TOPICS IN QUANTUM FIELD THEORY AND HOLOGRAPHY

BY

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DISSERTATION

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Abstract

In the first part of this thesis, we will study free fermions as models for topological insulators, on gravitational backgrounds which include both torsion and curvature, in $d = 2 + 1$ and $d = 4 + 1$ dimensions. We compute the parity-odd effective actions for these systems, and use these effective actions to deduce the structure of anomalies (in particular, the torsional contributions) in the edge states which live on the boundary between two different bulk phases. We also give intrinsic, microscopic derivations of these torsional anomalies by considering Hamiltonian spectral flow for edge states in the presence of torsion. All of these calculations fit perfectly within the well-known framework of anomaly inflow, and extend the framework to include torsional contributions. Furthermore, our condensed-matter-inspired setup provides natural resolutions to some previously ill-understood ultraviolet divergences in intrinsic edge calculations of torsional anomalies.

In the second part of this thesis, we consider the Bosonic and Fermionic $U(N)$ vector models close to their free fixed points, with single-trace deformations turned on. We derive the higher-spin holographic duals corresponding to these vector models by first formulating these theories in terms of the geometry of infinite jet bundles, and then interpreting the renormalization group equations for single-trace deformations as Hamilton's equations of motion on a one-higher dimensional emergent spacetime. We evaluate the resulting bulk on-shell action explicitly, and show that it reproduces all the correlation functions of the vector models. Furthermore, we show that the linearized bulk equations of motion contain the Fronsdal equations of motion on Anti-de Sitter space, thus proving equivalence with Vasiliev higher-spin theories to linearized order. The bulk theory we derive is consistent with the known AdS/CFT framework, and gives a concrete boundary to bulk implementation of AdS/CFT as a geometrization of the renormalization group.

To my Parents.

Acknowledgements

"*If I have seen further, it is by standing on the shoulders of giants.*" – Issac Newton.

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Part I

Torsion, Parity-odd response and Anomalies

Chapter 1

Introduction

Quantum field theory anomalies imply that symmetries that were present in the classical Lagrangian are broken due to quantum effects. While at one time they might have been thought of as a sickness of certain field theories, anomalies lie at the heart of some of the most fundamental physical phenomena in real materials. The canonical example is the integer quantum Hall effect (IQHE) where a 2+1-dimensional electron gas in a large, uniform magnetic field exhibits a Hall conductance which is quantized in units of e^2/h when the chemical potential lies in a Landau level gap (and has been measured to be quantized up to 10 significant digits). The precise quantization arises from the connection between the Hall conductance and a topological invariant of 2+1-d electron systems called the first Chern number C_1. Since C_1 is a topological quantity which is determined by the ground state, it is not affected when the system is perturbed continuously, and is insensitive to the microscopic details of the sample as long as the bulk energy-gap is not destroyed. Thus, response coefficients which are determined by topological invariants are the most universal features of gapped systems.

For all understood topological response coefficients there is a complementary way to view the quantization by studying the properties of the gapless, fermionic modes that lie on the boundary of the system. There is a deep connection between topological transport in the bulk of a gapped material (say in 2+1-d) and field theory anomalies that are present for the (say 1+1-d) gapless boundary states[1, 2]. The connection between anomalous currents, topology, and index theorems underlies some of the most beautiful transport phenomena that have been predicted, and in some cases observed in real materials. For the IQHE this bulk-boundary correspondence connects the bulk Hall transport to the spectral flow of the boundary chiral modes due to the chiral anomaly. The edge anomaly provides a complementary picture of the origin of the Hall conductance quantization which is commonly known as Laughlin's gauge argument (though it was not originally written in terms of anomalies)[3].

While most anomalies connected with charge and spin currents are well understood, the anomalous thermal,

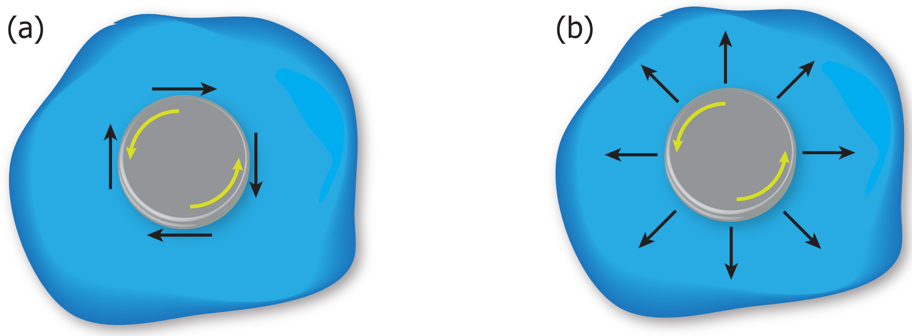

Figure 1.1: Fluid mechanics illustration of the viscous forces. A counter-clockwise rotating solid cylinder immersed in 2d liquid droplet with (a) non-zero shear viscosity (b) non-zero dissipationless viscosity. Note that the resulting forces (arrows outside cylinder) are tangent and perpendicular to the cylinder motion (arrows inside cylinder) respectively. The shear viscosity impedes the cylinder while the dissipationless viscosity pushes fluid toward or away from the cylinder depending on the rotation direction.

and visco-elastic responses (VE) are not. The thermal and VE responses lie at the intersection between geometry, topology, and quantum field theory as they are usually represented as topological phenomena associated to geometric deformations of a field theory. One example of such a novel effect is a dissipationless, electronic viscosity response in the 2+1-d topological Chern insulator with broken time-reversal symmetry[4, 5, 6]. While the ordinary shear viscosity generates a frictional force *tangent* to fluid motion, the dissipationless viscosity produces a *perpendicular* force (see Figure 1.1)[4, 7]. This viscosity is not clearly understood except in some special cases including the integer and fractional QHE with rotation[8, 9] and translation invariance[10], and chiral superconductors[11]. However, all of these models share the feature that they are Galilean invariant, and in relativistic systems, or lattice models with broken continuous translation symmetry, it is not clear if the topological viscosity is quantized, or even well-defined (for the lattice case)[6]. This is unusual as one would naively expect that it should be quantized like *all* of the other examples of topological response coefficients, such as the quantized Hall conductance (which is simultaneously present in the 2+1-d Chern insulator phase)[12].

In part I of this thesis, we will address these issues by constructing an explicit bulk-boundary correspondence in $d = 2+1$ and $d = 4+1$ *relativistic* systems, which allows us to understand the anomaly mechanism associated to the topological viscosity. The interplay of the *topological* response with the *geometric* deformations of the system makes this problem more subtle than previous known examples of topological responses, because, while topology does not care about the details of a shape, geometry does. The model we will focus on for most of this work is the massive Dirac model. This model represents the low-energy physics of topological insulators in various dimensions, and with various symmetries[13]. This model responds quite differently to geometric perturbations than typical non-relativistic electrons (*i.e.* systems with small spin-orbit coupling). To illustrate the underlying premise, we first note that conventional non-relativistic electrons in a crystal are described by the Schrodinger equation at low-energy, and are only elastically influenced by the stretching of bonds that is captured by the strain tensor[14]. However, spin-orbit coupled electrons described, for example, by the Dirac equation at low-energy, are also aware of the local orbital orientation, which is not contained in the strain tensor. Instead the Dirac model couples to geometric perturbations via a local "frame field" that we will introduce below. This additional sensitivity generates physical responses to shearing, twisting, and compressing/stretching that are not found in weakly spin-orbit coupled systems. These phenomena are the focus of our work and are connected with the idea of geometric *torsion* as we will discuss. The work presented in part I will draw heavily from [15, 16].

In this chapter, we will introduce the basic background material required to understand the details of these calculations, which will be presented in subsequent chapters. In section 1.1, we introduce some basic concepts of geometry and elasticity, with a special emphasis on torsion, from a condensed matter perspective that are relevant to our later discussions. We will follow this up with a more mathematically precise description in section 1.2, from the point of view of general relativity and high energy physics. Finally, in section 1.3, we introduce some basic aspects of fermions in the presence of background gauge and gravitational fields, again focussing on the role of torsion – we will then be ready to delve into detailed calculations of viscoelastic response and anomaly inflow for the model at hand.

1.1 Informal Preliminaries

Before we move on to a more precise description with which high-energy theorists will be more comfortable, we try to informally introduce the necessary background material for a condensed-matter audience using the language of elasticity theory. Conventional elasticity theory is one of the foundational underpinnings of solid state physics as it contains within it the physics of the lattice structure, including, for example, phonon fluctuations away from the ordered reference state. At a given time, one characterizes an elastic medium via a displacement field $\mathbf{u}(x_n)$ which gives the vector displacement of a lattice site n, away from the position x_n of a given reference state (note that we will take the continuum limit where n becomes a continuous label and thus x_n becomes a continuous coordinate yielding a *field* $\mathbf{u}(x)$) . If every lattice point is displaced by the same amount then the crystal has just been globally translated and does not feel any internal stress. However, if the displacements of lattice sites are not identical, the material will respond by generating a

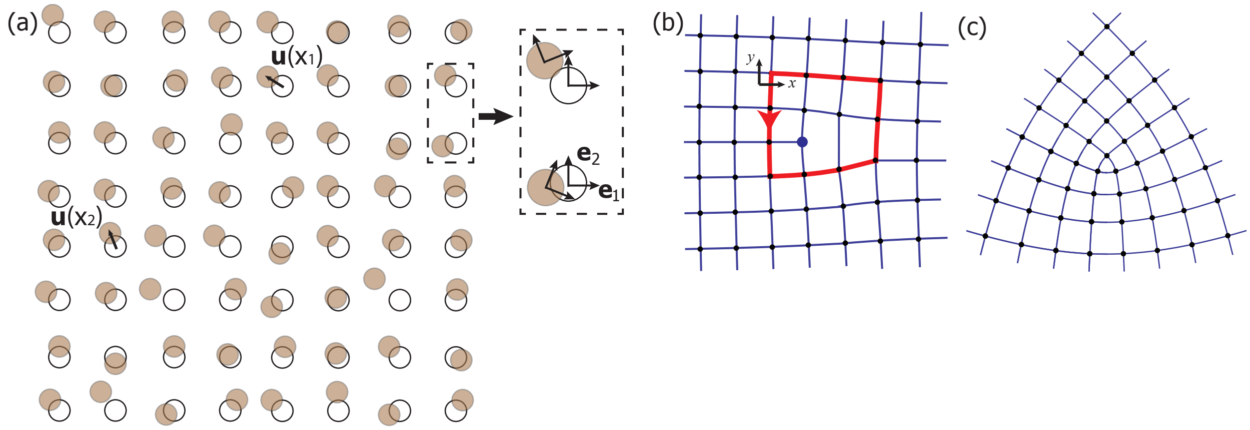

Figure 1.2: (a) Reference state (hollow circles) and displaced state (solid circles) for an elastic medium. Displacement vectors for each site n are denoted by $\mathbf{u}(x_n)$. Zoom-in shows frame field vectors $\mathbf{e}_1, \mathbf{e}_2$ in the reference state (aligned to crystal x, y-axes) and the displaced state (rotated with respect to crystal axes). (b) Edge dislocation representing the fundamental *torsion* lattice defect. An electron traveling the thick line surrounding the dislocation will be translated with respect to the same path in the reference state that does not enclose a dislocation. The Burgers vector is in the y-direction. (c) Disclination represented by a single triangular plaquette in a square lattice crystal. Gives rise to *curvature i.e.* objects that travel around a disclination are rotated with respect to the reference-state path.

stress (momentum-current density)

$$T^{ij} = \Lambda^{ijk\ell} u_{k\ell} + \eta^{ijk\ell} \dot{u}_{k\ell}, \qquad u_{k\ell} = 1/2(\partial_k u_\ell + \partial_\ell u_k) \tag{1.1}$$

where repeated indices are always summed, T^{ij} is the stress tensor (momentum current density), $\Lambda^{ijk\ell}$ is the elasticity tensor which relates stress to the strain $u_{k\ell}$ (*i.e.* a generalization of Hooke's law) and $\eta^{ijk\ell}$ is the viscosity tensor relating stress to the strain rate/velocity gradient $\dot{u}_{k\ell}$ (*i.e.* a velocity dependent frictional force). See Figure 1.2 (a) for an illustration of a lattice elastic medium and a displacement field.

A non-zero strain tensor indicates that the (spatial) geometry of the elastic medium has been distorted. The geometric characterization of the lattice is contained in the metric tensor which determines the distance between lattice points. In the ordered reference state shown in Figure 1.2 (a) the metric tensor is just $g_{ij} = \delta_{ij}$ which implies that distances between sites are calculated in the usual Euclidean way. When the material is strained, the spatial metric tensor is modified to become $g_{ij} = \delta_{ij} + 2u_{ij}$[14], which is what is meant when we say the geometry is deformed. Static lattice deformations affect the electronic behavior since the bonds are deformed. For electrons described by the Schrodinger equation at low-energy, the Hamiltonian is modified to become (to linear order in strain)

$$H = \frac{p^2}{2m} \to \frac{p_i g^{ij}(x) p_j}{2m} = \frac{p^2}{2m} - 2u^{ij}(x)\frac{p^2}{2m} + i\hbar(\partial_i u_{ij}(x))\frac{p_j}{m} \tag{1.2}$$

where $g^{ij}(x)$ is the inverse of the metric tensor which depends on position via the contribution of the strain tensor. Thus, depending on the spatial profile of the strain, the electron spectrum can get modified.

While the strain/metric based elasticity theory is quite successful, it is not general enough to model all of the electronic structure effects arising from the coupling of materials with spin-orbit coupling to geometric deformations. What is needed is a more fundamental field: the frame field $\underline{e}_a$ in d spatial dimensions where $a = 1, 2, \ldots d$ labels each vector of the frame (with components $\underline{e}_a^i$). The frame-field is a set of d vectors residing on *each* lattice site, and heuristically encodes the local bond stretching (through the vector lengths) *and* the local orbital orientation (through their relative angles on each site). As we will see later, in many instances it is more natural to consider the co-frame field e^a which is a local basis of 1-forms that are dual to the vectors $\underline{e}_b$ (*i.e.* they satisfy $e^a(\underline{e}_b) = \delta^a_b$). For the reference state shown in Figure 1.2 (a) the reference frame fields are orthonormal vectors which are aligned with the crystal axes. The distances between lattice sites, *i.e.* the (inverse) metric tensor is determined from the frame fields via $g^{ij}(x) = \delta^{ab}\underline{e}_a^i(x)\underline{e}_b^j(x)$[17]. The key relationship between the metric and the frame is that we can *locally* rotate the frame at each site by any $SO(d)$ rotation matrix R and we get the *same* metric back:

$$\tilde{g}^{ij} = \delta^{ab}(R_a^c(x)\underline{e}_c^i(x))(R_b^d(x)\underline{e}_d^j(x)) = R_a^c(x)R_a^d(x)\underline{e}_c^i\underline{e}_d^j = \delta^{cd}\underline{e}_c^i\underline{e}_d^j = g^{ij} \tag{1.3}$$

since $RR^T = \mathbb{I}$. This implies that an elasticity theory determined completely from the metric does not capture local orbital deformations since each different local orbital orientation yields the *same* metric tensor. However, electrons with SOC propagating in a lattice will be sensitive to the local orbital orientation, which is exactly why a frame field must be introduced to couple these materials to geometric perturbations. This modification to elasticity theory is closely related to so-called micro-polar or 'Cosserat' elasticity[18, 19].

At this point it is useful to explicitly show how the frame field enters spin-orbit coupled Hamiltonians. The low-energy description of two such systems are given by the Dirac Hamiltonian (which represents, for example, topological insulators) [20, 21] and the Luttinger Hamiltonian (which represents, for example, the upper-most valence bands of III-V semiconductors) [22, 23]:

$$H_D = v\sum_{i,a} p_i \underline{e}_a^i \Gamma^a + m\Gamma^0 \tag{1.4}$$

$$H_L = \delta^{ab}\frac{p_i \underline{e}_a^i \underline{e}_b^j p_j}{2m} + \alpha\left(p_k \underline{e}_a^k S^a\right)\left(p_\ell \underline{e}_b^\ell S^b\right) = \frac{p_i g^{ij} p_j}{2m} + \alpha(p_k \underline{e}_a^k)(p_\ell \underline{e}_b^\ell) S^a S^b \tag{1.5}$$

for Dirac matrices Γ^a, spin-3/2 matrices S^a, and parameters v, m, α. Hence, the prescription is to replace terms of the form $p_i M^i$ for a matrix M^i, which arise naturally in materials with SOC, with $\sum_a p_i \underline{e}_a^i M^a$. Note that for H_L, since $S^a S^b \neq \delta^{ab}$, the quadratically dispersing Luttinger model is indeed affected by the local orbital orientation since it couples to more than just the metric tensor. The effects of the frame field are thus not limited to the linearly dispersing Dirac equation and affect any coupling between the direction of electron propagation p_i and the spin/orbital degrees of freedom represented by M^i.

There are two complimentary interpretations of the (co-)frame-field which we will use. The first interpretation is in terms of familiar elasticity quantities, namely to first order in the displacement field, the co-frame and frame can be expanded as

$$e_i^a = \delta_i^a + \frac{\partial u^a}{\partial x^i}, \quad \underline{e}_a^i = \delta_a^i - \frac{\partial u_a}{\partial x_i} \tag{1.6}$$

where $\partial_i u^a \equiv w_i^a$ is the *distortion tensor* which is familiar from elasticity theory[14]. The quantity w_i^a is effectively the unsymmetrized strain tensor and contains information about local rotations through the anti-symmetric combination $M_{ij} = \delta_{ia} w_j^a - \delta_{ja} w_i^b$. The distortion tensor also contains information about dislocations through the line-integral

$$\oint_C w_i^a dx^i = \oint_C du^a = -b^a \tag{1.7}$$

where b^a are the components of the total Burgers vector of the dislocation(s) enclosed within the curve C (see Figure 1.2b for an example)[14].

For point-like dislocations in 2d we can write $de^a = -b^a\delta^{(2)}(x)$ from Stokes' theorem where de^a is the exterior derivative of the 1-form e^a. This formula suggests a second description of the e^a as a set of d vector potentials. As a comparison, we know that for electrons in an electromagnetic vector potential we use the minimal coupling replacement $p_i \to p_i + qA_i$ which *shifts* the momentum in the Hamiltonian, and we have already mentioned that the proper replacement for the frame field is to *scale* momentum

$$p_i \to p_i \underline{e}_a^i = p_i \delta_a^i - p_i w_i^a = p_a - p_i w_i^a. \tag{1.8}$$

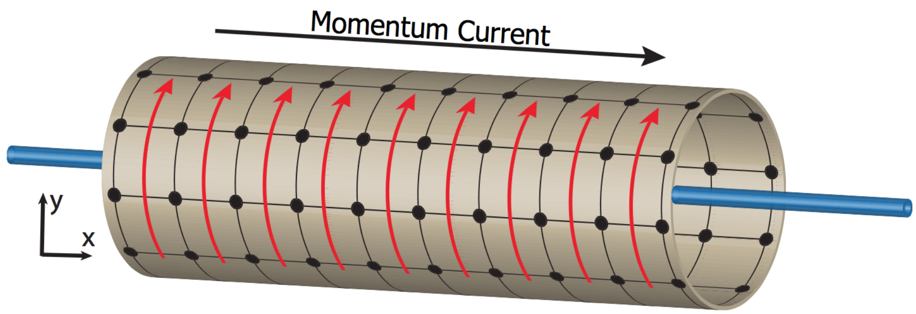

Figure 1.3: Laughlin gauge argument for torsion: Thought experiment with an insertion of torsion flux *i.e.* a dislocation into cylindrical hole, equivalent to twisting cylinder in the y-direction as a function of time. Non-zero dissipationless viscosity causes transfer of p_y-momentum in the x-direction, *i.e.* a momentum current perpendicular to time-dependent strain.

Comparing to the electromagnetic case, this shows that each frame-vector yields a vector potential that minimally couples to electrons via momentum *i.e.* the momentum components are the *charges* of these gauge fields. With this interpretation, dislocations are just the magnetic fluxes of these vector potentials, and the translation effect of a dislocation is just the Aharonov-Bohm effect for the co-frame vector potentials. In general we can construct the *torsion* tensor, which, in the absence of curvature can be chosen to take the simple form of a field strength tensor of the co-frame vector potentials ${T_{ij}}^a = \partial_i e^a_j - \partial_j e^a_i$. This has an extra index a compared to the electromagnetic version F_{ij}, which labels the particular vector potential/co-frame potential. This is how "torsion" naturally enters the discussion, and as we can see, it is intimately connected to dislocation density. We also mention that there exist other elastic defects such as disclination defects which represent sources of geometric curvature and also orbital-twisting defects that can be produced in a strain-free lattice with a trivial metric but non-trivial frame (*e.g.* a torsional skyrmion[24]), examples of the former are shown in Fig. 1.2c.

With the background theory now set up, we will move on to discuss the current state of the field of topological VE response, and some of the questions which we will address in this thesis. The first calculation of a topological VE response was the work of Avron *et al.* which showed that a dissipation-less viscosity is present in integer quantum Hall states[4, 25, 7]. The work was not followed up on until over a decade later when Read showed that, not surprisingly, fractional quantum Hall states could also exhibit such a viscosity, and that the response would be quantized if rotation symmetry were preserved[8, 9]. Soon after, Haldane showed that rotation symmetry is not a necessary ingredient and that the viscosity is related to a fundamental property of an unreconstructed quantum Hall edge: the edge dipole moment[10]. For these systems the viscosity, denoted ζ_H, is a quantized multiple of $\hbar/\ell_B^2$ where ℓ_B is the magnetic length, and was dubbed the *Hall viscosity.* This quantity has units of angular momentum density, or momentum per unit length, or dynamic viscosity (force/velocity), and interestingly, it depends on a non-universal length scale which varies when the magnetic field is tuned. In fact, one even can remain on the same Hall plateau with fixed Hall conductance, and tune the field so that the viscosity changes. When rotation symmetry is present, conserved angular momentum can be transferred between edges via an applied torque (*e.g.* due to the electric field generated from perpendicular applied flux). The amount of transferred angular momentum does not depend on ℓ_B, and is given by the quantized multiple of $\hbar$ appearing in ζ_H. The same is true of the edge dipole moment, which is also independent of ℓ_B for unreconstructed edges, and is the same universal number multiplying $\hbar$.

This quantization emerges quite naturally in the Landau level problem where the quantum Hall effect is generated by an external magnetic field. However, the situation is much more subtle and complicated when the quantum Hall effect is generated by a topological band structure which can naturally furnish multiple length scales. We will focus on this type of system to study the impact that a combination of geometry and topology will have in band theory. A dissipation-less viscosity response, analogous to the Hall viscosity, was shown to exist and was calculated in a (properly regularized) continuum model for the Chern insulator, *i.e.* the massive Dirac Hamiltonian in 2+1d [6]; much of the first part of this thesis will build on this observation, generalize it to higher dimensions, and flesh out many details related to their proposal. The regularized value was found to be $\zeta_H = \frac{\hbar}{8\pi\xi^2}$ where $\xi = \hbar v/2m$ is the length scale induced by the Dirac mass m (with units of energy) for a material with a Fermi-velocity (speed of light) v. In relation to the discussion of elasticity theory above, the non-zero viscosity coefficient produces a Chern-Simons response for the *co-frame fields*:

$$S_{eff}[e^a_\mu] = \frac{\zeta_H}{2}\int d^2x dt\ \epsilon^{\mu\nu\rho} e^a_\mu \partial_\nu e^b_\rho \eta_{ab} \tag{1.9}$$

where $a, b = 0, 1, 2$, and $\eta_{ab} = \mathrm{diag}[-1, 1, 1]$ is the flat-space Minkowski metric. This is essentially multiple copies of the conventional Abelian Chern-Simons term, one for each of the co-frame fields (including the co-frame in the time direction). As shown in Ref. [6], if we calculate the electronic contribution to the stress response $T^\mu_a = \frac{1}{\det(e)}\frac{\delta S_{eff}}{\delta e^a_\mu}$, one finds that electron *momentum-density* is bound at dislocation defects and *momentum-current* is generated perpendicular to any velocity-gradients/strain-rates (see Figure 1.3a for a picture of the latter). This is completely analogous to the *charge density* bound to magnetic flux and *charge current* produced by electric fields (or time-dependent fluxes) in the quantum Hall Chern-Simons response. The principal issue, however, is that the viscosity ζ_H does not appear to be quantized, or even universal, which might seem rather strange in light of all the previous results on topological responses in topological insulators[26, 12]. However, there is a natural resolution to this discrepancy, which we will explain in the next chapter.

