miniswaps are compatible.*

Proof. Suppose the lower of the two adjacent sections is $\texttt{head}(S)$. The only miniswap that introduces a label to the northeast edge (i.e., $\overline{\mathsf{x}}$ if $\texttt{head}(S) = \{\mathsf{x}\}$ or $\overline{\mathsf{x}^{\rightarrow}}$ if $\texttt{head}(S) = \{\mathsf{x}, \mathsf{x}^{\rightarrow}\}$) is $\mathsf{H1}$. However in that case $\texttt{head}(S) = S$ and the compatibility issue is moot. Since $\texttt{body}$ miniswaps do not affect edge labels, the remaining check is when a $\texttt{tail}$ miniswap involves $\underline{\mathsf{x}}$ where x is the left box of $\texttt{tail}(S)$. This only occurs in $\mathsf{T6}$. In this case $\texttt{tail}(S) = S$, so compatibility is again moot. □

Lemma 4.23 (Swap commutation)**.** *If S_1, S_2 are distinct snakes in a $\mathcal{G}$-good tableau T, then applying $\mathsf{swap}_{\mathcal{G}}$ to S_1 commutes with applying $\mathsf{swap}_{\mathcal{G}}$ to S_2.*

Proof. By definition, the locations of virtual labels in one snake are unaffected by swapping another snake. Hence if the snakes do not share a horizontal edge, there is no concern. If they do, this is the situation of Lemma 4.17(III). The northmost row r of the lower snake (say S_1) is $\{\mathsf{x}, \mathsf{x}^{\rightarrow}\}$ with $\mathcal{G}^+ \in \mathsf{x}^{\rightarrow}$. Hence by (G.4), $\mathcal{G}, \mathcal{G} \notin \overline{\mathsf{x}^{\rightarrow}}$. By inspection, no miniswap involving r affects $\overline{\mathsf{x}^{\rightarrow}}$. Now, the upper snake S_2 has a single row, which by the previous sentence is either an $\mathsf{H3}$ or $\mathsf{H8}$ $\texttt{head}$, irregardless of whether we have acted on r already. Therefore, $\mathsf{swap}_{\mathcal{G}}$ acts trivially on S_2 whether we act on S_1 or S_2 first. □

Lemma 4.23 permits us to define the **swap** operation $\mathsf{swap}_{\mathcal{G}}$ on a $\mathcal{G}$-good tableau as the result of applying $\mathsf{swap}_{\mathcal{G}}$ to all snakes (in arbitrary order). Extend $\mathsf{swap}_{\mathcal{G}}$ to a $\mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$-linear operator.

An **inner corner** of ν/λ is a maximally southeast box of λ. An **outer corner** of ν/λ is a maximally southeast box of ν/λ.

Let $T \in \texttt{Bundled}(\nu/\lambda)$ and $\{\mathsf{x}_i\}$ be a subset of the inner corners of ν/λ. Define $T^{(1_1)}$ to be T with $\bullet_{1_1}$ placed in each x_i.

Lemma 4.24. *Each $T^{(1_1)}$ is 1_1-good.*

Proof. (G.2) is clear. By Lemma 4.4, T is good; (G.1), (G.3)–(G.8) and (G.12) are unaffected by adding $\bullet_{1_1}$'s to inner corners. (G.9)–(G.11) and (G.13) hold vacuously. □

The **slide** of T at $\{\mathsf{x}_i\}$ is

$$\texttt{slide}_{\{\mathsf{x}_i\}}(T) := \mathsf{swap}_{\mathcal{G}_{\max}} \circ \mathsf{swap}_{\mathcal{G}_{\max}^-} \circ \cdots \circ \mathsf{swap}_{1_1}(T^{(1_1)}), \tag{4.8}$$

with all $\bullet_{\mathcal{G}_{\max}^+}$'s deleted. If Σ is a formal $\mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$-linear sum of tableaux we write $V \in \Sigma$ to mean V occurs in Σ with nonzero coefficient. The following proposition shows (4.8) is well-defined.

Proposition 4.4 (Swaps preserve goodness)**.** *If T is a $\mathcal{G}$-good tableau, then each $U \in \mathsf{swap}_{\mathcal{G}}(T)$ is $\mathcal{G}^+$-good.*

Proof. We omit the lengthy proof, which appears as [PeYo15b, Appendix A]. $\square$

Lemma 4.25 (Swaps preserve content)**.** *If T is a $\mathcal{G}$-good tableau of content μ, then each $U \in \mathsf{swap}_{\mathcal{G}}(T)$ has content μ.*

Proof. No miniswap eliminates genes in a section. We consider each miniswap that introduces a new gene to a section; this gene must be $\mathcal{G}$. We show that $\mathcal{G}$ appears elsewhere in T. The first case is $\mathsf{H2}$, which produces a $\mathcal{G}$ in its section, where there was only a $\mathcal{G}$ previously. $\mathcal{G}$ only appears if some $\mathcal{G}$ is west of it in T. The same analysis applies *verbatim* to $\mathsf{H7}$ and $\mathsf{T5}$. The remaining cases are $\mathsf{T1}$ and $\mathsf{T3}$. By Lemma 4.18(IV, V), the snake on which these miniswaps act has at least two rows. Moreover, there is a $\mathcal{G}$ directly below the $\bullet_{\mathcal{G}}$ under consideration. In particular, $\mathcal{G}$ already appeared in T. $\square$

Lemma 4.26. *No label is strictly southeast of a $\bullet_{\mathcal{G}^+_{\max}}$ in any $U \in \mathsf{swap}_{\mathcal{G}_{\max}} \circ \mathsf{swap}_{\mathcal{G}^-_{\max}} \circ \cdots \circ \mathsf{swap}_{1_1}(T^{(1_1)})$. In particular, all $\bullet_{\mathcal{G}^+_{\max}}$'s are at outer corners of ν/λ.*

Proof. By Proposition 4.4, U is $\mathcal{G}^+_{\max}$-good. Let x be a box of U and $\bullet_{\mathcal{G}^+_{\max}} \in \mathsf{x}$. There is no $\bullet_{\mathcal{G}^+_{\max}}$ strictly southeast of x by (G.2). By definition, there is no label $\mathcal{Q}$ in $T^{(1_1)}$ with $\mathtt{family}(\mathcal{Q}) \geq \mathtt{family}(\mathcal{G}^+_{\max})$. Hence by Lemma 4.25, there are no such labels in U. Therefore, any genetic label ℓ southeast of x is marked. Clearly, we may assume ℓ is in x's row or column. If ℓ is in x's column, we contradict (G.11). If ℓ is in x's row, we contradict Lemma 4.6. $\square$

Clearly,

Lemma 4.27. *If T is a good tableau with no genetic label southeast of a $\bullet$, then deleting all $\bullet$'s gives a bundled tableau.* $\square$

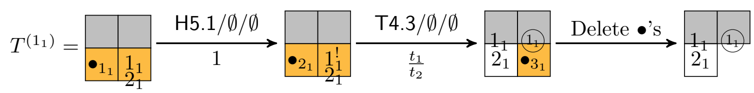
Corollary 4.2. *Given $\rho \in \lambda^+$ and a tableau $T \in B^{\nu}_{\rho,\mu}$, any tableau $U \in \mathtt{slide}_{\rho/\lambda}(T)$ is in either $B^{\nu}_{\lambda,\mu}$ or $B^{\delta}_{\lambda,\mu}$ for some $\delta \in \nu^-$.*

Proof. By Lemma 4.4, T is a good tableau. By Lemma 4.24, adding $\bullet_{1_1}$ to each box of ρ/λ gives a good tableau $T^{(1_1)}$. By Proposition 4.4, each swap gives a formal sum of good tableaux. By Lemma 4.26, after all swaps, $\bullet_{\mathcal{G}^+_{\max}}$'s are at outer corners with no labels strictly southeast. By Lemma 4.27, deleting these $\bullet_{\mathcal{G}^+_{\max}}$'s gives a bundled tableau (namely U). U has shape ν/λ or δ/λ for $\delta \in \nu^-$, since there is at most one $\bullet_{\mathcal{G}^+_{\max}}$ deleted in any row or column by (G.2). Content preservation is Lemma 4.25. $\square$

4.7.3 Examples

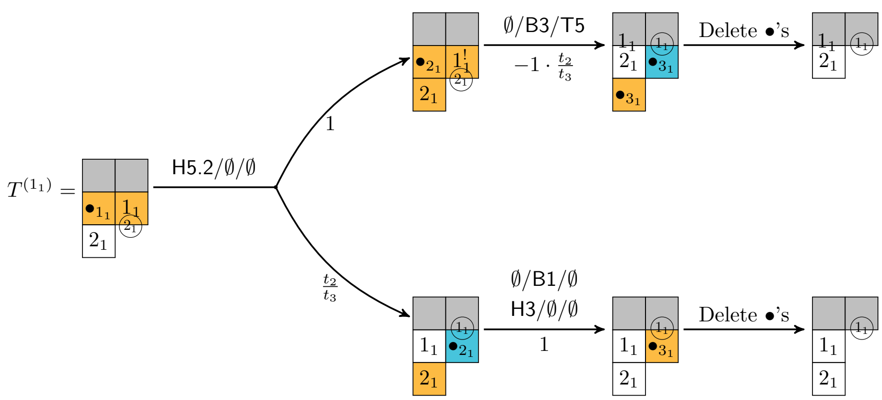
We give a number of examples of computing $\mathtt{slide}_{\{\mathsf{x}_i\}}(T)$. It is convenient to encode the computations in a diagram. Each non-terminal tableau has its snakes differentiated by color. The notation above each arrow indicates the types of the snakes from southwest to northeast, for example H5.3/∅/T2 means the head of the snake is H5.3, the body is empty and the tail is T2. The notation below arrows indicates the product of the coefficients coming from each miniswap (we will assume for this purpose that the lower left corner of T coincides with the lower left corner of $k \times (n-k)$). Each $U \in \mathtt{slide}_{\{x_i\}}(T)$ is a terminal tableau of the diagram. Moreover, $[U]\mathtt{slide}_{\{x_i\}}(T)$ is the sum of the products of the coefficients over all directed paths from T to U.

Example 4.12.



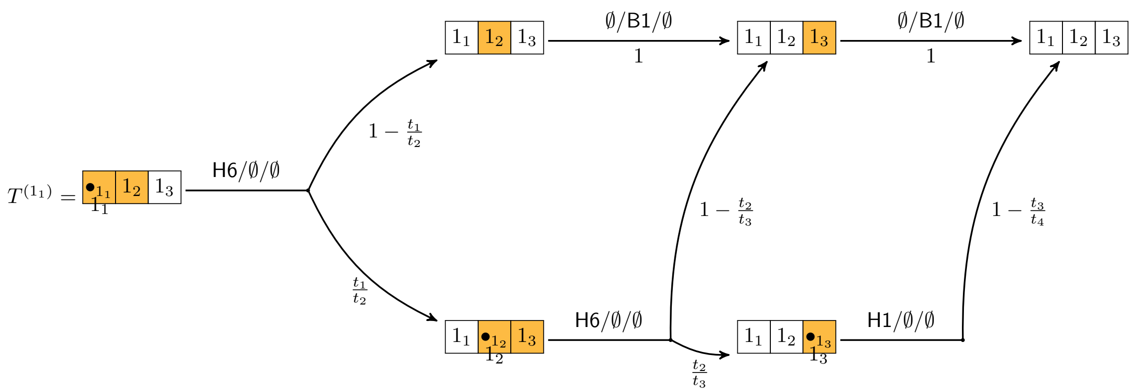
◇

Example 4.13.



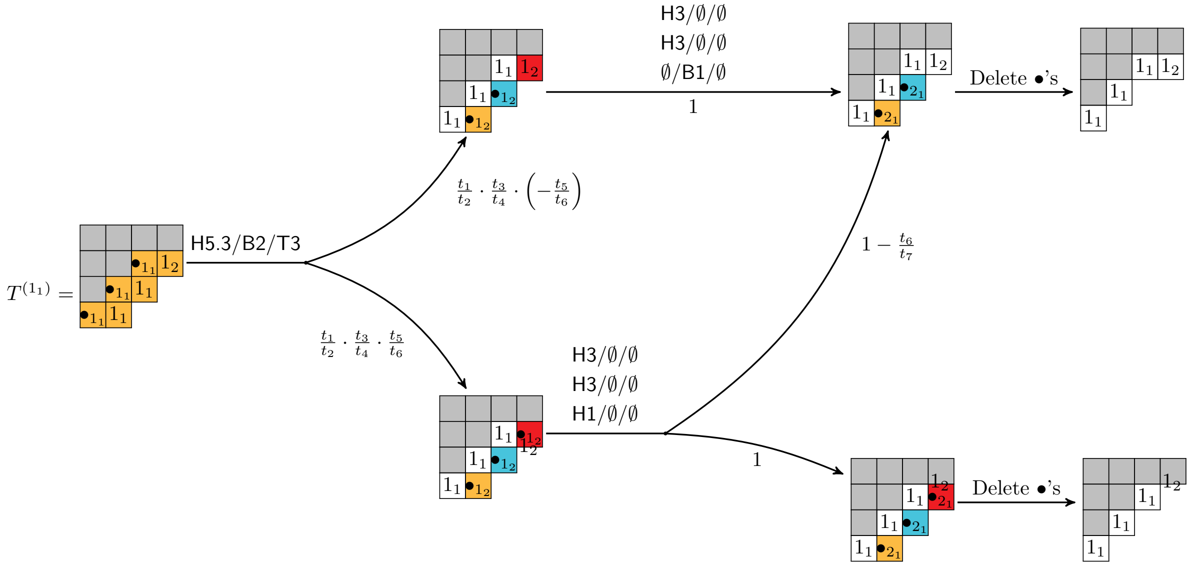
◇

Example 4.14.



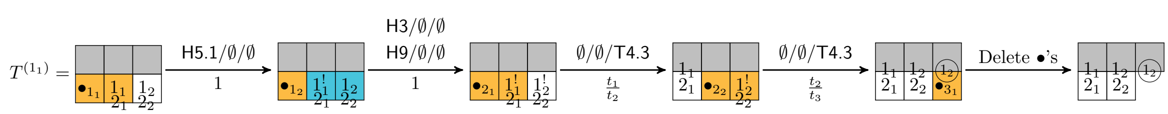
◇

Example 4.15.



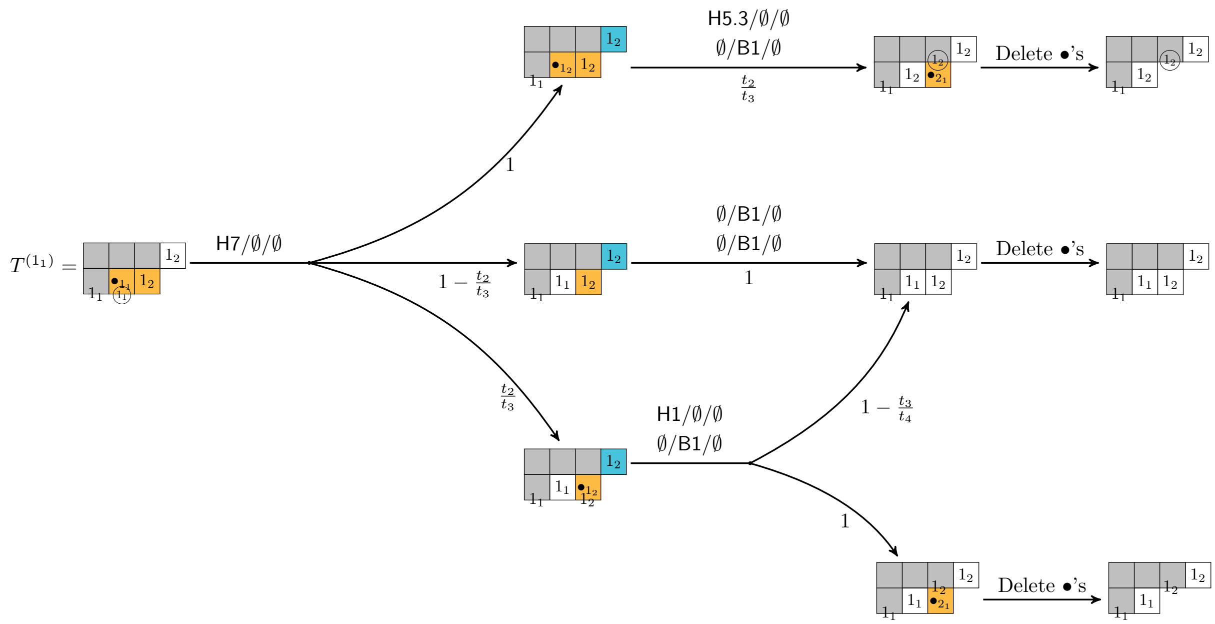
◇

Example 4.16.



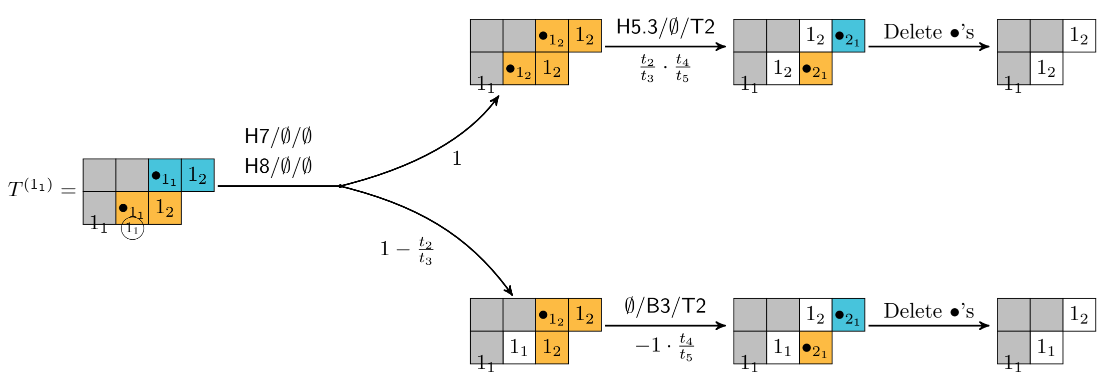
◇

Example 4.17.



♢

Example 4.18.

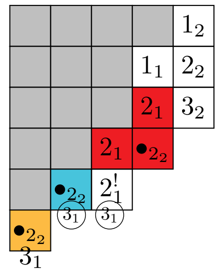


♢

4.8 Ladders

Let U be a $\mathcal{G}^+$-good tableau. Consider the boxes of U containing $\bullet_{\mathcal{G}^+}$ or unmarked $\mathcal{G}$. This set decomposes into maximal edge-connected components, which we call **ladders**.

Example 4.19.



This 2_2-good tableau has three ladders; we have given each ladder a separate color. (All virtual labels are depicted.)

$\diamondsuit$

Lemma 4.28. *A row r of a ladder L is one of the following (edge labels other than $\mathcal{G}$ and virtual labels are not shown):*

$$\text{(L1)}\ \boxed{\bullet} \qquad \text{(L2)}\ \boxed{\mathcal{G}} \qquad \text{(L3)}\ \overset{\mathcal{G}}{\boxed{\bullet}} \qquad \text{(L4)}\ \boxed{\mathcal{G}}\boxed{\bullet}$$

Proof. By (G.2), at most one $\bullet_{\mathcal{G}+}$ occurs in each row. By Lemma 4.11, at most one $\mathcal{G}$ appears in each row. Thus r has at most two boxes. If it has one box, r is clearly L1, L2 or L3. If r has two boxes, then it has one box label $\mathcal{G}$ and one box label $\bullet_{\mathcal{G}+}$. Since the $\mathcal{G}$ is not marked, it is West of the $\bullet_{\mathcal{G}+}$. By (G.4) and (G.7), no edge label $\mathcal{G}$ is possible in this two-box scenario. Thus L4 is the only two box possibility. $\square$

Lemma 4.29. *A ladder L is a short ribbon where each column with 2 boxes is* $\begin{array}{|c|}\hline \mathcal{G} \\ \hline \bullet \\ \hline \end{array}$.

Proof. In each column, there is at most one $\bullet_{\mathcal{G}+}$ by (G.2) and at most one $\mathcal{G}$ by (G.4). If the column consists of $\bullet_{\mathcal{G}+}$ and $\mathcal{G}$, then the $\mathcal{G}$ is North of the $\bullet_{\mathcal{G}+}$, since otherwise the $\mathcal{G}$ is marked. Therefore the columns are as described.

If L has a 2×2 subsquare the North box of each column must contain $\mathcal{G}$, violating (G.3). Each row has at most two boxes by Lemma 4.28. That L is a skew shape is now immediate from the descriptions of L's rows and columns. $\square$

Lemma 4.30 (Relative positioning of ladders)**.** *Suppose U is $\mathcal{G}^+$-good, and that L, M are distinct ladders of U. Then, up to relabeling of the ladders, L is entirely SouthWest of M (that is, if b, b' are boxes of L, M respectively, then b is SouthWest of b').*

Proof. Suppose not. There are three cases to consider:

Case 1: ($\mathsf{b} \in L$ is NorthWest of $\mathsf{b}' \in M$): By definition, b and b' contain either $\bullet_{\mathcal{G}+}$ or $\mathcal{G}$. By (G.2) and Lemmas 4.11, 4.12 and 4.13, we see that no combination of these choices is possible.

Case 2: (b is North and in the same column as b'): If $\bullet_{\mathcal{G}+} \in \mathsf{b}$ and $\bullet_{\mathcal{G}+} \in \mathsf{b}'$, we violate (G.2). If $\bullet_{\mathcal{G}+} \in \mathsf{b}$ and $\mathcal{G} \in \mathsf{b}'$, then the latter would be marked. Hence $\mathcal{G} \in \mathsf{b}$. Since $\mathcal{G} \in \mathsf{b}'$ or $\bullet_{\mathcal{G}+} \in \mathsf{b}'$, we have by (G.4) and (G.9) that $\mathsf{b}^{\downarrow} = \mathsf{b}'$ and so b, b' are in the same ladder, contradicting $L \neq M$.

Case 3: (b is West and in the same row as b′): By (G.2), at least one of b, b' contains $\mathcal{G}$. By Lemma 4.11, at least one of b, b' contains $\bullet_{\mathcal{G}^+}$. If $\mathcal{G} \in \mathsf{b}$ and $\bullet_{\mathcal{G}^+} \in \mathsf{b}'$, then by (G.3) and (G.9), $\mathsf{b}' = \mathsf{b}^{\rightarrow}$, contradicting $L \neq M$. If $\bullet_{\mathcal{G}^+} \in \mathsf{b}$ and $\mathcal{G} \in \mathsf{b}'$, then the latter is marked. $\square$

4.9 Reverse genomic jeu de taquin

Let r be a ladder row in a $\mathcal{G}^+$-good tableau U and let x be the westmost box in r. We define the **reverse miniswap** operation revmswap on r. The cases below are labeled in accordance with the classification of Lemma 4.28. Below, each $\bullet$ on the left of the "$\mapsto$" is a $\bullet_{\mathcal{G}^+}$, while on the right it is a $\bullet_{\mathcal{G}}$.

(Case L1):

(Subcase L1.1: $\mathcal{G} \in \mathsf{x}^{\uparrow}$):

$$r = \boxed{\bullet} \mapsto \mathsf{revmswap}(r) = \boxed{\mathcal{G}}$$

(Subcase L1.2: $\mathcal{G} \notin \mathsf{x}^{\uparrow}$):

$$r = \boxed{\bullet} \mapsto \mathsf{revmswap}(r) = \boxed{\bullet}$$

(Case L2):

(Subcase L2.1: $\bullet_{\mathcal{G}^+} \in \mathsf{x}^{\downarrow}$ or $\mathcal{G}^! \in \mathsf{x}^{\downarrow}$):

$$r = \boxed{\mathcal{G}} \mapsto \mathsf{revmswap}(r) = \boxed{\bullet}$$

(Subcase L2.2: $\bullet_{\mathcal{G}^+} \notin \mathsf{x}^{\downarrow}$, $\mathcal{G}^! \notin \mathsf{x}^{\downarrow}$, $\mathcal{G}^! \notin \mathsf{x}$, x contains the westmost $\mathcal{G}$):

$$r = \boxed{\mathcal{G}} \mapsto \mathsf{revmswap}(r) = \boxed{\mathcal{G}} + \boxed{\underset{\mathcal{G}}{\bullet}}.$$

(Subcase L2.3: $\bullet_{\mathcal{G}^+} \notin \mathsf{x}^{\downarrow}$, $\mathcal{G}^! \notin \mathsf{x}^{\downarrow}$, $\mathcal{G}^! \notin \mathsf{x}$, x does *not* contain the westmost $\mathcal{G}$):

$$r = \boxed{\mathcal{G}} \mapsto \mathsf{revmswap}(r) = \boxed{\mathcal{G}} + \boxed{\underset{\mathcal{G}}{\bullet}}.$$

(Case L3):

$$r = \overset{\mathcal{G}}{\boxed{\bullet}} \mapsto \mathsf{revmswap}(r) = \boxed{\underset{\mathcal{G}}{\bullet}}.$$

(Case L4):

(Subcase L4.1: $\mathcal{G}^+ \in \underline{\mathsf{x}^{\rightarrow}}$ with $\mathtt{family}(\mathcal{G}^+) = \mathtt{family}(\mathcal{G})$, and either $\bullet_{\mathcal{G}^+} \in \mathsf{x}^{\downarrow}$ or $\mathcal{G}^! \in \mathsf{x}^{\downarrow}$):

$$r = \boxed{\mathcal{G}}\boxed{\underset{\mathcal{G}^+}{\bullet}} \mapsto \mathsf{revmswap}(r) = \boxed{\bullet}\boxed{\mathcal{G}^+}$$

(Subcase L4.2: $\mathcal{G}^+ \in \underline{\mathsf{x}^{\rightarrow}}$ with $\mathtt{family}(\mathcal{G}^+) = \mathtt{family}(\mathcal{G})$, $\bullet_{\mathcal{G}^+} \notin \mathsf{x}^{\downarrow}$, $\mathcal{G}^! \notin \mathsf{x}^{\downarrow}$ and x contains the westmost $\mathcal{G}$):

$$r = \begin{array}{|c|c|}\hline \mathcal{G} & \underset{\mathcal{G}^+}{\bullet} \\ \hline \end{array} \mapsto \mathsf{revmswap}(r) = \begin{array}{|c|c|}\hline \underset{\mathcal{G}}{\bullet} & \mathcal{G}^+ \\ \hline \end{array}$$

(Subcase L4.3: $\mathcal{G}^+ \in \underline{\mathsf{x}^{\rightarrow}}$ with $\mathtt{family}(\mathcal{G}^+) = \mathtt{family}(\mathcal{G})$, $\bullet_{\mathcal{G}^+} \notin \mathsf{x}^{\downarrow}$, $\mathcal{G}^! \notin \mathsf{x}^{\downarrow}$ and x does *not* contain the westmost $\mathcal{G}$):

$$r = \begin{array}{|c|c|}\hline \mathcal{G} & \underset{\mathcal{G}^+}{\bullet} \\ \hline \end{array} \mapsto \mathsf{revmswap}(r) = \begin{array}{|c|c|}\hline \underset{\mathcal{G}}{\bullet} & \mathcal{G}^+ \\ \hline \end{array}$$

(Subcase L4.4: there is *no* $\mathcal{G}^+ \in \underline{\mathsf{x}^{\rightarrow}}$ with $\mathtt{family}(\mathcal{G}^+) = \mathtt{family}(\mathcal{G})$, and x contains the westmost $\mathcal{G}$): Let A be the labels in $\bar{\mathsf{x}}$, $Z = \{\mathcal{E} \in A \colon N_{\mathcal{G}} = N_{\mathcal{E}}\}$, $Z^\sharp = Z \cup \{\mathcal{G}\}$, $\mathcal{F} = \min Z^\sharp$, $A'' = Z^\sharp \backslash \{\mathcal{F}\}$, and $A' = A \backslash Z$.

$$r = \begin{array}{|c|c|}\hline \overset{A}{\mathcal{G}} & \bullet \\ \hline \end{array} \mapsto \mathsf{revmswap}(r) = \begin{array}{|c|c|}\hline \overset{A'}{\bullet} & \underset{A''}{\mathcal{F}} \\ \hline \end{array}.$$

(Subcase L4.5: there is *no* $\mathcal{G}^+ \in \underline{\mathsf{x}^{\rightarrow}}$ with $\mathtt{family}(\mathcal{G}^+) = \mathtt{family}(\mathcal{G})$, and x does *not* contain the westmost $\mathcal{G}$): Let $A, Z, Z^\sharp, \mathcal{F}$ and A' be as in L4.4; also let $A''' = Z \backslash \{\mathcal{F}\}$.

$$r = \begin{array}{|c|c|}\hline \overset{A}{\mathcal{G}} & \bullet \\ \hline \end{array} \mapsto \mathsf{revmswap}(r) = \begin{array}{|c|c|}\hline \overset{A'}{\bullet} & \underset{A''', \mathcal{G}}{\mathcal{F}} \\ \hline \end{array}.$$

Lemma 4.31. *Every ladder row falls into exactly one of the above cases.*

Proof. This is tautological, given Lemma 4.28. □

Lemma 4.32. *No* revmswap *affects an edge that is shared by two rows of the same ladder L.*

Proof. No revmswap affects the upper (virtual) edge labels of the right box of a ladder row. Hence it suffices to analyze those cases that affect the lower (virtual) edge labels of the left box of a ladder row. These are L2.2, L2.3, L3, L4.2 and L4.3. In each case there can be no ladder row of L below, by Lemma 4.29. Hence that edge is not shared. □

Thus it makes sense to define $\mathtt{revswap}_{\mathcal{G}^+}$ on a ladder L, by applying revmswap to each row of L simultaneously (where the conditions on each revmswap refer to the original ladder L).

Lemma 4.33. *If L_1, L_2 are distinct ladders in a $\mathcal{G}^+$-good tableau U, then applying $\mathtt{revswap}_{\mathcal{G}^+}$ to L_1 commutes with applying $\mathtt{revswap}_{\mathcal{G}^+}$ to L_2.*

Proof. This follows, since by definition L_1 and L_2 do not share any edges. □

Lemma 4.33 permits us to define the **reverse swap** $\mathtt{revswap}_{\mathcal{G}^+}$ on a $\mathcal{G}^+$-good tableau by applying $\mathtt{revswap}_{\mathcal{G}^+}$ to all ladders (in arbitrary order). We extend this to a $\mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$-linear operator.

Lemma 4.34 (Reverse swaps preserve content)**.** *If U is $\mathcal{G}^+$-good and of content μ, then each $T \in \texttt{revswap}_{\mathcal{G}^+}(U)$ has content μ.*

Proof. Let $\mathcal{H}$ be a gene in U. We must show $\mathcal{H} \in T$. Let ℓ be the westmost instance of $\mathcal{H}$ in U. If ℓ is not part of a ladder, $\mathcal{H}$ appears in the same location in T and we are done. Thus suppose ℓ is in a ladder row r. Consider the reverse miniswap applied to r. If it is anything but $\mathsf{L2.1}$ or $\mathsf{L4.1}$, then there is an $\mathcal{H}$ in that row of T. If it is $\mathsf{L2.1}$ or $\mathsf{L4.1}$, let x be the box containing ℓ. By definition, U has $\bullet_{\mathcal{H}^+} \in \mathsf{x}^{\downarrow}$ or $\mathcal{H}^! \in \mathsf{x}^{\downarrow}$. In the former case, the miniswap applied at $\mathsf{x}^{\downarrow}$ is $\mathsf{L1.1}$, so $\mathcal{H}$ appears in $\mathsf{x}^{\downarrow}$ in T. In the latter case, $\mathsf{x}^{\downarrow}$ is not in a ladder, so $\mathcal{H}$ appears in $\mathsf{x}^{\downarrow}$ in T.

Conversely suppose $\mathcal{H}$ is not a gene in U. We must show it does not appear in T. If it appeared in T, it must be created by some miniswap. Clearly no miniswap but $\mathsf{L1.1}$ could possibly introduce a new gene. But if we apply $\mathsf{L1.1}$ at some box x of U, introducing $\mathcal{H} \in \mathsf{x}$ in T, then U has $\mathcal{H} \in \mathsf{x}^{\uparrow}$ by definition, so $\mathcal{H}$ was indeed a gene of U. □

Proposition 4.5 (Reverse swaps preserve goodness)**.** *If U is $\mathcal{G}^+$-good, each $T \in \texttt{revswap}_{\mathcal{G}^+}(U)$ is $\mathcal{G}$-good.*

Proof. We omit this lengthy proof, which appears as [PeYo15b, Appendix B]. □

Lemma 4.35. *Let T be a $\mathcal{G}$-good tableau and $U \in \mathsf{swap}_{\mathcal{G}}(T)$.*

(I) If $\texttt{label}_U(\mathsf{x}) = \mathcal{G}$, then $\texttt{label}_T(\mathsf{x}) \in \{\bullet_{\mathcal{G}}, \mathcal{G}\}$.

(II) If $\texttt{label}_U(\mathsf{x}) = \bullet_{\mathcal{G}^+}$, then $\texttt{label}_T(\mathsf{x}) \in \{\bullet_{\mathcal{G}}, \mathcal{G}, \mathcal{F}^!, \mathcal{G}^+\}$.

(III) If $\texttt{label}_U(\mathsf{x}) = \mathcal{G}^!$, then $\texttt{label}_T(\mathsf{x}) = \mathcal{G}$.

Proof. By inspection of the miniswaps. □

Lemma 4.36. *Let U be a $\mathcal{G}^+$-good tableau and $T \in \texttt{revswap}_{\mathcal{G}^+}(U)$.*

(I) If $\texttt{label}_U(\mathsf{x}) = \mathcal{G}^!$, then $\texttt{label}_T(\mathsf{x}) = \mathcal{G}$.

(II) If $\texttt{label}_U(\mathsf{x}) = \mathcal{G}$, then $\texttt{label}_T(\mathsf{x}) \in \{\mathcal{G}, \bullet_{\mathcal{G}}\}$.

(III) If $\texttt{label}_U(\mathsf{x}) = \bullet_{\mathcal{G}^+}$, then $\texttt{label}_T(\mathsf{x}) \in \{\bullet_{\mathcal{G}}, \mathcal{G}, \mathcal{G}^+, \mathcal{F}^!\}$. If moreover $\texttt{label}_T(\mathsf{x}) = \mathcal{G}^+$, then $\texttt{label}_T(\mathsf{x}^{\leftarrow}) = \bullet_{\mathcal{G}}$, while if moreover $\texttt{label}_T(\mathsf{x}) = \mathcal{F}^!$, then $N_{\mathcal{F}} = N_{\mathcal{G}}$, $\texttt{label}_T(\mathsf{x}^{\leftarrow}) = \bullet_{\mathcal{G}}$ and either $\mathcal{G} \in \underline{\mathsf{x}}$ or $\mathcal{G} \in \underline{\mathsf{x}}$.

Proof. By inspection of the reverse miniswaps. □

For a good tableau T of shape ν/λ, define a T-**patch** of ν/λ as one of the following:

(Pat.1) A row of a snake of T (including both upper and lower edges of the row).

(Pat.2) A box not in a snake (the box excludes the edges).

(Pat.3) A horizontal edge not bounding a box of a snake.

Clearly, the set $\{P\}$ of T-patches covers ν/λ. Given a tableau W of shape ν/λ, let $W|_P$ be the tableau obtained by restricting W to P.

Proposition 4.6. *Let T, U be good. Then $U \in \mathsf{swap}_{\mathcal{G}}(T)$ if and only if $T \in \mathtt{revswap}_{\mathcal{G}^+}(U)$.*

Proof. ($\Rightarrow$) Suppose $U \in \mathsf{swap}_{\mathcal{G}}(T)$. We show $T \in \mathtt{revswap}_{\mathcal{G}^+}(U)$.

Claim 4.2. *Every ladder row r of U is contained in a distinct T-patch.*

Proof. Distinctness is clear. We now argue containment. If r has one box, containment is trivial. Otherwise, r has two boxes, and we are in case $\mathsf{L4}$ of the ladder row classification of Lemma 4.28. So, in U, each box of r contains $\bullet_{\mathcal{G}^+}$ or $\mathcal{G}$. One considers all possibilities, under Lemma 4.35, for the entries in T of the boxes of r. Since T is good, these boxes of T either form a row of a snake section or are $\boxed{\mathcal{G}}\boxed{\mathcal{G}^+}$. We are done by (Pat.1) in the former case. The latter case cannot occur, since by inspection of the miniswaps, this cannot swap to $\mathsf{L4}$. □

By the definitions, notice that $\mathtt{revswap}_{\mathcal{G}^+}(U) \neq 0$. Moreover:

Claim 4.3. *For each T-patch P, there exists $W \in \mathtt{revswap}_{\mathcal{G}^+}(U)$ such that $W|_P = T|_P$ (ignoring virtual labels).*

Proof. If P is type (Pat.2), then by definition $T|_P = U|_P$, since P is not part of a snake. In particular $U|_P$ does not contain $\mathcal{G}$ or $\bullet_{\mathcal{G}^+}$. So $U|_P$ is not part of a ladder of U. Hence for any $W \in \mathtt{revswap}_{\mathcal{G}^+}(U)$, $W|_P = U|_P = T|_P$ as desired.

If P is type (Pat.3), then $T|_P = U|_P$, since P is not part of a snake. Moreover, by definition, no box y bounded by the edge P is part of a snake in T. Therefore, $\bullet_{\mathcal{G}}, \mathcal{G} \notin \mathsf{y}$ in T. Hence $\bullet_{\mathcal{G}^+}, \mathcal{G} \notin \mathsf{y}$ in U. So P does not bound a box of a ladder of U. Thus for any $W \in \mathtt{revswap}_{\mathcal{G}^+}(U)$, $W|_P = U|_P = T|_P$.

Finally if P is type (Pat.1), by inspection of the miniswaps, combined with Claim 4.2, $U|_P$ contains at most one ladder row r, and possibly a non-ladder box y. Since $\mathtt{revswap}_{\mathcal{G}^+}$ does not affect y, it suffices to indicate the reverse miniswap on r to give our desired $W|_P = T|_P$. We refer to the list of outputs described in Section 4.7.

$\mathsf{H1}$: Use $\mathsf{L2.2}$ or $\mathsf{L3}$ respectively on the two mswap outputs.

$\mathsf{H2}$: Use $\mathsf{L1.2}$ or $\mathsf{L2.3}$ respectively on the two mswap outputs.

$\mathsf{H3}$: Use $\mathsf{L1.2}$: By T's (G.2) and (G.9) and Lemma 4.35(I) applied to T, we have $\mathcal{G} \notin \mathsf{x}^{\uparrow}$ in U.

H4: This case does not arise, since here U does not exist.

H5.1: Use L1.2.

H5.2: For the first output, use L1.2. For the second output, use L4.4 or L4.5. We must show in the latter cases that $Z = \emptyset$. Otherwise if $\mathcal{E} \in Z$, then $\mathcal{E} \in \overline{\mathsf{x}}$ in T. Since $N_{\mathcal{E}} = N_{\mathcal{G}}$ in both T and U, this contradicts Lemma 4.7 for T.

H5.3: Use L4.4 or L4.5. The argument that these apply is the same as for H5.2.

H6: Use L2.2 for the first output and L4.2 for the second. By Lemma 4.35(II) and T's (G.2) and (G.4), $\bullet_{\mathcal{G}^+} \notin \mathsf{x}^{\downarrow}$; by Lemma 4.35(I) and T's (G.2) or (G.4), $\mathcal{G}^! \notin \mathsf{x}^{\downarrow}$; that the $\mathcal{G} \in \mathsf{x}$ is westmost follows from T's (G.7) and [PeYo15b, Lemma A.3] applied to T.

H7: Use L1.2 for the first output: By Lemma 4.35(I) and T's (G.2) and (G.9), U has $\mathcal{G} \notin \mathsf{x}^{\uparrow}$. Use L2.3 for the second output and L4.3 for the third: By Lemma 4.35(II) and T's (G.2) or (G.4), $\bullet_{\mathcal{G}^+} \notin \mathsf{x}^{\downarrow}$; by Lemma 4.35(I) and T's (G.2) or (G.4), $\mathcal{G}^! \notin \mathsf{x}^{\downarrow}$; that the $\mathcal{G} \in \mathsf{x}$ is not westmost follows from T's $\mathcal{G} \in \underline{\mathsf{x}}$.

H8: Use L1.2: By T's (G.2) and (G.9) and Lemma 4.35(I), U has $\mathcal{G} \notin \mathsf{x}^{\uparrow}$.

H9: Here r does not exist.

B1: Use L2.2 or L2.3: By Lemma 4.35(II) and T's (G.2) or (G.4), $\bullet_{\mathcal{G}^+} \notin \mathsf{x}^{\downarrow}$; by Lemma 4.35(III) and T's (G.4), $\mathcal{G}^! \notin \mathsf{x}^{\downarrow}$.

B2: Use L4.4 or L4.5; applicability is as for H5.2.

B3: If we are not in the bottom row, we may use L4.4 or L4.5 as for B2. Otherwise, use L1.1.

T1: Use L2.1: By Lemma 4.18(IV, VII), T has $\mathcal{G} \in \mathsf{x}^{\downarrow}$, so the hypothesis holds by inspection of the miniswaps.

T2: Use L4.4 or L4.5; applicability is as for H5.2.

T3: Use L2.1 or L4.1; applicability is as for T1.

T4.1: Use L1.2: By T's (G.2) and (G.12) and Lemma 4.35(I), U has $\mathcal{G} \notin \mathsf{x}^{\uparrow}$ and $\mathcal{G} \notin \overline{\mathsf{x}}$.

T4.2: Use L1.2 on the first output; applicability is as for T4.1. Use L4.4 on the second output; applicability is as for H5.2.

T4.3: Use L4.4; applicability is as for H5.2.

T5: Use L4.5; applicability is as for H5.2.

T6: This case does not arise, since here U does not exist. $\square$

By definition, $\texttt{revswap}_{\mathcal{G}^+}(U)$ is obtained by acting on ladder rows of U independently. By Claim 4.3, it follows that $\texttt{revswap}_{\mathcal{G}^+}(U)$ is also obtained by acting on the T-patches of U independently. Thus ($\Leftarrow$) holds by Claim 4.2.

($\Leftarrow$) Suppose $T \in \texttt{revswap}_{\mathcal{G}^+}(U)$. We show U is in $\mathsf{swap}_{\mathcal{G}}(T)$.

Recall $\mathsf{swap}_{\mathcal{G}}(T)$ is a formal sum, given by independently replacing each snake section in each prescribed way. Trvially, by (Pat.1), each snake section is a union of T-patches. Moreover, if a snake section σ consists of more than one T-patch, then σ is a $\mathtt{body}$ with at least two rows, and hence either $\mathsf{B2}$ or $\mathsf{B3}$. Therefore $\mathsf{mswap}(\sigma)$ has a unique output in this case. Since $\mathsf{swap}_{\mathcal{G}}$ acts trivially on the T-patches of types (Pat.2) and (Pat.3), by Lemma 4.23, it follows that $\mathsf{swap}_{\mathcal{G}}(T)$ is also given by acting independently on the T-patches of T. It remains to show that locally at P, we may swap $T|_P$ to obtain $U|_P$.

To make these local verifications, we use:

Claim 4.4.

(I) Every ladder row of U sits in a distinct T-patch of type (Pat.1).

(II) Every T-patch P of type (Pat.1) not coming from an $\mathsf{H9}$ *snake section, contains a ladder row of U.*

Proof. (I): By Lemma 4.36, every ladder row of U is contained in a T-patch of type (Pat.1). Consider a T-patch P of type (Pat.1); P consists of at most two boxes. If P does not consist of two boxes, clearly at most one ladder row of U can be contained in it. If P consists of two boxes, they are joined by a vertical edge. Since distinct ladder rows do not share a vertical edge, it follows that distinct ladder rows of U are contained in distinct T-patches.

(II): By inspection of the reverse miniswaps. □

If P is type (Pat.2) or (Pat.3), then by Claim 4.4, P does not intersect any ladder row of U. Thus $T|_P = U|_P$. By definition, P is not part of any snake in T. Hence for any $V \in \mathsf{swap}_{\mathcal{G}}(T)$, $V|_P = T|_P = U|_P$ as desired.

Finally suppose P is a patch of type (Pat.1). If it comes from an $\mathsf{H9}$ snake section, then $V \in \mathsf{swap}_{\mathcal{G}}(T)$, $V|_P = T|_P = U|_P$. Otherwise, by Claim 4.4, P contains a unique ladder row in U. We consider each ladder row type in turn and indicate the miniswaps on $T|_P$ that give our desired $V|_P = U|_P$. We refer to the list of outputs described at the beginning of Section 4.9. The following case analysis completes the proof of ($\Rightarrow$).

$\mathsf{L1.1}$: Use $\mathsf{B3}$: Since $\mathtt{label}_U(\mathsf{x}^{\uparrow}) = \mathcal{G}$, we apply at $\mathsf{x}^{\uparrow}$ either $\mathsf{L2.1}$, $\mathsf{L4.1}$, $\mathsf{L4.4}$ or $\mathsf{L4.5}$. In each case $\mathtt{label}_T(\mathsf{x}^{\uparrow}) = \bullet_{\mathcal{G}+}$. Hence x and $\mathsf{x}^{\uparrow}$ are part of the southmost two rows of a snake of T. We claim $\mathsf{x}^{\leftarrow}$ is not part of this snake. Note that by assumption $\mathsf{x}^{\leftarrow}$ is not part of any ladder of U. Thus $\mathtt{label}_U(\mathsf{x}^{\leftarrow}) = \mathtt{label}_T(\mathsf{x}^{\leftarrow})$ and $\mathtt{label}_T(\mathsf{x}^{\leftarrow}) \notin \{\bullet_{\mathcal{G}}, \mathcal{G}\}$. If $\mathsf{x}^{\leftarrow}$ is part of x's snake in T, then $\mathtt{label}_T(\mathsf{x}^{\leftarrow}) = \mathcal{F}^! \prec \mathcal{G}$ and southeast of some $\bullet_{\mathcal{G}}$. Hence in U, $\mathsf{x}^{\leftarrow}$ is southeast of some $\bullet_{\mathcal{G}+}$; this contradicts U's (G.2) in view of U's $\bullet_{\mathcal{G}} \in \mathsf{x}$. Thus x is the unique box of the southmost row of its snake and by Definition-Lemma 4.1, it is the southmost row of a $\mathsf{B3}$ snake section.

L1.2: Use H2, H3, H7, H8, T4.1 or T4.2: Since $\mathtt{label}_T(\mathsf{x}^{\downarrow}) = \mathcal{G}$, $\mathtt{label}_U(\mathsf{x}^{\downarrow}) \in \{\bullet_{\mathcal{G}^+}, \mathcal{G}\}$. Hence by Lemma 4.29, $\mathsf{x}^{\downarrow}$ is not in x's snake in T. Since $\mathtt{label}_T(\mathsf{x}) = \bullet_{\mathcal{G}}$, $\mathsf{x}^{\uparrow}$ is not in x's snake in T. Hence x is in a one-row snake. Since L1.2 applies, $\mathtt{label}_U(\mathsf{x}^{\rightarrow}) \neq \mathcal{G}$, so $\mathtt{label}_T(\mathsf{x}^{\rightarrow}) \neq \mathcal{G}$. Thus x's snake in T is type (ii), (iv) or (vi) in Definition-Lemma 4.1(III). Type (ii) uses H2 or H3; type (iv) uses H7 or H8; type (vi) uses T4.1 or T4.2.

L2.1: Use T1 or T3: By assumption, $\mathtt{label}_U(\mathsf{x}^{\downarrow}) \in \{\bullet_{\mathcal{G}^+}, \mathcal{G}^!\}$. Hence by inspection of the reverse miniswaps, $\mathtt{label}_T(\mathsf{x}^{\downarrow}) = \mathcal{G}$. Since $\mathtt{label}_T(\mathsf{x}) = \bullet_{\mathcal{G}}$, $\mathsf{x}^{\uparrow}$ is not in x's snake. Hence by Definition-Lemma 4.1(I,II), x is in its snake's $\mathtt{tail}$. By T's (G.3), $\mathtt{label}_T(\mathsf{x}^{\rightarrow}) \succ \mathcal{G}$, so $\mathtt{label}_U(\mathsf{x}^{\rightarrow}) \neq \mathcal{F}^!$. Thus either T1 or T3 applies.

L2.2: Use B1 for the first output. By assumption and U's (G.9), U has no $\bullet_{\mathcal{G}^+}$ adjacent to x. Moreover by U's (G.4), no box adjacent to x is in any ladder. Hence T has no $\bullet_{\mathcal{G}}$ adjacent to x. If $\mathcal{F}^! \in \mathsf{x}^{\leftarrow}$ in T, then (possibly marked) $\mathcal{F} \in \mathsf{x}^{\leftarrow}$ in U. If $\mathtt{label}_U(\mathsf{x}^{\leftarrow}) = \mathcal{F}^!$, then we contradict unmarked $\mathcal{G} \in \mathsf{x}$ in U. If $\mathtt{label}_U(\mathsf{x}^{\leftarrow})$ is unmarked, then U has no $\bullet_{\mathcal{G}^+}$ northwest of $\mathsf{x}^{\leftarrow}$. By U's (G.3) and (G.4), U has no $\mathcal{G}$ northwest of $\mathsf{x}^{\leftarrow}$. But since $\mathcal{F}^! \in \mathsf{x}^{\leftarrow}$ in T, T has a $\bullet_{\mathcal{G}^+}$ northwest of $\mathsf{x}^{\leftarrow}$. Hence by [PeYo15b, Lemma A.3], U has a $\bullet_{\mathcal{G}^+}$ or $\mathcal{G}$ northwest of $\mathsf{x}^{\leftarrow}$, a contradiction.

Use H1 or H6 for the second output. Since $\mathsf{x}^{\rightarrow}$ is not in any ladder of U, $\mathtt{label}_U(\mathsf{x}^{\rightarrow}) = \mathtt{label}_T(\mathsf{x}^{\rightarrow})$. Moreover by U's (G.3), $\mathtt{label}_U(\mathsf{x}^{\rightarrow}) \succ \mathcal{G}$, so $\mathtt{label}_T(\mathsf{x}^{\rightarrow}) \succ \mathcal{G}$. If $\mathtt{label}_T(\mathsf{x}^{\rightarrow}) = \mathcal{G}^+$, H6 applies. Otherwise, H1 applies.

L2.3: Use B1 for the first output; applicability is as for the first output of L2.2. Use H2 or H7 for the second output. Since $\mathsf{x}^{\rightarrow}$ is not in any ladder of U, $\mathtt{label}_U(\mathsf{x}^{\rightarrow}) = \mathtt{label}_T(\mathsf{x}^{\rightarrow})$. Moreover by U's (G.3), $\mathtt{label}_U(\mathsf{x}^{\rightarrow}) \succ \mathcal{G}$, so $\mathtt{label}_T(\mathsf{x}^{\rightarrow}) \succ \mathcal{G}$. If $\mathtt{label}_T(\mathsf{x}^{\rightarrow}) = \mathcal{G}^+$, H7 applies. Otherwise, H2 applies.

L3: Use H1. By U's (G.12), $\mathtt{label}_U(\mathsf{x}^{\rightarrow}) \notin \{\mathcal{G}, \mathcal{G}^+\}$. Moreover by U's (G.13) and (G.12), $\mathtt{label}_U(\mathsf{x}^{\rightarrow})$ is not marked, so $\mathtt{label}_U(\mathsf{x}^{\rightarrow}) \succeq \mathcal{G}^+$. Thus $\mathtt{label}_U(\mathsf{x}^{\rightarrow}) \succ \mathcal{G}^+$. Since $\mathsf{x}^{\rightarrow}$ is not in any ladder of U, $\mathtt{label}_T(\mathsf{x}^{\rightarrow}) \succ \mathcal{G}^+$.

L4.1: Use T3. By inspection of the reverse miniswaps, T has $\mathcal{G} \in \mathsf{x}^{\downarrow}$. Hence x's snake in T has at least two rows. Hence x is part of its snake's $\mathtt{tail}$.

L4.2: Use H6.

L4.3: Use H7.

L4.4: If $Z \neq \emptyset$, use T4.2 or T4.3. Otherwise use H5.3, B2, B3 or T2. If $Z \neq \emptyset$, some T4 applies. If it is T4.1, T has $\mathcal{H} \in \underline{\mathsf{x}^{\rightarrow}}$ with $\mathtt{family}(\mathcal{H}) = \mathtt{family}(\mathcal{G}) + 1$ and $N_{\mathcal{H}} = N_{\mathcal{G}}$. Hence U also has $\mathcal{H} \in \underline{\mathsf{x}^{\rightarrow}}$, contradicting Lemma 4.7 for U. If $Z = \emptyset$, there is nothing to check.

L4.5: If $Z \neq \emptyset$, use T5. Otherwise use H5.3, B2, B3 or T2. $\square$

The following proposition characterizes good tableaux in terms of forward swapping.

Proposition 4.7. *A tableau U is $\mathcal{G}$-good if and only if $U \in \mathsf{swap}_{\mathcal{G}^-} \circ \cdots \circ \mathsf{swap}_{1_2} \circ \mathsf{swap}_{1_1}(T^{(1_1)})$ for some bundled tableau T and choice of inner corners of T to initially place $\bullet_{1_1}$'s in.*

Proof. ($\Rightarrow$) Given a $\mathcal{G}$-good tableau U, let $T^{(1_1)}$ be any tableau appearing in $\texttt{revswap}_{1_2} \circ \cdots \circ \texttt{revswap}_{\mathcal{G}^-} \circ \texttt{revswap}_{\mathcal{G}}(U)$. By Proposition 4.5, $T^{(1_1)}$ is a 1_1-good tableau. By $T^{(1_1)}$'s (G.2) and (G.9), the $\bullet_{1_1}$'s of $T^{(1_1)}$ are at inner corners and there is no genetic label northwest of a $\bullet_{1_1}$. Let T be obtained by removing the $\bullet_{1_1}$'s of $T^{(1_1)}$. Then it is clear T is a bundled tableau. Now $U \in \mathsf{swap}_{\mathcal{G}^-} \circ \cdots \circ \mathsf{swap}_{1_2} \circ \mathsf{swap}_{1_1}(T^{(1_1)})$ holds by Propositions 4.4 and 4.6.

($\Leftarrow$) Immediate from Lemma 4.24 and Proposition 4.4. $\square$

4.10 The reversal tree

4.10.1 Walkways

An i**-walkway** W in an $(i+1)_1$-good tableau T is an edge-connected component of the collection of boxes x in T such that:

(W.1) $\bullet_{(i+1)_1} \in \mathsf{x}$; or

(W.2) $i_k \in \mathsf{x}$ and x is not southeast of a $\bullet_{(i+1)_1}$ (equivalently, $i_k \in \mathsf{x}$ is not marked).

Lemma 4.37 (Structure of an i-walkway)**.** *Let W be an i-walkway.*

(I) Each column c of W has at most two boxes; if c has two boxes, the southern box contains $\bullet_{(i+1)_1}$.

(II) W has no 2×2 subsquare.

(III) W is an edge-connected skew shape.

(IV) The $\bullet_{(i+1)_1}$'s are at outer corners of W.

(V) The box and upper edge labels of family i form a $\prec$-interval in the set of genes.

Therefore, each i-walkway looks like:

$$\begin{array}{cccccccccccc} & & & & & & & \star & \star & \star & \star & \\ & & \star & \star & \star & \star & \star & \bullet & & & & \\ \star & \star & \bullet & & & & & & & & & \end{array}.$$

where each $\star$ is a genetic label and the blank box contains either $\bullet_{(i+1)_1}$ or a genetic label.

Proof of Lemma 4.37: (I): By (G.2), at most one box of c comes from (W.1). By (G.4), at most one box of c comes from (W.2). Thus the first assertion of (I) holds. The second assertion holds by (W.2).

(II): Suppose W contains a 2×2 subsquare. Then the two southern boxes of the subsquare contain $\bullet_{(i+1)_1}$'s by (I), contradicting (G.2).

(III): W is edge-connected by definition. In view of (II), it remains to show there are no two boxes y, z of W with y NorthWest of z. Suppose otherwise. By (G.2), at least one of y, z contains a genetic label. If $\bullet_{(i+1)_1} \in \mathsf{y}$ and $i_k \in \mathsf{z}$, we violate (W.2). If $\bullet_{(i+1)_1} \in \mathsf{z}$ and $i_k \in \mathsf{y}$, consider the box b in y's column and z's row. By (G.2), b contains a genetic label. By (G.4), $\mathtt{label}(\mathsf{b}) > i_k$. Since $\bullet_{(i+1)_1} \in \mathsf{z}$, this contradicts (G.9). Finally, if $i_k \in \mathsf{y}$ and $i_h \in \mathsf{z}$, then we contradict (G.12).

(IV): Immediate from (W.2) and (G.2).

(V): By the edge-connectedness of W we know that W occupies consecutive columns. Thus we are done by (G.4)–(G.6). □

4.10.2 Walkway reversal

Let $U \in B^{\alpha}_{\lambda,\mu}$ for some $\alpha \in \{\nu\} \cup \nu^{-}$. Obtain $U^{(0)}$ from U by placing $\bullet_{(\ell(\mu)+1)_1}$ in each box of ν/α. The root of the **reversal tree** $\mathfrak{T}_U$ is $U^{(0)}$. The children $\{U^{(1)}\}$ of $U^{(0)}$ are the tableaux in the formal sum $\mathtt{revswap}_{\ell(\mu)_1^+} \circ \cdots \circ \mathtt{revswap}_{(\ell(\mu)+1)_1}(U^{(0)})$. By Proposition 4.5, each $U^{(1)}$ is $\ell(\mu)_1$-good. We define the children $\{U^{(2)}\}$ of a $U^{(1)}$ by reverse swapping successively through labels of family $\ell(\mu) - 1$, etc. Similarly, all tableaux thus obtained are also good. (A tableau may have a copy of itself as a child; this occurs only if $U^{(0)}$ has no $\bullet_{(\ell(\mu)+1)_1}$'s.) After $\ell(\mu) - i$ steps, a descendant $U' = U^{(\ell(\mu)-i)}$ is an $(i+1)_1$-good tableau.

Lemma 4.38. *Let U' be an $(i+1)_1$-good tableau. If ℓ is a box or edge label that is not in an i-walkway, then ℓ appears in the same location in every $T \in \mathtt{revswap}_{i_1^+} \circ \cdots \circ \mathtt{revswap}_{(i+1)_1}(U')$.*

Proof. The case analysis is as follows:

Case 1: ($\ell \in \mathsf{x}$ is a box label in U'):

Subcase 1.1: ($\mathtt{family}(\ell) \neq i$): During the reversal process $\mathtt{revswap}_{i_1^+} \circ \cdots \circ \mathtt{revswap}_{(i+1)_1}$, the label ℓ is never part of any ladder consisting of $\mathcal{H}$ and $\bullet_{\mathcal{H}^+}$ where $\mathcal{H} \in \{i_1, \ldots, i_{\mu_i}\}$. Thus $\mathtt{revswap}_{\mathcal{H}^+}$ does not move ℓ.

Subcase 1.2: ($\mathtt{family}(\ell) = i$): Since x is not part of an i-walkway, by (W.2) it is southeast of a $\bullet_{(i+1)_1}$ in U'. By inspection of the reverse miniswaps, this remains true for each tableau V appearing in the reversal process $\mathtt{revswap}_{i_1^+} \circ \cdots \circ \mathtt{revswap}_{(i+1)_1}$. The box x is never part of a ladder during this process, for when we apply $\mathtt{revswap}_{\mathcal{H}^+}$, where $\mathcal{H}$ is ℓ's gene, $\bullet_{\mathcal{H}^+}$ is northwest of x and so $\ell^! \in \mathsf{x}$. The case then follows.

Case 2: (ℓ is an edge label in U'): Let x and $\mathsf{x}^{\downarrow}$ be the boxes adjacent to the edge.

Subcase 2.1: (x and $\mathsf{x}^{\downarrow}$ do not contain a label of family i in U'): As above, x and $\mathsf{x}^{\downarrow}$ are not part of a ladder consisting of $\mathcal{H}$ and $\bullet_{\mathcal{H}^+}$, where $\mathcal{H} \in \{i_1, \ldots, i_{\mu_i}\}$. Hence neither is the ℓ in question, and so this ℓ remains fixed throughout the reversal process.

Subcase 2.2: (x or $\mathsf{x}^{\downarrow}$ contains a label $\mathcal{H}$ of family i in U'): By (G.4), at most one of x or $\mathsf{x}^{\downarrow}$ contains such a label. Without loss of generality, suppose it is x (the argument in the other case is the same). Since $\ell \in \underline{\mathsf{x}}$ is not part of an i-walkway, neither is x. By the arguments of Subcase 1.2, x is never part of a ladder, since $\mathcal{H}^! \in \mathsf{x}$. Thus $\underline{\mathsf{x}}$ is unchanged. $\square$

Consider an i-walkway W of U'. By Lemma 4.37(V), the genes of family i in W form an interval; let it be $(w_1, \ldots, w_n)$ in increasing $\prec$-order.

Lemma 4.39 (Characterization of one-row walkway reversals)**.** *Let W be a 1-row i-walkway in an $(i+1)_1$-good tableau U'. Let a and z be the westmost and eastmost boxes of W, respectively. Consider the region $\mathcal{R}$ occupied by W.*

(I) Suppose U' has $\bullet_{(i+1)_1} \in \mathsf{z}$ and no *label of family i in $\overline{\mathsf{z}}$. Then there exists a filling R of $\mathcal{R}$ with $\bullet_{i_1} \in \mathsf{a}$ and $w_1 \notin \underline{\mathsf{a}}$ such that for any $V \in \mathtt{revswap}_{i_1^+} \circ \cdots \circ \mathtt{revswap}_{(i+1)_1}(U')$, $V|_{\mathcal{R}} = R$.*

(II) Suppose U' has $\bullet_{(i+1)_1} \in \mathsf{z}$ and a label of family i in $\overline{\mathsf{z}}$. Then there exists a filling R of $\mathcal{R}$ with $\bullet_{i_1} \in \mathsf{a}$ and either $w_1 \in \underline{\mathsf{a}}$ or $w_1 \in \underline{\mathsf{a}}$ such that for any $V \in \mathtt{revswap}_{i_1^+} \circ \cdots \circ \mathtt{revswap}_{(i+1)_1}(U')$, $V|_{\mathcal{R}} = R$.

(III) Suppose U' has a label of family i in z. Then there exist two fillings R, R' of $\mathcal{R}$ such that

- *(i) R has $w_1 \in \mathsf{a}$;*
- *(ii) R' has $\bullet_{i_1} \in \mathsf{a}$ and either $w_1 \in \underline{\mathsf{a}}$ or $w_1 \in \underline{\mathsf{a}}$;*
- *(iii) R and R' are otherwise identical; and*
- *(iv) for any $V \in \mathtt{revswap}_{i_1^+} \circ \cdots \circ \mathtt{revswap}_{(i+1)_1}(U')$, $V|_{\mathcal{R}} \in \{R, R'\}$.*

Proof. We argue (I)–(III) separately, by induction on the number of boxes of W. The base cases (where W consists of a single box $\mathsf{a} = \mathsf{z}$) are clear by Lemma 4.28 and inspection of the reverse miniswaps. Assume W has at least two boxes and let $\overline{W}$ be W with a removed.

(I): By induction, $\overline{W}$ reverses uniquely to some $\overline{R}$, which has a $\bullet_{w_2} \in \mathsf{a}^{\rightarrow}$ and $w_2 \notin \underline{\mathsf{a}^{\rightarrow}}$. (By a technical modification of the hypotheses, we may apply the inductive hypothesis to this partial walkway here and below.) This extends uniquely by L4.4 or L4.5 (followed by some number of applications of L1.2) to an R with the claimed properties.

(II): The unique reversal $\overline{R}$ of $\overline{W}$ has a $\bullet_{w_2} \in \mathsf{a}^{\rightarrow}$ and $w_2 \in \underline{\mathsf{a}^{\rightarrow}}$. (By (V.2), $w_2 \notin \underline{\mathsf{a}^{\rightarrow}}$.) We obtain the desired unique reversal R by applying L4.2 or L4.3 to $\{\mathsf{a}, \mathsf{a}^{\rightarrow}\}$ in $\overline{R} \cup \{\mathsf{a}\}$.

(III): There are precisely two reversals of $\overline{W}$: $\overline{R}$ and $\overline{R}'$. The former reversal has $w_2 \in \mathsf{a}^{\rightarrow}$, while the latter has $\bullet_{w_2} \in \mathsf{a}^{\rightarrow}$ and $w_2 \in \underline{\mathsf{a}^{\rightarrow}}$. (By (V.2), $w_2 \notin \underline{\mathsf{a}^{\rightarrow}}$.) Applying $\mathsf{L4.2}$ or $\mathsf{L4.3}$ (as appropriate) to $\{\mathsf{a}, \mathsf{a}^{\rightarrow}\}$ in $\overline{R}' \cup \{\mathsf{a}\}$ returns R' as described. Applying $\mathsf{L2.2}$ or $\mathsf{L2.3}$ (as appropriate) to a in $\overline{R} \cup \{\mathsf{a}\}$ returns precisely R and R'. (We apply $\mathsf{L4.2}$ to $\overline{R}' \cup \{\mathsf{a}\}$ exactly when we apply $\mathsf{L2.2}$ to $\overline{R} \cup \{\mathsf{a}\}$.) $\square$

Lemma 4.40 (Characterization of multirow walkway reversals)**.** *Let W be an i-walkway* with at least two rows *in an $(i+1)_1$-good tableau U'. Let a and z be the westmost and eastmost boxes, respectively, in its southmost row. Thus $\bullet_{(i+1)_1} \in \mathsf{z}$. Let $\mathcal{R}$ be the region occupied by W.*

(I) Suppose $\mathsf{a} = \mathsf{z}$. Then there exists a filling R of $\mathcal{R}$ with $w_1 \in \mathsf{a}$ such that for any $V \in \mathtt{revswap}_{i_1^+} \circ \cdots \circ \mathtt{revswap}_{(i+1)_1}(U')$, $V|_{\mathcal{R}} = R$.

(II) Suppose $\mathsf{a} \neq \mathsf{z}$ and $\mathtt{label}_W(\mathsf{z}^{\leftarrow}) = \mathtt{label}_W(\mathsf{z}^{\uparrow})$. Then there exists a filling R of $\mathcal{R}$ with $\bullet_{i_1} \in \mathsf{a}$ and no label of family i on $\underline{\mathsf{a}}$ such that for any $V \in \mathtt{revswap}_{i_1^+} \circ \cdots \circ \mathtt{revswap}_{(i+1)_1}(U')$, $V|_{\mathcal{R}} = R$.

(III) Suppose $\mathsf{a} \neq \mathsf{z}$ and $\mathtt{label}_W(\mathsf{z}^{\leftarrow}) \neq \mathtt{label}_W(\mathsf{z}^{\uparrow})$. Then there exist two fillings R, R' of $\mathcal{R}$ such that

- *(i) R has $w_1 \in \mathsf{a}$;*
- *(ii) R' has $\bullet_{i_1} \in \mathsf{a}$ and either $w_1 \in \underline{\mathsf{a}}$ or $w_1 \in \underline{\mathsf{a}}$;*
- *(iii) R and R' are otherwise identical; and*
- *(iv) for any $V \in \mathtt{revswap}_{i_1^+} \circ \cdots \circ \mathtt{revswap}_{(i+1)_1}(U')$, $V|_{\mathcal{R}} \in \{R, R'\}$.*

Proof. (I): Let $\overline{W}$ be W with the two boxes in the westmost column of W removed. If $\overline{W} = \emptyset$, then $W = \{\bullet_{(i+1)_1} \in \mathsf{z}, w_1 \in \mathsf{z}^{\uparrow}\}$; here we obtain the desired result by use of $\mathsf{L1.1}$ and $\mathsf{L2.1}$. Hence assume $\overline{W} \neq \emptyset$. Clearly,

$$\mathtt{label}_W(\mathsf{z}^{\uparrow\rightarrow}) \in \{w_2, \bullet_{(i+1)_1}\}. \tag{4.9}$$

Depending on whether $\overline{W}$ has multiple rows, by induction or by Lemma 4.39, there are at most two reversals of $\overline{W}$.

Case 1: ($\overline{W}$ has a unique reversal $\overline{R}$): By (4.9) and induction/Lemma 4.39, we have two scenarios possible:

Subcase 1.1: ($\overline{R}$ has $\bullet_{w_2} \in \mathsf{z}^{\uparrow\rightarrow}$ and no labels of family i appear on $\underline{\mathsf{z}^{\uparrow\rightarrow}}$): Here we extend to a unique reversal of W by applying $\mathsf{L4.4}$ or $\mathsf{L4.5}$ at $\mathsf{z}^{\uparrow}$ and $\mathsf{L1.1}$ at z. This results in $w_1 \in \mathsf{z} = \mathsf{a}$.

Subcase 1.2: ($\overline{R}$ has $w_2 \in \mathsf{z}^{\uparrow\rightarrow}$): We extend to a unique reversal of W by applying $\mathsf{L2.1}$ at $\mathsf{z}^{\uparrow}$ and $\mathsf{L1.1}$ at z. This results in $w_1 \in \mathsf{z} = \mathsf{a}$, as desired.

Case 2: ($\overline{W}$ has two reversals $\overline{R}$ and $\overline{R}'$): By (4.9) and induction/Lemma 4.39, $\overline{R}$ and $\overline{R}'$ differ only in $\mathsf{z}^{\uparrow\rightarrow}$: $\overline{R}$ has $w_2 \in \mathsf{z}^{\uparrow\rightarrow}$ whereas $\overline{R}'$ has $\bullet_{w_2} \in \mathsf{z}^{\uparrow\rightarrow}$ and $w_2 \in \underline{\mathsf{z}^{\uparrow\rightarrow}}$. By $\mathsf{L2.1}$ and $\mathsf{L1.1}$ in the $\overline{R}$ case and by $\mathsf{L4.1}$ and $\mathsf{L1.1}$ in the $\overline{R}'$ case, both extend to the same reversal R of W; here R has $w_1 \in \mathsf{z} = \mathsf{a}$, as claimed.

(II): We have some cases.

Case 1: (The southmost row of W has exactly two boxes $\{\mathsf{a} = \mathsf{z}^{\leftarrow}, \mathsf{z}\}$): Let $\overline{W}$ be W with $\{\mathsf{a}, \mathsf{z}, \mathsf{z}^{\uparrow}\}$ removed. If $\overline{W}$ is empty, the result is clear, so we may assume otherwise. Thus (4.9) still holds. Depending on whether $\overline{W}$ has multiple rows or not, either by induction or by Lemma 4.39, it follows there are at most two reversals of $\overline{W}$.

Subcase 1.1: ($\overline{W}$ has a unique reversal $\overline{R}$): By (4.9) and induction/Lemma 4.39, two scenarios are possible:

Subcase 1.1.1: ($\overline{R}$ has $\bullet_{w_2} \in \mathsf{z}^{\uparrow\rightarrow}$ and no label of family i on $\underline{\mathsf{z}^{\uparrow\rightarrow}}$): We extend to a unique reversal R of W by applying L4.5 at $\{\mathsf{z}^{\uparrow}, \mathsf{z}^{\uparrow\rightarrow}\}$ and either L4.4 or L4.5 (as required) at $\{\mathsf{a}, \mathsf{z}\}$; R has $\bullet_{w_1} \in \mathsf{a}$ and no label of family i on $\underline{\mathsf{a}}$.

Subcase 1.1.2: ($w_2 \in \mathsf{z}^{\uparrow\rightarrow}$): We extend to a unique reversal R of W by applying L2.1 at $\mathsf{z}^{\uparrow}$ and either L4.4 or L4.5 (as required) at z. This again results in $\bullet_{w_1} \in \mathsf{a}$ and no label of family i on $\underline{\mathsf{a}}$.

Subcase 1.2: ($\overline{W}$ has two reversals $\overline{R}$ and $\overline{R}'$): By (4.9) and induction/Lemma 4.39, $\overline{R}$ and $\overline{R}'$ differ only in $\mathsf{z}^{\uparrow\rightarrow}$: $\overline{R}$ has a $w_2 \in \mathsf{z}^{\uparrow\rightarrow}$ whereas $\overline{R}'$ has a $\bullet_{w_2} \in \mathsf{z}^{\uparrow\rightarrow}$ and $w_2 \in \underline{\mathsf{z}^{\uparrow\rightarrow}}$. By L2.1 and L4.4 or L4.5 in the $\overline{R}$ case and by L4.1 and L4.4 or L4.5 in the $\overline{R}'$ case, both extend to the same reversal R of W. R has $\bullet_{w_1} \in \mathsf{z} = \mathsf{a}$.

In each of the Subcases above, we are done after applying a sequence of L1.2's at a.

Case 2: (The southmost row of W contains at least three boxes): Let $\overline{W}$ be W with a removed. By induction, $\overline{W}$ has a unique reversal $\overline{R}$ with $\bullet_{w_2}$ in $\mathsf{a}^{\rightarrow}$ and no label of of family i on $\underline{\mathsf{a}^{\rightarrow}}$. Now we uniquely extend $\overline{R}$ to a reversal R of W by applying L4.4 or L4.5 at $\{\mathsf{a}, \mathsf{a}^{\rightarrow}\}$; R has $\bullet_{w_1} \in \mathsf{a}$ and no label of of family i on $\underline{\mathsf{a}}$, and the result follows after applying a sequence of L1.2's at a.

(III): Let $\overline{W}$ be W with the southmost row and $\mathsf{z}^{\uparrow}$ removed. Recall $\texttt{label}_W(\mathsf{z}) = \bullet_{(i+1)_1}$ and suppose W has $w_{q-1} \in \mathsf{z}^{\leftarrow}$ and $w_q \in \mathsf{z}^{\uparrow}$. If $\overline{W}$ is empty, we are done by applying L2.1 at $\mathsf{z}^{\uparrow}$ and L1.1 at z, followed by application of Lemma 4.39(III) to the southmost row. Thus assume $\overline{W}$ is not empty. By induction or Lemma 4.39, there are at most two reversals of $\overline{W}$:

Case 1: ($\overline{W}$ has a unique reversal $\overline{R}$): Observe that exactly one of the following two cases holds.