1.2 Formal Preliminaries

Gravity is usually described as a theory of metrics, corresponding to a measure of invariant distance

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{1.10}$$

where x^μ are local coordinates on a manifold. We can package the information contained in the metric (and more in fact) into the components of a co-frame, a *local* basis of 1-forms $e^a = e^a_\mu dx^\mu$ on the manifold. Equivalently, we can regard e^a as a local section of the oriented co-tangent bundle of the manifold. The metric is related to the components of the 1-forms via

$$e^a_\mu e^b_\nu \eta_{ab} = g_{\mu\nu} \tag{1.11}$$

where η_{ab} are the components of the Lorentz-invariant Minkowski metric. We will denote the dual set of *frame vector fields* as $\underline{e}_a$, with $e^a(\underline{e}_b) = \delta^a_b$. To translate (co-)tangent bundle data from point to point on the manifold, we need a connection or covariant derivative ∇. Conventionally we write the translation of the frame along a vector field $\underline{X}$ as

$$\nabla_{\underline{X}} e^a = -\omega^a{}_b(\underline{X}) e^b \tag{1.12}$$

where we have introduced the components of the spin connection $\omega^a{}_b$, which we regard as a set of 1-forms. In a basis of local coordinates this equation can be written as $X^\mu \nabla_\mu e^a_\nu = -X^\mu \omega_\mu{}^a{}_b e^b_\nu$. The spin-connection can be thought of as a non-Abelian gauge field that couples to the rotation and Lorentz transformation generators. Throughout our work, we will make one assumption about this connection, which is that it is *metric compatible*. In metric terms, this means that the metric is covariantly constant $\nabla_X g = 0$, but using the relationship between the metric and the co-frame and the definition (1.12), it also corresponds to the spin connection being valued in the orthogonal group,[1] *i.e.*, $\omega_{ab} = -\omega_{ba}$ (where $\omega_{ab} \equiv \eta_{ac}\omega^c{}_b$). Under a local

[1] We will work in $d = D + 1$ spacetime dimensions, so the relevant orthogonal group is $SO(1, D)$.

change of basis (*i.e.*, a local Lorentz transformation) $e^a \mapsto \Lambda^a{}_b e^b$, the connection transforms as[2]

$$\omega^a{}_b \quad \mapsto \quad (\Lambda\omega\Lambda^{-1} - d\Lambda\Lambda^{-1})^a{}_b \tag{1.13}$$

Thus $\omega^a{}_b$ is the 'gauge field' for local Lorentz transformations. The curvature 2-form, or field strength, of the connection

$$R^a{}_b \quad = \quad d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b. \tag{1.14}$$

transforms linearly

$$R^a{}_b \mapsto (\Lambda R \Lambda^{-1})^a{}_b. \tag{1.15}$$

The components of the curvature 2-form give the Riemann tensor, $R_{cd} = \frac{1}{2} R_{ab;cd} e^a \wedge e^b$. If we denote the covariant derivative acting on (local) Lorentz tensors by D, the torsion 2-form is defined as

$$T^a \quad = \quad De^a \equiv de^a + \omega^a{}_b \wedge e^b \tag{1.16}$$

Torsion also transforms linearly under local Lorentz transformations.[3]

$$T^a \mapsto \Lambda^a{}_b T^b. \tag{1.19}$$

We write the components of the torsion 2-form as $T^c = \frac{1}{2} T^c{}_{ab} e^a \wedge e^b$.

A very basic property of the connection, is that it satisfies the following translation algebra

$$[\nabla_a, \nabla_b] \quad = \quad -T^c_{ab} \nabla_c + R_{cd;ab} J^{cd} \tag{1.20}$$

where J^{cd} is the generator of rotations. We will see an explicit representation of this algebra in the next section. The left hand side can be interpreted as successive parallel translations along $\underline{e}_b, \underline{e}_a, -\underline{e}_b, -\underline{e}_a$, and thus we see that the components of the torsion tensor correspond to the non-closure of these successive translations by an extra translation, while the components of the Riemann tensor imply that a rotation is

[2] Note that we are reserving the term 'Lorentz transformation' for these local changes of basis for the orthonormal co-frame. These should not be confused with (linear) diffeomorphisms, which are local changes of the coordinates.

[3] The Bianchi identities are

$$DR^a{}_b \quad \equiv \quad dR^a{}_b - R^a{}_c \wedge \omega^c{}_b + \omega^a{}_c \wedge R^c{}_b = 0 \tag{1.17}$$

$$DT^a \quad \equiv \quad dT^a + \omega^a{}_d \wedge T^d = R^a{}_d \wedge e^d \tag{1.18}$$

If the torsion vanishes, the latter corresponds to a symmetry property of the Riemann tensor.

also involved.

In classical general relativity (GR), a basic property of the theory is that the torsion is taken to vanish; this is one manifestation of the equivalence principle. In fact, there is a *unique* connection, the Levi-Civita connection $\mathring{\omega}^a{}_b$, with this property which is determined entirely by the co-frame alone (*i.e.*, the metric). Indeed in the familiar Einstein-Hilbert Lagrangian formulation of GR, the torsion vanishes as a constraint. In other formulations (the first-order or Riemann-Cartan formulations), e^a and $\omega^a{}_b$ are regarded as independent degrees of freedom and the torsion may then vanish by equations of motion (for suitable choice of matter field configurations). In the latter formalism, one can envisage including sources that would induce torsion, much as the usual sources induce curvature. It should be emphasized though that in our context, we regard e^a and $\omega^a{}_b$ as background fields, with no dynamics of their own.

Given the form of the translation algebra (1.20), the vanishing of torsion in fact corresponds to a *choice of state*. As in the previous section we can consider an elastic medium given by a (space-time) lattice Λ. We will typically be interested in continuum limits, giving rise to a continuum quantum field theory, in the presence of a variety of background fields (so that we can study various transport properties). At each point in the lattice, we have defined a frame, whose magnitudes are tied to the (local) lattice spacing (see equation (1.6)). The commutator of translations on the lattice is defined by hopping along a square path; failure to return to the starting position corresponds to the path encircling a *dislocation* of the lattice, and the magnitude and direction of the translation determines the *Burgers' vector* $\underline{b}$ of the dislocation. There exist two primary types of dislocations: (i) an edge dislocation with $\underline{b}$ perpendicular to the tangent vector of the dislocation line (b) a screw dislocation with $\underline{b}$ parallel to the dislocation line (only exists in 3+1-d or higher). An example of the former is shown in Fig. 1.2b. Now consider the continuum limit. If the limit is taken in such a way that we obtain a *density of dislocations* $\underline{b}(x)$, then we should associate this with non-zero torsion in the continuum theory. Lattice dislocations correspond to point sources of torsion. The frame is rotated if the path encircles a *disclination* and continuum limits yielding a density of disclinations correspond to curvature. Disclinations are significant if and only if the field in question carries a non-trivial Lorentz representation (that is the generator J^{ab} is non-zero), *i.e.*, it carries spin. The effects of dislocations do not carry this requirement. We reiterate that here by torsion and curvature we mean non-dynamical background fields coupled to our dynamical fields of interest, such as free fermions.

Thus, in condensed matter systems coupled to elastic media, we conclude that lattice defects in the microscopic theory give rise to background curvature and torsion in the continuum limit, and thus the nature of the background is determined not just by the metric, but by both the co-frame and connection. As we will show in detail below, this corresponds to the presence of independent Lorentz and diffeomorphism currents (whereas in the absence of torsion, these reduce to just the conventional stress-energy tensor). However, even in the absence of torsion in a particular special choice of background, torsional perturbations should be also considered in the context of transport properties. Studying effective actions[4] of a given field theory in the presence of background co-frame and connections is equivalent to studying the correlation functions of the these currents, as the backgrounds correspond to sources for the current operators.

It is convenient to introduce some additional notation. As indicated above, given a co-frame e^a, there is a uniquely determined *Levi-Civita connection* $\mathring{\omega}^a{}_b$ whose torsion vanishes. We define the *contorsion* $C^a{}_b$ via

$$\omega^a{}_b = \mathring{\omega}^a{}_b + C^a{}_b \tag{1.21}$$

so[5]

$$\begin{aligned} T^a &= C^a{}_b \wedge e^b & (1.22) \\ R^a{}_b &= \mathring{R}^a{}_b + (\mathring{D}C)^a{}_b + C^a{}_c \wedge C^c{}_b & (1.23) \end{aligned}$$

Note also that the contorsion is a Lorentz tensor. Generally, we will regard e^a and $\omega^a{}_b$ as independent. For later use, we will also define

$$H = \frac{1}{3!} H_{abc} e^a \wedge e^b \wedge e^c = e^a \wedge T^b \eta_{ab} \tag{1.24}$$

where $H_{abc} = -3!C_{[a;bc]} = 3T_{[bc;a]}$. H is not in general a closed form, and hence we define the *Nieh-Yan 4-form*:

$$N = dH = T^a \wedge T^b \eta_{ab} - R_{ab} \wedge e^a \wedge e^b. \tag{1.25}$$

1.3 Dirac Fermions Coupled to Torsion

In this section, we discuss various aspects of free Dirac fermions on a generic gravitational background, mainly focussing on the role played by torsion. We are studying Dirac models since they represent the

[4] We use the term 'effective action' here interchangeably with 'generating functional'. The latter term is most appropriate, as indeed, the use of the effective action is that it encodes the correlation functions of currents.

[5] We define $\mathring{D}$ as the LC covariant derivative, $(\mathring{D}C)^a{}_b = dC^a{}_b + \mathring{\omega}^a{}_c \wedge C^c{}_b + C^a{}_c \wedge \mathring{\omega}^c{}_b$ and $\mathring{R}$ as the LC curvature $\mathring{R}^a{}_b = d\mathring{\omega}^a{}_b + \mathring{\omega}^a{}_c \wedge \mathring{\omega}^c{}_b$.

minimal continuum models of topological insulators in any dimension.

The Dirac action may be written as[6]

$$S[\psi; e, \omega] = \frac{1}{D!} \int \epsilon_{a_1 \ldots a_d} e^{a_1} \wedge \ldots \wedge e^{a_D} \wedge \left[\frac{1}{2} \overline{\psi} \gamma^{a_d} \nabla \psi - \frac{1}{2} \overline{\nabla \psi} \gamma^{a_d} \psi - e^{a_d} \overline{\psi} m \psi \right] \tag{1.26}$$

$$= \int d^d x \det e \left[\frac{1}{2} \overline{\psi} \gamma^a \nabla_{\underline{e}_a} \psi - \frac{1}{2} \overline{\nabla_{\underline{e}_a} \psi} \gamma^a \psi - \overline{\psi} m \psi \right] \tag{1.27}$$

We have written the action in this way as it is precisely real (written in other ways, the action might be real up to the addition of a boundary term). In odd space-time dimensions, m is real, and its sign will play a central role in determining the character of the resulting insulating state. In even space-time dimensions m is essentially complex if no additional discrete symmetries are imposed ($m \to m e^{i\theta\gamma_5}$, where γ_5 is the chirality operator). In addition, when torsion is non-zero, there is an additional term[7] that can be added to the action, of the form

$$S_T[e, \omega] = \frac{1}{16} \alpha \int \det e \; T^a(\underline{e}_b, \underline{e}_c) \overline{\psi} \{\gamma_a, \gamma^{bc}\} \psi. \tag{1.28}$$

The classical equation of motion for the spinor field involves the Dirac operator

$$\not{D} = \gamma^a e^\mu_a \left(\partial_\mu + A^A_\mu t_A + \frac{1}{4} \omega_{\mu;bc} \gamma^{bc} + B_\mu \right) + \frac{1}{8} \alpha T_{bc;a} \gamma^{abc} \tag{1.29}$$

where $B_a \equiv \frac{1}{2} T^b(\underline{e}_a, \underline{e}_b) = -\frac{1}{2} C^b{}_a(\underline{e}_b)$. The B term arises upon integration by parts in deriving the equations of motion. We have included here for completeness a non-Abelian gauge field (if the spinor is in a gauge representation t_A) and we note that the torsional B-term enters in such a way that it looks like it corresponds to an additional gauge field. It is not of course independent of the spin connection, but does vanish with the torsion. In fact, as explained in [27], the classical theory possesses a corresponding background scaling symmetry when $m = 0$ under which the fields and background transform as

$$e^a(x) \mapsto e^{\Lambda(x)} e^a(x), \qquad \omega^a{}_b(x) \mapsto \omega^a{}_b(x), \tag{1.30}$$

$$\psi(x) \mapsto e^{-(d-1)\Lambda(x)/2} \psi(x), \qquad \not{D} \mapsto e^{-\Lambda} (e^{-(d-1)\Lambda/2} \not{D} e^{(d-1)\Lambda/2}). \tag{1.31}$$

[6] The (Lorentz and gauge) covariant derivative of the Dirac spinor is $\nabla \psi = d\psi + \frac{1}{4} \omega_{ab} \gamma^{ab} \psi + A\psi$, where A is an appropriate (non-Abelian) gauge connection. We note also that the invariant form of the action, eq. (1.26), does not involve the frame $\underline{e}_a$ dual to e^a.

[7] There are actually two other terms at the same level of power counting. The first, of the form $i \int \det e \left[T^a(\underline{e}_b, \underline{e}_c) \overline{\psi} [\gamma_a, \gamma^{bc}] \psi - 2 \nabla_{\underline{e}_a} (\overline{\psi} \gamma^a \psi) \right]$ is Nieh-Yan-Weyl invariant (see below), but a total derivative. The second, of the form $i \int \det e \; T^a(\underline{e}_b, \underline{e}_c) \overline{\psi} [\gamma_a, \gamma^{bc}] \psi$, is redundant (it can be absorbed into the definition of a $U(1)$ gauge field).

We note that this implies

$$T^a \mapsto e^{\Lambda}\left(T^a + d\Lambda \wedge e^a\right) \tag{1.32}$$

and hence

$$B_a = \frac{1}{2}T^b(\underline{e}_a, \underline{e}_b) \mapsto e^{-\Lambda}\left(B_a + \frac{d-1}{2}\underline{e}_a(\Lambda)\right) \tag{1.33}$$

If we introduce a 1-form $B \equiv B_a e^a$, then this is equivalent to[8]

$$B \mapsto B + \frac{d-1}{2}d\Lambda \tag{1.35}$$

which is the transformation of an Abelian ($\mathbb{R}^+$, not $U(1)$) connection. We will refer to this as the Nieh-Yan-Weyl (NYW) symmetry. Note that this is *not* the Weyl symmetry of the metric theory, because in that case, ω must transform in order that the torsion remain zero. In our case, the Weyl symmetry (at least as far as the *Dirac operator* is concerned) corresponds to a complexification of a $U(1)$ symmetry. In addition, the classical Dirac theory also has the usual background diffeomorphism, local Lorentz, and gauge symmetries, which we will discuss below.

Another way to write the Dirac operator is in terms of the Levi-Civita connection, and the totally antisymmetric part of the contorsion

$$\begin{aligned} \not{\mathcal{D}} &= \gamma^a e_a^\mu \left(\partial_\mu + A_\mu^A t_A + \frac{1}{4}\overset{\circ}{\omega}_{\mu;bc}\gamma^{bc}\right) + \frac{1}{4}C_{a;bc}\gamma^a\gamma^{bc} + B_a\gamma^a + \frac{1}{8}\alpha T_{bc;a}\gamma^{abc} && (1.36)\\ &= \gamma^a e_a^\mu \left(\partial_\mu + A_\mu^A t_A + \frac{1}{4}\overset{\circ}{\omega}_{\mu;bc}\gamma^{bc}\right) - \frac{1-\alpha}{4}\frac{1}{3!}H_{abc}\gamma^{abc} && (1.37) \end{aligned}$$

where we have done some γ-matrix algebra and $H_{abc} = -3!C_{[a;bc]}$ was defined previously. We note that from this form, that only the completely anti-symmetric part of the torsion tensor couples to the Dirac operator, and the dimensionless parameter $1-\alpha$ determines the coupling strength. Thus we can regard it as 'torsional charge', and write $1-\alpha = q_T$. For convenience, we will set $q_T = 1$ throughout most of this section (equivalently, we could redefine H_{abc} by absorbing q_T into it), but we will resurrect it when required in the next chapter.

Since the Dirac theory is quadratic in fermion fields, the partition function in the quantum theory is obtained

[8]Note that the invariance of $\omega^a{}_b$ implies that the contorsion transforms as

$$C^a{}_b \mapsto C^a{}_b - d\Lambda(\underline{e}_b)e^a + \eta^{ac}\eta_{bd}d\Lambda(\underline{e}_c)e^d \tag{1.34}$$

and the LC connection transforms oppositely.

by performing a path integral over fermions

$$Z(A, e^a, \omega^a{}_b; m) = \det(\not{D} - m) \tag{1.38}$$

The diffeomorphism, local Lorentz, and gauge symmetries of the Dirac theory remain unaffected by perturbative (*i.e.* local) anomalies upon quantization in arbitrary dimension. In odd dimensions the NYW symmetry at $m = 0$ is also non-anomalous. At $m \neq 0$, the NYW symmetry is explicitly broken. Additionally, the mass term also breaks parity invariance. In this thesis, we will mainly be interested in the *quantum effective action* for odd-dimensional Dirac fermions

$$S_{eff}[e, \omega, A] = -\ln\ \det(\not{D} - m). \tag{1.39}$$

More precisely, we will be interested in the parity-violating piece of the effective action, which we denote as $S_{odd}[e, \omega, A]$. In the absence of torsion, symmetry considerations severely constrain the form of parity odd terms. For example in $d = 3$, we have the Chern Simons terms

$$S_{odd}[e, \mathring{\omega}, A] = \frac{1}{2}\int \left(\sigma_H\ A \wedge dA + \kappa_H \mathrm{tr}\ (\mathring{\omega} \wedge d\mathring{\omega} + \frac{2}{3}\mathring{\omega} \wedge \mathring{\omega} \wedge \mathring{\omega})\right) \tag{1.40}$$

The coefficient σ_H is called the *Hall conductivity*, while κ_H, the coefficient of the gravitational-Chern-Simons term, has no standard name (although it is loosely connected with thermal Hall conductivity). Non-zero torsion allows us to construct additional terms like

$$\frac{1}{2}\int \zeta_H\ e^a \wedge T_a + \frac{1}{2}\int \tilde{\kappa}_H\ \mathring{R}\, e^a \wedge T_a + \cdots \tag{1.41}$$

The first term above was discussed in a slightly different guise in section 1.1 (see Eq 1.9); the coefficient ζ_H is a dissipationless viscosity analogous to the *Hall viscosity*, and we will refer to it as such. The reader should bear in mind however, that traditionally the Hall viscosity is defined with respect to the symmetrized energy-momentum tensor in a metric theory, and as such the viscosity we're discussing here is a different response coefficient; perhaps, torsional Hall viscosity would be a better name [28]. The second term is also similar in form, and may be interpreted as a local curvature dependent contribution to the Hall viscosity. Similar terms can also be written in higher dimensions; we will discuss for instance the case of $d = 5$ in the next chapter.

Additionally, the effective action also has parity even terms. Returning to our $d = 3$ example, we could write for instance

$$\begin{aligned} S_{even}[e,\omega,A] &= \frac{1}{2\kappa_N}\int\left(\epsilon_{abc}e^a\wedge \mathring{R}^{bc}-\frac{3\gamma^2}{2}H\wedge *H-\frac{\Lambda}{3}\epsilon_{abc}e^a\wedge e^b\wedge e^c\right) \\ &= \frac{1}{2\kappa_N}\int d^3x\,\det(e)\left(\mathring{R}-\frac{\gamma^2}{4}H_{abc}H^{abc}-2\Lambda\right) \end{aligned} \tag{1.42}$$

where $\frac{\kappa_N}{8\pi}$ is the *Newton's constant*, Λ is the *cosmological constant*, and γ is a dimensionless parameter. Although parity even terms are out of the scope of this work, we will briefly examine them for the 2+1 Dirac model in the next chapter, because in this case, there is some interesting structure which emerges.

In even dimensions, it is also possible to couple chiral fermions to the frame and connection. The action is a straightforward modification of (1.26, 1.27)

$$S_\pm[\psi;e,\omega] = \frac{1}{D!}\int \epsilon_{a_1\ldots a_d}e^{a_1}\wedge\ldots\wedge e^{a_D}\wedge\left[\frac{1}{2}\overline{\psi}\gamma^{a_d}\nabla P_\pm\psi-\frac{1}{2}\overline{\nabla\psi}\gamma^{a_d}P_\pm\psi\right] \tag{1.43}$$

$$= \int d^dx\,\det e\left[\frac{1}{2}\overline{\psi}\gamma^a\nabla_{\underline{e}_a}P_\pm\psi-\frac{1}{2}\overline{\nabla_{\underline{e}_a}\psi}\gamma^aP_\pm\psi\right] \tag{1.44}$$

with $P_\pm = \frac{1\pm\gamma^5}{2}$ being the chirality projection operators. The chiral theory also has the symmetries of the Dirac theory. However, all the symmetries are spoilt by perturbative anomalies upon quantization on generic backgrounds. Later in this thesis, we will explore such chiral anomalous conservation laws for Lorentz, diffeomorphism, and gauge currents, and their connection with S_{odd} for the Dirac model. We will see that while torsional terms like (1.41) leave *consistent* anomalies unaffected, they do modify the *covariant* anomalies.

Classical Ward identities

In this section, we state the classical conservation laws for fermions coupled to the coframe, connection and a $U(1)$ gauge field.[9] Although we will discuss these in the context of Dirac fermions (for arbitrary d), the results generalize in a straightforward manner to chiral fermions in even dimensions. Let us begin by defining

[9] From now on, we will restrict the gauge group to $U(1)$ in favor of somewhat simpler notation. We will use the symbol q for the $U(1)$ charge.

the following currents

$$(J^\mu) = q\,\overline{\psi}\gamma^a \underline{e}^\mu_a \psi \tag{1.45}$$

$$(J_\mu)^a = \frac{1}{2}(\overline{\psi}\gamma^a\nabla_\mu\psi - \overline{\nabla_\mu\psi}\gamma^a\psi) \tag{1.46}$$

$$(J^\mu)^a{}_b = \frac{1}{4}\underline{e}^\mu_c\overline{\psi}\gamma^{ca}{}_b\psi \tag{1.47}$$

which couple respectively to the gauge field, coframe, and connection in the classical action. In the absence of torsion the last two currents are not independent. The components of the current J^a give the usual notion of the stress-energy tensor via

$$T_{\mu\nu} = J^a_\mu e^b_\nu \eta_{ab} \tag{1.48}$$

Also note that the spin current J^{ab}_μ vanishes in $d = 2$. It will be convenient to introduce the corresponding 1-forms $J = J_\mu dx^\mu$, $J^a = J^a_\mu dx^\mu$ and $J^{ab} = J^{ab}_\mu dx^\mu$. Invariance under $U(1)$ gauge transformations implies that J is conserved, *i.e.* $d * J = 0$, which in components is the usual $\partial_\mu(\det(e)J^\mu) = 0$.

Diffeomorphisms

The invariance of the classical action under local background diffeomorphisms follows immediately from writing it as the integral of a top form, as in (1.26). We will take the action of local diffeomorphisms on fermions and background fields as

$$\delta\psi = i_\xi\nabla\psi,\quad \delta e^a = D\xi^a + i_\xi T^a,\quad \delta\omega_{ab} = i_\xi R_{ab},\quad \delta A = i_\xi F \tag{1.49}$$

where ξ is a vector field with compact support and i_ξ is the interior product of ξ with a differential form. These transformations differ from ordinary diffeomorphisms by local gauge transformations, so we will refer to these as *covariant diffeomorphisms*. Using equations of motion for the fermions, the variation in the action under (1.49) is given by

$$\delta_{Diff.}S = \int \left[i_\xi F \wedge *J + (D\xi_a + i_\xi T_a)\wedge *J^a + i_\xi R_{ab}\wedge *J^{ab}\right] \tag{1.50}$$

and so invariance of the action implies the classical Ward identity

$$D * J^a - i_{\underline{e}^a}T_b \wedge *J^b - i_{\underline{e}^a}R_{bc}\wedge *J^{bc} - i_{\underline{e}^a}F\wedge *J = 0 \tag{1.51}$$

Local Lorentz transformations

The spinors and background fields transform under an infinitesimal Lorentz transformation as

$$\delta\psi = \frac{1}{4}\theta_{ab}\gamma^{ab}\psi,\ \delta e^a = -\theta^a{}_b e^b,\ \delta\omega^a{}_b = -(D\theta)^a{}_b \tag{1.52}$$

Under (1.52), the action changes by

$$\delta_{Lor.}S = -\int \left[D\theta_{ab} \wedge *J^{ab} + \theta_{ab}e^a \wedge *J^b\right] \tag{1.53}$$

The Ward identity is

$$D * J^{ab} - e^{[a} \wedge *J^{b]} = 0 \tag{1.54}$$

Nieh-Yan-Weyl transformations

The action on fermions and background fields is given by

$$\delta\psi = -\frac{d-1}{2}\Lambda\psi,\ \delta e^a = \Lambda e^a,\ \delta\omega_{ab} = 0 \tag{1.55}$$

Under $\delta\psi = -\frac{d-1}{2}\Lambda\psi$, the action transforms as

$$\delta S_{NYW} \quad = \quad -(d-1)\int \Lambda \left[\eta_{ab}e^a \wedge *J^b - m\ vol\ \overline{\psi}\psi\right] \tag{1.56}$$

The second term, where *vol* is the volume form, is present because the mass term explicitly violates the NYW symmetry. For $m = 0$, we have the Ward identity

$$\eta_{ab}e^a \wedge *J^b = 0 \tag{1.57}$$

In components, this is $T^\mu{}_\mu = g^{\mu\nu}e^a_\mu J^b_\nu\eta_{ab}$, the trace of the stress-energy tensor. Thus in this sense, the NYW symmetry gives rise to the same conservation law as does Weyl invariance of the second-order formalism.

Lichnerowicz-Weitzenbock Formula

We end this chapter with a quick derivation of the operator $\not{\mathcal{D}}^2$, which will play a central role in some of our computations in the next chapter. We begin by noting

$$\begin{aligned} \not{\mathcal{D}}^2 &= \gamma^a(\nabla_a + B_a)\gamma^b(\nabla_b + B_b) && (1.58) \\ &= \gamma^a\gamma^b(D_a + B_a)(D_b + B_b) && (1.59) \\ &= \gamma^a\gamma^b\mathcal{D}_a\mathcal{D}_b && (1.60) \end{aligned}$$

where D_a is fully (Lorentz) covariant and $\mathcal{D}_a = D_{\underline{e}_a} + B_a$. In manipulating this expression we need various facts about the Clifford algebra and we also encounter the commutators

$$\begin{aligned} [D_a, D_b] &= -T^c_{ab}D_c + \frac{1}{4}R_{cd;ab}\gamma^{cd} + iqF_{ab} && (1.61) \\ \left[D_{[a}, B_{b]}\right] &= -\frac{1}{2}T^c_{ab}B_c + \frac{1}{2}G_{ab} && (1.62) \end{aligned}$$

where $G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$. Consequently, the operator $\not{\mathcal{D}}^2$ takes the general form

$$\not{\mathcal{D}}^2 = \eta^{ab}\mathcal{D}_a\mathcal{D}_b - \frac{1}{4}R + \frac{iq}{2}F_{ab}\gamma^{ab} + \frac{1}{8}R_{cd;ab}\gamma^{abcd} + \frac{1}{2}\gamma^{ab}G_{ab} - \frac{1}{2}\gamma^{ab}T^c_{ab}\mathcal{D}_c - \frac{1}{2}R^b{}_{a;db}\gamma^{ad} \qquad (1.63)$$

This is called the Lichnerowicz-Weitzenbock formula. In the absence of torsion, the curvature tensors satisfy $\mathring{R}_{ab;cd} = \mathring{R}_{cd;ab}$ and $\mathring{R}^b{}_{a;bd} = \mathring{R}^b{}_{d;ba}$. Therefore the last four terms in (1.63) vanish in this case.

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Chapter 2

Parity odd effective actions

All types of free-fermion topological insulator/superconductor phases can be represented by massive Dirac Hamiltonians with various symmetries, i.e.,

$$H = \sum_{a=1}^{D} p_a \Gamma^a + m\Gamma^0 \tag{2.1}$$

where $\{\Gamma^A, \Gamma^B\} = 2\eta^{AB}$ for $A, B = 0, 1, 2, \ldots D$ and η^{AB} is the flat Lorentz metric. In odd space-time dimensions the Hamiltonians of insulators without additional symmetries (called the unitary A class) are classified by an integer topological invariant ν. Non-trivial insulators, i.e., insulators where $\nu \neq 0$ are said to exhibit the D-dimensional quantum Hall effect, or just the quantum Hall effect if $D = 2$. These systems are gapped in the bulk, but harbor $D-1$-dimensional chiral fermions on their boundaries ($D-1$ would give an even-dimensional boundary space-time). The bulk remains gapped, unless the mass vanishes, at which point there is a topological phase transition between insulating states where ν differs by one. The precise value of ν is not determined by Eq. (2.1) alone but requires information about the regularization scheme to uniquely define ν. Throughout this chapter we will use Pauli-Villars (spectator fermion) type regularization as it matches the structure of many simplified condensed matter lattice-Dirac models including lattice models with Wilson mass terms. Our convention is to choose the regularization such that $m < 0$ is the topological phase with $\nu = 1$ and $m > 0$ is the trivial phase with $\nu = 0$. We note that such a regularization is required even in the absence of all gravitational/torsional effects, as noted in Ref. [29], since otherwise a 2+1-d free-fermion model would give rise to a non-integer Hall conductivity.

The topological insulator phase with $\nu = 1$ will possess chiral boundary states that will produce anomalous currents in the presence of background electromagnetic and gravitational fields. These anomalous currents are matched by a bulk response of the topological insulating state where all anomalous current flowing from the boundary simply flows through the bulk to another boundary. Even without boundaries, the bulk of the material can respond similarly when background fields are present. The bulk response is captured by topological terms that appear in the effective action when the gapped fermions are integrated out in the presence of background fields. For instance as discussed in the previous chapter, the effective action for a massive Dirac fermion in $d = 2+1$ flat space-time in the presence of background electromagnetic fields, contains the parity-odd Chern-Simons term

$$S_{odd}[A] = \frac{\sigma_H}{2}\int_{M_3} A \wedge dA \tag{2.2}$$

where $\sigma_H = \frac{1}{2}(1-\text{sign}(m))\frac{q^2}{2\pi}$. The flow of the corresponding Hall current $*J_{bulk} = \sigma_H dA$ into the boundary between a trivial $\sigma_H = 0$ phase and a topological $\sigma_H = q^2/2\pi$ phase, precisely matches the $U(1)$ anomaly of the edge chiral fermion (to be discussed in the next chapter). In this chapter, we derive such topological response terms in the fermion effective action in odd-dimensional space-times with curvature and torsion, from an anomaly polynomial which is naturally defined in one higher dimension. The relevant terms are easily identified as they violate parity and can be easily extracted. In our discussion below, we will use the techniques presented in [30], albeit adapted to the case with non-zero torsion. Our main emphasis, as mentioned previously, will be on torsional terms and the corresponding transport physics. In particular, we will see that including torsion results in UV divergences in the effective action, which we will carefully regulate. Although such divergences represent non-universal effects, the *difference* of such quantities between distinct phases is finite and is captured by the boundary physics.

2.1 The anomaly polynomial

Let us consider massive Dirac fermions on a $d = D+1 = 2n-1$-dimensional manifold-without-boundary M_{2n-1}, endowed (locally) with the co-frame e^A, spin connection ω_{AB}, and a $U(1)$ connection A. In Euclidean signature, the fermionic quantum effective action is given by

$$S_{eff}[e,\omega,A] = -\ln\det\left(i\not{D}_{2n-1} + im\right). \tag{2.3}$$

Formally, we may rewrite the above as

$$S_{eff}[e,\omega,A] = -\sum_{\lambda_k}\frac{1}{2}\ln\left(\lambda_k^2 + m^2\right) - i\sum_{\lambda_k}\tan^{-1}\frac{m}{\lambda_k} \tag{2.4}$$

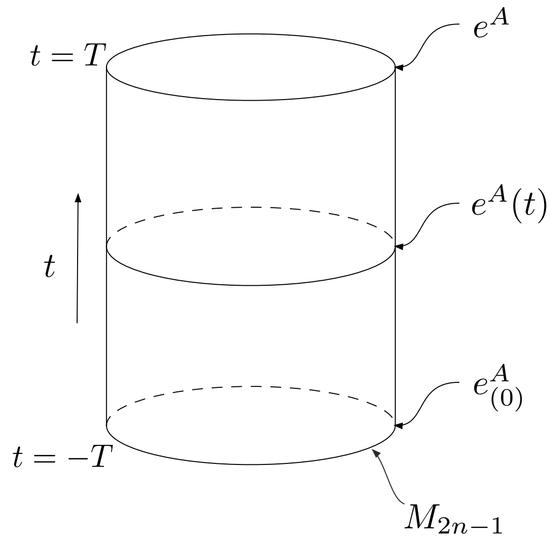

Figure 2.1: An illustration of the one-parameter family of background co-frames, which interpolates between the fiducial co-frame $e^A_{(0)}$ and the co-frame in which we are interested e^A.

where λ_k are the eigenvalues of the Dirac operator: $i\not{D}_{2n-1}|\psi_k\rangle = \lambda_k|\psi_k\rangle$, $|\psi_k\rangle$ being the eigenstates. The parity violating piece must come with odd powers of m

$$S_{odd}[e, \omega, A] = -i \sum_{\lambda_k} \tan^{-1} \frac{m}{\lambda_k}. \tag{2.5}$$

In order to compute (2.5) as a functional of the background gauge and gravitational sources (e^A, ω_{AB}, A), it is convenient to use the following strategy [30]: imagine a one-parameter family of backgrounds $(e^A(t), \omega_{AB}(t), A(t))$ which *adiabatically* interpolates between a fiducial background $(e^A_{(0)}, \omega_{(0)\ AB}, A_{(0)})$ and (e^A, ω_{AB}, A) (see Fig. 2.1).[1] For instance, we may choose the co-frame to be

$$e^A(t) = \begin{cases} e^A_{(0)}, & -\infty < t < -T \\ \frac{1}{2}\left[1 - \varphi(t)\right]\ e^A_{(0)} + \frac{1}{2}\left[1 + \varphi(t)\right]\ e^A, & -T \leq t \leq T \\ e^A, & T < t < \infty \end{cases} \tag{2.6}$$

where $\varphi(t)$ is an arbitrary function which smoothly interpolates between $[-1, 1]$ as t runs from $-T$ to T, for some large and positive T. The other sources $\omega_{AB}(t)$ and $A(t)$ may be chosen similarly. This gives us a one-parameter family of Dirac operators $\not{D}_{2n-1}(t) = \not{D}_{2n-1}[e^A(t), \omega_{AB}(t), A(t)]$ with eigenvalues $\lambda_k(t)$.

[1]Note that this is merely a technique which facilitates the computation. Also, t is an external parameter, and not to be confused with time.

Taking a t-derivative of equation (2.5), we obtain

$$\frac{dS_{odd}}{dt}(t) = im \sum_{\lambda_k} \frac{1}{\lambda_k^2(t) + m^2} \frac{d\lambda_k}{dt}. \tag{2.7}$$

Exponentiating the factor of $(\lambda_k^2 + m^2)^{-1}$ and using $\frac{d\lambda_k}{dt} = \langle \psi_k(t) | i \frac{d\not{D}_{2n-1}}{dt}(t) | \psi_k(t) \rangle$, we therefore find

$$\int_{-\infty}^{\infty} dt \, \frac{d}{dt} S_{odd}(t) = -m \int_{-\infty}^{\infty} dt \int_0^{\infty} ds \; \mathrm{Tr}_{2n-1} \frac{d\not{D}_{2n-1}}{dt} e^{-s\left(m^2 - \not{D}_{2n-1}^2(t)\right)} \tag{2.8}$$

where Tr_{2n-1} is the trace over the spectrum of $\not{D}_{2n-1}(t)$.

On the other hand, consider the $2n$-dimensional Dirac operator $\not{D}_{2n}$ on the space $M_{2n-1} \times \mathbb{R}$ given by[2]

$$\not{D}_{2n} = \sigma^1 \otimes \frac{d}{dt} + \sigma^2 \otimes \not{D}_{2n-1}(t). \tag{2.9}$$

The square of $\not{D}_{2n}$ is easily computed

$$\not{D}_{2n}^2 = \frac{d^2}{dt^2} + i\sigma^3 \otimes \frac{d\not{D}_{2n-1}}{dt} + \not{D}_{2n-1}^2(t). \tag{2.10}$$

Also note that the $2n$-dimensional chirality operator is given by $\Gamma^{2n+1} = \sigma^3 \otimes 1$. Now, define a $2n$-form $\mathcal{P}^{(0)}(m)$ on $M_{2n-1} \times \mathbb{R}$ by

$$\int_{M_{2n-1} \times \mathbb{R}} \mathcal{P}^{(0)}(m) = im\sqrt{\pi} \int_0^{\infty} ds \; s^{-1/2} \mathrm{Tr}_{2n} \Gamma^{2n+1} e^{-s\left(m^2 - \not{D}_{2n}^2\right)} \tag{2.11}$$

where Tr_{2n} is trace over the spectrum of $\not{D}_{2n}$ defined on $M_{2n-1} \times \mathbb{R}$. Notice that $\mathrm{Tr}_{2n} \Gamma^{2n+1} e^{s \not{D}_{2n}^2}$ is the integral over $M_{2n-1} \times \mathbb{R}$ of the Atiyah-Singer index density, which is *locally* exact. Since M_{2n-1} is taken to be without-boundary, $\mathcal{P}^{(0)}(m)$ is a total derivative in t. Using the assumption of adiabaticity we may carry out the trace in the t- direction to obtain

$$\int_{M_{2n-1} \times \mathbb{R}} \mathcal{P}^{(0)}(m) = -m \int_{-\infty}^{\infty} dt \int_0^{\infty} ds \; \mathrm{Tr}_{2n-1} \frac{d\not{D}_{2n-1}}{dt} e^{-s\left(m^2 - \not{D}_{2n-1}^2\right)} + \cdots \tag{2.12}$$

where $\cdots$ indicate terms with three or more t-derivatives. These terms drop out because the background

[2] Here we take the Clifford matrices on $M_{2n-1} \times \mathbb{R}$ to be $\Gamma^0 = \sigma^1 \otimes 1$, $\Gamma^A = \sigma^2 \otimes \gamma^A$

fields are asymptotically t-independent (see Eq. (2.6)). Comparing with (2.8), we conclude that

$$S_{odd}[e,\omega,A] - S_{odd}[e_{(0)},\omega_{(0)},A_{(0)}] = \int_{M_{2n-1}\times\mathbb{R}} \mathcal{P}^{(0)}(m). \tag{2.13}$$

Therefore, the parity odd fermion effective action $S_{odd}[e,\omega,A]$ in $d = 2n-1$ may be interpreted as the "Chern-Simons" form correponding to the locally exact index polynomial $\mathcal{P}^{(0)}(m)$ defined in $2n$ dimensions. We will refer to $\mathcal{P}^{(0)}(m)$ as the *anomaly polynomial*.

We will mainly focus on computing $S_{odd}[e,\omega,A]$ in the limit where the mass scale $|m|$ is taken to be much larger than all background curvature and torsion scales. Our general strategy to compute $\mathcal{P}^{(0)}(m)$ in this limit will be as follows: in the limit $s \mapsto 0$, there exists an *asymptotic expansion*

$$\mathrm{Tr}_{2n}\Gamma^{2n+1}e^{s\not{\mathcal{D}}_{2n}^2} \simeq \sum_{k=0}^{\infty} b_k s^{-n/2+k} \tag{2.14}$$

where the b_k are integrals over M_{2n} of polynomials in curvature, torsion, and their covariant derivatives. These asymptotic expansions in various dimensions can be computed efficiently using techniques from supersymmetric quantum mechanics, which are reviewed in detail in Appendix A (chapter A). The important point here is that it suffices to use this asymptotic expansion in order to extract terms in (2.11) which survive in the limit where $|m|$ is taken to be much larger than all background curvature and torsion scales. Unfortunately, as will become clear soon, the anomaly polynomial as defined above is divergent if the background spin connection is torsional. These are the same divergences that one would encounter in a direct computation of the $2n-1$ dimensional parity odd effective action (for instance, by using Feynman diagrams) in the presence of background torsion. In order to remedy the situation, we introduce N Pauli-Villar's regulator fermions with coefficients C_i and masses M_i, with $i = 1, 2\cdots N$. For convenience, we label $C_0 = 1$ and $M_0 = m$. We then define the regularized anomaly polynomial

$$\mathcal{P}(m) = \sum_{i=0}^{N} C_i \mathcal{P}^{(0)}(M_i). \tag{2.15}$$

The C_i's and M_i's may be determined by requiring UV finiteness. In a condensed matter context this type of regulator is natural in simple lattice Dirac models which are often used to describe topological insulators. These models contain massive spectator Dirac fermions at locations in the Brillouin zone far away from the region which contains the low-energy fermion(s). Indeed, upon including the spectator fermions of the lattice Dirac model (interpreted as Pauli-Villar's regulator fermions), the anomaly polynomial $\mathcal{P}(m)$ becomes finite in arbitrary even dimension; the proof is presented in Appendix B (section B.1).

Since the anomaly polynomial is the (exterior) derivative of the parity odd effective action in $2n-1$ dimensions, it encodes the $2n-1$ dimensional transport coefficients for the two gapped phases. Furthermore, as has been explained in [30, 15], covariant anomalies of the $2n-2$-dimensional edge theory can be extracted out of the fermion effective action in $d = 2n-1$ by computing Hall-type currents passing between the edges through the bulk. In this way, $\mathcal{P}(m)$ encodes all the anomalies of the $2n-2$ dimensional edge theory. Let us now apply the above formalism to explicitly compute the parity odd terms in the fermion effective actions in $d = 2+1$ and $d = 4+1$.

2.2 $d = 2 + 1$

We first begin with the asymptotic expansion (see Appendix A for details)

$$\mathrm{Tr}_4\, \Gamma^5 e^{s\not{\mathcal{D}}^2} \simeq \int_{M_3\times\mathbb{R}} \left(\frac{q_T}{16\pi^2 s} dH + \frac{1}{192\pi^2} \mathrm{tr}\, R^{(-q_T)} \wedge R^{(-q_T)} + \frac{1}{8\pi^2} F\wedge F + \frac{q_T}{96\pi^2} d*d*dH + O(s) \right) \tag{2.16}$$

where we recall that $H = e^A \wedge T_A$, and we have defined $R^{(-q_T)}_{AB}$ to be the curvature 2-form for the connection

$$\omega^{(-q_T)}_{AB} = \mathring{\omega}_{AB} + \frac{q_T}{2} H_{ABC} e^C. \tag{2.17}$$

The terms higher order in s may be ignored as they give rise to negative powers of m. We may also drop the last term in (2.16) as it necessarily contains three or more t-derivatives, and does not pull back to the boundary for asymptotically t-independent backgrounds, as explained in the previous section. The unregulated polynomial (2.11) is then given by

$$\mathcal{P}^{(0)}(m) = \frac{i\zeta^{(0)}_H}{2} dH + \frac{i\kappa^{(0)}_H}{2} \mathrm{tr}\, R^{(-q_T)} \wedge R^{(-q_T)} + \frac{i\sigma^{(0)}_H}{2} F\wedge F. \tag{2.18}$$

The unregulated transport coefficients may be computed from (2.11) and (2.16)

$$\begin{aligned} \zeta^{(0)}_H(m) &= -\frac{q_T}{4\pi}\left[-\frac{m}{\sqrt{\pi\epsilon}} + \sigma_0 m^2 \right] \\ \kappa^{(0)}_H(m) &= \frac{1}{96\pi}\sigma_0 \\ \sigma^{(0)}_H(m) &= \frac{q^2}{4\pi}\sigma_0 \end{aligned} \tag{2.19}$$

where $\sigma_0 = \text{sign}(m)$, and $\frac{1}{\sqrt{\epsilon}} \sim \Lambda$ is the UV cutoff. Introducing the Pauli-Villar's regulator fermions, and requiring finiteness in the limit $\epsilon \mapsto 0$, we are led to the constraints

$$\sum_{i=0}^{N} C_i = 0, \;\; \sum_{i=0}^{N} C_i M_i = 0. \tag{2.20}$$

Even without the UV divergent term this action would need to be regularized due to the fact that the Hall conductivity $\sigma_H^{(0)}(m)$ is not an integer multiple of $\frac{q^2}{2\pi}$ as it must be for a non-interacting system[29]. One possible choice for $\{C_i\}$ and $\{M_i\}$ that solves the constraints can be inferred from the spectator fermion structure of the 2+1-d lattice Dirac model[31] where

M_i	C_i
m	$+$
$m+2\Delta$	-
$m+2\Delta$	-
$m+4\Delta$	$+$

where the energy scale Δ is a large energy scale with $|m| << \Delta << \Lambda$. The regulated anomaly polynomial is then given by[3]

$$\mathcal{P}(m) = \frac{i\zeta_H}{2} dH + \frac{i\kappa_H}{2} \text{tr}\, R^{(-q_T)} \wedge R^{(-q_T)} + \frac{i\sigma_H}{2} F \wedge F \tag{2.21}$$

with the regulated transport coefficients

$$\begin{aligned} \zeta_H &= \frac{q_T m^2}{2\pi} \frac{1-\sigma_0}{2} \\ \kappa_H &= \frac{1}{48\pi} \frac{1-\sigma_0}{2} \\ \sigma_H &= \frac{q^2}{2\pi} \frac{1-\sigma_0}{2}. \end{aligned} \tag{2.22}$$

Since the anomaly polynomial is a total derivative, we may read off the parity odd effective action from the above as the corresponding Chern-Simons form

$$\begin{aligned} S_{odd}[e, \omega, A] &= \frac{i}{2} \int_{M_3} \big(\zeta_H \; e^A \wedge T_A + \sigma_H A \wedge dA \\ &+ \kappa_H \; \text{tr}(\omega^{(-q_T)} \wedge d\omega^{(-q_T)} + \frac{2}{3} \omega^{(-q_T)} \wedge \omega^{(-q_T)} \wedge \omega^{(-q_T)}) \Big) \end{aligned} \tag{2.23}$$

[3] We have also cancelled out a σ_0-independent (and hence independent of whether or not the system is in the topological or trivial phase) divergence proportional to dH by adding a counterterm. Such a counterterm is required only in $d = 2+1$, and not in higher dimensions.

Expanding S_{odd} to linear order in torsion, we find

$$\begin{aligned} S_{odd}[e,\omega,A] &= \frac{i}{2}\int_{M_3}\Big(\sigma_H A\wedge dA + \kappa_H \mathrm{tr}(\mathring{\omega}\wedge d\mathring{\omega} + \frac{2}{3}\mathring{\omega}\wedge\mathring{\omega}\wedge\mathring{\omega}) \\ &+ \zeta_H\; e^A\wedge T_A - q_T\kappa_H\;\mathring{R}e^A\wedge T_A + \cdots\Big). \end{aligned} \tag{2.24}$$

which is the same action that was derived in [15] by a more direct computation. It might seem odd that the coefficient of the $e^A\wedge T_A$ term is a dimensionful parameter, as opposed to the other coefficients, which are universal and quantized – this apparent discrepancy was also pointed out in the previous chapter. We are now in a position to understand this: the quantization of both σ_H and κ_H is forced upon us by the requirement of gauge invariance under large gauge transformations. The $e^A\wedge T_A$ term on the other hand, is *globally* well-defined (i.e., gauge, Lorentz, and diffeomorphism invariant), and hence requires no such quantization of it's coefficient. A second way to understand why σ_H is quantized, is that this coefficient can be shown to be equivalent to the Chern class of a certain Berry connection[4] over momentum space (or Brillouin zone in the case of lattice-calculations), which is quantized from standard arguments. A similar construction for ζ_H is also possible, in that we can interpret ζ_H as the averaged (over the Brillouin zone) Berry curvature in a larger parameter space which includes modular deformations of the Brillouin zone – this computation is presented in section B.2. From this point of view as well, we find that ζ_H is not quantized (as it does not have the interpretation of a topological invariant). It is interesting however to note that the coefficient of the $\mathring{R}\,e^A\wedge T_A$ term is universal (and quantized in the present calculation).