Subcase 1.1: ($\overline{R}$ has $\bullet_{w_{q+1}} \in \mathsf{z}^{\uparrow\rightarrow}$ and no label of family i on $\underline{\mathsf{z}^{\uparrow\rightarrow}}$): Apply L4.4 at $\mathsf{z}^{\uparrow}$ and L1.1 at z.

Subcase 1.2: ($\overline{R}$ has $w_{q+1} \in \mathsf{z}^{\uparrow\rightarrow}$): Apply L2.1 at $\mathsf{z}^{\uparrow}$ and L1.1 at z.

Case 2: ($\overline{W}$ has two reversals $\overline{R}$ and $\overline{R}'$): By induction/Lemma 4.39, $\overline{R}$ and $\overline{R}'$ differ only in $\mathsf{z}^{\uparrow\rightarrow}$: $\overline{R}$ has $w_{q+1} \in \mathsf{z}^{\uparrow\rightarrow}$ whereas $\overline{R}'$ has a $\bullet_{w_{q+1}} \in \mathsf{z}^{\uparrow\rightarrow}$ and $w_{q+1} \in \underline{\mathsf{z}^{\uparrow\rightarrow}}$. Apply L2.1 and L1.1 in the $\overline{R}$ case. Apply L4.1 and L1.1 in the $\overline{R}'$ case.

In each of the cases above, the indicated reverse miniswaps leave us with the southmost row having $w_1 \in \mathsf{a}$ and $w_q \in \mathsf{z}$. We complete the reversal using Lemma 4.39(III), yielding the desired conclusion. $\square$

Proposition 4.8. *The children of a node U' in $\mathfrak{T}_U$ are obtained by replacing each walkway W with R or R, R' (as defined in Lemmas 4.39 and 4.40) independently in all possible ways.*

Proof. That nothing changes outside the walkways is Lemma 4.38. Independence follows from walkways being edge-disjoint. $\square$

Proposition 4.9. *$\mathfrak{T}_U$ is a tree.*

Proof. Let U' and U'' be distinct i_1-good nodes of $\mathfrak{T}_U$. By induction and Lemmas 4.39 and 4.40, U' and U'' differ in the placement of a label of family strictly larger than i. This label is unaffected by later reverse swaps, so U' and U'' cannot have the same child. $\square$

Proposition 4.10 (Characterization of reversal tree leaves)**.**

(I) Let L be a leaf of $\mathfrak{T}_U$. Then if we ignore the $\bullet_{1_1}$'s, either $L = U$ or $L \in \Lambda^+$ and has shape ν/ρ for some $\rho \in \lambda^+$. Moreover, $[U]\mathtt{slide}_{\rho/\lambda}(L) \neq 0$.

(II) If $M \in \Lambda^+$ has shape ν/ρ and $[U]\mathtt{slide}_{\rho/\lambda}(M) \neq 0$, then M appears as a leaf of $\mathfrak{T}_U$. $\square$

Proof. (I): By Proposition 4.5, L is 1_1-good. By (G.9), there are no labels northwest of a $\bullet_{1_1}$. By (G.2), $\bullet_{1_1}$'s appear in distinct rows and columns. This proves the second sentence. The third sentence then follows from Proposition 4.6.

(II): Immediate from Proposition 4.6. $\square$

4.11 The recurrence coefficients

Given $U \in B^{\alpha}_{\lambda,\mu}$, where $\alpha \in \{\nu\} \cup \nu^-$, let $\mathtt{leaf}(\mathfrak{T}_U)$ be the collection of leaves of the tree $\mathfrak{T}_U$ defined in Section 4.10.

Let W be an i-walkway of shape $\overline{\nu}/\overline{\lambda}$ with $\bullet_{(i+1)_1}$'s in $\overline{\nu}/\overline{\alpha}$. Let S be a reversal of W, as defined by Lemmas 4.39 and 4.40. Let $\mathtt{a}$ be the southwestmost box of W, $\mathtt{b}$ be the northeastmost box of W and $\mathtt{z}$ the eastmost box of W's southmost row. By Lemma 4.37(V), the labels of family i of S form an interval $(w_1, \ldots, w_n)$ with respect to $\prec$. Let $\overline{\alpha}_\star$ denote $\overline{\alpha}$ with its southmost row deleted, and set $\overline{\lambda}_\star := \overline{\lambda} \cap \overline{\alpha}_\star$. Let $\Delta(S, W) := (\#\bullet_{i_1}\text{'s in } S) - (\#\bullet_{(i+1)_1}\text{'s in } W)$. For a tableau T, let $\widetilde{T}$ denote T excluding boxes containing w_1 and outer corners containing $\bullet_{w_1^+}$.

Claim 4.5.

(I.i) If S has $w_1 \notin \mathsf{a}^{\rightarrow}$ and w_1 or w_1 $\in \underline{\mathsf{a}}$, while W has either at least two rows or $w_n \in \mathsf{b}$, then $[W]\texttt{slide}_{\overline{\rho}/\overline{\lambda}}(S) = (-1)^{\Delta(S,W)-1}(1 - \texttt{wt}\,\overline{\alpha}/(\overline{\alpha}_\star \cup \overline{\lambda}))\,\texttt{wt}\,\overline{\alpha}_\star/\overline{\lambda}_\star$.

(I.ii) If S has $w_1 \notin \mathsf{a}^{\rightarrow}$ and w_1 or w_1 $\in \underline{\mathsf{a}}$, while W has exactly one row and $w_n \in \overline{\mathsf{b}}$, then $[W]\texttt{slide}_{\overline{\rho}/\overline{\lambda}}(S) = (-1)^{\Delta(S,W)}\,\texttt{wt}\,\overline{\alpha}/\overline{\lambda}$.

(II) If S has $\bullet_{i_1} \in \mathsf{a}$, $w_1 \in \mathsf{a}^{\rightarrow}$ and $w_1 \notin \underline{\mathsf{a}}$, then $[W]\texttt{slide}_{\overline{\rho}/\overline{\lambda}}(S) = (-1)^{\Delta(S,W)}\,\texttt{wt}\,\overline{\alpha}/\overline{\lambda}$.

(III) If S has $w_1 \in \mathsf{a}$, then $[W]\texttt{slide}_{\overline{\rho}/\overline{\lambda}}(S) = (-1)^{\Delta(S,W)}\,\texttt{wt}\,\overline{\alpha}_\star/\overline{\lambda}_\star$.

Proof. We simultaneously induct on the number of genes of family i in S. (We gloss over some technical reindexing in the arguments below.) We check the base case of one gene directly from the swapping rules of Section 4.7. Now let us assume that S has at least two genes of family i and the claims hold for situations with fewer genes of family i.

In the illustrative examples below that accompany the general analysis, we use for simplicity $1, 2, \dots$ to represent $w_1, w_2, \dots$ respectively. Also, for simplicity, our examples assume a is the southwest corner of $k \times (n-k)$, i.e., $\beta(\mathsf{a}) = 1 - \frac{t_1}{t_2}$.

Case (I.i).1: $(\mathsf{a}^{\rightarrow} \neq \mathsf{z})$: Consider $S = \begin{array}{|c|c|c|c|c|} \hline & & \bullet & 4 & 5 \\ \hline \underset{1}{\bullet} & 2 & 3 & & \\ \hline \end{array}$ and $W = \begin{array}{|c|c|c|c|c|} \hline & & 3 & 4 & 5 \\ \hline 1 & 2 & \bullet & & \\ \hline \end{array}$. Then

$$\mathsf{swap}_1(S) = \left(1 - \tfrac{t_1}{t_2}\right) \begin{array}{|c|c|c|c|c|} \hline & & \bullet & 4 & 5 \\ \hline 1 & 2 & 3 & & \\ \hline \end{array} + \tfrac{t_1}{t_2} \begin{array}{|c|c|c|c|c|} \hline & & \bullet & 4 & 5 \\ \hline 1 & \underset{2}{\bullet} & 3 & & \\ \hline \end{array} := \left(1 - \tfrac{t_1}{t_2}\right)S' + \tfrac{t_1}{t_2}S''.$$

Inductively by (III), $[W]\texttt{slide}(\widetilde{S'}) = \frac{t_4}{t_7}$. Inductively by (I.i), $[W]\texttt{slide}(\widetilde{S''}) = (1 - \frac{t_2}{t_3})\frac{t_4}{t_7}$. Hence $[W]\texttt{slide}(S) = \left(1 - \frac{t_1}{t_2}\right)\frac{t_4}{t_7} + \frac{t_1}{t_2}\left(1 - \frac{t_2}{t_3}\right)\frac{t_4}{t_7} = \left(1 - \frac{t_1}{t_3}\right)\frac{t_4}{t_7}$, as desired. In general,

$$\begin{aligned}[W]\texttt{slide}(S) &= \left(1 - \hat{\beta}(\mathsf{a})\right)(-1)^{\Delta(S,W)-1}\,\texttt{wt}\,\overline{\alpha}_\star/\overline{\lambda}_\star + \hat{\beta}(\mathsf{a})(-1)^{\Delta(S,W)-1}\left(1 - \frac{\texttt{wt}\,\overline{\alpha}/(\overline{\alpha}_\star \cup \overline{\lambda})}{\hat{\beta}(\mathsf{a})}\right)\texttt{wt}\,\overline{\alpha}_\star/\overline{\lambda}_\star \\ &= (-1)^{\Delta(S,W)-1}(1 - \texttt{wt}\,\overline{\alpha}/(\overline{\alpha}_\star \cup \overline{\lambda}))\,\texttt{wt}\,\overline{\alpha}_\star/\overline{\lambda}_\star.\end{aligned}$$

Case (I.i).2: $(\mathsf{a}^{\rightarrow} = \mathsf{z})$: Let $S = \begin{array}{|c|c|c|c|} \hline & & & \bullet \\ \hline & \bullet & 2 & 3 \\ \hline \underset{1}{\bullet} & 2 & & \\ \hline \end{array}$ and $W = \begin{array}{|c|c|c|c|} \hline & & & 3 \\ \hline & 2 & 3 & \bullet \\ \hline 1 & \bullet & & \\ \hline \end{array}$. Then

$$\mathsf{swap}_1(S) = \left(1 - \frac{t_1}{t_2}\right) \begin{array}{|c|c|c|c|} \hline & & & \bullet \\ \hline & \bullet & 2 & 3 \\ \hline 1 & 2 & & \\ \hline \end{array} := \left(1 - \frac{t_1}{t_2}\right)S'.$$

By (III), $[W]\texttt{slide}(\widetilde{S'}) = \frac{t_3}{t_5}\frac{t_6}{t_7}$. Hence $[W]\texttt{slide}(S) = (1 - \frac{t_1}{t_2})\frac{t_3}{t_5}\frac{t_6}{t_7}$, as desired. In general,

$$[W]\texttt{slide}(S) = \left(1 - \hat{\beta}(\mathsf{a})\right)(-1)^{\Delta(S,W)-1}\texttt{wt}\,\overline{\alpha}_\star/\overline{\lambda}_\star = (-1)^{\Delta(S,W)-1}(1 - \texttt{wt}\,\overline{\alpha}/(\overline{\alpha}_\star \cup \overline{\lambda}))\,\texttt{wt}\,\overline{\alpha}_\star/\overline{\lambda}_\star.$$

Case (I.ii): Let $S = \begin{ytableau} \bullet_1 & 2 & 3 \end{ytableau}$ and $W = \begin{ytableau} 1 & 2 & \overset{3}{\bullet} \end{ytableau}$. Then $\mathsf{swap}_1(S) = (1-\frac{t_1}{t_2})\begin{ytableau} 1 & 2 & 3 \end{ytableau} + \frac{t_1}{t_2}\begin{ytableau} 1 & \bullet_2 & 3 \end{ytableau} := (1-\frac{t_1}{t_2})S' + \frac{t_1}{t_2}S''$. By Lemma 4.39, $[W]\mathtt{slide}(\widetilde{S'}) = 0$. By (I.ii), $[W]\mathtt{slide}(\widetilde{S''}) = \frac{t_2}{t_3}$. Hence $[W]\mathtt{slide}(S) = \frac{t_1}{t_2}\frac{t_2}{t_3} = \frac{t_1}{t_3}$, as desired. In general,

$$[W]\mathtt{slide}(S) = \hat{\beta}(\mathsf{a})(-1)^{\Delta(S,W)}\frac{1}{\hat{\beta}(\mathsf{a})}\,\mathtt{wt}\,\overline{\alpha}/\overline{\lambda} = (-1)^{\Delta(S,W)}\,\mathtt{wt}\,\overline{\alpha}/\overline{\lambda}.$$

Case (II).1: ($\mathsf{a}^{\rightarrow} \neq \mathsf{z}$): Let $S = \begin{ytableau} \none & \none & \none & \bullet & 4 & 5 \\ \bullet & 1 & 2 & 3 \end{ytableau}$ and $W = \begin{ytableau} \none & \none & \none & 3 & 4 & 5 \\ 1 & 2 & 3 & \bullet \end{ytableau}$. Then $\mathsf{swap}_1(S) = \frac{t_1}{t_2}\begin{ytableau} \none & \none & \none & \bullet & 4 & 5 \\ 1 & \bullet & 2 & 3 \end{ytableau} := \frac{t_1}{t_2}S'$. By (II), $[W]\mathtt{slide}(\widetilde{S'}) = \frac{t_2}{t_4}\frac{t_5}{t_8}$. Hence $[W]\mathtt{slide}(S) = \frac{t_1}{t_2} \cdot \frac{t_2}{t_4}\frac{t_5}{t_8} = \frac{t_1}{t_4}\frac{t_5}{t_8}$, as desired. In general,

$$[W]\mathtt{slide}(S) = \hat{\beta}(\mathsf{a}) \cdot (-1)^{\Delta(S,W)}\frac{1}{\hat{\beta}(\mathsf{a})}\,\mathtt{wt}\,\overline{\alpha}/\overline{\lambda} = (-1)^{\Delta(S,W)}\,\mathtt{wt}\,\overline{\alpha}/\overline{\lambda}.$$

Case (II).2: ($\mathsf{a}^{\rightarrow} = \mathsf{z}$ and the northmost $w_1 \in S$ is *not* immediately below $\bullet_{i_1}$): Let $S = \begin{ytableau} \none & \none & \none & \none & \bullet & 4 & 5 \\ \none & \bullet & 1 & 2 & 3 \\ \bullet & 1 \end{ytableau}$ and $W = \begin{ytableau} \none & \none & \none & \none & 3 & 4 & 5 \\ \none & 1 & 2 & 3 & \bullet \\ 1 & \bullet \end{ytableau}$. Then $\mathsf{swap}_1(S) = \frac{t_1}{t_2}\frac{t_3}{t_4}\begin{ytableau} \none & \none & \none & \none & \bullet & 4 & 5 \\ \none & 1 & \bullet & 2 & 3 \\ 1 & \bullet \end{ytableau} := \frac{t_1}{t_2}\frac{t_3}{t_4}S'$. By (II), $[W]\mathtt{slide}(\widetilde{S'}) = -\frac{t_4}{t_6}\frac{t_7}{t_{10}}$. Hence $[W]\mathtt{slide}(S) = \frac{t_1}{t_2}\frac{t_3}{t_4} \cdot \left(-\frac{t_4}{t_6}\frac{t_7}{t_{10}}\right) = -\frac{t_1}{t_2}\frac{t_3}{t_6}\frac{t_7}{t_{10}}$, as desired. In general,

$$[W]\mathtt{slide}(S) = \prod_{\mathsf{x}:\mathtt{label}_W(\mathsf{x})=1} \hat{\beta}(\mathsf{x}) \cdot (-1)^{\Delta(S,W)} \prod_{\mathsf{y}:\mathtt{label}_W(\mathsf{y})>1} \hat{\beta}(\mathsf{y}) = (-1)^{\Delta(S,W)}\,\mathtt{wt}\,\overline{\alpha}/\overline{\lambda}.$$

Case (II).3: ($\mathsf{a}^{\rightarrow} = \mathsf{z}$ and the northmost $w_1 \in S$ is immediately below $\bullet_{i_1}$): Let $S = \begin{ytableau} \none & \none & \none & \none & \bullet & 5 & 6 \\ \none & \bullet & 2 & 3 & 4 \\ \bullet & 1 \end{ytableau}$ and $W = \begin{ytableau} \none & \none & \none & \none & 4 & 5 & 6 \\ \none & 1 & 2 & 3 & \bullet \\ 1 & \bullet \end{ytableau}$. Then

$$\mathsf{swap}_1(S) = -\frac{t_1}{t_2}\frac{t_3}{t_4}\begin{ytableau} \none & \none & \none & \none & \bullet & 5 & 6 \\ \none & 1 & 2 & 3 & 4 \\ 1 & \bullet \end{ytableau} + \frac{t_1}{t_2}\frac{t_3}{t_4}\begin{ytableau} \none & \none & \none & \none & \bullet & 5 & 6 \\ \none & 1 & \bullet_2 & 3 & 4 \\ 1 & \bullet \end{ytableau} := -\frac{t_1}{t_2}\frac{t_3}{t_4}S' + \frac{t_1}{t_2}\frac{t_3}{t_4}S''.$$

By (III), $[W]\mathtt{slide}(\widetilde{S'}) = \frac{t_7}{t_{10}}$. By (I.i), $[W]\mathtt{slide}(\widetilde{S''}) = \left(1 - \frac{t_4}{t_6}\right)\frac{t_7}{t_{10}}$. Hence $[W]\mathtt{slide}(S) = -\frac{t_1}{t_2}\frac{t_3}{t_4}\frac{t_7}{t_{10}} + \frac{t_1}{t_2}\frac{t_3}{t_4}\left(1 - \frac{t_4}{t_6}\right)\frac{t_7}{t_{10}} = -\frac{t_1}{t_2}\frac{t_3}{t_6}\frac{t_7}{t_{10}}$, as desired. Depending whether (I.i) or (I.ii) applies inductively, we have in general respectively

$$\begin{aligned}[W]\mathtt{slide}(S) &= -\prod_{\mathsf{x}:\mathtt{label}_W(\mathsf{x})=1} \hat{\beta}(\mathsf{x}) \cdot Y(-1)^{\Delta(S,W)-1} + \prod_{\mathsf{x}:\mathtt{label}_W(\mathsf{x})=1} \hat{\beta}(\mathsf{x}) \cdot (1-Z)Y(-1)^{\Delta(S,W)-1} \\ &= (-1)^{\Delta(S,W)}YZ \prod_{\mathsf{x}:\mathtt{label}_W(\mathsf{x})=1} \hat{\beta}(\mathsf{x}) = (-1)^{\Delta(S,W)}\,\mathtt{wt}\,\overline{\alpha}/\overline{\lambda}\end{aligned}$$

or

$$
\begin{aligned}
[W]\texttt{slide}(S) &= \prod_{\mathsf{x}:\texttt{label}_W(\mathsf{x})=1} \hat{\beta}(\mathsf{x}) \cdot Z(-1)^{\Delta(S,W)} \\
&= (-1)^{\Delta(S,W)} Z \prod_{\mathsf{x}:\texttt{label}_W(\mathsf{x})=1} \hat{\beta}(\mathsf{x}) = (-1)^{\Delta(S,W)}\, \texttt{wt}\, \overline{\alpha}/\overline{\lambda},
\end{aligned}
$$

where Y is the weight of the boxes of W that contain genetic labels and are North of all w_1's and Z is the weight of the boxes of W that contain genetic labels greater than w_1 and are not North of all w_1's.

Case (III).1: ($\mathsf{a} \neq \mathsf{z}$): Let $S = \begin{array}{ccccccc} & & & & \bullet & 5 & 6 \\ & \bullet & 2 & 3 & 4 & & \\ 1 & 2 & & & & & \end{array}$ and $W = \begin{array}{ccccccc} & & & & 4 & 5 & 6 \\ & 2 & 3 & 4 & \bullet & & \\ 1 & \bullet & & & & & \end{array}$. Then $\mathsf{swap}_1(S) = \begin{array}{ccccccc} & & & & \bullet & 5 & 6 \\ & \bullet & 2 & 3 & 4 & & \\ 1 & 2 & & & & & \end{array} := S'$. By (III), $[W]\texttt{slide}(\widetilde{S'}) = \frac{t_3}{t_6}\frac{t_7}{t_{10}}$. Hence $[W]\texttt{slide}(S) = \frac{t_3}{t_6}\frac{t_7}{t_{10}}$, as desired. In general, $[W]\texttt{slide}(S) = (-1)^{\Delta(S,W)}\, \texttt{wt}\, \overline{\alpha}_\star/\overline{\lambda}_\star$.

Case (III).2: ($\mathsf{a} = \mathsf{z}$ and the northmost $w_1 \in S$ is *not* immediately below $\bullet_{i_1}$): Let $S = \begin{array}{cccccc} & & & \bullet & 4 & 5 \\ \bullet & 1 & 2 & 3 & & \\ 1 & & & & & \end{array}$ and $W = \begin{array}{cccccc} & & & 3 & 4 & 5 \\ 1 & 2 & 3 & \bullet & & \\ \bullet & & & & & \end{array}$. Then $\mathsf{swap}_1(S) = -\frac{t_2}{t_3} \begin{array}{cccccc} & & & \bullet & 4 & 5 \\ 1 & \bullet & 2 & 3 & & \\ \bullet & & & & & \end{array} := -\frac{t_2}{t_3}S'$. By (II), $[W]\texttt{slide}(\tilde{S'}) = -\frac{t_3}{t_5}\frac{t_6}{t_9}$. Hence $[W]\texttt{slide}(S) = -\frac{t_2}{t_3} \cdot \left(-\frac{t_3}{t_5}\frac{t_6}{t_9}\right) = \frac{t_2}{t_5}\frac{t_6}{t_9}$, as desired. In general,

$$
[W]\texttt{slide}(S) = -\prod_{\mathsf{x}:\texttt{label}_W(\mathsf{x})=1} \hat{\beta}(\mathsf{x}) \cdot (-1)^{\Delta(S,W)-1} \prod_{\mathsf{y}:\texttt{label}_W(\mathsf{y})>1} \hat{\beta}(\mathsf{y}) = (-1)^{\Delta(S,W)}\, \texttt{wt}\, \overline{\alpha}_\star/\overline{\lambda}_\star.
$$

Case (III).3: ($\mathsf{a} = \mathsf{z}$ and the northmost $w_1 \in S$ is immediately below $\bullet_{i_1}$): Let $S = \begin{array}{cccccc} & & & \bullet & 4 & 5 \\ \bullet & 2 & 3 & 4 & & \\ 1 & & & & & \end{array}$ and $W = \begin{array}{cccccc} & & & 4 & 5 & \bullet \\ 1 & 2 & 3 & \bullet & & \\ \bullet & & & & & \end{array}$. Then $\mathsf{swap}_1(S) = \frac{t_2}{t_3} \begin{array}{cccccc} & & & \bullet & 4 & 5 \\ 1 & 2 & 3 & 4 & & \\ \bullet & & & & & \end{array} - \frac{t_2}{t_3} \begin{array}{cccccc} & & & \bullet & 4 & 5 \\ 1 & \bullet 2 & 3 & 4 & & \\ \bullet & & & & & \end{array} := \frac{t_2}{t_3}S' - \frac{t_2}{t_3}S''$. By (III), $[W]\texttt{slide}(\tilde{S'}) = -\frac{t_6}{t_8}$. By (I.i), $[W]\texttt{slide}(\tilde{S''}) = -(1-\frac{t_3}{t_5})\frac{t_6}{t_8}$. Hence $[W]\texttt{slide}(S) = \frac{t_2}{t_3} \cdot \left(-\frac{t_6}{t_8}\right) - \frac{t_2}{t_3} \cdot \left(-(1-\frac{t_3}{t_5})\frac{t_6}{t_8}\right) = -\frac{t_2}{t_5}\frac{t_6}{t_8}$, as desired. Depending whether (I.i) or (I.ii) applies inductively, we have in general respectively

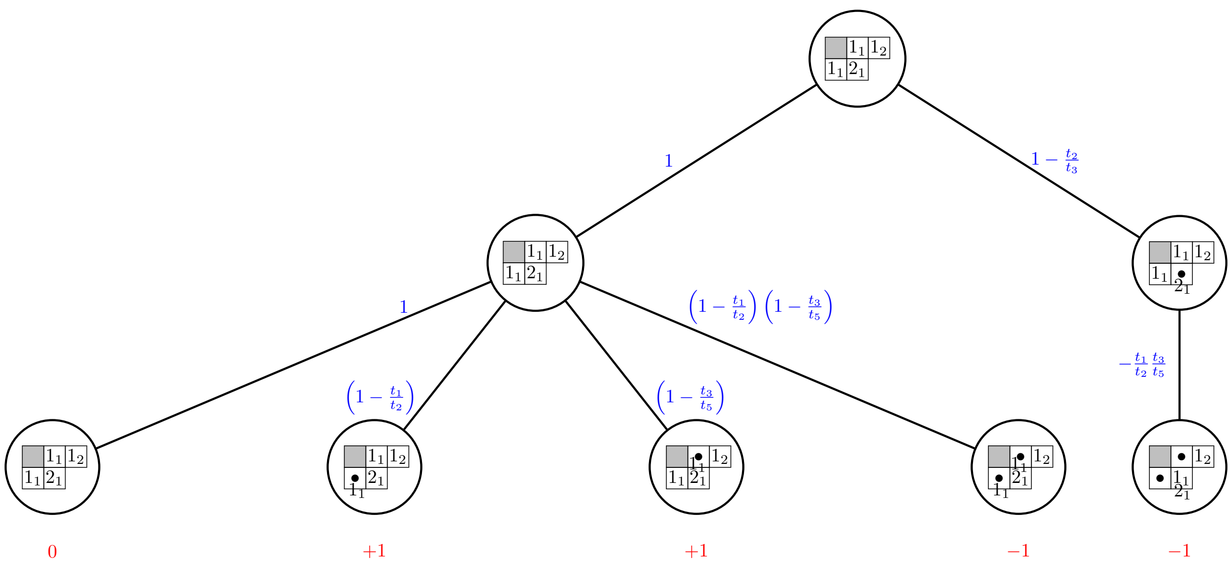
$$
\begin{aligned}
[W]\texttt{slide}(S) &= \prod_{\mathsf{x}:\texttt{label}_W(\mathsf{x})=1} \hat{\beta}(\mathsf{x}) \cdot (-1)^{\Delta(S,W)} Y - \prod_{\mathsf{x}:\texttt{label}_W(\mathsf{x})=1} \hat{\beta}(\mathsf{x}) \cdot (-1)^{\Delta(S,W)} (1-Z) Y \\
&= (-1)^{\Delta(S,W)} YZ \prod_{\mathsf{x}:\texttt{label}_W(\mathsf{x})=1} \hat{\beta}(\mathsf{x}) = (-1)^{\Delta(S,W)}\, \texttt{wt}\, \overline{\alpha}_\star/\overline{\lambda}_\star
\end{aligned}
$$

or

$$[W]\mathtt{slide}(S) = -\prod_{\mathsf{x}:\mathtt{label}_W(\mathsf{x})=1} \hat{\beta}(\mathsf{x}) \cdot (-1)^{\Delta(S,W)-1} Z$$
$$= (-1)^{\Delta(S,W)} Z \prod_{\mathsf{x}:\mathtt{label}_W(\mathsf{x})=1} \hat{\beta}(\mathsf{x}) = (-1)^{\Delta(S,W)}\, \mathtt{wt}\, \overline{\alpha}_\star/\overline{\lambda}_\star,$$

where Y is the weight of the boxes of W containing genetic labels and are North of all w_1's and Z is the weight of the boxes of W containing genetic labels greater than w_1 and are not North of all w_1's. $\square$

Example 4.20. Let $\lambda = (1)$, $\nu = (3,2)$ and $\mu = (2,1)$. Consider $U = \begin{array}{|c|c|c|}\hline & 1_1 & 1_2 \\ \hline 1_1 & 2_1 \\ \cline{1-2}\end{array} \in \Lambda$. Below, we give the reversal tree $\mathfrak{T}_U$.



Each edge is labeled (in blue) by $[U']\mathtt{swap}_{i_{\mu_i}} \circ \cdots \circ \mathtt{swap}_{i_1}(V')$ where U' is the parent of the i_1-good tableau V'. This label agrees with the application of Claim 4.5 to each i-walkway of V'. Below each leaf (in red) is the coefficient in Λ^+ (i.e., $(-1)^{|\rho/\lambda|+1}$ if nonzero). $\diamondsuit$

Lemma 4.41. *Suppose U' is an $(i+1)_1$-good node of $\mathfrak{T}_U$. Let Γ be the boxes of U containing labels of family i. Then $\sum_{V'} (-1)^{1\,+\,\#\bullet\text{'s in } V'} [U']\mathtt{swap}_{i_{\mu_i}} \circ \cdots \circ \mathtt{swap}_{i_1}(V') = (-1)^{1\,+\,\#\bullet\text{'s in } U'}\, \mathtt{wt}\,\Gamma$, where the sum is over all children V' of U' in $\mathfrak{T}_U$.*

Proof. Consider boxes of U' containing unmarked labels of family i or $\bullet_{(i+1)_1}$. By (W.1) and (W.2), these boxes decompose into an edge-disjoint union of i-walkways $W_1, W_2, \ldots, W_t$. Let Γ_j be the boxes of W_j in U containing labels of family i; thus $\Gamma = \Gamma_1 \sqcup \Gamma_2 \sqcup \cdots \sqcup \Gamma_t$. Let R_j and R'_j (if it exists) be the reversal(s) defined by Lemmas 4.39 and 4.40 with respect to the walkway W_j. As computed by Claim 4.5, let a_j be the coefficient of W_j obtained by sliding R_j. Let b_j be the coefficient of W_j obtained by sliding R'_j if it exists; set $b_j := 0$ if R'_j does not exist. We now assert that

$$(-1)^{\#\bullet\text{'s in } R_j} a_j + (-1)^{\#\bullet\text{'s in } R'_j} b_j = (-1)^{\#\bullet\text{'s in } W_j}\, \mathtt{wt}\, \Gamma_j. \tag{4.10}$$

Suppose there is a unique reversal (i.e., $b_j = 0$). This occurs under Lemma 4.39(I,II) and Lemma 4.40(I,II). In these four cases, R_j is respectively the S from (II), (I.ii), (III) and (II) of Claim 4.5. Hence in each of these cases, (4.10) is immediate from the apposite case of Claim 4.5 (note that for Lemma 4.40(I), the southmost row of W_j has a single box and $\overline{\alpha}_\star/\overline{\lambda}_\star = \overline{\alpha}/\overline{\lambda} = \Gamma_j$). Suppose there are two reversals. This occurs under Lemma 4.39(III) and Lemma 4.40(III), which show that R_j is the S from Claim 4.5(III) and R'_j is the S from Claim 4.5(I.i). Hence (4.10) also follows in these cases, by adding the two apposite coefficients given by Claim 4.5.

Since by Proposition 4.8 all V' are obtained by independent replacements of W_j by R_j and R'_j (if it exists),

$$\begin{aligned}
\sum_{V'} (-1)^{1\,+\,\#\bullet\text{'s in } V'} [U']\mathsf{swap}_{i_{\mu_i}} \circ \cdots \circ \mathsf{swap}_{i_1}(V') &= -\prod_{j=1}^{t} \left((-1)^{\#\bullet\text{'s in } R_j} a_j + (-1)^{\#\bullet\text{'s in } R'_j} b_j \right) \\
&= -\prod_{j=1}^{t} (-1)^{\#\bullet\text{'s in } W_j}\, \mathtt{wt}\, \Gamma_j \\
&= (-1)^{1\,+\,\#\bullet\text{'s in } U'}\, \mathtt{wt}\, \Gamma.
\end{aligned}$$

□

Lemma 4.42. *Let U' be an $(i{+}1)_1$-good node of $\mathfrak{T}_U$. Let $\Gamma^{(i)}$ be the set of boxes $\{\mathsf{x} \in \alpha/\lambda : \mathtt{family}(\mathtt{label}_U(\mathsf{x})) \leq i\}$. Then*

$$\sum_{T} (-1)^{1+\#\bullet\text{'s in } T} [U']\mathsf{swap}_{i_{\mu_i}} \circ \mathsf{swap}_{i^-_{\mu_i}} \circ \cdots \circ \mathsf{swap}_{1^+_1} \circ \mathsf{swap}_{1_1}(T) = \mathtt{wt}(\Gamma^{(i)})(-1)^{1+\#\bullet\text{'s in } U'},$$

where the sum is over all $T \in \mathtt{leaf}(\mathfrak{T}_U)$ that are descendants of U'.

Proof. We induct on $i \geq 0$. In the base case $i = 0$, $U' = T$ for $T \in \mathtt{leaf}(\mathfrak{T}_U)$ and the lefthand side equals $(-1)^{1+\#\bullet\text{'s in } T}$. This equals the righthand side since $\Gamma^{(0)} = \emptyset$ so $\mathtt{wt}\,\Gamma^{(0)} = 1$.

Now let $i > 0$. We have $\sum_T (-1)^{1+\#\bullet\text{'s in }T}[U']\mathsf{swap}_{i_{\mu_i}} \circ \mathsf{swap}_{i^-_{\mu_i}} \circ \cdots \circ \mathsf{swap}_{1^+_1} \circ \mathsf{swap}_{1_1}(T)$

$$= \sum_{V' \text{ a child of } U'} \sum_{T\in\mathtt{leaf}(\mathfrak{T}_{U'}) \text{ below } V'} (-1)^{1\,+\,\#\bullet\text{'s in }T}[U']\mathsf{swap}_{i_{\mu_i}} \circ \cdots \circ \mathsf{swap}_{i_1} \circ$$

$$\mathsf{swap}_{(i-1)_{\mu_{i-1}}} \circ \cdots \circ \mathsf{swap}_{1_1}(T)$$

$$= \sum_{V' \text{ a child of } U'} \sum_{T\in\mathtt{leaf}(\mathfrak{T}_{U'}) \text{ below } V'} (-1)^{1\,+\,\#\bullet\text{'s in }T}[U']\mathsf{swap}_{i_{\mu_i}} \circ \cdots \circ \mathsf{swap}_{i_1}(V')\cdot$$

$$[V']\mathsf{swap}_{(i-1)_{\mu_{i-1}}} \circ \cdots \circ \mathsf{swap}_{1_1}(T).$$

The previous equality is since $\mathfrak{T}_U$ is a tree (Proposition 4.9) and V' is the unique child of U' that is an ancestor of T. The previous summation equals

$$\sum_{V' \text{ a child of } U'} [U']\mathtt{swap}_{i_{\mu_i}} \circ \cdots \circ \mathtt{swap}_{i_1}(V') \sum_{T\in\mathtt{leaf}(\mathfrak{T}_{U'}) \text{ below } V'} (-1)^{1\,+\,\#\bullet\text{'s in }T}[V']\mathtt{swap}_{(i-1)_{\mu_{i-1}}} \circ \cdots \circ \mathtt{swap}_{1_1}(T)$$

$$= \sum_{V' \text{ a child of } U'} [U']\mathtt{swap}_{i_{\mu_i}} \circ \cdots \circ \mathtt{swap}_{i_1}(V') \cdot \mathtt{wt}(\Gamma^{(i-1)})(-1)^{1\,+\,\#\bullet\text{'s in }V'} \qquad \text{(by induction)}$$

$$= \mathtt{wt}(\Gamma^{(i-1)}) \sum_{V' \text{ a child of } U'} (-1)^{1\,+\,\#\bullet\text{'s in }V'}[U']\mathtt{swap}_{i_{\mu_i}} \circ \cdots \circ \mathtt{swap}_{i_1}(V')$$

$$= \mathtt{wt}(\Gamma^{(i-1)}) \cdot (-1)^{1\,+\,\#\bullet\text{'s in }U'}\mathtt{wt}(\Gamma) \qquad \text{(by Lemma 4.41)}$$

$$= (-1)^{1\,+\,\#\bullet\text{'s in }U'}\mathtt{wt}(\Gamma^{(i)}),$$

since by definition $\mathtt{wt}(\Gamma^{(i)}) = \mathtt{wt}(\Gamma) \cdot \mathtt{wt}(\Gamma^{(i-1)})$. ∎

Proposition 4.11. *For $U \in B^{\alpha}_{\lambda,\mu}$,*

$$\sum_{T\in\mathtt{leaf}(\mathfrak{T}_U)} (-1)^{|\rho(T)/\lambda|+1}[U]\mathtt{slide}_{\rho(T)/\lambda}(T) = \mathtt{wt}(\alpha/\lambda)(-1)^{|\nu/\alpha|+1} \tag{4.11}$$

where $\rho(T) \in \{\lambda\} \cup \lambda^+$ is the "inner shape" of T, i.e., T has shape $\nu/\rho(T)$.

Proof. Take $U' = U$ in Lemma 4.42. ∎

Now assume $U \in B^{\nu}_{\lambda,\mu}$. The root of $\mathfrak{T}_U$ contains no $\bullet_{(\ell(\mu)+1)_1}$'s. One leaf of $\mathfrak{T}_U$ is U itself. This is the unique leaf not in Λ^+. Let $\mathtt{leaf}^*(\mathfrak{T}_U)$ be the collection of all other leaves.

Proposition 4.12. *For $U \in B^{\nu}_{\lambda,\mu}$,*

$$\sum_{T\in\mathtt{leaf}^*(\mathfrak{T}_U)} (-1)^{|\rho(T)/\lambda|+1}[U]\mathtt{slide}_{\rho(T)/\lambda}(T) = 1 - \mathtt{wt}(\nu/\lambda) \tag{4.12}$$

where $\rho(T) \in \lambda^+$ is the "inner shape" of T, i.e., T has shape $\nu/\rho(T)$.

Proof. This is immediate from Proposition 4.11, since $\nu = \alpha$ and the contribution from the excluded leaf is 1. $\square$

Example 4.21. In Example 4.20, summing the weights below the left child of U gives $1 - \frac{t_1}{t_2}\frac{t_3}{t_5}$, in agreement with Lemma 4.41. Proposition 4.12 asserts in this case that

$$1 - \mathtt{wt}(\nu/\lambda) = 1 - \frac{t_1}{t_5} = \left(1 - \frac{t_1}{t_2}\right) + \left(1 - \frac{t_3}{t_5}\right) - \left(1 - \frac{t_1}{t_2}\right)\left(1 - \frac{t_3}{t_5}\right) + \frac{t_1}{t_2}\frac{t_3}{t_5} \cdot \left(1 - \frac{t_2}{t_3}\right),$$

as the reader may verify. $\diamondsuit$

Recall $\Lambda^+ = \sum_{\rho \in \lambda^+} (-1)^{|\rho/\lambda|+1} \sum_{T \in B^{\nu}_{\rho,\mu}} T$. For $T \in B^{\nu}_{\rho,\mu}$, write $T^{(1_1)}$ (cf. Section 4.7.2) for T with $\bullet_{1_1}$ in each box of ρ/λ.

Now set

$$P_{\mathcal{G}} := \sum_{\rho \in \lambda^+} (-1)^{|\rho/\lambda|-1} \sum_{T \in B^{\nu}_{\rho,\mu}} \mathsf{swap}_{\mathcal{G}^-} \circ \mathsf{swap}_{(\mathcal{G}^-)^-} \circ \cdots \circ \mathsf{swap}_{1_1}(T^{(1_1)}). \tag{4.13}$$

In particular, P_{1_1} is Λ^+ where each T is replaced by $T^{(1_1)}$. By Lemma 4.24 and Proposition 4.4, each $P_{\mathcal{G}}$ is a formal sum of $\mathcal{G}$-good tableaux.

The main conclusion of this section is

Proposition 4.13. *$P_{\mathcal{G}^+_{\max}}$ with all $\bullet_{\mathcal{G}^+_{\max}}$'s removed equals $\Lambda + \Lambda^-$.*

Proof. By Corollary 4.2 each tableau appearing in $P_{\mathcal{G}^+_{\max}}$ (with $\bullet_{\mathcal{G}^+_{\max}}$'s removed) is a tableau in $\Lambda + \Lambda^-$. On the other hand, given any U appearing in $\Lambda + \Lambda^-$, we constructed the tree $\mathfrak{T}_U$ in Section 4.10. By Proposition 4.10, the leaves of $\mathfrak{T}_U$ are exactly those tableaux $T \in \Lambda^+$ such that $U \in \mathtt{slide}_{\rho/\lambda}(T)$. It remains to show that $[U]P_{\mathcal{G}^+_{\max}} = 1 - \mathtt{wt}(\nu/\lambda)$ if $U \in \Lambda$ and $[U]P_{\mathcal{G}^+_{\max}} = (-1)^{|\nu/\delta|+1}\mathtt{wt}(\delta/\lambda)$ if $U \in \Lambda^-$ and the shape of U is δ/λ. These are precisely the statements of Propositions 4.12 and 4.11, respectively. $\square$

4.12 Weight preservation

4.12.1 Fine tableaux and their weights

A tableau is **fine** if it is good or can be obtained from a good tableau by swapping some subset of its snakes, i.e. it appears in the formal sum of tableaux resulting from this partial swap.

Let T be fine and fix $\mathsf{x} \in T$. Suppose $\ell \in \underline{\mathsf{x}}$. Define $\mathtt{edgefactor}(\ell)$ as in Section 4.1.4; see (4.1). The **edge weight** $\mathtt{edgewt}(T) := \prod_{\ell} \mathtt{edgefactor}(\ell)$, where the product is over all (non-virtual) edge labels of T.

Suppose T is obtained by swapping some of the snakes of the good tableau S and U is obtained from T by swapping the remaining snakes. We define the positions in T of a **virtual label** $(\mathcal{H})$ as follows. Consider a box x in column c. If c intersects a snake in S that has been swapped in T, and that snake is not the upper snake described in Lemma 4.17(III), then $(\mathcal{H}) \in \underline{\mathsf{x}}$ (in T) if and only if $(\mathcal{H}) \in \underline{\mathsf{x}}$ (in U). Otherwise, $(\mathcal{H}) \in \underline{\mathsf{x}}$ (in T) if and only if $(\mathcal{H}) \in \underline{\mathsf{x}}$ (in S). Observe that if T is indeed good, this definition is clearly consistent with the definition of virtual labels in a good tableau.

Suppose $(\mathcal{H}) \in \underline{\mathsf{x}}$. If $\mathtt{label}_T(\mathsf{x})$ is marked and each $\mathcal{F} \in \underline{\mathsf{x}}$ with $\mathcal{F} \prec \mathcal{G}$ is marked, then

$$\mathtt{virtualfactor}_{\underline{\mathsf{x}} \in T}((\mathcal{H})) := -\mathtt{edgefactor}_{\underline{\mathsf{x}} \in T}(\mathcal{H}) = \frac{t_{\mathsf{Man}(\mathsf{x})}}{t_{r+N_{\mathcal{H}}+1-\mathtt{family}(\mathcal{H})+\mathsf{Man}(\mathsf{x})}} - 1. \tag{4.14}$$

Otherwise

$$\mathtt{virtualfactor}_{\underline{\mathsf{x}} \in T}((\mathcal{H})) := 1 - \mathtt{edgefactor}_{\underline{\mathsf{x}} \in T}(\mathcal{H}) = \frac{t_{\mathsf{Man}(\mathsf{x})}}{t_{r+N_{\mathcal{H}}+1-\mathtt{family}(\mathcal{H})+\mathsf{Man}(\mathsf{x})}}. \tag{4.15}$$

The **virtual weight** $\mathtt{virtualwt}(T)$ is $\prod_{(\ell)} \mathtt{virtualfactor}((\ell))$, where the product is over all instances of virtual labels.

Call $\mathsf{x} \in T$ **productive** if any of the following hold:

(P.1) $\mathtt{label}_T(\mathsf{x}) < \mathtt{label}_T(\mathsf{x}^{\rightarrow})$ or $\mathsf{x}^{\rightarrow} \notin T$;

(P.2) $\bullet_{i_{k+1}} \in \mathsf{x}$, $i_k \in \mathsf{x}^{\leftarrow}$, $i_{k+1} \in \underline{\mathsf{x}}$, and either $\mathtt{family}(\mathtt{label}(\mathsf{x}^{\rightarrow})) \neq i$ or $\mathsf{x}^{\rightarrow} \notin T$;

(P.3) $\mathcal{H} \in \mathsf{x}$, $\bullet_{\mathcal{G}} \in \mathsf{x}^{\rightarrow}$, and $\underline{\mathsf{x}^{\rightarrow}}$ does not contain a label of the same family as $\mathcal{H}$; or

(P.4) $i_k \in \mathsf{x}$, $i_{k+1} \in \mathsf{x}^{\rightarrow}$ and $\bullet_{i_{k+1}} \in \mathsf{x}^{\rightarrow\uparrow}$, with x not SouthEast of a $\bullet_{i_{k+1}}$.

Define $\mathtt{boxfactor}(\mathsf{x})$ and **box weight** $\mathtt{boxwt}(T) = \prod_{\mathsf{x}} \mathtt{boxfactor}(\mathsf{x})$ as in Section 4.1.4, specifically (4.2), with the addendum that $\bullet_{\mathcal{H}} \in \mathsf{x}$ is evaluated like $\mathcal{H} \in \mathsf{x}$.

Example 4.22.

- The right two boxes of $\boxed{1_1}\boxed{1_2}\boxed{2_1}$ are productive by (P.1). The left box is not productive.
- The left box of $\boxed{1_1}\boxed{\bullet_{1_2} \, \underline{1_2}}$ is not productive. The right box is productive by (P.2).
- The first and third boxes of $\boxed{1_1}\boxed{\bullet_{1_2}}\boxed{2_1}$ are productive by (P.3) and (P.1) respectively. The middle box is not productive; a box with $\bullet_{1_2}$ is productive only if (P.2) holds.
- The right box of the second row in both $\begin{array}{|c|c|}\hline \, & \bullet_{1_2} \\ \hline 1_1 & 1_2 \\ \hline \end{array}$ and $\begin{array}{|c|c|}\hline \, & \bullet_{1_1} \\ \hline 1_1 & 1_2 \\ \hline \end{array}$ is productive by (P.1). The left box in the second row is productive only in the first case, by (P.4).

Finally the **weight** is

$$\texttt{wt}(T) := (-1)^{d(T)}\texttt{edgewt}(T) \cdot \texttt{virtualwt}(T) \cdot \texttt{boxwt}(T),$$

where $d(T) = \sum_{\mathcal{G}} (|\mathcal{G}| - 1)$, the sum is over genes $\mathcal{G}$, and $|\mathcal{G}|$ is the (multiset) cardinality of $\mathcal{G}$ (not including virtual labels). We will view $\texttt{wt}$ as a $\mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm n}]$-linear operator of formal sums of tableaux.

By Lemma 4.4, bundled tableaux are good and hence also fine. Hence for a bundled tableau B, we have two *a priori* distinct notions of $\texttt{wt}\, B$. The following lemma justifies our failure to distinguish these notationally:

Lemma 4.43. *For B a bundled tableau, $\texttt{wt}\, B$ as a fine tableau equals $\texttt{wt}\, B$ as a bundled tableau.*

Proof. By definition, the two notions of $\texttt{edgewt}(B)$ coincide, as do the two notions of $d(B)$. Since B has no $\bullet$'s, only (P.1) is available to effect productivity. Hence the two notions of productive boxes coincide, and thus, by definition, so too do the two notions of $\texttt{boxwt}(B)$. As remarked above, the locations of virtual labels are the same, whether we think of B as bundled or fine. By Lemma 4.1, $\texttt{wt}\, B$ as a bundled tableau is

$$(-1)^{d(B)}\texttt{edgewt}(B)\texttt{boxwt}(B) \prod_{\textcircled{\ell}} (1 - \texttt{edgefactor}(\ell)),$$

where the product is over all instances of virtual labels and $\texttt{edgefactor}(\ell)$ means the factor that would be given by ℓ in $\textcircled{\ell}$'s place. Since B is bundled, it has no marked labels. Hence $\texttt{virtualwt}(B)$ is calculated using only (4.15), not (4.14). Thus $\texttt{virtualwt}(B) = \prod_{\textcircled{\ell}} (1 - \texttt{edgefactor}(\ell))$, and the lemma follows. $\square$

4.12.2 Main claim about weight preservation

Proposition 4.14.

(I) $\texttt{wt}\, P_{1_1} = \texttt{wt}\, \Lambda^+$.

(II) For every $\mathcal{G}$, $\texttt{wt}\, P_{\mathcal{G}} = \texttt{wt}\, P_{1_1}$.

(III) $\texttt{wt}\, P_{\mathcal{G}^+_{\max}} = \texttt{wt}\, \Lambda + \texttt{wt}\, \Lambda^-$.

Proof. We will first prove the easier statements (I) and (III).

(I): Suppose $T \in B^{\nu}_{\rho,\mu}$ for some $\rho \in \lambda^+$. It is enough to show $\texttt{wt}\, T = \texttt{wt}\, T^{(1_1)}$. Certainly $\texttt{edgewt}(T) = \texttt{edgewt}(T^{(1_1)})$ and $d(T) = d(T^{(1_1)})$. Adding $\bullet_{1_1}$'s preserves the virtual labels' locations, so $\texttt{virtualwt}(T) = \texttt{virtualwt}(T^{(1_1)})$.

A productive box in T is also productive in $T^{(1_1)}$ and has the same $\texttt{boxfactor}$. Suppose x is a productive box of $T^{(1_1)}$ that is not productive in T. It satisfies one of (P.1)–(P.4). If x satisfies (P.1), it is productive in T. If it satisfies (P.2), then $\bullet_{1_1} \in \mathsf{x}$ and $\mathsf{x}^{\leftarrow}$ contains a label, contradicting $\mathsf{x}^{\leftarrow} \in \rho$. If it satisfies (P.3), then $\mathsf{x}^{\rightarrow} \in \rho$, contradicting that x contains a label. Finally if x satisfies (P.4), then $\bullet_{i_{k+1}} \in \mathsf{x}^{\rightarrow\uparrow}$ and $i_k \in \mathsf{x}$. But every $\bullet$ in $T^{(1_1)}$ is $\bullet_{1_1}$. Hence $i_{k+1} = 1_1$, which is impossible since 1_0 is not a label in our alphabet. Thus the productive boxes of T and $T^{(1_1)}$ are the same, and with the same respective $\texttt{boxfactor}$s. Therefore, $\texttt{wt}\, T = \texttt{wt}\, T^{(1_1)}$.

(III): Suppose $U \in P_{\mathcal{G}^+_{\max}}$ and let $\widetilde{U}$ be given by deleting each $\bullet_{\mathcal{G}^+_{\max}}$. Proposition 4.13 states $P_{\mathcal{G}^+_{\max}}$ with all $\bullet_{\mathcal{G}^+_{\max}}$'s removed equals $\Lambda + \Lambda^-$. Thus, it suffices to show $\texttt{wt}\, U = \texttt{wt}\, \widetilde{U}$. Clearly, $\texttt{edgewt}(U) = \texttt{edgewt}(\widetilde{U})$ and $d(U) = d(\widetilde{U})$. One checks that the virtual labels of U and the virtual labels of $\widetilde{U}$ appear in the same places. Hence $\texttt{virtualwt}(U) = \texttt{virtualwt}(\widetilde{U})$.

Suppose x is productive in U. Then it satisfies one of (P.1)–(P.4). If x satisfies (P.1) in U, then it satisfies (P.1) in $\widetilde{U}$. Now x cannot satisfy (P.2) in U, since if it did, $\bullet_{\mathcal{G}^+_{\max}} \in \mathsf{x}$ and $\underline{\mathsf{x}}$ contains a label, contradicting Lemma 4.26. If x satisfies (P.3) in U, then it satisfies (P.1) in $\widetilde{U}$. If x satisfies (P.4) in U, then $\bullet_{\mathcal{G}^+_{\max}} \in \mathsf{x}^{\rightarrow\uparrow}$ but is not an outer corner, again contradicting Lemma 4.26. Thus if x is productive in U, it is productive in $\widetilde{U}$. Conversely, if x is productive in $\widetilde{U}$, it satisfies (P.1), since there are no $\bullet_{\mathcal{G}^+_{\max}}$'s in $\widetilde{U}$. Hence x satisfies (P.1) or (P.3) in U. Thus the productive boxes of U and $\widetilde{U}$ are the same. These boxes have the same $\texttt{boxfactor}$s. Thus $\texttt{boxwt}(U) = \texttt{boxwt}(\widetilde{U})$.

(II): We induct on $\mathcal{G}$ with respect to $\prec$. The base case $\mathcal{G} = 1_1$ is trivial. The inductive hypothesis is that $\texttt{wt}\, P_{\mathcal{G}} = \texttt{wt}\, P_{1_1}$. Our inductive step is to show $\texttt{wt}\, P_{\mathcal{G}^+} = \texttt{wt}\, P_{\mathcal{G}}$.

Consider the set

$$\texttt{Snakes}_{\mathcal{G}} = \{S \text{ is a snake in } T \colon [T]P_{\mathcal{G}} \neq 0\}.$$

We emphasize that each $S \in \texttt{Snakes}_{\mathcal{G}}$ refers to a particular instance of a snake in a specific tableau $T \in P_{\mathcal{G}}$. In particular, $\texttt{Snakes}_{\mathcal{G}}$ is not a multiset.

For $\mathcal{B} \subseteq \texttt{Snakes}_{\mathcal{G}}$ define $\texttt{swapset}_{\mathcal{B}}(T)$ to be the formal sum of fine tableaux obtained by swapping each snake of $\mathcal{B}$ that appears in T (done in any order, as permitted by Lemma 4.23).

We will construct m subsets $\mathcal{B}_i$ such that (D.1) and (D.2) below hold:

(D.1) We have a disjoint union $\texttt{Snakes}_{\mathcal{G}} = \bigsqcup_{1 \leq i \leq m} \mathcal{B}_i$.

(D.2) For every $1 \leq i \leq m$ and $J \subseteq \{1, \ldots, \hat{i}, \ldots, m\}$, let $\mathcal{B}_J := \cup_{j \in J} \mathcal{B}_j$. Then

$$\sum_{T \in \Gamma_i} [T]P_{\mathcal{G}} \cdot \texttt{wt}(\texttt{swapset}_{\mathcal{B}_J}(T)) = \sum_{T \in \Gamma_i} [T]P_{\mathcal{G}} \cdot \texttt{wt}(\texttt{swapset}_{\mathcal{B}_i} \circ \texttt{swapset}_{\mathcal{B}_J}(T)), \tag{4.16}$$

where $\Gamma_i := \{T \in P_{\mathcal{G}} : T \text{ contains a snake from } \mathcal{B}_i\}$.

Claim 4.6. *The existence of* $\{\mathcal{B}_i\}$ *satisfying (D.1) and (D.2) implies* $\mathtt{wt}(P_{\mathcal{G}^+}) = \mathtt{wt}(P_{\mathcal{G}})$.

Proof. By Lemma 4.23, snakes may be swapped in any order, so choose an arbitrary ordering of the blocks $\mathcal{B}_i$. By (D.1), $P_{\mathcal{G}^+} := \mathtt{swap}_{\mathcal{G}}(P_{\mathcal{G}}) = \mathtt{swapset}_{\mathcal{B}_m} \circ \cdots \circ \mathtt{swapset}_{\mathcal{B}_1}(P_{\mathcal{G}})$. Thus

$$\begin{aligned}
\mathtt{wt}(P_{\mathcal{G}^+}) &= \mathtt{wt}(\mathtt{swap}_{\mathcal{G}}(P_{\mathcal{G}})) \\
&= \mathtt{wt}(\mathtt{swapset}_{\mathcal{B}_m} \circ \cdots \circ \mathtt{swapset}_{\mathcal{B}_1}(P_{\mathcal{G}})) \\
&= \mathtt{wt}(\mathtt{swapset}_{\mathcal{B}_{m-1}} \circ \cdots \circ \mathtt{swapset}_{\mathcal{B}_1}(P_{\mathcal{G}}))
\end{aligned}$$

Here we have just used (4.16) from (D.2) together with linearity of $\mathtt{wt}$ and $\mathtt{swapset}_{\mathcal{B}_i}$ and the triviality $\mathtt{swapset}_{\mathcal{B}_J}(T) = \mathtt{swapset}_{\mathcal{B}_i} \circ \mathtt{swapset}_{\mathcal{B}_J}(T)$ for $T \notin \Gamma_i$. Repeating this argument $m-1$ further times, we obtain the desired equality with $\mathtt{wt}(P_{\mathcal{G}})$. $\square$

In order to provide the desired decomposition, we need to first construct certain *"pairing" maps.* These are given in Section 4.12.3. Given these, the description of the decomposition satisfying (D.1) and (D.2) is relatively straightforward and is found in [PeYo15b, Appendix C]. $\square$

4.12.3 Pairing maps

Let $G^{\nu}_{\lambda,\mu}(\mathcal{G})$ be the set of $\mathcal{G}$-good tableaux of shape ν/λ and content μ. For $\mathcal{Q} \prec \mathcal{G}$ and $T \in G^{\nu}_{\lambda,\mu}(\mathcal{G})$, let $\mathfrak{R}_{\mathcal{Q}}(T) := \{V \in \mathtt{revswap}_{\mathcal{Q}^+} \circ \cdots \circ \mathtt{revswap}_{\mathcal{G}}(T)\}$.

Lemma 4.44. *For any genes* $\mathcal{Q} \prec \mathcal{G}$ *and any tableau* $T \in G^{\nu}_{\lambda,\mu}(\mathcal{G})$

$$\mathfrak{R}_{\mathcal{Q}}(T) = \{W \in G^{\nu}_{\lambda,\mu}(\mathcal{Q}) : T \in \mathsf{swap}_{\mathcal{G}^-} \circ \cdots \circ \mathsf{swap}_{\mathcal{Q}}(W)\}.$$

Proof. This is immediate from Proposition 4.6, noting that, by Lemmas 4.25 and 4.34 and Propositions 4.4 and 4.5, both forward and reverse swaps preserve goodness and content. $\square$

Let $\mathcal{S}_1$ be the subset of tableaux in $G^{\nu}_{\lambda,\mu}(i_k)$ with a box x such that for some $\ell \geq k$, $\bullet_{i_k} \in \mathsf{x}$, $\bullet_{i_k} \in \mathsf{x}^{\rightarrow\uparrow}$, $i_{\ell+1} \in \mathsf{x}^{\rightarrow}$ and $i_\ell \in \underline{\mathsf{x}}$, i.e. locally the tableau is $\mathcal{C}_1 = \begin{array}{|c|c|} \hline \multicolumn{1}{c|}{} & \bullet_{i_k} \\ \hline \bullet_{i_k} & i_{\ell+1} \\ \hline \end{array}_{\,i_\ell}$ (with possibly additional edge labels), where x southwestmost depicted box. Let $\mathcal{S}'_1$ be the subset of tableaux in $G^{\nu}_{\lambda,\mu}(i_k)$ with a box x such that $i_\ell \in \mathsf{x}$, $i_{\ell+1} \in \mathsf{x}^{\rightarrow}$, $\bullet_{i_k} \in \mathsf{x}^{\rightarrow\uparrow}$, i_ℓ appears outside of x and no $\bullet_{i_k}$ appears West of x in x's row. Locally the tableau is $\mathcal{C}'_1 = \begin{array}{|c|c|} \hline \multicolumn{1}{c|}{} & \bullet_{i_k} \\ \hline i_\ell & i_{\ell+1} \\ \hline \end{array}$ (with possibly additional edge labels).

Lemma 4.45. *If $T \in \mathcal{S}_1$ (respectively, $\mathcal{S}_1'$), there is a unique $\mathcal{C}_1$ (respectively, $\mathcal{C}_1'$) that it contains.*

Proof. Let x be the lower-left box of any fixed choice of $\mathcal{C}_1$ in T. Since $i_\ell \in \underline{\mathsf{x}}$, the $i_{\ell+1} \in \mathsf{x}^{\rightarrow}$ is westmost in T by (G.6). Hence this configuration is unique. The argument for the other claim is the same, except we replace "$i_\ell \in \underline{\mathsf{x}}$" with "$i_\ell \in \mathsf{x}$". $\square$

For $T \in \mathcal{S}_1$, let $\phi_1(T)$ to be the same tableau with the unique $\mathcal{C}_1$ replaced by $\mathcal{C}_1'$. (By this we mean that we delete the labels specified in $\mathcal{C}_1$ and add the labels specified in $\mathcal{C}_1'$; any additional edge labels in T are unchanged.)

Lemma 4.46. *$\phi_1 : \mathcal{S}_1 \to \mathcal{S}_1'$ is a bijection.*

Proof. Let $\phi_1^{-1} : \mathcal{S}_1' \to \mathcal{S}_1$ be the putative inverse of ϕ_1, defined by replacing $\mathcal{C}_1'$ in a $T \in \mathcal{S}_1'$ by $\mathcal{C}_1$. We are done once we show that ϕ_1 and ϕ_1^{-1} are well-defined since the maps are clearly injective and are mutually inverse.

Let x be the southwestmost box in the unique (by Lemma 4.45) $\mathcal{C}_1$ in T.

(ϕ_1 is well-defined): Let $T \in \mathcal{S}_1$. We only need that $\phi_1(T)$ is good. Conditions (G.1) and (G.2) hold trivially in $\phi_1(T)$. (G.3) holds if $\mathsf{x}^{\leftarrow}$ is empty. Suppose $\mathcal{F} \in \mathsf{x}^{\leftarrow}$. By T's (G.9), $\mathcal{F} \prec i_k$. Hence $\mathcal{F} \prec i_\ell$, and (G.3) holds in $\phi_1(T)$. The $i_\ell \in \underline{\mathsf{x}}$ in T shows that $\phi_1(T)$ satisfies (G.4), (G.6) and (G.8). (G.5), (G.7), (G.9), (G.11) hold trivially. Since $i_{\ell+1} \in \mathsf{x}^{\rightarrow}$ is not marked, by Lemma 4.5(II) there is no marked label in T in x's row, so (G.10) and (G.13) hold for $\phi_1(T)$. For (G.12), suppose T has labels ℓ, ℓ' that violate (G.12) in $\phi_1(T)$. Since ℓ must be northWest of x, by T's (G.9), $\ell \prec i_k$. Since ℓ' must be southeast of x, by T's (G.3), (G.4) and (G.11), $\ell' \succ i_{\ell+1}$. Hence $\texttt{family}(\ell) = \texttt{family}(\ell') = i$. If ℓ is North of x, then by (G.4) the box of x's row directly below ℓ contains a label that violates T's (G.9). By T's (G.4), ℓ' is not South of $\mathsf{x}^{\rightarrow}$. Hence ℓ, ℓ' are box labels in the row of x, and no violation of $\phi_1(T)$'s (G.12) occurs.

(ϕ_1^{-1} is well-defined): Let $T' \in \mathcal{S}_1'$. We must show that (G.7) and (G.13) hold in $\phi_1^{-1}(T')$ and that (G.1)–(G.6) and (G.8)–(G.12) hold even if the virtual label is replaced by a nonvirtual one (cf. (V.1)–(V.3)). (G.1), (G.3)–(G.10), (G.12) and (G.13) are trivial to verify. To verify (G.2) for $\phi_1^{-1}(T')$, it suffices to show T' has no $\bullet_{i_k}$ South of x in the same column, or West of x in the same row. (G.9) for T' rules out the possibility of $\bullet_{i_k}$ South of x in the same column of T'. By definition, there is no $\bullet_{i_k}$ West of x in the same row. To see (G.11) for $\phi_1^{-1}(T')$, we check there is no marked label $\mathcal{F}^!$ in the column of x. Such a label cannot appear North of x in T' by Lemma 4.5 and (G.2), considering the $\bullet_{i_k} \in \mathsf{x}^{\rightarrow\uparrow}$. By (G.4), it cannot appear South of x in T' either. $\square$

Proposition 4.15. *For each $T \in \mathcal{S}_1$, $[T]P_{i_k} = -[\phi_1(T)]P_{i_k}$.*

Proof. Let $T^{\dagger} := \phi_1(T)$.

Special case $k = 1$: Let $\widetilde{T}$ be the tableau obtained from T by deleting:

- all labels of family i and greater;
- all marked labels; and
- all boxes containing a deleted box label.

Notice that any label SouthEast of a deleted label or a $\bullet_{i_1}$ will have been deleted.

As well we reindex the genes so that the subscripts of each family form an initial segment of $\mathbb{Z}_{>0}$. (This reindexing is only possibly needed if T contained a marked label.) We leave $\bullet_{i_1}$'s in place. In the same way, produce $\widetilde{T^{\dagger}}$ from $T^{\dagger}$. By definition of ϕ_1, $\widetilde{T}$ has one more $\bullet_{i_1}$ than $\widetilde{T^{\dagger}}$ and otherwise the two tableaux are exactly the same (the family i labels of $\mathcal{C}_1$ and $\mathcal{C}'_1$ having been deleted).

Ignoring $\bullet_{i_1}$'s, $\widetilde{T}, \widetilde{T^{\dagger}}$ are of some common skew shape θ/λ. If we include the $\bullet_{i_1}$'s, their respective total shapes are some ω/λ and $\omega^{\dagger}/\lambda$ where $\omega, \omega^{\dagger} \in \theta^{+}$.

Claim 4.7. *$\widetilde{T} \in G^{\omega}_{\lambda,\widetilde{\mu}}(i_1)$ and $\widetilde{T^{\dagger}} \in G^{\omega^{\dagger}}_{\lambda,\widetilde{\mu}}(i_1)$ where $\widetilde{\mu}$ is a partition (e.g., if T has no marked labels then $\widetilde{\mu} := (\mu_1, \mu_2, \ldots, \mu_{i-1})$). Thus, $\widetilde{T}$ and $\widetilde{T^{\dagger}}$ (with $\bullet_{i_1}$'s removed) are in $B^{\theta}_{\lambda,\widetilde{\mu}}$.*

Proof. We prove the claim for $\widetilde{T}$; the proof for $\widetilde{T^{\dagger}}$ is essentially the same.

Clearly, (G.1)–(G.7), (G.9) and (G.12) for $\widetilde{T}$ are inherited from the assumption T is good. (G.10), (G.11) and (G.13) are vacuous for $\widetilde{T}$. It remains to show (G.8) holds for $\widetilde{T}$ (which moreover implies $\widetilde{\mu}$ is a partition).

Suppose $\widetilde{T}$ fails (G.8). Then there is a least q such that $\widetilde{T}$ has a ballotness violation between families q and $q+1$. That is, in some genotype G of $\widetilde{T}$ there are more labels of family $q+1$ than of family q in some initial segment of $\texttt{word}(G)$. Since we have deleted all labels of family i and greater, $q < i-1$. By failure of (G.8), either there exist labels q_r and $(q+1)_s$ of $\widetilde{T}$ with $N_{q_r} = N_{(q+1)_s}$ such that $(q+1)_s$ appears before q_r in $\texttt{word}(G)$, or else there is a label $(q+1)_s$ of $\widetilde{T}$ with $N_{(q+1)_s} > N_{q_v}$ for all v. Let $q_{r'}$ (if q_r exists) and $(q+1)_{s'}$ be the corresponding labels of T. We assert in the former case that $N_{q_{r'}} \leq N_{(q+1)_{s'}}$ in T. In the latter case, we assert $N_{(q+1)_{s'}} > N_{q_{v'}}$ in T for all v'. Either of these inequalities contradicts T's (G.8).

To see these assertions, suppose that q_h is a gene of T that is entirely deleted in the construction of $\widetilde{T}$ (i.e. every instance of q_h in T is marked). Consider an instance of q_h in T in x or $\underline{\mathsf{x}}$. Since this q_h is marked and $q < i-1$, by Lemma 4.6 we know T has some nonvirtual and marked $(q+1)^{!}_{z} \in \underline{\mathsf{x}}$ with $N_{q_h} = N_{(q+1)_z}$. By T's (G.7), there is no $(q+1)_z$ West of x in T. By T's Lemma 4.7, there is no $(q+1)_z$ East of x in T. Hence the $(q+1)^{!}_{z} \in \underline{\mathsf{x}}$ is the only $(q+1)_z$ in T. Since it is marked, the gene $(q+1)_z$ is entirely deleted in $\widetilde{T}$. By this argument, if $q_{\hat{h}}$ is any other gene of T that is entirely deleted in the construction of $\widetilde{T}$, there is a distinct

$(q+1)_{\hat{z}}$ with $N_{q_{\hat{h}}} = N_{(q+1)_{\hat{z}}}$ that is also entirely deleted in $\widetilde{T}$. Hence $N_{q_{r'}} \geq N_{(q+1)_{s'}}$ or $N_{(q+1)_{s'}} > N_{q_{v'}}$ in T for all v', as asserted.

The last sentence of the claim follows from the first by Lemma 4.27, since no genetic label is southeast of a $\bullet_{i_1}$. $\square$

In view of Claim 4.7, it makes sense to speak of $\mathfrak{T}_{\widetilde{T}}$ and of $\mathfrak{T}_{\widetilde{T^\dagger}}$. By Proposition 4.11,

$$\sum_{L \in \mathtt{leaf}(\mathfrak{T}_{\widetilde{T}})} (-1)^{|\rho(L)/\lambda|+1}[\widetilde{T}]\mathtt{slide}_{\rho(L)/\lambda}(L) = (-1)^{1+\#\text{ of }\bullet\text{'s in }\widetilde{T}} \cdot \mathtt{wt}(\theta/\lambda). \tag{4.17}$$

Similarly,

$$\sum_{L \in \mathtt{leaf}(\mathfrak{T}_{\widetilde{T^\dagger}})} (-1)^{|\rho(L)/\lambda|+1}[\phi_1(\widetilde{T})]\mathtt{slide}_{\rho(L)/\lambda}(L) = (-1)^{1+\#\text{ of }\bullet\text{'s in }\widetilde{T^\dagger}} \cdot \mathtt{wt}(\theta/\lambda). \tag{4.18}$$

In particular, these quantities differ by a factor of -1.

By inspection of the reverse miniswaps, $\mathtt{revswap}_{a_q}$ for $1 \leq a \leq i-1$ does not affect any labels of family i or greater or any labels that are marked in T. Hence one sees that $\mathtt{revswap}_{1_2} \circ \cdots \circ \mathtt{revswap}_{(i-1)_{\mu_{i-1}}} \circ \mathtt{revswap}_{i_1}(T)$ (respectively $T^\dagger$) is the same as $\mathtt{revswap}_{1_2} \circ \cdots \circ \mathtt{revswap}_{(i-1)_{\mu_{i-1}}} \circ \mathtt{revswap}_{i_1}(\widetilde{T})$ (respectively $\widetilde{T^\dagger}$) followed by adding back the labels of $T \setminus \widetilde{T}$ (respectively $T^\dagger \setminus \widetilde{T^\dagger}$). Therefore, by our comparison of (4.17) and (4.18) above,

$$[T]P_{i_1} = (-1)^{1+\#\text{ of }\bullet\text{'s in }\widetilde{T}} \cdot \mathtt{wt}(\theta/\lambda) = -[T^\dagger]P_{i_1},$$

as desired.

Reduction to the $k=1$ case: In the calculation of

$$\mathtt{revswap}_{i_2} \circ \mathtt{revswap}_{i_3} \circ \cdots \circ \mathtt{revswap}_{i_k}(T)$$

and

$$\mathtt{revswap}_{i_2} \circ \mathtt{revswap}_{i_3} \circ \cdots \circ \mathtt{revswap}_{i_k}(T^\dagger),$$

it is straightforward by inspection that each reverse miniswap involving either $\bullet$ of $\mathcal{C}_1$ or the $\bullet$ of $\mathcal{C}_1'$ is L1.2. Therefore there exists an instance of $\mathcal{C}_1$ in each $W \in \mathfrak{R}_{i_1}(T)$ and an instance of $\mathcal{C}_1'$ in each $W' \in \mathfrak{R}_{i_1}(T^\dagger)$. By Lemma 4.45, these instances are unique. Extending ϕ_1 linearly, since T and $T^\dagger$ are the same outside of the regions $\mathcal{C}_1$ and $\mathcal{C}_1'$, it is easy to see inductively that for all $2 \leq q \leq k$,

$$\phi_1(\mathtt{revswap}_{i_q} \circ \cdots \circ \mathtt{revswap}_{i_k}(T)) = \mathtt{revswap}_{i_q} \circ \cdots \circ \mathtt{revswap}_{i_k}(T^\dagger).$$

In particular, ϕ_1 bijects $\mathfrak{R}_{i_1}(T)$ with $\mathfrak{R}_{i_1}(T^\dagger)$.

Let $V \in \mathfrak{R}_{i_1}(T)$. By the $k=1$ case above, $[V]P_{i_1} = -[\phi_1(V)]P_{i_1}$. Moreover, when we apply $\mathsf{swap}_{i_k^-} \circ \cdots \circ \mathsf{swap}_{i_1}$ to V and $\phi_1(V)$, each miniswap involving a $\bullet$ of $\mathcal{C}_1$ or $\mathcal{C}_1'$ is H3. Hence, $[T]\mathsf{swap}_{i_k^-} \circ \cdots \circ \mathsf{swap}_{i_1}(V) = [T^\dagger]\mathsf{swap}_{i_k^-} \circ \cdots \circ \mathsf{swap}_{i_1}(\phi_1(V))$. Thus by Lemma 4.44, $[T]P_{i_k} = -[T^\dagger]P_{i_k}$. $\square$

Let $\mathcal{S}_2$ be the subset of tableaux in $G^\nu_{\lambda,\mu}(i_k)$ with a box x such that $\bullet_{i_k} \in \mathsf{x}$, $\bullet_{i_k} \in \mathsf{x}^{\rightarrow\uparrow}$, $i_{k+1} \in \mathsf{x}^{\rightarrow}$ and $i_k \in \underline{\mathsf{x}}$, i.e. locally the tableau is $\mathcal{C}_2 = \begin{array}{|c|c|}\hline \multicolumn{1}{c|}{} & \bullet_{i_k} \\ \hline \underset{i_k}{\bullet_{i_k}} & i_{k+1} \\ \hline \end{array}$ (with possibly additional edge labels). Let $\mathcal{S}_2'$ be the subset of tableaux in $G^\nu_{\lambda,\mu}(i_k)$ with a box x such that $i_k \in \mathsf{x}$, $i_{k+1} \in \mathsf{x}^{\rightarrow}$, $\bullet_{i_k} \in \mathsf{x}^{\rightarrow\uparrow}$, no i_k appears outside of x and no $\bullet_{i_k}$ appears West of x in x's row. Locally the tableau is $\mathcal{C}_2' = \begin{array}{|c|c|}\hline \multicolumn{1}{c|}{} & \bullet_{i_k} \\ \hline i_k & i_{k+1} \\ \hline \end{array}$ (with possibly additional edge labels).

Lemma 4.47. *If $T \in \mathcal{S}_2$ (respectively, $\mathcal{S}_2'$), there is a unique $\mathcal{C}_2$ (respectively, $\mathcal{C}_2'$) that it contains.*

Proof. Let x be the southwestmost box of a $\mathcal{C}_2$ in T. By (G.7), the $i_k \in \underline{\mathsf{x}}$ is the westmost i_k in T; hence this configuration is unique. The claim about $\mathcal{C}_2'$ is clear since the i_k is unique. $\square$

For $T \in \mathcal{S}_2$, let $\phi_2(T)$ be T with the unique $\mathcal{C}_2$ replaced by $\mathcal{C}_2'$.

Lemma 4.48. *$\phi_2 : \mathcal{S}_2 \to \mathcal{S}_2'$ is a bijection.*

Proof. This may be proved almost exactly as Lemma 4.46. $\square$

Proposition 4.16. *For each $T \in \mathcal{S}_2$, $[T]P_{i_k} = -[\phi_2(T)]P_{i_k}$.*

Proof. Let $T^\dagger := \phi_2(T)$. Let x be the southwestmost box of $\mathcal{C}_2$ in T. Then x is also the southwestmost box of $\mathcal{C}_1'$ in $T^\dagger$.

Special case $k=1$: The proof is *verbatim* the argument for the $k=1$ case of Proposition 4.15.

Reduction to the $k=1$ case: Suppose $k>1$. Let $\mathcal{Z}$ be the set of boxes in an i_k-good tableau that either (1) contain $\bullet_{i_k}$ or (2) contain a label $\mathcal{F}$ with $i_1 \preceq \mathcal{F} \preceq i_{k-1}$ and are not southeast of a $\bullet_{i_k}$. Call an edge connected component of $\mathcal{Z}$ an i_k**-walkway**. We will now apply the development of i-walkways, from Sections 4.10 and 4.11, in slightly modified form to the i_k**-walkways**. To be more precise, Lemmas 4.37, 4.38, 4.39 and 4.40 are true after replacing "$(i+1)_1$" with "i_k" and "i-walkway" with "i_k-walkway". In addition, Claim 4.5 holds *verbatim*. The proofs are trivial modifications of those given.

Let W be the i_k-walkway of T containing x (W includes all edges of boxes in W). Let $W^\dagger$ be the analogous i_k-walkway of $T^\dagger$. Note that W and $W^\dagger$ have the same skew shape.

Claim 4.8. *Let $\mathbb{S}$, $\mathbb{S}'$ and $\mathbb{T}$ be respectively the set of reversals of W, $W^\dagger$ and W^c (the complement of W) under $\texttt{revswap}_{i_2} \circ \cdots \circ \texttt{revswap}_{i_k}$. Then:*

(I) $\mathfrak{R}_{i_1}(T) = \{V \in G^{\nu}_{\lambda,\mu}(i_1) : V|_W \in \mathbb{S}, V|_{W^c} \in \mathbb{T}\}$

(II) $\mathfrak{R}_{i_1}(T^\dagger) = \{V' \in G^{\nu}_{\lambda,\mu}(i_1) : V'|_{W^\dagger} \in \mathbb{S}', V'|_{(W^\dagger)^c} \in \mathbb{T}\}$

Proof. We prove only (I), as the proof of (II) is similar (using $T|_{W^c} = T^\dagger|_{W^c}$). Fix $2 \leq h \leq k$ and let L be a ladder of $A \in \mathfrak{R}_{i_h}(T)$. L contains only $\bullet_{i_h}$ and unmarked i_{h-1}. Each of the boxes x of L is in $\mathcal{Z}$: This is clear if $h = k$ and follows for smaller h by induction. Thus $L \subseteq \mathcal{Z}$. Therefore, since L is edge connected, it sits inside an edge connected component of $\mathcal{Z}$. Thus, since W is one such component, reverse swapping acts independently on W and W^c. □

Case 1: (W (and hence $W^\dagger$) has a single row): By construction, x is the eastmost box of W and $W^\dagger$. By Lemma 4.39(II), for every $V \in \mathfrak{R}_{i_1}(T)$, $V|_W = R'$. By Lemma 4.39(III), for every $V' \in \mathfrak{R}_{i_1}(T^\dagger)$, $V'|_{W^\dagger} \in \{R, R'\}$ where this R' is the same as in the previous sentence.

Since R' is the unique reversal of W and is a reversal of $W^\dagger$, we have $\mathfrak{R}_{i_1}(T) \subseteq \mathfrak{R}_{i_1}(T^\dagger)$ by Claim 4.8. Let $\iota : \mathfrak{R}_{i_1}(T) \to \mathfrak{R}_{i_1}(T^\dagger)$ be the inclusion map. Let $f : \mathfrak{R}_{i_1}(T) \to \mathfrak{R}_{i_1}(T^\dagger)$ be the map given by replacing the R' occupying the region W with R. Again appealing to Claim 4.8 we see that these maps are well-defined, injective and $\mathfrak{R}_{i_1}(T^\dagger) = \operatorname{im} \iota \sqcup \operatorname{im} f$.

By Claim 4.5(III), forward swapping R produces $W^\dagger$ with coefficient 1. By Claim 4.5 (part (I.i) or (I.ii), as appropriate) forward swapping R' produces $\beta W + (1-\beta)W^\dagger$ for some β. Moreover, when applying $\mathsf{swap}_{i_{k-1}} \circ \cdots \circ \mathsf{swap}_{i_1}$ to $V \in \mathfrak{R}_{i_1}(T)$ or $V' \in \mathfrak{R}_{i_1}(T^\dagger)$, every snake lies entirely inside some edge-connected component of $\mathcal{Z}$. W is one of these components. Thus, for each $V \in \mathfrak{R}_{i_1}$, $[T]\mathsf{swap}_{i_{k-1}} \circ \cdots \circ \mathsf{swap}_{i_1}(V)$ factors as a contribution from the region W times a contribution from $\mathcal{Z} \setminus W$. That is, for the same α,

$$\sum_{V \in \mathfrak{R}_{i_1}(T)} [T]\mathsf{swap}_{i_{k-1}} \circ \cdots \circ \mathsf{swap}_{i_1}(V) = \alpha\beta, \quad \sum_{V' \in \mathfrak{R}_{i_1}(T^\dagger)} [T^\dagger]\mathsf{swap}_{i_{k-1}} \circ \cdots \circ \mathsf{swap}_{i_1}(V') = \alpha$$

$$\sum_{V' \in \mathfrak{R}_{i_1}(T^\dagger)} [T^\dagger]\mathsf{swap}_{i_{k-1}} \circ \cdots \circ \mathsf{swap}_{i_1}(V') = \alpha(1-\beta).$$

Therefore,

$$[T]P_{i_k} = \sum_{V \in \mathfrak{R}_{i_1}(T)} [V]P_{i_1} \cdot [T]\mathsf{swap}_{i_{k-1}} \circ \cdots \circ \mathsf{swap}_{i_1}(V) = [V]P_{i_1}\alpha\beta,$$

while

$$
\begin{aligned}
[T^\dagger]P_{i_k} &= \sum_{V' \in \mathfrak{R}_{i_1}(T^\dagger)} [V']P_{i_1} \cdot [T^\dagger]\mathsf{swap}_{i_{k-1}} \circ \cdots \circ \mathsf{swap}_{i_1}(V') \\
&= \sum_{V \in \mathfrak{R}_{i_1}(T)} [\iota(V)]P_{i_1} \cdot [T^\dagger]\mathsf{swap}_{i_{k-1}} \circ \cdots \circ \mathsf{swap}_{i_1}(\iota(V)) \\
&+ \sum_{V \in \mathfrak{R}_{i_1}(T)} [f(V)]P_{i_1} \cdot [T^\dagger]\mathsf{swap}_{i_{k-1}} \circ \cdots \circ \mathsf{swap}_{i_1}(f(V))
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{V \in \mathfrak{R}_{i_1}(T)} [V]P_{i_1} \cdot [T^\dagger]\mathsf{swap}_{i_{k-1}} \circ \cdots \circ \mathsf{swap}_{i_1}(V) \\
&- \sum_{V \in \mathfrak{R}_{i_1}(T)} [V]P_{i_1}\mathsf{swap}_{i_{k-1}} \circ \cdots \circ \mathsf{swap}_{i_1}(f(V)) \\
&= [V]P_{i_k}(\alpha(1-\beta)-\alpha).
\end{aligned}
$$

Now, $[T]P_{i_k} = -[T^\dagger]P_{i_k}$ follows.

Case 2: (W (and hence $W^\dagger$) has at least two rows): There are three cases to consider, corresponding to the case of Lemma 4.40.

In Cases (I) and (II) of Lemma 4.40, W and $W^\dagger$ have a unique reversal R. By Claim 4.5(III) or Claim 4.5(II) respectively, forward swapping R produces $\beta W - \beta W^\dagger$ for some β. In Case (III) of Lemma 4.40, W and $W^\dagger$ share the same pair of reversals R, R'. By Claim 4.5(III) and (I.i), forward swapping R produces $\beta W - \beta W^\dagger$ for some β, while forward swapping R' produces $\beta' W - \beta' W^\dagger$ for some β'. Using these facts, one may argue similarly to Case 1 to deduce $[T]P_{i_k} = -[T^\dagger]P_{i_k}$. $\square$

Let $\mathcal{S}_3$ be the subset of tableaux in $G^{\nu}_{\lambda,\mu}(i_k)$ with a box x such that $\bullet_{i_k} \in \mathsf{x}$, $i_k \in \mathsf{x}^{\rightarrow}$ and $i_k \in \underline{\mathsf{x}}$, i.e. locally the tableau is $\mathcal{C}_3 = \begin{array}{|c|c|}\hline \underset{i_k}{\bullet_{i_k}} & i_k \\ \hline \end{array}$. (with possibly additional edge labels). Let $\mathcal{S}_3'$ be the subset of tableaux in $G^{\nu}_{\lambda,\mu}(i_k)$ with a box x such that $\bullet_{i_k} \in \mathsf{x}$, $i_k \in \mathsf{x}^{\rightarrow}$, no i_k appears West of $\mathsf{x}^{\rightarrow}$, $i_{k-1} \notin \mathsf{x}^{\leftarrow}$, and $(i+1)_h \notin \underline{\mathsf{x}^{\rightarrow}}$ where $N_{i_k} = N_{(i+1)_h}$. Locally the tableau is $\mathcal{C}_3' = \begin{array}{|c|c|}\hline \bullet_{i_k} & i_k \\ \hline \end{array}$ (with possibly additional edge labels).

Lemma 4.49. *If $T \in \mathcal{S}_3$ (respectively, $\mathcal{S}_3'$), there is a unique $\mathcal{C}_3$ (respectively, $\mathcal{C}_3'$) that it contains.*

Proof. If $\mathcal{C}_3$ occurs in a good tableau, it is unique since the edge i_k is westmost in its gene by (G.7). Similarly $\mathcal{C}_3'$ is unique since the $i_k \in \mathsf{x}^{\rightarrow}$ is westmost by assumption. $\square$

Define $\phi_3(T)$ to be T with the unique $\mathcal{C}_3$ replaced by $\mathcal{C}_3'$.

Lemma 4.50. *$\phi_3 : \mathcal{S}_3 \to \mathcal{S}_3'$ is a bijection.*

Proof. Define the (putative) inverse ϕ_3^{-1} by replacing $\mathcal{C}_3'$ with $\mathcal{C}_3$. Once we establish that ϕ_3 and ϕ_3^{-1} are well-defined, we are done, since ϕ_3 and ϕ_3^{-1} are clearly mutually inverse.

Let $T \in \mathcal{S}_3$. Trivially, each (G.n) holds for $\phi_3(T)$. By T's (G.12), $i_{k-1} \notin \mathsf{x}^{\leftarrow}$. If $(i+1)_h \in \underline{\mathsf{x}^{\rightarrow}}$ in $\phi_3(T)$ with $N_{i_k} = N_{(i+1)_h}$, then T would violate Lemma 4.7. By T's (G.4) and (G.7), the $i_k \in \mathsf{x}^{\rightarrow}$ is westmost in $\phi_3(T)$.

Now let $T \in \mathcal{S}_3'$. We check the goodness conditions for $\phi_3^{-1}(T)$.

Claim 4.9. *No label of family i appears in x's column in T.*

Proof. By T's (G.12), there are no labels of family i North of x and in its column. By T's (G.11), a label ℓ South of x and in its column is not marked, i.e., $\ell \succeq i_k$. Since we assumed the $i_k \in \mathsf{x}^{\rightarrow}$ is westmost, $\ell \neq i_k$. By T's (G.6), $\ell \neq i_l$ for $l > k$. Hence $i_k < \ell$. ☐

(G.4) and (G.5): By T's (G.9), all labels North of x and in its column are of family at most i. By T's (G.11), all labels South of x and in its column are of family at least i. Hence by Claim 4.9, $\phi_3^{-1}(T)$'s (G.4) and (G.5) follow.

(G.8): If there is a genotype G of $\phi_3^{-1}(T)$ that is not ballot, then it uses the $i_k \in \underline{\mathsf{x}}$. Furthermore, since T is ballot, some $(i+1)_h$ with $N_{i_k} = N_{(i+1)_h}$ appears in $\mathtt{word}(G)$ before the $i_k \in \underline{\mathsf{x}}$. By Lemma 4.7 applied to T, this $(i+1)_h$ can only be South of $\mathsf{x}^{\rightarrow}$ and in $\mathsf{x}^{\rightarrow}$'s column or North of x and in x's column. By T's (G.9), it cannot be North of x and in its column. Suppose it appears South of $\mathsf{x}^{\rightarrow}$ and in its column. By assumption, $(i+1)_h \notin \underline{\mathsf{x}^{\rightarrow}}$. Hence suppose it appears south of $\mathsf{x}^{\rightarrow\downarrow}$, and consider $\mathtt{label}(\mathsf{x}^{\downarrow})$. By (G.11) $\mathtt{family}(\mathtt{label}(\mathsf{x}^{\downarrow})) \geq i$. By Claim 4.9, $\mathtt{family}(\mathtt{label}(\mathsf{x}^{\downarrow})) \neq i$. By T's (G.3) and (G.4), $\mathtt{label}(\mathsf{x}^{\downarrow}) \prec (i+1)_h$. Hence $\mathtt{family}(\mathtt{label}(\mathsf{x}^{\downarrow})) = i+1$. But by Lemma 4.11, $\mathtt{label}(\mathsf{x}^{\downarrow}) \neq (i+1)_h$. Hence by T's (G.6), $(i+1)_{h-1} \in \mathsf{x}^{\downarrow}$. This creates a (G.8) violation in T, as this label is read before any i_{k-1}.

(G.12): Since T is good, if $\phi_3^{-1}(T)$ violates (G.12), the violation involves the $i_k \in \underline{\mathsf{x}}$. Since by assumption $i_{k-1} \notin \mathsf{x}^{\leftarrow}$, the last sentence of (G.12) does not apply. Suppose i_j is SouthEast of $\underline{\mathsf{x}}$, then it is also SouthEast of $i_k \in \mathsf{x}^{\rightarrow}$, which will lead to a violation of T's (G.12). Suppose i_j is NorthWest of $\underline{\mathsf{x}}$, then to avoid a violation of T's (G.12) with the $i_k \in \mathsf{x}^{\rightarrow}$, i_j must be either in x's row or in an upper edge of that row. Since we have $\bullet_{i_k} \in \mathsf{x}$, this avoids violating $\phi_3^{-1}(T)$'s (G.12).

All of the remaining (G.n)-conditions are trivial to verify. ☐

Proposition 4.17. *For $T \in \mathcal{S}_3$, $[T]P_{i_k} = [\phi_3(T)]P_{i_k}$.*

Proof. Let $T^{\dagger} := \phi_3(T)$. By inspection of the reverse miniswaps, and downward induction on $\mathcal{Q}$, there is a bijection $f_{\mathcal{Q}} : \mathfrak{R}_{\mathcal{Q}}(T) \to \mathfrak{R}_{\mathcal{Q}}(T^{\dagger})$ given by deletion of the $i_k \in \underline{\mathsf{x}}$. If $L \in \mathfrak{R}_{1_1}(T)$, then L and $f_{1_1}(L)$ have the same number of $\bullet_{1_1}$'s. Hence, $[L]P_{1_1} = [f_{1_1}(L)]P_{1_1}$; cf. (4.13).

Extend $f_{\mathcal{Q}}$ linearly. By inspection of the miniswaps,

$$f_{i_k}(\mathsf{swap}_{i_k^-} \circ \cdots \circ \mathsf{swap}_{1_1}(L)) = \mathsf{swap}_{i_k^-} \circ \cdots \circ \mathsf{swap}_{1_1}(f_{1_1}(L)).$$

Hence by Lemma 4.44, $[T]P_{i_k} = [T^{\dagger}]P_{i_k}$. ∎

Let $\mathcal{S}_4$ be the subset of tableaux in $G^{\nu}_{\lambda,\mu}(i_k)$ with a box x such that $\bullet_{i_k} \in \mathsf{x}$, $\mathcal{F}^! \in \mathsf{x}^{\rightarrow}$, $i_k \in \underline{\mathsf{x}}$ and $i_k \in \underline{\mathsf{x}^{\rightarrow}}$, i.e. locally the tableau is $\mathcal{C}_4 = \begin{array}{|c|c|}\hline \bullet_{i_k} & \mathcal{F}^! \\ \hline \multicolumn{1}{c}{i_k} & \multicolumn{1}{c}{i_k} \end{array}$ (with possibly additional edge labels). Let $\mathcal{S}'_4$ be the subset of tableaux in $G^{\nu}_{\lambda,\mu}(i_k)$ with a box x such that $\bullet_{i_k} \in \mathsf{x}$, $\mathcal{F}^! \in \mathsf{x}^{\rightarrow}$, $i_k \in \underline{\mathsf{x}^{\rightarrow}}$, $(i+1)_h \notin \underline{\mathsf{x}^{\rightarrow}}$ if $N_{(i+1)_h} = N_{i_k}$, and $i_{k-1} \notin \mathsf{x}^{\leftarrow}$. Locally the tableau is $\mathcal{C}'_4 = \begin{array}{|c|c|}\hline \bullet_{i_k} & \mathcal{F}^! \\ \hline \multicolumn{1}{c}{} & \multicolumn{1}{c}{i_k} \end{array}$ (with possibly additional edge labels).

Lemma 4.51. *If $T \in \mathcal{S}_4$ (respectively, $\mathcal{S}'_4$), there is a unique $\mathcal{C}_4$ (respectively, $\mathcal{C}'_4$) that it contains.*

Proof. This follows since by (G.7), T contains at most one edge label i_k. ∎

Set $\phi_4 : \mathcal{S}_4 \to \mathcal{S}'_4$ by replacing $\mathcal{C}_4$ with $\mathcal{C}'_4$.

Lemma 4.52. *$\phi_4 : \mathcal{S}_4 \to \mathcal{S}'_4$ is a bijection.*

Proof. Define a putative inverse $\phi_4^{-1} : \mathcal{S}'_4 \to \mathcal{S}_4$ by replacing $\mathcal{C}'_4$ with $\mathcal{C}_4$. Clearly, ϕ_4 and ϕ_4^{-1} are mutually inverse. It remains to check well-definedness. Indeed, it is trivial to check each goodness condition holds for $\phi_4(T)$. By Lemma 4.7 for T, there is not $(i+1)_h \in \underline{\mathsf{x}^{\rightarrow}}$ with $N_{(i+1)_h} = N_{i_k}$. By T's (G.12), $i_{k-1} \notin \mathsf{x}^{\leftarrow}$. Thus ϕ_4 is well-defined.

Claim 4.10. *No label of family i appears in x's column in T.*

Proof. By T's (G.12), i_ℓ cannot appear North of x and in its column. If i_ℓ is South of x and in its column, then by T's (G.6) and (G.7), $\ell < k$, so this i_ℓ is marked, contradicting T's (G.11). ∎

Now let $T \in \mathcal{S}'_4$. We check the goodness conditions for $\phi_4^{-1}(T)$:

(G.4) and (G.5): By T's (G.9), every label North of x and in its column has family at most i. By T's (G.11), every label South of x and in its column has family at least i. Moreover, by Claim 4.10, no label of family i appears in x's column in T. Hence (G.4) and (G.5) hold in $\phi_4^{-1}(T)$.

(G.8): Suppose $\phi_4^{-1}(T)$ has a nonballot genotype G. By T's (G.8), G must use the $i_k \in \underline{\mathsf{x}}$. Also by T's (G.8), some $(i+1)_h$ with $N_{(i+1)_h} = N_{i_k}$ appears in $\mathtt{word}(G)$ before this $i_k \in \underline{\mathsf{x}}$. By T's (G.9) and (G.8), this $(i+1)_h$ appears South of $\mathsf{x}^{\rightarrow}$ and in $\mathsf{x}^{\rightarrow}$'s column. By T's (G.4) and the first hypothesis on $\mathcal{S}'_4$, in fact $(i+1)_h \in \mathsf{x}^{\rightarrow\downarrow}$. By T's (G.3), $\mathtt{family}(\mathtt{label}(\mathsf{x}^{\downarrow})) \leq i+1$. By (G.11) and the $\bullet_{i_k} \in \mathsf{x}$, $\mathtt{family}(\mathtt{label}(\mathsf{x}^{\downarrow})) \geq i$. By Claim 4.10, no label of family i appears in x's column in T. Thus $\mathtt{family}(\mathtt{label}(\mathsf{x}^{\downarrow})) = i+1$. Then by T's (G.3) and (G.6), $(i+1)_{h-1} \in \mathsf{x}^{\downarrow}$. Hence by Claim 4.10, this contradicts Lemma 4.7 for T.

(G.12): If there is an i_ℓ SouthEast of the $i_k \in \underline{\mathsf{x}}$ in $\phi_4^{-1}(T)$, then we either violate T's (G.2), (G.4) or (G.12). Now suppose there is an i_ℓ NorthWest of $i_k \in \underline{\mathsf{x}}$ in $\phi_4^{-1}(T)$. By T's (G.12), this i_ℓ is West and either in x's row or on the upper edge of that row. If $i_\ell \in \mathsf{x}^{\leftarrow}$, then $\ell = k-1$ by T's (G.6). However then we contradict the last hypothesis on $\mathcal{S}_4'$. So the i_ℓ and i_k satisfy (G.12).

The remaining goodness conditions are trivial to verify. $\square$

Proposition 4.18. *For each $T \in \mathcal{S}_4$, $[T]P_{i_k} = [\phi_4(T)]P_{i_k}$.*

Proof. Let $T^\dagger = \phi_4(T)$. Let $f_{\mathcal{Q}} : \mathfrak{R}_{\mathcal{Q}}(T) \to \mathfrak{R}_{\mathcal{Q}}(T^\dagger)$ be defined by deleting the $i_k \in \underline{\mathsf{x}}$ and replacing the $i_k \in \underline{\mathsf{x}}^{\rightarrow}$ by i_k. Now the proof proceeds exactly as that for Proposition 4.17. $\square$

4.13 Proof of the conjectural K_T rule from [ThYo13]

We briefly recap the conjectural rule for $K_{\lambda,\mu}^{\nu}$ from [ThYo13, Section 8]. An **equivariant increasing tableau** is an edge-labeled filling of ν/λ using the labels $1, 2, \ldots, |\mu|$ such that each label is strictly smaller than any label below it in its column and each *box* label is strictly smaller than the box label immediately to its right. Any subset of the boxes of ν/λ may be marked by $\star$'s, except that if i and $i+1$ are box labels in the same row, then the box containing i may *not* be $\star$-ed. Let $\mathtt{EqInc}(\nu/\lambda, |\mu|)$ denote the set of all such equivariant increasing tableaux.

An **alternating ribbon** R is a filling of a short ribbon by two symbols such that adjacent boxes are filled differently; all edges except the southwestmost edge are empty; and if this edge is filled, it is filled with the other symbol than in the box above it. Let $\mathtt{switch}(R)$ be the alternating ribbon of the same shape where each box is instead filled with the other symbol. If the southwestmost edge was filled by one of these symbols, that symbol is deleted. If R consists of a single box with only one symbol used, then $\mathtt{switch}$ does nothing to it. Define $\mathtt{switch}$ to act on an edge-disjoint union of alternating ribbons, by acting on each independently.

Example 4.23. Let $R = \begin{array}{|c|c|c|}\hline \ & \heartsuit & \spadesuit \\ \hline \heartsuit & \spadesuit & \ \\ \hline \spadesuit & & \end{array}$. Then $\mathtt{switch}(R) = \begin{array}{|c|c|c|}\hline \ & \spadesuit & \heartsuit \\ \hline \spadesuit & \heartsuit & \ \\ \hline \end{array}$. $\diamondsuit$

Given $T \in \mathtt{EqInc}(\nu/\lambda, |\mu|)$ and an inner corner $\mathsf{x} \in \lambda$, label x with $\bullet$ and erase all $\star$'s. Call this tableau V_1. Consider the alternating ribbons $\{R_1\}$ made of $\bullet$ and 1. V_2 is obtained by applying $\mathtt{switch}$ to each R_1. Now let $\{R_2\}$ be the collection of ribbons consisting of $\bullet$ and 2, and produce V_3 by applying $\mathtt{switch}$ to each R_2. Repeat until the $\bullet$'s have been switched past all the numerical labels in T; the final placement of numerical labels gives $\mathtt{KEqjdt}_{\mathsf{x}}(T)$, the **slide** of T into x. The sequence $V_1, V_2, \ldots$ is the **switch sequence** of (T, x). Finally, define $\mathtt{KEqrect}(T)$ by successively applying $\mathtt{KEqjdt}_{\mathsf{x}}$ in **column rectification order**, i.e., successively pick x to be the eastmost inner corner.

Lemma 4.53. *For V_j in the switch sequence of (T, x):*

(I) The numerical box labels strictly increase along rows from left to right (ignoring $\bullet$'s).

(II) The numerical labels strictly increase down columns (ignoring $\bullet$'s and reading labels of a given edge in increasing order).

(III) Every numerical label southeast of a $\bullet$ is at least j.

(IV) Every numerical label northwest of a $\bullet$ is strictly less than j.

Proof. These are proved by simultaneous induction on j. In the inductive step, one considers any 2×2 local piece of V_j and studies the possible cases that can arise as one transitions from $V_j \to V_{j+1}$; we leave the straightforward details to the reader. $\square$

A set of labels is a **horizontal strip** if they are arranged in increasing order from southwest to northeast, with no two labels of the set in the same column.

Lemma 4.54. *Let $T \in \mathtt{EqInc}(\nu/\lambda, |\mu|)$ and $\mathsf{x} \in \lambda$ be an inner corner. Then $\{i, i+1, \ldots, j\}$ forms a horizontal strip in V_k of the switch sequence of (T, x) if and only it does so in V_{k+1}.*

Proof. This quickly reduces to consideration of the possibilities in a 2×2 local piece of V_k. Then we proceed by straightforward case analysis using Lemma 4.53. $\square$

A label $\mathfrak{s} \in T$ is **special** if it is an edge label or lies in a $\star$-ed box. At most one $\mathfrak{s}$ appears in a column c. In column rectification order, each slide $\mathtt{KEqjdt}_{\mathsf{x}}$ for $\mathsf{x} \in c$ moves an $\mathfrak{s}$ in c at most one step North (and it remains in c). A special label $\mathfrak{s}$ in c **passes through** x if it occupies x at any point during c's rectification *and* initially $\mathfrak{s} \notin \mathsf{x}$. Let $\mathsf{x}_1, \ldots, \mathsf{x}_s$ be the boxes $\mathfrak{s}$ passes through and let $\mathsf{y}_1, \ldots, \mathsf{y}_t$ be the numerically labeled boxes East of x_s in the same row. Set $\mathtt{factor}_K(\mathfrak{s}) := 1 - \prod_{i=1}^{s} \hat{\beta}(\mathsf{x}_i) \prod_{j=1}^{t} \hat{\beta}(\mathsf{y}_j)$. If $\mathfrak{s}$ does not move during the rectification of c, then $\mathtt{factor}_K(\mathfrak{s}) := 0$. Now set $\mathtt{wt}_K(T) := \prod_{\mathfrak{s}} \mathtt{factor}_K(\mathfrak{s})$, where the product is over all special labels. Lastly, we define $\mathtt{sgn}(T) := (-1)^{|\mu| - \#\star\text{'s in } T - \#\text{labels in } T}$.

Let $\mu[1] = (1, 2, 3, \ldots, \mu_1)$, $\mu[2] = (\mu_1 + 1, \mu_1 + 2, \ldots, \mu_1 + \mu_2)$, etc. Let T_μ be the **superstandard tableau** of shape μ, i.e., row i is filled by $\mu[i]$. The following is the conjecture of [ThYo13]:

Theorem 4.2. $K^{\nu}_{\lambda,\mu} = \sum_T \mathtt{sgn}(T) \cdot \mathtt{wt}_K(T)$, *where the sum is over*

$$\mathcal{A}^{\nu}_{\lambda,\mu} := \{T \in \mathtt{EqInc}(\nu/\lambda, |\mu|) \colon \mathtt{KEqrect}(T) = T_\mu\}.$$

We will prove Theorem 4.2 (after some preparation) by connecting to Theorem 4.1.

Let $\mathcal{B}^{\nu}_{\lambda,\mu}$ be the set of all $T \in \mathtt{BallotGen}(\nu/\lambda)$ that have content μ. We need a **semistandardization map** $\Phi\colon \mathcal{A}^{\nu}_{\lambda,\mu} \to \mathcal{B}^{\nu}_{\lambda,\mu}$. Given $A \in \mathcal{A}^{\nu}_{\lambda,\mu}$, erase all $\star$'s and replace the labels $1, 2, \ldots, \mu_1$ with $1_1, 1_2, \ldots, 1_{\mu_1}$ respectively. Next, replace $\mu_1 + 1, \mu_1 + 2, \ldots, \mu_1 + \mu_2$ by $2_1, 2_2, \ldots, 2_{\mu_2}$ respectively, etc. The result is $\Phi(A)$. Note Φ is not bijective. Define a **standardization map** $\Psi\colon \mathcal{B}^{\nu}_{\lambda,\mu} \to \mathcal{A}^{\nu}_{\lambda,\mu}$ by reversing the above process in the obvious way; $\Psi(B)$ is $\star$-less.

Lemma 4.55. *For $B \in \mathcal{B}^{\nu}_{\lambda,\mu}$, $\Psi(B) \in \mathtt{EqInc}(\nu/\lambda, |\mu|)$.*

Proof. That $\Psi(B)$ has the desired shape and content is clear. Row strictness follows from (S.1), and column strictness from (S.2). $\square$

Lemma 4.56. *For $B \in \mathcal{B}^{\nu}_{\lambda,\mu}$ and for each i, $\mu[i]$ forms a horizontal strip in $\Psi(B)$ and also in each tableau of any switch sequence during the column rectification of $\Psi(B)$.*

Proof. By (S.2–4), the labels $i_1, \ldots, i_{\mu_i}$ form a horizontal strip of B. The claim for $\Psi(B)$ is then immediate by definition of Ψ. The claim for the tableaux of the switch sequences then follows by Lemma 4.54. $\square$

Lemma 4.57. *Let $B \in \mathcal{B}^{\nu}_{\lambda,\mu}$. Then*

(I) after column rectifying the eastmost j columns of $\Psi(B)$, there are no edge labels in these eastmost j columns; and

(II) while rectifying the next column, there is never an edge label north of a $\bullet$ and in the same column, in any tableau of any switch sequence.

Proof. (I): Suppose there were such an edge label $\ell \in \underline{\mathsf{x}}$ after rectifying the eastmost j columns. Then $\ell \in \underline{\mathsf{x}}$ in $\Psi(B)$, since rectification never adds a label to any edge. Suppose x is in the ith row from the top of $\Psi(B)$. Then since no label of B is too high, $\ell \in \mu[k]$ where $k \leq i$. Let the boxes North of $\underline{\mathsf{x}}$ and in the same column be $\mathsf{x}_1, \ldots, \mathsf{x}_i = \mathsf{x}$ from north to south. By Lemma 4.53(II), we have for each e that $\mathtt{label}(\mathsf{x}_e) \in \mu[f(e)]$ for some $f(e) \leq k$. But then by Lemma 4.56, $f : \{1, 2, \ldots, i\} \to \{1, 2, \ldots, k-1\}$ is injective, a contradiction.

(II): Let c be the column currently being rectified. For the columns East of c, the claim follows from part (I), noting that rectification never adds a label to any edge. For column c itself, the claim is vacuous if there is no $\bullet$ in c. If there is $\bullet \in c$, the claim follows from noting that every label of column c North of this $\bullet$ must have participated in some $\mathtt{switch}$ and that $\mathtt{switch}$ never outputs any edge labels. $\square$

An equivariant increasing tableau T is **ballot** if $\Phi(T)$ is ballot in the sense of Section 4.1.3. That is, for every $\widetilde{T}$ obtained by selecting one copy of each label in T, every initial segment of $\widetilde{T}$'s column reading word has, for each $i \geq 1$, at least as many labels from $\mu[i]$ as from $\mu[i+1]$. We extend this definition to tableaux with $\bullet$'s by ignoring the $\bullet$'s.

Lemma 4.58. *Let $B \in \mathcal{B}^{\nu}_{\lambda,\mu}$. Then $\Psi(B)$ is ballot, as is each tableau of any switch sequence during the column rectification of $\Psi(B)$.*

Proof. Let $A = \Psi(B)$. Since B is ballot and $\Phi(A) = B$, A is ballot by definition. Suppose that some V_q is ballot, but V_{q+1} is not. Then there exist i and a $\widetilde{V_{q+1}}$ with a ballotness violation between $\mu[i]$ and $\mu[i+1]$.

If $q \notin \mu[i] \cup \mu[i+1]$, then the labels of $\mu[i]$ and $\mu[i+1]$ appear in the same locations in V_q and V_{q+1}, contradicting that V_q is ballot.

If $q \in \mu[i+1]$, then no $\mu[i]$-label moves. For each $\ell \in \mu[i+1]$ appearing in $\widetilde{V_{q+1}}$, there is an ℓ east of that position in V_q. Hence we construct a nonballot $\widetilde{V_q}$ by choosing those corresponding ℓ's, the same labels from $\mu[i]$ as in $\widetilde{V_{q+1}}$, and all other labels arbitrarily. This contradicts that V_q is ballot.

Finally if $q \in \mu[i]$, then there is some x in column c of V_q with $\bullet \in \mathsf{x}$ and $q \in \mathsf{x}^{\rightarrow}$ such that the q moving into x violates ballotness in the columns East of c. That is, locally the `switch` is

$$V_q \supseteq \begin{array}{|c|c|}\hline a & b \\ \hline \bullet & q \\ \hline d & e \\ \hline \end{array} \mapsto \begin{array}{|c|c|}\hline a & b \\ \hline q & \bullet \\ \hline d & e \\ \hline \end{array} \subseteq V_{q+1} \quad \text{or} \quad V_q \supseteq \begin{array}{|c|c|}\hline a & \bullet \\ \hline \bullet & q \\ \hline d & e \\ \hline \end{array} \mapsto \begin{array}{|c|c|}\hline a & q \\ \hline q & \bullet \\ \hline d & e \\ \hline \end{array} \subseteq V_{q+1},$$

where the x is the left box of the second row. The $q \in \mathsf{x}^{\rightarrow}$ is Westmost in V_q, since otherwise the nonballotness of V_{q+1} contradicts that V_q is ballot. In particular, $q \neq d$. Hence by Lemma 4.53(III), $q < d$.

Since V_q is ballot but V_{q+1} is not, there is a $\bar{q} \in \mu[i+1]$ in $c^{\rightarrow}$ in V_q, and hence in V_{q+1}. By Lemma 4.53(II) applied to V_q, this $\bar{q}$ is below q in $c^{\rightarrow}$. By Lemma 4.57(I), there are no edge labels East of column c. So in fact e and hence d both exist. Indeed by Lemma 4.53(II) and Lemma 4.56, $e = \bar{q}$. By Lemma 4.56, q is the only label of $\mu[i]$ that appears in c in V_{q+1}. Hence $d \notin \mu[i]$. Thus by Lemma 4.53(I) applied to V_q, we conclude $d \in \mu[i+1]$. However this again contradicts that V_q is ballot. $\square$

For $A \in \texttt{EqInc}(\nu/\lambda, |\mu|)$, let $A^{(k)}$ be the "partial" tableau that is the column rectification of the eastmost k columns of A.

Lemma 4.59. *Let $B \in \mathcal{B}^{\nu}_{\lambda,\mu}$ and let $A = \Psi(B)$. For each i, the ith row of $A^{(k)}$ consists of a (possibly empty) final segment from $\mu[i]$.*

Proof. By Lemma 4.53(I, II), $A^{(k)}$ has strictly increasing rows and columns. By Lemma 4.56, the labels $\mu[i]$ form a horizontal strip in $A^{(k)}$ for each i; moreover the labels of $\mu[i]$ appearing in $A^{(k)}$ are a final segment of $\mu[i]$. By Lemma 4.57(I), there are no edge labels in $A^{(k)}$. By Lemma 4.58, $A^{(k)}$ is ballot. The lemma follows. $\square$

Corollary 4.3. *A rectifies to T_μ.*

Proof. Immediate from Lemma 4.59. $\square$

Proposition 4.19. *For $B \in \mathcal{B}^{\nu}_{\lambda,\mu}$, $\Psi(B) \in \mathcal{A}^{\nu}_{\lambda,\mu}$.*

Proof. By Lemma 4.55, $\Psi(B) \in \mathtt{EqInc}(\nu/\lambda, |\mu|)$. By Corollary 4.3, $\Psi(B)$ rectifies to T_μ. $\square$

Lemma 4.60. *For $A \in \mathcal{A}^{\nu}_{\lambda,\mu}$, $\mu[i]$ forms a horizontal strip in A and each A' in the column rectification of T.*

Proof. This is true for T_μ, and hence true for A and each A' by Lemma 4.54. $\square$

Lemma 4.61. *For $A \in \mathcal{A}^{\nu}_{\lambda,\mu}$, $\Phi(A)$ is semistandard.*

Proof. Row-strictness of A implies that $\Phi(A)$ satisfies (S.1). Since by Lemma 4.60, $\mu[i]$ is a horizontal strip in A for each i, (S.2)–(S.4) hold in $\Phi(A)$. $\square$

Lemma 4.62. *For $A \in \mathcal{A}^{\nu}_{\lambda,\mu}$, $\Phi(A)$ is ballot.*

Proof. Suppose $\Phi(A)$ is not ballot. Then by definition, A is not ballot. We assert that every tableau in every switch sequence in the column rectification of A is also *not* ballot, implying T_μ is not ballot, a contradiction.

Suppose V_ℓ is not ballot, but $V_{\ell+1}$ is. We derive a contradiction. Since V_ℓ is not ballot, we pick a nonballot $\widetilde{V}_\ell$. Suppose this nonballotness can be blamed on positions $\mathsf{a}_1, \ldots, \mathsf{a}_s$ containing labels of $\mu[i]$ and positions $\mathsf{b}_1, \ldots, \mathsf{b}_{s+1}$ containing labels of $\mu[i+1]$ (for some i). Suppose $\mathsf{a}_1, \ldots, \mathsf{a}_s$ and $\mathsf{b}_1, \ldots, \mathsf{b}_{s+1}$ are left to right in $\widetilde{V}_\ell$; no two a_j's (respectively b_j's) are in the same column by Lemma 4.60. We may assume b_1 is southwestmost among all these positions, say in column c and that among all offending choices of i and positions, we picked one so that c is eastmost.

Since $V_{\ell+1}$ is supposed ballot, there is a label $\ell \in \mu[i+1]$ in b_1 of V_ℓ that moved to column $c^{\leftarrow}$. Locally, the `switch` is $\begin{array}{|c|c|}\hline x & y \\ \hline \bullet & \ell \\ \hline\end{array} \mapsto \begin{array}{|c|c|}\hline x & y \\ \hline \ell & \bullet \\ \hline\end{array}$. By Lemma 4.60, $\mu[i+1]$ forms a horizontal strip in V_ℓ. Hence $x, y \notin \mu[i+1]$. Also, no label in column c is in $\mu[i]$ since otherwise we contradict that c is chosen eastmost. Now, there is some label $m \in \mu[i]$ above the $\bullet$ in column $c^{\leftarrow}$ of V_ℓ since $V_{\ell+1}$ is ballot. Using Lemma 4.57(II), it follows that $m = x$. Now, we have argued $y \notin \mu[i] \cup \mu[i+1]$. However, by Lemma 4.53(I, II) applied to V_ℓ, there are no other possibilities for y, a contradiction. $\square$

Proposition 4.20. *For $A \in \mathcal{A}^{\nu}_{\lambda,\mu}$, $\Phi(A) \in \mathcal{B}^{\nu}_{\lambda,\mu}$.*

Proof. By construction, $\Phi(A)$ is an edge-labeled genomic tableau of shape ν/λ and content μ. By Lemma 4.61, $\Phi(A)$ is semistandard. By Lemma 4.62, $\Phi(A)$ is ballot. Since A rectifies to T_μ, no label of $\Phi(A)$ is too high. $\square$

Given a label ℓ in $A \in \mathcal{A}^{\nu}_{\lambda,\mu}$, let $\Phi(\ell)$ be the corresponding label in $\Phi(A) \in \mathcal{B}^{\nu}_{\lambda,\mu}$. Recall the definitions of Section 4.1.4.

Lemma 4.63.

(I) If ℓ is an edge label, then $\mathtt{factor}_K(\ell) = \mathtt{edgefactor}(\Phi(\ell))$.

(II) If ℓ is in a $\star$-ed box, then $\mathtt{factor}_K(\ell) = 1 - \mathtt{boxfactor}(\Phi(\ell))$.

Proof. These follow from the definitions of the factors combined with Lemma 4.59. $\square$

Lemma 4.64. *If $B \in \mathcal{B}^{\nu}_{\lambda,\mu}$, then*

$$\mathtt{boxwt}(B) = \sum_{A \in \Phi^{-1}(B)} (-1)^{\#\star\text{'s in } A} \prod_{\text{special box label } \ell \text{ of } A} \mathtt{factor}_K(\ell).$$

Proof. A box x is productive in B if and only if it may be $\star$-ed in $\Psi(B)$. We are done by Lemma 4.63(II) and the "inclusion-exclusion" identity $\sum_{S \subset [N]} (-1)^{|S|} \prod_{s \in S} (1 - z_s) = z_1 z_2 \cdots z_N$. $\square$

Proof of Theorem 4.2. Recall Theorem 4.2 asserts $K^{\nu}_{\lambda,\mu} = \sum_{A \in \mathcal{A}^{\nu}_{\lambda,\mu}} \mathtt{sgn}(A)\mathtt{wt}_K(A)$. To see this, observe that by Propositions 4.19 and 4.20,

$$\begin{aligned}
\sum_{A \in \mathcal{A}^{\nu}_{\lambda,\mu}} \mathtt{sgn}(A)\mathtt{wt}_K(A) &= \sum_{B \in \mathcal{B}^{\nu}_{\lambda,\mu}} \sum_{A \in \Phi^{-1}(B)} \mathtt{sgn}(A)\mathtt{wt}_K(A) \\
&= \sum_{B \in \mathcal{B}^{\nu}_{\lambda,\mu}} \sum_{A \in \Phi^{-1}(B)} (-1)^{|\mu| - \#\star\text{'s in } A - \#\text{labels in } A} \prod_{\text{edge label } \ell \text{ of } A} \mathtt{factor}_K(\ell) \prod_{\text{special box label } \ell \text{ of } A} \mathtt{factor}_K(\ell) \\
&= \sum_{B \in \mathcal{B}^{\nu}_{\lambda,\mu}} \sum_{A \in \Phi^{-1}(B)} (-1)^{|\mu| - \#\text{labels in } A} \left(\prod_{\text{edge label } \ell \text{ of } A} \mathtt{factor}_K(\ell) \right) (-1)^{\#\star\text{'s in } A} \prod_{\text{special box label } \ell \text{ of } A} \mathtt{factor}_K(\ell).
\end{aligned}$$

The number of labels of A equals the number of labels of B for any $A \in \Phi^{-1}(B)$. Combining this with Lemma 4.63(I) shows the previous expression equals

$$= \sum_{B \in \mathcal{B}^{\nu}_{\lambda,\mu}} (-1)^{|\mu| - \#\text{labels in } B} \left(\prod_{\text{edge label } \ell \text{ of } B} \mathtt{edgefactor}(\ell) \right) \sum_{A \in \Phi^{-1}(B)} (-1)^{\#\star\text{'s in } A} \prod_{\text{special box label } \ell \text{ of } A} \mathtt{factor}_K(\ell).$$

By Lemma 4.64, this equals

$$= \sum_{B \in \mathcal{B}^{\nu}_{\lambda,\mu}} (-1)^{|\mu| - \#\text{labels in } B} \mathtt{edgewt}(B)\mathtt{boxwt}(B) := L^{\nu}_{\lambda,\mu}.$$

Since by Theorem 4.1, $L^{\nu}_{\lambda,\mu} = K^{\nu}_{\lambda,\mu}$, we are done. $\square$

Chapter 5

The Knutson-Vakil puzzle conjecture

This chapter derives from joint work with A. Yong [PeYo15c].

5.1 Introduction

A. Knutson–R. Vakil [CoVa05, §5] conjectured a combinatorial rule for the structure coefficients of the torus-equivariant K-theory ring of a Grassmannian. The structure coefficients are with respect to the basis of Schubert structure sheaves. Their rule extends *puzzles*, combinatorial objects founded in work of A. Knutson-T. Tao [KnTa03] and in their collaboration with C. Woodward [KnTaWo04]. The various puzzle rules play a prominent role in modern Schubert calculus, see e.g., [BuKrTa03, Va06, CoVa05], recent developments [Kn10, KnPu11, BKPT14, Bu15] and the references therein.

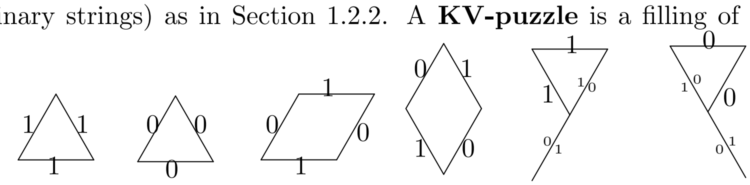
Here we use the results of Chapter 4 to prove a mild correction of the puzzle conjecture.

5.1.1 The puzzle conjecture

Recall that the structure coefficients $K_{\lambda,\mu}^{\nu} \in K_{\mathsf{T}}(\mathrm{pt})$ in the torus-equivariant K-theory of the Grassmannian are defined by

$$[\mathcal{O}_{X_\lambda}] \cdot [\mathcal{O}_{X_\mu}] = \sum_{\nu} K_{\lambda,\mu}^{\nu} [\mathcal{O}_{X_\nu}].$$

Consider the n-length equilateral triangle oriented as Δ. Let $\Delta_{\lambda,\mu,\nu}$ be Δ with the boundary given by λ, μ, ν (thought of as binary strings) as in Section 1.2.2. A **KV-puzzle** is a filling of $\Delta_{\lambda,\mu,\nu}$ with the following **puzzle pieces**:

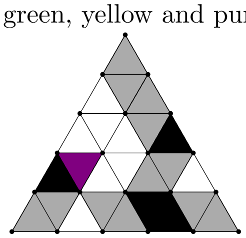


The double-labeled edges are **gashed**. A **filling** requires that the common (non-gashed) edges of adjacent puzzle pieces share the same label. Two gashed edges may not be overlayed. The pieces on either side of a gash must have the indicated labels. The first three may be rotated but the fourth (**equivariant piece**) may not [KnTa03]. We call the remainder **KV-pieces**; these may not be rotated. The fifth piece may only be placed if the equivariant piece is attached to its left. There is a "nonlocal" requirement [CoVa05, §5] for using the sixth piece: it "may only be placed (when completing the puzzle from top to bottom and left to right as usual) if the edges to its right are a (possibly empty) series of horizontal 0's followed by a 1."

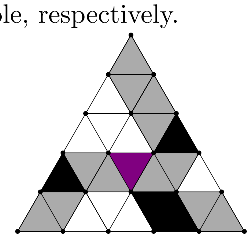
The **weight** $\mathrm{wt}(P)$ of a **KV-puzzle** P is a product of the following factors. Each KV-piece contributes a factor of -1. For each equivariant piece one draws a $\searrow$ diagonal arrow from the center of the piece to the ν-side of Δ; let a be the unit segment of the ν-boundary, as counted from the right. Similarly one determines b by drawing a $\swarrow$ antidiagonal arrow. The equivariant piece contributes a factor of $1 - \frac{t_a}{t_b}$.

Conjecture 5.1 (The Knutson-Vakil puzzle conjecture)**.** *$K_{\lambda,\mu}^{\nu} = \sum_P \mathrm{wt}(P)$ where the sum is over all KV-puzzles of $\Delta_{\lambda,\mu,\nu}$.*

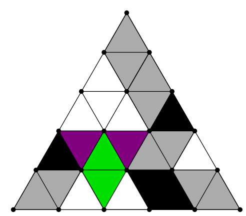
We consider the structure coefficient $K_{01001,00101}^{10010}$ for $\mathrm{Gr}_2(\mathbb{C}^5)$. The reader can check that there are six KV-puzzles $P_1, P_2, \ldots, P_6$ with the indicated weights. Henceforth, we color-code the six puzzle pieces black, white, grey, green, yellow and purple, respectively.



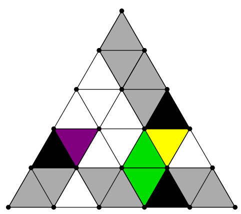
$\mathrm{wt}(P_1) = -1$



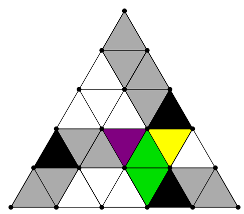
$\mathrm{wt}(P_2) = -1$



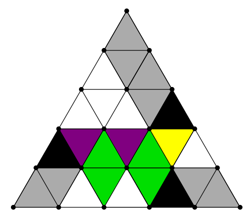
$\mathrm{wt}(P_3) = (-1)^2(1 - \frac{t_3}{t_4})$



$\mathrm{wt}(P_4) = (-1)^2(1 - \frac{t_2}{t_3})$



$\mathrm{wt}(P_5) = (-1)^2(1 - \frac{t_2}{t_3})$



$\mathrm{wt}(P_6) = (-1)^3(1 - \frac{t_3}{t_4})(1 - \frac{t_2}{t_3})$

Using double Grothendieck polynomials [LaSc82] (see also [FuLa94] and references therein), one computes $K_{01001,00101}^{10010} = -\frac{t_2}{t_4} = \mathrm{wt}(P_2) + \mathrm{wt}(P_3) + \mathrm{wt}(P_5) + \mathrm{wt}(P_6)$. This gives a counterexample to Conjecture 5.1. Actually, this subset of four puzzles witnesses the rule of Theorem 5.1 below.

5.1.2 A modified puzzle rule

We define a **modified KV-puzzle** to be a KV-puzzle with the nonlocal condition on the second KV-piece replaced by the requirement that the second KV-piece only appears in the combination pieces  or .

Theorem 5.1. *$K_{\lambda,\mu}^{\nu} = \sum_P \mathrm{wt}(P)$ where the sum is over all modified KV-puzzles of $\Delta_{\lambda,\mu,\nu}$.*

We have a few remarks. First, the rule of Theorem 5.1 is "positive" in the sense of D. Anderson-S. Griffeth-E. Miller's [AnGrMi11]; cf. the discussion in Section 4.1.4. Second, it is a natural objective to interpret Theorem 5.1 via geometric degeneration; see [CoVa05, Kn10]. Third, the author has found a tableau formulation similar to that of Chapter 4 to complement the puzzle rule of [Kn10] for the *different* Schubert calculus problem in $K_{\mathsf{T}}(\mathsf{X})$ of multiplying a class of a Schubert variety by that of an opposite Schubert variety; further discussion may appear elsewhere.

To prove Theorem 5.1, we first give a variant of the main theorem of Chapter 4; see Section 5.2. In Section 5.3, we then give a weight-preserving bijection between modified KV-puzzles and the objects of the rule of Section 5.2.

5.2 A tableau rule for $K_{\lambda,\mu}^{\nu}$

We need to briefly recall some definitions from Chapter 4; there the Schubert varieties X_λ are indexed by Young diagrams λ contained in a $k \times (n-k)$ rectangle. An **edge-labeled genomic tableau** is a filling of the boxes and horizontal edges of a skew diagram ν/λ with subscripted labels i_j, where i is a positive integer and the j's that appear for each i form an initial interval of positive integers. Each box of ν/λ contains one label, whereas the horizontal edges weakly between the southern border of λ and the northern border of ν are filled by (possibly empty) sets of labels. A genomic edge-labeled tableau T is **semistandard** if

(S.1) the box labels of each row strictly increase lexicographically from left to right;

(S.2) ignoring subscripts, each label is strictly less than any label strictly south in its column;

(S.3) ignoring subscripts, the labels appearing on a given edge are distinct;

(S.4) if i_j appears strictly west of i_k, then $j \leq k$.

Index the rows of ν from the top starting at 1. We say a label i_j is **too high** if it appears weakly above the north edge of row i. We refer to the collection of all i_j's (for fixed i, j) as a **gene**. The **content** of T is the composition $(\alpha_1, \alpha_2, \dots)$ where α_i is greatest so that i_{α_i} is a gene of T.

Recall from Chapter 1 that a semistandard tableau S is *ballot* if, reading the labels down columns from right to left, we obtain a word W with the following property: For each i, every initial segment of W contains at least as many i's as $(i+1)$'s. Given an edge-labeled genomic tableau T, choose one label from each gene and delete all others; now delete all subscripts. We say T is **ballot** if, regardless of our choices from genes, the resulting tableau (possibly containing holes) is necessarily ballot in the above classical sense. (In the case of multiple labels on a edge, read them from least to greatest.)

We now diverge slightly from the treatment of Chapter 4, borrowing notation from [ThYo13]. Given a box x in an edge-labeled genomic tableau T, we say x is **starrable** if it contains i_j, is in row $> i$, and i_{j+1} is not a box label to its immediate right. Let $\texttt{StarBallotGen}_\mu(\nu/\lambda)$ be the set of all ballot semistandard edge-labeled genomic tableaux of shape ν/λ and content μ with no label too high, where the label of each starrable box may freely be marked by $\star$ or not. The tableau T illustrated in Figure 5.2 is an element of $\texttt{StarBallotGen}_{(10,5,3)}((15,8,5)/(12,2,1))$. There are three starrable boxes in T, in only one of which the label has been starred.

Let $\mathsf{Man}(\mathsf{x})$ denote the length of any $\{\uparrow, \rightarrow\}$-lattice path from the southwest corner of $k \times (n-k)$ to the northwest corner of x. For x in row r containing $i_j^\star$, set $\texttt{starfactor}(\mathsf{x}) := 1 - \frac{t_{\mathsf{Man}(\mathsf{x})+1}}{t_{r-i+\mu_i-j+1+\mathsf{Man}(\mathsf{x})}}$. For an edge label $\ell = i_j$ in the southern edge of x in row r, set $\texttt{edgefactor} := 1 - \frac{t_{\mathsf{Man}(\mathsf{x})}}{t_{r-i+\mu_i-j+1+\mathsf{Man}(\mathsf{x})}}$. Finally for $T \in \texttt{StarBallotGen}_\mu(\nu/\lambda)$, define

$$\widehat{\mathrm{wt}}(T) := (-1)^{\hat{d}(T)} \times \prod_{\ell} \texttt{edgefactor}(\ell) \times \prod_{\mathsf{x}} \texttt{starfactor}(\mathsf{x});$$

here the products are respectively over edge labels ℓ and boxes x containing starred labels, while $\hat{d}(T) := \#(\text{labels in } T) + \#(\star\text{'s in } T) - |\mu|$. Let

$$\hat{L}_{\lambda,\mu}^{\nu} := \sum_T \widehat{\mathrm{wt}}(T),$$

where the sum is over all $T \in \texttt{StarBallotGen}_\mu(\nu/\lambda)$.

We need a reformulation of Theorem 4.1; the proof is a simple application of the "inclusion-exclusion" identity $\prod_{i\in[m]} a_i = \sum_{S\subseteq[m]} (-1)^{|S|} \prod_{i\in S}(1-a_i)$.

Theorem 5.2. $K_{\lambda,\mu}^{\nu} = \hat{L}_{\lambda,\mu}^{\nu}$. $\square$

Example 5.1. Let $k = 2, n = 5$ and $\lambda = (2,0)$, $\mu = (1,0)$ and $\nu = (3,1)$. The four tableaux contributing to $\hat{L}_{\lambda,\mu}^{\nu}$ are

$\widehat{\mathrm{wt}}(T_2) = -1$ $\quad\widehat{\mathrm{wt}}(T_3) = (-1)^2(1 - \frac{t_3}{t_4})$ $\quad\widehat{\mathrm{wt}}(T_5) = (-1)^2(1 - \frac{t_2}{t_3})$ $\quad\widehat{\mathrm{wt}}(T_6) = (-1)^3(1 - \frac{t_3}{t_4})(1 - \frac{t_2}{t_3})$

Our indexing of these tableaux alludes to the precise connection to the four puzzles P_2, P_3, P_5 and P_6 of Section 5.1.1, as explained in the next section. $\diamondsuit$

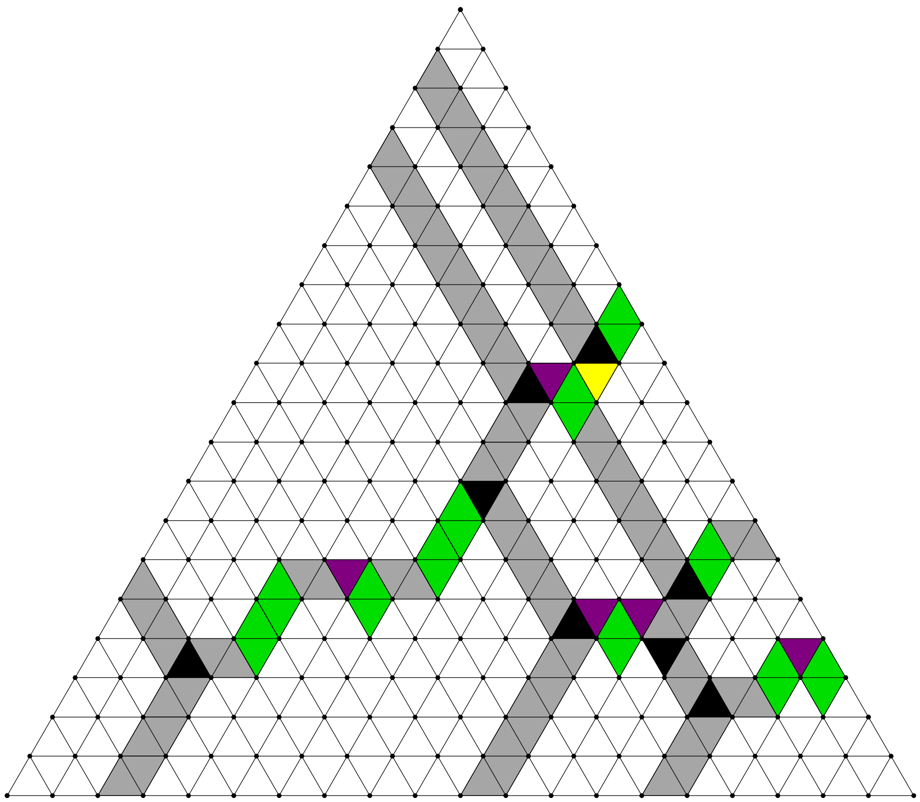


Figure 5.1: A "generic" modified KV-puzzle P ($k = 3, n = 20$).

5.3 Proof of Theorem 5.1: Bijecting the tableau and puzzle rules

5.3.1 Description of the bijection

To relate the modifed KV-puzzle rule of Theorem 5.1 with the tableau rule of Theorem 5.2, we give a variant of T. Tao's "proof without words" [Va06] (and its modification by K. Purbhoo [Pu08]) from Section 1.2.4 that bijects cohomological puzzles (using the first three pieces) and a tableau Littlewood-Richardson rule. An extension of this proof for equivariant puzzles (i.e., fillings that additionally use the equivariant piece) was given by V. Kreiman [Kre10]; we also encorporate elements of his bijection in our analysis.

Figure 5.1 gives a "generic" example of a (modified) KV-puzzle P. We will define a **track** π_i from the ith 1 (from the left) on the ν-boundary of $\Delta_{\lambda,\mu,\nu}$ to the ith 1 (from the top) on the μ-boundary. To do this, we describe the **flow** through the (oriented, non-KV) puzzle pieces that use a 1 and four **combination pieces** (possible ways one can use the KV-pieces under the rules for a modified KV-puzzle):

(A.1)  : go northeast

(A.2)  : go north then northeast

(A.3) : go left to right

(A.4) : go northeast

(A.5) : go in through the north \ of the purple triangle, come out northeast from the purple gash into the southwest \ of the green rhombus and pass northeast through this rhombus

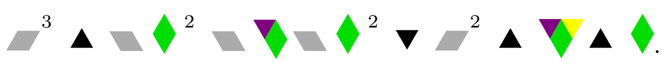
(A.6) : come in through the left side and out the top

(A.7) : come in through the southwest side of the green rhombus and out the top of the yellow triangle

(A.8) : come in through the north \ of the purple triangle, out the gash into the \ of the , out the — of into the bottom of the grey rhombus and out its top

(A.9) : come into the north \ of the purple triangle, out the gash into the southwest \ of the green rhombus and out the northeast \ into the left side of the yellow triangle and then go out the — of that triangle.

Thinking of the (combination) pieces in (A.1)–(A.9) as letters of an alphabet, we can encode the northmost track in P (from Figure 5.1) as the word

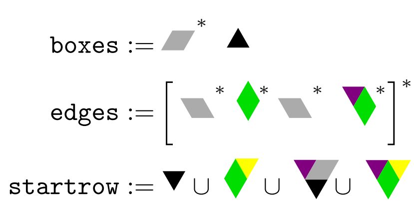


Recall, if κ is a letter/word in some alphabet, then the **Kleene star** is $\kappa^* := \{\emptyset, \kappa, \kappa\kappa, \ldots\}$.

Proposition 5.1 (Decomposition of π_i). *The list of (combination) pieces that appear in π_i, as read from southwest to northeast, is a word from the following formal grammar:*

$$\texttt{boxes}[\texttt{edges startrow boxes}]^* \ \texttt{edges} \tag{5.1}$$

where



Proof. By inspection of the rules for modified KV-puzzles. $\square$

The remaining filling of the puzzle is forced, which we explain in two steps. First there is the **NWray** of each $\blacktriangle$, i.e., the (possibly empty) path of upward pointing grey rhombi  growing from the / of this $\blacktriangle$.

Lemma 5.1. *The NWray of* $\blacktriangle$ *ends either at the* λ*-boundary of* $\Delta_{\lambda,\mu,\nu}$ *or with a piece from* `startrow`. *In the latter case, the shared edge is the south-then-eastmost edge of the (combination) piece.*

Proof. The north / of  is labeled 1. By inspection, the only (combination) pieces that can connect to this edge are  and those from `startrow` (at the stated shared edge). □

Second, pieces of the puzzle not in a track or NWray are 0-triangles (depicted white).

We correspond Young diagrams to $\{0,1\}$-sequences. Trace the $\{\leftarrow,\downarrow\}$-lattice path defined by the southern boundary of λ (as placed in the northwest corner of $k\times(n-k)$) starting from the northeast corner of $k\times(n-k)$ towards the southeast corner of $k\times(n-k)$. Record each $\leftarrow$ step with "0" and each $\downarrow$ step with "1".

We now convert P into (we claim) an edge-labeled starred genomic tableau $T := \phi(P)$ of shape ν/λ with content μ. The placement of the labels of family i is governed by the decomposition (5.1) of π_i. The initial sequence of k 's indicates the leftmost possible placement of box labels $i_{\mu_i}, i_{\mu_i-1},\ldots,i_{\mu_i-k+1}$ (from right to left) in row i of T. Continuing to read the sequence, one interprets:

(B.1)  $\leftrightarrow$ "place (unstarred) box label of next smaller gene"

(B.2) $\blacktriangle$ $\leftrightarrow$ "end placing box labels in current row"

(B.3)  $\leftrightarrow$ "skip to the next column left"

(B.4)  $\leftrightarrow$ "place lower edge label of the next smaller gene"

(B.5)  $\leftrightarrow$ "place lower edge label of the same gene last used"

(B.6) $\blacktriangledown$ $\leftrightarrow$ "go to next row"

(B.7)  $\leftrightarrow$ "go to next row and place $\star$-ed box label of the next smaller gene"

(B.8)  $\leftrightarrow$ "go to next row and place (unstarred) box label of the same gene last used"

(B.9)  $\leftrightarrow$ "go to next row and place $\star$-ed box label of the same gene last used".

Applying ϕ to the puzzle P of Figure 5.1 gives the tableau T of Figure 5.2. Here, $\lambda = 0^5 10^{10} 1010$, corresponding to the inner shape $(12,2,1)$ (which is shaded in grey). Since $\mu = 0^7 10^5 10^2 10^3$, the content of T is $(10,5,3)$. Finally, since $\nu = 0^2 10^7 10^3 10^5$, the outer shape of T is $(15,8,5)$. As another example, ϕ connects the puzzles P_2, P_3, P_5 and P_6 of Section 5.1 respectively with the tableaux T_2, T_3, T_5 and T_6 of Example 5.1.

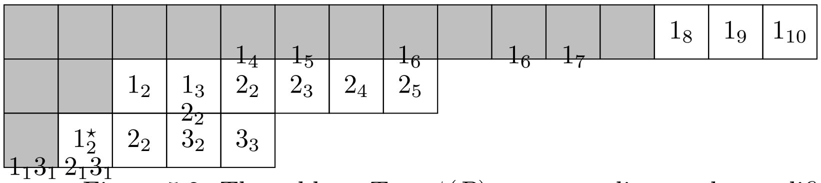


Figure 5.2: The tableau $T := \phi(P)$ corresponding to the modified KV-puzzle P of Figure 5.1.

Conversely, given $T \in \texttt{StarBallotGen}_\mu(\nu/\lambda)$, construct a word σ_i using the correspondences (B.1)–(B.9), for $1 \leq i \leq k$. That is, read the occurrences (possibly zero) of family i in T from right to left and from the ith row down. (Note about (B.6) in the degenerate case that there are no labels of family i in the next row: use ▲ after reading the leftmost column in ν/λ that does not have any labels of family $< i$.)

Lemma 5.2. *Each σ_i is of the form (5.1).*

Proof. Since T is semistandard, in any row, all box labels of family i are contiguous and strictly right of any (lower) edge labels of that family on that row. The lemma follows. □

We describe a claimed filling $P := \psi(T)$ of $\Delta_{\lambda,\mu,\nu}$. There are k 1's on each side of $\Delta_{\lambda,\mu,\nu}$; to the ith 1 from the left on the ν-boundary of $\Delta_{\lambda,\mu,\nu}$, place puzzle pieces in the order indicated by σ_i. That is attach the next (combination) piece using the northmost \ edge on its west side, if it exists. Otherwise attach at the piece's unique southern edge. We attach at the unique — or \ edge of the thus far constructed track. Fill in the order $i = 1, 2, 3, \ldots, k$. Now stack ◆'s northwest of each ▲ until (we claim) it reaches one of the pieces of (A.6)–(A.9) at the southmost / edge, or the λ-boundary of $\Delta_{\lambda,\mu,\nu}$. Complete using white triangles.

Sections 5.3.2–5.3.4 prove ϕ and ψ are well-defined and weight-preserving maps between

$$\mathcal{P} := \{\text{modified KV-puzzles of } \Delta_{\lambda,\mu,\nu}\} \text{ and } \mathcal{T} := \texttt{StarBallotGen}_\mu(\nu/\lambda).$$

Semistandardness (specifically (S.4)) implies that knowing the locations of labels of family i, and which labels are repeated or $\star$-ed, uniquely determines the gene(s) in each location. The injectivity of ϕ and ψ is easy from this. Moreover, by construction (cf. Lemma 5.2), the two maps are mutually reversing. Thus, Theorem 5.1 follows from Theorem 5.2. □

5.3.2 Well-definedness of $\phi : \mathcal{P} \to \mathcal{T}$

Let $P \in \mathcal{P}$ be a modified KV-puzzle for $\Delta_{\lambda,\mu,\nu}$. For the track π_i, let $▲_{i,j}$ refer to the jth black triangle seen along π_i (as read from southwest to northeast). Let $\mathbb{S}$ denote any of the (combination) pieces that appear in `startrow`. Similarly, we let $\mathbb{S}_{i,j}$ be the jth such piece on π_i.

Figure 5.1 illustrates the "ragged honeycomb" structure of modified KV-puzzles. To formalize this, first note by inspection that the π_i do not intersect. Second we have:

Claim 5.1. *There is a bijective correspondence between the* 1*'s on the* λ*-boundary and the* $\blacktriangle$*'s in* π_1*. Specifically, the* j*th* 1 *on the* λ*-boundary is the terminus of the NWray of* $\blacktriangle_{1,j}$*. Similarly, there is a bijective correspondence between* $\blacktriangle_{i+1,j}$ *and* $\mathbb{S}_{i,j}$ *in that the former's NWray terminates at the southmost* / *edge of the latter.*

Proof. Follows by combining Proposition 5.1 and Lemma 5.1. $\square$

Define $\mathcal{L}_i$ to be the **left sequence** of π_i: Start at the southwest corner of $\Delta_{\lambda,\mu,\nu}$ and read the $\{\rightarrow, \nearrow\}$-lattice path that starts along the ν-boundary and travels up the left boundary of π_i. The $\{0,1\}$-sequence records the labels of the edges seen. Similarly, define $\mathcal{R}_i$ to be the **right sequence** of π_i by travelling up the right side of π_i *but only reading the* $\rightarrow$ *and* $\nearrow$ *edges.* (In Figure 5.1, $\mathcal{L}_1 = 0^6 1 0^{10} 1010 (= \lambda)$ while $\mathcal{R}_1 = 0^2 1 0^{11} 1 0^2 1 0^2$.)

In view of Claim 5.1, the following is "graphically" clear by considering the n diagonal strips through P:

Claim 5.2. $\mathcal{L}_1 = \lambda$*,* $\mathcal{L}_{i+1} = \mathcal{R}_i$ *for* $1 \leq i \leq k-1$*, and* $R_k = \nu$*.*

Let $T^{(i)}$ be the tableau after adding labels of family $1, 2, \ldots, i$. We declare $T^{(0)}$ to be the empty tableau of shape λ/λ. Let $\nu^{(i)}$ be the outer shape of $T^{(i)}$ (interpreted as the $\{0,1\}$-sequence for its lattice path).

Claim 5.3. $\mathcal{L}_i = \nu^{(i-1)}$ *and* $\mathcal{R}_i = \nu^{(i)}$*.*

Proof. Both assertions follow by inspection of the correspondences (B.1)–(B.9). (Also the second follows from the first, by Claim 5.2.) $\square$

It is straightforward from Claims 5.2 and 5.3 that $T = \phi(P)$ is semistandard in the sense of (S.1)–(S.4) of Section 5.2. By Proposition 5.1, no label of T is $\star$-ed unless it is the rightmost box label of its family in a row ($> i$). Since labels of family i are placed in the boxes of row i or below, no label of T can be too high. Since $\mathcal{R}_k = \nu$, the shape of T is ν/λ.

Claim 5.4. T *has content* μ*.*

Proof. Let β be the content of T. Then β_i is the number of (distinct) genes of family i that appear in T, which, in terms of P, is the number of  and  in π_i minus the number of purple KV-pieces  in π_i. Thus the vertical height h_i of π_i (at its right end*point*) is $\beta_i + \#\blacktriangle$. However, h_i equals the number of line segments strictly below the ith 1 on the μ-boundary; i.e., $h_i = n - i - (n - k - \mu_i) = (k-i) + \mu_i$. By Claims 5.1 and 5.1, $\#\blacktriangle = (k-i)$, hence $\beta = \mu$, as desired. $\square$

Finally,

Claim 5.5. *T is ballot.*

Proof. The **height** of a (combination) piece is the distance of any northernmost point to the ν-boundary as measured along any (anti)diagonal. The height h of $\blacktriangle_{i+1,j}$ equals the number of ▰'s, $\blacktriangle$'s and $\blacklozenge$'s that appear weakly before $\blacktriangle_{i+1,j}$ in π_{i+1} minus the number of $\blacktriangledown$'s before $\blacktriangle_{i+1,j}$ in π_{i+1}. There are exactly j such $\blacktriangle$'s, while the number of ▰'s and $\blacklozenge$'s is the number of labels used and the number of $\blacktriangledown$'s is the number of these labels that are repeats. That is $h = j + (\#\text{distinct genes of family } i+1 \text{ in row } j+1 \text{ and above})$ where we do *not* include labels on the lower edges of row $j+1$. Similarly, the height h' of $\mathbb{S}_{i,j}$ is given by $h' = j + (\#\text{distinct genes of family } i \text{ in row } j \text{ and above})$ where we include labels on the lower edges of row j. By Claim 5.1, $h' - h \geq 0$ and so ballotness follows. □

5.3.3 Well-definedness of $\psi : \mathcal{T} \to \mathcal{P}$

Let $T \in \mathcal{T}$ be a starred ballot genomic tableau of shape ν/λ and content μ. Let $P = \psi(T)$. Let π_i be the track associated to σ_i. As in Section 5.3.2, we define the $\{0,1\}$-sequences $\mathcal{L}_i$ and $\mathcal{R}_i$ associated to π_i. Here, $T^{(i)}$ is *defined* as the subtableau of T using the labels of family $1, 2, \ldots, i$. Hence $T^{(0)}$ is the empty tableau of shape λ/λ. Let $\nu^{(i)}$ be the outer shape of $T^{(i)}$.