We now focus on the physics of the torsional terms. The $\zeta_H\;e^A\wedge T_A$ term has the interpretation of a relativistic version of the *Hall viscosity*, as has been explained in the previous chapter. Here we wish to delve a bit into the curvature correction $\mathring{R}\;e^A\wedge T_A$ since similar terms will appear in higher dimensions. We may loosely interpret this term as a local-curvature dependent Hall viscosity. On a space-time of the form $\mathbb{R}\times\Sigma$, with Σ a constant curvature Riemann surface of Euler characteristic χ_Σ and area A, terms linear in torsion in (2.24) become

$$S_{odd}[e,\omega,A] = \frac{i}{2}\left(\zeta_H - \frac{4\pi q_T\kappa_H\chi_\Sigma}{A}\right)\int e^A\wedge T_A. \tag{2.25}$$

For curvature and area preserving deformations of the co-frame, we thus find a *shift* in the effective Hall viscosity $\boldsymbol{\zeta_H}$ relative to its flat space value

$$\boldsymbol{\zeta_H} = \zeta_H - \frac{4\pi\kappa_H\chi_\Sigma}{A}. \tag{2.26}$$

[4]The idea is to think of the Brillouin zone as the parameter space. Then the ground state bundle of the Hamiltonian $H(k)$ over the Brillouin zone is a line-bundle, with a canonical connection 1-form called the Berry connection.

This effect is reminiscent of the Wen-Zee shift of the number density in a quantum Hall fluid in the presence of curvature. In fact, let us define the *spin density* $\mathfrak{s}$ of the Chern insulator as

$$\mathfrak{s} = \frac{1}{A}\int_{\Sigma} *J^{12} \tag{2.27}$$

where J^{12} is the spatial component of the spin current J^{AB}. To lowest order in torsion, this may be computed from the action[5] (2.25), and we see that the local spin density is also affected by the local curvature, and in fact satisfies

$$\boldsymbol{\zeta_H} = -\mathfrak{s}. \tag{2.28}$$

Thus, the shift due to curvature may be interpreted as a shift in the spin density relative to its flat space value. Equation (2.28) is similar to the relation between Hall viscosity and spin presented in [32, 33].

Although our main focus in this work is on parity-odd terms, we note that for $d = 2+1$, the parity-even terms can similarly be computed with careful regularization. The complete effective action then arranges into chiral gravity, namely an $SL(2,\mathbb{R})$ Chern-Simons term [15] – we take a brief detour to explain how this works. The parity even terms in the effective action are given by

$$S_{even}[e,\omega,A] = \lim_{\epsilon\to 0^+}\int_{\epsilon}^{\infty}\frac{dt}{2t}\mathrm{Tr}\; e^{-tm^2+t\not{D}^2} \tag{2.29}$$

Once again, it suffices to use the asymptotic expansion for Tr $e^{t\not{D}^2}$ in order to compute terms which survive in the large (ma) limit. The asymptotic expansion in this case is given by (see Appendix A)

$$\mathrm{Tr}\; e^{t\not{D}^2} \simeq \int_{M_3}\frac{2}{(4\pi t)^{3/2}}\left(1-\frac{t}{12}R^{(-q_T)}+O(t^2)\right)vol_{M_3} \tag{2.30}$$

where $R^{(-q_T)} = \mathring{R} - \frac{{q_T}^2}{4}H_{abc}H^{abc}$ is the scalar curvature constructed out of $\omega_{ab}^{(-q_T)}$. Using (2.29) and (2.30), we get

$$S_{even} \quad = \quad \int_{M_3}\left(-\frac{\Lambda^{(0)}}{\kappa_N^{(0)}}vol_{M_3} + \frac{1}{2\kappa_N^{(0)}}\epsilon_{abc}e^a\wedge R^{(-q_T),bc}+\cdots\right) \tag{2.31}$$

where the ellipsis indicates terms of order $(ma)^{-1}$, and the coefficients as written are divergent. In order to regulate all the divergences, we need to introduce more Pauli-Villar's regulators; we list out all the regulated coefficients below:

[5]In particular, J^{AB} is obtained by varying with respect to ω_{AB}, holding e^A fixed.

$$\zeta_H = -\frac{q_T}{4\pi}\sum_{i=0}^{N} C_i\left[-\frac{M_i}{\sqrt{\pi\epsilon}} + \sigma_i|M_i|^2 + \cdots\right] \tag{2.32}$$

$$\sigma_H = \frac{e^2}{4\pi}\sum_{i=0}^{N} C_i\sigma_i + \cdots \tag{2.33}$$

$$i\kappa_H = \frac{1}{96\pi}\sum_{i=0}^{N} C_i\sigma_i + \cdots \tag{2.34}$$

$$\frac{1}{2\kappa_N} = \frac{1}{48\pi}\sum_{i=0}^{N} C_i\left[-\frac{1}{\sqrt{\pi\epsilon}} + |M_i| + \cdots\right] \tag{2.35}$$

$$-\frac{\Lambda}{\kappa_N} = \sum_{i=0}^{N} C_i\left[\frac{2}{3(4\pi\epsilon)^{3/2}} - \frac{M_i^2}{4\pi\sqrt{\pi\epsilon}} + \frac{|M_i|^3}{6\pi} + \cdots\right] \tag{2.36}$$

We require that the terms that diverge as $\epsilon \to 0$ have zero coefficients. This implies

$$\sum_{i=0}^{N} C_i = 0, \qquad \sum_{i=0}^{N} C_i M_i = 0, \qquad \sum_{i=0}^{N} C_i|M_i|^2 = 0 \tag{2.37}$$

Thus, we have one new condition from the parity even sector and we now see that the first condition (we used this above) is also required by the parity even sector. If we assume for simplicity that all of the regulator masses are positive,[6] then we arrive at

$$\begin{aligned}
\zeta_H &= q_T\frac{m^2}{2\pi}\frac{1-\sigma_0}{2} && (2.38)\\
\sigma_H &= -\frac{q^2}{2\pi}\frac{1-\sigma_0}{2} && (2.39)\\
\kappa_H &= -\frac{1}{48\pi}\frac{1-\sigma_0}{2} && (2.40)\\
\frac{1}{\kappa_N} &= \frac{|m|}{12\pi}\frac{1-\sigma_0}{2} && (2.41)\\
\frac{\Lambda}{\kappa_N} &= -\frac{1}{6\pi}\sum_{i=0}^{N} C_i|M_i|^3 && (2.42)
\end{aligned}$$

where again, $\sigma_0 \equiv sign(m)$. If we examine the conditions (2.37), we can furthermore establish that

$$\frac{\Lambda}{\kappa_N} = \Lambda_0^3 - \frac{1}{3\pi}|m|^3\frac{1-\sigma_0}{2} \tag{2.43}$$

for a quantity Λ_0 that generally scales with the regulator masses, but is independent of σ_0. We thus see that apart from the Λ_0^3 term, all of these coefficients vanish in the trivial phase ($\sigma_0 = 1$) and the effective action

[6]This assumption leads to the $m > 0$ phase being trivial. Another choice would make the $m < 0$ phase trivial.

there is just $S^+_{eff} = \Lambda_0^3 \int vol_M$, a pure cosmological term (of course, there are also higher order terms in curvature and torsion, which decay exponentially or as negative powers of (ma), that we have not included here; those terms then determine the transport properties of the trivial phase). The non-trivial phase has an action consisting of the same Λ_0^3 volume term, plus an action that is known as *chiral gravity* (as well as the usual $U(1)$ gauge Chern-Simons term). In other words, the difference of the gravitational actions between the two phases can be written in terms of the Chern-Simons form of a single $SL(2,\mathbb{R})$ connection.[7] Indeed, writing $\omega^a = \frac{1}{2}\epsilon^a{}_{bc}\omega^{bc}$, we define a connection $\mathcal{A}^a = \omega^{(\beta),a} - i\frac{1}{\ell}e^a$. One then finds

$$\begin{aligned} iCS[\mathcal{A}^a] &= 2i\left(\mathcal{A}^a \wedge d\mathcal{A}_a - \frac{1}{3}\epsilon_{abc}\mathcal{A}^a \wedge \mathcal{A}^b \wedge \mathcal{A}^c\right) && (2.44)\\ &= -iCS[\omega^{(\beta)}_{ab}] + \frac{4}{\ell^3}vol + \frac{2}{\ell}\epsilon_{abc}e^a \wedge R^{(\beta),bc} - i\frac{2}{\ell^2}e^a \wedge T^{(\beta)}_a && (2.45)\\ &= -iCS[\omega^{(\beta)}_{ab}] + \frac{4}{\ell^3}vol + \frac{2}{\ell}\epsilon_{abc}e^a \wedge R^{(\beta),bc} - i\frac{6\beta}{\ell^2}e^a \wedge T_a && (2.46) \end{aligned}$$

and we thus see that if we identify $\beta = -q_T$ as above, $\ell = (2|m|)^{-1}$ and $\Lambda = -1/\ell^2$, the action in the non-trivial topological phase is

$$S^-_{eff} = S^+_{eff} + \frac{ik}{4\pi}\int_{M_3} CS[\mathcal{A}^a] - \frac{iq^2}{4\pi}\int_{M_3} CS[A] \tag{2.47}$$

The Chern-Simons level k evaluates to $1/24$.[8] Incidentally, chiral gravity has been studied in the context of holography[35], in which the gravitational fields are dynamical. Indeed if we introduce the notation $\mu = \frac{1}{2\kappa_N \kappa_H}$ (here $\mu\ell = -1$), the Brown-Henneaux formula in asymptotically-AdS_3 geometries gives the central charges of the dual $1+1$-dimensional theory as

$$c_L = \frac{12\pi\ell}{\kappa_N}\left(1 - \frac{1}{\mu\ell}\right),\ c_R = \frac{12\pi\ell}{\kappa_N}\left(1 + \frac{1}{\mu\ell}\right). \tag{2.48}$$

Thus in the holographic case, we have $c_L = 1, c_R = 0$. In the present case, this $1+1$-dimensional matter is supported on the interface between the topological insulator phase and the trivial phase.

[7] Here we are using the language of real time. The group theory involved here is that the isometry group of AdS_3 is $\sim SO(2,2) \sim SO(2,1) \times SO(2,1) \sim SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$. See Ref. [34] for details.

[8] This result satisfies the quantization condition $k \in \frac{1}{48}\mathbb{Z}$ given in [34] for manifolds which admit a spin-structure. Here we get twice that result, because we have a full Dirac fermion in 3d.

2.3 $d = 4 + 1$

Let us now return to parity odd physics, and repeat the above analysis for $d = 4+1$. We begin with the corresponding 6-dimensional asymptotic expansion

$$\begin{aligned} \mathrm{Tr}_6\, \Gamma^7 e^{s\not{D}_6^2} \simeq & \int_{\mathbb{R}\times M_5} \Big(-\frac{q_T}{32\pi^3 s} F\wedge dH - \frac{1}{384\pi^3} F\wedge \mathrm{tr}\, R^{(-q_T)}\wedge R^{(-q_T)} - \frac{1}{48\pi^3} F\wedge F\wedge F \\ & - \frac{q_T}{192\pi^3} d\left(F\wedge *d*dH\right) + \frac{q_T}{384\pi^3} d*d*\left(F\wedge dH\right) + O(s)\Big) . \end{aligned} \tag{2.49}$$

We do not consider $O(s)$ terms as they lead to inverse powers of m, and are generally of higher order in the curvature/torsion expansion. As before, we may also drop the last term in (2.49), as it does not pull back to the boundary effective action. The unregulated anomaly polynomial is then easily obtained

$$\mathcal{P}^{(0)}(m) = \frac{i\zeta_H^{(0)}}{2} F\wedge dH + \frac{i\kappa_H^{(0)}}{2} F\wedge \mathrm{tr}\, R^{(-q_T)}\wedge R^{(-q_T)} + \frac{i\sigma_H^{(0)}}{3} F\wedge F\wedge F + \frac{i\lambda^{(0)}}{2} d\left(F\wedge *d*dH\right) \tag{2.50}$$

with the unregulated transport coefficients

$$\begin{aligned} \zeta_H^{(0)}(m) &= -\frac{qq_T}{8\pi^2}\left[-\frac{m}{\sqrt{\pi\epsilon}} + \sigma_0 m^2\right] \\ \kappa_H^{(0)}(m) &= \frac{q}{192\pi^2}\sigma_0 \\ \sigma_H^{(0)}(m) &= \frac{q^3}{16\pi^2}\sigma_0 \\ \lambda^{(0)}(m) &= \frac{qq_T}{96\pi^2}\sigma_0 . \end{aligned} \tag{2.51}$$

The structure of divergences is the same as previously encountered in $2+1$ dimensions - namely a linear divergence. In fact, more generally the structure of divergences (i.e. linear, quadratic etc.) of the parity-odd effective action is identical in $d = 4n-1$ and $d = 4n+1$. Therefore, it suffices to use the Pauli-Villar's regulators we used in $d = 2+1$, which gives the regulated anomaly polynomial

$$\mathcal{P}(m) = \frac{i\zeta_H}{2} F\wedge dH + \frac{i\kappa_H}{2} F\wedge \mathrm{tr}\, R^{(-q_T)}\wedge R^{(-q_T)} + \frac{i\sigma_H}{3} F\wedge F\wedge F + \frac{i\lambda}{2} d\left(F\wedge *d*dH\right) \tag{2.52}$$

with the regulated transport coefficients

$$\begin{aligned}
\zeta_H &= \frac{qq_T m^2}{4\pi^2}\frac{1-\sigma_0}{2} \\
\kappa_H &= \frac{q}{96\pi^2}\frac{1-\sigma_0}{2} \\
\sigma_H &= \frac{q^3}{8\pi^2}\frac{1-\sigma_0}{2} \\
\lambda &= \frac{qq_T}{48\pi^2}\frac{1-\sigma_0}{2}.
\end{aligned} \tag{2.53}$$

The parity odd effective action in $d = 4+1$ is then given by

$$\begin{aligned}
S_{odd}[e,\omega,A] &= \frac{i}{2}\int_{M_5}\Big(\zeta_H\ F\wedge e^A\wedge T_A + \frac{2\sigma_H}{3}\ A\wedge F\wedge F \qquad (2.54) \\
&+ \kappa_H\ F\wedge \mathrm{tr}\left(\omega^{(-q_T)}\wedge d\omega^{(-q_T)} + \frac{2}{3}\omega^{(-q_T)}\wedge\omega^{(-q_T)}\wedge\omega^{(-q_T)}\right) + \lambda\ F\wedge *d*dH\Big).
\end{aligned}$$

As before, we stress that this should be regarded as giving rise to the leading (in powers of $|m|$) parity-violating terms in correlation functions of the charge, stress, and spin currents. Once again, we may expand this to linear order in torsion to obtain

$$\begin{aligned}
&= \frac{i}{2}\int_{M_5}\Big(\frac{2\sigma_H}{3}\ A\wedge F\wedge F + \kappa_H\ F\wedge \mathrm{tr}(\mathring{\omega}\wedge d\mathring{\omega} + \frac{2}{3}\mathring{\omega}\wedge\mathring{\omega}\wedge\mathring{\omega}) \qquad (2.55) \\
&+ \zeta_H\ F\wedge e^A\wedge T_A - q_T\kappa_H\ (\mathring{R}\ F + 2F_C\wedge \mathring{R}^C + F^{CD}\mathring{R}_{CD})\wedge e^A\wedge T_A + \lambda\ F\wedge *d*dH + \cdots\Big)
\end{aligned}$$

where we have introduced the notation $F_A = F(\underline{e}_A),\ F_{AB} = F(\underline{e}_A, \underline{e}_B),\ \mathring{R}_B = \mathring{R}_{AB}(\underline{e}^A)$ and so on.

Let us focus on the second line above. The term proportional to ζ_H now represents a *magneto-Hall viscosity*, which is to say a dissipationless viscosity in the presence of a magnetic flux through perpendicular spatial dimensions. To be more explicit, let us consider a simple example where we take the space-time manifold to be of the form $M_5 = \mathbb{R}\times\Sigma\times\widetilde{\Sigma}$, with Σ and $\widetilde{\Sigma}$ being two constant curvature Riemann surfaces with areas A and $\widetilde{A}$. If we turn on a $U(1)$ magnetic flux of $F = \frac{2\pi n}{q\widetilde{A}} vol_{\widetilde{\Sigma}}$ through $\widetilde{\Sigma}$ (for $n\in\mathbb{Z}$), then the effective dissipationless viscosity for co-frame deformations in the orthogonal surface Σ is given by

$$\boldsymbol{\zeta}_H = n\frac{q_T m^2}{2\pi}\frac{1-\sigma_0}{2}. \tag{2.56}$$

Just as in $2+1$-d, we also have curvature dependent corrections to the effective magneto-Hall viscosity. For the choice of M_5 and F we are working with, the terms linear in torsion simplify to give us the following

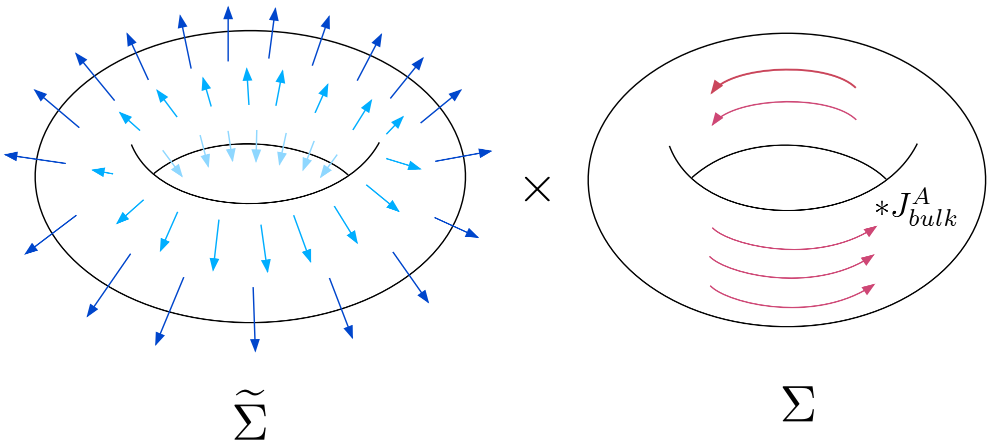

Figure 2.2: An illustration describing the field setup for a magneto-Hall viscosity response: turning on a $U(1)$ flux through $\widetilde{\Sigma}$ gives rise to a Hall viscosity response on Σ.

effective action on the subspace Σ

$$S_{odd}(\Sigma) = \frac{i}{2}\int_{\mathbb{R}\times\Sigma}\left\{\boldsymbol{\zeta}_H - \frac{q_T}{q}\kappa_H\left(2\pi n\mathring{R} + \frac{32\pi^2 n\chi_{\widetilde{\Sigma}}}{\widetilde{A}}\right)\right\} e^A \wedge T_A. \tag{2.57}$$

As before, if we restrict ourselves to curvature and area-preserving co-frame deformations on Σ, we find that the effective magneto-Hall viscosity gets shifted from its flat space value to

$$\boldsymbol{\zeta}_H \mapsto \boldsymbol{\zeta}_H - \frac{q_T}{q}\kappa_H\left(\frac{32\pi^2 n\chi_{\widetilde{\Sigma}}}{\widetilde{A}} + \frac{8\pi^2 n\chi_\Sigma}{A}\right). \tag{2.58}$$

Once again, the shift in the magneto-Hall viscosity may be interpreted as a shift in the spin density on Σ relative to the flat space value.

With the completed derivation of the $2+1$-d and $4+1$-d parity-violating terms in the effective action we are now ready to explore measurable consequences in real condensed matter systems. In the next chapter, we will consider the properties of chiral fermions which live on the boundaries of the phases discussed above, and the corresponding bulk-boundary anomaly inflow. We will later also understand these phenomena from a microscopic spectral flow point of view.

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Chapter 3

Callan-Harvey Anomaly Inflow and Boundary Chiral Anomalies

To study the properties of isolated chiral fermions (or pairs of chiral fermions in a Weyl semi-metal) we must consider their anomaly structure. One nice way to organize the anomalous currents is to consider the low-energy chiral modes which are localized on an interface between topological and trivial phases in odd space-time dimensions. Here we will deal with the case of 1+1 and 3+1 dimensional edge modes, and their relationship with the 2+1 and 4+1 dimensional parity-odd transport coefficients described in the previous chapter.

3.1 $d = 2 + 1$

Consider the non-trivial phase labelled by non-vanishing parity odd coefficients $(\sigma_H, \zeta_H, \kappa_H)$ on a 2+1 dimensional manifold M_3, separated from the trivial phase by the 1+1 dimensional boundary $\Sigma_2 = \partial M_3$. This can be thought of in terms of a 2+1 dimensional Dirac fermion with mass $m < 0$ on M_3, and $m > 0$ outside, with some interpolation region, the interface Σ_2, which we refer to as the domain wall. In general, there could be multiple fermions with mass domain walls along Σ_2, and their number decides $(\sigma_H, \zeta_H, \kappa_H)$. The domain wall hosts 1+1 dimensional chiral fermions, whose anomalies will encode the shifts in $(\sigma_H, \zeta_H, \kappa_H)$ between opposite sides of the domain wall[1]. In the absence of curvature (we will return to the general case later), the parity odd effective action can be taken to be [1,2]

$$S_{odd,bulk}[e, \omega, A] = \frac{\zeta_H}{2} \int_{M_3} e^A \wedge T_A + \frac{\sigma_H}{2} \int_{M_3} A \wedge dA \tag{3.1}$$

Let us first focus on the gauge Chern-Simons term and review its relationship with anomalies in the boundary. In the presence of a boundary, the $U(1)$ Chern-Simons term is diffeomorphism and Lorentz invariant, but

[1] The gravitational Chern Simons terms (proportional to κ_H) lead to currents which are proportional to the Levi-Civita scalar curvature. Hence we ignore these terms temporarily.

[2] In this section, we will use upper case letters for Lorentz indices in the bulk and lower case letters for Lorentz indices on the boundary/domain-wall Σ_2.

not gauge invariant. Under a gauge transformation we have

$$\delta_\alpha S_{odd,bulk} = \frac{\sigma_H}{2}\int_{M_3} d\alpha \wedge F = \frac{\sigma_H}{2}\int_{\Sigma_2} \alpha F \tag{3.2}$$

Gauge invariance implies that this should be accounted for by the $U(1)$ anomaly of chiral fermions localized on Σ_2. For n_L left-handed and n_R right-handed chiral fermions on the edge, the anomaly is given by

$$\delta_\alpha S_{\Sigma_2} = \frac{n_L - n_R}{4\pi} q^2 \int_{\Sigma_2} \alpha F \tag{3.3}$$

This cancels the variation of the bulk action provided $q^2(n_L - n_R) = -2\pi\sigma_H$, which is indeed the case as can be checked by constructing the localized zero modes of the bulk Dirac operator (see [1] for details). The anomaly in (3.3) is called a *consistent* anomaly, because it is obtained by the variation of the chiral effective action in 1+1 dimensions. We refer to the corresponding non-conserved current as J_{cons}, with the anomalous Ward identity

$$d * J_{cons} = \frac{n_R - n_L}{4\pi} q^2\ F = \frac{\sigma_H}{2}\ F \tag{3.4}$$

Returning to the bulk, the variation of the effective action with respect to the gauge field determines the current

$$\delta S_{odd,bulk} = \sigma_H \int_{M_3} \delta A \wedge F + \frac{\sigma_H}{2}\int_{\Sigma_2} \delta A \wedge A \tag{3.5}$$

We can read off the bulk $U(1)$ current from here

$$* J_{bulk} = \sigma_H\ F \tag{3.6}$$

which is conserved by virtue of the Bianchi identity, i.e. $d * J_{bulk} = 0$. However, the flux of the bulk current into Σ_2 is non-trivial and is given by

$$\Delta Q = \int_{\Sigma_2} * J_{bulk} = \sigma_H \int_{\Sigma_2} F = \frac{n_R - n_L}{2\pi} q^2 \int_{\Sigma_2} F \tag{3.7}$$

We can interpret this as the charge injected into the edge from the bulk, but notice that it is twice as much as the consistent anomaly in (3.3). To explain this apparent discrepancy, notice from (3.5) that there is an additional boundary current induced from the bulk

$$* j = \frac{\sigma_H}{2} A \tag{3.8}$$

This prompts us to define the net boundary current $J_{cov} = J_{cons} + j$, which we will call the *covariant current*. The conservation equation for J_{cov} is now

$$d * J_{cov} = \sigma_H \; F = \frac{n_R - n_L}{2\pi} F \tag{3.9}$$

which agrees with (3.7). The anomaly in the form (3.9) is called the *covariant anomaly*.[3] We see therefore, that the covariant current in the boundary carries the charge which is injected into it from the bulk.

The Hall viscosity term in (3.1) on the other hand is invariantly defined under diffeomorphisms, Lorentz, and $U(1)$ gauge transformations, and thus does not lead to consistent anomalies in the edge theory. The consistent frame current J^a_{cons} in the edge is therefore symmetric, and suffers only from the anomaly due to the $U(1)$ Chern-Simons term

$$D * J^a_{cons} - i_{e^a} T_b \wedge *J^b_{cons} - i_{e^a} F \wedge *J_{cons} = -q^2 \frac{n_R - n_L}{4\pi} (i_{\underline{e}^a} A) F \tag{3.10}$$

$$e^{[a} \wedge *J^{b]}_{cons} = 0 \tag{3.11}$$

(Recall that since the domain wall is $1+1$-dimensional, the Lorentz current J^{ab} vanishes.) Equation (3.10) is not gauge covariant. However, it is clear what we must do - we shift to the covariant currents.

Consider then, the variation of the bulk action under a change in the frame and connection

$$\delta S_{odd,bulk} = \zeta_H \int_{M_3} \delta e^A \wedge T_A - \frac{\zeta_H}{2} \int_{M_3} \delta\omega_{AB} e^A \wedge e^B + \frac{\zeta_H}{2} \int_{\Sigma_2} \delta e^A \wedge e_A \tag{3.12}$$

In the boundary term, we should interpret the result in terms of fields defined on the boundary. Generally (as was implied in the discussion of the gauge case above), p-forms will pull back to the boundary. In the case of vector-valued forms e^A and ω^{AB}, we also must decompose the pullbacks in representations of the boundary Lorentz group. Generally, we are free to choose independently a co-frame E^a and spin connection Ω^{ab} in the boundary. These can be identified with the pull-backs of e^a and ω^{ab} up to a Lorentz transformation. The normal components e^n and ω^{na} represent extrinsic effects. Conventionally, the pullback of e^n to Σ vanishes, which can be achieved by a local bulk Lorentz transformation of the frame.

[3] The reason for the terminology *consistent* and *covariant* comes from the more general case of non-Abelian gauge anomalies. In that case, the consistent anomaly satisfies the Wess-Zumino consistency condition, but fails to be gauge covariant (it involves dA rather than F). The covariant anomaly on the other hand, does not satisfy the Wess-Zumino consistency condition, but is fully gauge covariant. The consistent and covariant versions of the anomaly differ by current redefinitions which do not come from local counterterms, but are equivalent so far as anomaly cancellation is concerned. The difference between the covariant and consistent currents is usually referred to as the *Bardeen-Zumino polynomial*.

We read off the bulk frame current and spin current from (3.12)

$$* J^A_{bulk} = \zeta_H \, T^A, \;\; *J^{AB}_{bulk} = -\frac{\zeta_H}{2} e^A \wedge e^B \tag{3.13}$$

while the frame current induced in the edge theory is

$$* j^a = \frac{\zeta_H}{2} \, e^a \tag{3.14}$$

It is easy to check that the bulk currents satisfy the proper (non-anomalous) Ward identities

$$D * J^A_{bulk} - i_{e^A} T_B \wedge *J^B_{bulk} - i_{e^A} R_{BC} \wedge *J^{BC}_{bulk} - i_{e^A} F \wedge *J_{bulk} = 0 \tag{3.15}$$

$$D * J^{AB}_{bulk} - e^{[A} \wedge *J^{B]}_{bulk} = 0$$

But once again, the fluxes into Σ are non-trivial. These are easily computed[4]

$$\Delta Q^a = \zeta_H \int_{\Sigma_2} T^a \tag{3.16}$$

$$\Delta Q^{ab} = -\frac{\zeta_H}{2} \int_{\Sigma_2} e^a \wedge e^b \tag{3.17}$$

We now write the Ward identities in the edge for the covariant currents $J^a_{cov} = J^a_{cons} + j^a$ and J_{cov}

$$\begin{aligned} d * J_{cov} &= \sigma_H \, F && (3.18) \\ D * J^a_{cov} - i_{e^a} T_b \wedge *J^b_{cov} - i_{e^a} F \wedge *J_{cov} &= \zeta_H \, T^a && (3.19) \\ e^{[a} \wedge *J^{b]}_{cov} &= \frac{\zeta_H}{2} e^a \wedge e^b && (3.20) \end{aligned}$$

Notice that the right-hand sides precisely agree with the charge (in this case energy-momentum) entering the edge from the bulk.

Let us now extend the above analysis to include curvature. One finds that the covariant anomalies, when

[4]Here, we disregard the normal component ΔQ^n_Σ. Since this is related to a bulk diffeomorphism normal to the edge, we expect that it is related to *extrinsic* rather than intrinsic edge properties.

written in terms of torsion and Levi-Civita curvature, become

$$
\begin{aligned}
d * J_{cov} &= \sigma_H \; F && (3.21)\\
D * J^a_{cov} - i_{e^a} T_b \wedge * J^b_{cov} - i_{e^a} F \wedge * J_{cov} &= \zeta_H \; T^a + \kappa_H \left(e^a \wedge d\mathring{R} - \mathring{R} \; T^a \right) && (3.22)\\
e^{[a} \wedge * J^{b]}_{cov} &= \frac{1}{2} \left(\zeta_H - \kappa_H \mathring{R} \right) e^a \wedge e^b && (3.23)
\end{aligned}
$$

It is also possible to derive these identities from the intrinsic edge point of view using the Fujikawa method with a suitable choice of regularization; we will show how this works below.

We emphasize again, that *the torsional anomalies appearing above are not obstructions to defining gauge, or diffeomorphism-invariant partition functions.* Given this understanding, it is natural to ask if the torsion terms in the diffeomorphism Ward identity could be removed by the addition of local counterterms. Indeed, a shift of the frame current

$$
* J^a_{cov} \to * J^a_{cov} - \frac{1}{2} \left(\zeta_H - \kappa_H \; \mathring{R} \right) e^a \tag{3.24}
$$

would make it symmetric. This shift however *does not* come from a local counterterm in the boundary theory – which means that no intrinsic $1+1$ dimensional UV completion of the theory is capable of providing such shifts, nor is it an ordinary improvement term.[5] In fact, it amounts to shifting the bulk effective action by

$$
\Delta S_{odd,bulk} = -\frac{1}{2} \int_{M_3} \zeta_H \; e^A \wedge T_A + \frac{1}{2} \int_{M_3} \kappa_H \; \mathring{R} e^A \wedge T_A \tag{3.25}
$$

One of our main precepts is that divergences that appear in the bulk are common to all phases. Thus shifting the values of ζ_H, κ_H, etc. by finite counterterms simultaneously in all phases is allowed, but this does not change the differences in their values between phases. Therefore, we cannot avoid having a torsional response in one of the two phases. It is this invariant information that is encoded in the covariant anomalies of the edge theory, and these are the important physical effects.

Fujikawa method

We now take a short detour to derive the covariant Ward identities discussed above from the intrinsic edge point of view. We will use standard methods that produce the covariant anomalies, and the novelty of the calculation is that we will produce the torsional contributions to the anomalies. In so doing, we will

[5] An *improvement* of the frame current is a current redefinition which makes it symmetric, but does not modify it's conservation equation.

encounter divergences associated with the torsional terms. Our context provides these divergences with a clear interpretation, as the ultraviolet cutoff of the edge theory is determined by the mass gap in the bulk, and their presence is linked with bulk transport properties.

The chiral fermions localized on a 1+1-d space-time manifold Σ_2 coming from the boundary of the manifold M_3 couple to the frame e^a on Σ_2. For simplicity, we will assume that the geometry near Σ_2 is separable, with a frame of the form $e^A = (Ndx, e^a)$. For our purposes, it will also suffice to ignore extrinsic couplings to the chiral fermions because these do not affect the covariant anomaly computations which are of interest here.[6]

Let us quickly review the Fujikawa method for computing covariant anomalies. Our discussion here mainly follows [36, 37]. The main point of the Fujikawa method is that the variation of the (chiral) fermion measure under symmetry transformations leads to anomalous Ward identities. For a Dirac fermion Ψ, one defines the measure as follows: expand Ψ and $\bar{\Psi}$ in terms of eigenfunctions Φ_m of a self adjoint operator, conventionally chosen to be the Dirac operator

$$\not{D}\Phi_m = \lambda_m \Phi_m \tag{3.27}$$

$$\Psi = \sum_m a_m \Phi_m, \quad \bar{\Psi} = \sum_n b_n \bar{\Phi}_n \tag{3.28}$$

and define the measure as $[d\Psi d\overline{\Psi}] = \prod_{m,n} db_m da_n$. However for a left-chiral fermion ψ, the operator $\not{D}_L = \not{D}\frac{1}{2}(1-\gamma^5)$ is not self-adjoint. Thus, ψ must be expanded in terms of eigenfunctions ϕ_m of $\not{D}_L^\dagger \not{D}_L$ and $\bar{\psi}$ must be expanded in terms of eigenfunctions χ_n of $\not{D}_L \not{D}_L^\dagger$. Under a symmetry transformation $T : \psi \to \psi' = \psi + \delta_T \psi$, the measure could transform in general

$$[d\psi' d\overline{\psi}'] = e^{-i\int_{\Sigma_2} \mathcal{A}_T}[d\psi d\overline{\psi}] \tag{3.29}$$

When this happens, the classical Ward identity gets modified by the anomalous correction $\mathcal{A}_T$. Let us now study this in detail for diffeomorphisms and local Lorentz transformations.

[6] The only extrinsic coupling to the chiral states is through a term in the effective action of the form

$$S_{ext} \simeq \frac{1}{48\pi}\int_{\Sigma_2} ln(N)\mathring{R}\, vol_\Sigma \tag{3.26}$$

where $\mathring{R}$ is the Ricci scalar on Σ. This term ensures that while the edge theory is anomalous under a Nieh-Yan-Weyl transformation of the frame on Σ, there is no anomaly due to a Nieh-Yan-Weyl transformation of the bulk frame.

Diffeomorphisms

Recall from (1.49) that under a covariant diffeomorphism generated by a vector field $\underline{\xi}$, we have

$$\delta\psi = \nabla_{\underline{\xi}}\psi \tag{3.30}$$

For simplicity, we restrict ourselves to volume preserving diffeomorphisms. Then to linear order in $\underline{\xi}$, it is easy to check that the measure transforms as

$$[d\psi' d\overline{\psi}'] = \exp\left(-\sum_m \int_{\Sigma_2} vol_{\Sigma_2}\bar{\phi}_m \xi^i \nabla_i \phi_m + \sum_n \int_{\Sigma_2} vol_{\Sigma_2}\bar{\chi}_n \xi^i \nabla_i \chi_n\right)[d\psi d\overline{\psi}] \tag{3.31}$$

Thus the corresponding Ward identity (1.51) gets modified to

$$i\int_{\Sigma_2} \xi^a (D * J_a - i_{e_a} T^b \wedge * J_b - i_{e_a} F \wedge * J) = \sum_m \int_{\Sigma_2} vol_{\Sigma_2}\left(\bar{\phi}_m \xi^a \nabla_a \phi_m - \bar{\chi}_n \xi^a \nabla_a \chi_n\right) = -\mathrm{Tr}_2\, \gamma^5 \xi^a \nabla_a \tag{3.32}$$

Clearly, the trace is ill-defined by itself, and needs to be regulated. Customarily, it is regulated using the heat-kernel regularization in Euclidean space

$$-\mathrm{Tr}_2\, \gamma^5 \xi^a \nabla_a e^{\not{D}^2/\Lambda^2} \tag{3.33}$$

where Λ the ultraviolet cutoff is taken off to infinity. However for the edge theory we consider, the ultraviolet cutoff Λ is of order m, the bulk mass gap, since the spectrum of *localized edge modes* of the bulk Dirac operator only exists for energies $E < |m|$. This issue is irrelevant in the torsionless case, because the leading terms in the anomaly are finite and cutoff independent. However, in the presence of torsion we find a quadratic divergence in the anomaly if regulated naively. Moreover, the divergent term cannot be removed by a local counterterm.

The guiding principle in choosing the appropriate regularization must then be bulk-boundary matching — the non-conservation of charge as manifested in the covariant anomaly must match the influx of charge due to the parity violating terms in the bulk action. Fortunately, a minor generalization of the results in [37] readily implies that (in a separable geometry) the flux of the bulk frame current in a given phase into the edge is given by

$$-\sum_{i=0}^{N} \frac{1}{2} C_i \mathrm{sign}(M_i)\mathrm{Tr}_2\, \gamma^5 \xi^a \nabla_a \frac{1}{(1-\not{D}^2/M_i^2)^{1/2}} \tag{3.34}$$

where M_i are the masses of the bulk fermions, including Pauli-Villars regulators. Notice that this is exactly the trace that was obtained in the Fujikawa formalism, albeit in a regulated form. Therefore, it is clear that we must regulate the Ward identity (3.32) as

$$i \int_{\Sigma_2} \xi^a (D * J_a - i_{e_a} T^b \wedge *J_b - i_{e_a} F \wedge *J) = -\Delta \sum_{i=0}^{N} \frac{1}{2} C_i \mathrm{sign}(M_i) \mathrm{Tr}_2\ \gamma^5 \xi^a \nabla_a \frac{1}{(1 - \not{D}^2/M_i^2)^{1/2}} \tag{3.35}$$

with $M_0 = m$ the mass gap in the bulk, and M_i for $i = 1, \cdots N$ being the masses of the Pauli-Villars regulator fermions in the bulk. The symbol Δ indicates that the anomaly is the difference between the flux from the non-trivial phase and the flux from the trivial phase.

In order to compute the trace (in Euclidean space), we can rewrite it as

$$-\Delta \sum_{i=0}^{N} \frac{1}{2\Gamma(\frac{1}{2})} C_i M_i \int_{\epsilon}^{\infty} dt\ t^{-1/2} \mathrm{Tr}_2\ \gamma^5 \nabla_a e^{-t(-\not{D}^2 + M_i^2)} \tag{3.36}$$

The asymptotic expansion corresponding to this trace in 2 dimensions is given by (see Appendix A)

$$\mathrm{Tr}_2\ \gamma^5 \xi^a \nabla_a e^{t\not{D}^2} \simeq -i \int_{\Sigma_2} \xi^a \left(\frac{1}{4\pi t} T_a + \frac{1}{48\pi} e_a \wedge d\mathring{R} - \frac{1}{48\pi} \mathring{R}\, T_a + \cdots \right) \tag{3.37}$$

where the ellipsis denote terms higher order in t (which are unimportant here as they will give higher order terms suppressed by inverse powers of the cutoff). The integral over t in (3.36) diverges as $\epsilon \to 0$ for the first term above, but the divergence is cancelled by the condition $\sum_{i=0}^{N} C_i M_i = 0$ on the regulator masses. Using the expressions for regulated bulk coefficients (2.32, 2.33, 2.34), we get the Ward identity

$$(D * J_a - i_{e_a} T^b \wedge *J_b - i_{e_a} F \wedge *J) = \zeta_H\ T_a + \kappa_H\ (e_a \wedge d\mathring{R} - \ \mathring{R}\, T_a) \tag{3.38}$$

where ζ_H and κ_H are the regulated coefficients in the non-trivial phase. Note that this is exactly what we found in our analysis in the previous section (see (3.22)).

Local Lorentz transformations

The change in the measure corresponding to Lorentz transformations $\delta\psi = \frac{1}{4}\theta_{ab}\gamma^{ab}\psi$ is given by

$$[d\psi' d\overline{\psi}'] = \exp\left(-\frac{1}{4} \mathrm{Tr}_2\ \theta_{ab} \gamma^5 \gamma^{ab} \right) [d\psi d\overline{\psi}'] \tag{3.39}$$

Following the discussion of the diffeomorphism anomaly in the previous section, we regulate the Ward identity as

$$i\int_{\Sigma_2} \theta_{ab}\, e^{[a} \wedge *J^{b]} = -\frac{1}{4}\Delta \sum_{i=0}^{N} \frac{1}{2} C_i \mathrm{sign}(M_i) \mathrm{Tr}_2\, \theta_{ab}\gamma^5\gamma^{ab} \frac{1}{(1-\not{D}^2/M_i^2)^{1/2}} \tag{3.40}$$

Using the asymptotic expansion in 2 dimensions

$$\mathrm{Tr}_2\, \gamma^5\gamma^{ab} e^{t\not{D}^2} \simeq \int_\Sigma -i\epsilon^{ab}\left(\frac{1}{2\pi t} - \frac{1}{24\pi}\mathring{R} + \cdots\right) vol_\Sigma \tag{3.41}$$

we find (3.23)

$$e^{[a} \wedge *J^{b]}_{cov} = \frac{1}{2}\left(\zeta_H - \kappa_H \mathring{R}\right) e^a \wedge e^b \tag{3.42}$$

So, in summary, it is possible to derive the anomalies of the edge theory from an intrinsic calculation; provided we use consistent regulators, we find a precise match between the bulk and the boundary computations.

3.2 $d = 4 + 1$

Let us now turn to the $d = 4 + 1$ case. Consider then the non-trivial phase labelled by transport coefficients $(\sigma_H, \zeta_H, \kappa_H, \lambda)$ on a 4+1 dimensional manifold M_5, separated from the trivial phase by a 3+1 dimensional interface $\Sigma_4 = \partial M_5$. As before, one model for this system is a 4+1 dimensional Dirac fermion with mass $m < 0$ on M_5, and $m > 0$ outside, with some interpolation region, the interface Σ_4, which we refer to as the domain wall. In general, there could be multiple fermions with mass domain walls along Σ_4, and their number decides $(\sigma_H, \zeta_H, \kappa_H, \lambda)$. The domain wall hosts 3+1-d chiral fermions, whose anomalies will encode the differences in $(\sigma_H, \zeta_H, \kappa_H, \lambda)$ between opposite sides of the domain wall.

In order to avoid complicating our discussion, we will follow our previous strategy and first explain the anomaly inflow only focusing on the first two terms in (2.54), and later present the more general result. We start with the 4+1-d bulk effective action

$$S_{bulk} = i\frac{\sigma_H}{3}\int_{M_5} A \wedge F \wedge F + i\frac{\zeta_H}{2}\int_{M_5} F \wedge H \tag{3.43}$$

where we recall the notation $H = e^A \wedge T_A$. The first term is the second (Abelian) Chern-Simons form and is diffeomorphism and Lorentz invariant, but not $U(1)$ invariant. This gauge non-invariance must be compensated by the *consistent* anomaly of the boundary/interface theory. This means that the boundary effective action S_{bdry} cannot be gauge invariant either. In fact, under a $U(1)$ gauge transformation $\delta A = d\alpha$, we must have

$$\delta_\alpha S_{bdry} = -\frac{i\sigma_H}{3}\int_{\Sigma_4} \alpha F \wedge F. \tag{3.44}$$

in order to cancel the gauge variation of the bulk Chern-Simons term. Interestingly, the second term in (3.43) is gauge, diffeomorphism, and Lorentz invariant despite its similarity to the first term, and hence we do not expect it to contribute to consistent anomalies in the boundary. This is an important distinction between the two terms. Using these constraints, the consistent Ward identities on the boundary are[7]

$$d * J_{cons} = \frac{\sigma_H}{3} F \wedge F \tag{3.45}$$

$$D * J^a_{cons} - i_{\underline{e}^a} T_b \wedge *J^b_{cons} - i_{\underline{e}^a} R_{bc} \wedge *J^{bc}_{cons} - i_{\underline{e}^a} F \wedge *J_{cons} = -\frac{\sigma_H}{3} i_{\underline{e}^a} A \wedge F \wedge F \tag{3.46}$$

$$D * J^{ab}_{cons} + e^{[b} \wedge *J^{a]}_{cons} = 0 \tag{3.47}$$

where lower-case Latin indices are local Lorentz indices on the boundary manifold Σ_4. The Ward identities written in terms of consistent currents are clearly not gauge covariant since they depend on gauge-variant fields like the vector-potential A. To remedy the situation, we must write these in terms of *covariant currents*. Consider then, the variation of the bulk response action[8]

$$\delta S_{bulk} = \int_{M_5} \left(\delta A \wedge *J_{bulk} + \delta e_A \wedge *J^A_{bulk} + \delta\omega_{AB} \wedge *J^{AB}_{bulk}\right) + \int_{\Sigma_4} \left(\delta A \wedge *j + \delta e^a \wedge *j_a + \delta\omega_{ab} \wedge *j^{ab}\right) \tag{3.48}$$

The conserved Hall currents in the bulk are given by

$$*J_{bulk} = \sigma_H F \wedge F + \frac{\zeta_H}{2} dH \tag{3.49a}$$

$$*J^A_{bulk} = \zeta_H F \wedge T^A \tag{3.49b}$$

$$*J^{AB}_{bulk} = -\frac{\zeta_H}{2} F \wedge e^A \wedge e^B \tag{3.49c}$$

[7]Note that the right hand side of equation (3.46) originates from the fact that this Ward identity corresponds to a *covariant* diffeomorphism, which involves an ordinary diffeomorphism plus a $U(1)$ and local Lorentz gauge transformation.

[8]Here we will assume that the boundary values of the variations δe^A and $\delta\omega_{AB}$ are non-zero only when the Lorentz indices are those of the boundary. In other words, we are ignoring extrinsic effects here.

while the induced currents in the boundary are

$$*j = \frac{2}{3}\sigma_H A \wedge F + \frac{\zeta_H}{2} H \tag{3.50a}$$

$$*j^a = \frac{\zeta_H}{2} F \wedge e^a \tag{3.50b}$$

$$*j^{ab} = 0. \tag{3.50c}$$

Define the covariant boundary currents $J_{cov} = J_{cons} + j$, $J^a_{cov} = J^a_{cons} + j^a$, and $J^{ab}_{cov} = J^{ab}_{cons} + j^{ab}$. Then the Ward identities written in terms of these are

$$d * J_{cov} = \sigma_H F \wedge F + \frac{\zeta_H}{2} dH \tag{3.51}$$

$$D * J^a_{cov} - i_{\underline{e}^a} T_b \wedge *J^b_{cov} - i_{\underline{e}^a} R_{bc} \wedge *J^{bc}_{cov} - i_{\underline{e}^a} F \wedge *J_{cov} = \zeta_H F \wedge T^a \tag{3.52}$$

$$D * J^{ab}_{cov} + e^{[b} \wedge *J^{a]}_{cov} = -\frac{\zeta_H}{2} F \wedge e^a \wedge e^b. \tag{3.53}$$

These are referred to as the *covariant anomalies* in the boundary theory. Notice that these precisely match the fluxes of bulk Hall currents (3.49) into Σ_4

$$\Delta Q = \sigma_H \int_{\Sigma_4} F \wedge F + \frac{\zeta_H}{2} \int_{\Sigma} dH \tag{3.54a}$$

$$\Delta Q^a = \zeta_H \int_{\Sigma_4} F \wedge T^a \tag{3.54b}$$

$$\Delta Q^{ab} = -\zeta_H \int_{\Sigma_4} F \wedge e^a \wedge e^b. \tag{3.54c}$$

Thus, the charge, momentum, and spin injected into the edge from the bulk are carried by the *covariant* currents J_{cov}, J^a_{cov}, and J^{ab}_{cov} respectively.

Having described the general idea of anomaly inflow in a simpler setting, we now give the full result for edge anomalies. Applying the same ideas discussed above to the full effective action (2.54), we get the flux of bulk charge, stress, and spin currents into the edge

$$\Delta Q = \int_{\Sigma_4} \left(\sigma_H F \wedge F + \frac{\zeta_H}{2} dH + \frac{\kappa_H}{2} \mathrm{tr}\, R^{(-q_T)} \wedge R^{(-q_T)} + \frac{\lambda}{2} d * d * dH \right) \tag{3.55a}$$

$$\Delta Q^a = \int_{\Sigma_4} \left(\zeta_H F \wedge T^a + \kappa_H\, e^a \wedge d\mathcal{A}_2 - q_T \kappa_H\, \mathcal{A}_2 \wedge T^a + \lambda\, d * d * F \wedge T^a \right) \tag{3.55b}$$

$$\Delta Q^{ab} = -\int_{\Sigma_4} \left(\frac{\zeta_H}{2} F - \frac{q_T \kappa_H}{2}\, \mathcal{A}_2 + \frac{\lambda}{2}\, d * d * F \right) \wedge e^a \wedge e^b \tag{3.55c}$$

where we have defined

$$\mathcal{A}_2 = \left(F \wedge R^{(-q_T)}_{ab}\right)(\underline{e}^a, \underline{e}^b) = \left(F^{ab} R^{(-q_T)}_{ab} + 2F^a \wedge R^{(-q_T)}_a + R^{(-q_T)} F\right). \tag{3.56}$$

These are the covariant $U(1)$, diffeomorphism, and Lorentz anomalies of the edge theory in the presence of curvature. Note the appearance of the dimensionful viscosity term $\frac{\zeta_H}{2}\, dH$ in the chiral $U(1)$ anomaly. This might seem problematic given the topological character of the (integrated) chiral anomaly. However, note that H is a globally well defined 3-form (unlike, for instance $A \wedge dA$), and dH is truly a total derivative. On compact 4-manifolds then, this term drops out. On the physics side, we are interested in the *local* anomaly densities – which is why it is important for us to keep this term. In fact, this term is precisely the Nieh-Yan term discussed earlier, and it now has a clear meaning in the present context: its coefficient is the difference of magneto-Hall viscosities across a 3+1-d interface between two different topological phases.