Claim 5.6 (cf. Claim 5.3)**.** *$\mathcal{L}_i = \nu^{(i-1)}$ and $\mathcal{R}_i = \nu^{(i)}$.*

Proof. By inspection of the correspondences (B.1)-(B.9). □

By the lattice path definition, each $\nu^{(j)}$ is a length n sequence. So π_i is a track that (by definition) starts at the south border of $\Delta_{\lambda,\mu,\nu}$ and terminates at the east border of $\Delta_{\lambda,\mu,\nu}$. Also, define $\blacktriangle_{i,j}$ and $\mathbb{S}_{i,j}$ as before.

Claim 5.7. *$\mathbb{S}_{i,j}$ and $\blacktriangle_{i+1,j}$ share a diagonal with the former strictly northwest of the latter.*

Proof. The 1's in $\mathcal{L}_{i+1}$ result solely from the $\blacktriangle$'s appearing in π_{i+1} while the 1's appearing in $\mathcal{R}_i$ result solely from the $\mathbb{S}$ (combination) pieces. Thus, that the pieces share a diagonal follows from Claim 5.6. For the "northwest" assertion, repeat Claim 5.5's argument but reverse the logic of the final sentence: since by assumption T is ballot, it follows that $h' \geq h$. □

Since Claims 5.6 and 5.7 combine to imply that the π_i are non-intersecting, attaching NWrays to each $\blacktriangle$ and filling with white 0-triangles as prescribed, we have a filling P of $\Delta_{\widetilde{\lambda},\widetilde{\mu},\nu}$ satisfying the modified KV-puzzle rule. It remains to check the λ- and μ-boundaries.

Claim 5.8. $\widetilde{\lambda} = \lambda$.

Proof. Graphically, $\widetilde{\lambda} = \mathcal{L}_1$. On the other hand, by Claim 5.6, we know that $\mathcal{L}_1 = \lambda$. $\square$

Claim 5.9. $\widetilde{\mu} = \mu$.

Proof. This is given by reversing the logic of the proof of Claim 5.4; here we are given the content of T and are determining the heights of the tracks π_i. $\square$

5.3.4 Weight-preservation

We wish to show:

Claim 5.10. *ϕ is weight-preserving, i.e.,* $\mathrm{wt}(P) = \widehat{\mathrm{wt}}(T)$.

Proof. The ± 1 sign associated to P and T is the same since each usage of a KV-piece in P corresponds to a $\star$-ed label or a repetition of a gene in T.

Now consider the weight $1 - \frac{t_a}{t_b}$ assigned to an equivariant piece p in P. Here a is the ordinal (counted from the right) of the line segment s on the ν-boundary hit by the diagonal "right leg" emanating from p. Then b equals $a+h-1$ where h is the height of the piece p. Suppose p lies in track π_i, and corresponds either to i_j on the lower edge of box x in row r or to $i_j^\star \in \mathsf{x}$ in row r. Consider the edge e on the left boundary of π_i that is on the same diagonal as s. If p is not attached to the first KV-piece, so it corresponds to an edge label, then e's index from the right in the string $\mathcal{L}_i$ equals $\mathsf{Man}(\mathsf{x})$. Otherwise e's index from the right in the string $\mathcal{L}_i$ equals $\mathsf{Man}(\mathsf{x}) + 1$.

Note that h equals the number of $\blacksquare$'s, $\blacktriangle$'s and $\blacklozenge$'s appearing weakly before p in π_i minus the number of $\blacktriangledown$'s appearing before p in π_i. The number of such $\blacktriangle$'s equals $1 + r - i$ if p corresponds to an edge label and equals $r - i$ if p corresponds to a starred label. The number of such $\blacksquare$'s and $\blacklozenge$'s minus the number of such $\blacktriangledown$'s equals $\mu_i - j + 1$. Weight preservation follows. $\square$

Chapter 6

Isotropic Grassmannians

This chapter derives from joint work with A. Yong [PeYo15a, PeYo16].

6.1 Shifted genomic tableaux

The **shifted diagram** of a strictly decreasing partition is given by taking the ordinary Young diagram and indenting row i (from the top) $i-1$ positions to the right. Let

$$\mathcal{D} := \{1' < 1 < 2' < 2 < \cdots\}.$$

A P**-tableau** is a filling of shifted shape ν/λ with entries from $\mathcal{D}$ such that:

(P.1) rows and columns weakly increase (left to right, top to bottom);

(P.2) each unprimed letter appears at most once in any column;

(P.3) each primed letter appears at most once in any row; and

(P.4) every primed letter k' has an unprimed k southwest of it.

The *Schur P-function* P_λ is a generating function over these tableaux (for more history and development of these functions, see e.g., [HoHu92] or[Ste89]).

Example 6.1. $\begin{array}{|c|c|c|}\hline 1 & 2' & 3 \\ \hline \multicolumn{1}{c|}{} & 2 & \multicolumn{1}{c}{} \\ \cline{2-2}\end{array}$ is a P-tableau of shape $\lambda = (3,1)$. The tableau $\begin{array}{|c|c|c|c|}\hline 2 & 3' & 4 & 4 \\ \hline \multicolumn{1}{c|}{} & 3' & 6 & \multicolumn{1}{|c}{} \\ \cline{2-3} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 7 & \multicolumn{1}{|c}{} \\ \cline{3-3}\end{array}$ is not a P-tableau because it violates both (P.3) and (P.4). However, if the lower $3'$ changes to 3, the result is a P-tableau. $\diamondsuit$

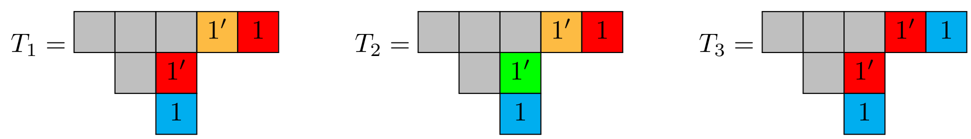
For $\alpha \in \mathcal{D}$, write $|\alpha| = k$ if $\alpha \in \{k', k\}$. We use initial letters of the Greek alphabet $(\alpha, \beta, \gamma, \ldots)$ for elements of $\mathcal{D}$, reserving Roman letters for elements of $\mathbb{Z}$.

For fixed $k \in \mathbb{Z}_{\geq 0}$, place a total order $\prec$ on those boxes with entry k' in top to bottom order and on those boxes with entry k using left to right order; declare the boxes containing k' to precede those containing k. A **gene** (of family k) in a P-tableau T is a set $\mathcal{G}$ of boxes of T such that:

- each entry in $\mathcal{G}$ is k' or k;
- the boxes of $\mathcal{G}$ are consecutive in the $\prec$-order; and
- no two boxes of $\mathcal{G}$ appear in the same row or the same column.

We write $\texttt{family}(\mathcal{G}) = k$.

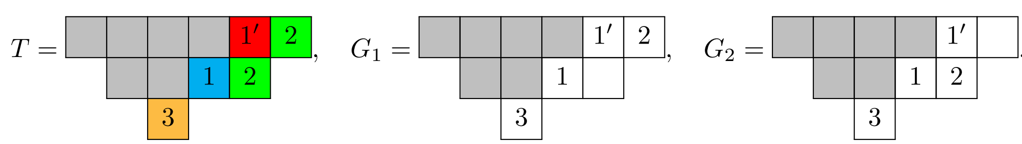
Example 6.2. Consider the following three colorings of the same P-tableau:



The red boxes in T_1 do not form a gene, since they are not consecutive in $\prec$-order (in view of the blue 1). In T_2 and T_3, the boxes of each color form valid genes. $\diamondsuit$

A **genomic P-tableau** is a P-tableau T together with a partition of its boxes into genes such that for every primed box b, there is an box c that is weakly southwest of b from a different gene than b but of the same family. The **content** of T is the number of genes of each family. A **genotype** G of T is a choice of a single box from each gene. Depict G by erasing the entries in boxes that are not chosen. A P-tableau may be identified with the genomic P-tableau where each box is its own gene.

Example 6.3. Let $\nu = (6, 4, 1)$ and $\lambda = (4, 2)$. Then a genomic P-tableau T of shape ν/λ and its two genotypes G_1, G_2 are



The content of T is $\mu = (2, 1, 1)$. $\diamondsuit$

Given a word w using the alphabet $\mathcal{D}$, $\hat{w}$ is the word obtained by writing w backwards, and replacing each k' with k while simultaneously replacing each k with $(k+1)'$. Let

$$\texttt{doubleseq}(G) := \texttt{seq}(G)\widehat{\texttt{seq}(G)}.$$

Say $\texttt{doubleseq}(G)$ is **locally ballot** at the letter $\alpha \in \mathcal{D}$, if $|\alpha| = 1$ or if in $\texttt{doubleseq}(G)$ the number of $|\alpha|$'s appearing strictly before that α is strictly less than the number of $(|\alpha|-1)$'s appearing strictly before that α. Declare $\texttt{doubleseq}(G)$ to be **ballot** if it is locally ballot at each letter. Finally, G is **ballot** if $\texttt{doubleseq}(G)$ is ballot, and the genomic P-tableau T is **ballot** if every genotype of T is ballot.

Example 6.4. Let G_1 and G_2 be as in Example 6.3. Then

$$\texttt{doubleseq}(G_1) = 21'134'2'13' \text{ and } \texttt{doubleseq}(G_2) = 1'2134'2'3'1.$$

The former is not ballot, as it starts with 2. Hence the genomic P-tableau T of Example 6.3 is not ballot. G_2 is also not ballot: $\texttt{doubleseq}(G_2)$ is locally ballot at every position except the 2 in second position; although there is a $1'$ before this 2, there is no 1. To emphasize the differences between ballotness in this section versus ballotness in Section 3.3, note that deleting the primes gives 12134231, which *is* ballot in the earlier sense. ♢

A **Q-tableau** is a filling of ν/λ with entries from $\mathcal{D}$ satisfying (P.1)–(P.3) and

(Q.4) no primed letters appear on the main diagonal.

(Observe that (Q.4) is a weakening of (P.4), so a P-tableau is a Q-tableau.)

A **gene** (of family k) in a Q-tableau T is a set $\mathcal{G}$ of boxes such that:

- each entry of $\mathcal{G}$ is k' or k,
- the boxes of $\mathcal{G}$ are consecutive in the $\prec$-order, and
- no two boxes of $\mathcal{G}$ *with the same label* appear in the same row or the same column.

We write $\texttt{family}(\mathcal{G}) = k$ as before.

A **genomic Q-tableau** is a Q-tableau T together with a partition of its boxes into genes. The definition of **ballotness** for genomic Q-tableaux is the same as for genomic P-tableaux. Let $\texttt{PGen}_\mu(\nu/\lambda)$ and $\texttt{QGen}_\mu(\nu/\lambda)$ respectively denote the sets of genomic P- and Q-tableaux of shape ν/λ and content μ.

Lemma 6.1. $\texttt{PGen}_\mu(\nu/\lambda) \subseteq \texttt{QGen}_\mu(\nu/\lambda)$.

Proof. Let $T \in \texttt{PGen}_\mu(\nu/\lambda)$. The definition of a gene in a Q-tableau differs from that for P-tableaux only in that it allows k' and k in the same row or column to be in the same gene. Hence each gene of T is a gene in the Q-tableau sense. Thus $T \in \texttt{QGen}_\mu(\nu/\lambda)$. □

6.2 Maximal orthogonal and Lagrangian Grassmannians

Let G/P be a **generalized flag variety**, where G is a complex, connected, reductive Lie group and P is a parabolic subgroup containing a Borel subgroup B. Let B_- be the opposite Borel to B with respect to a choice of maximal torus $\mathsf{T} \subseteq \mathsf{B}$. The **Schubert cells** of G/P are the B_--orbits, and the **Schubert varieties**

V_λ are their closures. Here $\lambda \in W/W_{\mathsf{P}}$ where W is the Weyl group of G and W_{P} is the parabolic subgroup of W corresponding to P. The classes of Schubert structure sheaves $\{[\mathcal{O}_{V_\lambda}]\}$ form a $\mathbb{Z}$-linear basis of the **Grothendieck ring** $K^0(\mathsf{G}/\mathsf{P})$. Let $t_{\lambda,\mu}^{\nu}$ be the structure constants with respect to this basis. A. Buch [Bu02, Conjecture 9.2] conjectured the sign-alternation:

$$(-1)^{\mathrm{codim}_{\mathsf{G}/\mathsf{P}}(V_\nu)-\mathrm{codim}_{\mathsf{G}/\mathsf{P}}(V_\lambda)-\mathrm{codim}_{\mathsf{G}/\mathsf{P}}(V_\mu)} t_{\lambda,\mu}^{\nu} \geq 0.$$

This was subsequently proved by M. Brion [Br02]. While the Grassmannian X is the most well-studied case of G/P, we now turn to an investigation of the next two most important cases when P is maximal parabolic.

Fix a non-degenerate, symmetric bilinear form $\beta(\cdot,\cdot)$ on $\mathbb{C}^{2n+1}$. A subspace $V \subseteq \mathbb{C}^{2n+1}$ is **isotropic** with respect to β if $\beta(\vec{v},\vec{w}) = 0$ for all $\vec{v},\vec{w} \in V$. Let

$$Y = \mathrm{OG}(n, 2n+1)$$

be the **maximal orthogonal Grassmannian**, i.e., the parameter space of all such isotropic n-dimensional subspaces in $\mathbb{C}^{2n+1}$. Define the **shifted staircase** δ_n to be the shifted shape whose ith row is of length i for $1 \leq i \leq n$. The Schubert varieties Y_λ of Y are indexed by shifted Young diagrams

$$\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_n)$$

contained in δ_n, i.e.,

$$\lambda_k \leq n - k + 1 \text{ for } 1 \leq k \leq n.$$

We have

$$\mathrm{codim}_Y(Y_\lambda) = |\lambda|.$$

Let $b_{\lambda,\mu}^{\nu}$ be $t_{\lambda,\mu}^{\nu}$ in this case. The first combinatorial rule for $b_{\lambda,\mu}^{\nu}$ was conjectured in [ThYo09b] and proved in [ClThYo14], using [BuRa12].

The following is a new rule for these structure coefficients. This rule directly extends the rule of J. Stembridge [Ste89, Theorem 8.3] for the ordinary cohomological structure constants of Y. (J. Stembridge's rule is stated in terms of projective representation theory of symmetric groups; the application to $H^\star(Y)$ is due to P. Pragacz [Pr89].)

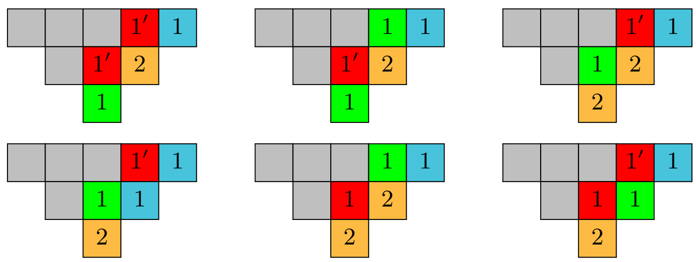
Theorem 6.1 (OG Genomic Littlewood-Richardson rule)**.**

$$b_{\lambda,\mu}^{\nu} = (-1)^{|\nu|-|\lambda|-|\mu|} \textit{ times the number of ballot genomic } P\textit{-tableaux of shape } \nu/\lambda \textit{ with content } \mu.$$

Example 6.5. (cf. [ClThYo14, Example 1.3]) That

$$b_{(3,1),(3,1)}^{(5,3,1)}(\mathrm{OG}(n, 2n+1)) = -6$$

is witnessed by:



$\diamondsuit$

Fix a symplectic bilinear form $\omega(\cdot,\cdot)$ on $\mathbb{C}^{2n}$. The **Lagrangian Grassmannian**

$$Z = \mathrm{LG}(n, 2n)$$

is the parameter space of n-dimensional linear subspaces of $\mathbb{C}^{2n}$ that are isotropic with respect to ω. The Schubert varieties $\{Z_\lambda\}$ of Z are indexed by the same shifted Young diagrams λ as above; also, $\mathrm{codim}_Z(Z_\lambda) = |\lambda|$. Let $c_{\lambda,\mu}^{\nu}$ be $t_{\lambda,\mu}^{\nu}$ in this case.

There is a well-known relationship in the "cohomological case", i.e., when $|\lambda| + |\mu| = |\nu|$, between the structure constants for Y and Z:

$$c_{\lambda,\mu}^{\nu} = 2^{\ell(\lambda)+\ell(\mu)-\ell(\nu)} b_{\lambda,\mu}^{\nu}, \tag{6.1}$$

where $\ell(\pi)$ denotes the number of nonzero parts of π. We are not aware of any generalization of (6.1); cf. [BuRa12, Examples 4.9 and 5.8]. On the other hand, we propose the following extension of this relationship:

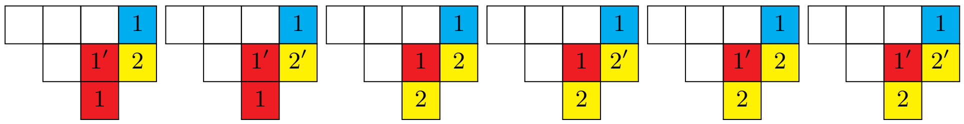
Conjecture 6.1. *For any strict partitions λ, μ, ν, we have $|b_{\lambda,\mu}^{\nu}| \leq |c_{\lambda,\mu}^{\nu}|$.* $\square$

This conjecture is true in the cohomological case since it is known that $\ell(\lambda) + \ell(\mu) \geq \ell(\nu)$ whenever $b_{\lambda,\mu}^{\nu} > 0$. Moreover, we have verified this conjecture by computer for $n \leq 7$. In addition, by [BuRa12], this conjecture holds whenever μ has a single part.

Let $\texttt{QBallot}_\mu(\nu/\lambda) := \{$ballot genomic Q-tableaux of shape ν/λ with content $\mu\}$.

Conjecture 6.2. $|c^{\nu}_{\lambda,\mu}| \leq \#\texttt{QBallot}_\mu(\nu/\lambda)$. □

Example 6.6. Let $\lambda = (3,1), \mu = (2,1)$ and $\nu = (4,3,1)$. Then $\#\texttt{QBallot}_\mu(\nu/\lambda) = 6$:



The third tableau above is the only one that is a genomic P-tableau; hence $b^{\nu}_{\lambda,\mu} = -1$. Therefore Conjectures 6.2 and 6.1 predict $1 \leq |c^{\nu}_{\lambda,\mu}| \leq 6$. Indeed, $c^{\nu}_{\lambda,\mu} = -5$. ♢

We have computer verified Conjecture 6.2 for $n \leq 6$. Moreover, the bound is sharp, as indicated in the two propositions below.

Proposition 6.1. *For* $\mu = (p)$, $|c^{\nu}_{\lambda,\mu}| = \#\texttt{QBallot}_\mu(\nu/\lambda)$.

Proof. By applying Γ (defined in Section 6.3.1) to the tableaux in $\texttt{QBallot}_{(p)}(\nu/\lambda)$ and retaining the primes, one obtains precisely the *KLG-tableaux* of A. Buch–V. Ravikumar [BuRa12, §5]. By [BuRa12, Corollary 5.6], the number of the latter is $(-1)^{|\nu|-|\lambda|-p} c^{\nu}_{\lambda,(p)}$. □

Proposition 6.2. *For* $|\nu| \leq |\lambda| + |\mu|$, $|c^{\nu}_{\lambda,\mu}| = \#\texttt{QBallot}_\mu(\nu/\lambda)$.

Proof. When $|\nu| < |\lambda| + |\mu|$, $c^{\nu}_{\lambda,\mu} = 0$ for geometric reasons. Clearly in this case also $\texttt{QBallot}_\mu(\nu/\lambda) = \emptyset$.

Suppose $|\nu| = |\lambda|+|\mu|$ and $T \in \texttt{QBallot}_\mu(\nu/\lambda)$. The number of boxes of ν/λ on the main diagonal is $\ell(\nu) - \ell(\lambda)$. By pigeonhole, each gene of T is a single box. Hence these tableaux are exactly the tableaux of [Ste89, Theorem 8.3] with condition (2) removed. Therefore by the discussion of [Ste89, p. 126], $\#\texttt{QBallot}_\mu(\nu/\lambda)$ is the coefficient of the Schur Q-function Q_μ in the expansion of the skew Schur Q-function $Q_{\nu/\lambda}$. It is well known that these coefficients agree with the structure constants for Z in this case. □

That is, we conjecturally have combinatorially-related upper and lower bounds for $|c^{\nu}_{\lambda,\mu}|$ in terms of genomic tableaux. Let

$$\texttt{PBallot}_\mu(\nu/\lambda) := \{\text{ballot genomic } P\text{-tableaux of shape } \nu/\lambda \text{ with content } \mu\}.$$

Naturally, one seeks a set $\texttt{QBallot}^{\star}_\mu(\nu/\lambda)$ satisfying

$$\texttt{PBallot}_\mu(\nu/\lambda) \subseteq \texttt{QBallot}^{\star}_\mu(\nu/\lambda) \subseteq \texttt{QBallot}_\mu(\nu/\lambda),$$

such that $\#\texttt{QBallot}^{\star}_{\mu}(\nu/\lambda) = |c^{\nu}_{\lambda,\mu}|$. Let

$$\texttt{QBallot}^{\dagger}_{\mu}(\nu/\lambda) :=$$
$$\{T \in \texttt{QBallot}_{\mu}(\nu/\lambda) : \text{no gene contains both primed and unprimed labels}\}.$$

Conjecture 6.3. $\#\texttt{QBallot}^{\dagger}_{\mu}(\nu/\lambda) \leq |c^{\nu}_{\lambda,\mu}|$. □

This has also been computer-checked for $n \leq 6$. It suggests that one should look to define $\texttt{QBallot}^{\star}_{\mu}(\nu/\lambda)$ from $\texttt{QBallot}_{\mu}(\nu/\lambda)$ by imposing a condition on genes with both primed and unprimed labels.

6.3 Proof of OG Genomic Littlewood-Richardson rule (Theorem 6.1)

Our proof of Theorem 6.1 proceeds parallel to the first proof of Theorem 3.2. (We are not aware of any set-valued tableau or puzzle formulation of Theorem 6.1.)

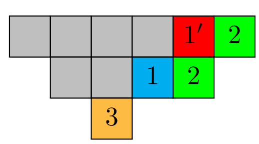
6.3.1 Shifted K-(semi)standardization maps

Let T be a genomic P-tableau. Impose a total order on genes of T by $\mathcal{G}_1 < \mathcal{G}_2$ if $\mathsf{b}_1 \prec \mathsf{b}_2$, for b_i a box of $\mathcal{G}_i$. (Note that since the boxes of a gene form a $\prec$-interval, this order is well-defined.)

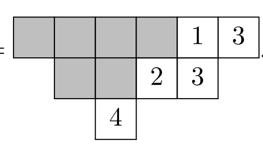
A **shifted increasing tableau** is a filling of a shifted shape that strictly increases along rows and down columns (see [ThYo09b, §7] and [ClThYo14]). Define the **shifted K-standardization map**

$$\Gamma : \texttt{PGen}_{\mu}(\nu/\lambda) \to \texttt{Inc}(\nu/\lambda)$$

by filling the ith gene in $<$-order with the entry i.

Example 6.7. If T is the genomic P-tableau 

in Example 6.3, then

$\Gamma(T) =$ 

. ♢

Recall

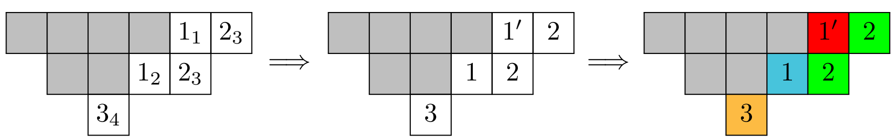
$$\mathcal{P}_k(\mu) := \left\{ 1 + \sum_{i<k} \mu_i, 2 + \sum_{i<k} \mu_i, \ldots, \sum_{j \leq k} \mu_j \right\}.$$

and let $S \in \texttt{Inc}(\nu/\lambda)$ have largest entry n. Let

$$\mu = (\mu_1, \mu_2, \ldots, \mu_h)$$

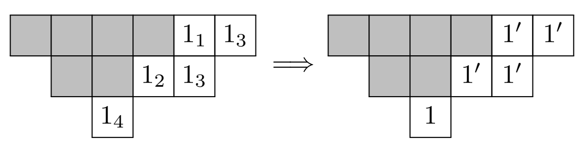
be a composition of n. The **shifted K-semistandardization** $\Delta_\mu(S)$ with respect to μ is defined as follows. Replace each entry i in S with k_i for the unique k such that $i \in \mathcal{P}_k(\mu)$. For each k_h, replace it with k' if there is a k_j southwest of it with $h < j$; otherwise replace it with k. If the result is a P-tableau, define a (putative) genomic P-tableau structure by putting all boxes that have the same entry in S into the same gene. If the result is a P-genomic tableau, we say μ is **admissible for** S; otherwise $\Delta_\mu(S)$ is not defined. Clearly, if $\Delta_\mu(S)$ is defined, it has content μ.

Example 6.8. Let S be the increasing tableau of Example 6.7. Let $\eta = (2, 1, 1)$. We compute $\Delta_\eta(S)$ in stages:

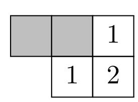


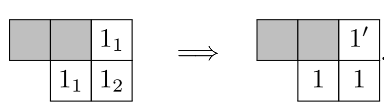
Observe that we obtain the genomic P-tableau T of Example 6.3.

Compare this to the computation of $\Delta_\theta(S)$, where $\theta = (4)$:

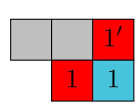


Since the tableau obtained is not a P-tableau (it violates (P.3)), $\Delta_\theta(S)$ is undefined. $\diamondsuit$

Example 6.9. Let V be the increasing tableau  and let $\kappa = (2)$. Then in the construction of $\Delta_\kappa(V)$, we first obtain a valid P-tableau:



However the putative genomic structure



is invalid, so $\Delta_\kappa(V)$ is undefined. $\diamondsuit$

An increasing tableau S is **μ-Pieri-filled** if μ is admissible for S and $\Gamma(\Delta_\mu(S)) = S$.

Remark 6.1. It is easy to check that for $\mu = (\mu_1, \mu_2, \ldots, \mu_h)$, an increasing tableau S is μ-Pieri-filled if and only if for each $k \leq h$, the entries of S in $\mathcal{P}_k(\mu)$ form a *Pieri filling* of a *ribbon* in the sense of [ClThYo14, §4]. $\square$

Lemma 6.2. *Let $T \in \mathtt{PGen}_\mu(\nu/\lambda)$. Then μ is admissible for $\Gamma(T)$ and $\Delta_\mu(\Gamma(T)) = T$. Hence $\Gamma(T)$ is μ-Pieri-filled.*

Proof. The construction of $\Delta_\mu(\Gamma(T))$ is in stages. First we construct the underlying putative P-tableau structure for $\Delta_\mu(\Gamma(T))$. We will show that this is the same as the underlying P-tableau of T. Consider a box b in ν/λ. Suppose the box b in T contains $\alpha \in \mathcal{D}$ (the color being irrelevant for now). Then it is clear that $\Delta_\mu(\Gamma(T))$ has $\beta \in \mathsf{b}$ with $|\beta| = |\alpha|$. The letter β is primed if and only if there is γ in box c southwest of b in $\Delta_\mu(\Gamma(T))$ with $|\gamma| = |\beta|$ and the entry of c in $\Gamma(T)$ strictly greater than the entry of b in $\Gamma(T)$. The entry of c in $\Gamma(T)$ is strictly greater than the entry of b in $\Gamma(T)$ exactly when $\mathsf{b} \prec \mathsf{c}$. By definition, this happens if and only if α is primed. Thus $\alpha = \beta$. Therefore T and (the partially constructed tableau) $\Delta_\mu(\Gamma(T))$ have the same underlying P-tableau structure.

In the next stage of constructing $\Delta_\mu(\Gamma(T))$, we attempt to partition the boxes into genes to produce a genomic P-tableau. By construction, T and $\Delta_\mu(\Gamma(T))$ have the same partition of labels into genes; hence $\Delta_\mu(\Gamma(T))$ is defined and the first claim of the lemma holds. The second claim follows from the first by applying Γ. $\square$

Let $\mathtt{PF}_\mu(\nu/\lambda) := \{S : S \text{ is increasing of shape } \nu/\lambda \text{ and } \mu\text{-Pieri filled}\}$.

Theorem 6.2. $\Gamma : \mathtt{PGen}_\mu(\nu/\lambda) \to \mathtt{PF}_\mu(\nu/\lambda)$ *and* $\Delta_\mu : \mathtt{PF}_\mu(\nu/\lambda) \to \mathtt{PGen}_\mu(\nu/\lambda)$ *are mutually inverse bijections.*

Proof. Immediate by definition and Lemma 6.2. $\square$

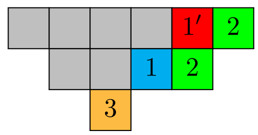
6.3.2 Genomic P-Knuth equivalence

Given a colored sequence w of symbols from $\mathcal{D}$, write $\hat{w}$ for the sequence given by writing w backwards, replacing each k' with k and each k with $(k+1)'$ and preserving the colors (cf. the *uncolored* definition of $\hat{w}$ after Example 6.3). A **genomic P-word** is a word s of colored symbols from $\mathcal{D}$ such that in the concatenation $s\hat{s}$ all unprimed i's of a fixed color are consecutive among the set of all unprimed i's. Let $\mathtt{genomicseq}(T)$ denote the colored row reading word (right to left, and top to bottom) of a genomic P-tableau T, as for genomic tableaux in Section 3.3.

Lemma 6.3. *Let T be a genomic P-tableau. Then* $\texttt{genomicseq}(T)$ *is a genomic P-word.*

Proof. The follows from the fact that T is a P-tableau together with the condition that the boxes of each gene of family i are consecutive in $\prec$-order. $\square$

A **genotype** of a genomic P-word w is an uncolored subword given by choosing one letter of each color. A **P-genotype** of the double sequence $w\hat{w}$ is a word of the form $x\hat{x}$ where x is any genotype of w. We say $w\hat{w}$ is **locally ballot** at the letter α if every P-genotype of $w\hat{w}$ that includes that α is locally ballot there. Finally $w\hat{w}$ is **ballot** if every P-genotype of $w\hat{w}$ is ballot, equivalently if $w\hat{w}$ is locally ballot at each letter. In particular, the genomic P-tableau T is ballot exactly when $\texttt{genomicseq}(T)\widehat{\texttt{genomicseq}}(T)$ is.

Example 6.10. Let T be the genomic P-tableau  of Example 6.3. Then

$$\texttt{genomicseq}(T)\widehat{\texttt{genomicseq}}(T) = 21'2134'2'3'13'.$$

It has exactly two P-genotypes:

$$21'134'2'13' \text{ and } 1'2134'2'3'1.$$

Neither P-genotype is ballot. $\diamondsuit$

We define the equivalence relation $\equiv_{GP}$ of **genomic P-Knuth equivalence** on genomic P-words as the transitive closure of the following relations:

$$\mathbf{u}\alpha\alpha\mathbf{v} \equiv_{GP} \mathbf{u}\alpha\mathbf{v}, \tag{GP.1}$$

$$\mathbf{u}\alpha\beta\alpha\mathbf{v} \equiv_{GP} \mathbf{u}\beta\alpha\beta\mathbf{v}, \tag{GP.2}$$

$$\mathbf{u}\beta\alpha\gamma\mathbf{v} \equiv_{GP} \mathbf{u}\beta\gamma\alpha\mathbf{v} \quad \text{if } \alpha \leq \beta < \gamma \text{ and } \beta = |\beta|, \text{ or } \alpha < \beta \leq \gamma \text{ and } \beta = |\beta|', \tag{GP.3}$$

$$\mathbf{u}\alpha\gamma\beta\mathbf{v} \equiv_{GP} \mathbf{u}\gamma\alpha\beta\mathbf{v} \quad \text{if } \alpha \leq \beta < \gamma \text{ and } \beta = |\beta|', \text{ or } \alpha < \beta \leq \gamma \text{ and } \beta = |\beta|, \tag{GP.4}$$

$$\mathbf{u}ij \equiv_{GP} \mathbf{u}j^{\dagger}i, \quad \text{where } j^{\dagger} = j' \text{ if } i = j, \text{ and } j^{\dagger} = j \text{ otherwise,} \tag{GP.5}$$

where red, blue, green represent distinct colors.

Theorem 6.3. *If $w_1 \equiv_{GP} w_2$, then $w_1\hat{w}_1$ is ballot if and only if $w_2\hat{w}_2$ is ballot.*

Proof. Let w be a genomic P-word. We need that (GP.1)–(GP.5) preserve ballotness of $w\hat{w}$.

(GP.1) and (GP.2): These relations change w without changing the set of genotypes of $w\hat{w}$. Hence they do not affect ballotness of the latter.

(GP.3): Suppose $w = \mathbf{u}\beta\alpha\gamma\mathbf{v}$ and that $w^* = \mathbf{u}\beta\gamma\alpha\mathbf{v}$ is obtained by (GP.3).

("$\Rightarrow$" for (GP.3)): We assume $w\hat{w}$ is ballot, and we must show $w^*\hat{w^*}$ is ballot, i.e., locally ballot (henceforth abbreviated "LB") at each letter.

(Case 1: $\alpha = \beta$): We have $i := |\alpha| = \alpha$ and $i < \gamma$.

(Case 1.1: $\gamma = |\gamma|$): Let $k := \gamma$. Then

$$w\hat{w} = \mathbf{u}iik\mathbf{v}\hat{\mathbf{v}}(k+1)'(i+1)'(i+1)'\hat{\mathbf{u}}$$

and

$$w^*\hat{w^*} = \mathbf{u}iki\mathbf{v}\hat{\mathbf{v}}(i+1)'(k+1)'(i+1)'\hat{\mathbf{u}}.$$

It suffices to show that $w^*\hat{w^*}$ is LB at k and $(k+1)'$. LBness at the latter is clear from the ballotness of $w\hat{w}$.

If $k > i+1$, then LBness at k is also clear from the ballotness of $w\hat{w}$. Hence assume $k = i+1$. The proof is now the same is for the corresponding case of Theorem 3.6.

(Case 1.2: $\gamma = |\gamma|'$): Let $k' := \gamma$. Then

$$w\hat{w} = \mathbf{u}iik'\mathbf{v}\hat{\mathbf{v}}k(i+1)'(i+1)'\hat{\mathbf{u}}$$

and

$$w^*\hat{w^*} = \mathbf{u}ik'i\mathbf{v}\hat{\mathbf{v}}(i+1)'k(i+1)'\hat{\mathbf{u}}.$$

It suffices to show that $w^*\hat{w^*}$ is LB at k' and k. LBness at the latter is clear from the ballotness of $w\hat{w}$. LBness at the former may be argued exactly as in Case 1.1.

(Case 2: $\beta = \gamma$): We have $j' := |\beta|' = \beta$ and $\alpha < j'$.

(Case 2.1: $\alpha = |\alpha|$): Let $i = \alpha$. Then

$$w\hat{w} = \mathbf{u}j'ij'\mathbf{v}\hat{\mathbf{v}}j(i+1)'j\hat{\mathbf{u}}$$

and

$$w^*\hat{w^*} = \mathbf{u}j'j'i\mathbf{v}\hat{\mathbf{v}}(i+1)'jj\hat{\mathbf{u}}.$$

It suffices to show that $w^*\hat{w^*}$ is LB at j' and j. That $w^*\hat{w^*}$ is LB at j' follows from the LBness of $w\hat{w}$ at j'. LBness at j is trivial.

(Case 2.2: $\alpha = |\alpha|'$): Let $i' = \alpha$. Then

$$w\hat{w} = \mathbf{u}j'i'j'\mathbf{v}\hat{\mathbf{v}}jij\hat{\mathbf{u}}$$

and

$$w^*\hat{w^*} = \mathbf{u}j'j'i'\mathbf{v}\hat{\mathbf{v}}ijj\hat{\mathbf{u}}.$$

It suffices to check that $w^*\hat{w^*}$ is LB at j' and j. This is clear from the ballotness of $w\hat{w}$.

(Case 3: $\alpha < \beta < \gamma$):

(Case 3.1: $\alpha = |\alpha|$): Let $i = \alpha$. If $\gamma > i+1$, ballotness is clear. Otherwise, by the assumptions of Case 3, $\beta = (i+1)'$ and $\gamma = i+1$. So

$$w\hat{w} = \mathbf{u}(i+1)'i(i+1)\mathbf{v}\hat{\mathbf{v}}(i+2)'(i+1)'(i+1)\hat{\mathbf{u}}$$

and

$$w^*\hat{w^*} = \mathbf{u}(i+1)'(i+1)i\mathbf{v}\hat{\mathbf{v}}(i+1)'(i+2)'(i+1)\hat{\mathbf{u}}.$$

It suffices to check that $w^*\hat{w^*}$ is LB at $(i+1)$ and $(i+2)'$. The latter is clear from ballotness of $w\hat{w}$. The LBness at $(i+1)$ follows from the LBness of $w\hat{w}$ at $(i+1)'$.

(Case 3.2: $\alpha = |\alpha|'$): Let $i' = \alpha$. If $\gamma > i+1$, ballotness is clear. Otherwise we have either $\gamma = (i+1)'$ or $\gamma = i+1$.

(Case 3.2.1: $\gamma = (i+1)'$): We have $\beta = i$. Then

$$w\hat{w} = \mathbf{u}ii'(i+1)'\mathbf{v}\hat{\mathbf{v}}(i+1)i(i+1)'\hat{\mathbf{u}}$$

and

$$w^*\hat{w^*} = \mathbf{u}i(i+1)'i'\mathbf{v}\hat{\mathbf{v}}i(i+1)(i+1)'\hat{\mathbf{u}}.$$

It suffices to check LBness at the two green letters. These checks hold by the ballotness of $w\hat{w}$.

(Case 3.2.2: $\gamma = i+1$): Here

$$w\hat{w} = \mathbf{u}\beta i'(i+1)\mathbf{v}\hat{\mathbf{v}}(i+2)'i\hat{\beta}\hat{\mathbf{u}}$$

and

$$w^*\hat{w^*} = \mathbf{u}\beta(i+1)i'\mathbf{v}\hat{\mathbf{v}}i(i+2)'\hat{\beta}\hat{\mathbf{u}}.$$

It suffices to check LBness at the two green letters. These checks are both direct from the ballotness of $w\hat{w}$.

("$\Leftarrow$" for (GP.3)): Conversely, assume $w^*\hat{w^*}$ is ballot. We need to show that $w\hat{w}$ is ballot. As with the arguments for $\Rightarrow$, we need to establish LBness at each letter. In brief, it suffices to check this in each case below at the green letters. In each of these situations, this is immediate from the assumption $w^*\hat{w^*}$ is ballot.

(Case 1: $\alpha = \beta$): We have $i := |\alpha| = \alpha$ and $i < \gamma$.

(Case 1.1: $\gamma = |\gamma|$): Let $k := \gamma$. Then

$$w^*\hat{w^*} = \mathbf{u}iki\mathbf{v}\hat{\mathbf{v}}(i+1)'(k+1)'(i+1)'\hat{\mathbf{u}}$$

and

$$w\hat{w} = \mathbf{u}iik\mathbf{v}\hat{\mathbf{v}}(k+1)'(i+1)'(i+1)'\hat{\mathbf{u}}.$$

(Case 1.2: $\gamma = |\gamma|'$): Let $k' := \gamma$. Then

$$w^*\hat{w^*} = \mathbf{u}ik'i\mathbf{v}\hat{\mathbf{v}}(i+1)'k(i+1)'\hat{\mathbf{u}}$$

and

$$w\hat{w} = \mathbf{u}iik'\mathbf{v}\hat{\mathbf{v}}k(i+1)'(i+1)'\hat{\mathbf{u}}.$$

(Case 2: $\beta = \gamma$): We have $j' := |\beta|' = \beta$ and $\alpha < j'$.

(Case 2.1: $\alpha = |\alpha|$): Let $i = \alpha$. Then

$$w^*\hat{w^*} = \mathbf{u}j'j'i\mathbf{v}\hat{\mathbf{v}}(i+1)'jj\hat{\mathbf{u}}$$

and

$$w\hat{w} = \mathbf{u}j'ij'\mathbf{v}\hat{\mathbf{v}}j(i+1)'j\hat{\mathbf{u}}.$$

(Case 2.2: $\alpha = |\alpha|'$): Let $i' = \alpha$. Then

$$w^*\hat{w^*} = \mathbf{u}j'j'i'\mathbf{v}\hat{\mathbf{v}}ijj\hat{\mathbf{u}}$$

and

$$w\hat{w} = \mathbf{u}j'i'j'\mathbf{v}\hat{\mathbf{v}}jij\hat{\mathbf{u}}.$$

(Case 3: $\alpha < \beta < \gamma$):

(Case 3.1: $\alpha = |\alpha|$): Let $i = \alpha$. If $\gamma > i + 1$, ballotness is clear. Otherwise, by the assumptions of Case

3, we have $\beta = (i+1)'$ and $\gamma = i+1$. Thus

$$w^*\hat{w^*} = \mathbf{u}(i+1)'(i+1)i\mathbf{v}\hat{\mathbf{v}}(i+1)'(i+2)'(i+1)\hat{\mathbf{u}}$$

and

$$w\hat{w} = \mathbf{u}(i+1)'i(i+1)\mathbf{v}\hat{\mathbf{v}}(i+2)'(i+1)'(i+1)\hat{\mathbf{u}}.$$

(Case 3.2: $\alpha = |\alpha|'$): Let $i' = \alpha$. If $\gamma > i+1$, ballotness is clear. Otherwise we have either $\gamma = (i+1)'$ or $\gamma = i+1$.

(Case 3.2.1: $\gamma = (i+1)'$): We have $\beta = i$. Then

$$w^*\hat{w^*} = \mathbf{u}i(i+1)'i'\mathbf{v}\hat{\mathbf{v}}i(i+1)(i+1)'\hat{\mathbf{u}}$$

and

$$w\hat{w} = \mathbf{u}ii'(i+1)'\mathbf{v}\hat{\mathbf{v}}(i+1)i(i+1)'\hat{\mathbf{u}}.$$

(Case 3.2.2: $\gamma = i+1$): Here

$$w^*\hat{w^*} = \mathbf{u}\beta(i+1)i'\mathbf{v}\hat{\mathbf{v}}i(i+2)'\hat{\beta}\hat{\mathbf{u}}$$

and

$$w\hat{w} = \mathbf{u}\beta i'(i+1)\mathbf{v}\hat{\mathbf{v}}(i+2)'i\hat{\beta}\hat{\mathbf{u}}.$$

(GP.4): This may be argued exactly as for (GP.3).

(GP.5): Suppose $w = \mathbf{u}ij$ and that $w^* = \mathbf{u}j^{\dagger}i$ is obtained by (GP.5). By symmetry, we may assume $i \leq j$. We must show $w\hat{w}$ is ballot if and only if $w^*\widehat{w^*}$ is.

(Case 1: $i < j$): Then

$$w\hat{w} = \mathbf{u}ij(j+1)'(i+1)'\hat{\mathbf{u}},$$

while

$$w^*\widehat{w^*} = \mathbf{u}ji(i+1)'(j+1)'\hat{\mathbf{u}}.$$

Suppose $w\hat{w}$ is ballot. It suffices to check LBness of $w^*\widehat{w^*}$ at the two blue letters. LBness at $(j+1)'$ is clear from the assumed ballotness of $w\hat{w}$. LBness at j, for $j = i+1$, follows from the LBness of $w\hat{w}$ at $(i+1)'$ (when $j \neq i+1$, the claim is clear).

Conversely suppose $w^*\widehat{w^*}$ is ballot. It suffices to check LBness of $w\hat{w}$ at the two blue letters; this is immediate.

(Case 2: $i = j$): Then

$$w\hat{w} = \mathbf{u}ii(i+1)'(i+1)'\hat{\mathbf{u}},$$

while

$$w^*\widehat{w^*} = \mathbf{u}i'i(i+1)'i\hat{\mathbf{u}}.$$

It is straightforward that $w\hat{w}$ is ballot if and only if $w^*\widehat{w^*}$ is. $\square$

Weak K-Knuth equivalence on words is the symmetric, transitive closure of these relations [BuSa13, Definition 7.6]:

$$\begin{aligned}
\mathbf{u}aa\mathbf{v} &\equiv_{wK} \mathbf{u}a\mathbf{v}, \\
\mathbf{u}aba\mathbf{v} &\equiv_{wK} \mathbf{u}bab\mathbf{v}, \\
\mathbf{u}bac\mathbf{v} &\equiv_{wK} \mathbf{u}bca\mathbf{v} \quad \text{if } a < b < c, \\
\mathbf{u}acb\mathbf{v} &\equiv_{wK} \mathbf{u}cab\mathbf{v} \quad \text{if } a < b < c, \\
\mathbf{u}ab &\equiv_{wK} \mathbf{u}ba.
\end{aligned}$$

Lemma 6.4. *For genomic P-words u, v we have $u \equiv_{GP} v$ if and only if $\Gamma(u) \equiv_{wK} \Gamma(v)$.*

Proof. This follows from applying Δ_μ to the generating relations for weak K-Knuth equivalence for Pieri-filled words. $\square$

6.3.3 Shifted jeu de taquin and the conclusion of the proof

The definitions of genomic *jeu de taquin* and K-*jeu de taquin* for shifted tableaux are analogous to the unshifted case. For details of shifted K-*jeu de taquin*, see [ClThYo14]. We sketch the modifications necessary for shifted genomic *jeu de taquin* and give an illustrative example. For each gene $\mathcal{G}$ of family k, define the operator $\mathsf{swap}_{\mathcal{G},\bullet}$ as follows: If b is a box of $\mathcal{G}$ in the tableau T with a neighbor containing a $\bullet$, replace the k or $k' \in \mathsf{b}$ with $\bullet$ and remove it from $\mathcal{G}$. If c is a box of T containing a $\bullet$ and with a $\mathcal{G}$ neighbor, c is a box of $\mathcal{G}$ in $\mathsf{swap}_{\mathcal{G},\bullet}(T)$; c has entry k in $\mathsf{swap}_{\mathcal{G},\bullet}(T)$ if either of its $\mathcal{G}$ neighbors in T have entry k or if c lies on the main diagonal; otherwise c has entry k' in $\mathsf{swap}_{\mathcal{G},\bullet}(T)$. The other boxes of T are the same in $\mathsf{swap}_{\mathcal{G},\bullet}(T)$.

Index the genes of T as

$$\mathcal{G}_1 < \mathcal{G}_2 < \cdots < \mathcal{G}_{|\mu|}$$

according to the total order on genes from Section 6.3.1. Then

$$\mathsf{jdt}_I(T) := \mathsf{DelBullets} \circ \mathsf{swap}_{\mathcal{G}_{|\mu|},\bullet} \circ \cdots \circ \mathsf{swap}_{\mathcal{G}_2,\bullet} \circ \mathsf{swap}_{\mathcal{G}_1,\bullet}(T^\bullet).$$

(This algorithm reduces to the classical *jeu de taquin* for semistandard P-tableaux in the case each gene contains only a single box.)

Example 6.11. Suppose $T^\bullet$ is the genomic tableau $\begin{array}{|c|c|c|c|c|} \hline & & \bullet & {\color{red}1'} & {\color{cyan}1} \\ \hline \multicolumn{1}{c|}{} & \bullet & {\color{red}1'} & {\color{orange}2} & \multicolumn{1}{c}{} \\ \cline{2-4} \multicolumn{2}{c|}{} & {\color{green}1} & \multicolumn{2}{c}{} \\ \cline{3-3} \end{array}$. Then

$$\mathtt{switch}_{\color{red}1}^{\bullet}(T^\bullet) = \begin{array}{|c|c|c|c|c|} \hline & & {\color{red}1'} & \bullet & {\color{cyan}1} \\ \hline \multicolumn{1}{c|}{} & {\color{red}1} & \bullet & {\color{orange}2} & \multicolumn{1}{c}{} \\ \cline{2-4} \multicolumn{2}{c|}{} & {\color{green}1} & \multicolumn{2}{c}{} \\ \cline{3-3} \end{array}, \quad \mathtt{switch}_{\color{green}1}^{\bullet} \circ \mathtt{switch}_{\color{red}1}^{\bullet}(T^\bullet) = \begin{array}{|c|c|c|c|c|} \hline & & {\color{red}1'} & \bullet & {\color{cyan}1} \\ \hline \multicolumn{1}{c|}{} & {\color{red}1} & {\color{green}1} & {\color{orange}2} & \multicolumn{1}{c}{} \\ \cline{2-4} \multicolumn{2}{c|}{} & \bullet & \multicolumn{2}{c}{} \\ \cline{3-3} \end{array}, \quad \text{and}$$

$$\begin{aligned} \mathtt{switch}_{\color{orange}2}^{\bullet} \circ \mathtt{switch}_{\color{cyan}1}^{\bullet} \circ \mathtt{switch}_{\color{green}1}^{\bullet} \circ \mathtt{switch}_{\color{red}1}^{\bullet}(T^\bullet) &= \mathtt{switch}_{\color{cyan}1}^{\bullet} \circ \mathtt{switch}_{\color{green}1}^{\bullet} \circ \mathtt{switch}_{\color{red}1}^{\bullet}(T^\bullet) \\ &= \begin{array}{|c|c|c|c|c|} \hline & & {\color{red}1'} & {\color{cyan}1} & \bullet \\ \hline \multicolumn{1}{c|}{} & {\color{red}1} & {\color{green}1} & {\color{orange}2} & \multicolumn{1}{c}{} \\ \cline{2-4} \multicolumn{2}{c|}{} & \bullet & \multicolumn{2}{c}{} \\ \cline{3-3} \end{array}. \end{aligned}$$

$$\text{So } \mathsf{jdt}_I(T) = \begin{array}{|c|c|c|c|} \hline & & {\color{red}1'} & {\color{cyan}1} \\ \hline \multicolumn{1}{c|}{} & {\color{red}1} & {\color{green}1} & {\color{orange}2} \\ \cline{2-4} \end{array}. \qquad \diamondsuit$$

Using this shifted genomic *jeu de taquin*, one can obtain shifted versions of *genomic infusion* and *genomic Bender-Knuth involutions*, analogous to the discussion of Section 3.6. This leads to a definition of *genomic P-Schur functions*, symmetric functions that deform the classical P-Schur functions just as the genomic Schur functions of Section 3.6 deform the classical Schur functions. We do not pursue these ideas further here.

Let S_μ denote the **row superstandard tableau** of shifted shape μ (that is, the tableau whose first row has entries $1, 2, 3, \ldots, \mu_1$, and whose second row has entries $\mu_1 + 1, \mu_2 + 2, \ldots, \mu_1 + \mu_2$ etc.).

Example 6.12. For $\mu = (4, 2)$, $S_\mu = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline \multicolumn{1}{c|}{} & 5 & 6 & \multicolumn{1}{c}{} \\ \cline{2-3} \end{array}$. $\diamondsuit$

Let

$$T_\mu := \Delta_\mu(S_\mu)$$

be the unique genomic P-tableau whose underlying P-tableau is the highest weight tableau of shifted shape

μ. We recall some results that we need.

Theorem 6.4 ([BuSa13, Theorem 7.8]). *Let S be a shifted increasing tableaux. Then S rectifies to S_μ if and only if* $\texttt{seq}(S) \equiv_{wK} \texttt{seq}(S_\mu)$. $\square$

Let $\texttt{IncRect}_\mu(\nu/\lambda) := \{$shifted increasing tableaux of shape ν/λ that rectify to $S_\mu\}$.

Theorem 6.5 ([ClThYo14, Theorem 1.2]). $b^\nu_{\lambda,\mu} = (-1)^{|\nu|-|\lambda|-|\mu|} \times \#\texttt{IncRect}_\mu(\nu/\lambda)$. $\square$

By Theorem 6.5, it is enough to biject $\texttt{IncRect}_\mu(\nu/\lambda)$ and $\texttt{PBallot}_\mu(\nu/\lambda)$. We claim that the maps Γ and Δ_μ give the desired bijections. It follows from Remark 6.1 and [ClThYo14, Proof of Theorem 1.1] that Δ_μ is well-defined on $\texttt{IncRect}_\mu(\nu/\lambda)$.

Let $S \in \texttt{IncRect}_\mu(\nu/\lambda)$. By Theorem 6.4,

$$\texttt{seq}(S) \equiv_{wK} \texttt{seq}(S_\mu).$$

By Lemma 6.4,

$$\texttt{genomicseq}(\Delta_\mu(S)) \equiv_{GP} \texttt{genomicseq}(\Delta_\mu(S_\mu)) = \texttt{genomicseq}(T_\mu).$$

Note $\texttt{genomicseq}(T_\mu)$ is ballot. Hence by Theorem 6.3, $\texttt{genomicseq}(\Delta_\mu(S))$ is ballot. Thus

$$\Delta_\mu(S) \in \texttt{PBallot}_\mu(\nu/\lambda).$$

Conversely, if $T \in \texttt{PBallot}_\mu(\nu/\lambda)$, then its genomic rectification is also ballot by Theorem 3.7. Hence its genomic rectification is T_μ. Therefore

$$\Gamma(T) \in \texttt{IncRect}_\mu(\nu/\lambda).$$

This completes the proof. $\square$

Chapter 7

K-promotion and cyclic sieving of increasing tableaux

This chapter is based on [Pe14], except for Section 7.7, which derives from joint work with J. Bloom and D. Saracino [BlPeSa16].

7.1 Introduction

An *increasing tableau*, as described in Section 3.1.4 is a semistandard tableau such that all rows and columns are strictly increasing. In this chapter, **we also assume that the set of entries is an initial segment of** $\mathbb{Z}_{>0}$, that is there are no missing values. For λ a partition of N, we write $|\lambda| = N$. We denote by $\mathrm{Inc}_k(\lambda)$ the set of increasing tableaux of shape λ with maximum value $|\lambda| - k$. Similarly $\mathrm{SYT}(\lambda)$ denotes standard Young tableaux of shape λ. Notice $\mathrm{Inc}_0(\lambda) = \mathrm{SYT}(\lambda)$. We routinely identify a partition λ with its Young diagram; hence for us the notations $\mathrm{SYT}(m \times n)$ and $\mathrm{SYT}(n^m)$ are equivalent.

A *small Schröder path* is a planar path from the origin to $(2n, 0)$ that is constructed from three types of line segment: *upsteps* by $(1, 1)$, *downsteps* by $(1, -1)$, and *horizontal steps* by $(2, 0)$, so that the path never falls below the horizontal axis and no horizontal step lies on the axis. The nth *small Schröder number* is defined to be the number of such paths. A *Dyck path* is a small Schröder path without horizontal steps.

Our first result is an extension of the classical fact that Catalan numbers enumerate both Dyck paths and rectangular standard Young tableaux of two rows, $\mathrm{SYT}(2 \times n)$. For $T \in \mathrm{Inc}_k(2 \times n)$, let $\mathrm{maj}(T)$ be the sum of all i in row 1 such that $i + 1$ appears in row 2.

Theorem 7.1. *There are explicit bijections between* $\mathrm{Inc}_k(2 \times n)$*, small Schröder paths with* k *horizontal steps, and* $\mathrm{SYT}(n - k, n - k, 1^k)$*. This implies the identity*

$$\sum_{T \in \mathrm{Inc}_k(2 \times n)} q^{\mathrm{maj}(T)} = q^{n + \binom{k}{2}} \frac{\begin{bmatrix} n-1 \\ k \end{bmatrix}_q \begin{bmatrix} 2n-k \\ n-k-1 \end{bmatrix}_q}{[n-k]_q}. \tag{7.1}$$

In particular, the total number of increasing tableaux of shape $2 \times n$ *is the* n*th small Schröder number.*

The "flag-shaped" standard Young tableaux of Theorem 7.1 were previously considered by R. Stanley [Sta96] in relation to polygon dissections.

Suppose X is a finite set, $\mathcal{C}_n = \langle c \rangle$ a cyclic group acting on X, and $f \in \mathbb{Z}[q]$ a polynomial. The triple $(X, \mathcal{C}_n, f)$ has the *cyclic sieving phenomenon* [ReStWh04] if for all m, the number of elements of X fixed by c^m is $f(\zeta^m)$, where ζ is any primitive nth root of unity. D. White [Wh07] discovered a cyclic sieving for $2 \times n$ standard Young tableaux. For this, he used a q-analogue of the hook-length formula (that is, a q-analogue of the Catalan numbers) and a group action by jeu de taquin promotion. B. Rhoades [Rh10, Theorem 1.3] generalized this result from $\mathrm{Inc}_0(2 \times n)$ to $\mathrm{Inc}_0(m \times n)$. Our main result is a generalization of D. White's result in another direction, from $\mathrm{Inc}_0(2 \times n)$ to $\mathrm{Inc}_k(2 \times n)$.

We first define *K-promotion* for increasing tableaux. Define the *SE-neighbors* of a box to be the (at most two) boxes immediately below it or right of it. Let T be an increasing tableau with maximum entry M. Delete the entry 1 from T, leaving an empty box. Repeatedly perform the following operation simultaneously on all empty boxes until no empty box has a SE-neighbor: Label each empty box by the minimal label of its SE-neighbors and then remove that label from the SE-neighbor(s) in which it appears. If an empty box has no SE-neighbors, it remains unchanged. We illustrate the local changes in Figure 7.1.

$$\begin{array}{|c|c|}\hline & i \\ \hline j \\ \cline{1-1}\end{array} \mapsto \begin{array}{|c|c|}\hline i & \\ \hline j \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline & j \\ \hline i \\ \cline{1-1}\end{array} \mapsto \begin{array}{|c|c|}\hline i & j \\ \hline \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline & i \\ \hline i \\ \cline{1-1}\end{array} \mapsto \begin{array}{|c|c|}\hline i & \\ \hline \\ \cline{1-1}\end{array} \qquad \begin{array}{c|c|}\cline{2-2} & \\ \hline \multicolumn{1}{|c|}{} & i \\ \hline\end{array} \mapsto \begin{array}{c|c|}\cline{2-2} & i \\ \hline \multicolumn{1}{|c|}{i} & \\ \hline\end{array}$$

Figure 7.1: Local changes during K-promotion for $i < j$.

Notice that the number of empty boxes may change during this process. Finally we obtain the K-promotion $\mathcal{P}(T)$ by labeling all empty boxes by $M + 1$ and then subtracting one from every label. Figure 7.2 shows a full example of K-promotion.

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 4 & 5 \\ \hline\end{array} \mapsto \begin{array}{|c|c|c|}\hline & 2 & 4 \\ \hline 3 & 4 & 5 \\ \hline\end{array} \mapsto \begin{array}{|c|c|c|}\hline 2 & & 4 \\ \hline 3 & 4 & 5 \\ \hline\end{array} \mapsto \begin{array}{|c|c|c|}\hline 2 & 4 & \\ \hline 3 & & 5 \\ \hline\end{array} \mapsto \begin{array}{|c|c|c|}\hline 2 & 4 & 5 \\ \hline 3 & 5 & \\ \hline\end{array} \mapsto \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 4 & 5 \\ \hline\end{array}$$

Figure 7.2: K-promotion.

Our definition of K-promotion is analogous to that of ordinary promotion, but uses the jeu de taquin for increasing tableaux introduced by H. Thomas–A. Yong [ThYo09b] in place of ordinary jeu de taquin. (The 'K' reflects the relations to K-theoretic Schubert calculus discussed in Section 3.1.4.) That is, we may also describe K-promotion as follows: Delete the entry 1, rectify the resulting skew increasing tableau as in Section 3.1.4, decrement each entry, and fill the empty outer corners with M. Observe that on standard Young tableaux, promotion and K-promotion coincide.

K-evacuation [ThYo09b, §4] is defined as follows. Let T be a increasing tableau with maximum entry M, and let $[T]_j$ denote the Young diagram consisting of those boxes of T with entry $i \leq j$. Then the K-evacuation $\mathcal{E}(T)$ is the increasing tableau encoded by the chain in Young's lattice $([\mathcal{P}^{M-j}(T)]_j)_{0 \leq j \leq M}$.

Like ordinary evacuation as described in Section 2.2.2, $\mathcal{E}$ is an involution.

Let the non-identity element of $\mathcal{C}_2$ act on $\mathrm{Inc}_k(2 \times n)$ by K-evacuation. We prove the following cyclic sieving, generalizing a result of J. Stembridge [Ste95].

Theorem 7.2. *For all n and k, the triple $\left(\mathrm{Inc}_k(2 \times n), \mathcal{C}_2, f\right)$ has the cyclic sieving phenomenon, where*

$$f(q) := \frac{\begin{bmatrix} n-1 \\ k \end{bmatrix}_q \begin{bmatrix} 2n-k \\ n-k-1 \end{bmatrix}_q}{[n-k]_q} \tag{7.2}$$

is the q-enumerator from Theorem 7.1.

We will then need:

Theorem 7.3. *For all n and k, there is an action of the cyclic group $\mathcal{C}_{2n-k}$ on $T \in \mathrm{Inc}_k(2 \times n)$, where a generator acts by K-promotion.*

In the case $k = 0$, Theorem 7.3 is implicit in work of M.-P. Schützenberger (*cf.* [Ha92, Sta09]). The bulk of this paper is devoted to proofs of Theorem 7.3, which we believe provide different insights. Finally we construct the following cyclic sieving.

Theorem 7.4. *For all n and k, the triple $\left(\mathrm{Inc}_k(2 \times n), \mathcal{C}_{2n-k}, f\right)$ has the cyclic sieving phenomenon.*

An analogous result for hook-shapes has been found by T. Pressey, A. Stokke, and T. Visentin [PrStVi14].

Our proof of Theorem 7.2 is by reduction to a result of J. Stembridge [Ste95], which relies on results about the Kazhdan–Luszig cellular representation of the symmetric group. Similarly, all proofs [Rh10, Pu13, FoKa14] of B. Rhoades' theorem for standard Young tableaux use representation theory or geometry. (Also [PePyRh09], giving new proofs of the 2- and 3-row cases of B. Rhoades' result, uses representation theory.) In contrast, our proof of Theorem 7.4 is completely elementary. It is natural to ask also for such representation-theoretic or geometric proofs of Theorem 7.4. In Section 7.5, we discuss obstacles to an approach based on Kazhdan–Lusztig bases and briefly describe a different representation-theoretic argument discovered by B. Rhoades [Rh15]. We do not know a common generalization of our Theorem 7.4 and B. Rhoades' theorem to $\mathrm{Inc}_k(m \times n)$. One obstruction is that for $k > 0$, Theorem 7.3 does not generalize in the obvious way to tableaux of more than 3 rows (*cf.* Example 7.1). We will relate this fact to results of P. Cameron and D. Fon-der-Flaass on plane partitions [CaFo95] in Chapter 8.

This chapter is organized as follows. In Section 7.2, we prove Theorem 7.1. We include an additional bijection (to be used in Section 7.6) between $\mathrm{Inc}_k(2 \times n)$ and certain noncrossing partitions that we interpret as generalized noncrossing matchings. In Section 7.3, we use the combinatorics of small Schröder paths to prove Theorem 7.3 and a characterization of K-evacuation necessary for Theorem 7.2. We also provide a counterexample to the naive generalization of Theorem 7.3 to 4-row increasing tableaux. In Section 7.4, we make connections with tropicalizations of Conway–Coxeter frieze patterns and demostrate a frieze-diagrammatic approach to some of the key steps in the previous section. In Section 7.5, we prove Theorem 7.2 by interpreting it representation-theoretically in the spirit of [Ste95] and [Rh10], and discuss representation-theoretic approaches to Theorem 7.4. In Section 7.6, we use noncrossing partitions to give another proof of Theorem 7.3 and to prove Theorem 7.4.

7.2 Bijections and Enumeration

Proposition 7.1. *There is an explicit bijection between* $\mathrm{Inc}_k(2 \times n)$ *and* $\mathrm{SYT}(n-k, n-k, 1^k)$.

Proof. Let $T \in \mathrm{Inc}_k(2 \times n)$. The following algorithm produces a corresponding $S \in \mathrm{SYT}(n-k, n-k, 1^k)$. Observe that every value in $\{1, \ldots, 2n-k\}$ appears in T either once or twice. Let A be the set of numbers that appear twice. Let B be the set of numbers that appear in the second row immediately right of an element of A. Note $|A| = |B| = k$.

Let T' be the tableau of shape $(n-k, n-k)$ formed by deleting all elements of A from the first row of T and all elements of B from the second. The standard Young tableau S is given by appending B to the first column. An example is shown in Figure 7.3.

This algorithm is reversible. Given the standard Young tableau S of shape $(n-k, n-k, 1^k)$, let B be the set of entries below the first two rows. By inserting B into the second row of S while maintaining increasingness, we reconstruct the second row of T. Let A be the set of elements immediately left of an element of B in this reconstructed row. By inserting A into the first row of S while maintaining increasingness, we reconstruct the first row of T. □

Corollary 7.1. *For all n and k the identity (7.3) holds:*

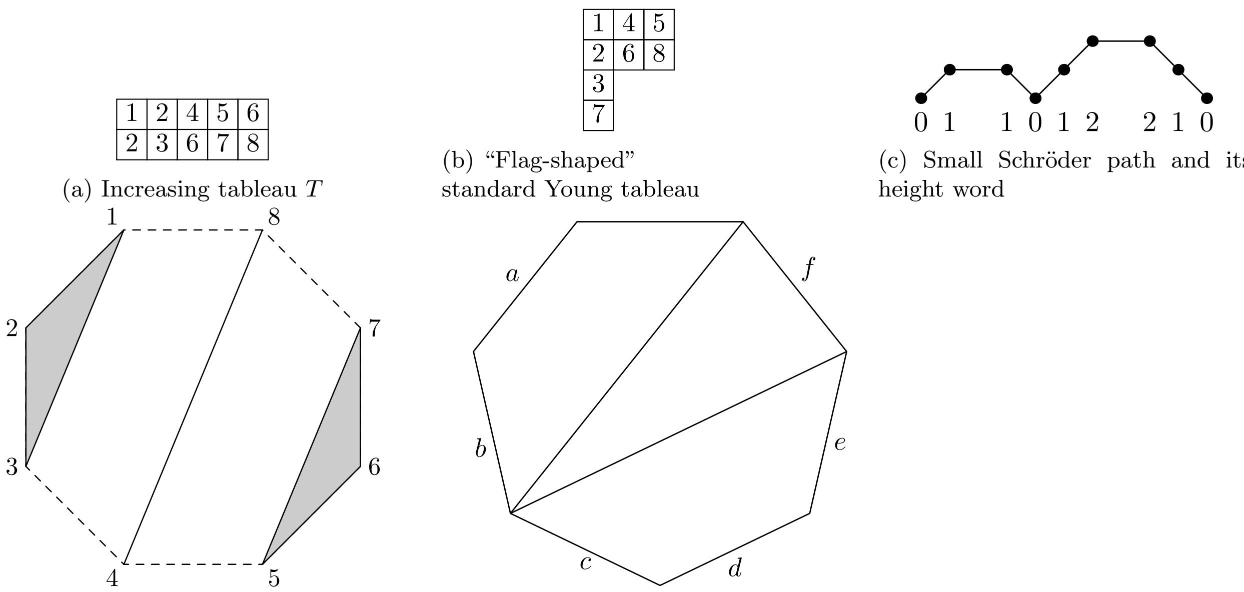
$$\sum_{T \in \mathrm{Inc}_k(2\times n)} q^{\mathrm{maj}(T)} = q^{n+\binom{k}{2}} \frac{\begin{bmatrix} n-1 \\ k \end{bmatrix}_q \begin{bmatrix} 2n-k \\ n-k-1 \end{bmatrix}_q}{[n-k]_q}. \tag{7.3}$$

Proof. Observe that $\mathrm{maj}(T)$ for a 2-row rectangular increasing tableau T is the same as the major index of the corresponding standard Young tableau. The desired q-enumerator follows by applying the q-hook-length formula to those standard Young tableaux (*cf.* [Sta99, Corollary 7.21.5]). □

Proof of Theorem 7.1. The bijection between $\mathrm{Inc}_k(2 \times n)$ and $\mathrm{SYT}(n-k, n-k, 1^k)$ is given by Proposition 7.1. The q-enumeration (7.1) is exactly Corollary 7.1.

We now give a bijection between $\mathrm{Inc}_k(2 \times n)$ and small Schröder paths with k horizontal steps. Let $T \in \mathrm{Inc}_k(2 \times n)$. For each integer j from 1 to $2n-k$, we create one segment of a small Schröder path P_T. If j appears only in the first row, then the jth segment of P_T is an upstep. If j appears only in the second row of T, the jth segment of P_T is a downstep. If j appears in both rows of T, the jth segment of P_T is horizontal. It is clear that the tableau T can be reconstructed from the small Schröder path P_T, so this operation gives a bijection. Thus increasing tableaux of shape (n, n) are counted by small Schröder numbers.

A bijection between small Schröder paths with k horizontal steps and $\mathrm{SYT}(n-k, n-k, 1^k)$ may be obtained by composing the two previously described bijections. $\square$



(a) Increasing tableau T

(b) "Flag-shaped" standard Young tableau

(c) Small Schröder path and its height word

(d) Noncrossing partition

(e) Polygon dissection

Figure 7.3: A rectangular increasing tableau $T \in \mathrm{Inc}5, 52$ with its corresponding standard Young tableau of shape $(3, 3, 1, 1)$, small Schröder path, noncrossing partition of $\{1, \dots, 8\}$ with all blocks of size at least two, and heptagon dissection.

For increasing tableaux of arbitrary shape, there is unlikely to be a product formula like the hook-length formula for standard Young tableaux or our Theorem 7.1 for the 2-row rectangular case. For example, we compute that $\mathrm{Inc}_2(4, 4, 4) = 2^2 \cdot 3 \cdot 7 \cdot 19$ and that there are $3 \cdot 1531$ increasing tableaux of shape $(4, 4, 4)$ in total. Summation formulas for counting increasing tableaux of rectangular shapes have recently been given by T. Pressey, A. Stokke, and T. Visentin [PrStVi14].

The following bijection will play an important role in our proof of Theorem 7.4 in Section 7.6. A partition of $\{1, \dots, N\}$ is *noncrossing* if the convex hulls of the blocks are pairwise disjoint when the values $1, \dots, N$ are equally spaced around a circle with 1 in the upper left and values increasing counterclockwise (*cf.* Figure 7.3(D)).

Proposition 7.2. *There is an explicit bijection between* $\mathrm{Inc}_k(2 \times n)$ *and noncrossing partitions of* $2n-k$ *into* $n-k$ *blocks all of size at least 2.*

Proof. Let $T \in \mathrm{Inc}_k(2 \times n)$. For each i in the second row of T, let s_i be the largest number in the first row that is less than i and that is not s_j for some $j < i$. Form a partition of $2n-k$ by declaring, for every i, that i and s_i are in the same block. We see this partition has $n-k$ blocks by observing that the largest elements of the blocks are precisely the numbers in the second row of T that do not also appear in the first row. Clearly there are no singleton blocks.

If the partition were *not* noncrossing, there would exist some elements $a < b < c < d$ with a, c in a block B and b, d in a distinct block B'. Observe that b must appear in the first row of T and c must appear in the second row of T (not necessarily exclusively). We may assume c to be the least element of B that is greater than b. We may then assume b to be the greatest element of B' that is less than c. Now consider s_c, which must exist since c appears in the second row of T. By definition, s_c is the largest number in the first row that is less than c and that is not s_j for some $j < c$. By assumption, b appears in the first row, is less than c, and is not s_j for any $j < c$; hence $s_c \geq b$. Since however b and c lie in distinct blocks, $s_c \neq b$, whence $b < s_c < c$. This is impossible, since we took c to be the least element of B greater than b. Thus the partition is necessarily noncrossing.

To reconstruct the increasing tableau, read the partition from 1 to $2n-k$. Place the smallest elements of blocks in only the first row, place the largest elements of blocks in only the second row, and place intermediate elements in both rows. □

The set $\mathrm{Inc}_k(2 \times n)$ is also in bijection with $(n+2)$-gon dissections by $n-k-1$ diagonals. We do not describe this bijection, as it is well known (*cf.* [Sta96]) and will not be used except in Section 7.6 for comparison with previous results. The existence of a connection between increasing tableaux and polygon dissections was first suggested in [ThYo11]. An example of all these bijections is shown in Figure 7.3.

Remark 7.1. A *noncrossing matching* is a noncrossing partition with all blocks of size two. Like Dyck paths, polygon triangulations, and 2-row rectangular standard Young tableaux, noncrossing matchings are enumerated by the Catalan numbers. Since increasing tableaux were developed as a K-theoretic analogue of standard Young tableaux, it is tempting also to regard small Schröder paths, polygon dissections, and noncrossing partitions without singletons as K-theory analogues of Dyck paths, polygon triangulations, and noncrossing matchings, respectively. In particular, by analogy with [PePyRh09], it is tempting to think of noncrossing partitions without singletons as "K-webs" for $\mathfrak{sl}_2$, although their representation-theoretic significance is unknown.