Using the structure of the anomalous terms presented here, we will show in the next chapter the microscopic origin of a subset of the anomalous currents using spectral-flow type arguments in the Hamiltonian formalism of the chiral boundary states. This will clarify the physical origin of the terms in which we are most interested, and will give a nice interpretation for some of the torsional contributions to the anomalous currents.

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Chapter 4

Spectral flow

In this chapter we will discuss the covariant anomalies of the boundary theory from the point of view of adiabatic spectral flow of the single-particle Hamiltonian spectrum of the chiral boundary states. In $d = 2+1$, we will first review the well-known case of the Hall conductivity and spectral flow induced by $U(1)$ electric field, and then move on to Hall viscosity. Similarly, in $d = 4 + 1$ we will review the well-known case of the Hall conductivity in presence of a background magnetic flux, and then move on to magneto-Hall viscosity, and the torsional chiral anomaly.

4.1 Hall Conductivity in $d = 2 + 1$

We consider a gauge field on a spatial cylinder of length L in the x-direction and radius R in the y-direction

$$A = E_y t dy \tag{4.1}$$

where E_y is a constant. This is equivalent to

$$F = E_y dt \wedge dy \tag{4.2}$$

$$*_3 F = -E_y dx \tag{4.3}$$

Thus we have a constant electric field in the y-direction which we imagine resulting from the threading of electromagnetic flux along the cylinder. We can parameterize $E_y = -\frac{h}{2\pi q R T}$ where q is the charge and T is the time it takes to thread one flux quantum into the hole of the cylinder.

Given the effective action for the bulk charge response $S_{eff}[A] = \int_M \left(\frac{1}{2}\sigma_H A \wedge dA\right)$, where $\sigma_H = -\frac{q^2}{h}$ and A is the electro-magnetic gauge field, the expectation value for the current is given by $J = \sigma_H *_3 F$. Thus we find the bulk current response to the electric field is

$$J = \frac{q^2}{2\pi\hbar}\frac{\hbar}{qRT}dx = \frac{q}{2\pi RT}dx \tag{4.4}$$

This means there is a constant current density in the x-direction, and over the time T we build up charge

$$\Delta Q = \int_0^T dt \int_0^{2\pi R} dy J_x = q. \tag{4.5}$$

From the point of view of the intrinsic boundary theory, which consists of chiral fermions of one chirality or the other, this increase in charge is an anomalous process. In $1+1$, the chiral symmetry is anomalous $d *_2 J_5 = \frac{q}{2\pi\hbar}F_{(2)}$, where $F_{(2)}$ is the gauge curvature in $1+1$. In the present case, this is the pull-back of the bulk field, which is just $F_{(2)} = E_y dt \wedge dy$. So the axial charge changes in this process by $\Delta Q_5 = \int_{\Sigma_x} d *_2 J_5 = -1$. This change occurs as a chiral fermion is pumped from one edge of the cylinder to the other through the bulk Chern-Simons response.

We can get a simpler pictorial understanding of the anomaly by considering the energy spectrum of the chiral fermions. The Hamiltonians for the left- and right-handed chiral fermions are

$$H_R = v(p - qA) \qquad H_L = -v(p - qA) \tag{4.6}$$

where the vector potential is $A = \frac{\hbar t}{qRT}$. Substituting this form into Eq. (4.6) we find

$$H_R = \frac{\hbar v}{R}\left(n - \frac{t}{T}\right) \qquad H_L = -\frac{\hbar v}{R}\left(n - \frac{t}{T}\right) \tag{4.7}$$

where n is an integer labeling the discrete momentum modes $p = \frac{2\pi n\hbar}{2\pi R}$. Assuming that T is very large so that the spectrum changes adiabatically, we find that the spectrum *flows* as time increases. At a time $t = T$, or in fact at any multiple of T, the spectrum returns to its initial configuration, yet the system as a whole has changed because the state occupation changes. When $t = rT$ for integer r there have been r flux-quanta threaded into the circle on which the chiral fermions live. For each flux quanta threaded an electron is transferred from the left movers to the right movers as illustrated in Fig. 4.1. Thus we reproduce our calculations from above by observing the transfer of electrons during the spectral flow process.

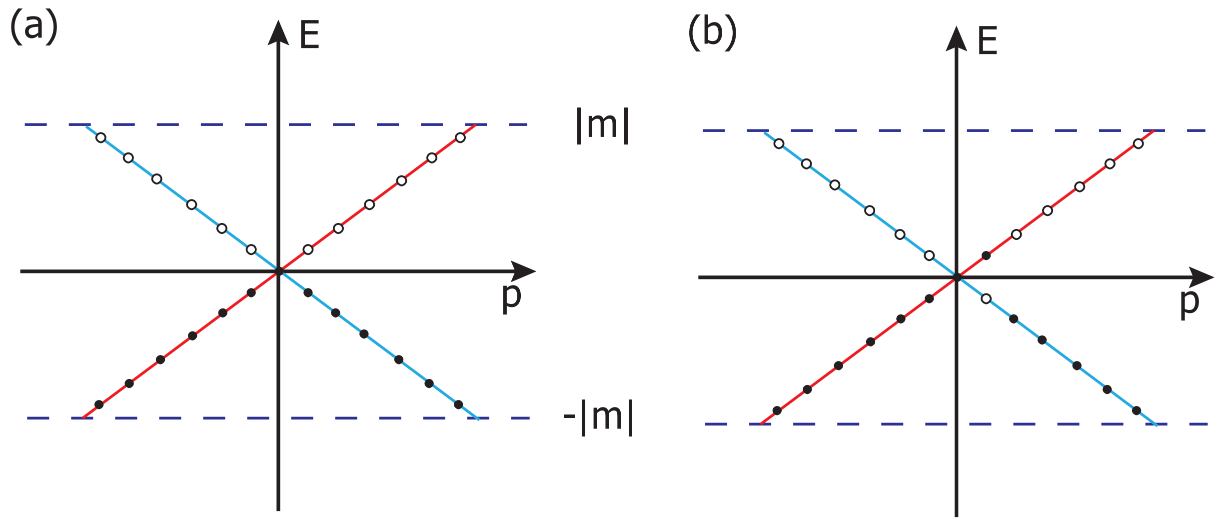

Figure 4.1: (a) Energy spectrum from Eq. (4.6) at time $t = 0$. Right/left handed fermion spectra are represented by positively/negatively sloped lines. The filled/empty circles represent occupied/unoccupied states. (b) Energy spectrum at time $t = T$ where one flux quantum has been threaded through the spatial ring. The spectrum returns to itself but the state occupation changes. One electron has been added to the right movers, and one has been removed from the left movers.

4.2 Hall Viscosity in $d = 2 + 1$

Now we would like to understand the momentum transport due to the Hall viscosity using a spectral-flow type argument similar to the case of charge transport. On the spatial cylinder of length L in the x-direction and radius R in the y-direction, consider the co-frame

$$e^0 = dt,\ e^1 = dx,\ e^2 = (1 + h(t))dy \tag{4.8}$$

where we will parameterize $h(t) = \frac{bt}{2\pi RT}$ where T is a very large time-scale so that the change is adiabatic, and b has units of length. For simplicity, we will choose the connection $\omega_{AB} = 0$, for which the given frame is torsional. This configuration represents the threading of torsion flux $T^2 = \frac{b}{2\pi RT} dt \wedge dy$, *i.e.* a dislocation into the ring with a time-dependent Burgers' vector tangent to the ring with length bt/T at time t. To calculate the bulk energy-momentum flow we must introduce a *covariant Killing vector field*[1] $\xi = \xi_a\, \underline{e}^a = \partial_y$. From our previous discussion of Hall viscosity response, the energy-momentum flux along ξ through a constant x

[1] We call a vector field ξ *covariantly Killing* if the co-frame is preserved under a covariant diffeomorphism along ξ, i.e. if $D\xi^a + i_\xi T^a = 0$.

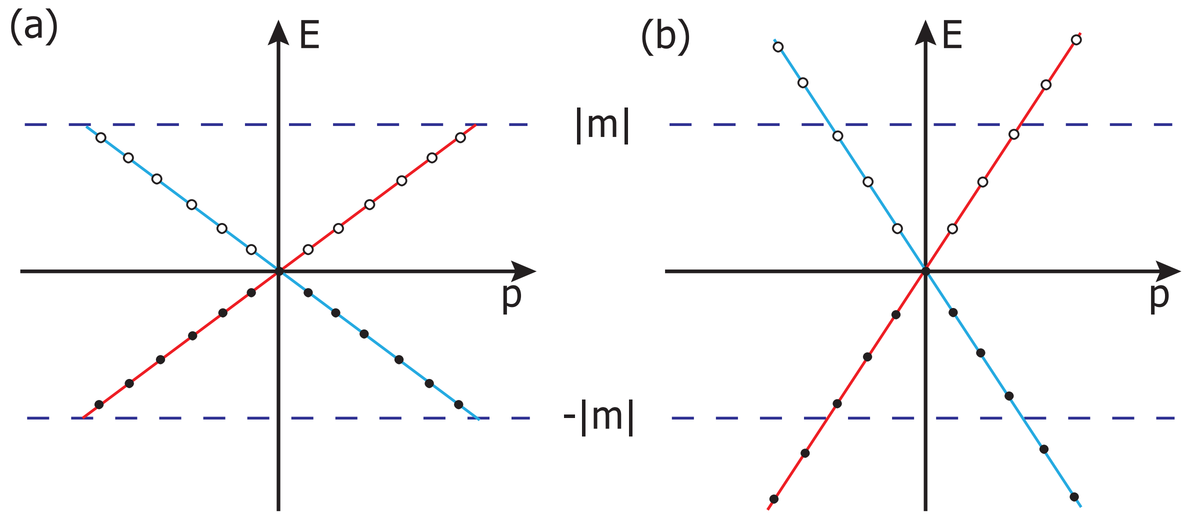

Figure 4.2: (a)Energy spectra for left (blue) and right (red) handed chiral fermions from Eq. 4.12 at $t = 0$ (b) Energy spectra for $t > 0$ assuming $b_L = b_R = b < 0$ which gives an increase to the velocity for both branches of chiral fermions. Note that when compared to the $t = 0$ case there are states that were occupied chiral fermion states that have been pushed past the cut-off scale and the states at $p = 0$ are unchanged. During this process no states cross $E = 0$ and there is not a conventional notion of spectral flow at low-energy. Both figures have the same momentum discretization spacing, but different velocities which leads to a different number of states within the cut-off window.

slice Σ_x is given by

$$\int_{\Sigma_x} \xi_a * J^a = \int_{\Sigma_x} \xi_a T^a = \zeta_H \int_{\Sigma_x} \left(1 + \frac{bt}{2\pi RT}\right) \frac{b}{2\pi RT} dt \wedge dy \tag{4.9}$$

This leads to a transfer of momentum from one edge to the other through the bulk of the cylinder. From the point of view of the edge chiral fermions localized at $x = 0$ and $x = L$, this is an anomalous process. For instance at $x = 0$, define the chiral momentum $P_L(t) = \int_{\gamma_t} \xi_a * J^a$, where γ_t is the spatial circle at time t and J^a the intrinsic 1+1d frame current on $\Sigma_{x=0}$. Then the anomalous conservation law (3.19) becomes (in the absence of $U(1)$ gauge fields)

$$\frac{dP_L}{dt} = \zeta_H \int_0^{2\pi R} \left(1 + \frac{bt}{2\pi RT}\right) \frac{b}{2\pi RT} dy = \zeta_H \left(1 + \frac{bt}{2\pi RT}\right) \frac{b}{T} \tag{4.10}$$

Let us now understand the anomalous momentum transfer in terms of the spectra of the left and right handed chiral fermions localized at $x = 0$ and $x = L$ respectively, with co-frame fields parameterized by b_L

and b_R with the Hamiltonians

$$H_R = \frac{\hbar v k}{1+\frac{b_R t}{2\pi R T}} \tag{4.11}$$

$$H_L = -\frac{\hbar v k}{1+\frac{b_L t}{2\pi R T}} \tag{4.12}$$

where v is the chiral fermion velocity and we have assumed the Hamiltonian is acting on translationally invariant plane-wave states. We show the energy spectra at two different times in Fig. 4.2(a) and (b). In the figure we have indicated a high-energy cut-off governed by the scale $|m|$. This scale represents the energy at which the edge states of a topological insulator merge with bulk states and are no longer localized on the edge. In fact, for such topological insulators like the lattice Dirac model, the cut-off is *exactly* the insulating mass scale $|m|$. We have assumed that the energy states are filled up to energy $E=0$ as indicated by the filled circles in Fig. 4.2. The range of momenta that is occupied by right (left) movers is between $p \in \left[-\frac{m}{v}\left(1+\frac{b_L t}{2\pi RT}\right), 0\right]$ $\left(p \in \left[0, \frac{m}{v}\left(1+\frac{b_R t}{2\pi RT}\right)\right]\right)$. The total momenta of the right and movers at time t is

$$\begin{aligned} P_R^{(tot)} &= \frac{2\pi R}{2\pi\hbar}\int_{-\frac{m}{v}\left(1+\frac{b_R t}{2\pi RT}\right)}^{0} p dp = -\frac{R}{\hbar}\left[\frac{m}{v}\left(1+\frac{b_R t}{2\pi RT}\right)\right]^2 \\ P_L^{(tot)} &= \frac{2\pi R}{2\pi\hbar}\int_{0}^{\frac{m}{v}\left(1+\frac{b_L t}{2\pi RT}\right)} p dp = \frac{R}{\hbar}\left[\frac{m}{v}\left(1+\frac{b_L t}{2\pi RT}\right)\right]^2 \end{aligned} \tag{4.13}$$

As we have seen, the Hall viscosity is related to a stress-energy response and thus to the rate of change of momentum. We find

$$\begin{aligned} \dot{P}_R^{(tot)} &= -\left(\frac{m}{\hbar v}\right)^2\left(1+\frac{b_L t}{2\pi RT}\right)\frac{\hbar b_L}{2\pi T} \\ \dot{P}_L^{(tot)} &= \left(\frac{m}{\hbar v}\right)^2\left(1+\frac{b_R t}{2\pi RT}\right)\frac{\hbar b_R}{2\pi T} \end{aligned} \tag{4.14}$$

We see that if we choose $b_L \neq b_R$ then momentum is not conserved at all if we only consider the edge states and take into account transfers between the edges. Momentum of course is still conserved globally because the excess/deficient amount of momentum gets trapped on some extra torsional flux that will appear in the gapped bulk region away from the edges when $b_L \neq b_R$. For now we will fix $b_L = b_R = b$ to avoid this extra complication.

Comparing equations (4.10) and (4.14), we see that the bulk and boundary momentum transport only matches for

$$\zeta_H = \frac{\hbar}{2\pi}\left(\frac{m}{\hbar v}\right)^2 \tag{4.15}$$

which is the same result we calculated earlier for the regulated Hall viscosity albeit with all the factors of $\hbar$ and velocity (speed of light) added back in.

While at first glance it appears strange that the viscosity depends on the mass, we can clearly see the reason why this dependence is necessary by examining Fig. 4.2. The effect of threading a torsional flux (*i.e.* threading a dislocation) into the loop on which the chiral fermions propagate can be interpreted in one of two ways. Our choice of torsional flux (*i.e.* our specific choice of frame) means that if we travel around the loop at time $t = T$ we enclose a Burgers' vector that is tangent to the ring of length b. Depending on the sign of b this implies that the ring looks either shorter or longer than its original length at $t = 0$. From this perspective we would think of chiral fermions with a fixed velocity but propagating on a ring with a time-dependent length (which will re-discretize the momentum modes as a function of time). The other interpretation is that the length stays fixed at $2\pi R$ but the chiral fermions are either traveling faster or slower depending on the sign of b. This is the interpretation represented in Fig. 4.2 where the velocity of the chiral fermions has increased at a later time but the momenta retain the original quantization scale. Thus we see that coupling to the $U(1)$ electromagnetic field causes a *translation* in the spectrum, but the coupling to torsion causes a *scaling* of the spectrum. As a function of time the two chiral branches rotate in opposite directions around the fixed point where $p = 0$. This is because $p = 0$ does not feel any effects of torsion since it is *uncharged* as far as torsion is concerned. So the torsional response is given by spectral scaling/rotation instead of spectral flow/translation. In terms of the discussion we used in the introduction this occurs because each momentum mode carries a different charge under torsion, while they all carry the same $U(1)$ gauge charge. In fact, the state at $p = 0$ does not even see the torsional flux and is unmodified since it carries zero torsional charge.

4.3 Magneto-Hall conductivity in $d = 4 + 1$

Let us now move on to $d = 4+1$. First we will study the effects of the $U(1)$ second Chern-Simons term that enters the response action

$$S_{bulk} = \frac{\sigma_H}{3}\int_{M_5} A \wedge F \wedge F. \tag{4.16}$$

This term gives rise to the 4+1-d quantum Hall effect in which a charge current is carried through the bulk in a direction perpendicular to applied electric and magnetic fields. This is reminiscent of the 2+1-d effect where a current is generated perpendicular to an applied electric field. Here we have a non-linear topological response which requires simultaneous electric and magnetic fields. The reason, of course, is well-known: the bulk current is intertwined with the boundary chiral anomalies which require parallel electric and magnetic fields on the 3+1-d surface. In 2+1-d the bulk Hall current is also connected with the 1+1-d chiral anomaly on the edge, but in this case the anomalous current is generated in the presence of an electric field alone.

To simplify our discussion let us consider the spatial geometry to be $\Sigma_3 \times [0, L]$, where $\Sigma_3 = \mathbb{R} \times S^1 \times S^1$. We will label the bulk direction by $w \in [0, L]$, while the coordinates on Σ_3 will be labelled by (x, y, z) with x being the non-compact direction. The edge states will be localized at $w = 0$ and $w = L$. We turn on a magnetic field B perpendicular to the surface of the (x, y)-cylinder, and an electric field $E_z = \frac{2\pi}{qL_zT}$ (for some large and positive time scale T and with $\hbar = 1$). This electric field can be generated by slowly threading magnetic flux through the hole of the (z, w) cylinder. The corresponding gauge field configuration will be chosen to be

$$A = Bxdy + E_z tdz \tag{4.17}$$

where the $U(1)$ flux is then given by

$$F = Bdx \wedge dy + E_z dt \wedge dz. \tag{4.18}$$

From the bulk Chern-Simons response we have the bulk Hall current

$$* J_{bulk} = \sigma_H F \wedge F = \frac{q^3}{8\pi^2} BE_z dt \wedge dx \wedge dy \wedge dz. \tag{4.19}$$

This yields a constant current density through the bulk in the w-direction and leads to a charge transfer over a time period T of

$$\Delta Q = \int_0^T \int_{\Sigma_3} *J_{bulk} = q^2 \frac{BL_x L_y}{2\pi} \tag{4.20}$$

from one edge to the other. Given that the system is in the non-trivial topologically insulating phase, we have a left-handed chiral fermion localized at $w = 0$ and a right-handed chiral fermion localized at $w = L$. From the boundary point of view, the above charge transfer is an anomalous process, which corresponds to the $U(1)$ chiral anomaly in the boundary theory

$$d * J_{cov} = \sigma_H F \wedge F. \tag{4.21}$$

Indeed, the anomalous charge created or destroyed on a boundary during the above process is precisely equal to the charge transferred across the bulk of the insulator by the Hall-current, as expected.

We can develop a more intuitive, microscopic picture of the anomaly from the Hamiltonian energy spectra of the chiral boundary states during the adiabatic flux threading process. In the presence of the above gauge field configuration, the low-energy spectrum on the boundary consists of two types of states (see Appendix B, section B.3): (i) positive and negative energy towers of gapped states

$$E(\ell, p_z, \sigma) = \pm \left\{ (p_z - qA_z)^2 + 2|qB| \left(\ell + \frac{1+\sigma}{2} \right) \right\}^{1/2}, \quad \ell = 1, 2, 3 \cdots, \quad \sigma = \pm 1 \tag{4.22}$$

and (ii) one gapless branch which depends on the chirality

$$E_L(p_z, t) = -\text{sign}(qB)\ (p_z - qA_z(t)), \quad E_R(p_z, t) = \text{sign}(qB)\ (p_z - qA_z(t)) \tag{4.23}$$

all of which have a degeneracy of $N = \frac{|q\Phi_B|}{2\pi}$ for every p_z, where $\Phi_B = BL_xL_y$ is the flux through the surface of the (x, y)-cylinder. For the purpose of our discussion, it suffices to concentrate on the gapless states. Since the z-direction is compactified on a circle, we may take $p_z = \frac{2\pi n}{L_z}$ $(n \in \mathbb{Z})$ and re-write the gapless branches as

$$E_L(p_z, t) = -\text{sign}(qB)\ \frac{2\pi}{L_z} \left(n - \frac{t}{T} \right), \quad E_R(p_z, t) = \text{sign}(qB)\ \frac{2\pi}{L_z} \left(n - \frac{t}{T} \right). \tag{4.24}$$

Here T is taken to be large, and we assume that the spectrum flows *adiabatically* as a function of time. We will put the chemical potential at $E = 0$ for convenience. If $\psi(\vec{x}, t)$ is the boundary-fermion field operator (with $\vec{x} = (x, y, z)$) then the net charge may be defined as

$$Q(t) = q \int_{\Sigma_3} d^3\vec{x}\ \frac{1}{2}\ \langle vac| \left[\psi^\dagger(\vec{x}, t), \psi(\vec{x}, t) \right] |vac\rangle = \frac{q}{2} \sum_{\{|E_n| \le |m|\}} \text{sign}(E_n) \tag{4.25}$$

where the summation is over all the Hamiltonian eigenstates with $|E_n| \le |m|$. The sum only includes these states because at energies beyond the mass gap of the bulk insulator there are no *localized* chiral modes on the boundary. During the flux threading, we find that after a period of time $t = rT$ for integral r, the spectrum returns to itself, but after a translation by r units with respect to the chemical potential. In fact, r is the number of magnetic flux quanta which have been threaded through the hole of the (w, z)-cylinder. For each flux quantum that is threaded, $N = \frac{|q\Phi_B|}{2\pi}$ states cross the chemical potential, and the charge jumps by Nq - either increasing or decreasing depending on the chirality. Taking into account the factor of sign(qB) in (4.24), we therefore reproduce precisely the charge transfer in Eq (4.20) due to the $U(1)$ chiral anomaly.

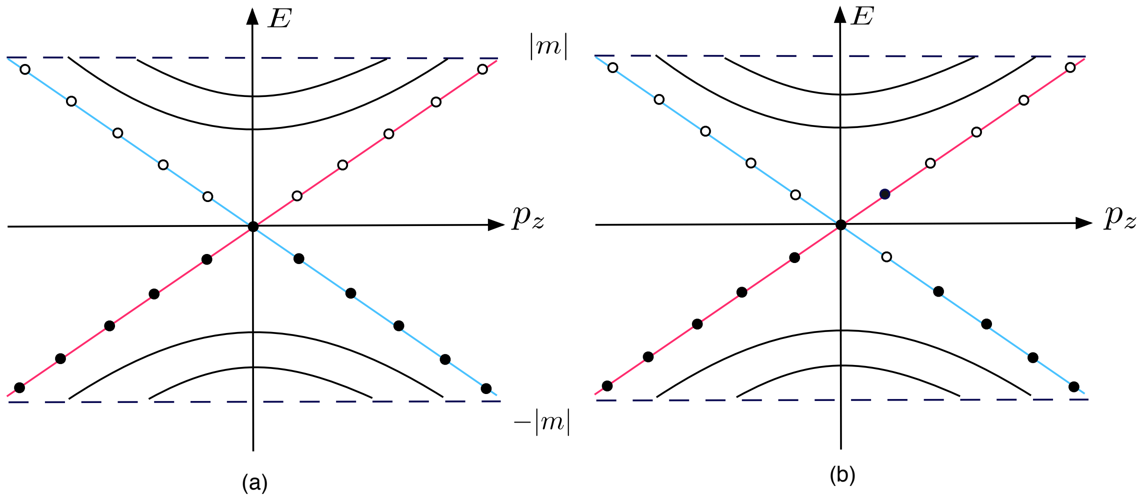

Figure 4.3: The Hamiltonian energy spectrum for chiral fermions in the presence of a uniform background magnetic field in the z-direction. The (black) gapped states are higher Landau levels, while the linear gapless (blue, red) curves are the zeroth Landau levels for left and right handed fermions respectively. We can consider the left and right handed fermions to exist on opposite boundaries of a cylinder. Once the energies of the linearly dispersing modes reach $\pm|m|$ these states are no longer localized on the boundary and lose their sense of chirality. (a) Before an electric field is turned on the states are filled to $E = 0$ on both boundaries. (b) After an electric field has acted and a single magnetic flux quantum has been threaded into the cylinder. Spectral flow has modified the level occupations such that one additional level of fermions appear in the right-handed branch and one level of fermions are missing from the left handed branch.