7.3 K-Promotion and K-Evacuation

In this section, we prove Theorem 7.3, as well as a proposition important for Theorem 7.2. Let $\max(T)$ denote the largest entry in a tableau T. For a tableau T, we write $\texttt{rot}(T)$ for the (possibly skew) tableau formed by rotating 180 degrees and reversing the alphabet, so that label x becomes $\max(T)+1-x$. Define *dual K-evacuation* $\mathcal{E}^*$ by $\mathcal{E}^* := \texttt{rot} \circ \mathcal{E} \circ \texttt{rot}$. (This definition of $\mathcal{E}^*$ strictly makes sense only for rectangular tableaux. For a tableau T of general shape λ, in place of applying $\texttt{rot}$, one should dualize λ (thought of as a poset) and reverse the alphabet. We will not make any essential use of this more general definition.)

Towards Theorem 7.3, we first prove basic combinatorics of the above operators that are well-known in the standard Young tableau case (*cf.* [Sta09]). These early proofs are all straightforward modifications of those for the standard case. From these results, we observe that Theorem 7.3 follows from the claim that $\texttt{rot}(T) = \mathcal{E}(T)$ for every $T \in \mathrm{Inc}_k(2 \times n)$. We first saw this approach in [Wh10] for the standard Young tableau case, although similar ideas appear for example in [Ha92, Sta09]; we are not sure where it first appeared.

Finally, beginning at Lemma 7.2, we prove that for $T \in \mathrm{Inc}_k(2\times n)$, $\texttt{rot}(T) = \mathcal{E}(T)$. Here the situation is more subtle than in the standard case. (For example, we will show that the claim is not generally true for T a rectangular increasing tableau with more than 2 rows.) We proceed by careful analysis of how $\texttt{rot}, \mathcal{E}, \mathcal{E}^*$, and $\mathcal{P}$ act on the corresponding small Schröder paths.

Remark 7.2. It is not hard to see that K-promotion is reversible, and hence permutes the set of increasing tableaux.

Lemma 7.1. *K-evacuation and dual K-evacuation are involutions, $\mathcal{P} \circ \mathcal{E} = \mathcal{E} \circ \mathcal{P}^{-1}$, and for any increasing tableau T, $(\mathcal{E}^* \circ \mathcal{E})(T) = \mathcal{P}^{\max(T)}(T)$.*

Before proving Lemma 7.1, we briefly recall the *K-theory growth diagrams* of [ThYo09b, §2, 4], which extend the standard Young tableau growth diagrams of S. Fomin (*cf.* [Sta99, Appendix 1]). For $T \in \mathrm{Inc}\lambda k$, consider the sequence of Young diagrams $([T]_j)_{0 \leq j \leq |\lambda|-k}$. Note that this sequence of diagrams uniquely encodes T. We draw this sequence of Young diagrams horizontally from left to right. Below this sequence, we draw, in successive rows, the sequences of Young diagrams associated to $\mathcal{P}^i(T)$ for $1 \leq i \leq |\lambda| - k$. Hence each row encodes the K-promotion of the row above it. We offset each row one space to the right. We will refer to this entire array as the *K-theory growth diagram for T*. (There are other K-theory growth diagrams for T that one might consider, but this is the only one we will need.) Figure 7.4 shows an example. We will write YD_{ij} for the Young diagram $[\mathcal{P}^{i-1}(T)]_{j-i}$. This indexing is nothing more than imposing "matrix-style" or "English" coordinates on the K-theory growth diagram. For example in Figure 7.4, YD_{58}

denotes , the Young diagram in the fifth row from the top and the eighth column from the left.

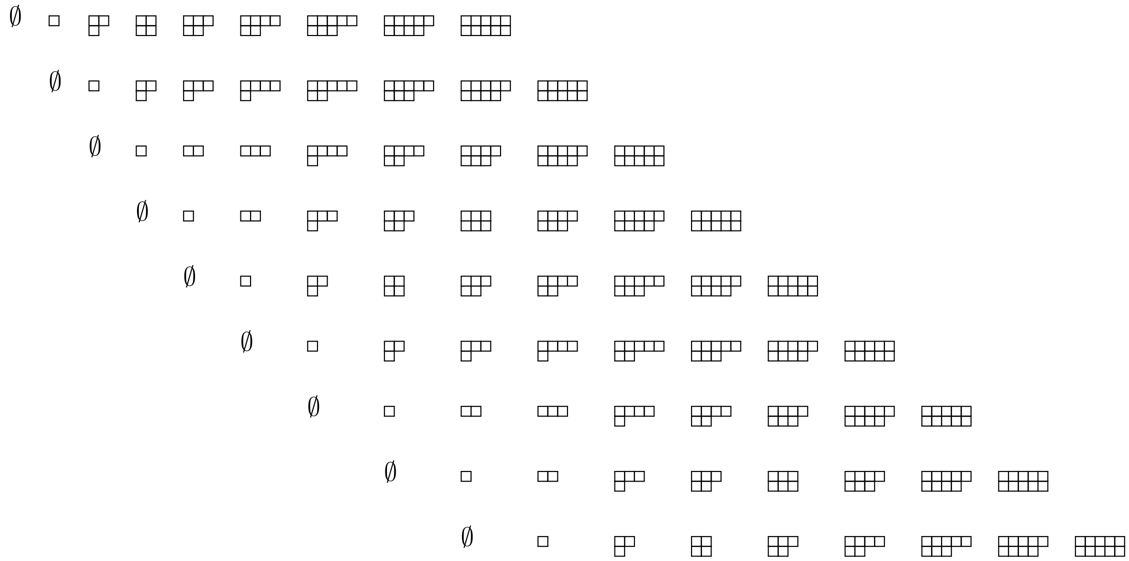


Figure 7.4: The K-theory growth diagram for the tableau T of Figure 7.3(A).

Remark 7.3. [ThYo09b, Proposition 2.2] In any 2×2 square $\begin{smallmatrix} \lambda & \mu \\ \nu & \xi \end{smallmatrix}$ of Young diagrams in a K-theory growth diagram, ξ is uniquely and explicitly determined by λ, μ and ν. Similarly λ is uniquely and explicitly determined by μ, ν and ξ. Furthermore these rules are symmetric, in the sense that if $\begin{smallmatrix} \lambda & \mu \\ \nu & \xi \end{smallmatrix}$ and $\begin{smallmatrix} \xi & \mu \\ \nu & \rho \end{smallmatrix}$ are both 2×2 squares of Young diagrams in K-theory growth diagrams, then $\lambda = \rho$.

Proof of Lemma 7.1. Fix a tableau $T \in \mathrm{Inc}\lambda k$. All of these facts are proven as in the standard case (*cf.* [Sta09, §5]), except one uses K-theory growth diagrams instead of ordinary growth diagrams. We omit some details from these easy arguments. The proof that K-evacuation is an involution appears in greater detail as [ThYo09b, Theorem 4.1]. For rectangular shapes, the fact that dual K-evacuation is an involution follows from the fact that K-evacuation is, since $\mathcal{E}^* = \mathtt{rot} \circ \mathcal{E} \circ \mathtt{rot}$.

Briefly one observes the following. Essentially by definition, the central column (the column containing the rightmost $\emptyset$) of the K-theory growth diagram for T encodes the K-evacuation of the first row as well as the dual K-evacuation of the last row. The first row encodes T and the last row encodes $\mathcal{P}^{|\lambda|-k}(T)$. Hence $\mathcal{E}(T) = \mathcal{E}^*(\mathcal{P}^{|\lambda|-k}(T))$.

By the symmetry mentioned in Remark 7.3, one also observes that the first row encodes the K-evacuation of the central column and that the last row encodes the dual K-evacuation of the central column. This yields $\mathcal{E}(\mathcal{E}(T)) = T$ and $\mathcal{E}^*(\mathcal{E}^*(\mathcal{P}^{|\lambda|-k}(T))) = \mathcal{P}^{|\lambda|-k}(T)$, showing that K-evacuation and dual K-evacuation are involutions. Combining the above observations, yields $(\mathcal{E}^* \circ \mathcal{E})(T) = \mathcal{P}^{|\lambda|-k}(T)$.

Finally to show $\mathcal{P} \circ \mathcal{E} = \mathcal{E} \circ \mathcal{P}^{-1}$, it is easiest to append an extra $\emptyset$ to the lower-right of the diagonal line of $\emptyset$s that appears in the K-theory growth diagram. This extra $\emptyset$ lies in the column just right of the central one. This column now encodes the K-evacuation of the second row. Hence by the symmetry mentioned in Remark 7.3, the K-promotion of this column is encoded by the central column. Thus if $S = \mathcal{P}(T)$, the central column encodes $\mathcal{P}(\mathcal{E}(S))$. But certainly $\mathcal{P}^{-1}(S) = T$ is encoded by the first row, and we have already observed that the central column encodes $\mathcal{E}(T)$. Therefore $\mathcal{P}(\mathcal{E}(S)) = \mathcal{E}(\mathcal{P}^{-1}(S))$. □

Let $\mathrm{er}(T)$ be the least positive integer such that $(\mathcal{E}^* \circ \mathcal{E})^{\mathrm{er}(T)}(T) = T$. We call this number the *evacuation rank* of T. Similarly we define the *promotion rank* $\mathrm{pr}(T)$ to be the least positive integer such that $\mathcal{P}^{\mathrm{pr}(T)}(T) = T$.

Corollary 7.2. *Let T be a increasing tableau. Then $\mathrm{er}(T)$ divides $\mathrm{pr}(T)$, $\mathrm{pr}(T)$ divides $\max(T) \cdot \mathrm{er}(T)$, and the following are equivalent:*

(a) $\mathcal{E}(T) = \mathcal{E}^*(T)$,

(b) $\mathrm{er}(T) = 1$,

(c) $\mathrm{pr}(T)$ *divides* $\max(T)$.

Moreover if T is rectangular and $\mathcal{E}(T) = \mathtt{rot}(T)$, then $\mathcal{E}(T) = \mathcal{E}^*(T)$.

Proof. Since, by Lemma 7.1, we have $(\mathcal{E}^* \circ \mathcal{E})(T) = \mathcal{P}^{\max(T)}(T)$, the evacuation rank of T is the order of $c^{\max(T)}$ in the cyclic group $\mathcal{C}_{\mathrm{pr}(T)} = \langle c \rangle$. In particular, $\mathrm{er}(T)$ divides $\mathrm{pr}(T)$. Since $T = (\mathcal{E}^* \circ \mathcal{E})^{\mathrm{er}(T)}(T) = (\mathcal{P}^{\max(T)})^{\mathrm{er}(T)}(T) = \mathcal{P}^{\max(T)\cdot \mathrm{er}(T)}(T)$, we have that $\max(T) \cdot \mathrm{er}(T)$ is a multiple of $\mathrm{pr}(T)$.

The equivalence of (a) and (b) is immediate from dual evacuation being an involution. These imply (c), since $(\mathcal{E}^* \circ \mathcal{E})(T) = \mathcal{P}^{\max(T)}(T)$. If $\mathrm{pr}(T)$ divides $\max(T)$, then $\mathcal{P}^{\max(T)}(T) = T$, so $(\mathcal{E}^* \circ \mathcal{E})(T) = T$, showing that (c) implies (b).

By definition, for rectangular T, $\mathcal{E}^*(T) = (\mathtt{rot} \circ \mathcal{E} \circ \mathtt{rot})(T)$, so if $\mathtt{rot}(T) = \mathcal{E}(T)$, then $\mathcal{E}^*(T) = (\mathcal{E} \circ \mathcal{E} \circ \mathcal{E})(T) = \mathcal{E}(T)$. □

Thus to prove Theorem 7.3, it suffices to show the following proposition:

Proposition 7.3. *Let $T \in \mathrm{Inc}_k(2 \times n)$. Then $\mathcal{E}(T) = \mathtt{rot}(T)$.*

We will also need Proposition 7.3 in the proof of Theorem 7.2, and additionally it has recently found application in [BlPeSa16, Theorem 5.3] which demonstrates homomesy (as defined by [PrRo13a]) on $\mathrm{Inc}_k(2 \times n)$. To prove Proposition 7.3, we use the bijection between $\mathrm{Inc}_k(2 \times n)$ and small Schröder paths from Theorem 7.1. These paths are themselves in bijection with the sequence of their node heights, which we call the *height word*. Figure 7.3(C) shows an example. For $T \in \mathrm{Inc}_k(2 \times n)$, we write P_T for the corresponding small Schröder path and S_T for the corresponding height word.

Lemma 7.2. *For $T \in \mathrm{Inc}_k(2 \times n)$, the ith letter of the height word S_T is the difference between the lengths of the first and second rows of the Young diagram $[T]_{i-1}$.*

Proof. By induction on i. For $i = 1$, both quantities equal 0. The ith segment of P_T is an upstep if and only if $[T]_i \setminus [T]_{i-1}$ is a single box in the first row. The ith segment of P_T is an downstep if and only if $[T]_i \setminus [T]_{i-1}$ is a single box in the second row. The ith segment of P_T is horizontal if and only if $[T]_i \setminus [T]_{i-1}$ is two boxes, one in each row. $\square$

Lemma 7.3. *Let $T \in \mathrm{Inc}_k(2 \times n)$. Then $P_{\mathtt{rot}(T)}$ is the reflection of P_T across a vertical line and $S_{\mathtt{rot}(T)}$ is the word formed by reversing S_T.*

Proof. Rotating T by 180 degrees corresponds to reflecting P_T across the horizontal axis. Reversing the alphabet corresponds to rotating P_T by 180 degrees. Thus $\mathtt{rot}(T)$ corresponds to the path given by reflecting P_T across a vertical line.

The correspondence between reflecting P_T and reversing S_T is clear. $\square$

Lemma 7.4. *Let $T \in \mathrm{Inc}_k(2 \times n)$ and $M = 2n - k$. Let x_i denote the $(M + 2 - i)$th letter of the height word $S_{\mathcal{P}^{i-1}(T)}$. Then $S_{\mathcal{E}(T)} = x_{M+1} x_M \ldots x_1$.*

Proof. Consider the K-theory growth diagram for T. Observe that $YD_{i,M+1}$ is the $(M + 2 - i)$th Young diagram in the ith row. Hence by Lemma 7.2, x_i is the difference between the lengths of the rows of $YD_{i,M+1}$. But $YD_{i,M+1}$ is also the ith Young diagram from the top in the central column. The lemma follows by recalling that the central column encodes $\mathcal{E}(T)$. $\square$

We define the *flow path* $\phi(T)$ of an increasing tableau T to be the set of all boxes that are ever empty during the K-promotion that forms $\mathcal{P}(T)$ from T.

Lemma 7.5. *Let $T \in \mathrm{Inc}_k(2 \times n)$.*

(a) The word S_T may be written in exactly one way as $0w_10w_3$ or $0w_11w_20w_3$, where w_1 is a sequence of strictly positive integers that ends in 1 and contains no consecutive 1s, w_2 is a (possibly empty) sequence of strictly positive integers, and w_3 is a (possibly empty) sequence of nonnegative integers.

(b) Let w_1^- be the sequence formed by decrementing each letter of w_1 by 1. Similarly, let w_3^+ be formed by incrementing each letter of w_3 by 1.

If S_T is of the form $0w_10w_3$, then $S_{\mathcal{P}(T)} = w_1^- 1 w_3^+ 0$. If S_T is of the form $0w_11w_20w_3$, then $S_{\mathcal{P}(T)} = w_1^- 1 w_2 1 w_3^+ 0$.

Proof. It is clear that S_T may be written in exactly one of the two forms. Write ℓ_i for the length of w_i. Suppose first that S_T is of the form $0w_10w_3$. By the correspondence between tableaux and height sequences, $[T]_{\ell_1+1}$ is a rectangle, and for no $0 < x < \ell_1 + 1$ is $[T]_x$ a rectangle. Say $[T]_{\ell_1+1} = (m, m)$. The flow path $\phi(T)$ contains precisely the first m boxes of the first row and the last $n - m + 1$ boxes of the second row. Only the entry in box $(2, m)$ changes row during K-promotion. It is clear then that $S_{\mathcal{P}(T)} = w_1^- 1 w_3^+ 0$.

Suppose now that S_T is of the form $0w_11w_20w_3$. Then $[T]_{\ell_1+1} = (p+1, p)$ for some p, and $[T]_{\ell_1+\ell_2+2} = (m, m)$ for some m. The flow path $\phi(T)$ contains precisely the first m boxes of the first row and the last $n - p + 1$ boxes of the second row. It is clear then that $S_{\mathcal{P}(T)} = w_1^- 1 w_2 1 w_3^+ 0$. □

Notice that when $T \in \mathrm{SYT}(2 \times n)$, S_T can always be written as $0w_10w_3$. Hence by Lemma 7.5(b), the promotion $S_{\mathcal{P}(T)}$ takes the particularly simple form $w_1^- 1 w_3^+ 0$.

For $T \in \mathrm{Inc}_k(2 \times n)$, take the first $2n - k + 1$ columns of the K-theory growth diagram for T. Replace each Young diagram in the resulting array by the difference between the lengths of its first and second rows. Figure 7.5 shows an example. We write a_{ij} for the number corresponding to the Young diagram YD_{ij}. By Lemma 7.2, we see that the ith row of this array of nonnegative integers is exactly the first $2n - k + 2 - i$ letters of $S_{\mathcal{P}^{i-1}(T)}$. Therefore we will refer to this array as the *height growth diagram for* T, and denote it by $\mathrm{hgd}(T)$. Observe that the rightmost column of $\mathrm{hgd}(T)$ corresponds to the central column of the K-theory growth diagram for T.

$$\begin{array}{ccccccccc}
0 & 1 & 1 & 0 & 1 & 2 & 2 & 1 & 0 \\
 & 0 & 1 & 1 & 2 & 3 & 3 & 2 & 1 \\
 & & 0 & 1 & 2 & 3 & 3 & 2 & 1 \\
 & & & 0 & 1 & 2 & 2 & 1 & 0 \\
 & & & & 0 & 1 & 1 & 0 & 1 \\
 & & & & & 0 & 1 & 1 & 2 \\
 & & & & & & 0 & 1 & 2 \\
 & & & & & & & 0 & 1 \\
 & & & & & & & & 0
\end{array}$$

Figure 7.5: The height growth diagram $\mathrm{hgd}(T)$ for the tableau T shown in Figure 7.3(A). The ith row shows the first $10 - i$ letters of $S_{\mathcal{P}^{i-1}(T)}$. Lemma 7.6 says that row 1 is the same as column 9, read from top to bottom.

We will sometimes write $\mathcal{P}(S_T)$ for $S_{\mathcal{P}(T)}$.

Lemma 7.6. *In $\mathrm{hgd}(T)$ for $T \in \mathrm{Inc}_k(2 \times n)$, we have for all j that $a_{1j} = a_{j,2n-k+1}$.*

Proof. Let $M = 2n - k$. We induct on the length of the height word. (The length of S_T is $M + 1$.)

Case 1. The height word S_T contains an internal 0.

Let the first internal 0 be the jth letter of S_T. Then the first j letters of S_T are themselves the height word of some smaller rectangular increasing tableau T'. Because of the local properties of K-theory growth diagrams mentioned in Remark 7.3, we observe that the jth column of $\text{hgd}(T)$ is the same as the rightmost column of $\text{hgd}(T')$. The height word $S_{T'}$ is shorter than the height word S_T, so by inductive hypothesis, the first row of $\text{hgd}(T')$ is the same as its rightmost column, read from top to bottom. Thus in $\text{hgd}(T)$, the first j letters of row 1 are the same as column j.

According to Lemma 7.5(b), in each of the first j rows of $\text{hgd}(T)$, the letter in column j is less than or equal to all letters to its right. Furthermore the letters in columns $j+1$ through $M+1$ are incremented, decremented, or unchanged from one row to the next in exactly the same way as the letter in column j. That is to say, for any $g \leq j \leq h$, $a_{gh} - a_{1h} = a_{gj} - a_{1j}$. Since $a_{1j} = a_{1,M+1} = 0$, this yields $a_{gj} = a_{g,M+1}$, so column j is the same as the first j letters of column $M+1$, read from top to bottom. Thus the first j letters of row 1 are the same as the first j letters of column $M+1$.

Now since $a_{j,M+1} = 0$, row j of $\text{hgd}(T)$ is itself the height word of some smaller tableau $T^\dagger$. Again by inductive hypothesis, we conclude that in $\text{hgd}(T)$, row j is the same as the last $M+2-j$ letters of column $M+1$, read from top to bottom. But as previously argued, the letters in columns $j+1$ through $M+1$ are incremented, decremented, or unchanged from one row to the next in the same way as the letter in column j. Hence row j agrees with the last $M+2-j$ letters of row 1, and so the last $M+2-j$ letters of row 1 agree with the last $M+2-j$ letters of column $M+1$. Thus, as desired, row 1 of $\text{hgd}(T)$ is the same as column $M+1$, read from top to bottom.

Case 2. The height word S_T contains no internal 0.

Notice that $s_{1M} = 1$. Hence by Lemma 7.5(b), there will be an internal 0 in the K-promotion $S_{\mathcal{P}(T)}$, unless S_T is the word 010 or begins 011.

Case 2.1. The height word $S_{\mathcal{P}(T)}$ contains an internal 0.

Let the first internal 0 be in column j of $\text{hgd}(T)$. Then by Lemma 7.5(b), the first $j-1$ letters of row 2 of $\text{hgd}(T)$ are all exactly one less than the letters directly above them in row 1. That is for $2 \leq h \leq j$, we have $a_{2h} = a_{1h} - 1$. Also observe $a_{2,M+1} = 1$.

The first $j-1$ letters of row 2 are the height sequence of some tableau T' with $S_{T'}$ shorter than S_T. So by inductive hypothesis, the first $j-1$ letters of row 2 of $\text{hgd}(T)$ are the same as the last $j-1$ letters of column j, read from top to bottom. That is to say $a_{2h} = a_{hj}$, for all $2 \leq h \leq j$.

Since $a_{2j} = 0$ and $a_{2,M+1} = 1$, and since the letters below the first row in columns $j+1$ through $M+1$

are incremented, decremented, or unchanged in the same way as the letter in column j, it follows that for $2 \leq h \leq j$, $a_{hj} = a_{h,M+1} - 1$. Therefore the first j letters of row 1 are the same as the first j letters of column $M+1$.

Consider the height word S' formed by prepending a 0 to the last $M+2-j$ letters of row 1. The last $M+2-j$ letters of row 2 are the same as the first $M+2-j$ letters of $\mathcal{P}(S')$. But the last $M+2-j$ letters of row 2 are the same as row j. Therefore by inductive hypothesis, the last $M+2-j$ letters of column $M+1$ are the same as the last $M+2-j$ letters of S', which are by construction exactly the last $M+2-j$ letters of row 1. Thus row 1 is exactly the same as column $M+1$.

Case 2.2. $S_T = 010$.

Trivially verified by hand.

Case 2.3. S_T begins 011.

Row 2 of hgd(T) is produced from row 1 by deleting the initial 0, changing the first 1 into a 0, and changing the final 0 into a 1. Let S' be the word formed by replacing the final 1 of row 2 with a 0. Note that $a_{3,M+1} = 1$. Therefore row 3 agrees with the first $M-1$ letters of $\mathcal{P}(S')$. Therefore by inductive hypothesis, the last $M-1$ letters of S' are the same as the last $M-1$ letters of column $M+1$. Hence the last M letters of column $M+1$ are the same as row 2, except for having a 1 at the beginning instead of a 0 and a 0 at the end instead of a 1. But these are exactly the changes we made to produce row 2 from row 1. Thus row 1 is the same as column $M+1$. □

Corollary 7.3. *In the notation of Lemma 7.4,* $S_T = x_1 x_2 \ldots x_{M+1}$. □

Proof of Proposition 7.3. By Corollary 7.3, $S_T = x_1 x_2 \ldots x_{2n-k+1}$. Hence by Lemma 7.3, we have $S_{\texttt{rot}(T)} = x_{2n-k+1} x_{2n-k} \ldots x_1$. However Lemma 7.4 says also $S_{\mathcal{E}(T)} = x_{2n-k+1} x_{2n-k} \ldots x_1$. By the bijective correspondence between tableaux and height words, this yields $\mathcal{E}(T) = \texttt{rot}(T)$. □

This completes our first proof of Theorem 7.3. We will obtain alternate proofs in Sections 7.4 and 7.6. We now show a counterexample to the obvious generalization of Theorem 7.3 to increasing tableaux of more than two rows.

Example 7.1. If T is the increasing tableau $\begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 7 \\ \hline 3 & \underline{5} & 6 & 8 \\ \hline 5 & \underline{7} & 8 & 10 \\ \hline 7 & 9 & 10 & 11 \\ \hline \end{array}$, then $\mathcal{P}^{11}(T) = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 7 \\ \hline 3 & \underline{4} & 6 & 8 \\ \hline 5 & \underline{6} & 8 & 10 \\ \hline 7 & 9 & 10 & 11 \\ \hline \end{array}$. (The underscores mark entries that differ between the two tableaux.) It can be verified that the promotion rank of T is 33. ◊

Computer checks of small examples (including all with at most seven columns) did not identify such a counterexample for T a 3-row rectangular increasing tableau. Indeed Conjecture 8.1 would imply that there are none. However it is not generally true that $\mathcal{E}(T) = \texttt{rot}(T)$ for $T \in \mathrm{Inc}_k(3 \times n)$.

Example 7.2. If T is the increasing tableau $\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & \underline{4} & 6 \\ \hline 5 & 7 & 8 \\ \hline \end{array}$ then $\mathcal{E}(T) = T$, while $\texttt{rot}(T) = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & \underline{5} & 6 \\ \hline 5 & 7 & 8 \\ \hline \end{array}$. Nonetheless the promotion rank of T is 2, which divides 8, so the obvious generalization of Theorem 7.3 holds in this example. ♢

7.4 Tropical frieze patterns

In this section, we make connections with tropical frieze patterns, which we use to give an alternate proof of Proposition 7.3 and Theorem 7.3.

Frieze patterns are simple cluster algebras introduced in [CoCo73]. They are infinite arrays of real numbers bounded between two parallel diagonal lines of 1s, satisfying the property that for each 2×2 subarray $\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}$ the relation $d = (bc + 1)/a$ holds. Figure 7.6 shows an example. Notice that by this local defining relation, the frieze pattern is determined by any one of its rows.

$$
\begin{array}{ccccccccccccc}
 & & \ddots & & & & & & & & & & \\
1 & 3 & 5 & 4 & 2 & 1 & & & & & & & \\
 & 1 & 2 & \frac{9}{5} & \frac{23}{20} & \frac{43}{40} & 1 & & & & & & \\
 & & 1 & \frac{7}{5} & \frac{29}{20} & \frac{89}{40} & 3 & 1 & & & & & \\
 & & & 1 & \frac{7}{4} & \frac{27}{8} & 5 & 2 & 1 & & & & \\
 & & & & 1 & \frac{5}{2} & 4 & \frac{9}{5} & \frac{7}{5} & 1 & & & \\
 & & & & & 1 & 2 & \frac{23}{20} & \frac{29}{20} & \frac{7}{4} & 1 & & \\
 & & & & & & 1 & \frac{43}{40} & \frac{89}{40} & \frac{27}{8} & \frac{5}{2} & 1 & \\
 & & & & & & & 1 & 3 & 5 & 4 & 2 & 1 \\
 & & & & & & & & & & \ddots & &
\end{array}
$$

Figure 7.6: A classical Conway–Coxeter frieze pattern.

A tropical analogue of frieze patterns may be defined by replacing the bounding 1s by 0s and imposing the tropicalized relation $d = \max(b + c, 0) - a$ on each 2×2 subarray. Such *tropical frieze patterns* have attracted some interest lately (e.g., [Pr05, Gu13, AsDu13, Gr15]).

One of the key results of [CoCo73] is that, if rows of a frieze pattern have length ℓ, then each row is equal to the row $\ell + 1$ rows below it, as well as to the central column between these two rows, read from top to bottom (*cf.* Figure 7.6). That the same periodicity occurs in tropical frieze patterns may be proved directly by imitating the classical proof, or it may be easily derived from the classical periodicity by taking logarithms. We do the latter.

Lemma 7.7. *If $\mathcal{TF}$ is a tropical frieze diagram with rows of length ℓ, then each row is equal to the row $\ell+1$ rows below it, as well as to the central column between these two rows, read from top to bottom.*

Proof. Pick a row $R = (a_0, a_1, \ldots, a_\ell)$ of $\mathcal{TF}$. Let e be the base of the natural logarithm.

For $t \in \mathbb{R}_{>0}$, construct the classical frieze pattern containing the row $R' = (e^{a_0/t}, \ldots, e^{a_n/t})$. Now take the logarithm of each entry of this frieze pattern and multiply each entry by t. Call the result $\mathcal{F}_t$. Note that $\mathcal{F}_t$ is not in general a frieze pattern; however, it does have the desired periodicity. Also observe that the row R appears in each $\mathcal{F}_t$ as the image of R'. Now take $\lim_{t\to 0} \mathcal{F}_t$. This limit also contains the row R. This process converts the relation $d = (bc+1)/a$ into the relation $d = \max(b+c, 0) - a$, so $\lim_{t\to 0} \mathcal{F}_t = \mathcal{TF}$. Since each $\mathcal{F}_t$ has the desired periodicities, so does $\mathcal{TF}$. $\square$

Let $T \in \mathrm{Inc}_k(2 \times n)$. Recall from Section 7.3 the K-theory growth diagram for T. Replace each Young diagram by 1 less than the difference between the lengths of its first and second rows. Delete the first and last number in each row (necessarily -1). We call the resulting array the *jeu de taquin frieze pattern* of T. (It is obviously closely related to the height growth diagram.) Each row is an integer sequence encoding the tableau corresponding to that row of the K-theory growth diagram. (Indeed it is the height word with all terms decremented by 1, and the first and last terms removed.)

Remark 7.4. Observe that this map from increasing tableaux to integer sequences is injective. The image is exactly those sequences such that

(1) the first and last terms are 0,

(2) every term is ≥ -1,

(3) successive terms differ by at most 1, and

(4) there are no consecutive -1s.

An integer sequence is the image of a standard Young tableau if it satisfies the stronger

(3′) successive terms differ by exactly 1,

in place of condition (3).

Example 7.3. For $T = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 5 \\ \hline 4 & 5 & 6 & 7 \\ \hline \end{array}$, we obtain the jeu de taquin frieze pattern

$$
\begin{array}{ccccccccccccc}
0 & 1 & 2 & 1 & 1 & 0 & & & & & & & \\
 & 0 & 1 & 0 & 0 & -1 & 0 & & & & & & \\
 & & 0 & -1 & 0 & 0 & 1 & 0 & & & & & \\
 & & & 0 & 1 & 1 & 2 & 1 & 0 & & & & \\
 & & & & 0 & 0 & 1 & 0 & -1 & 0 & & & \\
 & & & & & 0 & 1 & 0 & 0 & 1 & 0 & & \\
 & & & & & & 0 & -1 & 0 & 1 & 0 & 0 & \\
 & & & & & & & 0 & 1 & 2 & 1 & 1 & 0
\end{array}
.
$$
◇

The fact that the first row, last row, and central column are all equal is equivalent to Lemma 7.6 and Theorem 7.3. The following lemma gives an alternate approach.

Lemma 7.8. *A jeu de taquin frieze pattern is a subarray of a tropical frieze pattern.*

Proof. It is clear that we have bounding diagonals of 0s. It suffices to verify the local defining relation $d = \max(b+c, 0) - a$ on 2×2 subarrays. This follows fairly easily from the algorithmic relation of Lemma 7.5. □

The next corollary follows immediately from the above; although it can be proven directly, the derivation from results on K-promotion seems more enlightening.

Corollary 7.4. *Let $\mathcal{TF}$ be a tropical frieze diagram.*

(a) If any row of $\mathcal{TF}$ satisfies the conditions (1), (2), (3′) of Remark 7.4, then every row of $\mathcal{TF}$ does.

(b) If any row of $\mathcal{TF}$ satisfies the four conditions (1), (2), (3), (4) of Remark 7.4, then every row of $\mathcal{TF}$ does. □

Remark 7.5. We speculate ahistorically that one could have discovered K-promotion for increasing tableaux in the following manner. First one could have found a proof of Theorem 7.3 for standard Young tableaux along the lines of this section (indeed similar ideas appear in [KiBe96]), proving Corollary 7.4(a) in the process. Looking for similar results, one might observe Corollary 7.4(b) experimentally and be lead to discover K-promotion in proving it.

Are there other special tropical frieze patterns hinting at a promotion theory for other classes of tableaux? For example, tropical friezes containing a row satisfying conditions (1), (2), (3) of Remark 7.4 seem experimentally to be well-behaved, with all rows having successive terms that differ by at most 2.

We are able to prove the order of promotion on $\mathrm{SYT}(3 \times n)$ in a similar fashion, using tropicalizations of the 2-frieze patterns of [MoOvTa12]. Unfortunately we have been unable to extend this argument to $\mathrm{Inc}_k(3 \times n)$ for $k > 0$. Example 7.2 suggests that such an extension would be difficult.

7.5 Representation-theoretic interpretations

In [Ste95], J. Stembridge proved that, for every λ, $(\mathrm{SYT}(\lambda), \mathcal{C}_2, f^\lambda(q))$ exhibits cyclic sieving, where the non-identity element of $\mathcal{C}_2$ acts by evacuation and $f^\lambda(q)$ is the standard q-analogue of the hook-length formula. We briefly recall the outline of this argument. Considering the Kazhdan–Lusztig cellular basis for the Specht module V^λ, the long element $w_0 \in \mathfrak{S}_{|\lambda|}$ acts (up to a controllable sign) on V^λ by permuting the basis elements. Moreover under a natural indexing of the basis by $\mathrm{SYT}(\lambda)$, the permutation is exactly evacuation. The cyclic sieving then follows by evaluating the character of V^λ at w_0.

We can give an analogous proof of Theorem 7.2. However to avoid redundancy we do not do so here, and instead derive Theorem 7.2 by direct reduction to J. Stembridge's result.

Let $\mathcal{F}$ denote the map from $\mathrm{Inc}_k(2\times n)$ to $\mathrm{SYT}(n-k, n-k, 1^k)$ from Proposition 7.1. Theorem 7.2 follows immediately from the following proposition combined with J. Stembridge's previously-described result.

Proposition 7.4. *For all $T \in \mathrm{Inc}_k(2 \times n)$, $\mathcal{E}(\mathcal{F}(T)) = \mathcal{F}(\mathcal{E}(T))$.*

Proposition 7.4 was first suggested to the author by B. Rhoades, who also gave some ideas to the proof. Before we prove this result, we introduce some additional notation. Observe that if $S \in \mathrm{SYT}(n-k, n-k, 1^k)$, then $\mathtt{rot}(S)$ has skew shape $((n-k)^{k+2})/((n-k-1)^k)$. Let $T \in \mathrm{Inc}_k(2 \times n)$. Let A be the set of numbers that appear twice in T. Let C be the set of numbers that appear in the first row immediately left of an element of A. Let $d(T)$ be the tableau of shape $(n-k, n-k)$ formed by deleting all elements of A from the second row of T and all elements of C from the first. A tableau $\dashv(T)$ of skew shape $((n-k)^{k+2})/((n-k-1)^k)$ is given by attaching C in the kth column of $d(T)$. Figure 7.7 illustrates these maps. It is immediate from comparing the definitions that $\mathtt{rot}(\mathcal{F}(T)) = \dashv(\mathtt{rot}(T))$.

Lemma 7.9. *For all $T \in \mathrm{Inc}_k(2 \times n)$, $\mathcal{F}(T)$ is the rectification of $\dashv(T)$.*

Proof. It is enough to show that $\mathcal{F}(T)$ and $\dashv(T)$ lie in the same plactic class. Consider the row reading word of $\dashv(T)$. Applying the Robinson-Schensted-Knuth algorithm (see Section 1.4) to the first $2(n-k)$ letters of this word, we obtain the tableau $d(T)$. The remaining letters are those that appear in the first row of T immediately left of a element that appears in both rows. These letters are in strictly decreasing order. It remains to Schensted bump these remaining letters into $d(T)$ in strictly decreasing order, and observe that we obtain the tableau $\mathcal{F}(T)$.

$$\begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & 4 & 7 \\ \hline 4 & 5 & 6 & 7 & 8 \\ \hline \end{array}$$

(a) Increasing tableau T

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 6 & 7 \\ \hline 5 \\ \cline{1-1} 8 \\ \cline{1-1} \end{array}$$

(b) Flag-shaped tableau $\mathcal{F}(T)$

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 7 \\ \hline 5 & 6 & 8 \\ \hline \end{array}$$

(c) $d(T)$

$$\begin{array}{ccc} \cline{3-3} & & \multicolumn{1}{|c|}{3} \\ \cline{3-3} & & \multicolumn{1}{|c|}{4} \\ \hline \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{7} \\ \hline \multicolumn{1}{|c|}{5} & \multicolumn{1}{c|}{6} & \multicolumn{1}{c|}{8} \\ \hline \end{array}$$

(d) $\Finv(T)$

Figure 7.7: An illustration of the maps $\mathcal{F}$, d, and $\Finv$ on 2-row rectangular increasing tableaux.

Suppose we first bump in the letter i. By assumption i appears in the first row of T immediately left of an element j that appears in both rows. Since i is the biggest such letter, j appears in the first row of $d(T)$. Hence i bumps j out of the first row. We then bump j into the second row. The element that it bumps out of the second row is the least element greater than it. This is precisely the element h immediately to the right of j in the second row of T.

Repeating this process, since the elements we bump into $d(T)$ are those that appear in the first row of T immediately left of elements that appear in both rows, we observe that the elements that are bumped out of the first row are precisely those that appear in both rows of T. Hence the first row of the resulting tableau consists exactly of those elements that appear only in the first row of T. Thus the first row of the rectification of $\Finv(T)$ is the same as the first row of $\mathcal{F}(T)$.

Moreover, since the elements bumped out of the first row are precisely those that appear in both rows of T, these are also exactly the elements bumped into the second row. Therefore the elements bumped out of the second row are exactly those that appear in the second row of T immediately right of an element that appears twice. Thus the second row of the rectification of $\Finv(T)$ is also the same as the second row of $\mathcal{F}(T)$.

Finally, since elements are bumped out of the second row in strictly decreasing order, the resulting tableau has the desired shape $(n-k, n-k, 1^k)$. Thus $\mathcal{F}(T)$ is the rectification of $\Finv(T)$. $\square$

Proof of Proposition 7.4. Fix $T \in \mathrm{Inc}_k(2 \times n)$. Recall from Proposition 7.3 that $\mathcal{E}(T) = \texttt{rot}(T)$. Evacuation of standard Young tableaux can be defined as applying `rot` followed by rectification to a straight shape. Hence to prove $\mathcal{E}(\mathcal{F}(T)) = \mathcal{F}(\mathcal{E}(T))$, it suffices to show that $\mathcal{F}(\texttt{rot}(T))$ is the rectification of $\texttt{rot}(\mathcal{F}(T))$. We observed previously that $\texttt{rot}(\mathcal{F}(T)) = \Finv(\texttt{rot}(T))$. Hence Lemma 7.9 completes the proof by showing that $\Finv(\texttt{rot}(T))$ rectifies to $\mathcal{F}(\texttt{rot}(T))$. $\square$

B. Rhoades' proof of cyclic sieving for $\mathrm{SYT}(m \times n)$ under promotion follows the same general structure as J. Stembridge's proof for $\mathrm{SYT}(\lambda)$ under evacuation. That is, he considers the Kazhdan–Lusztig cellular representation $V^{m\times n}$ with basis indexed by $\mathrm{SYT}(m\times n)$ and looks for an element $w\in\mathfrak{S}_{mn}$ that acts (up to scalar multiplication) by sending each basis element to its promotion. It turns out that the long cycle $w=(123\ldots mn)$ suffices.

Given our success interpreting Theorem 7.2 along these lines, one might hope to prove Theorem 7.4 as follows. Take the Kazhdan–Lusztig cellular representation $V^{(n-k,n-k,1^k)}$ and index the basis by $\mathrm{Inc}_k(2\times n)$ via the bijection of Proposition 7.1. Then look for an element $w\in\mathfrak{S}_{2n-k}$ that acts (up to scalar multiplication) by sending each basis element to its K-promotion. With B. Rhoades, the author investigated this approach. Unfortunately we found by explicit computation that id and w_0 are generally the only elements of $\mathfrak{S}_{2n-k}$ acting on $V^{(n-k,n-k,1^k)}$ as permutation matrices (even up to scalar multiplications). This does not necessarily mean that no element of the group algebra could play the role of w. However prospects for this approach seem to us dim. We prove Theorem 7.4 in the next section by elementary combinatorial methods.

B. Rhoades [Rh15] however has recently given a representation-theoretic proof of Theorem 7.4 (and some related results) by introducing yet another basis for the Specht module corresponding to the shape $(n-k,n-k,1^k)$. The basis is indexed by the noncrossing partitions of $2n-k$ into $n-k$ blocks all of size at least 2. Here the action of $\mathfrak{S}_{2n-k}$ is by the defining permutation representation on the $2n-k$ elements being partitioned, but with any introduced crossings being resolved by intricate two-dimensional analogues of *skein relations*. In general, this skein basis is distinct from both the standard Specht module basis and the Kazhdan–Lusztig basis. B. Rhoades finishes the proof of Theorem 7.4 by showing that with respect to this new basis, the long cycle $w=(123\ldots(2n-k))$ acts (up to scalars) by rotation, so the result follows from some character evaluations.

7.6 Proof of Theorem 7.4

Recall definition (7.2) of $f(q)$. Our strategy (modeled throughout on [ReStWh04, §7]) is to explicitly evaluate f at roots of unity and compare the result with a count of increasing tableaux. To count tableaux, we use the bijection with noncrossing partitions given in Proposition 7.2. We will find that the symmetries of these partitions more transparently encode the promotion ranks of the corresponding tableaux.

Lemma 7.10. *Let ζ be any primitive dth root of unity, for d dividing $2n-k$. Then*

$$f(\zeta) = \begin{cases} \dfrac{(\frac{2n-k}{d})!}{(\frac{k}{d})!(\frac{n-k}{d})!(\frac{n-k}{d}-1)!\frac{n}{d}}, & \text{if } d|n \\[2em] \dfrac{(\frac{2n-k}{d})!}{(\frac{k+2}{d}-1)!(\frac{n-k-1}{d})!(\frac{n-k-1}{d})!\frac{n+1}{d}}, & \text{if } d|n+1 \\[2em] 0, & \text{otherwise.} \end{cases}$$

Proof. As in [ReStWh04, §7], we observe that

- for all $j \in \mathbb{N}$, ζ is a root of $[j]_q$ if and only if $d > 1$ divides j, and
- for all $j, j' \in \mathbb{N}$ with $j \equiv j' \mod d$,

$$\lim_{q \to \zeta} \frac{[j]_q}{[j']_q} = \begin{cases} \dfrac{j}{j'}, & \text{if } j \equiv 0 \mod d \\[2em] 1, & \text{if } j \not\equiv 0 \mod d. \end{cases}$$

The desired formula is then an easy but tedious calculation analogous to those carried out in [ReStWh04, §7]. □

We will write π for the bijection of Proposition 7.2 from $\mathrm{Inc}_k(2 \times n)$ to noncrossing partitions of $2n-k$ into $n-k$ blocks all of size at least 2. For Π a noncrossing partition of N, we write $\mathcal{R}(\Pi)$ for the noncrossing partition given by rotating Π clockwise by $2\pi/N$.

Lemma 7.11. *For any $T \in \mathrm{Inc}_k(2 \times n)$, $\pi(\mathcal{P}(T)) = \mathcal{R}(\pi(T))$.*

Proof. We think of K-promotion as taking place in two steps. In the first step, we remove the label 1 from the tableau T, perform K-jeu de taquin, and label the now vacated lower-right corner by $2n-k+1$. Call this intermediate filling T'. In the second step, we decrement each entry by one to obtain $\mathcal{P}(T)$.

The filling T' is not strictly an increasing tableau, as no box is labeled 1. However, by analogy with the construction of Proposition 7.2, we may associate to T' a noncrossing partition of $\{2, \ldots, 2n-k+1\}$. For each i in the second row of T', let s_i be the largest number in the first row that is less than i and that is not s_j for some $j < i$. The noncrossing partition $\pi(T')$ is formed by declaring i and s_i to be in the same block.

Claim 7.1. *$\pi(T')$ may be obtained from $\pi(T)$ merely by renaming the element 1 as $2n-k+1$.*

In T, there are three types of elements:

(A) those that appear only in the first row,

(B) those that appear only in the second, and

(C) those that appear in both.

Most elements of T' are of the same type as they were in T. In fact, the only elements that change type are in the block of $\pi(T)$ containing 1. If that block contains only one element other than 1, this element changes from type (B) to type (A). If the block contains several elements besides 1, the least of these changes from type (C) to type (A) and the greatest changes from type (B) to type (C). All other elements remain the same type. Observing that the element 1 was of type (A) in T and that $2n-k+1$ is of type (B) in T', this proves the claim. $\square$

By definition, $\pi(\mathcal{P}(T))$ is obtained from $\pi(T')$ merely by decrementing each element by one. However by the claim, $\mathcal{R}(\pi(T))$ is also obtained from $\pi(T')$ by decrementing each element by one. Thus $\pi(\mathcal{P}(T)) = \mathcal{R}(\pi(T))$. $\square$

It remains now to count noncrossing partitions of $2n-k$ into $n-k$ blocks all of size at least 2 that are invariant under rotation by $2\pi/d$, and to show that we obtain the formula of Lemma 7.10. We observe that for such rotationally symmetric noncrossing partitions, the cyclic group $\mathcal{C}_d$ acts freely on all blocks, except the central block (the necessarily unique block whose convex hull contains the center of the circle) if it exists. Hence there are no such invariant partitions unless $n-k \equiv 0$ or $1 \mod d$, in agreement with Lemma 7.10.

Arrange the numbers $1, 2, \ldots, n, -1, \ldots, -n$ counterclockwise, equally-spaced around a circle. Consider a partition of these points such that, for every block B, the set formed by negating all elements of B is also a block. If the convex hulls of the blocks are pairwise nonintersecting, we call such a partition a *noncrossing B_n-partition* or *type-B noncrossing partition* (*cf.* [Rei97]). Whenever we say that a type-B noncrossing partition has p pairs of blocks, we do not count the central block. There is an obvious bijection between noncrossing partitions of $2n-k$ that are invariant under rotation by $2\pi/d$ and noncrossing $B_{(2n-k)/d}$-partitions. Under this bijection a noncrossing partition Π with $n-k$ blocks corresponds to a type-B noncrossing partition with $\frac{n-k}{d}$ pairs of blocks if d divides $n-k$ (that is, if Π has no central block), and corresponds to a type-B noncrossing partition with $\frac{n-k-1}{d}$ pairs of blocks if d divides $n-k-1$ (that is, if Π has a central block). The partition Π has singleton blocks if and only if the corresponding type-B partition does.

Lemma 7.12. *The number of noncrossing B_N-partitions with p pairs of blocks without singletons and* without *a central block is*

$$\sum_{i=0}^{p}(-1)^i\binom{N}{i}\binom{N-i}{p-i}^2\frac{p-i}{N-i}.$$

The number of such partitions with *a central block is*

$$\sum_{i=0}^{p}(-1)^i\binom{N}{i}\binom{N-i}{p-i}^2\frac{N-p}{N-i}.$$

Proof. It was shown in [Rei97] that the number of noncrossing B_N-partitions with p pairs of blocks is $\binom{N}{p}^2$. In [ReStWh04], it was observed that [AtRe04, Lemma 4.4] implies that exactly $\frac{N-p}{N}\binom{N}{p}^2$ of these have a central block. Our formulas for partitions without singleton blocks follow immediately from these observations by Inclusion–Exclusion. □

It remains to prove the following pair of combinatorial identities:

$$\sum_{i=0}^{p}(-1)^i\binom{N}{i}\binom{N-i}{p-i}^2\frac{p-i}{N-i}=\frac{N!}{(N-2p)!p!(p-1)!(N-p)}$$

and

$$\sum_{i=0}^{p}(-1)^i\binom{N}{i}\binom{N-i}{p-i}^2\frac{N-p}{N-i}=\frac{N!}{(N-2p-1)!p!p!(N-p)}.$$

This is a straightforward exercise in hypergeometric series (*cf.*, e.g., [AnAsRo01, §2.7]). For example, the first sum is

$$\binom{N}{p}\sum_{i=0}^{p}(-1)^i\binom{p}{i}\binom{N-i-1}{p-i-1}=\binom{N}{p}\binom{N-1}{p-1}{}_2F_1\left(\begin{matrix}-p,\ 1-p\\ 1-N\end{matrix};1\right),$$

which may be evaluated by the Chu–Vandermonde identity. This completes the proof of Theorem 7.4. □

Recently, C. Athanasiadis–C. Savvidou [AtSa12, Proposition 3.2] independently enumerated noncrossing B_N-partitions with p pairs of blocks without singletons and without a central block.

Lemma 7.11 yields a second proof of Theorem 7.3. We observe that under the reformulation of Lemma 7.11, Theorem 7.4 bears a striking similarity to [ReStWh04, Theorem 7.2] which gives a cyclic sieving on the set of *all* noncrossing partitions of $2n-k$ into $n-k$ parts with respect to the same cyclic group action.

Additionally, under the correspondence mentioned in Section 7.2 between $\mathrm{Inc}_k(2\times n)$ and dissections of an $(n+2)$-gon with $n-k-1$ diagonals, Theorem 7.4 bears a strong resemblance to [ReStWh04, Theorem 7.1], which gives a cyclic sieving on the same set with the same q-enumerator, but with respect to an action by $\mathcal{C}_{n+2}$ instead of $\mathcal{C}_{2n-k}$. S.-P. Eu–T.-S. Fu [EuFu08] reinterpret the $\mathcal{C}_{n+2}$-action as the action of a Coxeter element on the k-faces of an associahedron. We do not know such an interpretation of our action by $\mathcal{C}_{2n-k}$. In [ReStWh04], the authors note many similarities between their Theorems 7.1 and 7.2 and ask for a unified proof. It would be very satisfying if such a proof could also account for our Theorem 7.4.

7.7 Homomesy of increasing tableaux under K-promotion

This section is based on joint work with J. Bloom and D. Saracino [BlPeSa16].

One might hope for Theorem 2.4 on homomesy of standard tableaux under promotion to generalize to $\mathrm{Inc}_k(m \times n)$ under K-promotion. However this is not the case:

Example 7.4. Consider

$$T = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 5 \\ \hline 2 & 4 & 5 & 7 \\ \hline 3 & 6 & 8 & 9 \\ \hline \end{array} \qquad \text{and} \qquad U = \begin{array}{|c|c|c|c|} \hline 1 & 4 & 5 & 6 \\ \hline 2 & 6 & 7 & 8 \\ \hline 3 & 7 & 8 & 9 \\ \hline \end{array}$$

and let S be the `rotate`-fixed set of black boxes. The K-promotion orbits $\mathcal{O}_T, \mathcal{O}_U$ of T, U respectively are both of size 9. However it may be computed that

$$\frac{1}{9} \sum_{A \in \mathcal{O}_T} \sigma_S(A) = \frac{91}{9}, \text{ while } \frac{1}{9} \sum_{B \in \mathcal{O}_U} \sigma_S(B) = 10. \qquad \diamondsuit$$

Say a pair $(m \times n, k)$ is *homomesic* if for any $S \subseteq m \times n$ fixed under `rotate`, $(\mathrm{Inc}_k(m \times n), \mathcal{C}, \sigma_S)$ is homomesic. It seems an interesting question to characterize homomesic pairs $(m \times n, k)$. Theorem 2.4 shows that $(m \times n, 0)$ is homomesic for all m, n. Example 7.4 shows that $(3 \times 4, 3)$ is *not* homomesic. The following theorem shows, however, that $(2 \times n, k)$ is always homomesic.

Theorem 7.5. *Let $S \subseteq 2 \times n$ be a set of elements fixed under $180°$ rotation. Then for any k,*

$$(\mathrm{Inc}_k(2 \times n), \mathcal{C}, \sigma_S)$$

is homomesic.

Proof. By Theorem 7.3, the order of $\mathcal{P}$ on $\mathrm{Inc}_k(2 \times n)$ is $2n - k$. The fact that in this case K-evacuation is the same as $180°$ rotation plus alphabet reversal is Proposition 7.3. The theorem then follows from an analogue of Lemma 2.1. For this the growth diagram proof of Lemma 2.1 may be copied nearly verbatim, using the K-theory growth diagrams described in Section 7.3 and replacing every instance of k in the proof with $2n - q$ (the order of $\mathcal{P}$). $\square$

Chapter 8

Resonance of plane partitions and increasing tableaux

This chapter derives from joint work with K. Dilks and J. Striker [DiPeSt15].

8.1 Introduction

The introduction to this chapter is in two parts. The first subsection defines resonance and gives our prototypical example on alternating sign matrices. The second subsection describes our main results; these include two instances of resonance (on plane partitions and increasing tableaux), an equivariant bijection between these two sets with a number of new consequences, and a higher-dimensional analogue of N. Williams and J. Striker's result on the equivariance of (poset-)promotion and rowmotion [StWi12].

8.1.1 Resonance

We introduce the following concept of *resonance*[1].

Definition 8.1. Suppose $G = \langle g \rangle$ is a cyclic group acting on a set X, $\mathcal{C}_\omega = \langle c \rangle$ a cyclic group of order ω acting nontrivially on a set Y, and $f : X \to Y$ a surjection. We say the triple (X, G, f) exhibits **resonance with frequency** ω if, for all $x \in X$, $c \cdot f(x) = f(g \cdot x)$, that is, the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "g\cdot"'] \arrow[d, "f"] & X \arrow[d, "f"] \\ Y \arrow[r, "c\cdot"'] & Y \end{tikzcd}$$

or a set of words with c acting by a cyclic shift. Resonance is a "pseudo-periodicity" property of the G-action, in that the resonant frequency ω is generally less than the order of the G-action. Note that, in

[1]The mathematically precise definition of resonance given here was new in [DiPeSt15], though the phenomenon had been discussed by various people over the previous year or more, in particular, at the 2015 "Dynamical Algebraic Combinatorics" workshop at the American Institute of Mathematics where the work described here began. Thanks to Jim Propp for coining the term "resonance" which so nicely encapsulates the idea.

general, (X, G, id_X) satisfies the definition of resonance with frequency $|G|$; we call this an instance of *trivial resonance*.

Our prototypical example of resonance is the action of *gyration* on *alternating sign matrices*.

Definition 8.2. An **alternating sign matrix (ASM)** is a square matrix with the following properties: the entries are in the set $\{0, 1, -1\}$, each row and each column sums to one, and the nonzero entries along each row or column alternate in sign. Let ASM_n denote the set of $n \times n$ alternating sign matrices. (See Figure 8.2 for examples of ASMs.)

Alternating sign matrices were introduced by D. Robbins–H. Rumsey [RoRu86] as part of their study of the lambda-determinant. With W. Mills [MiRoRu83], they then conjectured an enumeration for ASM_n, which was proved by D. Zeilberger [Ze96] and G. Kuperberg [Ku96] (cf. [Br99] for a detailed exposition of this history). Alternating sign matrices are known to be in bijection with *fully-packed loop configurations* [Wi00, Pr01]; see Figures 8.2 and 8.3.

Definition 8.3. Consider an $[n] \times [n]$ grid of dots in $\mathbb{Z}^2$. For each dot in the top row, draw an edge from that dot up one unit. Similarly for each dot on each of the other three sides of the grid, draw an edge one unit away from the grid and perpendicular to that edge. (Corner dots are thereby attached two edges, one horizontal and one vertical.) Beginning with the vertical edge at the top left of the grid, alternately color the edges blue and red; then delete all red edges. A **fully-packed loop configuration (FPL)** of order n is a set of paths and loops on the $[n] \times [n]$ grid such that the paths end with the blue boundary edges constructed above and each of the n^2 vertices within the grid has exactly two incident edges.

Definition 8.4. Given a fully-packed loop configuration, number the external edges clockwise, starting with the upper left external edge. Each external edge will be connected by a path to another external edge, and these paths will never cross. This matching on the external edges is a noncrossing matching on $2n$ points, and is called the **link pattern** of the FPL. Let rot denote the operator on link patterns that rotates them by an angle of $2\pi/2n$.

Note that this map from fully-packed loop configurations to link patterns is not injective.

Definition 8.5. Given an $[n] \times [n]$ grid of dots, color the interiors of the squares in a checkerboard pattern. Given an FPL of order n drawn on this grid, its **gyration**, Gyr, is computed by first visiting all squares of one color then all squares of the other color, applying at each visited square the "local move" that swaps the edges around a square if the edges are parallel and otherwise leaves them fixed. Figure 8.1 shows an example of the operation.

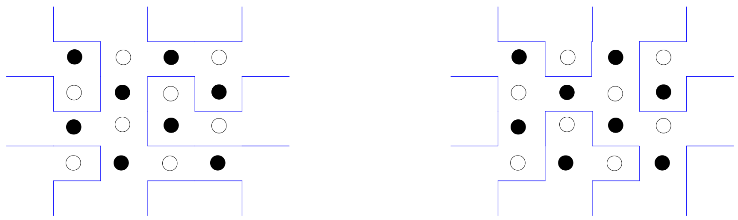


Figure 8.1: An example of gyration on the fully-packed loop configuration shown at left. First at each each square marked with $\bullet$, we replace the local configuration $\overline{\underline{\bullet}}$ with $|\bullet|$ and vice versa, obtaining the picture on the right. Then we perform the same local switch at each square marked with $\circ$. In this case, there are no local configurations $\overline{\underline{\circ}}$ with $|\circ|$ in the picture on the right, so we obtain the picture on the right as the final result of gyration.

The following theorem of B. Wieland gives a remarkable property of gyration.

Theorem 8.1 (B. Wieland [Wi00]). *Gyration of an FPL rotates the associated link pattern by an angle of* $2\pi/2n$.

We reformulate this theorem into a statement of resonance.

Corollary 8.1. *Let f be the map from an alternating sign matrix through its FPL to the link pattern. Then,* $(\mathrm{ASM}_n, \langle \mathrm{Gyr} \rangle, f)$ *exhibits resonance with frequency* $2n$.

For example, consider gyration on 5×5 alternating sign matrices. Gyration has orbits of size 2, 4, 5, and 10. So the order of gyration (i.e., the smallest positive k such that $\mathrm{Gyr}^k = \mathrm{id}$) in this case is 20, but (ASM_5, Gyr, f) exhibits resonance with frequency 10. Consider the orbit of gyration in Figure 8.2. This orbit is of size 4, while the link pattern orbit is of size 2. So even though $\mathrm{Gyr}^{10}(A) \neq A$ for A an ASM in this orbit, $\mathrm{rot}^{10}(f(A)) = f(A)$ (since, in this case, $\mathrm{rot}^2(f(A)) = f(A)$).

As another example, the ASM in Figure 8.3 is in a gyration orbit of size 84 ($= 12 \cdot 7$), while (ASM_6, Gyr, f) exhibits resonance with frequency 12. So even though $\mathrm{Gyr}^{12}(A) \neq A$ for A an ASM in this orbit, $\mathrm{rot}^{12}(f(A)) = f(A)$.

We think of the property of resonance as somewhat analogous to the *cyclic sieving phenomenon* (introduced by V. Reiner–D. Stanton–D. White [ReStWh04], generalizing the $q = -1$ phenomenon of J. Stembridge [Ste94a]) and the *homomesy property* (isolated by J. Propp–T. Roby [PrRo15], inspired by observations of D. Panyushev [Pa09]) in being a somewhat subtle 'niceness' property of a cyclic group action. We suspect that the phenomenon of resonance, like those of cyclic sieving and homomesy, is significantly more common than previously realized. Heuristically, one is led to suspect the presence of resonance in a system by observing that many orbit cardinalities are multiples or divisors (or multiples of divisors) of ω.

$$
\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}
\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}
\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}
\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}
$$

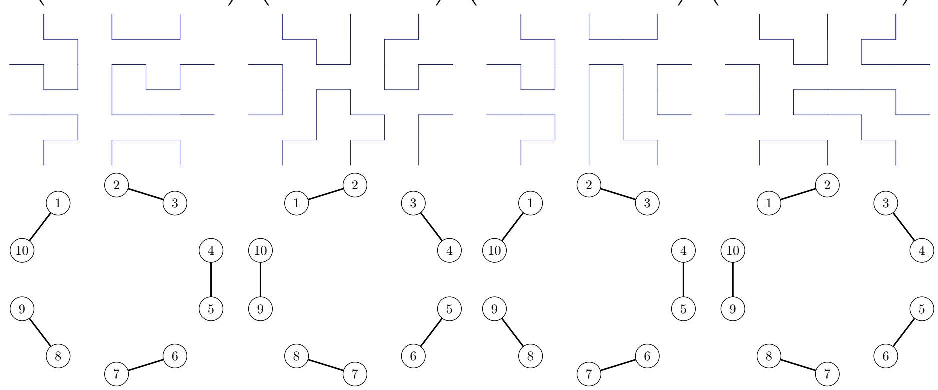
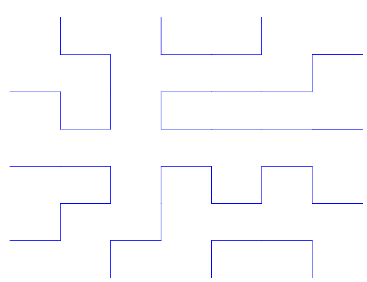


Figure 8.2: A length 4 gyration orbit in ASM_5; Top Row: ASM, Middle Row: FPL, Bottom Row: link pattern

$$
\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}
$$



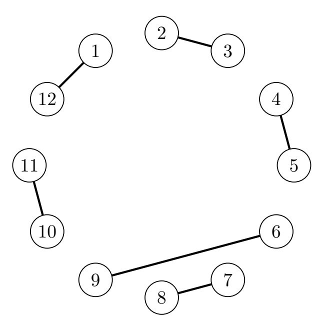


Figure 8.3: A 6×6 ASM with gyration orbit of length 84, with its corresponding FPL and link pattern

8.1.2 Main results

The remainder of this chapter centers around two new examples of resonance on *plane partitions* under *rowmotion* and *increasing tableaux* under *K-promotion*, as well as a new equivariant bijection relating these phenomena. Here we summarize our main results; see the referenced sections for relevant definitions.

Rowmotion has attracted much attention since it was first studied (under another name) by A. Brouwer–A. Schrijver [BrSc74] in 1974; see for example [Fo93, CaFo95, Pa09, StWi12, ArStTh13, RuSh13, PrRo15, RuWa15]. More recently, several authors have studied a *birational* lift of rowmotion [EiPr14, GrRo16, GrRo15], with some relations to Zamolodchikov periodicity.

Let $J(\mathbf{a} \times \mathbf{b} \times \mathbf{c})$ denote the set of plane partitions inside an $a \times b \times c$ box and Row denote rowmotion; see Section 8.3.3 for the definitions of $X_{\max}$ and D. Our first main resonance result is the following.

Theorem 8.8. $(J(\mathbf{a} \times \mathbf{b} \times \mathbf{c}), \langle \mathrm{Row} \rangle, X_{\max} \circ D)$ *exhibits resonance with frequency* $a + b + c - 1$.

To better study plane partitions, we introduce and develop the machinery of *affine hyperplane toggles* and *n-dimensional lattice projections*, including a higher-dimensional analogue of N. Williams and J. Striker's result on the equivariance of (poset-)promotion and rowmotion [StWi12]. We obtain a large family of toggling actions $\{\mathcal{P}^{\sigma}_{\pi,v}\}$ whose orbit structures are equivalent to that of rowmotion. See Sections 8.3.4 and 8.3.5 for further details.

Theorem 8.9. *Let P be a finite poset with an n-dimensional lattice projection π. Let $v = (v_1, v_2, v_3, \ldots, v_n)$ and $w = (w_1, w_2, w_3, \ldots, w_n)$, where $v_j, w_j \in \{\pm 1\}$. Finally suppose that $\sigma : \mathrm{supp}(P, \pi, v) \to \mathrm{supp}(P, \pi, v)$ and $\tau : \mathrm{supp}(P, \pi, w) \to \mathrm{supp}(P, \pi, w)$ are bijections. Then there is an equivariant bijection between $J(P)$ under $\mathcal{P}^{\sigma}_{\pi,v}$ and $J(P)$ under $\mathcal{P}^{\tau}_{\pi,w}$.*

Our other main object of study is K-promotion on increasing tableaux, as described in Chapter 7. Here however, **we do not require that the set of entries is an initial segment of $\mathbb{Z}_{>0}$**. Let $\mathrm{Inc}^q(\lambda)$ denote the set of increasing tableaux of shape λ and entries at most q, and let Con be the content map. In Section 8.2.2, we prove the following, our first main result on K-promotion.

Theorem 8.2. $(\mathrm{Inc}^q(\lambda), \langle \mathcal{P} \rangle, \mathrm{Con})$ *exhibits resonance with frequency q.*

This similarity of Theorems 8.2 and 8.8 leads us to establish an equivariant bijection between plane partitions under rowmotion and increasing tableaux under K-promotion.

Theorem 8.11. $J(\mathbf{a} \times \mathbf{b} \times \mathbf{c})$ *under* Row *is in equivariant bijection with* $\mathrm{Inc}^{a+b+c-1}(a \times b)$ *under* $\mathcal{P}$.

Part of our approach to establishing this equivariant bijection involves the reinterpretation of K-promotion in terms of K-Bender-Knuth involutions, which we introduce; see Proposition 8.1. We also extend, in Section 8.2.4, a result of B. Rhoades on *descent cycling* to the K-promotion setting.

We obtain a variety of corollaries of this equivariant bijection. Many of these corollaries are new proofs of previously discovered results on the order of Row and $\mathcal{P}$. We highlight here only those results that are new.

Corollary 8.6. *The order of* $\mathcal{P}$ *on* $\mathrm{Inc}^{a+b}(a \times b)$ *is* $a+b$.

Corollary 8.7. *The order of* $\mathcal{P}$ *on* $\mathrm{Inc}^{a+b+1}(a \times b)$ *is* $a+b+1$.

We also obtain the following strengthening of a theorem of P. Cameron–D. Fon-der-Flaass [CaFo95, Theorem 6(a)]. The original theorem had the more stringent hypothesis $c > ab - a - b + 1$.

Theorem 8.12. *If* $a+b+c-1$ *is prime and* $c > \frac{2ab-2}{3} - a - b + 2$, *then the cardinality of every orbit of* Row *on* $J(\mathbf{a} \times \mathbf{b} \times \mathbf{c})$ *is a multiple of* $a+b+c-1$.

The rest of this chapter is structured as follows. In Section 8.2, we recall the K-promotion operator on increasing tableaux and establish a number of new properties (including resonance) that we will use. In Section 8.3, we establish resonance of plane partitions under rowmotion, and extend machinery developed by N. Williams and J. Striker [StWi12], to introduce the family of toggle group actions $\{\mathcal{P}^{\sigma}_{\pi,v}\}$ and show that each $\mathcal{P}^{\sigma}_{\pi,v}$ acts with the same cycle structure as rowmotion. In Section 8.4, we give an equivariant bijection between increasing tableaux under K-promotion and plane partitions under $\mathcal{P}_{(1,1,-1)}$ and Row. We then extract a number of corollaries from this equivariant bijection, including new proofs of theorems of A. Brouwer–A. Schrijver [BrSc74] and P. Cameron–D. Fon-der-Flaass [CaFo95], a strengthening of a theorem of P. Cameron–D. Fon-der-Flaass [CaFo95], and several new results on the order of K-promotion. Finally, we conjecture the order of rowmotion on plane partitions of height 3 (which we have shown to be also the order of K-promotion on certain classes of increasing tableaux). In Section 8.5, we propose additional instances of resonance related to alternating sign matrices and *totally symmetric self-complementary plane partitions*.

8.2 K-Promotion on increasing tableaux

In this section, we study increasing tableaux, the first of the objects in our main bijection (Theorem 8.10). After recalling the basic concepts, we establish resonance of increasing tableaux under K-promotion in Theorem 8.2. In Section 8.2.3, we reinterpret K-promotion in terms of K-Bender-Knuth involutions, which we introduce; this interpretation plays an important role in Section 8.4.2 in establishing equivariance of our main bijection. In Section 8.2.4, we extend a *descent cycling* result of B. Rhoades [Rh10, Lemma 3.3] from standard Young tableaux to increasing tableaux; this extension is used in Theorem 8.12 to improve on a theorem of P. Cameron–D. Fon-der-Flaass [CaFo95].

8.2.1 Increasing tableaux

Identify a partition λ with its Young diagram. An **increasing tableau** of shape λ is a filling of λ with positive integers such that labels strictly increase from left to right across rows and from top to bottom down columns. An example appears in Figure 8.4. We write $\mathrm{Inc}^q(\lambda)$ for the set of all increasing tableaux of shape λ with all entries at most q. **In contrast to the definition in Chapter 7, we do not assume here that every integer between** 1 **and** q **appears.** The K-promotion operator $\mathcal{P}$ is defined exactly as in Chapter 7, with the caveats that there may be no label 1 to delete and that after swapping the empty box (if any) should be filled with $q+1$ before decrementing.

1	4	5	8
2	5	7	9
6	7	9	10
8	10		

Figure 8.4: An increasing tableau T of shape $\lambda = (4, 4, 4, 2)$.

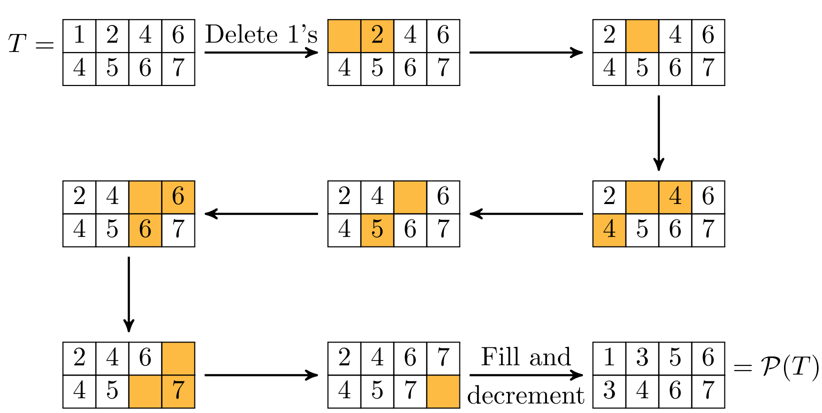


Figure 8.5: Calculating the K-promotion of $T \in \mathrm{Inc}^7(2 \times 4)$. In each intermediate step, we have colored the short ribbons on which we are about to act.

8.2.2 Content cycling

Define the **content** of an increasing tableau $T \in \mathrm{Inc}^q(\lambda)$ to be the binary sequence $\mathrm{Con}(T) = (a_1, a_2, \ldots, a_q)$, where $a_i = 1$ if i is an entry of T and $a_i = 0$ if it is not. That is, $a_i := \chi_i(T)$ where χ_i denotes the indicator function for the label i.

Lemma 8.1. *Let $T \in \mathrm{Inc}^q(\lambda)$. If $\mathrm{Con}(T) = (a_1, a_2, \ldots, a_q)$, then $\mathrm{Con}(\mathcal{P}(T))$ is the cyclic shift $(a_2, \ldots, a_q, a_1)$.*

Proof. Case 1: $(\chi_1(T) := a_1 = 0)$: Then T has no labels 1. Hence the first step of K-promotion is trivial, deleting no labels. The ribbon switching process is also trivial, as there are no empty boxes. Therefore, at the final step, there are no boxes to fill. Thus, in this case, the total effect of K-promotion is merely to subtract 1 from each entry. The lemma is then immediate in this case.

Case 2: $(\chi_1(T) := a_1 = 1)$: Then the first step of K-promotion is to delete a nonempty collection of labels 1. Hence there are a nonzero number of empty boxes. The ribbon switching process may change the number of empty boxes, but clearly preserves its nonzeroness. Hence in the final step of K-promotion, there will be a nonzero number of boxes filled with $q+1$ and then decremented by 1. Hence $\chi_q(\mathcal{P}(T)) = 1$.

Let $i > 1$ and suppose $\chi_i(T) = 1$. Then i appears as an entry of T. The ribbon switching process preserves this property (though not in general the number of entries i). Hence after subtracting one from each entry, this yields $\chi_{i-1}(\mathcal{P}(T)) = 1$. If instead $\chi_i(T) = 0$, then i does not appear in T. Hence during the ribbon switching process, when we consider the ribbons consisting of i's and empty boxes, each is a single empty box and by definition we make no change. Hence the ribbon switching process preserves the absence of i. After decrementing, this yields $\chi_{i-1}(\mathcal{P}(T)) = 0$. □

The following instance of resonance follows directly from Lemma 8.1.

Theorem 8.2. $(\mathrm{Inc}^q(\lambda), \langle\mathcal{P}\rangle, \mathrm{Con})$ *exhibits resonance with frequency q.*

This leads to the following corollary.

Corollary 8.2. *Suppose q is prime and $T \in \mathrm{Inc}^q(\lambda)$ does not have full content. Then the size of the K-promotion orbit of T is a multiple of q.*

8.2.3 K-Bender-Knuth involutions

In this subsection, we reinterpret K-promotion as a product of involutions, which we will need in our proof of Theorem 8.8. We define operators K-BK_i on $\mathrm{Inc}^q(\lambda)$ for each $1 \leq i \leq q$. Take $T \in \mathrm{Inc}^q(\lambda)$. We compute K-$\mathrm{BK}_i(T)$ as follows: Consider the set of boxes in T that contain either i or $i+1$. This set decomposes into connected components that are short ribbons. On each nontrivial such component, we do nothing. On each component that is a single box, replace the symbol i by $i+1$ or *vice versa*. The result is $K\text{-BK}_i(T)$. That is, the action of $K\text{-BK}_i$ on T is to increment i and/or decrement $i+1$, *wherever possible*. These operators are illustrated in Figure 8.6.