4.4 Momentum and Charge Transport from Magneto-Hall Viscosity

In this section, we will consider the momentum and charge transport due to torsion flux. These transport processes both arise from the term

$$S_{bulk} = \frac{\zeta_H}{2} \int_{M_5} F \wedge e^A \wedge T_A. \tag{4.26}$$

To simplify the discussion of Hamiltonian spectral flow, we will set $q_T = 1$ throughout this section. We can determine the momentum current by varying with respect to e^A and the charge current by varying with respect to A. We focus first on the momentum transport by turning on a $U(1)$ magnetic flux and torsion electric field. To generate the necessary background fields we turn on a $U(1)$ magnetic field through the (x, y) cylinder using $A = Bxdy$. We can thread torsion magnetic flux through the hole of the (z, w) cylinder, represented by the co-frame

$$e^0 = dt, \;\; e^1 = dx, \;\; e^2 = dy, \;\; e^3 = (1 + h(t))dz, \;\; e^5 = dw \tag{4.27}$$

where we take $h(t) = \frac{bt}{L_z T}$, for some large and positive time-scale T. The time-dependent torsion flux threading will generate a circulating torsion electric field in the z-direction. For simplicity, we will set the spin connection[2] $\omega_{AB} = 0$. As a result, the above configuration is torsional with the torsion electric field given by $T^3 = \frac{b}{L_z T} dt \wedge dz$. The bulk stress current from the term (4.26) in the action, in the presence of our set background fields, is

$$* J^3_{bulk} = \zeta_H \; F \wedge T^3 = q \frac{m^2 Bb}{4\pi^2 L_z T} dt \wedge dz \wedge dx \wedge dy. \tag{4.28}$$

In order to compute the momentum transferred due to this current over a time-period t, we introduce a *covariant Killing vector field* $\xi^A \underline{e}_A = \partial_z$. Then the rate of momentum transfer from one edge to the other due to the constant stress-current density is

$$\frac{dP^3}{dt} = \int_{\Sigma_3} \xi_A * J^A_{cov} = \text{sign}(qB) \frac{m^2 N}{2\pi} \left(1 + \frac{bt}{L_z T}\right) \frac{b}{T} \tag{4.29}$$

where $N = \frac{|q\Phi_B|}{2\pi} = \frac{|qB| L_x L_y}{2\pi}$. From the boundary point of view, this set of background fields gives rise to the diffeomorphism anomaly

$$d * (\xi_A J^A_{cov}) = \zeta_H \; F \wedge \xi_A T^A. \tag{4.30}$$

In order to understand this from the Hamiltonian point of view, it suffices once again to focus on the gapless boundary state branches for left- and right-handed chiral fermions in the presence of the uniform background magnetic field:

$$E_L(p_z, t) = -\text{sign}(qB) \frac{p_z}{\left(1 + \frac{bt}{L_z T}\right)}, \quad E_R(p_z, t) = \text{sign}(qB) \frac{p_z}{\left(1 + \frac{bt}{L_z T}\right)} \tag{4.31}$$

with degeneracy of $N = \frac{|q\Phi_B|}{2\pi}$ for every p_z. Note that these Hamiltonian spectra differ from the usual spectra (for a trivial co-frame field) via a scaling of the momenta (or from another point of view a scaling of the velocity), on account of the torsional electric field. In analogy with the boundary charge, we define the boundary momentum by

$$P^3(t) = \int_{\Sigma_3} d^3\vec{x} \, \frac{1}{2} \, \langle vac| \left[\psi^\dagger(\vec{x}, t), \hat{P}_3 \psi(\vec{x}, t)\right] |vac\rangle = \frac{1}{2} \sum_{\{|E_n| \leq |m|\}} \text{sign}(E_n) p^z_n \tag{4.32}$$

[2] In particular, we are supposing that the curvature $R^A{}_B$ vanishes. Consequently, $\omega^A{}_B$ is pure gauge, and we are choosing it to be zero here.

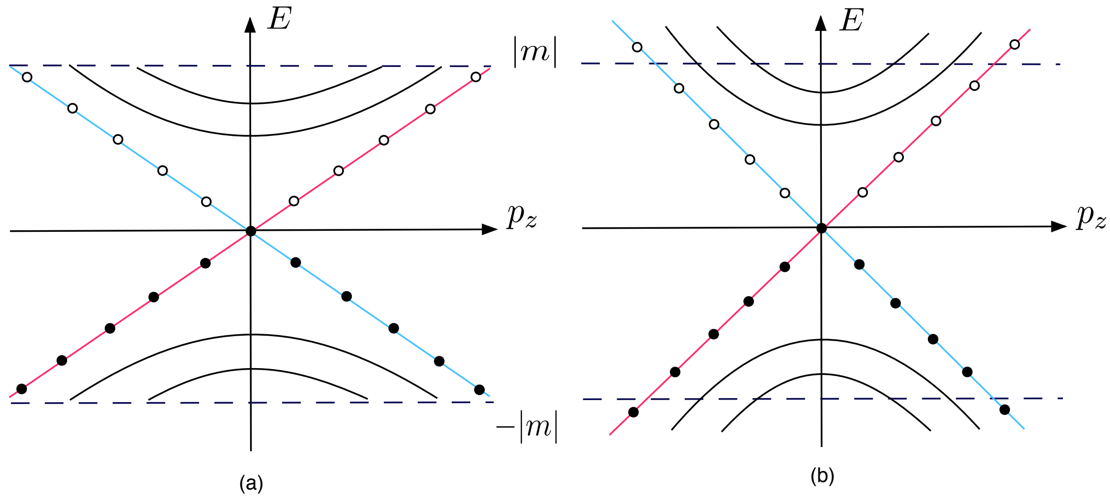

Figure 4.4: The Hamiltonian energy spectrum for chiral fermions in the presence of a uniform background magnetic field in the z-direction. The (black) gapped states are higher Landau levels, while the linear gapless (blue, red) curves are the zeroth Landau levels for left and right handed fermions respectively. We can consider the left and right handed fermions to exist on opposite boundaries of a cylinder. Once the energies of the linearly dispersing modes reach $\pm|m|$ these states are no longer localized on the boundary and lose their sense of chirality. (a) The initial state before the torsion electric field is applied. (b) A later state after some amount of torsional flux is threaded through the cylinder and the torsion electric field has had time to act on the system. The spectral rotation/stretching around $E = 0$ pushes some occupied chiral modes outside of the topological insulator mass gap which causes them to be lost into the sea of gapped bulk states. The overall process changes the momentum localized on each edge since each chiral fermion state lost to the bulk carries momentum that originally was localized on the boundary.

where we recall that the summation is over all Hamiltonian eigenstates with $|E_n| \leq |m|$. Using this, we can compute the net momentum along ξ on both the edges at a time t

$$P^3_L(t) = -\text{sign}(qB)\frac{m^2 N L_z}{4\pi}\left(1 + \frac{bt}{L_z T}\right)^2, \quad P^3_R(t) = \text{sign}(qB)\frac{m^2 N L_z}{4\pi}\left(1 + \frac{bt}{L_z T}\right)^2 \tag{4.33}$$

where now we have taken L_z to be large. From here, we get the rate of momentum change

$$\frac{dP^3_L}{dt} = -\text{sign}(qB)\frac{m^2 N}{2\pi}\left(1 + \frac{bt}{L_z T}\right)\frac{b}{T} \tag{4.34}$$

$$\frac{dP^3_R}{dt} = \text{sign}(qB)\frac{m^2 N}{2\pi}\left(1 + \frac{bt}{L_z T}\right)\frac{b}{T}. \tag{4.35}$$

Comparing with Eq (4.29), we find a precise agreement of the momentum transfer rates. Note that in contrast with the charge anomaly discussed in the previous section, the momentum anomaly in the present case is generated by a spectral rotation/stretching about $E = 0$ which pushes some edge states to energies $|E| > |m|$, thus causing them to get lost into the sea of gapped bulk states (see figure 4.4).

We will now look at one final anomalous transport process. Interestingly, because of the mixed dependence of $S_{bulk} = \frac{\zeta_H}{2}\int_{M_5} F\wedge e^A \wedge T_A$ on e^A, ω^{AB} and A, we can also generate a *charge* current with a certain arrangement of background geometry fields. This is unusual as this type of transport does not occur in the 2+1-d effective action. Let us turn on a torsion magnetic field $T^3 = Cdx\wedge dy$ on the (x,y) cylinder, and thread torsion magnetic flux (i.e., a dislocation) through the hole of the (z,w) cylinder to generate the torsion electric field $T^3 = \frac{b}{L_z T}dt\wedge dz$. This can be achieved through the co-frame

$$e^0 = dt,\ \ e^1 = dx,\ \ e^2 = dy,\ \ e^3 = \left(1+\frac{bt}{L_z T}\right)dz + Cxdy,\ \ e^4 = dw \tag{4.36}$$

upon choosing $\omega_{AB} = 0$. From the bulk response action we get the bulk charge current

$$* J_{bulk} = \frac{\zeta_H}{2}d(e^A\wedge T_A) = \frac{qm^2}{8\pi^2}\frac{bC}{L_z T}\ dt\wedge dx\wedge dy\wedge dz. \tag{4.37}$$

Just like in the case of the 4+1-d quantum Hall effect this gives a constant current density in the w-direction which transfers charge from one boundary to the other at a rate

$$\frac{dQ}{dt} = \frac{qm^2 b\Phi_T}{8\pi^2 T}. \tag{4.38}$$

From the perspective of the boundary fermions, this current is due to another manifestation of the $U(1)$ chiral anomaly $d * J_{cov} = \frac{\zeta_H}{2}T^a\wedge T_a$ for the chiral boundary states. This is of course the Nieh-Yan contribution to the (covariant) chiral anomaly, discussed previously.

Let us now explore how the anomaly can be understood microscopically from a Hamiltonian point of view. Once again, it suffices to focus on the lowest energy part of the spectrum of the chiral fermions in the background frame field (see Appendix B, section B.3 for a derivation):

$$E_L(t) = -\mathrm{sign}(Cp_z)\frac{p_z}{\left(1+\frac{bt}{L_zT}\right)},\ \ E_R(t) = \mathrm{sign}(Cp_z)\frac{p_z}{\left(1+\frac{bt}{L_zT}\right)} \tag{4.39}$$

with degeneracy $N(p_z,t) = \frac{|p_z\Phi_T|}{2\pi\left(1+\frac{bt}{L_zT}\right)}$. From the definition

$$Q = q\int_{\Sigma_3} d^3\vec{x}\ \frac{1}{2}\ \langle vac|\left[\psi^\dagger(\vec{x}),\psi(\vec{x})\right]|vac\rangle = \frac{q}{2}\sum_{\{|E_n|\le|m|\}}\mathrm{sign}(E_n) \tag{4.40}$$

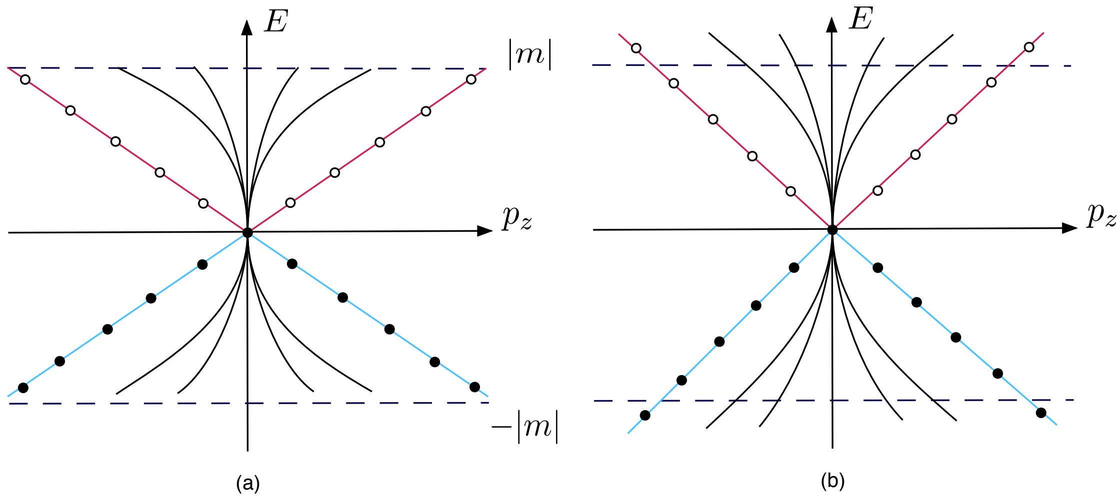

Figure 4.5: The Hamiltonian energy spectrum for chiral fermions in the presence of a uniform background *torsion* magnetic field in the z-direction. The (black) states are higher torsion Landau levels, while the linear gapless (blue, red) curves are the zeroth Landau levels for left and right handed fermions respectively. We can consider the left and right handed fermions to exist on opposite boundaries of a cylinder. Once the energies of the linearly dispersing modes reach $\pm|m|$ these states are no longer localized on the boundary and lose their sense of chirality. Note that something unusual happens here compared to the previous two figures. In a torsion magnetic field one chirality disperses upward while the other disperses downward. (a) The Hamiltonian spectrum before the application of a torsion electric field. (b) The spectral modification induced by an additional torsion electric field along the z direction.

we see that the net left- and right-handed charges at a time t are given by (taking the large L_z limit)

$$Q_L = -\frac{qL_z}{2\pi}\int_0^{m\left(1+\frac{bt}{L_zT}\right)} dp_z\ \frac{\Phi_T}{2\pi}\frac{p_z}{\left(1+\frac{bt}{L_zT}\right)} = -\frac{qm^2\Phi_T L_z}{8\pi^2}\left(1+\frac{bt}{L_zT}\right) \tag{4.41}$$

$$Q_R = \frac{qL_z}{2\pi}\int_0^{m\left(1+\frac{bt}{L_zT}\right)} dp_z\ \frac{\Phi_T}{2\pi}\frac{p_z}{\left(1+\frac{bt}{L_zT}\right)} = \frac{qm^2\Phi_T L_z}{8\pi^2}\left(1+\frac{bt}{L_zT}\right). \tag{4.42}$$

From here, we find the rates of change of net charge are given by

$$\frac{dQ_L}{dt} = -\frac{qm^2 b\Phi_T}{8\pi^2 T}, \quad \frac{dQ_R}{dt} = \frac{qm^2 b\Phi_T}{8\pi^2 T} \tag{4.43}$$

which precisely agrees with the previous result in Eq. (4.38).

We see here that the reason that the Nieh-Yan term can contribute to the covariant $U(1)$ anomaly is due to the structure of the low-energy chiral fermion branches in the presence of a uniform torsional magnetic field (see Appendix B, section B.3). As a comparison, we know that in the case of a conventional $U(1)$ magnetic field the low energy states of a single Weyl node become quasi-1D branches that disperse chirally, i.e., the states coming from a left-handed (right-handed) Weyl node have a positive (negative) group velocity (if $qB < 0$) $E = \pm v p_z$. Heuristically, the magnetic field acts to convert a 3+1-d Weyl fermion into a highly degenerate quasi-1D Weyl fermion at low-energy which only disperses along the direction of the applied uniform magnetic field. The torsional magnetic field (which for instance can be thought of as a density of screw dislocations) acts differently. Instead it generates quasi-1D upward dispersing or downward dispersing branches depending on the chirality of the 3+1-d Weyl node $E = \pm v|p_z|$. These branches contain both left- and right-movers but they have a fixed chirality. For example, for torsional field $C > 0$ the downward dispersing branch of the low-energy modes are made up of left-handed modes alone, whereas the upward dispersing branch contains only right-handed modes. The degeneracy also depends on the value of the momentum p_z as the torsional magnetic field is effectively stronger for larger p_z charge. This seems a bit strange at first, but we can see that the microscopic calculation precisely matches the bulk anomaly calculation.

With this, we conclude our discussion of parity-odd response and anomalies in free fermion systems on torsional backgrounds. To recapitulate, we have computed the parity-odd effective actions for free fermions on torsional geometries in $d = 2+1$ and $d = 4+1$ dimensions. We used these effective actions to deduce the structure of anomalies (in particular, the torsional contributions) in the edge states which live on the boundary between two different bulk phases. We then gave intrinsic, microscopic derivations of these anomalies by considering spectral flow in torsional backgrounds (and in the case of $d = 1+1$, also using the Fujikawa method). As explained previously, all of these calculations fit perfectly within the well-known framework of anomaly inflow, and indeed extend the framework to include torsional contributions. Furthermore, our condensed-matter-inspired picture provides natural resolutions to some previously ill-understood issues involving UV divergences – any intrinsic calculation of torsional anomalies suffers from UV divergences, which cannot be cured by intrinsic UV completions. Indeed, our calculations show that it is the bulk which provides the required UV completion and cures the UV divergences appearing in the torsional anomalies of the edge theory.

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Part II

The Exact Renormalization Group and Higher-Spin AdS/CFT

Chapter 5

Introduction

The second part of this thesis is dedicated to the AdS/CFT correspondence and its interpretation as a geometric formulation of the Wilsonian renormalization group, largely based on [38, 39, 40]. In this chapter, we begin with a lightning review of AdS/CFT and its interpretation in terms of the renormalization group. This introduction is by no means meant to be complete; instead we will only give sufficient details for the reader to be able to appreciate the discussion in the following chapters.

5.1 The AdS/CFT correspondence

The AdS/CFT correspondence [41, 42, 43, 44], sometimes also called gauge/gravity duality or holography, is a deep conjecture about the exact equivalence between two very different types of theories. The CFT side of the correspondence refers to a Conformal Field Theory (CFT) on a *fixed* d dimensional spacetime, with no quantum gravitational degrees of freedom. For the purposes of this thesis, it suffices to restrict the spacetime in question to be the d-dimensional Minkowski spacetime $\mathbb{R}^{1,d-1}$. In this case, a conformal field theory is defined as a relativistic quantum field theory which is invariant under the *conformal* isometries of $\mathbb{R}^{1,d-1}$, namely diffeomorphisms $f : \mathbb{R}^{1,d-1} \to \mathbb{R}^{1,d-1}$ which preserve the metric up to an overall (spacetime dependent) Weyl factor:

$$\eta_{\mu\nu} \to g_{\mu\nu} = \partial_\mu f^\lambda(x)\partial_\nu f^\sigma(x)\eta_{\lambda\sigma} = \Omega^2(x)\eta_{\mu\nu} \tag{5.1}$$

Since the only effect of such diffeomorphisms is to rescale the metric locally, we can give another equivalent definition for a CFT as a relativistic quantum field theory which satisfies the additional constraint

$$T^\mu{}_\mu(x) = 0 \tag{5.2}$$

where $T_{\mu\nu}(x)$ is the stress tensor (also known as the energy-momentum tensor). The group of conformal isometries of Minkowski spacetime, henceforth called the conformal group, is $SO(2,d)$.[1] The constraints imposed by conformal symmetry turn out to be quite strong. In fact, conformal symmetry completely determines the structural form of all the two and three point correlation functions of the theory up to overall constant coefficients. In fact, the operator content of the theory together with the *operator product expansion* (OPE), which is equivalent to knowing the three-point functions of the theory, in principle completely determines all the correlation functions of the theory.[2] Additionally, in most known examples of the AdS/CFT correspondence, the CFT has a large number of degrees of freedom (loosely speaking, a large "central charge"), which we denote as N for vector-like models and N^2 for gauge theories; for example, N could be the number of colors in the gauge group. While the conjecture is independent of N, it is most tractable and useful in the case where $N \to \infty$. We should note here that in gauge theories, the large N limit is more precisely taken to be the t'Hooft limit, where we fix $\lambda = g_{YM}^2 N$.

On the other hand, the AdS side of this correspondence refers to a theory of quantum gravity coupled to other matter and gauge fields, on a $d+1$ dimensional asymptotically Anti-de Sitter (AdS) spacetime. The metric on AdS spacetime (more precisely, a certain coordinate patch of the spacetime, referred to as the Poincare patch) is given by

$$g = \frac{\ell_{AdS}^2}{z^2}\left(dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu\right) \tag{5.3}$$

Note that Minkowski spacetime appears as the *conformal boundary* of AdS, which in the above coordinate patch corresponds to $z \to 0$. As a sanity check, note that the conformal group of the CFT is realized in the bulk as the isometry group of AdS. Each primary operator in the CFT corresponds to a dual field propagating on the *bulk* AdS spacetime. For example, the stress tensor of the CFT corresponds to the graviton propagating on AdS, if the CFT has a conserved $U(1)$ current, then the dual bulk field is a $U(1)$ gauge field etc. Of course, quantum gravity is notoriously difficult to make sense of in terms effective quantum field theory. In the context of AdS/CFT, the bulk theory is typically a theory of closed superstrings propagating on AdS spacetime; for instance one of the most well-studied example is the duality between Type IIB superstring theory on $AdS_5 \times S_5$ and $\mathcal{N} = 4$ super-Yang-Mills theory on $d = 4$ Minkowski space. However, for the most part we will confine our attention to the low-energy effective field theory, or *supergravity* description – in simple terms, superstring theories have an infinite tower of excitations, but in the limit where the string tension $1/\alpha'$ is large (in units of the AdS radius ℓ_{AdS}), we can ignore the massive states and focus on the massless states in the spectrum, the effective field theory description of which is supergravity. In the

[1]Note that isometry group $SO(1, d-1)$, also known as the Lorentz group, is a subgroup of the bigger conformal group.

[2]The structural form of the four and higher-point functions are not determined universally by conformal symmetry, but are in fact theory dependent. Nevertheless, they can in principle be computed from the operator content and OPE data.

AdS/CFT correspondence, ℓ^2_{AdS}/α' in the bulk is typically $O(\lambda^k)$ (where λ is the t'Hooft coupling in the CFT and k is some rational number); so one expects the supergravity approximation to be meaningful in the $\lambda \gg 1$ limit.[3] In this sense, the AdS/CFT correspondence relates a strongly coupled, large N CFT to a weakly coupled, semi-classical gravitational theory on AdS. We say semi-classical, because the gravitational coupling constant in the bulk is related to the central charge of the CFT as $G_N \sim \frac{1}{N^2}$ for gauge theories (the analogous coupling constant in the vector model case is proportional to $\frac{1}{N}$), and so the $N \to \infty$ limit corresponds to the semi-classical limit in the dual AdS spacetime.

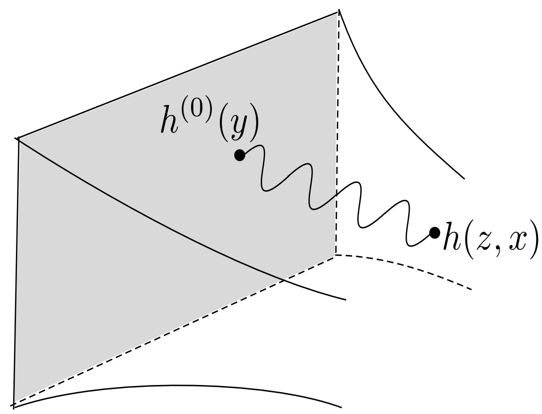

Figure 5.1: A cartoon picture of Anti-de Sitter space: the shaded region is the conformal boundary, i.e. Minkowski space, on which the CFT lives. A source $h^{(0)}(y)$ in the boundary turns on the corresponding bulk field $h(z,x)$ propagating in AdS.