Clearly each $K\text{-BK}_i$ is an involution. We call it the ith **K-Bender-Knuth involution** because in the case T is standard, $K\text{-BK}_i$ coincides with the classical involution introduced by E. Bender–D. Knuth [BeKn72] and discussed in Section 1.3.

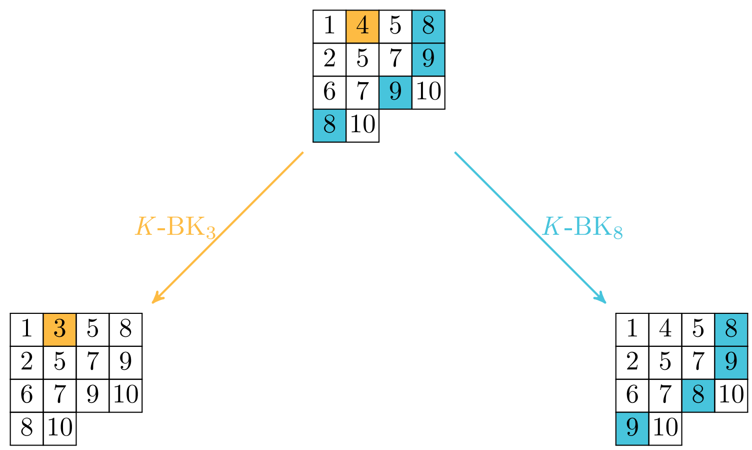


Figure 8.6: The action of some K-Bender-Knuth involutions on the tableau T from Figure 8.4.

Proposition 8.1. *For* $T \in \mathrm{Inc}^q(\lambda)$, $\mathcal{P}(T) = K\text{-BK}_{q-1} \circ \cdots \circ K\text{-BK}_1(T)$.

Proof. Another way to think of $K\text{-BK}_i$ is as the K-infusion (see Section 3.1.4) of the labels i through the labels $i+1$. That is, treat the labels i as empty boxes and swap the short ribbons of empty boxes and $(i+1)$'s as in the definition of K-promotion; then relabel each $i+1$ as i and each empty box as $i+1$.

From this characterization, it is clear that $K\text{-BK}_{q-1} \circ \cdots \circ K\text{-BK}_1$ amounts to deleting the 1's and swapping the empty boxes successively through each other label in order, decrementing each other label as the empty boxes swap through it, and finally labeling the empty boxes at outer corners by q. This is transparently the same as K-promotion, except that the decrementing of labels happens throughout the process instead of all at the end. $\square$

8.2.4 Descent cycling

In this subsection, we restrict consideration to increasing tableaux of rectangular shape. We extend a result of B. Rhoades [Rh10, Lemma 3.3] from standard Young tableaux to increasing tableaux. Our proof is a elaboration of his argument. We will use this result in Theorem 8.12 to improve on a theorem of

P. Cameron–D. Fon-der-Flaass [CaFo95, Theorem 6(a)]. Throughout this subsection, we write "East", "east" and "southEast" to mean "strictly east", "weakly east" and "weakly south and strictly east" respectively, etc.

Definition 8.6. Let $T \in \mathrm{Inc}^q(a \times b)$. For $1 \leq i < q$, the symbol i is a **descent** of T if some instance of i appears in a higher row than some instance of $i+1$. Additionally, q is a **descent** of T if $q-1$ is a descent of $\mathcal{P}(T)$.

Lemma 8.2. *Suppose i is a descent of $T \in \mathrm{Inc}^q(a \times b)$. Then $i - 1 \mod q$ is a descent of $\mathcal{P}(T)$.*

Proof. Throughout this proof, we use the original definition of K-promotion involving swaps, instead of the K-Bender-Knuth alternative.

Case 1: $(1 < i < q)$: T has an instance of i in row h and an instance of $i+1$ in row k with $h < k$. In $\mathcal{P}(T)$, there is an $i-1$ in row h or $h-1$ and there is an i in row k or $k-1$. Hence $i-1$ is a descent in $\mathcal{P}(T)$ if $k - h > 1$. Thus assume $k = h+1$.

Restrict attention to rows h and $h+1$ of T. T has a unique i in row h and a unique $i+1$ in row $h+1$. By increasingness, this $i+1$ is not East of this i.

Suppose the $i+1$ is West of the i. Then T contains the local configuration $\begin{array}{|c|c|}\hline y & z \\ \hline i{+}1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$. Since $z \leq i < i+1$, the $i+1$ cannot move North during this application of K-promotion. Hence $\mathcal{P}(T)$ has i in row $h+1$, and $i-1$ is a descent of $\mathcal{P}(T)$.

Thus, it remains to consider the case that i and $i+1$ are in the same column of T. The $i+1$ can only move North if the i moves. If the i moves North, we are done, so assume i moves West. Then T has the local configuration $\begin{array}{|c|c|}\cline{2-2} \multicolumn{1}{c|}{} & i \\ \hline y & i{+}1 \\ \hline\end{array}$ where $y \geq i$. But by increasingness, $y < i+1$. Hence $y = i$, so T has the local configuration $\begin{array}{|c|c|}\cline{2-2} \multicolumn{1}{c|}{} & i \\ \hline i & i{+}1 \\ \hline\end{array}$. Therefore, $\mathcal{P}(T)$ has the local configuration $\begin{array}{|c|c|}\hline i{-}1 & i \\ \hline i & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$ and thus $i-1$ is a descent of $\mathcal{P}(T)$.

Case 2: $(i = 1)$: We must show that q is a descent of $\mathcal{P}(T)$, that is, $q-1$ is a descent of $\mathcal{P}^2(T)$.

For $V \in \mathrm{Inc}^q(a \times b)$, let $\mathcal{F}(V)$ be the **flow path** of V, that is the set of pairs of adjacent boxes $\{B, B'\}$ of V such that B and B' are at some point part of the same short ribbon during the application of $\mathcal{P}$ to V. For B a box of $a \times b$, we write $B^{\uparrow}$ for the box immediately North of B, $B^{\rightarrow}$ for the box immediately East of B, etc. Define the **upper flow path** $\overline{\mathcal{F}}(V)$ to be those $\{B, B^{\rightarrow}\} \in \mathcal{F}(V)$ such that $\{B, B^{\rightarrow}\}$ is northmost in its columns among $\mathcal{F}(V)$ together with those $\{B, B^{\downarrow}\}$ such that $\{B, B^{\downarrow}\}$ is eastmost in its rows among $\mathcal{F}(V)$. Similarly define the **lower flow path** $\underline{\mathcal{F}}(V)$ to be those pairs in $\mathcal{F}(V)$ that are southmost or westmost. Figure 8.7 shows an example of these flow paths.

Let Q be the box in the lower right corner of $a \times b$. By Proposition 8.1, q appears in $\mathcal{P}(T)$. Hence by increasingness, $\mathcal{P}(T)$ has $q \in Q$. Thus it suffices to show that $\{Q^{\uparrow}, Q\} \in \mathcal{F}(\mathcal{P}(T))$. The proof proceeds by

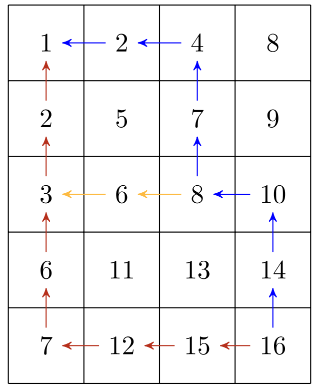


Figure 8.7: The flow path of a tableau $V \in \mathrm{Inc}^{16}(5 \times 4)$. Elements of the lower flow path are shown in red, while elements of the upper flow path are shown in blue and the remaining elements of the flow path are shown in yellow-orange.

comparing $\mathcal{F}(T)$ and $\mathcal{F}(\mathcal{P}(T))$.

Let $\overline{S} = \{B \in a \times b : \{B, B^{\rightarrow}\} \in \overline{\mathcal{F}}(T)\}$. It is clear that $\overline{S}$ contains exactly one box from each column of $a \times b$, except the eastmost column.

If $\{Q^{\uparrow}, Q\} \notin \mathcal{F}(\mathcal{P}(T))$, then there is some $B \in \bigcup \overline{\mathcal{F}}(\mathcal{P}(T))$ such that $B \in \overline{S}$. Choose B to be maximally west among such boxes.

Since B is chosen maximally west, $\{B^{\leftarrow}, B\} \notin \overline{\mathcal{F}}(\mathcal{P}(T))$. Suppose $\{B^{\uparrow}, B\} \in \overline{\mathcal{F}}(\mathcal{P}(T))$. Then in $\mathcal{P}(T)$, the entry of B is strictly less than the entry of $B^{\uparrow\rightarrow}$. That is, if h is the entry of B and k is the entry of $B^{\uparrow\rightarrow}$ then $h < k$. However, in T we have $k + 1 \in B^{\uparrow\rightarrow}$ and $h + 1 \in B^{\rightarrow}$; this contradicts the increasingness of T. Thus B is the northwestmost box of $a \times b$.

Since 1 is a descent of T and $B \in \overline{S}$, T has $1 \in B$, $2 \in B^{\downarrow}$ and $2 \in B^{\rightarrow}$.

Let $\underline{S} = \{B \in a \times b : \{B, B^{\rightarrow}\} \in \underline{\mathcal{F}}(T)\}$. We claim that if $\{B, B^{\rightarrow}\} \in \underline{\mathcal{F}}(T)$, then there is a pair $\{A, A^{\rightarrow}\} \in \mathcal{F}(\mathcal{P}(T))$ with A North of B in the same column. To see this, first observe by local analysis that if $\{B, B^{\rightarrow}\} \in \mathcal{F}(T)$ and $B^{\uparrow} \in \bigcup \mathcal{F}(\mathcal{P}(T))$, then $\{B^{\uparrow}, B^{\uparrow\rightarrow}\} \in \mathcal{F}(\mathcal{P}(T))$. Now recall that $\underline{S}$ contains exactly one box from each column of $a \times b$, except the eastmost column. Moreover since T has $2 \in B^{\downarrow}$, no box of $\underline{S}$ is in the northmost row. The claim follows. Thus $Q^{\uparrow} \in \bigcup \mathcal{F}(\mathcal{P}(T))$ and we are done.

Case 3: ($i = q$): By definition. $\square$

Proposition 8.2. *The symbol i is a descent of $T \in \mathrm{Inc}^q(a \times b)$ if and only if $i - 1 \mod q$ is a descent of $\mathcal{P}(T)$.*

Proof. Suppose i is a descent of T. By Lemma 8.2, $i - 1 \mod q$ is a descent of $\mathcal{P}(T)$. Since $\mathrm{Inc}^q(a \times b)$ is finite, there is some M such that $\mathcal{P}^M(T) = \mathcal{P}^{-1}(T)$. Hence by M applications of Lemma 8.2, $i + 1$ is a descent of $\mathcal{P}^{-1}(T)$. $\square$

Definition 8.7. Let $T \in \mathrm{Inc}^q(a \times b)$. For $1 \leq i < q$, i is **transpose descent** of T if some instance of i appears in a lower indexed column than some instance of $i+1$. Additionally q is a **transpose descent** of T if $q-1$ is a transpose descent of $\mathcal{P}(T)$.

Equivalently, j is a transpose descent of T if and only if j is a descent of the transpose of T.

Proposition 8.3. *The symbol i is a transpose descent of $T \in \mathrm{Inc}^q(a \times b)$ if and only if $i - 1 \mod q$ is a transpose descent of $\mathcal{P}(T)$.*

Proof. Since clearly K-promotion commutes with transposing, the proposition is immediate from Proposition 8.2. $\square$

The following is an enriched version of Corollary 8.2 for rectangular tableaux.

Proposition 8.4. *Let $T \in \mathrm{Inc}^q(a \times b)$ with q prime. Suppose at least one of the following is true:*

- *T does not have full content,*
- *some $1 \leq i \leq q$ is not a descent in T, or*
- *some $1 \leq i \leq q$ is not a transpose descent in T.*

Then, the K-promotion orbit of T has cardinality a multiple of q.

Proof. If T does not have full content, Corollary 8.2 applies. Otherwise, some $1 \leq i \leq q$ is not a (transpose) descent in T. The proposition is then immediate by Proposition 8.2 or 8.3. $\square$

Finally, we prove the following lemma, which we will use in Section 8.4.3.

Lemma 8.3. *Let $T \in \mathrm{Inc}^q(a \times b)$ and suppose that $1 \leq i < q$ is both a descent and a transpose descent in T. Then the number of i's in T plus the number of $(i+1)$'s in T is at least 3.*

Proof. Since i is a descent, both i and $i+1$ must appear in T. Hence if i appears at least twice in T, we are done. Thus assume i appears exactly once in T. Since i is a descent, some $i+1$ appears South of this i. Since i is a transpose descent, some $i+1$ appears East of this i.

We claim these instances of $i+1$ are distinct, completing the proof of the lemma. Otherwise, we have $i+1$ SouthEast of i. Consider the label z of the box that is in the row of the i and in the column of the $i+1$. By the increasingness conditions on T, $i < z < i+1$, contradicting that z is an integer. $\square$

In Section 8.4.3, we will use Proposition 8.4, Lemma 8.3, and our main results, Theorems 8.9 and 8.11, to give in Theorem 8.12 a strengthening of a theorem of P. Cameron–D. Fon-der-Flaass on plane partitions in a box.

8.3 Promotion and rowmotion, revisited

In this section, we switch our focus from increasing tableaux to our other main objects of study: *plane partitions.* A **plane partition** is a stack of unit cubes in the positive orthant, justified toward the origin in all three directions. Plane partitions inside a box with side lengths a, b, and c, are counted by P. MacMahon's box formula: $\prod \frac{i+j+k-1}{i+j+k-2}$ where the product is over all $1 \le i \le a$, $1 \le j \le b$, $1 \le k \le c$ [Mac1915].

Plane partitions inside an $a \times b \times c$ box can be seen as *order ideals* in the product of three chains poset $\mathbf{a} \times \mathbf{b} \times \mathbf{c}$. Thus, most of our discussion in this section centers on posets and order ideals, keeping in mind that all such general results can be applied to plane partitions.

We begin in Section 8.3.1 by discussing the *rowmotion* action on order ideals and some results on the order of this action on products of two and three chains. In Section 8.3.2, we discuss the *toggle group*, first defined by P. Cameron–D. Fon-der-Flaass [CaFo95] and further studied by J. Striker and N. Williams [StWi12]. In Section 8.3.3, we use the main theorem of [StWi12] to prove resonance of plane partitions under rowmotion. The toggle group will be the algebraic structure underlying Sections 8.3.4 and 8.3.5, in which we revisit this main result of [StWi12] by proving, in Theorem 8.9, a generalization in the setting of *n-dimensional lattice projections.*

8.3.1 Rowmotion

Let P be a finite partially ordered set (poset). P is a **chain** if all its elements are mutually comparable. Let $\mathbf{n}$ denote the n-element chain. The **product of k chains** poset, $P = \mathbf{n_1} \times \mathbf{n_2} \times \cdots \mathbf{n_k}$, has as elements ordered integer k-tuples $(x_1, x_2, \ldots, x_k)$ such that $0 \le x_i \le n_i - 1$ with partial order given by componentwise comparison.

A subset $I \subseteq P$ is an **order ideal** if it is closed downward, i.e. if $y \in I$ and $x \le y$, then $x \in I$. Denote the set of order ideals of P as $J(P)$. An order ideal in P is uniquely determined by its set of maximal elements, or alternatively by the set of minimal elements of its complement in P. We study the orbit structure of **rowmotion**, $\mathrm{Row}\colon J(P) \to J(P)$, defined as the order ideal whose maximal elements are the minimal elements of $P \setminus I$.

The function Row has a long history of rediscovery and has appeared under many names. A partial summary of previous work follows; for a more complete discussion, see [StWi12]. A. Brouwer–A. Schrijver [BrSc74] studied Row for $P = \mathbf{a} \times \mathbf{b}$, the product of two chains. They discovered that this action has much smaller orbits than one naively expects:

Theorem 8.3 ([BrSc74, Theorem 3.6])**.** *The order of* Row *on* $J(\mathbf{a} \times \mathbf{b})$ *is* $a + b$.

P. Cameron–D. Fon-der-Flaass [CaFo95] studied the same question on plane partitions, that is, $J(\mathbf{a} \times \mathbf{b} \times \mathbf{c})$.

Theorem 8.4 ([CaFo95, Theorem 6(b)]). *The order of* Row *on* $J(\mathbf{a} \times \mathbf{b} \times \mathbf{2})$ *is* $a+b+1$.

Extrapolating from Theorems 8.3 and 8.4, one might speculate that Row has order $a+b+c-1$ on $J(\mathbf{a}\times\mathbf{b}\times\mathbf{c})$. In general, the order is unknown but often significantly greater than this naive guess. However, P. Cameron–D. Fon-der-Flaass established the following related fact.

Theorem 8.5 ([CaFo95, Theorem 6(a)]). *If* $a+b+c-1$ *is prime and* $c > ab-a-b+1$*, then the cardinality of every orbit of* Row *on* $J(\mathbf{a} \times \mathbf{b} \times \mathbf{c})$ *is a multiple of* $a+b+c-1$.

We will revisit Theorems 8.3 and 8.4 in Remark 8.2. In Section 8.3.3, we give a new proof of Theorem 8.5. Furthermore, as a consequence of our main equivariant bijection between plane partitions and increasing tableaux (Theorem 8.11), we will show, in Theorem 8.12, that in Theorem 8.5 the condition $c > ab-a-b+1$ may be relaxed to $c > \frac{2ab-2}{3} - a - b + 2$. This is evidence toward the conjecture of P. Cameron–D. Fon-der-Flaass [CaFo95] that this condition may be dropped entirely.

The approach of P. Cameron–D. Fon-der-Flaass was to reinterpret rowmotion as a *toggle group action.* We describe the toggle group in the next subsection.

8.3.2 The toggle group

The **toggle group** was first studied by P. Cameron–D. Fon-der-Flaass [CaFo95] and subsequently by J. Striker–N. Williams [StWi12]. It is the subgroup of the symmetric group on all order ideals $\mathfrak{S}_{J(P)}$ generated by certain involutions, called **toggles**. For each element $e \in P$ define its **toggle** $t_e : J(P) \to J(P)$ as follows.

$$t_e(X) = \begin{cases} X \cup \{e\} & \text{if } e \notin X \text{ and } X \cup \{e\} \in J(P) \\ X \setminus \{e\} & \text{if } e \in X \text{ and } X \setminus \{e\} \in J(P) \\ X & \text{otherwise} \end{cases}$$

Remark 8.1. Observe that t_e, t_f commute whenever neither e nor f covers the other.

The following theorem interprets rowmotion as a toggle group action.

Theorem 8.6 ([CaFo95]). *Given any poset* P*,* Row *is the toggle group element that toggles the elements of* P *in the reverse order of any linear extension. If* P *is ranked, this is the same as toggling the ranks (rows) from top to bottom.*

In 2012 [StWi12], J. Striker–N. Williams built on the work of P. Cameron–D. Fon-der-Flaass, showing that rowmotion is conjugate to the toggle group action they called **promotion**, or Pro, defined as toggling the elements of the poset from left to right (given a suitable notion of left-to-right, for which they used the term *rc-poset*).

Theorem 8.7 ([StWi12, Theorem 5.2])**.** *For any rc-poset P, there is an equivariant bijection between $J(P)$ under* Pro *and $J(P)$ under* Row*.*

We discuss this result in further detail in Sections 8.3.4 and 8.3.5 and give a multidimensional generalization of it in Theorem 8.9.

Remark 8.2. For many posets, the orbit structure of promotion is easier to study than that of rowmotion. Thus Theorem 8.7 yielded many results on the orbit structure of rowmotion by translating from the analogous result on promotion. Theorem 8.7 was applied in [StWi12] to give simple new proofs of Theorem 8.3 of A. Brouwer–A. Schrijver and Theorem 8.4 of P. Cameron–D. Fon-der-Flaass (discussed in Section 8.3.1), as well as easy proofs of the *cyclic sieving phenomenon* of V. Reiner, D. Stanton, and D. White [ReStWh04] in these cases and a few others.

In the next subsection, we use Theorem 8.7 to prove resonance of rowmotion on plane partitions.

8.3.3 Resonance of plane partitions

In this subsection, we prove our second main resonance result, Theorem 8.8. We also give a new proof of Theorem 8.5.

In [StWi12, Section 7.2], J. Striker–N. Williams applied their theory to plane partitions, that is, the order ideals $J(\mathbf{a} \times \mathbf{b} \times \mathbf{c})$. They characterized $J(\mathbf{a} \times \mathbf{b} \times \mathbf{c})$ in terms of *boundary path matrices*. We give a sketch of this characterization here; for futher details, see [StWi12]. Given an order ideal in a special kind of planar poset (in the language of [StWi12], an rc-poset of height 1, or in the language of the next section, a poset with a 2-dimensional lattice projection), its **boundary path** is a binary sequence that encodes the path that separates the order ideal from the rest of the poset. Given a plane partition $P \in J(\mathbf{a} \times \mathbf{b} \times \mathbf{c})$, its **boundary path matrix** is a $b \times (a + b + c - 1)$ matrix $\{X_{i,j}\}$ with entries in $\{0, 1\}$ such that the ith row consists of the boundary path of layer i preceded by $i - 1$ zeros and succeeded by $b - i$ zeros. The rows of a boundary path matrix each sum to a and the entries obey the condition

$$\text{if } \sum_{j=1}^{k} X_{i,j} = \sum_{j=1}^{k} X_{i+1,j}, \text{ then } X_{i+1,j+1} \neq 1.$$

It was noted in [StWi12, Section 7.2] that Pro traces from left to right through the columns of the boundary path matrix, swapping each pair of entries in adjacent columns and the same row that result in a matrix still satisfying the condition above.

Given $P \in J(\mathbf{a} \times \mathbf{b} \times \mathbf{c})$ with boundary path matrix $\{X_{i,j}\}$, define $X_{\max}(P)$ to be the vector of length $a+b+c-1$ whose jth entry is $\max(X_{i,j})_{1 \leq i \leq b}$.

Lemma 8.4. *Let $P \in J(\mathbf{a} \times \mathbf{b} \times \mathbf{c})$. If $X_{\max}(P) = (x_1, x_2, \ldots, x_{a+b+c-1})$, then $X_{\max}(\mathrm{Pro}(P))$ is the cyclic shift $(x_2, \ldots, x_{a+b+c-1}, x_1)$.*

Proof. For $i > 1$, if column i of the boundary path matrix is all zeros, then in the application of Pro, all of these entries swap with the entries of column $i-1$, since the condition on the partial row sums is not violated.

If $i = 1$, the column of all zeros swaps all the way through the matrix, from the first column to the last column.

Thus, under Pro, a column of all zeros cyclically shifts to the left. □

The following instance of resonance follows directly from Lemma 8.4.

Proposition 8.5. *$(J(\mathbf{a} \times \mathbf{b} \times \mathbf{c}), \langle \mathrm{Pro} \rangle, X_{\max})$ exhibits resonance with frequency $a+b+c-1$.*

Let D be the conjugating toggle group element between rowmotion and promotion given in [StWi12, Theorem 5.4]. By the equivariance of Pro and Row in [StWi12], we have the following statement of resonance on rowmotion, which follows directly from Proposition 8.5 and [StWi12, Theorem 5.4].

Theorem 8.8. *$(J(\mathbf{a} \times \mathbf{b} \times \mathbf{c}), \langle \mathrm{Row} \rangle, X_{\max} \circ D)$ exhibits resonance with frequency $a+b+c-1$.*

This leads to the following corollary.

Corollary 8.3. *Suppose $a+b+c-1$ is prime and $P \in J(\mathbf{a} \times \mathbf{b} \times \mathbf{c})$. Suppose there is a zero in $X_{\max}(P)$. Then the size of the promotion orbit of P is a multiple of $a+b+c-1$.*

Using Corollary 8.3, we have a new proof of Theorem 8.5 of P. Cameron–D. Fon-der-Flaass.

Proof of Theorem 8.5. If $a+b+c-1$ is prime and $c > ab-a-b+1$, then there are a total of ab ones in the boundary path matrix, but a total of $a+b+c-1 > ab$ columns in the matrix, so there must be a column of all zeros. Thus, there is a zero in $X_{\max}(P)$ for any plane partition P in an $a \times b \times c$ box, and the promotion orbit is a multiple of $a+b+c-1$ by Corollary 8.3. Then by Theorem 8.7, the orbits of rowmotion are also multiples of $a+b+c-1$. □

P. Cameron–D. Fon-der-Flaass's proof of Theorem 8.5 is similar, but analyzes rowmotion directly rather than conjugating to promotion.

8.3.4 n-dimensional lattice projections

In this and the next subsections, we adapt the proof of the conjugacy of promotion and rowmotion from [StWi12] to give a generalization in the setting of *n-dimensional lattice projections*, which we introduce in Definition 8.9. (This new perspective includes the original theorem as the case $n = 2$.) We prove, in Theorem 8.9, the equivariance of the 2^{n-1} toggle group actions given in Definition 8.10.

Definition 8.8. We say that a poset P is **ranked** if it admits a rank function $\mathrm{rk}\colon P \to \mathbb{Z}$ satsifying $\mathrm{rk}(y) = \mathrm{rk}(x) + 1$ when y covers x.

Definition 8.9. We say that an **(n-dimensional) lattice projection** of a ranked poset P is an order and rank preserving map $\pi : P \to \mathbb{Z}^n$, where the rank function on $\mathbb{Z}^n$ is the sum of the coordinates and $x \leq y$ in $\mathbb{Z}^n$ if and only if the componentwise difference $y - x$ is in $(\mathbb{Z}_{\geq 0})^n$.

In light of Remark 8.1, the key feature of π is that it preserves cover relations. That is, if y covers x in P, then $\pi(y)$ covers $\pi(x)$ in $\mathbb{Z}^n$. However, since $\mathbb{Z}^n$ is ranked, π being cover-relation preserving would make $\mathrm{rk} \circ \pi$ a rank function for P. And if P is ranked, then a map $\pi : P \to \mathbb{Z}^n$ being cover-relation preserving is equivalent to it being order and rank preserving (up to a shift of the rank functions).

In [StWi12], the definition of an rc-poset was a poset that had a 2-dimensional lattice projection (albeit to a slightly different lattice). However, E. Sawin noted that every ranked poset P with rank function ρ has such an embedding given by $\pi(x) = (\rho(x), 0)$ for $x \in P$ [Sa13]. Similarly, any poset P with a lattice projection π has a rank function given by the sum of the coordinates in $\pi(x)$ for $x \in P$.

Additionally, a ranked poset may have multiple distinct projections. For example, in Figure 8.8, we have the boolean lattice on three elements, which we can think of a product of three chains of length 2. In Figure 8.9, we have the standard three-dimensional lattice projection of this poset we get by thinking of it as a product of three chains. In Figure 8.10, we show two different two-dimensional lattice projections of this poset. In the projection on the right, we assign every element of the same rank to the same point, but instead of doing so along the x-axis as in the previous paragraph, we do this diagonally in a zig-zag pattern. Therefore, instead of considering rc-posets, we consider any ranked poset, but with respect to a given lattice projection.

8.3.5 Promotion via affine hyperplane toggles

We now define a toggling order on our poset with respect an n-dimensional lattice projection, and with respect to a distinguished direction.

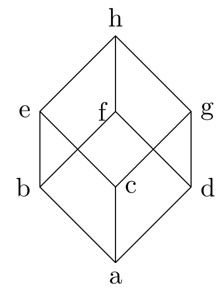


Figure 8.8: A product of three chains.

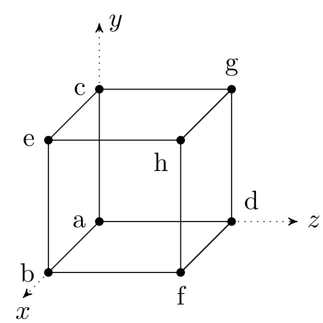


Figure 8.9: The standard three-dimensional lattice projection of a product of three two-element chains.

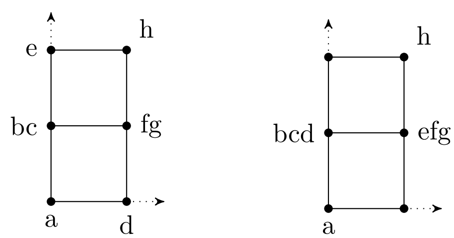


Figure 8.10: Two distinct two-dimensional lattice projections of a product of three two-element chains.

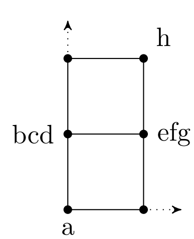


Figure 8.11: Gyration lattice projections of a product of three chains.

Definition 8.10. Let P be a poset with an n-dimensional lattice projection π, and let $v = (v_1, v_2, v_3, \ldots, v_n)$, where $v_j \in \{\pm 1\}$. Let $T^i_{\pi,v}$ be the product of toggles t_x for all elements x of P that lie on the affine hyperplane $\langle \pi(x), v\rangle = i$. If there is no such x, then this is the empty product, considered to be the identity. Then define **promotion with respect to** π **and** v as the toggle product $\mathrm{Pro}_{\pi,v} = \ldots T^{-2}_{\pi,v} T^{-1}_{\pi,v} T^{0}_{\pi,v} T^{1}_{\pi,v} T^{2}_{\pi,v} \ldots$.

See Figure 8.12 for an example.

Remark 8.3. Note that $\mathrm{Pro}_{\pi,-v} = (\mathrm{Pro}_{\pi,v})^{-1}$, so we will generally only consider distinguished vectors with $v_1 = 1$, as all promotion operators are either of this form, or the inverse of something of this form.

Lemma 8.5. *Two elements of the poset that lie on the same affine hyperplane $\langle \pi(x), v\rangle = i$ cannot be part of a covering relation, so by Remark 8.1, the operator $T^i_{\pi,v}$ is well-defined and $(T^i_{\pi,v})^2 = 1$.*

Proof. Assume that y covers x, and they both lie on the same affine hyperplane ($\langle \pi(x), v\rangle = \langle \pi(y), v\rangle = i$). Then $\langle \pi(y), v\rangle - \langle \pi(x), v\rangle = \langle \pi(y) - \pi(x), v\rangle = 0$. But since y covers x, $\pi(y) - \pi(x) = e_i$ for some i. And since v has all coordinates ± 1, then $\langle e_i, v\rangle = \pm 1$, a contradiction. $\square$

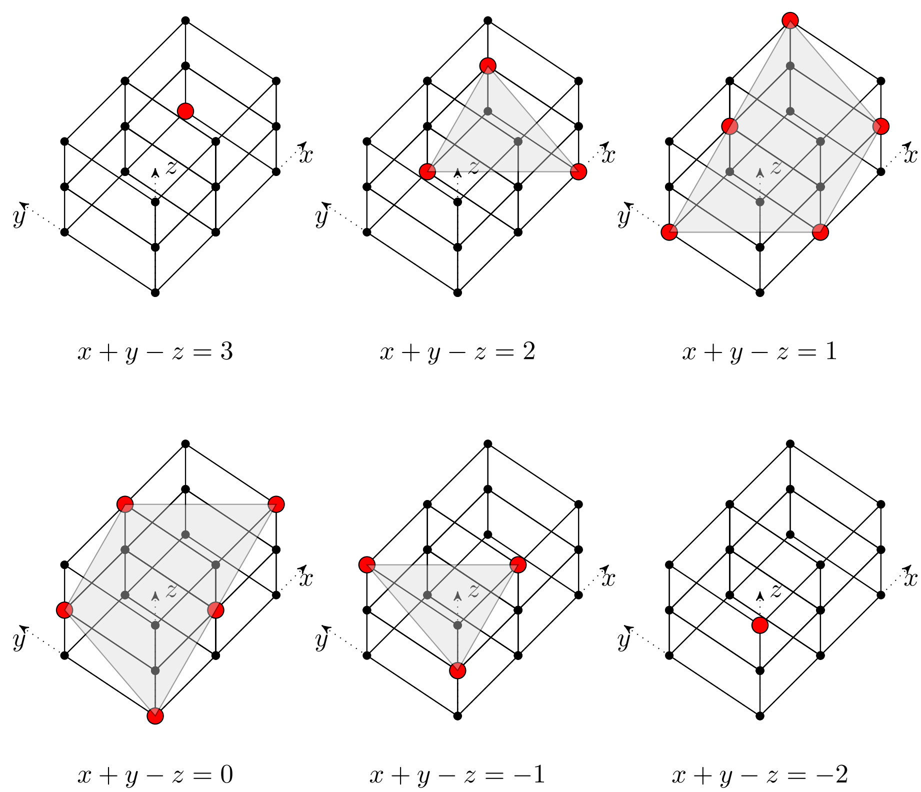


Figure 8.12: The affine hyperplane toggles corresponding to $\mathrm{Pro}_{\mathrm{id},(1,1,-1)}$ for the identity three-dimensional lattice projection of the poset $J(\mathbf{3} \times \mathbf{2} \times \mathbf{3})$

For ease of notation, we may suppress explicitly listing the lattice projection map π or the direction v when referring to the generalized promotion operator, if it is clear from context. Note that for a finite poset P, $T^i_{\pi,v}$ will be the identity operator for all but finitely many i.

Remark 8.4. To compare with the notion of promotion and rowmotion given in [StWi12], for a given 2-dimensional lattice projection π of a finite poset P, rowmotion corresponds to $\mathrm{Pro}_{\pi,(1,1)}$, and promotion corresponds to $\mathrm{Pro}_{\pi,(1,-1)}$.

Proposition 8.6. *For any finite ranked poset P and lattice projection π,* $\mathrm{Pro}_{\pi,(1,1,\ldots,1)} = \mathrm{Row}$.

Proof. $\mathrm{Pro}_{\pi,(1,1,\ldots,1)}$ sweeps through P from top to bottom (in the reverse order of a linear extension), so by Theorem 8.6, this is rowmotion. □

We give some further definitions and lemmas, in order to state and prove Theorem 8.9 in full generality.

Definition 8.11. Let P be a poset, and let π, v, and $T^i_{\pi,v}$ be as in Definition 8.10. Define the **support** of (P, π, v), denoted $\mathrm{supp}(P, \pi, v)$, to be the smallest interval $[a, b] \subseteq \mathbb{Z}$ such that $T^i_{\pi,v}$ is the identity operator for all $i \in \mathbb{Z} \setminus [a, b]$.

Definition 8.12. If (P, π, v) has finite support, that is, $\mathrm{supp}(P, \pi, v) = [a, b] \subset \mathbb{Z}$, let $\sigma : [a, b] \to [a, b]$ be a bijection. Then define **promotion with respect to P, π, v, and σ** as the following product of hyperplane-toggles:

$$\mathrm{Pro}^{\sigma}_{\pi,v} = T^{\sigma(a)}_{\pi,v} T^{\sigma(a+1)}_{\pi,v} \ldots T^{\sigma(b-1)}_{\pi,v} T^{\sigma(b)}_{\pi,v}.$$

We will use the following toggle group element in the proof of Theorem 8.9.

Definition 8.13. For a poset P, define the **parity** of $p \in P$ as even (resp. odd) if the parity of $\mathrm{rk}(p)$ is even (resp. odd). Define **gyration** Gyr as the toggle group element which first toggles all $p \in P$ with even parity, then all p with odd parity.

Remark 8.5. There is a connection between the toggle group element Gyr defined above and the gyration action on alternating sign matrices discussed in Section 8.1.1. Namely, there is a poset $\mathbf{A}_n$ whose order ideals are in bijection with alternating sign matrices, and for which the gyration action of Definition 8.5 is the action of Gyr. For details, see [StWi12, Section 8] and [Str15].

Remark 8.6. Given a lattice projection π, the rank of p is the same as the rank of $\pi(p) = (x_1, x_2, \ldots x_n)$, which is $\sum_i x_i$. Since all the coordinates in v are ± 1, the parity of $\pi(p)$ will be the same as the parity of $\langle \pi(p), v \rangle$. Thus, all elements lying on the same affine hyperplane with respect to v will have the same parity.

Lemma 8.6. *If (P,π,v) has finite support $[a,b]$, then for any bijection $\sigma:[a,b]\to[a,b]$ such that $\sigma(k)$ is odd if $k<\frac{a+b}{2}$ and even if $k>\frac{a+b}{2}$, we have $\mathrm{Pro}^{\sigma}_{\pi,v}=\mathrm{Gyr}$.*

We are now nearly ready to state and prove the main theorem of this section. We will need the following lemma, which appears as [StWi12, Lemma 5.1].

Lemma 8.7 ([HoHu92])**.** *Let G be a group whose generators $g_1,\ldots,g_n$ satisfy $g_i^2=1$ and $(g_ig_j)^2=1$ if $|i-j|>1$. Then for any $\sigma,\tau\in\mathfrak{S}_n$, $\prod_i g_{\sigma(i)}$ and $\prod_i g_{\tau(i)}$ are conjugate.*

The main theorem of this section is below, whose proof follows that of [StWi12, Theorem 5.2].

Theorem 8.9. *Let P be a finite poset with an n-dimensional lattice projection π. Let $v=(v_1,v_2,v_3,\ldots,v_n)$ and $w=(w_1,w_2,w_3,\ldots,w_n)$, where $v_j,w_j\in\{\pm1\}$. Finally suppose that $\sigma:\mathrm{supp}(P,\pi,v)\to\mathrm{supp}(P,\pi,v)$ and $\tau:\mathrm{supp}(P,\pi,w)\to\mathrm{supp}(P,\pi,w)$ are bijections. Then there is an equivariant bijection between $J(P)$ under $\mathrm{Pro}^{\sigma}_{\pi,v}$ and $J(P)$ under $\mathrm{Pro}^{\tau}_{\pi,w}$.*

Proof. Suppose P is a finite poset with an n-dimensional lattice projection π. Let $v=(v_1,v_2,v_3,\ldots,v_n)$, where $v_j\in\{\pm1\}$. We claim the toggles $T^i_{\pi,v}$ for $i\in\mathrm{supp}(P,\pi,v)$ satisfy the conditions of Lemma 8.7. By Lemma 8.5, $(T^i_{\pi,v})^2=1$. Also, if $\langle\pi(x),v\rangle=i$ and $\langle\pi(y),v\rangle=j$, then $\langle\pi(y)-\pi(x),v\rangle=j-i$. So if $|j-i|>1$, as all the coefficients in v are ±1, then $\pi(y)-\pi(x)$ cannot be e_i for any i, and y and x cannot be part of a covering relation. Thus, toggles on non-adjacent hyperplanes commute, and we have $(T^i_{\pi,v}T^j_{\pi,v})^2=1$ when $|j-i|>1$. So by Lemma 8.7, for any bijections $\sigma,\sigma':\mathrm{supp}(P,\pi,v)\to\mathrm{supp}(P,\pi,v)$, there is an equivariant bijection between $J(P)$ under $\mathrm{Pro}^{\sigma}_{\pi,v}$ and $J(P)$ under $\mathrm{Pro}^{\sigma'}_{\pi,v}$ (since such bijections can be considered as permutations in $\mathfrak{S}_{b-a+1}$ if $\mathrm{supp}(P,\pi,v)=[a,b]$).

Consider Gyr of Definition 8.13. By Lemma 8.6, for every v there exists a σ_v such that Gyr can be realized as $\mathrm{Pro}^{\sigma_v}_{\pi,v}$. Therefore, there is an equivariant bijection between $J(P)$ under $\mathrm{Pro}^{\sigma}_{\pi,v}$ and under $\mathrm{Pro}^{\sigma_v}_{\pi,v}=\mathrm{Gyr}$, from which the theorem follows. □

After we see a bijection between increasing tableaux and plane partitions given in the next section, we will use Theorem 8.9 to give an improvement on Theorem 8.5 of P. Cameron–D. Fon-der-Flaass (discussed in Section 8.3.1).

8.4 An equivariant bijection between plane partitions and increasing tableaux

8.4.1 Bijections between plane partitions and increasing tableaux

In this section, we introduce bijections between increasing tableaux and plane partitions. The existence of these bijections should not be at all surprising. However, these maps are key to many of our results. These maps are also fundamental to [HPPW16] (the basis of Chapter 9), where they are used to give the first bijective proofs of various results on plane partitions, including R. Proctor's main result from [Pr83].

We define a map $\Psi_3 : J(\mathbf{a} \times \mathbf{b} \times \mathbf{c}) \to \mathrm{Inc}^{a+b+c-1}(a \times b)$ as follows. Let $P \in J(\mathbf{a} \times \mathbf{b} \times \mathbf{c})$. Thinking of P in the standard way as a pile of small cubes in an $a \times b \times c$ box, project onto the $a \times b$ face. Record in position (i, j) the number of boxes of P with coordinate (i, j, k) for some $0 \leq k \leq c - 1$. The result is a standard plane partition representation of P, as a filling of the Young diagram $a \times b$ with nonnegative integers such that rows weakly decrease from left to right and columns weakly decrease from top to bottom. Rotate this plane partition 180°, so that rows and columns become weakly *increasing*. Now thinking of $a \times b$ as a graded poset with the upper left corner box the unique element of rank 0, add to each label its rank plus 1. That is, increase each label by one more than its distance from the upper left corner box. (This is just the standard way of converting a *weakly* increasing sequence into a *strictly* increasing one.) The result is the increasing tableau $\Psi_3(P)$. For an example of this transformation, see Figure 8.13.

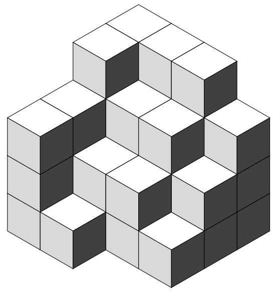
Theorem 8.10. $\Psi_3 : J(\mathbf{a} \times \mathbf{b} \times \mathbf{c}) \to \mathrm{Inc}^{a+b+c-1}(a \times b)$ *gives a bijection between plane partitions inside an* $a \times b \times c$ *box and increasing tableaux of shape* $a \times b$ *and entries at most* $a + b + c - 1$.

Proof. The map is defined as the composition of a projection, a rotation, and entrywise addition, all of which are clearly invertible. ☐

Similarly, define bijections $\Psi_2 : J(\mathbf{a} \times \mathbf{b} \times \mathbf{c}) \to \mathrm{Inc}^{a+b+c-1}(a \times c)$ and $\Psi_1 : J(\mathbf{a} \times \mathbf{b} \times \mathbf{c}) \to \mathrm{Inc}^{a+b+c-1}(b \times c)$ projecting onto the $\mathbf{a} \times \mathbf{c}$ and $\mathbf{b} \times \mathbf{c}$ faces, respectively (cf. Figure 8.14).

Given the simplicity of the bijection of Theorem 8.10, one might wonder why it was previously overlooked. The set of increasing tableaux in bijection with plane partitions includes those with gaps in the content. However much previous research on increasing tableaux was motivated by K-theoretic geometry, and in this context there is little reason to consider increasing tableaux without full content. Moreover, by restricting to tableaux of full content, one obtains some attractive enumerations [Pe14, PrStVi14]; for instance, the number of increasing tableaux with shape $2 \times n$ and full content is the nth *small Schröder number* [Pe14,

$P =$



Project to bottom face $\rightarrow$

4	4	4	3
4	3	3	2
3	2	2	1
3	1	0	0

Rotate 180°

0	0	1	3
1	2	2	3
2	3	3	4
3	4	4	4

Add 1+rank $\rightarrow$

1	2	4	7
3	5	6	8
5	7	8	10
7	9	10	11

$= \Psi_3(P)$

Figure 8.13: The process of applying Ψ_3 to the illustrated $P \in J(\mathbf{4} \times \mathbf{4} \times \mathbf{4})$. Here we think of Ψ_3 as projecting onto the bottom face of the large bounding box.

Theorem 1.1]. It was the equivariance of the actions of $\mathcal{P}$ and Row, discussed in the next section, which led us to observe the bijection of Theorem 8.10.

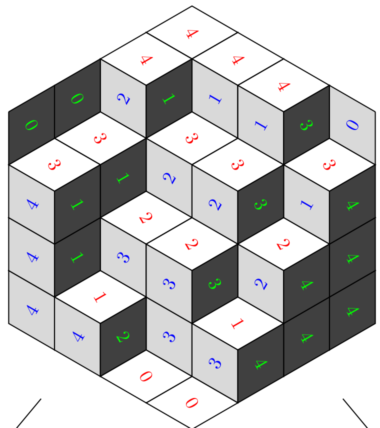
8.4.2 The equivariance of $\mathcal{P}$ and Row

Our first main result was Theorem 8.9, that given a poset P with lattice projection π, there is an equivariant bijection between the order ideals $J(P)$ under $\mathrm{Pro}^{\sigma}_{\pi,v}$ and $\mathrm{Pro}^{\tau}_{\pi,w}$, where σ, τ are any permutations of the hyperplane toggles associated to the $\{-1, 1\}$-vectors v, w. In this section, we use Theorem 8.9 in our proof of our second main result, Theorem 8.11, that $\mathcal{P}$ and Row are in equivariant bijection.

Lemma 8.8. Ψ_3 *intertwines* $\mathrm{Pro}_{\mathrm{id},(1,1,-1)}$ *and* $\mathcal{P}$. *That is, the following diagram commutes:*

$$\begin{tikzcd}
J(\mathbf{a} \times \mathbf{b} \times \mathbf{c}) \arrow[r, "\Psi_3"'] \arrow[d, "\mathrm{Pro}_{\mathrm{id},(1,1,-1)}"] & \mathrm{Inc}^{a+b+c-1}(a \times b) \arrow[d, "\mathcal{P}"] \\
J(\mathbf{a} \times \mathbf{b} \times \mathbf{c}) \arrow[r, "\Psi_3"'] & \mathrm{Inc}^{a+b+c-1}(a \times b)
\end{tikzcd}$$

Proof. Let $P \in J(\mathbf{a} \times \mathbf{b} \times \mathbf{c})$ and let $T = \Psi_3(P)$. Note that the poset $\mathbf{a} \times \mathbf{b} \times \mathbf{c}$ has a 3-dimensional lattice



4	4	4	3
4	3	3	1
4	3	1	0
2	1	0	0

4	4	4	3
4	3	3	2
3	2	2	1
3	1	0	0

4	4	4	2
4	3	2	1
3	3	2	1
3	2	1	0

1	2	4	6
2	4	7	9
4	7	8	10
7	9	10	11

1	2	4	7
3	5	6	8
5	7	8	10
7	9	10	11

1	3	5	7
3	5	7	8
4	6	8	10
6	9	10	11

Figure 8.14: The three bijections, Ψ_1, Ψ_3, and Ψ_2

projection, in the sense of Definition 8.9, given by the identity map.

By Proposition 8.1,

$$\mathcal{P}(T) = K\text{-BK}_{a+b+c-2} \circ \cdots \circ K\text{-BK}_1(T).$$

Similarly,

$$\text{Pro}_{\text{id},(1,1,-1)} = T_{\text{id},(1,1,-1)}^{(a-1)+(b-1)-(a+b+c-2)} \circ \cdots \circ T_{\text{id},(1,1,-1)}^{(a-1)+(b-1)-1}.$$

Thus, it suffices to show that

$$\Psi_3\left(T_{\text{id},(1,1,-1)}^{(a-1)+(b-1)-\ell}(P)\right) = K\text{-BK}_\ell(T).$$

By Definition 8.10, $T_{\text{id},(1,1,-1)}^{(a-1)+(b-1)-\ell}$ is the product of the the toggles t_x for all $x \in \mathbf{a} \times \mathbf{b} \times \mathbf{c}$ lying on the affine hyperplane determined by $\langle x, (1,1,-1)\rangle = (a-1)+(b-1)-\ell$. Consider $x = (i,j,k)$ on this hyperplane. Then $i+j-k = (a-1)+(b-1)-\ell$.

We have $x = (i,j,k) \in P$ if and only if the $(a-i, b-j)$ entry of T is at least $k+(a-i)+(b-j)-1 = k+a+b-i-j-1$. Since $k = i+j-(a-1)-(b-1)+\ell$, we can rewrite this condition as the $(a-i,b-j)$ entry of T being at least $(i+j-(a-1)-(b-1)+\ell)+a+b-i-j-1 = \ell+1$. Hence $x \in P$ if and only if the $(a-i,b-j)$ entry of T is at least $\ell+1$.

(Case 1: $x \in P$): If $(i,j,k+1) \in P$, then x is unaffected by the toggle and the $(a-i,b-j)$ entry of T is at least $\ell+2$ and so unaffected by K-BK$_\ell$.

Otherwise $(i,j,k+1) \notin P$ and the $(a-i,b-j)$ entry of T equals $\ell+1$. K-BK$_\ell$ will turn this $\ell+1$ into ℓ exactly when neither the $(a-i-1,b-j)$ nor the $(a-i,b-j-1)$ entry of T equals ℓ. By increasingness of T, neither entry is greater than ℓ. The $(a-i-1,b-j)$ entry of T is at least ℓ exactly when $(i+1,j,k) \in P$. Similarly the $(a-i,b-j-1)$ entry of T is at least ℓ exactly when $(i,j+1,k) \in P$. Hence K-BK$_\ell$ will turn this $\ell+1$ into ℓ exactly when neither $(i+1,j,k)$ nor $(i,j+1,k)$ is in P. But this is exactly when the hyperplane toggle removes x from P. Since P is an order ideal, $(i,j,k-1) \in P$, so if $T_{\text{id},(1,1,-1)}^{(a-1)+(b-1)-\ell}$ removes x from P, then the $(a-i,b-j)$ entry of $\Psi_3\left(T_{\text{id},(1,1,-1)}^{(a-1)+(b-1)-\ell}(P)\right)$ equals ℓ, as desired.

(Case 2: $x \notin P$): The $(a-i,b-j)$ entry of T is at most ℓ. If it is less than ℓ, then $(i,j,k-1) \notin P$. Hence x is unaffected by the hyperplane toggle and the $(a-i,b-j)$ entry of T is unaffected by K-BK$_\ell$.

Otherwise, the $(a-i,b-j)$ entry of T equals ℓ and $(i,j,k-1) \in P$. K-BK$_\ell$ will turn this ℓ into $\ell+1$ exactly when neither the $(a-i+1,b-j)$ nor the $(a-i,b-j+1)$ entry of T equals $\ell+1$. This happens exactly when both $(i-1,j,k) \in P$ and $(i,j-1,k) \in P$. Thus $T_{\text{id},(1,1,-1)}^{(a-1)+(b-1)-\ell}$ toggles x into P exactly when K-BK$_\ell$ turns this ℓ into $\ell+1$. $\square$

Remark 8.7. By symmetry of $J(\mathbf{a} \times \mathbf{b} \times \mathbf{c})$, we obtain analogous results for Ψ_1 and Ψ_2.

As a consequence of the above lemma and Theorem 8.9, we obtain the following.

Theorem 8.11. *$J(\mathbf{a} \times \mathbf{b} \times \mathbf{c})$ under Row is in equivariant bijection with $\mathrm{Inc}^{a+b+c-1}(a \times b)$ under $\mathcal{P}$.*

8.4.3 Consequences of the bijection

In this subsection, we give a number of consequences of Theorem 8.11. We first give another statement of resonance on plane partitions in Corollary 8.4 (cf. Theorem 8.8). In Corollary 8.5, we give $\mathcal{P}$-equivariant bijections between various sets of increasing tableaux using the tri-fold symmetry of $J(\mathbf{a}\times\mathbf{b}\times\mathbf{c})$. We exploit this symmetry to prove Corollaries 8.6 and 8.7. We make a conjecture about the order of Row on $J(\mathbf{a}\times\mathbf{b}\times\mathbf{3})$. Finally, in Theorem 8.12, we improve the bound of Theorem 8.5 of P. Cameron–D. Fon-der-Flaass.

We obtain the following statement of resonance of rowmotion on plane partitions as a consequence of Theorems 8.11 and 8.2.

Corollary 8.4. *$(J(\mathbf{a} \times \mathbf{b} \times \mathbf{c}), \mathrm{Row}, \mathrm{Con} \circ \Psi_3)$ exhibits resonance with frequency $a + b + c - 1$.*

We furthermore obtain the following corollary via the tri-fold symmetry of $J(\mathbf{a} \times \mathbf{b} \times \mathbf{c})$.

Corollary 8.5. *There are $\mathcal{P}$-equivariant bijections between the sets $\mathrm{Inc}^{a+b+c-1}(a \times b)$, $\mathrm{Inc}^{a+b+c-1}(a \times c)$, and $\mathrm{Inc}^{a+b+c-1}(b \times c)$.*

Proof. By Lemma 8.8 and Remark 8.7, $\Psi_2 \circ \Psi_3^{-1}$ is a $\mathcal{P}$-equivariant bijection between $\mathrm{Inc}^{a+b+c-1}(a \times b)$ and $\mathrm{Inc}^{a+b+c-1}(a \times c)$. Similarly, $\Psi_1 \circ \Psi_3^{-1}$ is an equivariant bijection between $\mathrm{Inc}^{a+b+c-1}(a \times b)$ and $\mathrm{Inc}^{a+b+c-1}(b \times c)$. □

Theorem 8.11 and Corollary 8.5 allow us to obtain a number of results for small values of c. We obtain new proofs of known results Theorems 8.3 and 8.4, while Corollaries 8.6 and 8.7 are new.

We use the following trivial fact about the order of $\mathcal{P}$ on increasing tableaux of one row.

Fact 8.1. The order of $\mathcal{P}$ on $\mathrm{Inc}^q(1 \times a)$ is q.

The following is a new proof of Theorem 8.3 of A. Brouwer–A. Schrijver [BrSc74], which we restate for convenience.

Theorem 8.3. *The order of Row on $J(\mathbf{a} \times \mathbf{b})$ is $a + b$.*

Proof. The order of Row on $J(\mathbf{a} \times \mathbf{b})$ is the same as the order of Row on $J(\mathbf{a} \times \mathbf{b} \times \mathbf{1})$. By Corollary 8.11, the order of Row on $J(\mathbf{a} \times \mathbf{b} \times \mathbf{1})$ equals the order of $\mathcal{P}$ on $\mathrm{Inc}^{a+b}(a \times 1)$. By Fact 8.1, the order of $\mathcal{P}$ on $\mathrm{Inc}^{a+b}(a \times 1)$ is $a + b$. □

The following result is new.

Corollary 8.6. *The order of* $\mathcal{P}$ *on* $\mathrm{Inc}^{a+b}(a \times b)$ *is* $a+b$*.*

Proof. By the tri-fold symmetry of Corollary 8.5, there is a $\mathcal{P}$-equivariant bijection between the sets $\mathrm{Inc}^{a+b}(a \times b)$ and $\mathrm{Inc}^{a+b}(1 \times a)$. The result is then immediate by Fact 8.1. $\square$

We can also use Theorem 8.11 and Corollary 8.5 to show that the Theorem 8.4 of P. Cameron–D. Fon-der-Flaass [CaFo95] is equivalent to Theorem 7.3, thus providing a new proof of Theorem 8.4. Alternatively, one may use Theorem 8.4 along with Theorem 8.11 and Corollary 8.5 to give a new proof of Theorem 7.3.

Theorem 8.4. *The order of* Row *on* $J(\mathbf{a} \times \mathbf{b} \times \mathbf{2})$ *is* $a+b+1$*.*

Proof. By Theorem 8.11 and Corollary 8.5, the order of Row on $J(\mathbf{a} \times \mathbf{b} \times \mathbf{2})$ equals the order of $\mathcal{P}$ on $\mathrm{Inc}^{a+b+1}(2 \times a)$. By Theorem 7.3, the latter is $a+b+1$. (Theorem 7.3 only considers those increasing tableaux of complete content; however its proof extends easily to the case of general content.) $\square$

The following result is new.

Corollary 8.7. *The order of* $\mathcal{P}$ *on* $\mathrm{Inc}^{a+b+1}(a \times b)$ *is* $a+b+1$*.*

Proof. By Corollary 8.5, there is a $\mathcal{P}$-equivariant bijection between the sets $\mathrm{Inc}^{a+b+1}(a \times b)$ and $\mathrm{Inc}^{a+b+1}(2 \times a)$. The result is then immediate from Theorem 7.3. $\square$

Recall that for $c > 3$, the order of Row on $J(\mathbf{a} \times \mathbf{b} \times \mathbf{c})$ is generally greater than $a+b+c-1$. Nonetheless, we make the following conjecture.

Conjecture 8.1. *The order of* Row *on* $J(\mathbf{a} \times \mathbf{b} \times \mathbf{3})$ *is* $a+b+2$*.*

As with the above corollaries, the results of this chapter show that Conjecture 8.1 is equivalent to the order of K-promotion being $a+b+2$ on either $\mathrm{Inc}^{a+b+2}(a \times b)$ or $\mathrm{Inc}^{a+b+2}(3 \times a)$. We have verified Conjecture 8.1 for $a \leq 7$ and b arbitrary.

Finally, we improve the bound in Theorem 8.5 of P. Cameron–D. Fon-der-Flaass [CaFo95] by more than a factor of $\frac{2}{3}$. This is evidence toward the conjecture of P. Cameron–D. Fon-der-Flaass [CaFo95] that this condition may be dropped entirely.

Theorem 8.12. *If* $a+b+c-1$ *is prime and* $c > \dfrac{2ab-2}{3} - a - b + 2$*, then the cardinality of every orbit of* Row *on* $J(\mathbf{a} \times \mathbf{b} \times \mathbf{c})$ *is a multiple of* $a+b+c-1$*.*

Proof. Let $q = a + b + c - 1$. The case $q = 2$ is trivial, so assume q is odd.

Consider $P \in J(\mathbf{a} \times \mathbf{b} \times \mathbf{c})$ and let $T = \Psi_3(P) \in \mathrm{Inc}^q(a \times b)$. If T does not have full content, then by Corollary 8.2, the K-promotion orbit of T has cardinality a multiple of q. Hence by Theorem 8.11, the rowmotion orbit of P has cardinality a multiple of q, as claimed. Thus, we may assume T has full content.

Similarly, by Proposition 8.4, we may assume that every $1 \leq i \leq q$ is both a descent and a transpose descent in T. Hence by Lemma 8.3, for $1 \leq j \leq \frac{q-1}{2}$, the number of $(2j-1)$'s in T plus the number of $2j$'s in T is at least 3. By the increasingness conditions on T, there is exactly 1 instance of q in T. Thus the total number of labels in T is at least $3\frac{q-1}{2} + 1$.

Since $T \in \mathrm{Inc}^q(a \times b)$, this forces $3\frac{a+b+c-2}{2} + 1 \leq ab$. Thus $c \leq \frac{2ab-2}{3} - a - b + 2$, contradicting the assumed bound on c. □

8.5 Open problems

We conclude by reformulating some observations from [StWi12] in terms of resonance; for further details, see [StWi12, Sections 8.3 and 8.4].

Recall from Remark 8.5 that there is a poset $\mathbf{A}_n$ whose order ideals are in bijection with $n \times n$ alternating sign matrices such that gyration (of Definition 8.5) is equivalent to the action of the toggle group element Gyr of Definition 8.13. Another element, SPro, of the toggle group on $\mathbf{A}_n$ was introduced in [StWi12, Definition 8.14]. It is shown in [StWi12, Theorem 8.15] that the orbit of the empty order ideal in $J(\mathbf{A}_n)$ under SPro has cardinality $3n - 2$. Further data contained in [StWi12, Figure 22] leads us to propose the following.

Problem 8.1. *Construct a natural map f such that $(\mathrm{ASM}_n, \mathrm{SPro}, f)$ exhibits resonance with frequency $3n - 2$.*

Similarly, there is a poset $\mathbf{T}_n$ whose order ideals are in bijection with *totally symmetric self-complementary plane partitions* inside a $2n \times 2n \times 2n$ box (denote this set as TSSCPP_n). It is shown in [StWi12, Theorem 8.19] that the cardinality of the rowmotion-orbit of the empty order ideal in $J(\mathbf{T}_n)$ is $3n-2$. Further data contained in [StWi12, Figure 22] suggests the following.

Problem 8.2. *Construct a natural map f such that $(\mathrm{TSSCPP}_n, \mathrm{Row}, f)$ exhibits resonance with frequency $3n - 2$.*

We suspect that a solution to the above problems would be a major step towards exhibiting an explicit bijection between ASM_n and TSSCPP_n, which are known (non-bijectively) to be equinumerous.

Chapter 9

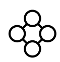
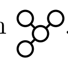
Doppelgängers and bijections of plane partitions

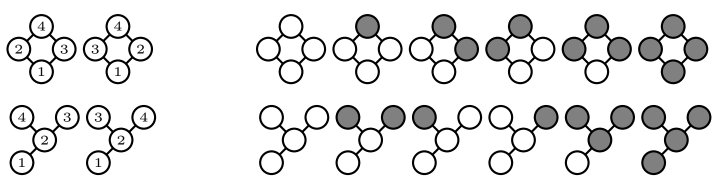
This chapter derives from joint work with Z. Hamaker, R. Patrias, and N. Williams [HPPW16].

9.1 Introduction

9.1.1 Doppelgängers

Let $n \in \mathbb{N}$ be a positive integer, and let $[n] := \{1, 2, \ldots, n\}$. If $\mathcal{P}$ is a poset with ℓ elements, its **linear extensions** or **standard Young tableaux of shape** $\mathcal{P}$ (written $\mathsf{SYT}(\mathcal{P})$) are all (strictly) order-preserving bijections from $\mathcal{P} \to [\ell]$, while for $p \in \mathbb{N}$ its **$\mathcal{P}$-partitions of height** p (written $\mathsf{PP}^{[p]}(\mathcal{P})$) are all weakly order-preserving maps from $\mathcal{P} \to \{0\} \cup [p]$.

Example 9.1. Let $\mathcal{P}$ have Hasse diagram  and let $\mathcal{Q}$ have Hasse diagram . The posets $\mathcal{P}$ and $\mathcal{Q}$ each have two standard Young tableaux, and six plane partitions of height one, as illustrated below.



$\diamondsuit$

Definition 9.1. Let $\mathcal{P} \neq \mathcal{Q}$ be two finite posets. We say that $\mathcal{P}$ and $\mathcal{Q}$ are **doppelgängers** if $\left|\mathsf{PP}^{[p]}(\mathcal{P})\right| = \left|\mathsf{PP}^{[p]}(\mathcal{Q})\right|$ for all nonnegative integers p.

The equality of the number of $\mathcal{P}$- and $\mathcal{Q}$-partitions in Definition 9.1 forces the corresponding equality of the number of standard Young tableaux.

Proposition 9.1. *If $\mathcal{P}$ and $\mathcal{Q}$ are doppelgängers, then* $|\mathsf{SYT}(\mathcal{P})| = |\mathsf{SYT}(\mathcal{Q})|$.

Proof. The quantity $\left|\mathsf{PP}^{[p]}(\mathcal{P})\right|$—the **order polynomial** of $\mathcal{P}$—is a polynomial in p with leading coefficient $\frac{1}{\ell!}\,|\mathsf{SYT}(\mathcal{P})|$ [Sta72]. $\square$

As a trivial example, any poset and its dual (obtained by turning its Hasse diagram upside down) are doppelgängers. As a less trivial example, the posets $\mathcal{P}$ and $\mathcal{Q}$ of Example 9.1 are doppelgängers because they both have order polynomial

$$\left|\mathsf{PP}^{[p]}(\mathcal{P})\right| = \left|\mathsf{PP}^{[p]}(\mathcal{Q})\right| = \frac{1}{12}p(p+1)^2(p+2).$$

We are interested in non-trivial examples of doppelgängers that arise naturally from root systems. We establish our examples through consideration of the K-theoretic Schubert calculus of minuscule varieties.

9.1.2 Root-Theoretic Posets

We briefly describe the posets required to state our main results. We name and informally describe these posets in Figure 9.1, and give examples in Figure 9.2. Note that Example 9.1 is an instance of the first row of Figure 9.1.

Label	Name	Description	Description	Name
B	$\Lambda_{\mathsf{Gr}(k,n)}$	$k \times (n-k)$ rectangle	shifted trapezoid ($n \geq 2k$)	$\Phi^+(B_{k,n})$
H	$\Lambda_{\mathsf{OG}(6,12)}$	shifted 5-staircase	See Figure 9.2	$\Phi^+(H_3)$
I	$\Lambda_{\mathcal{Q}^m}$	propeller	snake	$\Phi^+(I_2(m))$
A	$\Lambda_{\mathsf{LG}(n,2n)}$	shifted n-staircase	n-staircase	$\Phi^+(A_n)$

Figure 9.1: The eight posets used in Theorem 9.1, Theorem 9.2, and Theorem 9.1. Row B is a doubly infinite family, row H is a single example, and rows I and A are infinite families. Figure 9.2 illustrates particular examples.

Minuscule posets

Let Φ be an irreducible crystallographic root system with Weyl group W and weights Λ. A **minuscule** weight is a dominant weight $\omega \neq 0$ such that $\langle \omega, \alpha^\vee \rangle \in \{-1, 0, 1\}$ for all $\alpha \in \Phi$. The **weight poset** is the partial order on the orbit $\{w(\omega) : w \in W\}$ given by the transitive closure of the relations $\omega \prec \mu$ if and only if $\mu - \omega$ is a simple root; when ω is minuscule, this poset is a distributive lattice, and we call its poset of join-irreducibles a **minuscule poset**. A minuscule poset is therefore characterized by a Cartan type and the index of a minuscule weight (see Figure 9.6 for our indexing conventions). The posets on the left-hand side of Figure 9.1 are all (co)minuscule posets (except $\Lambda_{\mathcal{Q}^{2m-1}}$, which we define to be $\Lambda_{\mathcal{Q}^{2m}}$ without its top element[1]):

- $\Lambda_{\mathsf{Gr}(k,n)}$ is the minuscule poset of type (A_{n-1}, k);

[1]Warning: Our $\Lambda_{\mathcal{Q}^{2m-1}}$ conflicts with notation usually reserved for odd orthogonal Grassmannians.

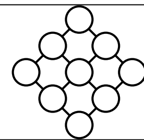
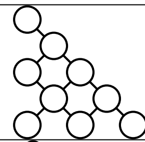
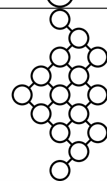
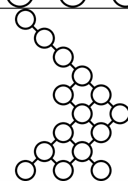
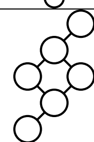
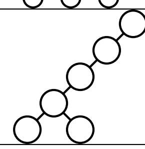
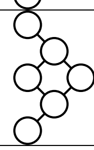
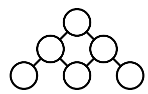
Label	Λ		Φ^+
B	$\Lambda_{\mathsf{Gr}(3,6)}$ 		$\Phi^+(B_{3,6})$
H	$\Lambda_{\mathsf{OG}(6,12)}$ 		$\Phi^+(H_3)$
I	$\Lambda_{\mathcal{Q}^6}$ 		$\Phi^+\left(I_2(6)\right)$
A	$\Lambda_{\mathsf{LG}(3,6)}$ 		$\Phi^+(A_3)$

Figure 9.2: Examples of the eight posets of Figure 9.1.

- $\Lambda_{\mathsf{OG}(6,12)}$ is the minuscule poset of type $(D_6, 1)$;
- $\Lambda_{\mathcal{Q}^{2m}}$ is the minuscule poset of type $(D_{m+1}, m+1)$; and
- $\Lambda_{\mathsf{LG}(n,2n)}$ is the cominuscule poset of type $(C_n, 1)$.

These objects are reviewed in more detail in Section 9.4.

Coincidental root posets

The **coincidental types** A_n, B_n, H_3, and $I_2(m)$ are those real reflection groups whose degrees $d_1 \leq d_2 \leq \cdots \leq d_n$ form an arithmetic sequence. Many natural enumerations are "more uniform" in these types. In crystallographic type, the **positive root poset** is the partial order on the positive roots Φ^+ given by the transitive closure of the relations $\alpha \prec \beta$ if and only if $\beta - \alpha$ is a simple root:

- $\Phi^+(A_n)$ is the positive root poset of type A_n; and
- for $n \geq 2k$, $\Phi^+(B_{k,n})$ is the restriction of $\Phi^+(B_{n-k})$ to the inversions of $b_{k,n} := (s_1 s_2 \cdots s_{n-k})^k \in W(B_{n-k})$, so that $\Phi^+(B_{n,2n}) = \Phi^+(B_n)$ is the positive root poset of type B_n.

For the noncrystallographic H_3 and $I_2(m)$ ($m \neq 2, 3, 4, 6$), we use the surrogate root posets constructed by D. Armstrong [Ar09]:

- $\Phi^+(H_3)$ is drawn in Figure 9.2; and

- $\Phi^+(I_2(m))$ is a chain of length $m-2$ with two minimal elements appended.

We further discuss the coincidental types in Section 9.5.

9.1.3 Main Results

For each coincidental type, Theorem 9.1 and Theorem 9.2 bijectively establish a pair of non-trivial doppelgängers. Theorem 9.1 addresses the top three rows of Figure 9.1, corresponding to the coincidental types B_n, H_3, and $I_2(m)$.

Theorem 9.1. *The pairs $\Lambda_{\mathsf{Gr}(k,n)}$ and $\Phi^+(B_{n,k})$, $\Lambda_{\mathsf{OG}(7,14)}$ and $\Phi^+(H_3)$, and $\Lambda_{\mathcal{Q}^m}$ and $\Phi^+(I_2(m))$ are each doppelgängers.*

Indeed, there are explicit, type-uniform (K-theoretic jeu de taquin) bijections

$$\text{(BP)} \qquad \mathsf{PP}^{[p]}\left(\Lambda_{\mathsf{Gr}(k,n)}\right) \simeq \mathsf{PP}^{[p]}\left(\Phi^+(B_{n,k})\right),$$

$$\text{(HP)} \qquad \mathsf{PP}^{[p]}\left(\Lambda_{\mathsf{OG}(7,14)}\right) \simeq \mathsf{PP}^{[p]}\left(\Phi^+(H_3)\right), \text{ and}$$

$$\text{(IP)} \qquad \mathsf{PP}^{[p]}\left(\Lambda_{\mathcal{Q}^m}\right) \simeq \mathsf{PP}^{[p]}\left(\Phi^+(I_2(m))\right),$$

which restrict to explicit, type-uniform bijections

$$\text{(BS)} \qquad \mathsf{SYT}\left(\Lambda_{\mathsf{Gr}(k,n)}\right) \simeq \mathsf{SYT}\left(\Phi^+(B_{n,k})\right), \qquad \textit{(M. Haiman [Ha92])}$$

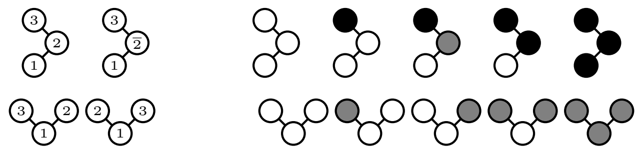
$$\text{(HS)} \qquad \mathsf{SYT}\left(\Lambda_{\mathsf{OG}(7,14)}\right) \simeq \mathsf{SYT}\left(\Phi^+(H_3)\right), \text{ and}$$

$$\text{(IS)} \qquad \mathsf{SYT}\left(\Lambda_{\mathcal{Q}^m}\right) \simeq \Phi^+\left(I_2(m)\right),$$

Although the bijections are type-uniform, our proofs are only partially so.

Theorem 9.2 establishes a relationship between $\Lambda_{\mathsf{LG}(n,2n)}$ and $\Phi^+(A_n)$ (from the last row of Figure 9.1), although these two posets are not quite doppelgängers. A minuscule poset may be identified with an order filter in the corresponding root poset, and therefore is naturally labeled by certain positive roots (see Equation (9.5)). The **diagonal** of the minuscule poset $\Lambda_{\mathsf{LG}(n,2n)}$ is the set of its elements labeled by long roots of $\Phi^+(C_n)$; this is the leftmost column of the posets illustrated in the top row of Example 9.2. Let $\overline{\mathsf{SYT}}(\Lambda_{\mathsf{LG}(n,2n)})$ denote the product $[2^{n(n-1)/2}] \times \mathsf{SYT}(\Lambda_{\mathsf{LG}(n,2n)})$ (with elements represented as shifted standard Young tableaux with any set of off-diagonal entries barred), and let $\overline{\overline{\mathsf{PP}}}^{[2p]}(\Lambda_{\mathsf{LG}(n,2n)})$ be the subset of $\mathsf{PP}^{[2p]}(\Lambda_{\mathsf{LG}(n,2n)})$ with only even heights on the diagonal. These definitions are illustrated in Example 9.2 for $n=2$ and $p=1$.

Example 9.2. For $n = 2$ and $p = 1$, $\overline{\mathsf{SYT}}(\Lambda_{\mathsf{LG}(2,4)})$ and $\overline{\mathsf{PP}}^{[2]}(\Lambda_{\mathsf{LG}(2,4)})$ (first row), and (the duals of) $\mathsf{SYT}(\Phi^+(A_2))$ and $\mathsf{PP}^{[1]}(\Phi^+(A_2))$ (second row) are illustrated below. The color white stands for height zero, gray for height one, and black for height two. The modified fillings for $\Lambda_{\mathsf{LG}(n,2n)}$ have no barred element nor the color gray in their leftmost columns.



$\diamondsuit$

The next theorem summarizes work in [Sh99] and [Pu14].

Theorem 9.2. *There is an explicit (symplectic jeu de taquin) bijection between*

(AP) $$\overline{\mathsf{PP}}^{[2p]}\left(\Lambda_{\mathsf{LG}(n,2n)}\right) \simeq \mathsf{PP}^{[p]}\left(\Phi^+(A_n)\right),$$ *(J. Sheats [Sh99])*

and there is also an explicit (jeu de taquin) bijection between

(AS) $$\overline{\mathsf{SYT}}\left(\Lambda_{\mathsf{LG}(n,2n)}\right) \simeq \mathsf{SYT}\left(\Phi^+(A_n)\right).$$ *(K. Purbhoo [Pu14])*

Our contribution is to conjecture that a simple generalization of K. Purbhoo's bijection also establishes (AP).

Conjecture 9.1. *There is an explicit (K-theoretic jeu de taquin) bijection for* (AP) *that restricts to K. Purbhoo's bijection for (AS).* $\square$

The precise version of this conjecture appears in Section 9.10.3.

9.1.4 Previous Work

We outline previous work on Theorems 9.1 and 9.2.

(BS) and (BP)

(BS) In his study of dual equivalence [Ha92], M. Haiman gave an elegant jeu de taquin bijection called *rectification* for (BS). We will review rectification in a more general context in Section 9.8.2.

(BP) In [Pr83], R. Proctor used a branching rule due to R. King ([Ki75, Li50]) from the Lie algebra inclusion $\mathfrak{sp}_{2n}(\mathbb{C}) \hookrightarrow \mathfrak{sl}_{2n}(\mathbb{C})$ to prove the identity (BP) *non-bijectively.* Indeed, he remarks that "the question of a combinatorial correspondence for [the identity (BP)] seems to be a complete mystery."

For $p = 1$, J. Stembridge produced a jeu de taquin bijection [Ste86], while V. Reiner [Rei97] gave an argument using type B noncrossing partitions. For $p = 2$, S. Elizalde gave an bijection in the language of pairs of lattice paths [El15]. The restriction of our bijection to these special cases is not immediately equivalent to any of these. No bijection was previously known for $p > 2$.

Our bijection simultaneously generalizes M. Haiman's bijection and provides the sought-after bijective proof of R. Proctor's result for arbitrary p.

(HS),(HP),(IS), and (IP)

These four identities (noted in [Wi13, Theorems 3.1.24 and 3.1.27]) are easy to establish, as one can explicitly compute the relevant order polynomials. We provide the first natural bijections.

(AS) and (AP)

(AS) This identity can be proven bijectively using properties of Sagan-Worley insertion [Wo84, Sa87]. K. Purbhoo [Pu14] gave an embedding of $\mathsf{LG}(n, 2n) \hookrightarrow \mathsf{Gr}(n, 2n)$, which he used to give an simple, explicit jeu de taquin bijection by *folding.* We will review this bijection in Section 9.10.3.

(AP) R. Proctor [Pr90] related Young tableaux indexing bases of certain highest weight representations of $\mathfrak{sp}_{2n}(\mathbb{C})$ with Gelfand patterns indexing the same representations to prove the identity (AP) *non-bijectively.*

J. Sheats [Sh99] defined *symplectic jeu de taquin* to give a combinatorial proof. His bijection is not immediately equivalent to that of our Conjecture 9.1. Further combinatorics based on the Garsia-Milne involution principle appears in [FuKr96, FuKr97].

Our Conjecture 9.1 would simultaneously generalize K. Purbhoo's bijection and give a new and simpler bijective proof of R. Proctor's result.

9.1.5 Bijections

Let (X, Y, Z) be a triple from a row of Figure 9.3.

Label	X	Y	Z
B	$\mathsf{Gr}(k,n)$	$B_{k,n}$	$\mathsf{OG}(n,2n)$
H	$\mathsf{OG}(6,12)$	H_3	$\mathsf{G}_\omega(\mathbb{O}^3,\mathbb{O}^6)$
I	$\mathcal{Q}^m$	$I_2(m)$	$\mathbb{Q}^{2m-2}$
A	$\mathsf{LG}(n,2n)$	A_n	$\mathsf{Gr}(n,2n)$

Figure 9.3: As illustrated in Figure 9.4, both the minuscule poset Λ_X and the coincidental root poset $\Phi^+(Y)$ embed in the ambient minuscule poset Λ_Z.

The posets $\Phi^+(Y)$ have an odd feature—besides having the minuscule doppelgänger Λ_X, the dual poset of each is an order ideal in a *second*, ambient minuscule poset Λ_Z.[2] The minuscule doppelgänger poset Λ_X also occurs as a subposet of Λ_Z. These observations are the key to our bijections.

We specify the shapes inside Λ_Z corresponding to Λ_X and $\Phi^+(Y)$ as

$$\mathsf{v}/\mathsf{u} := \Theta(\Lambda_X) \subseteq \Lambda_Z \qquad \text{and} \qquad \mathsf{w} := \chi(\Phi^+(Y)) \subseteq \Lambda_Z.$$

These embeddings are illustrated in Figure 9.4, and fully discussed in Section 9.6.

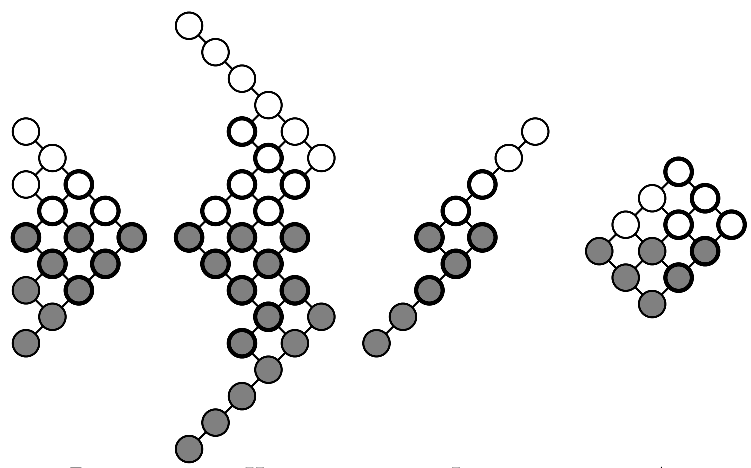


Figure 9.4: The minuscule posets Λ_Z in which the doppelgänger pairs of Figure 9.2 are embedded. The nodes with thick borders correspond to $\Theta(\Lambda_X) = \mathsf{v}/\mathsf{u}$, while the gray nodes represent $\chi(\Phi^+(Y)) = \mathsf{w}$.

The fundamental object that we borrow from the combinatorics of K-theoretic Schubert calculus is a natural generalization of standard Young tableaux. The **increasing tableaux of shape v/u and height k**—written $\mathsf{IT}^{[k]}(\mathsf{v}/\mathsf{u})$—are strictly order-preserving maps from $\mathsf{v}/\mathsf{u} \to [k]$. Note that here as in Chapter 8, unlike in Chapter 7, we do not require that all entries between 1 and k appear. For a ranked poset $\mathcal{P}$ whose maximal chains are all of the same length $\mathsf{ht}(\mathcal{P})$ (and, in particular, for all the posets of Figure 9.1), there

[2]We thank R. Proctor for pointing this out for H_3, long before we had any idea how to make sense of it.

is a simple bijection

$$\mathsf{IT}^{[k]}(\mathcal{P}) \simeq \mathsf{PP}^{[p]}(\mathcal{P}),$$

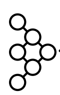
where $k = p + \mathsf{ht}(\mathcal{P})$ (see Proposition 9.6). We may therefore approach Theorem 9.1 by considering increasing tableaux instead of $\mathcal{P}$-partitions.

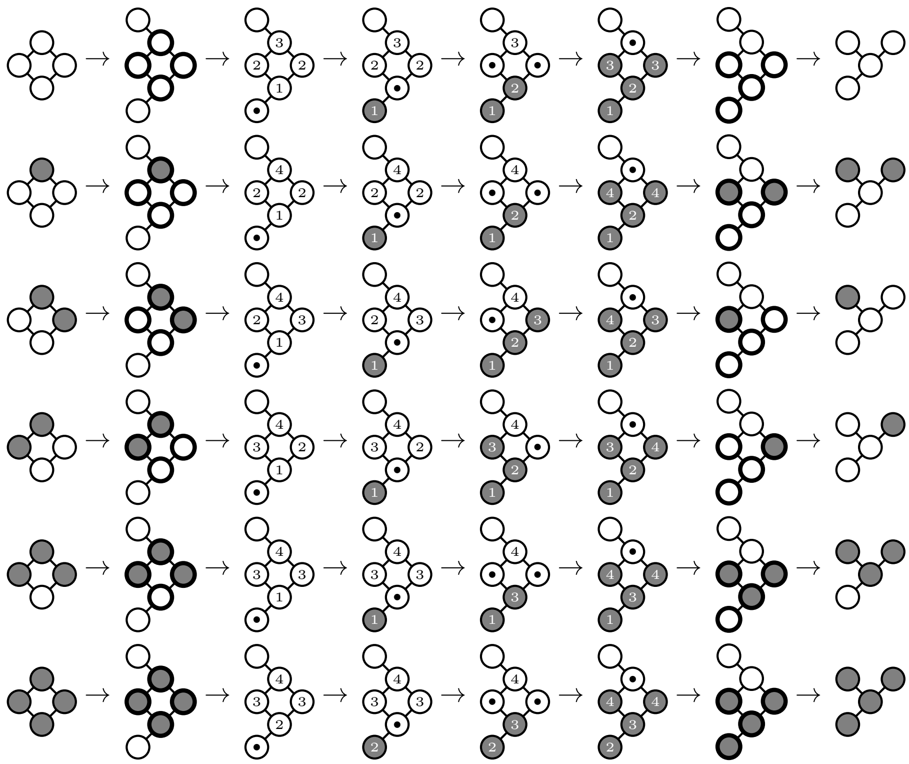
The significant advantage that increasing tableaux enjoy over $\mathcal{P}$-partitions is a well-developed theory of (K-theoretic) jeu de taquin initiated by H. Thomas and A. Yong in [ThYo09b] and further developed by A. Buch, E. Clifford, H. Thomas, M. Samuel, and A. Yong in [ClThYo14, BuSa13]. When we restrict to the minuscule posets Λ_Z, there are well-behaved jeu de taquin-like operations called *K-rectifications* that produce a tableau $\mathrm{Rect}(\mathsf{T}) \in \mathsf{IT}^{[k]}(\mathsf{w})$ from a given increasing tableau $\mathsf{T} \in \mathsf{IT}^{[k]}(\mathsf{v}/\mathsf{u})$ (see Theorem 9.11). These rectifications restrict to the set of standard Young tableaux.

Under the identification of increasing tableaux and $\mathcal{P}$-partitions, the bijections of Theorem 9.1 may be uniformly and simultaneously described as

$$\begin{aligned} \mathsf{IT}^{[k]}(\Lambda_X) &\to \mathsf{IT}^{[k]}(\Phi^+(Y)) \\ \mathsf{T} &\mapsto \chi^{-1}\left(\mathrm{Rect}\left(\Theta(\mathsf{T})\right)\right). \end{aligned}$$

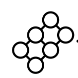
We illustrate these bijections in Example 9.3.

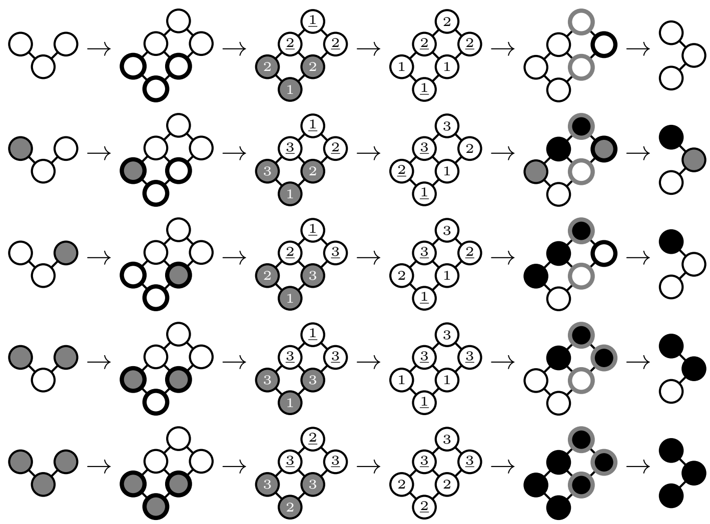
Example 9.3. Let $\mathcal{P}$ and $\mathcal{Q}$ be as in Example 9.1. These are an example of row (B) of Figure 9.3 (for $k = 2$ and $n = 4$), and so both embed in the poset . Rectification is the following jeu de taquin computation (see Section 9.8.2), which gives a bijection from $\mathsf{PP}^{[1]}(\mathcal{P})$ to $\mathsf{PP}^{[1]}(\mathcal{Q})$:



Restricting to the standard fillings in the middle two rows illustrated above recovers M. Haiman's bijection [Ha92] of standard Young tableaux of $\mathcal{P}$ and $\mathcal{Q}$. $\diamondsuit$

Conjecture 9.1 similarly extends K. Purbhoo's folding map.

Example 9.4. For $n = 2$ and $p = 1$, let $\Lambda_{\mathsf{LG}(2,4)}$ and the dual of $\Phi^+(A_2)$ be as in Example 9.2. These are an example of row (A) of Figure 9.3, and so both embed in the poset . Folding is the following alphabet-reordering jeu de taquin computation (see Section 9.8.2), bijecting $\mathsf{PP}^{[1]}(\Phi^+(A_2))$ to $\overline{\mathsf{PP}}^{[2]}(\Lambda_{\mathsf{LG}(2,4)})$:



Restricting to the standard fillings (in their respective alphabets) of the second and third rows above recovers K. Purbhoo's bijection of standard tableaux [Pu14]. ♢

9.2 Philosophy of the Proof

The philosophy behind our proof may be summarized as follows: Given a ring with a basis indexed by combinatorial objects, one may deduce bijections of the combinatorial objects from multiplicity-free products in the ring. In this section, we give a short example of this philosophy by sketching a parallel argument due to R. Stanley.

Theorem 9.3 (R. Stanley [Sta86, §3])**.** *The number of self-complementary plane partitions in a $(2r) \times (2s) \times (2t)$ box is equal to the number of pairs of plane partitions, each fitting inside an $s \times r \times t$ box.*

9.2.1 A Ring

Recall from Section 1.3 that the **ring of symmetric polynomials in n variables Λ_n** has a basis of **Schur functions** $\{s_\lambda : \lambda \text{ a partition with at most } n \text{ parts}\}$, where

$$s_\lambda := \sum_{\mathsf{T} \in \mathrm{SSYT}_n(\lambda)} \prod_{i=1}^{n} x_i^{\text{number of times } i \text{ appears in } \mathsf{T}}.$$

9.2.2 A Multiplicity-Free Product

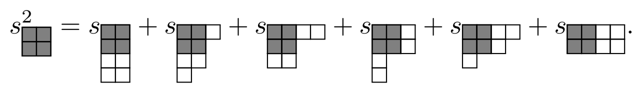
R. Stanley's argument hinges on the following multiplicity-free identity in Λ_{t+r}, expressing the square of a Schur function indexed by the rectangular partition (s^r) in the Schur basis:

$$s_{(s^r)}^2 = \sum_{\gamma} s_\gamma, \tag{9.1}$$

where γ ranges over the explicit set of partitions

$$\Big\{(s+\delta_1, \ldots, s+\delta_r, s-\delta_r, \ldots, s-\delta_1) : \delta = (\delta_1, \ldots, \delta_r) \subseteq (s^r)\Big\}.$$

Example 9.5. For $s = r = 2$, we have the $\binom{s+r}{s} = \binom{4}{2} = 6$-term expansion



$\diamondsuit$

Theorem 9.3 follows from Equation (9.1) as follows. The terms in the product on the left-hand side of Equation (9.1) are indexed by pairs of rectangular semistandard tableaux with entries in $[t+r]$ (top left of Figure 9.5). By subtracting i from the ith row, we produce a pair of plane partitions, each fitting inside an $s \times r \times t$ box, from this pair of semistandard tableaux (bottom left of Figure 9.5).

For the right-hand side, we again have a semistandard tableaux (top right of Figure 9.5). As before, we subtract i from the ith row to produce a plane partition. Now observe that the partitions λ occuring in the sum on the right-hand side are exactly of the form required so that λ and its rotation by 180° may be placed together to form a rectangular partition of size $((2s)^{2r})$. The interpretation in Theorem 9.3 is completed by noting that the filling of this rotation is specified by the self-complementarity condition (bottom right of Figure 9.5).

9.2.3 A Bijection

But, as sketched at the end of [Sta86], Theorem 9.3 can be realized with a simple bijection, guided by the multiplicity-free product above. Semistandard tableaux (unlike plane partitions) come with a theory of jeu de taquin as discussed in Section 1.3. By a standard combinatorial realization of the Littlewood-Richardson rule (see, for example, [Fu97]), placing our initial pair of semistandard tableaux "kitty-corner" from each other and applying jeu de taquin until arriving at a north-west-justified ("straight") shape gives a bijection from the pairs of tableaux representing the left-hand side of Equation (9.1) to the semistandard tableaux representing the terms of the right-hand side. That is, we use the combinatorics of the ring to extract a bijection from a multiplicity-free product.

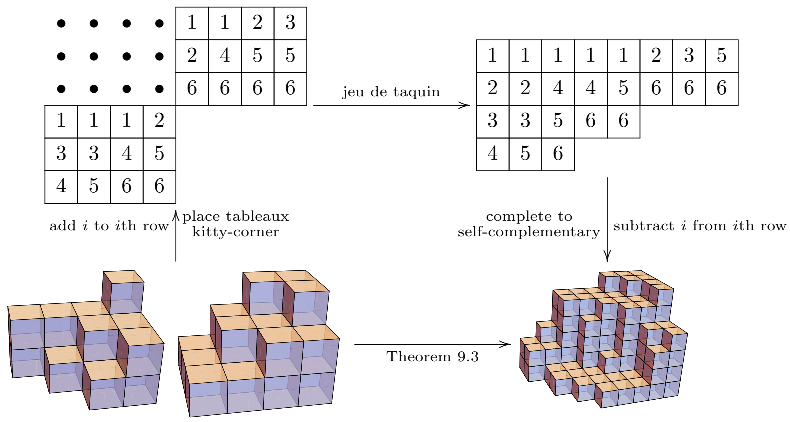


Figure 9.5: An illustration of a bijective proof of Theorem 9.3.

9.2.4 Outline of the Chapter

In summary, a bijection arises from a multiplicity-free identity in a ring with a basis indexed by combinatorial objects. In this chapter, we apply this philosophy using the objects and tools of minuscule K-theoretic Schubert calculus. To obtain the bijections of Theorem 9.1, we therefore need:

- combinatorial objects (Sections 9.4 to 9.6);
- rings with bases indexed by those objects (Section 9.7);
- combinatorial rules to compute structure coefficients in those rings (Section 9.8); along with
- interesting multiplicity-free formulas in those rings (Theorems 9.8 and 9.10).

The remainder of the chapter is structured as follows. In Section 9.3, we review required background and fix notation for root systems, Coxeter groups, and flag varieties. In Section 9.4, we discuss minuscule (co)weights and their associated posets. We then recall the coincidental types and their root posets in Section 9.5. Section 9.6 is devoted to certain embeddings of the minuscule posets and coincidental root posets of Figure 9.3 inside ambient minuscule posets. In Section 9.7, we recall the basic notions of (cohomological and K-theoretic) Schubert calculus, building to the powerful combinatorial toolkit of Section 9.8. In Section 9.9, we state and prove our main theorem (Theorem 9.16), which we specialize in Section 9.10

to conclude Theorem 9.1. Section 9.10 also contains the precise statement of Conjecture 9.1. Finally, in Section 9.11, we outline some related open problems and place our results in a larger framework.

9.3 Root Data

In this section, we review background and fix notation. We refer the reader to [Hi82, Hu92] for a more comprehensive treatment.

9.3.1 Root Systems

Let V be a real Euclidean space of rank n with a nondegenerate symmetric inner product $\langle \cdot, \cdot \rangle$. Fix an irreducible root system $\Phi \subset V$ with positive roots Φ^+ and choose a set of simple roots $\Delta := \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. When we wish to differentiate types, we will write $\Phi = \Phi(X_n)$ for the root system of type X_n (and similarly for other objects). For a root α, let $\alpha^\vee := 2\frac{\alpha}{\langle \alpha, \alpha \rangle}$ be the corresponding **coroot** and let $\Phi^\vee = \{\alpha^\vee : \alpha \in \Phi\}$ be the **dual root system**. The root system Φ is **crystallographic** if $\langle \alpha, \beta^\vee \rangle \in \mathbb{Z}$ for all $\alpha, \beta \in \Phi$; in this case we define the **positive root poset** to be the partial order on Φ^+ given by

$$\alpha \prec \beta \text{ if and only if } \beta - \alpha \text{ is a nonnegative sum of simple roots.} \tag{9.2}$$

The **height** of a positive root $\alpha = \sum_{i=1}^n a_i \alpha_i$ is the integer $\mathsf{ht}(\alpha) = \sum_{i=1}^n a_i$. We will abuse notation and write Φ^+ for the positive root poset, and we note that Φ^+ has is a unique maximal element $\widetilde{\alpha}$ called the **highest root**.

For Φ crystallographic, we let $Q := \mathbb{Z}\Phi$ be the **root lattice**, $Q^\vee := \mathbb{Z}\Phi^\vee$ the **coroot lattice**, and we set $\Lambda := \{\omega : \langle \omega, \alpha^\vee \rangle \in \mathbb{Z} \text{ for all } \alpha \in \Delta\}$ to be the **weight lattice** (whose elements are **weights**) and $\Lambda^\vee := \{\omega : \langle \omega, \alpha \rangle \in \mathbb{Z} \text{ for all } \alpha \in \Delta\}$ to be the **coweight lattice**. Then Λ contains the root lattice as a subgroup; the finite index $f = |\Lambda/Q|$ is called the **index of connection**. The **dominance order** is the order on Λ given by

$$\lambda \prec \omega \text{ if and only if } \omega - \lambda \text{ is a nonnegative sum of simple roots.}$$

We define the reflection $s_\alpha(v) := v - \langle v, \alpha^\vee \rangle \alpha$ for $\alpha \in \Phi^+$. The **Coxeter group** is the group W generated by these reflections; W has a smaller generating set called the **simple reflections** $S = \{s_i := s_{\alpha_i} : \alpha_i \in \Delta\}$, and the **Coxeter system** (W,S) has the presentation

$$\langle s_1, s_2, \cdots, s_n : s_i^2 = (s_i s_j)^{m_{ij}} = e \rangle,$$

where $m_{ij} = m_{ji}$ and e is the identity of W. The relations $(s_i s_j)^2 = e$ are called **commutation relations**, while higher order relations $(s_i s_j)^{m_{ij}}$ for $m_{ij} > 2$ are called **braid relations**. Associated to this presentation is the **Coxeter-Dynkin diagram**, a graph with vertices s_i and edges from s_i to s_j labeled by m_{ij} (we omit edges for commutations and omit labels for which $m_{ij} = 3$). In crystallographic type, we will use the standard convention of multiple bonds with arrows indicating long and short roots (see Figure 9.6 for some examples).

With some low-dimensional redundancy, finite irreducible Coxeter groups are classified as the crystallographic types $A_n, B_n, D_n, E_6, E_7, E_8, F_4$, and G_2, and the noncrystallographic types H_3, H_4, and $I_2(m)$. We shall refer to these symbols as **Coxeter-Cartan types**. Each crystallographic Coxeter group has an affine extension $\widetilde{W} = W \ltimes Q^\vee$ obtained by adding a new affine simple reflection parallel to $\widetilde{\alpha}$.

A **standard parabolic subgroup** $W_J \subset W$ is a group generated by a subset of the simple reflections $J \subset S$; a **maximal parabolic subgroup** is one for which $J = S \setminus \{s_i\}$—we shall denote such subgroups by $W_{\langle i \rangle}$. The set $W^J := W/W_J$ is called a **parabolic quotient**; we shall identify W^J with its minimal coset representatives and write $W^{\langle i \rangle} = W/W_{\langle i \rangle}$. Any $w \in W$ can be written as $w = w_J w^J$ with $w_J \in W_J$ and $w^J \in W^J$.

For any $w \in W$, we let

$$\mathsf{Red}(w) := \{(s_{i_1}, s_{i_2}, \ldots, s_{i_\ell}) : w = s_{i_1} s_{i_2} \cdots s_{i_\ell} \text{ and } \ell \text{ is minimal}\}$$

be its set of **reduced words**; by Matsumoto's theorem, $\mathsf{Red}(w)$ is connected under commutations and braid relations.

For $w \in W$ we let

$$\mathsf{w} := \big(-w(\Phi^+)\big) \cap \Phi^+ = \{\alpha_{i_1}, s_{i_1}(\alpha_{i_2}), \ldots, s_{i_1} s_{i_2} \cdots s_{i_{\ell-1}}(\alpha_{i_\ell})\}$$

be its **inversion set**, where $s_{i_1} s_{i_2} \cdots s_{i_\ell}$ is any reduced word for w. We write $\mathsf{len}(w) := |\mathsf{w}| = \ell$ for the **length** of w. The **Demazure product** is defined by $s_i \bullet w := \begin{cases} s_i w & \text{if } \mathsf{len}(s_i w) > \mathsf{len}(w) \\ w & \text{otherwise} \end{cases}$ and then extended to arbitrary words (the Demazure product corresponds to the product in the 0-Hecke algebra, and so gives a monoid structure on W.)

The **weak order** is the order on W defined by $w \leq v$ if and only if $\mathsf{w} \subseteq \mathsf{v}$. The group W has a unique longest element $w_\circ$ that is maximal in the weak order, and we write $w_\circ(J)$ for the longest element of W_J. Each parabolic quotient W^J also has a longest element $w_\circ^J$, and W^J consists of the elements in the interval $[e, w_\circ^J]$ (and so inherits the partial order from W). The map $\check{w} := w_\circ w w_\circ(J)$ gives an antiautomorphism of W^J.

9.3.2 Flag Varieties

Fix G a semisimple complex Lie group with Borel subgroup B, opposite Borel B_- and maximal torus $\mathsf{T} := \mathsf{B} \cap \mathsf{B}_-$.

For example, we denote the classical groups (over $\mathbb{C}$) by:

- $\mathsf{SL}(n) := \{A \text{ an } n \times n \text{ matrix} : \det(A) = 1\}$;
- $\mathsf{Sp}(2n) := \{A \text{ a } (2n) \times (2n) \text{ matrix} : A^{\mathrm{tr}} J A = J\}$, where $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$; and
- $\mathsf{SO}(n) := \{A \in \mathsf{SL}(n) : A^{-1} = A^{\mathrm{tr}}\}$.

One recovers the data of Section 9.3.1 in the following way. The Weyl group $W := \mathsf{N}(\mathsf{T})/\mathsf{T}$, where $\mathsf{N}(\mathsf{T})$ is the normalizer of T in G. The complex Lie algebra $\mathfrak{g}$ of G decomposes under the adjoint action as

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha,$$

where $\mathfrak{t}$ is the Lie algebra of the maximal torus and each $\mathfrak{g}_\alpha$ is one-dimensional. From this decomposition, we recover the root system Φ, the positive roots Φ^+, and the simple roots Δ.

Then G decomposes as the disjoint union (the **Bruhat decomposition**)

$$\mathsf{G} = \bigsqcup_{w \in W} \mathsf{B}_- w \mathsf{B}.$$

More generally, let P be a parabolic subgroup of G; we write $W^{\mathsf{P}} = W/W_{\mathsf{P}}$ for the corresponding parabolic quotient and subgroup of W. Then

$$\mathsf{G} = \bigsqcup_{w \in W^{\mathsf{P}}} \mathsf{B}_- w \mathsf{P}$$

and the **generalized flag variety** G/P has the decomposition

$$\mathsf{G}/\mathsf{P} = \bigsqcup_{w \in W^{\mathsf{P}}} \mathsf{B}_{-} w \mathsf{P}/\mathsf{P}. \tag{9.3}$$

9.4 Minuscule Posets

Definition 9.2. A weight $\omega \in \Lambda$ is called **minuscule** if $\omega \neq 0$ and $\langle \omega, \alpha^\vee \rangle \in \{-1, 0, 1\}$ for $\alpha \in \Phi$.

A minuscule weight for the dual root system $\Phi^\vee$ is called a **minuscule coweight**; the corresponding weight of the original root system is called **cominuscule**. Minuscule (co)weights frequently occur as simple examples of general constructions [Gr13, Pr84a, Pr84b]. In Schubert calculus, a (co)minuscule weight ω corresponds to a flag variety G/P, whose structure coefficients are nicely computable in terms of the poset $\Lambda_{\mathsf{G}/\mathsf{P}}$ (defined below) (cf., e.g., [ThYo09a]). From the point of view of the Weyl group W, a minuscule weight is a point whose W-orbit is small relative to $|W|$—such points can be used to construct small permutation representations of W. In the representation theory of Lie algebras, the highest weight representation V_ω associated to a minuscule weight ω consists only of the weights in this W-orbit, and bases for $V_{k\omega}$ are indexed by $\mathsf{PP}^{(k)}(\Lambda_{\mathsf{G}/\mathsf{P}})$.

Theorem 9.4. *For ω a dominant coweight in crystallographic Coxeter-Cartan type, the following are equivalent:*

1. *ω is minuscule—that is, $\omega \neq 0$ and $\langle \omega, \alpha \rangle \in \{-1, 0, 1\}$ for all $\alpha \in \Phi$;*
2. *$\omega = \omega_i$ is a fundamental coweight, and $c_i = 1$ in the expansion $\widetilde{\alpha} = \sum_{j=1}^{n} c_j \alpha_j$ of the highest root in the simple root basis;*
3. *$\omega = \omega_i$ is a fundamental coweight, and there is an automorphism of the affine Dynkin diagram sending α_0 to α_i;*
4. *ω is a nonzero minimal representative of $\Lambda^\vee/Q^\vee$ in the dominance order; and*
5. *$\omega = \omega_i$ is a fundamental coweight, and the corresponding node of the Dynkin diagram is marked in gray in Figure 9.6.*

Proof. See, for example, [Ste94b, Appendix: A Minuscule Atlas], [Ste98, Proposition 1.12], and [Gr13] □

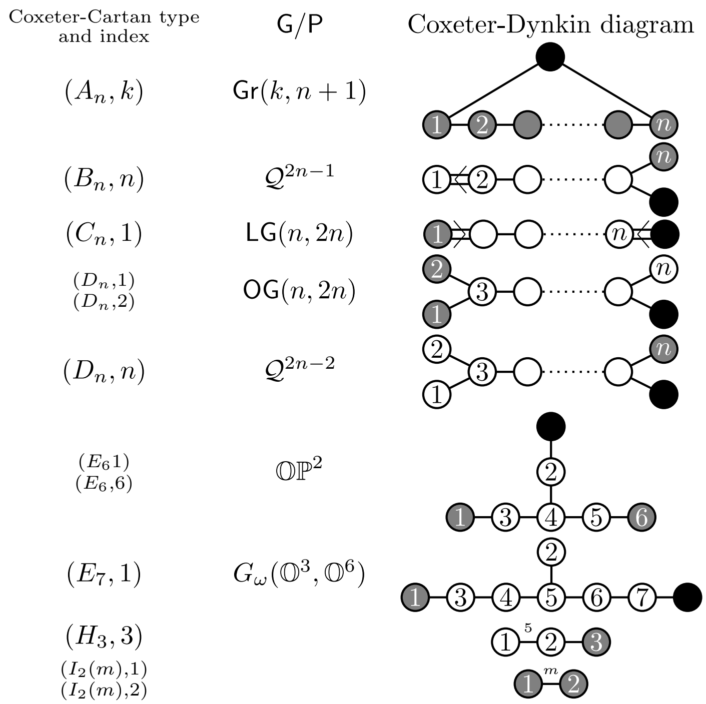


Figure 9.6: In crystallographic type, the roots α_i marked in gray have a corresponding cominuscule fundamental weight ω_i; the affine simple root is marked in black. For H_3 and $I_2(m)$, the roots marked in gray correspond to maximal parabolic quotients $W^{\langle i \rangle}$ whose longest element is fully commutative.

9.4.1 General Construction

The stabilizer

$$W_{\langle i \rangle} := \{w \in W : w(\omega_i) = \omega_i\}$$

is the maximal parabolic subgroup of W generated by $\Delta \backslash \{\alpha_i\}$; let P_i be the corresponding maximal parabolic subgroup of G. By the orbit-stabilizer theorem, the minimal coset representatives w of the parabolic quotient $W^{\langle i \rangle} := W/W_{\langle i \rangle}$ are in bijection with the weights in the orbit $\{w(\omega_i) : w \in W\}$. We will now explicitly describe the elements of these quotients in the case ω_i is minuscule.

Fix $w \in W$ and let $\mathbf{w} = (s_{k_1}, s_{k_2}, \ldots, s_{k_\ell})$ be a reduced word for w. Define a partial order $\prec_{\mathbf{w}}$ on $[\ell]$ by the transitive closure of the relations

$$i \prec_{\mathbf{w}} j \text{ if } i < j \text{ and } s_{k_i} s_{k_j} \neq s_{k_j} s_{k_i}. \tag{9.4}$$

This partial ordering defines an ordering on $[\ell]$ called a **heap** [Vi86, Ste96], and hence gives an ordering of the roots in the inversion set w of w.

We recall that a **fully commutative** element $w \in W$ is one whose set $\mathsf{Red}(w)$ of reduced words is connected using only commutations. For any two reduced words of a fully commutative w, it is then not difficult to see that the two induced partial orderings on w are isomorphic. We may therefore unambiguously refer to *the* heap w of $w \in W$, when w is fully commutative.

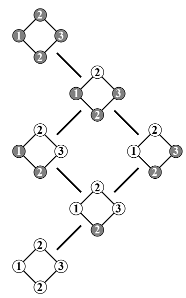


Figure 9.7: For $W = W(A_3)$, the minuscule weight ω_2 is fixed by the parabolic subgroup $W_{\langle 2 \rangle}$. The corresponding quotient $W^{\langle 2 \rangle}$ has a fully commutative longest element $w_\circ^{\langle 2 \rangle} = s_2 s_1 s_3 s_2$, whose heap is $\mathsf{w}_\circ^{\langle 2 \rangle} = [2] \times [2] \simeq \Lambda_{\mathsf{Gr}(2,\mathbb{C}^4)}$.

Theorem 9.5 ([Ste96, Proposition 2.2 and Lemma 3.1])**.** *For w fully commutative, there is a bijection*

$$\mathsf{SYT}(\mathsf{w}) \simeq \mathsf{Red}(w).$$

This induces a bijection

$$\mathsf{PP}^{[1]}(\mathsf{w}) \simeq [e, w].$$

In [Ste96], J. Stembridge classified all maximal parabolic quotients whose longest element $w_\circ^{\langle i \rangle}$ is fully commutative. This classification is summarized in Figure 9.6, and is intimately related to finding a natural subgroup of W isomorphic to Λ/Q. When W is a Weyl group, this classification essentially coincides with the classification of minuscule representations of the corresponding Lie algebra.[3]

By Theorem 9.5 and J. Stembridge's classification, when ω_i is minuscule the inversion sets of the elements in $W^{\langle i \rangle}$—which, *a priori* are just biclosed subsets of positive roots not lying in the root system corresponding to P_i—are *order ideals* in the heap for the longest element $w_\circ^{\langle i \rangle}$ of $W^{\langle i \rangle}$. This heap may be simply described as the order filter in Φ^+ generated by α_i:

$$\Lambda_{\mathsf{G}/\mathsf{P}_i} := \{\alpha \in \Phi^+ : \text{ if } \alpha = \sum_{j=1}^{n} a_j \alpha_j, \text{ then } a_i \neq 0\}. \tag{9.5}$$

We summarize this discussion with the theorem below.

[3]The Weyl group is not sensitive to the difference between long and short roots, and so confuses types B and C.

Theorem 9.6 (R. Proctor [Pr84a]). *When ω is a minuscule coweight, there is a bijection*

$$W^{\mathsf{P}} \simeq \mathsf{PP}^{[1]}(\Lambda_{\mathsf{G/P}})$$
$$u \mapsto \mathsf{u},$$

where for $W_{\mathsf{P}} := \{w \in W : w(\omega) = \omega\}$, W^{P} is the set of minimal coset representatives of W/W_{P}.

In particular, the weak order on W^{P} is a distributive lattice. When ω is a (co)minuscule weight, we shall also use the term **(co)minuscule** to describe the corresponding flag variety $\mathsf{G/P}$ and poset $\Lambda_{\mathsf{G/P}}$.

9.4.2 Explicit Constructions

Expanding on Section 9.1.2, we explicitly identify the posets from Figure 9.6 that we require by giving reduced words for $w_\circ^{\langle i \rangle}$ (the corresponding posets can then be built as heaps using Equation (9.4)). We have seen these posets before, for example in Section 2.4.

The Grassmannian $\mathsf{Gr}(k,n)$

In type A_{n-1}, any fundamental weight ω_k is minuscule. The **Grassmannian** is

$$\mathsf{Gr}(k,n) := \mathsf{SL}(n)/\mathsf{SL}(n)_k.$$

Let $W = W(A_{n-1})$. The image of Λ/Q in W is given by the cyclic group $\langle c \rangle$, where $c = s_1 s_2 \cdots s_{n-1}$, and the corresponding parabolic quotient $W^{\langle k \rangle}$ may be identified as the weak order interval $[e, c^k]$. Although c^k is not reduced for $k > 1$, one can check that

$$w_\circ^{\langle k \rangle} = \prod_{j=1}^{n-k} \prod_{i=k-j+1}^{n-j} s_i$$

is a reduced word. The poset $\Lambda_{\mathsf{Gr}(k,n)}$ is commonly described as a $[k] \times [n-k]$ rectangle, represented as a partition by $\underbrace{(n-k, n-k, \ldots, n-k)}_{k \text{ times}}$.

$\mathsf{LG}(n, 2n)$

In type C_n, ω_1 is a cominuscule weight. The **Lagrangian Grassmannian** is

$$\mathsf{LG}(n, 2n) := \mathsf{Sp}(2n)/\mathsf{Sp}(2n)_1.$$

Let $W = W(C_n)$. The longest element $w_\circ^{\langle 1\rangle}$ of $W^{\langle 1\rangle}$ is an involution—reflecting the fact that $|\Lambda/Q| = 2$—with reduced word

$$w_\circ^{\langle 1\rangle} = \prod_{i=1}^{n} \prod_{j=1}^{n-i+1} s_j,$$

so that—when drawn as a shifted Young diagram—$\Lambda_{\mathsf{LG}(n,2n)}$ is a shifted staircase of order n. We write this as the shifted partition $(n, n-1, \ldots, 1)_*$.

$\mathsf{OG}(n, 2n)$ and $\mathcal{Q}^{2n-2}$

In type D_n, ω_1 and ω_2 are minuscule weights with isomorphic minuscule posets $\Lambda_{\mathsf{OG}(n,2n)}$, while ω_n is also minuscule but with poset $\Lambda_{\mathcal{Q}^{2n-2}}$.

The **even Orthogonal Grassmannian** is

$$\mathsf{OG}(n, 2n) := \mathsf{SO}(2n)/\mathsf{SO}(2n)_1 = \mathsf{SO}(2n)/\mathsf{SO}(2n)_2,$$

and the **even dimensional quadric** is

$$\mathbb{Q}^{2n-2} := \mathsf{SO}(2n)/\mathsf{SO}(2n)_n.$$

Let $W = W(D_n)$ and write $s_{1,2}(j) = \begin{cases} s_1 & \text{if } j \text{ is odd} \\ s_2 & \text{if } j \text{ is even} \end{cases}$. One can check that the corresponding longest elements of $W^{\langle i\rangle}$ for $i \in \{1, 2, n\}$ have reduced words

$$\begin{aligned}
w_\circ^{\langle 1\rangle} &= \prod_{j=1}^{n} \left(s_{1,2}(j) \prod_{k=3}^{n-j+1} s_k \right) \\
w_\circ^{\langle 2\rangle} &= \prod_{j=1}^{n} \left(s_{1,2}(j+1) \prod_{k=3}^{n-j+1} s_k \right) \\
w_\circ^{\langle n\rangle} &= \left(\prod_{j=3}^{n} s_j \right)^{-1} (s_1 s_2) \left(\prod_{j=3}^{n} s_j \right).
\end{aligned}$$

The image of Λ/Q in W is then given by the elements $\{e, w_\circ^{\langle 1\rangle}, w_\circ^{\langle 2\rangle}, w_\circ^{\langle n\rangle}\}$, with multiplicative structure depending on the parity of n.

The poset $\Lambda_{\mathsf{OG}(n,2n)}$ is a shifted staircase of size $n-1$. The poset $\Lambda_{\mathcal{Q}^{2n-2}}$ can be compactly described as the iterated distributive lattice of order ideals $\mathcal{J}^{n-3}([2] \times [2])$, and we define $\Lambda_{\mathcal{Q}^{2n-3}}$ to be $\Lambda_{\mathcal{Q}^{2n-2}}$ without its top element.

$\mathsf{G}_\omega(\mathbb{O}^3, \mathbb{O}^6)$

In type E_7, only ω_1 is a minuscule weight. For $W = W(E_7)$, a reduced word for the longest element of $W^{\langle 1 \rangle}$ is

$$w_\circ^{\langle 1 \rangle} = s_1 s_3 s_4 s_5 s_6 s_2 s_5 s_4 s_3 s_1 s_7 s_6 s_5 s_4 s_3 s_2 s_5 s_4 s_6 s_5 s_7 s_6 s_2 s_4 s_3 s_1.$$

The poset $\Lambda_{\mathsf{G}_\omega(\mathbb{O}^3,\mathbb{O}^6)}$ is the second poset from the left in Figure 9.4.

9.5 Coincidental Root Posets

Definition 9.3. We call the Coxeter-Cartan types A_n, B_n, H_3, and $I_2(m)$ the **coincidental types**.

A. Miller observed that these are exactly those types for which the degrees $d_1 < d_2 < \cdots < d_n$ of the Coxeter group W form an arithmetic sequence [Mi15]. The coincidental types have many remarkable properties, and many enumerative questions are "more uniform" when restricted from all Coxeter-Cartan types to just the coincidental types. Such enumerative results include:

- the number of k-dimensional faces of the generalized cluster complex [FoRe05];
- the number of saturated chains of length k in the noncrossing partition lattice [Rea08];
- the number of reduced words for $w_\circ$ [Sta84, EdGr87, Ha92, Wi13];
- the number of multitriangulations [CeLaSt14]; and
- the Coxeter-biCatalan numbers [BaRe15].

9.5.1 Crystallographic

Since A_n and B_n are crystallographic, the root posets for those types are defined by Equation (9.2). Examples are given in Figure 9.2—when drawn as a Young diagram, $\Phi^+(A_n)$ is a **staircase** $(n, n-1, \ldots, 1)$ of order n, while $\Phi^+(B_n)$ is a **shifted double staircase** when drawn as a shifted Young diagram (see Figures 9.13 and 9.16 for examples).

More generally, using the conventions of Figure 9.6, let[4]

$$b_{k,n} := (s_1 s_2 \cdots s_{n-k})^k \in W(B_{n-k})$$

[4]The subposet specified by $\mathsf{b}_{k,n}$ in the root system of type C_{n-k} is isomorphic to $\Phi^+(B_{k,n})$, although it does not sit inside the root poset in quite the same way.

for $n \geq 2k$. The poset $\Phi^+(B_{k,n})$ is defined to be

$$\Phi^+(B_{k,n}) := \Phi^+(B_{n-k}) \cap \mathsf{b}_{k,n},$$

the restriction of $\Phi^+(B_{n-k})$ to the roots that are inversions of $b_{k,n}$. As a special case, since $b_{n,2n} = w_\circ \in W(B_n)$, we have $\Phi^+(B_n) = \Phi^+(B_{n,2n})$. When drawn as a shifted Young diagram, this poset may be described as the **shifted trapezoid**, with shifted partition shape $(n-1, n-3, \ldots, n-2k+1)_*$.

9.5.2 Non-Crystallographic

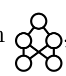
It remains to construct "root posets" in the noncrystallographic types H_3 and $I_2(m)$. For convenience, we work with reflections instead of roots.

It is a fact, uniformly proven by B. Kostant [Ko59], that the sizes of the ranks of Φ^+ and the degrees $d_1, d_2, \ldots, d_n$ of W form conjugate partitions under the identity [Hu92, Theorem 3.20]

$$\left|\{\alpha \in \Phi^+ : \mathsf{ht}(\alpha) = i\}\right| = |\{j : d_j > i\}|. \tag{9.6}$$

The obvious application of Equation (9.2) does not yield a root poset satisfying this condition in the non-crystallographic types. For example, if $\phi := \frac{1+\sqrt{5}}{2}$, then in the basis of simple roots,

$$\Phi^+(I_2(5)) = \{(1,0), (0,1), (\phi, 1), (1, \phi), (\phi, \phi)\},$$

which would be ordered by Equation (9.2) to have Hasse diagram , so that it has two elements of rank one, two elements of rank two, and one element of rank three. On the other hand, since the degrees of $I_2(5)$ are 2 and 5, Equation (9.6) predicts two elements of rank one, and one element for each rank greater than one.

On the basis of Equation (9.6) and a few other criteria from Coxeter-Catalan combinatorics, D. Armstrong constructed surrogate root posets in types H_3 and $I_2(m)$ with desirable behavior [Ar09]. For more details, see [CuSt15, Section 3] (which includes a construction of $\Phi^+(H_3)$ using a folding argument).

We will construct these root posets using the fully commutative theory reviewed in Section 9.4, and refer the reader to the noncrystallographic part of Figure 9.6 for the labeling conventions of the Coxeter-Dynkin diagram.

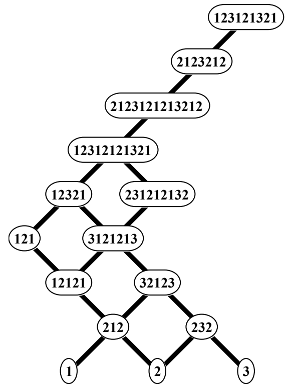


Figure 9.8: The nodes of the root poset of type H_3 labeled using the method of Section 9.5.2.

$I_2(m)$

In type $I_2(m)$, the root poset is a natural generalization of the root posets for the crystallographic dihedral types $A_1 \times A_1$, A_2, B_2, and G_2.

The Coxeter group $W = W(I_2(m))$ has two generators, $s = s_1$ and $t = s_2$. W has a fully-commutative maximal parabolic quotient W^J, where $J = \{t\}$. The longest element of W^J has one reduced word: $w_\circ^J = \underbrace{sts\cdots}_{m-1 \text{ letters}}$. The heap for $w_\circ^J$ is therefore a chain of length $m-1$, whose vertices are canonically labeled by the reflections coming from the corresponding letter of the word for $w_\circ^J$: $s, sts, ststs, \ldots$. We now apply Equation (9.6) to conclude that sts covers t.

H_3

We note that $W(I_2(5))$ is the maximal parabolic subgroup of $W = W(H_3)$ generated by $J = \{s_1, s_2\}$. By the previous section, we therefore obtain the root poset of the parabolic subgroup $W(I_2(5))$, which ought to be the restriction of the full root poset of H_3 to that parabolic subgroup. Now the parabolic quotient $W^J = W(H_3)/W(I_2(5))$ has a maximal element that is fully commutative (see the classification in Figure 9.6), which allows us to canonically label the heap of

$$w_\circ^J = s_3s_2s_1s_2s_1s_3s_2s_1s_2s_3$$

by the corresponding reflections. Letting $t \lessdot s_3ts_3$ for all reflections $t \in W_J$ such that $s_3ts_3 \neq t$, we obtain the desired poset, which is illustrated in Figure 9.8.

We write

$$w_\circ(Y) := \begin{cases} w_\circ \in W(Y) & \text{if } Y \in \{A_n, H_3, I_2(m)\} \\ b_{k,n} \in W(B_{n-k}) & \text{if } Y = B_{k,n} \end{cases}. \tag{9.7}$$

Note that the root posets of types D_n $(n \geq 4)$, E_6, E_7, E_8, and F_4 are each non-planar, in contrast to the coincidental root posets constructed above, and hence cannot embed in any minuscule poset.

9.6 Poset Embeddings

Let (X, Y, Z) be a triple in Figure 9.3. Following [ThYo09a] and [Pu14], we formalize Figure 9.4 by embedding the doppelgängers Λ_X and $\Phi^+(Y)$ into the ambient minuscule posets Λ_Z. That is, we explicitly characterize

$$\mathsf{v}/\mathsf{u} := \Theta(\Lambda_X) \subseteq \Lambda_Z \qquad \text{and} \qquad \mathsf{w} := \chi(\Phi^+(Y)) \subseteq \Lambda_Z.$$

9.6.1 X in Z: Embedding Minuscule Varieties

A minuscule flag variety is specified by a Cartan type and a minuscule weight. For X a minuscule flag variety, let $\mathrm{Cart}(X)$ be the corresponding Cartan type and let W^X be the corresponding parabolic quotient of the Weyl group $W(\mathrm{Cart}(X))$, as in Figure 9.6. For example, when $X = \mathsf{Gr}(k, n)$ we have $\mathrm{Cart}(X) = A_{n-1}$ and $W^X = A_{n-1}^{\langle k \rangle}$.

Rows (B),(H), and (I)

For each of the first three rows of Figure 9.3, define injections

$$\begin{aligned} \Theta_B : \Delta(A_{n-1}) &\hookrightarrow \Delta(D_n) \\ \alpha_i &\mapsto \alpha_{i+1}, \\ \Theta_H : \Delta(D_6) &\hookrightarrow \Delta(E_7) \\ {\scriptstyle \begin{matrix}\alpha_1\mapsto\alpha_6 \\ \alpha_6\mapsto\alpha_1\end{matrix}, \begin{matrix}\alpha_2\mapsto\alpha_2 \\ \alpha_4\mapsto\alpha_4\end{matrix}, \begin{matrix}\alpha_3\mapsto\alpha_5 \\ \alpha_5\mapsto\alpha_3\end{matrix},} & \\ \Theta_I : \Delta(D_m) &\hookrightarrow \Delta(D_{2m-2}) \\ \alpha_i &\mapsto \alpha_i. \end{aligned}$$

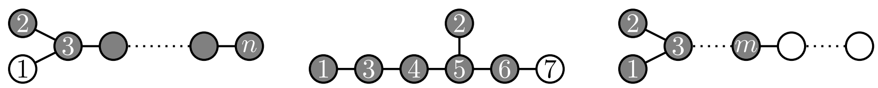
We drop the subscript on Θ when context is clear. These embeddings are illustrated in Figure 9.9 by drawing the Dynkin diagram of $\mathrm{Cart}(X)$ as a subdiagram of the Dynkin diagram of $\mathrm{Cart}(Z)$. They extend by linearity to injections of the full root system

$$\Theta : \Phi(\mathrm{Cart}(X)) \hookrightarrow \Phi(\mathrm{Cart}(Z))$$

and, under the correspondence between roots and reflections, to injections of the associated Weyl groups

$$\Theta : W(\mathrm{Cart}(X)) \hookrightarrow W(\mathrm{Cart}(Z)).$$

Following [ThYo09a], the next proposition states that these embeddings of Dynkin diagrams actually induce embeddings of the minuscule flag varieties. Using the cell decomposition of Equation (9.3) and results of [ThYo09a], it suffices to embed the parabolic Weyl group quotients in a sufficiently nice way.



$\Theta_B : \Delta(A_{n-1}) \hookrightarrow \Delta(D_n)$ $\Theta_H : \Delta(D_6) \hookrightarrow \Delta(E_7)$ $\Theta_I : \Delta(D_m) \hookrightarrow \Delta(D_{2m-2})$

Figure 9.9: Embedding $\Delta(\mathrm{Cart}(X))$ into $\Delta(\mathrm{Cart}(Z))$.

Proposition 9.2 (After [ThYo09a]). *For (X, Z) from the first three rows of Figure 9.3 (with m even in the third row), there is an embedding $\Theta : X \hookrightarrow Z$ such that*

$$\Theta(W^X) = \{ux : x \in W^X\} \subseteq W^Z,$$

for some $u \in W^Z$.

Proof. We first characterize $u \in W^Z$ in each of the three cases.

(B) u is the longest element of the parabolic quotient $W(D_k)^{\langle 1 \rangle}$, explicitly identified in Section 9.4.2.

(H) $u := s_7 s_6 s_5 s_4 s_3 s_1$; and

(I) $u := s_{2m} s_{2m-1} \cdots s_{m+2}$.

One checks case-by-case that if $x \in W^X$, then $u_X x \in W^Z$ by examining the corresponding heaps. This proves the second part of the proposition, which establishes the analogues of [ThYo09a, Corollary 6.7, Lemma 6.8] in these settings. The first part now follows from [ThYo09a, Proposition 6.1]. $\square$

We deduce that there is an embedding of the corresponding minuscule posets.

Corollary 9.1. *For (X, Z) from the first three rows of Figure 9.3, there are poset embeddings $\Theta : \Lambda_X \hookrightarrow \Lambda_Z$.*

Proof. For $X \neq \mathbb{Q}^{2m-1}$, let $v := uw_\circ^X \in W^Z$, where $w_\circ^X$ is the longest element of W^X. Since $w_\circ^X$ is fully commutative, by Theorem 9.6 and Proposition 9.2, Λ_X embeds in Λ_Z as the poset v/u. For $X = \mathbb{Q}^{2m-1}$, since $\Lambda_{\mathbb{Q}^{2m-1}}$ is defined to be $\Lambda_{\mathbb{Q}^{2m}}$ without its top element, we can use the previous embedding for $X = \mathbb{Q}^{2m}$. $\square$

Remark 9.1. We have not compiled here an exhaustive list of all such embeddings, but have limited ourselves to those that have connections to the coincidental types.

Row (A)

In [Pu14], K. Purbhoo describes the following embedding

$$\Theta_A : \mathsf{LG}(n, 2n) \hookrightarrow \mathsf{Gr}(n, 2n+1).$$

Index the coordinates of a vector $x \in \mathbb{C}^{2n+1}$ by $x = (x_{-n}, x_{-n+1}, \ldots, x_n)$ and let $\mathbb{V} := \{x : x_0 = 0\} \subset \mathbb{C}^{2n+1}$. We write $\mathsf{Gr}(n, \mathbb{V})$ for the Grassmannian of n-dimensional subspaces of $\mathbb{V}$. Define a symplectic form by

$$[x, y] := \sum_{\substack{i=-n \\ i \neq 0}}^{n} \frac{(n+i)!(n-i)!}{i} x_i y_{-i},$$

for $x, y \in \mathbb{V}$. For a subspace $V \subseteq \mathbb{V}$, define $V^\perp := \{x \in \mathbb{V} : [x, v] = 0 \text{ for all } v \in V\}$. Then Ω is

$$\Omega := \{V \in \mathsf{Gr}(n, \mathbb{V}) : V = V^\perp\} \subset \mathsf{Gr}(n, \mathbb{V}) \subset \mathsf{Gr}(n, 2n+1).$$

Proposition 9.3 (K. Purbhoo [Pu14])**.** *The space Ω is an embedding*

$$\Theta_A : \mathsf{LG}(n, 2n) \hookrightarrow \mathsf{Gr}(n, 2n+1).$$

K. Purbhoo interprets this as an embedding of $\Lambda_{\mathsf{LG}(n,2n)}$ in $\Lambda_{\mathsf{Gr}(n,2n+1)}$ in [Pu14, Lemma 3.11]. In the style of Proposition 9.2, this can be phrased as follows, though we do not entirely understand the relation to the geometry.

Proposition 9.4. *There is a poset embedding $\Theta_A : \Lambda_{\mathsf{LG}(n,2n)} \hookrightarrow \Lambda_{\mathsf{Gr}(n,2n+1)}$.*

Proof. Given a reduced word $\mathbf{x}$ for an element $x \in W^{\mathsf{LG}(n,2n)}$, we create a reduced word $\Theta_A(\mathbf{x})$ for an element $\Theta_A(x) \in W^{\mathsf{Gr}(n,2n+1)}$ by mapping a simple reflection of type C_n to the product of two simple reflections of type A_{2n} as follows:

$$s_i \mapsto s_{n+1-i}s_{n+i}.$$

This map of simple reflections induces the desired poset embedding by "doubling" the poset $\Lambda_{\mathsf{LG}(n,2n)}$ inside $\Lambda_{\mathsf{Gr}(n,2n+1)}$ (note that this differs from the construction in Section 9.8.3 by repeating the diagonal). $\square$

Note that given a reduced word for the longest element $w_\circ^{\langle 1\rangle}$ in $W^{\mathsf{LG}(n,2n)}$ (see Section 9.4.2), under the map to the element of $W^{\mathsf{Gr}(n,2n+1)}$, we have a choice of the order $s_{n+1-i}s_{n+i}$ or $s_{n+i}s_{n+1-i}$ for $i \neq 1$, while for $i = 1$ we must order the reflections $s_n s_{n+1}$. This would appear to account for the possibility of barring (or leaving unbarred) the off-diagonal entries in Section 9.10.3.

9.6.2 Y in Z: Embedding Root Posets

We now embed the root posets of the coincidental types $\Phi^+(Y)$ into the ambient minuscule posets Λ_Z. We find the existence of these embeddings mysterious—unexpectedly, the element $w \in W(\Lambda_Z)$ whose heap w coincides with $\Phi^+(Y)$ has the same number of reduced words as $w_\circ(Y) \in W(Y)$.

Proposition 9.5. *For (Y, Z) from any row of Figure 9.3, there is a poset embedding $\chi : \Phi^+(Y) \hookrightarrow \Lambda_Z$, so that $\chi(\Phi^+(Y)) = \mathsf{w}$ for some $w \in W(Z)$, and such that*

$$\mathsf{Red}(w_\circ(Y)) \simeq \mathsf{Red}(w),$$

where $w_\circ(Y)$ is the element defined in Equation (9.7) with inversion set $\Phi^+(Y)$.

Proof. We first characterize the elements $w \in W(Z)$ in each of the four cases.

(B) $w := \prod_{j=1}^{k}\left(s_j^*\prod_{i=3}^{n-2j+2} s_i\right)$, where $s_j^* = \begin{cases} s_1 & \text{if } j \text{ is odd} \\ s_2 & \text{if } j \text{ is even}\end{cases}$.

(H) $w := s_1s_3s_4s_5s_6s_7s_2s_5s_6s_4s_5s_2s_3s_4s_1$.

(I) $w := \left(\prod_{j=3}^{m} s_j\right)^{-1}(s_1s_2)$.

(A) $w := \prod_{j=1}^{n}\prod_{i=n-j+1}^{2n-2j+1} s_i$.

The statement that $\mathsf{Red}(w_\circ(Y)) \simeq \mathsf{Red}(w_Y)$ follows for (A) by [Sta84, EdGr87, HaYo13], for (B) by [Kra89, Ha92, BHRY14] (we use the poset isomorphism between $\Lambda_{\mathsf{OG}(n,2n)}$ and $\Lambda_{\mathsf{LG}(n-1,2n22)}$), and for (I) and (H) this is an easy check [Wi13]. $\square$

In types (A) and (B), the relation $\mathsf{Red}(w_\circ(Y)) \simeq \mathsf{Red}(w_Y)$ in Proposition 9.5 has two different combinatorial proofs, which are related in [HaYo13, BHRY14]. We refer the reader to [Las95] for additional historical context.

- One proof is via modified RSK insertion algorithms due to P. Edelman and C. Greene in type A_n, and W. Kraskiewicz in type B_n [EdGr87, Kra89, Lam95]—these insertions read and insert a reduced word for $w_\circ(A_n)$ or $w_\circ(B_{n,k})$ letter by letter to produce a standard Young tableau of shape w (a staircase or trapezoid). The bijection is concluded using Theorem 9.5, which canonically bijects $\mathsf{SYT}(\mathsf{w})$ with $\mathsf{Red}(w)$. The map backwards proceeds via *promotion* on the standard Young tableau encoding a reduced word of w.
- Another proof is via *Little bumps* and *signed Little bumps* [Li03, BHRY14]. Thinking of a reduced word as a wiring diagram, these methods take a reduced word for $w_\circ(A_n)$ or $w_\circ(B_{n,k})$ and systematically eliminate all braid moves—by introducing additional strands—to obtain a reduced word for w. Little bumps may be viewed as a combinatorialization of transition for Schubert polynomials (due in type A_n to A. Lascoux and M.-P. Schützenberger) [BiHa95, Bi98].

The Edelman-Greene bijection from $\mathsf{SYT}(\mathsf{w})$ to $\mathsf{Red}(w_\circ(Y))$ using promotion works in all coincidental types, which suggests the following open problem.

Problem 9.1. Uniformly develop a theory of insertion algorithms and Little bumps to explain the relation $\mathsf{Red}(w_\circ(Y)) \simeq \mathsf{Red}(w)$ in the coincidental types.

Remark 9.2. The first step towards a theory of Little bumps in types $I_2(m)$ and H_3 would be the representation of reduced words using wiring diagrams. Such representations exist, since both $W(I_2(m))$ and $W(H_3) \simeq \mathsf{Alt}_5 \times \mathbb{Z}/2\mathbb{Z}$ have (small) permutation representations coming from their actions on the parabolic subgroups identified in Section 9.5.2; recall that the usual permutation representations of $W(A_n)$ and $W(B_n)$ may be obtained in a similar manner.

For example, if we write $((i,j)) := (i,j)(-i,-j)$ for the transposition of $\pm i$ with $\pm j$, we have a representation of $W(H_3)$ on $\pm[1,2,3,4,5,6]$ defined by:

$$s_1 \mapsto ((1,2))((3,4))$$

$$s_2 \mapsto ((2,3))((4,5))$$

$$s_3 \mapsto ((1,-2))((5,6)).$$

9.7 Schubert Calculus

We now turn to the algebro-geometric context for the combinatorial objects of Sections 9.4 to 9.6. This section sets up the rings necessary to state the multiplicity-free identities that correspond to the bijections of Theorems 9.1 and 9.2. The corresponding combinatorics of structure coefficients is deferred to the next section.

9.7.1 Cohomology

Recall from Section 9.3.2 that for P a parabolic subgroup of G, the generalized flag variety G/P has the Bruhat decomposition

$$\mathsf{G}/\mathsf{P} = \bigsqcup_{w \in W^{\mathsf{P}}} \mathsf{B}_{-} w \mathsf{P}/\mathsf{P}.$$

The **Schubert classes** σ_w are the Poincaré duals of the **Schubert varieties**, which are the closures $X_w := \overline{\mathsf{B}_{-} w \mathsf{P}/\mathsf{P}}$. Since the Bruhat decomposition is a cell decomposition, the set $\{\sigma_w\}_{w \in W^{\mathsf{P}}}$ is a $\mathbb{Z}$-linear basis of the cohomology ring $H^\star(\mathsf{G}/\mathsf{P}, \mathbb{Z})$. As such, any cup product $\sigma_w \cdot \sigma_u$ of basis elements can be expressed in the basis:

$$\sigma_w \cdot \sigma_u = \sum_{v \in W^{\mathsf{P}}} c_{w,u}^{v} \sigma_v.$$

The Borel isomorphism from $H^\star(\mathsf{Gr}(k,n))$ to the coinvariant ring identifies Schubert classes with Schur functions, and the $c_{w,u}^{v}$ are the Littlewood-Richardson coefficients in this case [Le47]. This setup therefore generalizes the specific example discussed in Sections 1.3 and 9.2. Since Schur functions are commonly indexed by partitions, the (minuscule) Schubert class σ_w is often indexed by the inversion set w (an order ideal in $\Lambda_{\mathsf{Gr}(k,n)}$, by Theorem 9.6), which we call a **straight shape**. An **anti-normal shape** is then an order filter. Similarly, if $\mathsf{w} \subseteq \mathsf{v}$ we write v/w for the subposet $\mathsf{v} \setminus \mathsf{w} \subseteq \Lambda_{\mathsf{G}/\mathsf{P}}$, and call v/w a **skew shape**. Recall that for $w \in W^J$, we write $\breve{w} := w_\circ w w_\circ(J)$.

For G/P minuscule, H. Thomas and A. Yong have given a uniform combinatorial formula for $c_{w,u}^{v}$ [ThYo09a]. Their formula generalizes M.-P. Schützenberger's well-known rule for $\mathsf{G}/\mathsf{P} = \mathsf{Gr}(k,n)$. Given a standard tableau $T \in \mathsf{SYT}(\mathsf{v}/\mathsf{w})$, there is a map **rectification** (whose definition we defer until Section 9.8.2, where it will be given in greater generality, though we saw a special case in Section 1.2.3) that produces a tableau $\mathrm{Rect}(T) \in \mathsf{SYT}(\mathsf{u}')$, for some $u' \in W^{\mathsf{P}}$.

Theorem 9.7 ([ThYo09a]). *For G/P minuscule, the coefficient $c_{w,u}^{v}$ equals the number of standard tableaux $T \in \mathsf{SYT}(\mathsf{v}/\mathsf{w})$ whose rectification is any fixed standard tableaux of shape u.*

The following identities in $H^\star(Z)$ follow from Theorem 9.1 (which will be proven in Section 9.10) using the well-known S_3-symmetry of Littlewood-Richardson coefficients (see, e.g., [ThYo08]) and the combinatorial interpretation of the Littlewood-Richardson coefficients.

Theorem 9.8. *For (X, Y, Z) from the first three rows of Figure 9.3, with u, v, and w as defined in Section 9.6, the following identity holds in $H^\star(Z)$:*

$$\sigma_u \cdot \sigma_{\check{v}} = \sigma_{\check{w}}.$$

9.7.2 K-Theory

K-theoretic Schubert calculus turns to the Grothendieck ring $K(\mathsf{G}/\mathsf{P})$ of algebraic vector bundles over G/P as a richer analogue of the cohomology ring $H^\star(\mathsf{G}/\mathsf{P})$. $K(\mathsf{G}/\mathsf{P})$ has a $\mathbb{Z}$-linear basis given by the classes of the Schubert varieties' structure sheaves $\{[\mathcal{O}_{X_w}]\}_{w\in W/W_P}$. As before, we have an expansion:

$$[\mathcal{O}_{X_w}] \cdot [\mathcal{O}_{X_u}] = \sum_{v \in W/W_{\mathsf{P}}} C^v_{w,u} [\mathcal{O}_{X_v}], \tag{9.8}$$

where now now $(-1)^{|\mathsf{v}|-|\mathsf{w}|-|\mathsf{u}|} C^v_{w,u} \in \mathbb{Z}_{\geq 0}$ [Br02]. These K-theoretic structure constants generalize their cohomological counterparts—$C^v_{w,u} = c^v_{w,u}$ whenever $|\mathsf{v}| = |\mathsf{w}| + |\mathsf{u}|$, but when $|\mathsf{v}| > |\mathsf{w}| + |\mathsf{u}|$, while $C^v_{w,u}$ can be *nonzero.*

When $\sigma_w \cdot \sigma_u$ expands as a multiplicty-free sum of Schubert classes in $H^\star(\mathsf{G}/\mathsf{P})$, a result of A. Knutson determines the corresponding expansion of $[\mathcal{O}_{X_w}] \cdot [\mathcal{O}_{X_u}]$ in $K(\mathsf{G}/\mathsf{P})$. Recall that the **Möbius function** of a poset $\mathcal{P}$ is the function $\mu_{\mathcal{P}} : \mathcal{P} \times \mathcal{P} \to \mathbb{Z}$ uniquely characterized by

$$\mu_{\mathcal{P}}(x,y) := \sum_{x \leq z < y} \mu_{\mathcal{P}}(x,z) = 0 \tag{9.9}$$

for all $x < y \in \mathcal{P}$. Given a poset $\mathcal{P}$, we shall adjoin a minimal element $\hat{0}$ and write $\hat{\mu}_{\mathcal{P}}(x) := -\mu_{\mathcal{P}}(\hat{0}, x)$.

Theorem 9.9 (A. Knutson [Kn09, Theorem 3])**.** *Suppose*

$$\sigma_w \cdot \sigma_u = \sum_{v \in D} \sigma_v$$

is multiplicity-free. Write $\mathcal{P} := \{y \in W^{\mathsf{G}/\mathsf{P}} : y \geq v,$ for some $v \in D\}$. Then the corresponding expansion in

K-theory is

$$[\mathcal{O}_{X_w}] \cdot [\mathcal{O}_{X_u}] = \sum_{y \in \mathcal{P}} \hat{\mu}_{\mathcal{P}}(y) [\mathcal{O}_{X_y}].$$

Example 9.6. Continuing Example 9.5, for $s = r = 2$, we have

$$\begin{aligned}
\left[\mathcal{O}_{\ydiagram[*(gray)]{2,2}}\right]^2 = {} & \left[\mathcal{O}_{\ydiagram[*(gray)]{2,2}*{2,2,2,2}}\right] + \left[\mathcal{O}_{\ydiagram[*(gray)]{2,2}*{3,2,2,1}}\right] + \left[\mathcal{O}_{\ydiagram[*(gray)]{2,2}*{4,2,2}}\right] + \left[\mathcal{O}_{\ydiagram[*(gray)]{2,2}*{3,3,1,1}}\right] + \left[\mathcal{O}_{\ydiagram[*(gray)]{2,2}*{4,3,1}}\right] + \left[\mathcal{O}_{\ydiagram[*(gray)]{2,2}*{4,4}}\right] \\
& - \left[\mathcal{O}_{\ydiagram[*(gray)]{2,2}*{3,2,2,2}}\right] - \left[\mathcal{O}_{\ydiagram[*(gray)]{2,2}*{3,3,2,1}}\right] - \left[\mathcal{O}_{\ydiagram[*(gray)]{2,2}*{4,2,2,1}}\right] - \left[\mathcal{O}_{\ydiagram[*(gray)]{2,2}*{4,3,1,1}}\right] - \left[\mathcal{O}_{\ydiagram[*(gray)]{2,2}*{4,3,2}}\right] - \left[\mathcal{O}_{\ydiagram[*(gray)]{2,2}*{4,4,1}}\right] \\
& + \left[\mathcal{O}_{\ydiagram[*(gray)]{2,2}*{4,3,2,1}}\right].
\end{aligned}$$

$\diamondsuit$

When determining multiplicity-freeness in $K(\mathsf{G}/\mathsf{P})$, it therefore suffices to check the corresponding statement in $H^\star(\mathsf{G}/\mathsf{P})$ and then apply Theorem 9.9.

Remark 9.3. It is *not* the case that a multiplicity-free product in cohomology necessarily yields a multiplicity-free product in K-theory. For example, in $\mathsf{Gr}(3, 6)$, we have

$$\sigma_{\ydiagram{2,1}} \cdot \sigma_{\ydiagram{1,1}} = \sigma_{\ydiagram{2,2,1}} + \sigma_{\ydiagram{3,1,1}} + \sigma_{\ydiagram{3,2}}, ,$$

but

$$\left[\mathcal{O}_{\ydiagram{2,1}}\right] \cdot \left[\mathcal{O}_{\ydiagram{1,1}}\right] = \left[\mathcal{O}_{\ydiagram{2,2,1}}\right] + \left[\mathcal{O}_{\ydiagram{3,1,1}}\right] + \left[\mathcal{O}_{\ydiagram{3,2}}\right] - 2\left[\mathcal{O}_{\ydiagram{3,2,1}}\right]$$

The following theorem is the K-theoretic analogue of Theorem 9.8.

Theorem 9.10. *For (X, Y, Z) from the first three rows of Figure 9.3, with u, v, and w as defined in Section 9.6, the following identity holds in $K(Z)$:*

$$[\mathcal{O}_{X_u}] \cdot [\mathcal{O}_{X_{\check{v}}}] = [\mathcal{O}_{X_{\check{w}}}].$$

Proof. This follows from Theorem 9.8 and Theorem 9.9—since the product in cohomology has exactly one term, so does the product in K-theory. $\square$

9.8 Combinatorics of Structure Coefficients

In this section we introduce the combinatorial tools developed in the study of K-theoretic Schubert calculus by H. Thomas and A. Yong [ThYo09b], E. Clifford, H. Thomas, and A. Yong [ClThYo14], and A. Buch and M. Samuel [BuSa13]. The main result we wish to review is a combinatorial formula for the K-theoretic structure coefficients $C^v_{w,u}$ in the style of Theorem 9.7. We have seen special cases of this theory (for type A and maximal orthogonal Grassmannians in Chapters 3 and 6, but we review all the definitions here, since we will need them in full generality.

9.8.1 Increasing Tableaux

Following [BuSa13], we generalize the language of the introduction in this section to accommodate the embeddings of Section 9.6. Fix a finite poset $\mathcal{P}$ with order relation $\prec$ and an alphabet $\mathcal{A}$ (assume the symbol $\bullet \notin \mathcal{A}$). We call an order ideal w of $\mathcal{P}$ a **straight shape**, and the difference of two straight shapes $\mathsf{w} \subseteq \mathsf{v}$ a **skew shape** v/w. Note that v is the special case of a skew shape for $\mathsf{w} = \emptyset$. These shapes inherit the partial order from $\mathcal{P}$. A **tableau of shape** v/w **on the alphabet** $\mathcal{A}$ is a map $\mathsf{T} : \mathsf{v}/\mathsf{w} \to \mathcal{A}$.

Definition 9.4. Let $\mathcal{A}$ be a totally-ordered alphabet with order relation $<$. An **increasing tableau of shape** v/w **on the alphabet** $\mathcal{A}$ is a strictly order-preserving map $\mathsf{T} : \mathsf{v}/\mathsf{w} \to \mathcal{A}$, that is if $\alpha \prec \beta$ in v/w then $\mathsf{T}(\alpha) < \mathsf{T}(\beta)$. We write $\mathsf{IT}^{\mathcal{A}}(\mathsf{v}/\mathsf{w})$ for the set of all such maps.

For two disjoint alphabets $\mathcal{A}, \mathcal{B}$ with $\mathsf{T} \in \mathsf{IT}^{\mathcal{B}}(\mathsf{w})$ ($\mathcal{B}$ for below) and $\mathsf{U} \in \mathsf{IT}^{\mathcal{A}}(\mathsf{v}/\mathsf{w})$ ($\mathcal{A}$ for above), we write $\mathsf{T} \sqcup \mathsf{U}$ for the increasing tableau in $\mathsf{IT}^{\mathcal{B}\sqcup\mathcal{A}}(\mathsf{v})$, where $\mathcal{B} \sqcup \mathcal{A}$ is totally ordered so that $b < a$ for all $b \in \mathcal{B}$ and $a \in \mathcal{A}$. We define $\mathsf{IT}(\mathsf{v}/\mathsf{w}) := \bigcup_{k=1}^{\infty} \mathsf{IT}^{[k]}(\mathsf{v}/\mathsf{w})$ and set $\mathsf{T}^{\min}_{\mathsf{v}/\mathsf{w}}$ to be the componentwise minimal increasing tableau in $\mathsf{IT}(\mathsf{v}/\mathsf{w})$. We will call $\mathsf{T}^{\min}_{\mathsf{v}/\mathsf{w}}$ the **minimal increasing tableau of shape** v/w (see Figure 9.11 for an example), and it will play an important role in the sequel.

In special cases, the notions of increasing tableaux and $\mathcal{P}$-partitions are easily related, as was first observed in [DiPeSt15].

Proposition 9.6 ([DiPeSt15, Theorem 4.1])**.** *For a ranked poset $\mathcal{P}$ with all maximal chains of the same length $\mathsf{ht}(\mathcal{P})$, there is a bijection $\mathsf{PP}^{[p]}(\mathcal{P}) \simeq \mathsf{IT}^{[k]}(\mathcal{P})$, where $k = p + \mathsf{ht}(\mathcal{P})$.*

Proof. With our conventions, a bijection is evidently given by adding i to the ith rank. $\square$

Since all of the posets in Theorems 9.1 and 9.2 are of the required form, by Proposition 9.6 we may henceforth deal only with increasing tableaux. The significant advantage that increasing tableaux enjoy over $\mathcal{P}$-partitions is that increasing tableaux are equipped with a well-developed theory of jeu de taquin [ThYo09b, ThYo09a, ClThYo14, BuSa13].

9.8.2 Jeu de taquin and Other Games

Jeu de taquin

Given a shape $\mathsf{v}/\mathsf{w} \subseteq \mathcal{P}$, a tableau T of shape v/w on $\mathcal{A}$, and $a \in \mathcal{A}$, we let

$$\mathsf{T}_a := \{\alpha \in \mathsf{v}/\mathsf{w} : \alpha \text{ covers or is covered by some } \beta \text{ for which } \mathsf{T}(\beta) = a\}.$$

For two letters $a, b \in \mathcal{A}$, we may "exchange" them in T to obtain a new tableau

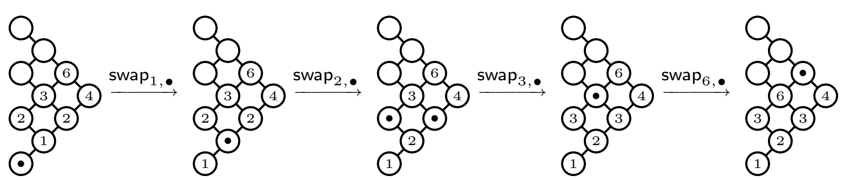
$$\mathsf{swap}_{a,b}(\mathsf{T})(\alpha) := \begin{cases} a & \text{if } \mathsf{T}(\alpha) = b \text{ and } \alpha \in \mathsf{T}_a; \\ b & \text{if } \mathsf{T}(\alpha) = a \text{ and } \alpha \in \mathsf{T}_b; \\ \mathsf{T}(\alpha) & \text{otherwise.} \end{cases}$$

If we remove a set of maximal elements from w to obtain w', we may extend the definition of T from v/w to a tableau T' of shape v/w' by setting $\mathsf{T}'(\alpha) := \bullet$ for $\alpha \in \mathsf{w}/\mathsf{w}'$. Given an increasing tableau T of shape v/w on the totally-ordered alphabet $\mathcal{A}$, the **slide** of T into w/w' is given by

$$\mathsf{jdt}_{\mathsf{w}/\mathsf{w}'}(\mathsf{T}) := \left(\prod_{a \in \mathcal{A}} \mathsf{swap}_{a,\bullet} \right) (\mathsf{T}'),$$

where the product is in the given linear ordering for $\mathcal{A}$, and where we restrict the domain of $\mathsf{jdt}_{\mathsf{w}/\mathsf{w}'}(\mathsf{T})$ to the subset $\mathsf{v}'/\mathsf{w}' := \{\alpha \subseteq \mathsf{v}/\mathsf{w}' : \mathsf{jdt}_{\mathsf{w}/\mathsf{w}'}(\mathsf{T})(\alpha) \neq \bullet\}$. This procedure is bijective.

Example 9.7. The following illustration is an example of a slide for $\mathcal{A} = 1 < 2 < 3 < 4 < 5 < 6$ (see Example 9.3 for several others).



$\diamondsuit$

When $\mathsf{T} \in \mathsf{SYT}(\mathsf{v}/\mathsf{w})$ and w/w' is a single box, this process recovers the usual notion of **jeu de taquin**. Two tableaux T and T' are called **jeu de taquin equivalent** if they are related by a sequence of slides.

Rectification

Let $\mathsf{T} \in \mathsf{IT}^{[k]}(\mathsf{w})$ and set $\mathsf{w}_i := \{\alpha \in \mathsf{w} : \mathsf{U}(\alpha) \leq k - i\}$, so that $\mathsf{w}_0 = \mathsf{w}$ and $\mathsf{w}_k = \emptyset$. The **T-rectification** of $\mathsf{U} \in \mathsf{IT}^{\mathcal{A}}(\mathsf{v}/\mathsf{w})$ is the tableau

$$\mathrm{Rect}_{\mathsf{T}}(\mathsf{U}) := \left(\prod_{i=0}^{k-1} \mathsf{jdt}_{\mathsf{w}_i/\mathsf{w}_{i+1}} \right) (\mathsf{U}).$$

A **unique rectification target (URT)** is an increasing tableau R of straight shape such that if $\mathrm{Rect}_{\mathsf{T}}(\mathsf{U}) = \mathsf{R}$ for *some* $\mathsf{T} \in \mathsf{IT}(\mathsf{w})$, then $\mathrm{Rect}_{\mathsf{T}'}(\mathsf{U}) = \mathsf{R}$ for *all* $\mathsf{T}' \in \mathsf{IT}(\mathsf{w})$. When the tableau dictating rectification order does not matter, we may simply write $\mathrm{Rect}(\mathsf{U}) = R$. Example 9.7 is an example of rectification.

Theorem 9.11 ([BuSa13, Theorem 3.12])**.** *For G/P minuscule and w any straight shape, $\mathsf{T}^{\mathsf{min}}_{\mathsf{w}}$ is a URT.*

We can now state the rule, generalizing Theorem 9.7, for the structure coefficients $C^v_{w,u}$ of Equation (9.8).

Theorem 9.12 ([BuSa13, Corollary 4.8])**.** *For G/P minuscule,*

$$(-1)^{|\mathsf{v}|-|\mathsf{w}|-|\mathsf{u}|} C^v_{w,u} = \left| \left\{ \mathsf{T} \in \mathsf{IT}(\mathsf{v}/\mathsf{w}) : \mathrm{Rect}(\mathsf{T}) = \mathsf{T}^{\min}_u \right\} \right|.$$

The Infusion Involution

Instead of discarding the rectification order T when performing rectification, we can consider what happens to the *pair* (T, U) as we move U past T. We keep track of the two tableaux using two different alphabets: $[k]$, and $[\overline{k}] := \{\overline{1} < \overline{2} < \cdots < \overline{k}\}$. For $\mathsf{U} \in \mathsf{IT}^{[k]}(\mathsf{v}/\mathsf{w})$, we will write $\overline{\mathsf{U}}$ to denote the increasing tableau in $\mathsf{IT}^{[\overline{k}]}(\mathsf{v}/\mathsf{w})$ obtained by sending $i \mapsto \overline{i}$.

Let $\mathsf{T} \in \mathsf{IT}^{[i]}(\mathsf{w})$ and $\mathsf{U} \in \mathsf{IT}^{[j]}(\mathsf{v}/\mathsf{w})$. Informally, we will glue the tableau $\overline{\mathsf{T}}$ on the alphabet $[\overline{i}]$ to the bottom of the tableau U on $[j]$ and then slide one alphabet past the other, so that the total ordering

$$[\overline{i}] \sqcup [j] = \overline{1} < \overline{2} < \cdots < \overline{i} < 1 < 2 < \cdots < j$$

becomes

$$[j] \sqcup [\overline{i}] = 1 < 2 < \cdots < j < \overline{1} < \overline{2} < \cdots < \overline{i}.$$

Formally, the **infusion involution** of $(\mathsf{T}, \mathsf{U}) \in \mathsf{IT}^{[i]}(\mathsf{w}) \times \mathsf{IT}^{[j]}(\mathsf{v}/\mathsf{w})$ is the pair of tableaux $(\mathsf{U}', \mathsf{T}') \in \mathsf{IT}^{[j]}(\mathsf{u}) \times$

$\mathsf{IT}^{[i]}(\mathsf{v}/\mathsf{u})$ defined by

$$\mathsf{U}' \sqcup \overline{\mathsf{T}'} = \left(\prod_{a=1}^{i} \prod_{b=j}^{1} \mathsf{swap}_{\overline{a},b} \right) (\overline{\mathsf{T}} \sqcup \mathsf{U}).$$

Theorem 9.13 ([ThYo09b, Theorem 3.1])**.** *For* $\mathsf{T} \in \mathsf{IT}(\mathsf{w})$ *and* $\mathsf{U} \in \mathsf{IT}(\mathsf{v}/\mathsf{w})$*,*

$$(\mathsf{infusion} \circ \mathsf{infusion})(\mathsf{T}, \mathsf{U}) = (\mathsf{T}, \mathsf{U}).$$

Folding

We introduce the alphabet $[\underline{k}] := \{\overline{k} < \cdots < \overline{2} < \overline{1}\}$ and for $\mathsf{U} \in \mathsf{IT}^{[k]}(\mathsf{v}/\mathsf{w})$ we let $\overline{\mathsf{U}}$ denote the increasing tableau in $\mathsf{IT}^{[\underline{k}]}(\mathsf{v}/\mathsf{w})$ obtained by sending $(k+1-i) \mapsto \bar{i}$.

Let $\mathsf{T} \in \mathsf{IT}^{[k]}(\mathsf{w})$ and $\mathsf{U} \in \mathsf{IT}^{[k]}(\mathsf{v}/\mathsf{w})$. Similarly to Section 9.8.2, we glue T to the bottom of $\overline{\mathsf{U}}$—but rather than completely slide one alphabet past another, we instead **fold** the two alphabets together, so that the total ordering

$$[k] \sqcup [\underline{k}] = 1 < 2 < \cdots < k < \overline{k} < \cdots < \overline{2} < \overline{1}$$

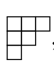
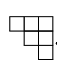
becomes the total ordering

$$[\overline{k}] \sqcup\!\sqcup [k] := \overline{1} < 1 < \overline{2} < 2 < \cdots < \overline{k} < k.$$

Following K. Purbhoo [Pu14, vL01], the **folding** of $(\mathsf{T}, \mathsf{U}) \in \mathsf{IT}^{[k]}(\mathsf{w}) \times \mathsf{IT}^{[k]}(\mathsf{v}/\mathsf{w})$ is the tableau $\mathsf{fold}(\mathsf{T}, \mathsf{U}) \in \mathsf{IT}^{[\underline{k}] \sqcup\!\sqcup [k]}(\mathsf{v})$ defined by

$$\mathsf{fold}(\mathsf{T}, \mathsf{U}) := \left(\prod_{b=k}^{1} \left(\prod_{a=b}^{k} \mathsf{swap}_{a,\overline{b}} \right) \circ \left(\prod_{a=k}^{b+1} \mathsf{swap}_{\overline{a},\overline{b}} \right) \right) (\mathsf{T} \cup \overline{\mathsf{U}}).$$

9.8.3 Relations and Equivalence

We write "Ferrers shape" for an order ideal in $\Lambda_{\mathsf{Gr}(k,n)}$ (for some k, n) and "shifted shape" for an order ideal in $\Lambda_{\mathsf{OG}(n,2n)}$, and the tableaux of the corresponding shape are denoted similarly. When working with Ferrers and shifted shapes, we will find it convenient to switch to the English convention on tableau orientation. That is, we vertically reflect and then rotate our tableaux 135° clockwise so "gravity" now points north-west. We differentiate shifted partitions from partitions using a subscripted "$*$"—thus $(3, 2, 1)$ stands for the Ferrers shape , while $(3, 2, 1)_*$ is the shifted shape .

For T a standard or increasing tableau, let $\mathsf{read}(\mathsf{T})$ be the **column reading word** obtained by reading

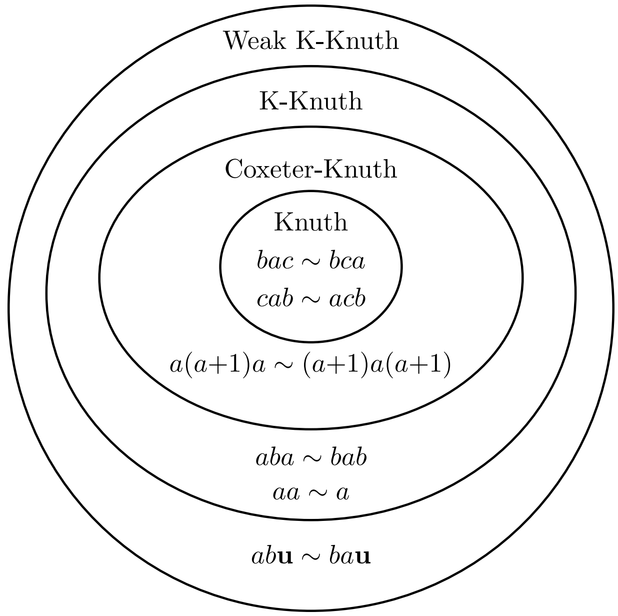


Figure 9.10: (Standard) Knuth-like relations, where $a < b < c$ are distinct positive integers and $\mathbf{u}$ is a word of positive integers.

the columns of T from left to right and bottom to top; where it will not cause confusion, we will abbreviate this to **reading word**. We wish to consider the set of words on the alphabet of positive integers, up to Knuth, Coxeter-Knuth, K-Knuth, or weak K-Knuth equivalences—summarized in Figure 9.10. We note that the "Knuth equivalence" in Figure 9.10 applies to words with distinct letters.

Remarkably, as summarized in Theorem 9.14, these relations on reading words of tableaux exactly mirror jeu de taquin slides on the tableaux themselves.

Theorem 9.14 ([BuSa13, Theorems 6.2 and 7.8])**.**

$$\textit{Two} \left\{ \begin{matrix} \textit{increasing (skew) Ferrers} \\ \textit{increasing (skew) shifted} \end{matrix} \right\} \textit{tableaux } \mathsf{T}, \mathsf{T}' \textit{ are jeu de taquin equivalent}$$

$$\textit{if and only if } \mathsf{read}(\mathsf{T}) \textit{ and } \mathsf{read}(\mathsf{T}') \textit{ are } \left\{ \begin{matrix} K\textit{-Knuth} \\ \textit{weakly } K\textit{-Knuth} \end{matrix} \right\} \textit{equivalent.}$$

The following proposition records two facts about K-Knuth equivalence for use in Section 9.10.1.

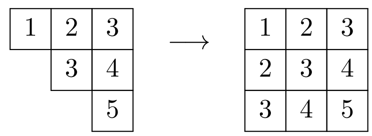
Proposition 9.7 ([ThYo09b, Theorem 6.1] and [BuSa13, Lemma 5.4 and Corollary 6.8])**.**

1. *The longest strictly increasing subsequences of K-Knuth equivalent words have the same length.*

2. *The length of the first row of an increasing Ferrers tableau* T *is the length of the longest strictly increasing subsequence of* $\mathsf{read}(\mathsf{T})$*.*

The **doubling** T^D of a shifted tableau T is the Ferrers tableau obtained by reflecting T across the shifted diagonal—note that the shifted diagonal itself is *not* duplicated [BuSa13, Section 7.1]. This construction is illustrated in Figure 9.11.

(a) For $\mathsf{u} = (3, 2, 1)_*$, the minimal increasing tableau $\mathsf{T}_{\mathsf{u}}^{\min}$ and $(\mathsf{T}_{\mathsf{u}}^{\min})^D$:



(b) A partially filled skew shape $\widetilde{U}$ and its doubling $\widetilde{\mathsf{U}}^D$:

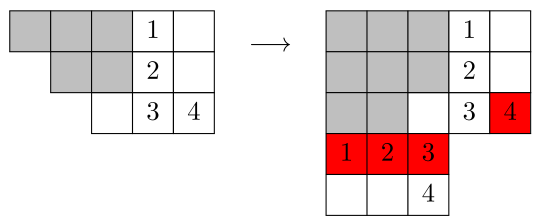


Figure 9.11: Examples of doubling. In red, we have marked the strictly increasing subsequence of length at least k from which we derive a contradiction in Proposition 9.9.

The operation of doubling allows us to relate weak K-Knuth equivalence to K-Knuth equivalence.

Proposition 9.8 ([BuSa13, Proposition 7.1])**.** *If* T *and* U *are weakly* K*-Knuth equivalent, then* $\mathsf{read}(\mathsf{T}^D)$ *and* $\mathsf{read}(\mathsf{U}^D)$ *are* K*-Knuth equivalent.*

9.9 Main Theorem

Using the techniques of Section 9.7.2, the bijections of Theorem 9.1 may now be given. The idea behind these bijections is illustrated in Figure 9.12. We first give a more general statement, and then specialize to the cases of interest.

For $\mathsf{x}, \mathsf{u} \subseteq \mathsf{v}$ order ideals in a minuscule poset, let

$$R^{\mathsf{v}}_{\mathsf{x},\mathsf{u}} := \{\mathsf{U} \in \mathsf{IT}(\mathsf{v}/\mathsf{x}) : \mathrm{Rect}_{\mathsf{T}}(\mathsf{U}) = \mathsf{T}_{\mathsf{u}}^{\mathsf{min}} \text{ for some } \mathsf{T} \in \mathsf{IT}^{[k]}(\mathsf{x})\}. \tag{9.10}$$

By Theorem 9.11, $\mathsf{T}_{\mathsf{u}}^{\mathsf{min}}$ is a URT. Therefore, if $\mathsf{U} \in R^{\mathsf{v}}_{\mathsf{x},\mathsf{u}}$, then the choice of rectification order T is irrelevant—$\mathrm{Rect}_{\mathsf{T}}(\mathsf{U}) = \mathsf{T}_{\mathsf{u}}^{\mathsf{min}}$ for *every* $\mathsf{T} \in \mathsf{IT}^{[k]}(\mathsf{x})$.

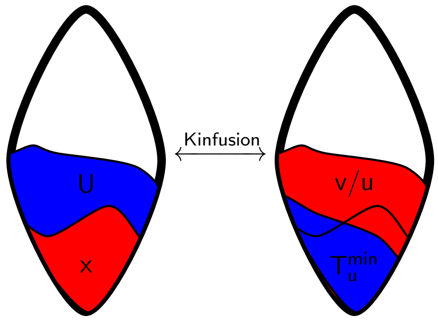


Figure 9.12: An illustration of the idea (and notation) behind the bijections of Theorem 9.15. To conclude Theorem 9.1, we wish to show that $\mathsf{x} = \chi(\Phi^+(Y))$ and that U is unique.

Theorem 9.15. *Fix $k \in \mathbb{N}$ a positive integer and $\mathsf{u} \subseteq \mathsf{v}$. Then* $\mathrm{Rect}_{\mathsf{T}_\mathsf{u}^{\mathsf{min}}}$ *gives a bijection*

$$\mathsf{IT}^{[k]}(\mathsf{v}/\mathsf{u}) \simeq \bigsqcup_{\mathsf{x} \subseteq \mathsf{v}} \left(R^{\mathsf{v}}_{\mathsf{x},\mathsf{u}} \times \mathsf{IT}^{[k]}(\mathsf{x}) \right).$$

Proof. Consider $\mathsf{T}' \in \mathsf{IT}^{[k]}(\mathsf{v}/\mathsf{u})$. We have

$$\mathsf{infusion}(\mathsf{T}_\mathsf{u}^{\mathsf{min}}, \mathsf{T}') = (\mathsf{T}, \mathsf{U}),$$

for some $\mathsf{x} \subseteq \mathsf{v}$, some $\mathsf{T} \in \mathsf{IT}^{[k]}(\mathsf{x})$ and some $\mathsf{U} \in \mathsf{IT}^{[k]}(\mathsf{v}/\mathsf{x})$. Since $\mathrm{Rect}_\mathsf{T}(\mathsf{U}) = \mathsf{T}_\mathsf{u}^{\mathsf{min}}$, we have $\mathsf{U} \in R^{\mathsf{v}}_{\mathsf{x},\mathsf{u}}$. Since infusion is an involution by Theorem 9.13, this means that $\mathrm{Rect}_{\mathsf{T}_\mathsf{u}^{\mathsf{min}}}$ is an injection

$$\mathsf{IT}^{[k]}(\mathsf{v}/\mathsf{u}) \hookrightarrow \bigsqcup_{\mathsf{x} \subseteq \mathsf{v}} \left(R^{\mathsf{v}}_{\mathsf{x},\mathsf{u}} \times \mathsf{IT}^{[k]}(\mathsf{x}) \right).$$

Conversely, for $\mathsf{x} \subseteq \mathsf{v}$, $\mathsf{U} \in R^{\mathsf{v}}_{\mathsf{x},\mathsf{u}}$ and $\mathsf{T} \in \mathsf{IT}^{[k]}(\mathsf{x})$, we have

$$\mathsf{infusion}(\mathsf{T}, \mathsf{U}) = (\mathsf{T}_\mathsf{u}^{\mathsf{min}}, \mathsf{T}') \in \mathsf{IT}(\mathsf{u}) \times \mathsf{IT}^{[k]}(\mathsf{v}/\mathsf{u}),$$

for a unique T'. This shows that $\mathrm{Rect}_{\mathsf{T}_\mathsf{u}^{\mathsf{min}}}$ is also surjective, and hence bijective. $\square$

In the special case when $R^{\mathsf{v}}_{\mathsf{x},\mathsf{u}}$ only has one element, we obtain a bijective statement involving only sets of increasing tableaux.

Theorem 9.16. *Fix $k \in \mathbb{N}$ a positive integer and $\mathsf{u} \subseteq \mathsf{v}$. Suppose $|R^{\mathsf{v}}_{\mathsf{x},\mathsf{u}}| \leq 1$ for every $\mathsf{x} \subseteq \mathsf{v}$. Then* $\mathrm{Rect}_{\mathsf{T}_\mathsf{u}^{\mathsf{min}}}$

gives a bijection

$$\mathsf{IT}^{[k]}(\mathsf{v}/\mathsf{u}) \simeq \bigsqcup_{\mathsf{x}:|R^{\mathsf{v}}_{\mathsf{x},\mathsf{u}}|\neq 0} \mathsf{IT}^{[k]}(\mathsf{x}).$$

This map restricts to a bijection

$$\mathsf{SYT}(\mathsf{v}/\mathsf{u}) \simeq \bigsqcup_{\mathsf{x}:|R^{\mathsf{v}}_{\mathsf{x},\mathsf{u}}|\neq 0} \mathsf{SYT}(\mathsf{x}).$$

Let $k = p + \mathsf{ht}(\mathsf{v}/\mathsf{u})$. *If all maximal chains of* v/u *are of equal length, and the same is true for each* x *with* $|R^{\mathsf{v}}_{\mathsf{x},\mathsf{u}}| \neq 0$, *then there is a bijection*

$$\mathsf{PP}^{[p]}(\mathsf{v}/\mathsf{u}) \simeq \bigsqcup_{\mathsf{x}:|R^{\mathsf{v}}_{\mathsf{x},\mathsf{u}}|\neq 0} \mathsf{PP}^{[p+\mathsf{ht}(\mathsf{v}/\mathsf{u})-\mathsf{ht}(\mathsf{x})]}(\mathsf{x}).$$

Proof. The first statement is immediate from Theorem 9.15. Furthermore, the map clearly restricts to the set of standard Young tableaux. The last statement then follows by Proposition 9.6. □

This theorem is illustrated in Example 9.3.

9.10 Applications: Doppelgängers

We fix (X, Y, Z) as in Figure 9.3. We recall that χ is the embedding of the coincidental root poset $\Phi^+(Y)$, Θ is the embedding of the doppelgänger minuscule poset Λ_X, and that we specify the corresponding shapes inside the ambient minuscule poset Λ_Z as

$$\mathsf{w} := \chi(\Phi^+(Y)) \qquad \text{and} \qquad \mathsf{v}/\mathsf{u} := \Theta(\Lambda_X).$$

We deduce Theorem 9.1 from Theorem 9.16 by showing that $R^{\mathsf{v}}_{\mathsf{w},\mathsf{u}}$ has a unique element and that $R^{\mathsf{v}}_{\mathsf{x},\mathsf{u}} = \emptyset$ for $\mathsf{x} \neq \mathsf{w}$.

9.10.1 (B) Rectangles and Trapezoids

Sections 9.4 to 9.6 identify the following shapes:

$\mathsf{w} = (n-1, n-3, \ldots, n-2k+1)_*$ is a shifted trapezoid and

$\mathsf{v}/\mathsf{u} = (n-k, n-k, \ldots, n-k)$ is a $k \times (n-k)$ rectangle. Then

$\mathsf{u} = (k-1, k-2, \ldots, 1)_*$ is a shifted staircase and

$\mathsf{v} = (2m-1, 2m-2, \ldots, m)_*$.

Figure 9.13 illustrates examples of v/u and w.

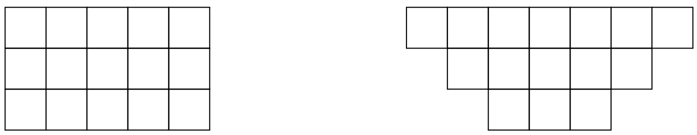


Figure 9.13: $\mathsf{v}/\mathsf{u} = \Theta(\Lambda_{\mathsf{Gr}(k,n)})$ and $\mathsf{w} = \chi(\Phi^+(B_{k,n}))$ for $k = 3$ and $n = 8$.

Write $a := 2k-3$ and let U be the increasing antinormal Ferrers tableau of shape v/w obtained by labeling each southwest-to-northeast diagonal of the staircase v/w with consecutive increasing integers, where the bottom row is labeled with the odd numbers $1, 3, \ldots, a-2, a$. Figure 9.11(a) and Figure 9.14 illustrates examples of $\mathsf{T}_{\mathsf{u}}^{\mathsf{min}}$ and U.

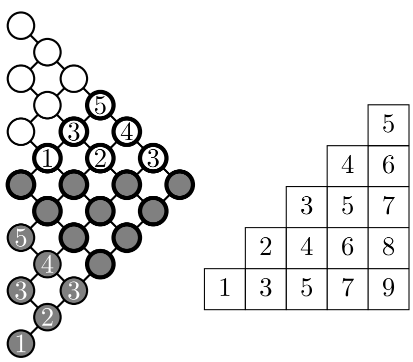


Figure 9.14: On the left are the tableau U (top, black numbers in white nodes) and $\mathsf{T}_{\mathsf{u}}^{\mathsf{min}}$ (bottom, white numbers in gray nodes) for $k = 3$ and $n = 6$. On the right is the increasing anti-normal Ferrers tableau U for $k = 6$—inserting spaces for clarity, it has reading word $\mathsf{read}(\mathsf{U}) = 1\ 32\ 543\ 7654\ 98765$.

We begin by characterizing a property of the reading word of any tableau that rectifies to $\mathsf{T}_{\mathsf{u}}^{\mathsf{min}}$.

Lemma 9.1. *For $a = 2k-3$, let $\pi \in S_a$ be the permutation with one-line notation*

$$[2, 4, \ldots, a{+}1, 1, 3, \ldots, a].$$

Then any tableau $\widetilde{U}$ that rectifies to $\mathsf{T}_{\mathsf{u}}^{\mathsf{min}}$ has a reading word $\mathsf{read}(\widetilde{U})$ whose Demazure product is π. In particular, since $\mathsf{len}(\pi) = \binom{k}{2}$, any such $\widetilde{U}$ must have at least $\binom{k}{2}$ cells.

Proof. It is easy to see that the reading word $\mathsf{read}(\mathsf{T}_{\mathsf{u}}^{\mathsf{min}})$ is a reduced word of the permutation π. Since $\mathsf{read}(\mathsf{T}_{\mathsf{u}}^{\mathsf{min}})$ is a reduced word, any weakly K-Knuth equivalent word is at least as long (see the weak K-Knuth relations in Figure 9.10). Furthermore, since every reduced word for π begins with two commuting letters, the words that are weakly K-Knuth equivalent to $\mathsf{read}(\mathsf{T}_{\mathsf{u}}^{\mathsf{min}})$ have Demazure product π. We conclude the statement using the equivalence of K-jeu de taquin equivalence of tableaux and K-Knuth equivalence of their reading words Theorem 9.14. □

We now consider tableaux whose reading word can be π. Recall that a permutation $\pi \in \mathfrak{S}_a$ is **vexillary** if its one-line notation avoids the pattern 2143; π is fully-commutative if and only if it avoids the pattern 321; and π is **Grassmannian** if it has at most one descent. A Grassmannian permutation is both vexillary *and* fully-commutative. In particular, the π of Lemma 9.1 is Grassmannian.

Lemma 9.2. *For $a = 2k-3$, let $\pi \in S_a$ be the permutation with one-line notation*

$$[2, 4, \ldots, a{+}1, 1, 3, \ldots, a].$$

Then there is a unique increasing anti-normal Ferrers tableau T_π such that $\mathsf{read}(\mathsf{T}_\pi) \in \mathsf{Red}(\pi)$.

Proof. We shall prove the statement more generally for Grassmannian permutations π. As suggested by V. Reiner, it suffices to prove that there is a unique such (straight) Ferrers tableau, since if T'_π is the tableau obtained from T_σ by reflecting across the diagonal and replacing $s_i \mapsto s_{a-i}$, then $\mathsf{read}(\mathsf{T}'_\pi)$ is a reduced word for $w_\circ \pi w_\circ$ (and both of the patterns 321 and 2143 are stable under conjugation by $w_\circ$).

The reduced words of a vexillary permutation form a single Coxeter-Knuth equivalence class; the additional assumption of fully-commutative implies that in the absence of braid moves, this Coxeter-Knuth class reduces to an ordinary Knuth equivalence. But any (semistandard) Knuth equivalence class contains a unique word that is the reading word of a Ferrers tableau [Fu97, Section 2], from which we conclude the lemma. □

Using the constraints provided by Lemma 9.1 and Lemma 9.2, we can now show that U is the unique tableau whose shape is an order filter of v rectifying to $\mathsf{T}_{\mathsf{u}}^{\mathsf{min}}$.

Proposition 9.9. *$R^{\mathsf{v}}_{\mathsf{w},\mathsf{u}} = \{\mathsf{U}\}$ and $R^{\mathsf{v}}_{\mathsf{x},\mathsf{u}} = \emptyset$ for $\mathsf{x} \neq \mathsf{w}$.*

Proof. We first show that $\mathsf{U} \in R^{\mathsf{v}}_{\mathsf{w},\mathsf{u}}$. Since $\mathsf{read}(\mathsf{T}_{\mathsf{u}}^{\mathsf{min}}) = \mathsf{read}(\mathsf{U})$ and $\mathsf{T}_{\mathsf{u}}^{\mathsf{min}}$ is a URT, $\mathrm{Rect}_{\mathsf{T}}(\mathsf{U}) = \mathsf{T}_{\mathsf{u}}^{\mathsf{min}}$ for any $\mathsf{T} \in \mathsf{IT}(\mathsf{w})$ by [BuSa13, Theorem 7.8]. By definition, we conclude $\mathsf{U} \in R^{\mathsf{v}}_{\mathsf{w},\mathsf{u}}$.

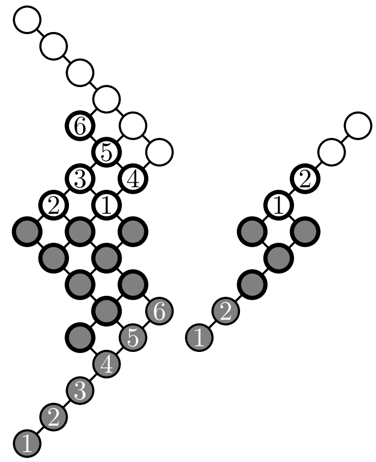


Figure 9.15: The shape v is the set of all boxes that are either gray or thick-bordered; the shape w is the set of gray boxes; and the shape v/u is the set of all thick-bordered boxes. $\mathsf{T}_{\mathsf{u}}^{\mathsf{min}}$ is the bottom tableau consisting of the gray boxes with white numbers. The top tableau consisting of the white boxes with black numbers is the unique tableau U whose shape is an order filter in v that rectifies to $\mathsf{T}_{\mathsf{u}}^{\mathsf{min}}$.

Let $\widetilde{\mathsf{U}} \in R_{\mathsf{x},\mathsf{u}}^{\mathsf{v}}$ for some x. We now argue that $\widetilde{\mathsf{U}}$ is necessarily of shape v/w. By Propositions 9.7 and 9.8, since the shape of $(\mathsf{T}_{\mathsf{u}}^{\mathsf{min}})^D$ is a $(k-1)\times(k-1)$ square (see Figure 9.11 (a)), the longest strictly increasing subsequence in $\mathsf{read}(\widetilde{\mathsf{U}}^D)$ is of length $k-1$. We claim that this forces the rth column of $\widetilde{\mathsf{U}}$ (from the right) to have at most $k-r$ cells: if the rth column of $\widetilde{\mathsf{U}}$ has more than $k-r$ cells, then the rth row of $\widetilde{\mathsf{U}}^D$—along with the last $r-1$ entries in the bottom row of $\widetilde{\mathsf{U}}$—form a strictly increasing sequence of length at least k in $\mathsf{read}(\widetilde{\mathsf{U}}^D)$. Since there are at least $\binom{k}{2}$ entries in $\widetilde{\mathsf{U}}$ by Lemma 9.1, the shape of $\widetilde{\mathsf{U}}$ is v/w. This construction is illustrated in Figure 9.11 (b).

It remains to show that the fillings of $\widetilde{\mathsf{U}}$ and U are equal. By Lemma 9.1, the Demazure product of $\mathsf{read}(\widetilde{\mathsf{U}})$ is π. But since $\mathsf{read}(\widetilde{\mathsf{U}})$ has length $\binom{k}{2} = \mathsf{len}(\pi)$, $\mathsf{read}(\widetilde{\mathsf{U}})$ is then a reduced word for π. Since both $\mathsf{read}(\widetilde{\mathsf{U}})$ and $\mathsf{read}(\mathsf{U})$ are reduced words for the Grassmannian permutation π, and both $\widetilde{\mathsf{U}}$ and U have antinormal shapes, Lemma 9.2 implies $\widetilde{\mathsf{U}} = \mathsf{U}$. □

By combining Theorem 9.16 and Proposition 9.9, we conclude Theorem 9.1 (B).

9.10.2 (H) and (I)

For row (H), let the tableau U and its rectification be as illustrated on the left in Figure 9.15. It is a straightforward but lengthy calculation to verify that $R_{\mathsf{w},\mathsf{u}}^{\mathsf{v}} = \{\mathsf{U}\}$ and that $R_{\mathsf{x},\mathsf{u}}^{\mathsf{v}} = \emptyset$ for $\mathsf{x} \neq \mathsf{w}$. We performed this calculation via computer, explicitly rectifying all applicable tableaux.

For row (I), the shape x must be an order filter of size $\lfloor \frac{m-2}{2} \rfloor$ in v. There is a unique such order filter—the shape v/w—and (since v/w is a chain) it has a unique filling that rectifies to $T_{\mathsf{u}}^{\mathsf{min}}$. We refer the reader the

the illustration on the right in Figure 9.15.

By Theorem 9.16, these observations prove Theorem 9.1 (H) and (I).

9.10.3 (A) Staircases and Shifted Staircases

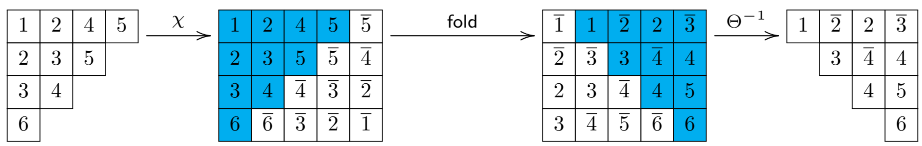


Figure 9.16: For $n = 4$, the leftmost two diagrams show how to embed a tableau in $\mathsf{IT}^{[k]}(\Phi^+(A_n))$ as a tableau in $\mathsf{IT}^{[k]\sqcup[\overline{k}]}(\Lambda_{\mathsf{Gr}(n,2n)})$. The middle two show the result after performing fold. The rightmost two diagrams show how to extract a tableau in $\overline{\mathsf{IT}}^{[k]}(\Lambda_{\mathsf{LG}(n,2n)})$ from a tableau in $\mathsf{IT}^{[\overline{k}]\sqcup[k]}(\Lambda_{\mathsf{Gr}(n,2n)})$.

We begin by recalling the modified definitions of tableaux for $\Lambda_{\mathsf{LG}(n,2n)}$. Let $N := \frac{n(n+1)}{2}$ and write $\overline{\mathsf{SYT}}(\Lambda_{\mathsf{LG}(n,2n)})$ for the set of injections from $\Lambda_{\mathsf{LG}(n,2n)}$ to the alphabet $[\overline{N}] \sqcup [N]$ with no barred letter on the (shifted) diagonal and with exactly one of i or $\overline{i}$ in the image. Similarly, let $\overline{\mathsf{IT}}^{[k]}(\Lambda_{\mathsf{LG}(n,2n)})$ denote the set of increasing tableaux of shape $\Lambda_{\mathsf{LG}(n,2n)}$ on the alphabet $[\overline{k}] \sqcup [k]$.

As in Proposition 9.5—and illustrated in the leftmost two diagrams of Figure 9.16—we can embed an element of $\mathsf{IT}^{[k]}(\Phi^+(A_n))$ on the standard alphabet $[k]$ as an element of $\mathsf{IT}^{[k]\sqcup[\underline{k}]}(\mathsf{Gr}(n,2n))$ by reflecting across the diagonal anry—extending the standard alphabet to $[k] \sqcup [\underline{k}]$. In [Pu14], K. Purbhoo used this embedding to give the bijection (see Section 9.8.2)

$$\mathsf{fold} : \mathsf{SYT}(\Phi^+(A_n)) \to \overline{\mathsf{SYT}}(\Lambda_{\mathsf{LG}(n,2n)}).$$

We conjecture that the increasing modification of K. Purbhoo's folding is also a bijection between the corresponding sets of increasing tableaux.

Conjecture 9.1. *The map* fold *is a bijection*

$$\mathsf{fold} : \mathsf{IT}^{[k]}\left(\Phi^+(A_n)\right) \to \overline{\mathsf{IT}}^{[k]}\left(\Lambda_{\mathsf{LG}(n,2n)}\right).$$

Standardizing the alphabet $[\overline{k}] \sqcup [k]$ by replacing $i \mapsto 2i$ and $\overline{i} \mapsto 2i-1$, Proposition 9.6 gives a bijection (subtracting $j+1$ from the jth rank)

$$\overline{\mathsf{IT}}^{[k]}\left(\Lambda_{\mathsf{LG}(n,2n)}\right) \simeq \overline{\mathsf{PP}}^{[2p]}(\Lambda_{\mathsf{LG}(n,2n)}),$$

where $2k = 2p + n$ and $\overline{\mathsf{PP}}^{[2p]}(\Lambda_{\mathsf{LG}(n,2n)})$ is the subset of $\mathsf{PP}^{[2p]}(\Lambda_{\mathsf{LG}(n,2n)})$ with only even heights on the diagonal. So Conjecture 9.1 gives an explicit conjectural bijection between

$$\overline{\mathsf{PP}}^{[2p]}\left(\Lambda_{\mathsf{LG}(n,2n)}\right) \simeq \mathsf{PP}^{[p]}\left(\Phi^+(A_n)\right).$$

This bijection is illustrated for $n = 2$ and $p = 1$ in Example 9.4.

Remark 9.4. The rightmost two diagrams of Figure 9.16 show how to extract an element of $\overline{\mathsf{IT}}^{[k]}(\Lambda_{\mathsf{LG}(n,2n)})$ from an element of $\mathsf{IT}^{[\overline{k}]\sqcup[k]}(\Lambda_{\mathsf{Gr}(n,2n))}$, using the embedding in Proposition 9.4. We do not currently understand (even conjecturally) how to recover the forgotten half of $\mathsf{IT}^{[\overline{k}]\sqcup[k]}(\Lambda_{\mathsf{Gr}(n,2n))}$ from the tableau in $\overline{\mathsf{IT}}^{[k]}(\Lambda_{\mathsf{LG}(n,2n)})$. In the standard case, the forgotten tableau is recovered by sending barred elements to unbarred elements, and vice-versa; this does not generalize in the obvious way to the increasing setting.

9.11 Future Work

In [ThYo05], H. Thomas and A. Yong characterize all multiplicity-free products of Schubert classes in $\mathsf{Gr}(k,n)$. It is natural to wish to extend this to all minuscule flag varieties; except for the single remaining infinite family (up to isomorphism), this is a finite check.

Problem 9.2. *Classify all multiplicity-free products of cohomological Schubert classes in all minuscule flag varieties.*

As pointed out Remark 9.3, multiplicity-free products in cohomology are not necessarily multiplicity-free in K-theory. It would be interesting to apply A. Knutson's Theorem 9.9 to classify the latter products. To our knowledge, this is open even in the Grassmannian case (although additional combinatorial tools are available in that case, e.g. [Sn09]).

Problem 9.3. *Classify all multiplicity-free products of K-theoretic Schubert classes in all minuscule flag varieties.*

Given any multiplicity-free product from Problem 9.2 or Problem 9.3, Theorem 9.16 then gives a combinatorial identity. We have *not* recorded in this chapter all such identities—or even all such identities that lead to a pair of doppelgängers. For example, [ThYo09a, Figure 9] specifies an embedding of $\mathsf{OG}(5,10)$ in the minuscule poset of type E_6 and an embedding of the minuscule poset of type E_6 inside $G_\omega(\mathbb{O}^3,\mathbb{O}^6)$ that lead to the (not especially exciting) doppelgängers of Figure 9.17.

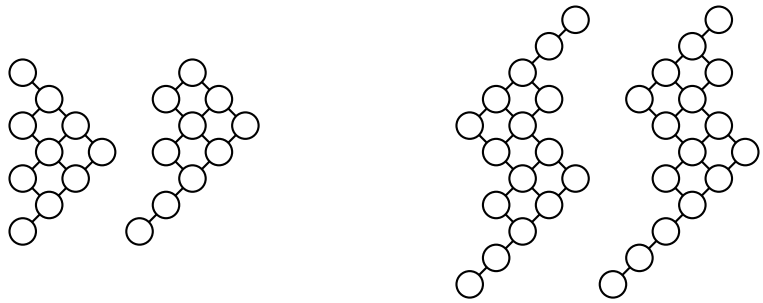


Figure 9.17: Two other pairs of doppelgängers (cf. [ThYo09a, Figure 9]).

More generally, it is possible to derive poset identities (relating the number of standard or increasing fillings) by comparing Richardson varieties. V. Reiner, K. Shaw, and S. van Willigenburg have partial results in this direction for Grassmannians [RSvW07].[5]

Problem 9.4. *When do the (cohomological or K-theoretic) classes of two Richardson varieties have the same expansion into Schubert classes?*

[5]We thank F. Bergeron for pointing out that [RSvW07] was of the same spirit as the problems we have been considering.

References

[AnGrMi11] D. Anderson, S. Griffeth and E. Miller, *Positivity and Kleiman transversality in equivariant K-theory of homogeneous spaces*, J. Eur. Math. Soc. **13** (2011), 57–84.

[AnAsRo01] G.E. Andrews, R. Askey, and R. Roy, *Special functions*, Encyclopedia Math. Appl., Cambridge University Press, 2001.

[Ar09] D. Armstrong, *Generalized noncrossing partitions and combinatorics of Coxeter groups*, Mem. Amer. Math. Soc. **202** (2009), no. 949, x+159 pp.

[ArStTh13] D. Armstrong, C. Stump, and H. Thomas, *A uniform bijection between nonnesting and noncrossing partitions*, Trans. Amer. Math. Soc. **365** (2013), 4121–4151.

[AsDu13] I. Assem and G. Dupont, *Friezes over semirings and tropicalisations*, preprint, 2013. Available at `http://prospero.dmi.usherb.ca/ibrahim/Publications.aspx`

[AtRe04] C.A. Athanasiadis and V. Reiner, *Noncrossing partitions for the group D_n*, SIAM J. Discrete Math. **18** (2004), 397–417.

[AtSa12] C.A. Athanasiadis and C. Savvidou, *The local h-vector of the cluster subdivision of a simplex*, Sém. Lothar. Combin. **66** (2012).

[BaRe15] E. Barnard and N. Reading, *Coxeter-biCatalan combinatorics*, Discrete Math. Theor. Comput. Sci. Proc. **FPSAC'15** (2015), 157–168. FPSAC 2015, Daejeon, South Korea.

[BeKn72] E. Bender and D. Knuth, *Enumeration of plane partitions*, J. Combin. Theory Ser. A **13** (1972), 40–54.

[Bi98] S. Billey, *Transition equations for isotropic flag manifolds*, Discrete Math. **193** (1998), 69–84.

[BiHa95] S. Billey and M. Haiman, *Schubert polynomials for the classical groups*, J. Amer. Math. Soc. **8** (1995), 443–482.

[BHRY14] S. Billey, Z. Hamaker, A. Roberts, and B. Young, *Coxeter-Knuth graphs and a signed Little map for type B reduced words*, Electron. J. Combin. **21** (2014), 1–39.

[BiLa00] S. Billey, V. Lakshmibai, *Singular loci of Schubert varieties*, Progr. Math., vol. 182, Birkhäuser, 2000.

[BlPeSa16] J. Bloom, O. Pechenik and D. Saracino, *Proofs and generalizations of a homomesy conjecture of Propp and Roby*, Discrete Math. **339** (2016), 194–206.

[Br99] D.M. Bressoud, *Proofs and confirmations: The story of the alternating sign matrix conjecture*, MAA Spectrum, Mathematical Association of America, Washington, DC, 1999.

[Br02] M. Brion, *Positivity in the Grothendieck group of complex flag varieties*, J. Algebra **258** (2002), 137–159.

[Br05] M. Brion, *Lectures on the geometry of flag varieties*, Topics in cohomological studies of algebraic varieties, 33–85, Trends Math., Birkhäuser, Basel, 2005.

[BrSc74] A. Brouwer and A. Schrijver, *On the period of an operator, defined on antichains*, Math. Centrum report ZW **24/74** (1974).

[Bu02] A. Buch, *A Littlewood-Richardson rule for the K-theory of Grassmannians*, Acta Math. **189** (2002), 37–78.

[Bu05] A. Buch, *Combinatorial K-theory*, Topics in cohomological studies of algebraic varieties, 87–103, Trends Math., Birkhäuser, Basel, 2005.

[Bu15] A. Buch, *Mutations of puzzles and equivariant cohomology of two-step flag varieties*, Ann. of Math. **182** (2015), 173–220.

[BKPT14] A. Buch, A. Kresch, K. Purbhoo and H. Tamvakis, *The puzzle conjecture for the cohomology of two-step flag manifolds*, J. Algebraic Combin., to appear, 2014. arXiv:1401.1725

[BKSTY08] A. Buch, A. Kresch, M. Shimozono, H. Tamvakis and A. Yong, *Stable Grothendieck polynomials and K-theoretic factor sequences*, Math. Ann. **340** (2008), 359–382.

[BuKrTa03] A. Buch, A. Kresch and H. Tamvakis, *Gromov-Witten invariants on Grassmannians*, J. Amer. Math. Soc. **16** (2003), 901–915.

[BuMi11] A. Buch and L. Mihalcea, *Quantum K-theory of Grassmannians*, Duke Math. J. **156** (2011), no. 3, 501–538.

[BuRa12] A. Buch and V. Ravikumar, *Pieri rules for the K-theory of cominuscule Grassmannians*, J. Reine Angew. Math. (Crelle's J.) **668** (2012), 109–132.

[BuSa13] A. Buch and M. Samuel, *K-theory of minuscule varieties*, J. Reine Angew. Math. (Crelle's J.), to appear, 2014. arXiv:1306.5419

[CaFo95] P. Cameron and D. Fon-der-Flaass, *Orbits of antichains revisited*, Eur. J. Combin. **16** (1995), 545–554.

[CeLaSt14] C. Ceballos, J.-P. Labbé, and C. Stump, *Subword complexes, cluster complexes, and generalized multi-associahedra*, J. Algebraic Combin. **39** (2014), 17–51.

[ClThYo14] E. Clifford, H. Thomas and A. Yong, *K-theoretic Schubert calculus for $OG(n, 2n+1)$ and jeu de taquin for shifted increasing tableaux*, J. Reine Angew. Math. (Crelle's J.) **690** (2014), 51–63.

[CoCo73] J.H. Conway and H.S.M. Coxeter, *Triangulated polygons and frieze patterns*, Math. Gaz. **57** (1973), 87–94.

[Co09] I. Coşkun, *A Littlewood-Richardson rule for two-step flag varieties*, Invent. Math. **176** (2009), 325–395.

[CoVa05] I. Coşkun and R. Vakil, *Geometric positivity in the cohomology of homogeneous spaces and generalized Schubert calculus*, in "Algebraic Geometry — Seattle 2005" Part 1, 77–124, Proc. Sympos. Pure Math., 80, Amer. Math. Soc., Providence, RI, 2009.

[CuSt15] M. Cuntz and C. Stump, *On root posets for noncrystallographic root systems*, Math. Comp. **84** (2015), 485–503.

[DiPeSt15] K. Dilks, O. Pechenik and J. Striker, *Resonance in orbits of plane partitions and increasing tableaux*, preprint, 2015. arXiv:1512.00365

[EdGr87] P. Edelman and C. Greene, *Balanced tableaux*, Adv. Math. **63** (1987), 42–99.

[EiPr14] D. Einstein and J. Propp, *Piecewise-linear and birational toggling*, Discrete Math. Theor. Comput. Sci. Proc. **AT** (2014), 513–524. FPSAC 2014, Chicago, IL, USA.

[El15] S. Elizalde, *Bijections for pairs of non-crossing lattice paths and walks in the plane*, Eur. J. Combin. **49** (2015), 25–41.

[EuFu08] S.-P. Eu and T.-S. Fu, *The cyclic sieving phenomenon for faces of generalized cluster complexes*, Adv. in Appl. Math. **40** (2008), 350–376.

[FoKi94] S. Fomin and A. N. Kirillov, *Grothendieck polynomials and the Yang-Baxter equation*, Proc. 6th Intern. Conf. on Formal Power Series and Algebraic Combinatorics, DIMACS, 1994, 183–190.

[FoRe05] S. Fomin and N. Reading, *Generalized cluster complexes and Coxeter combinatorics*, Int. Math. Res. Not. **2005** (2005), no. 44, 2709–2757.

[Fo93] D. Fon-der-Flaass, *Orbits of antichains in ranked posets*, Eur. J. Combin. **14** (1993), 17–22.

[FoKa14] B. Fontaine and J. Kamnitzer, *Cyclic sieving, rotation, and geometric representation theory*, Selecta Math. **20** (2014), 609–625.

[FuKr96] M. Fulmek and C. Krattenthaler, *A bijection between Proctor's and Sundaram's odd orthogonal tableaux*, Discrete Math. **161** (1996), 101–120.

[FuKr97] M. Fulmek and C. Krattenthaler, *Lattice path proofs for determinantal formulas for symplectic and orthogonal characters*, J. Combin. Theory Ser. A **77** (1997), 3–50.

[Fu97] W. Fulton, *Young tableaux: With applications to representation theory and geometry*, London Mathematical Society Student Texts, **35**, Cambridge University Press, Cambridge, 1997.

[FuLa94] W. Fulton and A. Lascoux, *A Pieri formula in the Grothendieck ring of a flag bundle*, Duke Math. J. **76** (1994), no. 3, 711–729.

[GMPPRST16] C. Gaetz, M. Mastrianni, R. Patrias, H. Peck, C. Robichaux, D. Schwein and K. Tam, *K-Knuth equivalence for increasing tableaux*, Electron. J. Combin. **23** (2016), 1–37.

[GiLe16] M. Gillespie and J. Levinson, *Monodromy and K-theory of Schubert curves via generalized jeu de taquin*, preprint, 2016. arXiv:1602.02375

[Gr15] J. Grabowski, *Graded cluster algebras*, J. Algebraic Combin. **42** (2015), 1111–1134.

[Gr01] W. Graham, *Positivity in equivariant Schubert calculus*, Duke Math. J. **109** (2001), 599–614.

[GrKu08] W. Graham and S. Kumar, *On positivity in T-equivariant K-theory of flag varieties*, Int. Math. Res. Not. **2008** (2008), 1–43.

[Gr13] R. Green, *Combinatorics of minuscule representations*, vol. 199, Cambridge University Press, Cambridge, 2013.

[GrRo15] D. Grinberg and T. Roby, *Iterative properties of birational rowmotion II: rectangles and triangles*, Electron. J. Combin. **22** (2015), 1–49.

[GrRo16] D. Grinberg and T. Roby, *Iterative properties of birational rowmotion I: generalities and skeletal posets*, Electron. J. Combin. **23** (2016), 1–40.

[Gu13] L. Guo, *On tropical friezes associated with Dynkin diagrams*, Int. Math. Res. Not. **2013** (2013), 4243–4284.

[HLMvW11] J. Haglund, K. Luoto, S. Mason and S. van Willigenburg. *Refinements of the Littlewood-Richardson rule*, Trans. Amer. Math. Soc. **363** (2011), 1665–1686.

[Ha92] M. Haiman, *Dual equivalence with applications, including a conjecture of Proctor*, Discrete Math. **99** (1992), 79–113.

[HKPWZZ15] Z. Hamaker, A. Keilthy, R. Patrias, L. Webster, Y. Zhang and S. Zhou, *Shifted Hecke insertion and the K-theory of* $\mathrm{OG}(n, 2n+1)$, preprint, 2015. arXiv:1510.08972

[HPPW16] Z. Hamaker, R. Patrias, O. Pechenik and N. Williams, *Doppelgängers: Bijections of plane partitions*, preprint, 2016. arXiv:1602.05535

[HaYo13] Z. Hamaker and B. Young, *Relating Edelman-Greene insertion to the Little map*, J. Algebraic Combin. **40** (2014), 693–710.

[Hi82] H. Hiller, *Geometry of Coxeter groups*, vol. 54, Pitman Publishing, 1982.

[HoHu92] P. Hoffman and J. Humphreys, *Projective representations of the symmetric group: Q-functions and shifted tableaux*, Oxford Mathematical Monographs, Oxford University Press, New York, 1992.

[Hu92] J. Humphreys, *Reflection groups and Coxeter groups*, vol. 29, Cambridge University Press, 1992.

[IkNa13] T. Ikeda and H. Naruse, *K-theoretic analogue of factorial Schur P- and Q- functions*, Adv. Math. **243** (2013), 22–66.

[IkNaNu11] T. Ikeda, H. Naruse and Y. Numata, *Bumping algorithm for set-valued shifted tableaux*, Discrete Math. Theor. Comput. Sci. Proc. **AO** (2011), 527–538. FPSAC 2011, Reykjavik, Iceland.

[IkSh14] T. Ikeda and T. Shimazaki, *A proof of K-theoretic Littlewood-Richardson rules by Bender-Knuth-type involutions*, Math. Res. Lett. **21** (2014), 333–339.

[JoPrSh98] W. Jockusch, J. Propp and P. Shor, *Random domino tilings and the arctic circle theorem*, preprint, 1998. math/9801068

[KaMo16+] R. Kaliszewski and J. Morse. *Airports for equivariant K-theory of Grassmannians*, in preparation, 2016.

[Ki75] R.C. King, *Branching rules for classical Lie groups using tensor and spinor methods*, J. Phys. A **8** (1975), 429–449.

[KiBe96] A.N. Kirillov and A.D. Berenstein, *Groups generated by involutions, Gelfand-Tsetlin patterns and combinatorics of Young tableaux*, St. Petersburg Math. J. **7** (1996), 77–127.

[Kn70] D. Knuth, *Permutations, matrices, and generalized Young tableaux*, Pacific J. Math. **34** (1970), 709–727.

[Kn09] A. Knutson, *Frobenius splitting and Möbius inversion*, preprint, 2009. arXiv:0902.1930

[Kn10] A. Knutson, *Puzzles, positroid varieties, and equivariant K-theory of Grassmannians*, preprint, 2010. arXiv:1008.4302

[Kn14] A. Knutson, *Schubert calculus and shifting of interval positroid varieties*, preprint, 2014. arXiv:1408.1261

[KnMiYo09] A. Knutson, E. Miller and A. Yong, *Gröbner geometry of vertex decompositions and of flagged tableaux*, J. Reine Angew. Math. (Crelle's J.) **630** (2009), 1–31.

[KnPu11] A. Knutson and K. Purbhoo, *Product and puzzle formulae for $GL(n)$ Belkale-Kumar coefficients*, Electron. J. Combin. **18**(1) (2011), P76.

[KnTa03] A. Knutson and T. Tao, *Puzzles and (equivariant) cohomology of Grassmannians*, Duke Math. J. **119** (2003), no. 2, 221–260.

[KnTaWo04] A. Knutson, T. Tao and C. Woodward, *The honeycomb model of $GL_n(\mathbb{C})$ tensor products II: puzzles determine facets of the Littlewood-Richardson cone*, J. Amer. Math. Soc. **17** (2004), 19–48.

[Ko59] B. Kostant, *The principal three-dimensional subgroup and the Betti numbers of a complex simple lie group*, Amer. J. Math. **81** (1959), 973–1032.

[KoKu90] B. Kostant and S. Kumar, *T-equivariant K-theory of generalized flag varieties*, J. Differential Geom. **32** (1990), 549–603.

[Kra89] W. Kraśkiewicz, *Reduced decompositions in hyperoctahedral groups*, C. R. Acad. Sci. Paris Sér. I Math. **309** (1989), no. 16, 903–907.

[Kre05] V. Kreiman, *Schubert classes in the equivariant K-theory and equivariant cohomology of the Grassmannian*, preprint, 2005. arXiv:math/0512204

[Kre10] V. Kreiman, *Equivariant Littlewood-Richardson skew tableaux*, Trans. Amer. Math. Soc. **362** (2010), 2589–2617.

[Ku96] G. Kuperberg. *Another proof of the alternating sign matrix conjecture*, Int. Math. Res. Not. **1996** (1996), 139–150.

[Lam95] T.-K. Lam, *B and D analogues of stable Schubert polynomials and related insertion algorithms*, Ph.D. thesis, Massachusetts Institute of Technology, 1995.

[LaPy07] T. Lam and P. Pylyavskyy, *Combinatorial Hopf algebras and K-homology of Grassmannians*, Int. Math. Res. Not. **2007** (2007), 1–48.

[Las95] A. Lascoux, *Polynômes de Schubert: Une approche historique*, Discrete Math. **139** (1995), 303–317.

[LaSc82] A. Lascoux and M.-P. Schützenberger, *Structure de Hopf de l'anneau de cohomologie et de l'anneau de Grothendieck d'une variété de drapeaux*, C. R. Acad. Sci. Paris, **295** (1982), 629–633.

[Le00] C. Lenart, *Combinatorial aspects of the K-theory of Grassmannians*, Ann. Comb. **4** (2000), 67–82.

[LePo07] C. Lenart and A. Postnikov, *Affine Weyl groups in K-theory and representation theory*, Int. Math. Res. Not. **2007** (2007), 1–65.

[Le47] L. Lesieur, *Les problemes d'intersection sur une variété de Grassmann*, C. R. Acad. Sci. Paris **225** (1947), 916–917.

[LiMoSh16] H. Li, J. Morse and P. Shields, *Structure constants for K-theory of Grassmannians revisited*, preprint, 2016. arXiv:1601.04509

[Li50] D.E. Littlewood, *The theory of group characters and matrix representations of groups*, vol. 357, American Mathematical Soc., 1950.

[Li03] D. Little, *Combinatorial aspects of the Lascoux-Schützenberger tree*, Adv. Math. **174** (2003), 236–253.

[LiRi34] D. E. Littlewood and A. R. Richardson, *Group characters and algebra*, Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **233** (1934), 99–141.

[Mac95] I.G. Macdonald, *Symmetric functions and Hall polynomials*, 2nd ed., Oxford Mathematical Monographs, Oxford University Press, 1995.

[Mac98] I.G. Macdonald, *Symmetric functions and orthogonal polynomials*, Dean Jacqueline B. Lewis Memorial Lectures presented at Rutgers University, New Brunswick, NJ, University Lecture Series, American Mathematical Soc., 1998.

[Mac1915] P.A. MacMahon, *Combinatory analysis*, 2 vols, Cambridge University Press, 1915-16.

[Man01] L. Manivel, *Symmetric functions, Schubert polynomials and degeneracy loci*, trans. by J. Swallow, Amer. Math. Soc., 2001.

[Mi15] A. Miller, *Foulkes characters for complex reflection groups*, Proc. Amer. Math. Soc. **143** (2015), 3281–3293.

[MiRoRu83] W. Mills, D. Robbins, and H. Rumsey, Jr., *Alternating sign matrices and descending plane partitions*, J. Combin. Theory Ser. A **34** (1983), no. 3, 340–359.

[MoSa99] A. Molev and B. Sagan, *A Littlewood-Richardson rule for factorial Schur functions*, Trans. Amer. Math. Soc. **351** (1999), no. 11, 4429–4443.

[Mo16+] C. Monical, *Combinatorial K-theoretic Demazure atoms*, in preparation, 2016.

[MoOvTa12] S. Morier-Genoud, V. Ovsienko, S. Tabachnikov, *2-frieze patterns and the cluster structure of the space of polygons*, Ann. Inst. Fourier **62** (2012), 937–987.

[Pa09] D. Panyushev, *On orbits of antichains of positive roots*, Eur. J. Combin. **30** (2009), 586–594.

[Pa15] R. Patrias, *Antipode formulas for combinatorial Hopf algebras*, preprint, 2015. arXiv:1501.00710

[PaPy14] R. Patrias and P. Pylyavskyy, *Dual filtered graphs*, preprint, 2014. arXiv:1410.7683.

[PaPy16] R. Patrias and P. Pylyavskyy, *Combinatorics of K-theory via a K-theoretic Poirier–Reutenauer bialgebra*, Discrete Math. **339** (2016), 1095–1115.

[Pe14] O. Pechenik, *Cyclic sieving of increasing tableaux and small Schröder paths*, J. Combin. Theory Ser. A, **125** (2014), 357–378.

[PeYo15a] O. Pechenik and A. Yong, *Genomic tableaux and combinatorial K-theory*, Discrete Math. Theor. Comput. Sci. Proc. **FPSAC'15** (2015), 37–48. FPSAC 2015, Daejeon, South Korea.

[PeYo15b] O. Pechenik and A. Yong, *Equivariant K-theory of Grassmannians*, preprint, 2015. arXiv:1506.01992

[PeYo15c] O. Pechenik and A. Yong, *Equivariant K-theory of Grassmannians II: The Knutson-Vakil conjecture*, preprint, 2015. arXiv:1508.00446

[PeYo16] O. Pechenik and A. Yong, *Genomic tableaux*, preprint, 2016. arXiv:1603.08490

[PePyRh09] T.K. Petersen, P. Pylyavskyy and B. Rhoades, *Promotion and cyclic sieving via webs*, J. Algebraic Combin. **30** (2009), 19–41.

[Pr89] P. Pragacz, *Algebro-geometric applications of Schur S- and Q-polynomials*, Topics in invariant theory (Paris, 1989/1990), 130–191, Lecture Notes in Math., Springer, Berlin, 1991.

[PrStVi14] T. Pressey, A. Stokke and T. Visentin, *Increasing tableaux and generalized Narayana numbers*, Ann. Comb., accepted, 2015. arXiv:1409.2841

[Pr83] R. Proctor, *Shifted plane partitions of trapezoidal shape*, Proc. Amer. Math. Soc. **89** (1983), 553–559.

[Pr84a] R. Proctor, *Bruhat lattices, plane partition generating functions, and minuscule representations*, Eur. J. Combin. **5** (1984), 331–350.

[Pr84b] R. Proctor, *The geometry of Coxeter groups, by H. Hiller (Book review)*, Bull. Amer. Math. Soc. (N.S.) **10** (1984), 142–150.

[Pr90] R. Proctor, *New symmetric plane partition identities from invariant theory work of De Concini and Procesi*, Eur. J. Combin. **11** (1990), 289–300.

[Pr01] J. Propp, *The many faces of alternating-sign matrices*, Discrete Math. Theor. Comput. Sci. Proc. **AA** (2001), 43–58. FPSAC 2001, Tempe, AZ, USA.

[Pr05] J. Propp, *The combinatorics of frieze patterns and Markoff numbers*, preprint, 2005. arXiv:math/0511633

[PrRo13a] J. Propp and T. Roby, *Homomesy in products of two chains*, Discrete Math. Theor. Comput. Sci. Proc. **AS** (2013) 975–986. FPSAC 2013, Paris, France.

[PrRo13b] J. Propp and T. Roby, *Combinatorial actions & homomesic orbit averages*, 23 May 2013, talk at Dartmouth College Mathematics Colloquium. Slides available at `http://www.math.uconn.edu/ troby/homomesyActions2013DartmouthColloq.pdf`

[PrRo15] J. Propp and T. Roby, *Homomesy in products of two chains*, Electron. J. Combin. **22** (2015), 1–29.

[Pu08] K. Purbhoo, *Puzzles, tableaux, and mosaics*, J. Algebraic Combin. **28** (2008), 461–480.

[Pu13] K. Purbhoo, *Wronskians, cyclic group actions, and ribbon tableaux*, Trans. Amer. Math. Soc. **365** (2013), 1977–2030.

[Pu14] K. Purbhoo, *A marvellous embedding of the Lagrangian Grassmannian*, preprint, 2014. arXiv:1403.0984

[Rea08] N. Reading, *Chains in the noncrossing partition lattice*, SIAM J. Discrete Math. **22** (2008), no. 3, 875–886.

[Rei97] V. Reiner, *Non-crossing partitions for classical reflection groups*, Discrete Math.**177** (1997), 195–222.

[RSvW07] V. Reiner, K. Shaw, and S. van Willigenburg, *Coincidences among skew Schur functions*, Adv. Math. **216** (2007), 118–152.

[ReStWh04] V. Reiner, D. Stanton and D. White, *The cyclic sieving phenomenon*, J. Combin. Theory Ser. A **108** (2004), 17–50.

[ReTeYo16] V. Reiner, B. Tenner and A. Yong, *Poset edge densities, nearly reduced words and barely set-valued tableaux*, preprint, 2016. arXiv:1603.09589

[Rh10] B. Rhoades, *Cyclic sieving, promotion, and representation theory*, J. Combin. Theory Ser. A **117** (2010), 38–76.

[Rh15] B. Rhoades, *A skein action of the symmetric group on noncrossing partitions*, preprint, 2015. arXiv:1501.04680

[RoRu86] D. Robbins and H. Rumsey, Jr., *Determinants and alternating sign matrices*, Adv. Math. **62** (1986), 169–184.

[RoYo15] C. Ross and A. Yong, *Combinatorial rules for three bases of polynomials*, Sém. Lothar. Combin. **74** (2015), 1–11.

[RuSh13] D. Rush and X. Shi, *On orbits of order ideals of minuscule posets*, J. Algebraic Combin. **37** (2013), 545–569.

[RuWa15] D. Rush and K. Wang, *On orbits of order ideals of minuscule posets II: Homomesy*, preprint, 2015. arXiv:1509.08047

[Sa87] B. Sagan, *Shifted tableaux, Schur Q-functions, and a conjecture of R. Stanley*, J. Combin. Theory Ser. A **45** (1987), 62–103.

[Sa13] E. Sawin, personal communication to Jessica Striker (2013).

[Sc72] M.-P. Schützenberger, *Promotion des morphismes d'ensembles ordonnés*, Discrete Math. **2** (1972), 73–94.

[Sc77] M.-P. Schützenberger, *Combinatoire et représentation du groupe symétrique* (Actes Table Ronde CNRS, Univ. Louis-Pasteur Strasbourg, Strasbourg, 1976), pp. 59–113. Lecture Notes in Math., Vol. 579, Springer, Berlin, 1977.

[Sh99] J. Sheats, *A symplectic jeu de taquin bijection between the tableaux of King and of De Concini*, Trans. Amer. Math. Soc. **351** (1999), no. 9, 3569–3607.

[Sn09] M. Snider, *A combinatorial approach to multiplicity-free Richardson subvarieties of the Grassmannian*, preprint, 2009. arXiv:0902.1931

[Sta72] R. Stanley, *Ordered structures and partitions*, American Mathematical Soc., 1972.

[Sta84] R. Stanley, *On the number of reduced decompositions of elements of Coxeter groups*, Eur. J. Combin. **5** (1984), 359–372.

[Sta86] R. Stanley, *Symmetries of plane partitions*, J. Combin. Theory Ser. A **43** (1986), 103–113.

[Sta96] R. Stanley, *Polygon dissections and standard Young tableaux*, J. Combin. Theory Ser. A **76** (1996), 175–177.

[Sta99] R. Stanley, *Enumerative Combinatorics*, vol. 2, Cambridge University Press, 1999, with Appendix 1 by S. Fomin.

[Sta09] R. Stanley, *Promotion and evacuation*, Electron. J. Combin. **16** (2009), 1–24.

[Ste86] J. Stembridge, *Trapezoidal chains and antichains*, Eur. J. Combin. **7** (1986), no. 4, 377–387.

[Ste89] J. Stembridge, *Shifted tableaux and the projective representations of symmetric groups*, Adv. Math. **74** (1989), 87–134.

[Ste94a] J. Stembridge, *Some hidden relations involving the ten symmetry classes of plane partitions*, J. Combin. Theory Ser. A **68** (1994), 372–409.

[Ste94b] J. Stembridge, *On minuscule representations, plane partitions and involutions in complex Lie groups*, Duke Math. J. **73** (1994), 469–490.

[Ste95] J. Stembridge, *Canonical bases and self-evacuating tableaux*, Duke Math. J. **82** (1995), 473–754.

[Ste96] J. Stembridge, *On the fully commutative elements of Coxeter groups*, J. Algebraic Combin. **5** (1996), 353–385.

[Ste98] J. Stembridge, *The partial order of dominant weights*, Adv. Math. **136** (1998), 340–364.

[Str15] J. Striker, *The toggle group, homomesy, and the Razumov-Stroganov correspondence*, Electron. J. Combin. **22** (2015), no. 2.

[StWi12] J. Striker and N. Williams, *Promotion and rowmotion*, Eur. J. Combin. **33** (2012), 1919–1942.

[ThYo05] H. Thomas and A. Yong, *Multiplicity-free Schubert calculus*, Canad. Math. Bull. **51** (2005), 171–186.

[ThYo08] H. Thomas and A. Yong, *An S_3-symmetric Littlewood-Richardson rule*, Math. Res. Lett. **15** (2008), 1027–1037.

[ThYo09a] H. Thomas and A. Yong, *A combinatorial rule for (co)minuscule Schubert calculus*, Adv. Math. **222** (2009), no. 2, 596–620.

[ThYo09b] H. Thomas and A. Yong, *A jeu de taquin theory for increasing tableaux, with applications to K-theoretic Schubert calculus*, Algebra Number Theory **3** (2009), 121–148.

[ThYo11] H. Thomas and A. Yong, *Longest increasing subsequences, Plancherel-type measure and the Hecke insertion algorithm*, Adv. in Appl. Math. **46** (2011), 610–642.

[ThYo13] H. Thomas and A. Yong, *Equivariant Schubert calculus and jeu de taquin*, Ann. Inst. Fourier (Grenoble), to appear, 2013. arXiv:1207.3209

[ThYo15] H. Thomas and A. Yong, *Cominuscule tableau combinatorics*, Adv. Stud. Pure Math. (2015) (in press). (from 5th MSJ-SI “Schubert calculus”) arXiv:math/0701215v2

[Wi00] B. Wieland, *A large dihedral symmetry of the set of alternating sign matrices*, Electron. J. Combin. **7** (2000).

[Wi06] M. Willems, *K-théorie équivariante des tours de Bott. Application à la structure multiplicative de la K-théorie équivariante des variétés de drapeaux*, Duke Math. J. **132** (2006), no. 2, 271–309.

[WoYo12] A. Woo and A. Yong, *A Gröbner basis for Kazhdan-Lusztig ideals*, Amer. J. Math. **134** (2012), 1089–1137.

[Va06] R. Vakil, *A geometric Littlewood-Richardson rule*, Ann. of Math. **164** (2006), 371–422.

[vL01] M. van Leeuwen, *Some bijective correspondences involving domino tableaux*, Electron. J. Comb. **7** (2000), 1–25.

[Vi86] G. Viennot, *Heaps of pieces, I: Basic definitions and combinatorial lemmas*, Combinatoire énumérative, Springer, 1986, pp. 321–350.

[Wh07] D. White, personal communication to B. Rhoades (see [Rh10]), 2007.

[Wh10] D. White, *Proof that promotion in a SYT of shape a^b has order ab*, unpublished, 2010.

[Wi13] N. Williams, *Cataland*, Ph.D. thesis, University of Minnesota, 2013.

[Wo84] D. Worley, *A theory of shifted young tableaux*, Ph.D. thesis, Massachusetts Institute of Technology, 1984.

[Ze96] D. Zeilberger, *Proof of the alternating sign matrix conjecture*, Electron. J. Combin. **3** (1996).