Having described both sides of the correspondence, AdS/CFT provides us with a precise dictionary to translate from one side to the other. The most important element of this dictionary, which will be relevant in our context is the computation of correlation functions in the CFT, which corresponds to a very simple and natural calculation in the bulk. This calculation is easiest to describe in the Euclidean signature for scalar operators, although the generalization to higher-spins is straightforward for the most part. To this effect, let $\mathcal{O}$ be a scalar, primary operator of dimension Δ in the CFT, and let us suppose we wish to compute their correlations functions. We construct the Euclidean generating functional

$$S_{CFT}[h^{(0)}] = -\ln Z_{CFT}[h^{(0)}], \quad Z_{CFT}[h^{(0)}] \sim \int [D\phi] \exp\left(-S_0[\phi] + \int d^d x\, h^{(0)}(x)\mathcal{O}(x)\right) \tag{5.4}$$

where we have collectively denoted all the fields integrated over in the CFT path-integral as ϕ, and $S_0[\phi]$ is the classical action for the CFT. All *connected* correlation functions of the operator $\mathcal{O}$ can be obtained

[3]For instance in the type II superstring on $AdS_5 \times S_5$, the masses of higher-oscillator string modes are of the order $m_s^2 \sim 1/\alpha' = \lambda^{1/2}/\ell^2_{AdS}$, while the Kaluza-Klein modes of fields reduced over S^5 are of order $m^2_{KK} \sim 1/\ell^2_{AdS}$. In the large λ limit, all the higher-oscillator string modes decouple.

from the generating functional S_{CFT} by taking derivatives with respect to the sources h, as usual. Now the prescription for computing S_{CFT} in the large N limit from the dual AdS theory is as follows: as discussed above, corresponding to each operator $\mathcal{O}$, there is a bulk field $h(z,x)$ propagating on AdS spacetime. The linearized equation of motion for the bulk field is completely fixed by $SO(2,d)$ once the dimension Δ of $\mathcal{O}$ is specified,

$$\nabla^2 h - m^2 h = 0, \quad m^2 = \Delta(d-\Delta) \tag{5.5}$$

where ∇^2 is the Laplace operator on AdS; the full non-linear equation depends on further details such as the OPE content of the CFT. We are instructed to solve the *classical* equation of motion for h subject to the boundary conditions

$$\lim_{z\to 0} h(z,x) \sim z^{d-\Delta} h^{(0)}(x) \tag{5.6}$$

and evaluate the corresponding bulk on-shell action $S_{bulk,o.s.}[h^{(0)}]$. Remarkably, the AdS/CFT dictionary tells us that

$$S_{CFT}[h^{(0)}] = S_{bulk,o.s}[h^{(0)}] \tag{5.7}$$

Equation (5.7) will be the central equation we will focus on in our discussion in the following chapters. Of course, in practice for the generating functional S_{CFT} to exist, one must impose an ultraviolet cutoff on the path-integral. In the dual bulk picture, this corresponds to the fact that AdS space has an infinite volume in the limit $z \to 0$, and thus $S_{bulk,o.s}$ suffers from an infrared divergence. In order to make sense of this, one must impose an IR cutoff in the bulk. In most standard treatments, one simply cuts off the bulk spacetime at $z = z_0$; the cutoff z_0 can then be interpreted as a UV cutoff from the CFT point of view; however the dictionary does not tell us which specific cutoff this corresponds to in the CFT. Another important feature of the dictionary which will be relevant for our purposes, is that conserved current operators in the CFT are dual to massless gauge fields in the bulk – for example, a conserved spin-one current J^μ is dual to the photon A_μ in the bulk, the stress tensor $T^{\mu\nu}$ of the CFT is dual to the graviton $h_{\mu\nu}$ in the bulk and so on. As we will show in Appendix C, the bulk dynamics of these gauge fields is once again entirely fixed by the conformal invariance of the CFT to linear order (and by the OPE data at higher-orders, although this is harder to show).

There are of course a vast number of other elements in the AdS/CFT dictionary which allow us to compute, for instance, expectation values of non-local operators (such as Wilson loops) in the CFT, entanglement entropies of subregions in the CFT, correlation functions for the thermal state or highly excited states in the CFT (which correspond to blackhole solutions in the bulk description) etc. It is not possible to go into those details here; the interested reader should look up the review [45], and references there-in.

5.2 The Renormalization group

It is widely believed that the AdS/CFT correspondence should be interpreted as a geometrization of the renormalization group (RG) of quantum field theories. In this picture, scale transformations in the field theory correspond to movement in the radial direction (parametrized above by the coordinate z); accordingly, changing the bulk cutoff from $z = z_0$ to $z = z_0 + \delta z_0$ corresponds to lowering the UV cutoff in the quantum field theory. Given equation (5.7), it is then natural to expect that tracking the bulk on-shell action as a function of the bulk cutoff z_0 is equivalent to computing the Wilsonian effective action for the CFT along the renormalization group flow. Indeed, starting from a regulated version of equation (5.7), it is possible to derive the RG equations for the boundary sources from a bulk calculation – this process usually goes under the name of *holographic renormalization*. Early papers [46, 47] on this subject noted the relationship between RG flow and Hamilton-Jacobi theory of the bulk radial evolution. Additional contributions were made for example by [48, 49, 50, 51, 52, 53] and more recent discussions include [54, 55, 56, 57, 58, 59].

One of the main goals of this thesis will be to go in the opposite direction – we will start from a well-defined, UV regulated CFT and try to deduce the dual AdS dynamics from the renormalization group equations of the CFT. From the perspective of quantum field theory, considerations of the renormalization group usually begin within the context of perturbation theory, naturally interpreted in terms of deformations away from a free RG fixed point. Indeed, the 'exact renormalization group' (ERG) originally formulated by Polchinski [60] was constructed within the confines of a path integral over bare elementary fields with (regulated) canonical kinetic terms corresponding to the free fixed point. The word exact here refers to the fact that one keeps track of all possible operators which are generated along the RG flow, without truncating to the relevant and marginal ones (which is typical of most treatments of RG). Both the power and the curse of ERG is that it is formulated in terms of the free fixed point. One of the hallmarks of the AdS/CFT correspondence is that the bulk description is simplest, and most useful in a quite opposite limit – it is the strong (t'Hooft) coupling limit in the CFT where simple geometric constructions in the bulk are possible (as discussed in the previous section). So on the face of it, one might expect the program of going from the CFT to the bulk in a meaningful way to be hopeless.

However, there exists a conjectured duality [61, 62, 63] between free vector models in $d = 2 + 1$ and certain types of higher-spin theories on AdS_4 (for a preliminary introduction to higher-spin theories, see Appendix C; more details can be found in for e.g., [64, 65, 66] and the reviews [67, 68]). While the field theory side in this case is completely under control, the bulk is a far more complicated, and highly non-linear theory involving fields of arbitrarily high spin. This is to be expected – the weak-coupling limit in the CFT is expected to correspond to the limit where the higher-spin string states (assuming that there is an underlying stringy duality) become massless. From the field theory side, the signature of this is that free theories have a large symmetry structure; for instance, in the free, Bosonic, $U(N)$ vector model, one has an infinite tower of higher-spin *conserved* currents of the form:

$$J_{\mu_1 \cdots \mu_s}(x) \sim \sum_{m=1}^{N} \phi_m^* \partial_{(\mu_1} \cdots \partial_{\mu_s)} \phi_m \tag{5.8}$$

In the language introduced in the previous section, this infinite tower of primary operators in the CFT are dual to a corresponding infinite tower of massless *gauge* fields $h_{\mu_1 \cdots \mu_s}(z, x)$ propagating on AdS. Of course, any theory involving an infinite tower of massless fields will inevitably turn out to be highly non-linear (and possibly non-local). Nevertheless, one might hope that the free vector model provides an accessible testing ground for the holography/RG correspondence, especially in the large N limit. This is primarily the case we will work with in the following chapters. The non-linearities will manifest themselves in keeping track of the infinite tower of primary operators and their descendents along an RG flow, but the Polchinski exact RG makes this process tractable.

A useful way to think of these vector model/higher-spin dualities can be illustrated by considering 3d Chern-Simons theories, known to be 'dual' to 2d Wess-Zumino-Witten models. In this case, the theory is topological in the bulk (thus giving rise to a sort of holography long appreciated by condensed matter theorists (and experimentalists!)). What this means is that the theory does not depend on a bulk metric, and diffeomorphism invariance is broken only on the boundary through the introduction of boundary terms that explicitly involve a boundary metric. In particular, it is conjectured that 3d gravity[69, 70] (and higher spin generalizations[71, 72, 73]) can be thought of in these terms. Here, the dynamical degrees of freedom do not include a metric in the bulk, but at least a wide class of classical solutions have a geometric interpretation, the Chern-Simons gauge fields recast in terms of a co-frame and spin connection (or higher spin generalizations thereof). The equations of motion are first order (i.e., classical solutions are flat connections). In fact, it is widely believed that a consistent field theoretic formulation of theories with arbitrarily high spins must be similar in spirit – namely, that it should be formulation in the language of connections and flatness conditions. As we will see in this work, the picture that emerges from a study of exact RG equations of vector models is precisely of this nature.

In the next chapter, we will introduce the bosonic $U(N)$ vector model, and present a framework to deal with its higher-spin symmetries. Then in chapter 7, we will study the renormalization group flow of this model, and cast the RG equations in the language of Hamiltonian dynamics. The corresponding equations of motion will be interpreted as the higher-spin equations of motion. In chapter 8, we will then explicitly evaluate the on-shell bulk action and show that it reproduces the generating function of connected correlation functions of the vector model, thus proving equation (5.7). In chapter 9, we will show how to recover the massless higher-spin gauge fields on AdS and their linearized dynamics from the renormalization group equations of the vector model. Finally, in chapter 10 we will extend this discussion to include multi-trace interactions in the vector model, and show that in the large N limit, this merely corresponds to a change in the boundary conditions on the equations derived previously.

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Chapter 6

The $U(N)$ vector models

In this section, we introduce the Bosonic $U(N)$ vector model, which consists of N complex scalar fields in d space-time dimensions (the Fermionic vector model will be introduced in chapter 10). The action at the free fixed point is given by

$$S^0_{Bos.} = -\int d^d x \; \phi^*_m(x)\Box_{(x)}\phi^m(x) \tag{6.1}$$

where we have taken the space-time metric to be $g_{\mu\nu} = \eta_{\mu\nu}$ and $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$. The index m runs from 1 to N, and repeated indices implies summation. We are interested in deforming the theory away from the free fixed point with generic "single-trace" operators[1] of the schematic form

$$\phi^*_m\phi^m, \;\; \phi^*_m\partial_\mu\phi^m, \;\; \phi^*_m\partial_\mu\partial_\nu\phi^m, \cdots \tag{6.2}$$

with no prejudice towards the number of derivatives. (As mentioned in the previous chapter, these operators are representatives of the conserved, higher-spin current operators at the free fixed point − the precise definition of these operators requires them to be traceless, symmetric and conserved, but these complications will be irrelevant for our purposes.) In order to do so, it is most convenient to introduce two bi-local sources $B(x,y)$ and $W_\mu(x,y)$, and write the full action for the $U(N)$ model as

$$S^{reg.}_{Bos.} = -\int_{x,u,y} \phi^*_m(x)\eta^{\mu\nu}D_\mu(x,u)D_\nu(u,y)\phi^m(y) + \int_{x,y} \phi^*_m(x)B(x,y)\phi^m(y) \tag{6.3}$$

where we have introduced the new "covariant" derivative

$$D_\mu(x,y) = \partial^{(x)}_\mu \delta^d(x,y) + W_\mu(x,y) \tag{6.4}$$

[1] Restricting to single-trace operators of course is not very general, but it is a *consistent truncation* of the full set of ERG equations in which sources for all "multi-trace" operators are included. We will return to this more general system in chapter 10.

One can easily check that this action sources all possible single-trace operators. We have written the action in this precise form, because (as we will see shortly) D_μ is a covariant derivative for a background gauge symmetry. While we will work with arbitrary bi-local sources B and W_μ for the most part, it is convenient to think in terms of a *quasi-local* expansion for these in the form

$$B(x,y) \sim \sum_{s=0}^{\infty} B^{a_1\cdots a_s}(x)\, \partial^{(x)}_{a_1}\cdots\partial^{(x)}_{a_s}\delta^d(x-y) \tag{6.5}$$

$$W_\mu(x,y) \sim \sum_{s=0}^{\infty} W_\mu{}^{a_1\cdots a_s}(x)\, \partial^{(x)}_{a_1}\cdots\partial^{(x)}_{a_s}\delta^d(x-y) \tag{6.6}$$

Putting these expressions into the action, we see that they amount to sourcing arbitrary single-trace operators with any number of derivatives; such operators can be organized into conformal modules, represented by lowest weight primary operators, which in this case turn out to be the conserved higher-spin currents. It is clear from equations (6.5) and (6.6) that the above bilocal sources are really "matrices" acting on L_2 functions over spacetime, and the bilocal representation is merely convenient notation which makes this manifest. Given this matrix notation, it is also convenient to introduce the following product and trace on such bilocal sources:

$$(f{\cdot}g)(x,y) = \int_u f(x,u)g(u,y), \qquad \mathrm{Tr.}(f) = \int_x f(x,x) \tag{6.7}$$

The sources B and W_μ that we have introduced above couple, respectively, to the following bi-local operators in the CFT

$$\hat{\Pi}(x,y) = \phi^*_m(y)\phi^m(x), \quad \hat{\Pi}^\mu(x,y) = \int_u \Big(\phi^*_m(y)D^\mu(x,u)\phi^m(u) - D^\mu(y,u)\phi^*_m(u)\phi^m(x)\Big) \tag{6.8}$$

We will see below that $\hat{\Pi}^\mu(x,y)$ can be interpreted as a bi-local current operator, which packages together the higher-spin currents mentioned previously. There is a minor subtlety in defining $U(N)$ singlet bilocal operators – since $\phi^m(x)$ is a section of a $U(N)$ vector bundle, the only natural contraction between $\phi^*_m(y)$ and $\phi^m(x)$ should involve a $U(N)$ Wilson line. For instance,

$$\hat{\Pi}(x,y) = \phi^*_m(y)\Big(\mathscr{P}\, e^{\int_y^x A^{(0)}}\Big)^m{}_n\phi^n(x) \tag{6.9}$$

where $A^{(0)}$ is a background $U(N)$ connection. By not including the Wilson lines explicitly, we are assuming that the $U(N)$ vector bundle is trivial – this means that $A^{(0)}$ can be taken to be flat, and in particular we make the choice $A^{(0)} = 0$. Another way of saying this which is particularly natural in $d = 3$, is that we can introduce a dynamical $U(N)$ gauge field $A_\mu(x)$ into the theory, with the Chern-Simons interaction

$$S_{CS} = \frac{k}{4\pi}\int \mathrm{tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right), \quad k = \frac{N}{\lambda} \tag{6.10}$$

minimally coupled to the vector model in question. Keeping N fixed (but large) and sending $\lambda \to 0$ corresponds to localizing on a flat connection $A_\mu^{(0)}$, which in the present case we may take to be pure gauge. Note that λ here is analogous to the t'Hooft coupling discussed in the previous chapter; we therefore see that the vector model is the limit of $\lambda \to 0$, which is consistent with the existence of higher-spin symmetry.

The (unregulated) generating function (or partition function) is obtained by performing the path integral

$$Z[B,W] \sim \int [d\phi d\phi^*]\, e^{-S_{Bos.}} \tag{6.11}$$

The path integration in (6.11) is over the set of all square integrable complex scalar functions over the space-time $\mathbb{R}^{1,d-1}$, where the measure is conventionally written formally as

$$[d\phi d\phi^*] = \prod_{m=1}^{N}\prod_{x\in\mathbb{R}^{1,d-1}} d\phi_m(x) d\phi_m^*(x) \tag{6.12}$$

6.1 The $U(L_2)$ symmetry

Given the measure in equation (6.12), it is natural to ask what a general linear transformation in function space would do to the path integral. To that end, consider a general linear bi-local field redefinition

$$\phi(x) \mapsto \int_y \mathcal{L}(x,y)\phi(y) \tag{6.13}$$

where $\mathcal{L} : L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)$ is a *unitary* map of square integrable functions, i.e.

$$\mathcal{L}^\dagger\text{•}\mathcal{L}(x,y) \equiv \int_u \mathcal{L}^*(u,x)\mathcal{L}(u,y) = \delta^d(x-y). \tag{6.14}$$

We will refer to the group of such transformations as $U(L_2(\mathbb{R}^d))$, or simply $U(L_2)$ for short.[2] If we consider an infinitesimal version of the above transformation

$$\mathcal{L}(x,y) \simeq \delta(x-y) + \epsilon(x,y) \tag{6.15}$$

then the $U(L_2)$ condition implies

$$\epsilon^*(x,y) + \epsilon(y,x) = 0 \tag{6.16}$$

For example, consider an ϵ of the form

$$\epsilon(x,y) = i\xi(x)\,\delta(x-y) + \xi^\mu(x)\,\partial_\mu^{(x)}\delta(x-y) + i\xi^{\mu\nu}(x)\,\partial_\mu^{(x)}\partial_\nu^{(x)}\delta(x-y) + \cdots \tag{6.17}$$

where $\xi,\ \xi^\mu,\ \xi^{\mu\nu}\cdots$ are all real. This satisfies the $U(L_2)$ condition provided $\partial_\mu\xi^\mu = 0,\ \partial_\mu\xi^{\mu\nu} = 0$ and so on. The first term above is an infinitesimal $U(1)$ gauge transformation (generated by the spin-1 current), the second term is a volume-preserving diffeomorphism (generated by the spin-2 current), while the rest are higher-spin transformations (generated by higher-spin currents).

Formally, the measure (6.12) is invariant under $U(L_2)$ transformations, i.e. the Jacobian is unity. Coming to the action (6.3), we obtain

$$S_{Bos.}[\mathcal{L}\cdot\phi, B, W_\mu] = S_{Bos.}\left[\phi,\ \mathcal{L}^{-1}\cdot B\cdot\mathcal{L},\ \mathcal{L}^{-1}\cdot W_\mu\cdot\mathcal{L} + \mathcal{L}^{-1}\cdot[\partial_\mu,\mathcal{L}]_\cdot\right] \tag{6.18}$$

Thus, we find that W_μ acts like a *background* gauge field for unitary bi-local field redefinitions, while B conjugates tensorially. This is the reason the derivative $D_\mu(x,y)$ defined previously acts as a covariant derivative with respect to $U(L_2)$ transformations. In the infinitesimal case, the transformation properties of B and W can be written as

$$\delta B = [B,\epsilon]_\cdot\,, \quad \delta W_\mu = [D_\mu,\epsilon]_\cdot \tag{6.19}$$

where we have defined the '$\cdot$-bracket' $[f,g]_\cdot = f\cdot g - g\cdot f$. Given the formal invariance of the path integral measure, we obtain the Ward identity

$$Z[B,W_\mu] = Z\left[\mathcal{L}^{-1}\cdot B\cdot\mathcal{L},\ \mathcal{L}^{-1}\cdot W_\mu\cdot\mathcal{L} + \mathcal{L}^{-1}\cdot[\partial_\mu,\mathcal{L}]_\cdot\right] \tag{6.20}$$

[2]Here we are considering such transformations that commute with the $U(N)$; this is appropriate since we are sourcing $U(N)$-singlets.

The infinitesimal version of this Ward identity is given by

$$\text{Tr.}\left\{[D_\mu, \epsilon]_{\cdot}\boldsymbol{\cdot}\frac{\delta}{\delta W_\mu} + [\epsilon, B]_{\cdot}\boldsymbol{\cdot}\frac{\delta}{\delta B}\right\} Z[B, W_\mu] = 0. \tag{6.21}$$

which directly implies the conservation equation

$$[D_\mu, \Pi^\mu]_{\cdot} + [B, \Pi]_{\cdot} = 0. \tag{6.22}$$

where Π and Π^μ are vacuum expectation values of corresponding the operators defined previously. The above equation, evaluated at the fixed point $B = W_\mu = 0$, gives a bilocal conservation equation, which from the discussion below equation (6.17), is equivalent to the conservation equations for all the higher-spin currents. In other words, the $U(L_2)$ Ward identity is a convenient way to package the higher-spin symmetries of the free fixed point.

In fact, $U(L_2)$ symmetry teaches us a vital lesson about the fixed point – since W_μ behaves like a background connection under $U(L_2)$, the configuration $W_\mu = 0$ is gauge equivalent to the pure-gauge configuration $W_\mu = \mathcal{L}^{-1}\boldsymbol{\cdot}[\partial_\mu, \mathcal{L}]_{\boldsymbol{\cdot}}$. Thus, for any flat connection $W^{(0)}$ satisfying

$$dW^{(0)} + W^{(0)} \wedge W^{(0)} = 0 \tag{6.23}$$

with $d = dx^\mu\,[\partial_\mu, \,]_{\boldsymbol{\cdot}}$, the partition function $Z[B = 0, W^{(0)}]$ describes the free-fixed point. For this reason, we will find it convenient to pull out a flat piece from the full source W and write it as

$$W = W^{(0)} + \widehat{W} \tag{6.24}$$

Indeed, it is $\widehat{W}$ and B which represent arbitrary single-trace, *tensorial* deformations away from the free-fixed point, and thus parametrize single-trace RG flows away from the free CFT. We will return to this point in the next chapter.

The group $U(L_2)$ does not exhaust the background symmetries of the free bosonic $U(N)$ vector model. We can further enlarge this group, by considering transformations of the form

$$\mathcal{L}^\dagger\boldsymbol{\cdot}\mathcal{L}(x, y) \equiv \int_u \mathcal{L}^*(u, x)\mathcal{L}(u, y) = \Omega^2\delta^d(x - y) \tag{6.25}$$

where Ω is some positive constant.[3] We will call this larger group $CU(L_2)$. Let us now think about how the enlarged symmetry acts on the path-integral. Firstly, in order to continue thinking of the field theory as living on Minkowski spacetime, we introduce a conformal factor z in the background metric:

$$\eta_{\mu\nu} \mapsto \frac{1}{z^2}\eta_{\mu\nu}, \tag{6.26}$$

and redefine the sources by rescaling them: $B_{old} = z^{d+2}B_{new}$ and $W_{old} = z^d W_{new}$.[4] For simplicity, we will drop the subscript *new* from here on. With these changes, the action takes the form

$$S_{Bos.}[\phi, z, B, W] = -\frac{1}{z^{d-2}}\int_{x,y}\phi^*_m(x)D_\mu \cdot D^\mu(x,y)\phi^m(y) + \frac{1}{z^{d-2}}\int_{x,y}\phi^*_m(x)B(x,y)\phi^m(y) \tag{6.27}$$

Having made these changes, we find straightforwardly

$$S_{Bos.}\Big[\mathcal{L}\cdot\phi, z, B, W_\mu\Big] = S_{Bos.}\Big[\phi, \lambda^{-1}z,\ \mathcal{L}^{-1}\cdot B\cdot\mathcal{L},\ \mathcal{L}^{-1}\cdot W_\mu\cdot\mathcal{L} + \mathcal{L}^{-1}\cdot[\partial_\mu, \mathcal{L}]\Big] \tag{6.28}$$

where $\mathcal{L}$ is a $CU(L_2)$ element satisfying equation (6.25), with $\Omega = \lambda^{\frac{d-2}{2}}$. Once again, we find that the 1-form W_μ transforms like a gauge field, while the 0-form B conjugates tensorially. Note further, that the conformal factor z rescales to $\lambda^{-1}z$. Thus, we conclude that $CU(L_2)$ is a background symmetry of the action up to a conformal rescaling of the background metric. In terms of the quantum partition function, we have the Ward identity

$$Z[z, B, W_\mu] = Z[\lambda^{-1}z,\ \mathcal{L}^{-1}\cdot B\cdot\mathcal{L},\ \mathcal{L}^{-1}\cdot W_\mu\cdot\mathcal{L} + \mathcal{L}^{-1}\cdot[\partial_\mu, \mathcal{L}]] \tag{6.29}$$

Since the $CU(L_2)$ transformations involve a rescaling of the background metric, we expect them to play an important role in the renormalization group analysis; we will see this explicitly in the next chapter. We also remark that the measure of the path integral is, in general, not invariant under these transformations (unless $\Omega = 1$), but will pick up an overall normalization factor, which one might think of as an anomaly. This can be absorbed into the source for the identity operator.

We have one more background symmetry to discuss before we move on to the renormalization group. Recall that we have split the 1-form W_μ into a flat connection and a tensorial piece – see eq. (6.24). With this

[3] In particular, for diffeomorphism invariance in the dual theory, it might be important to generalize this to the case where Ω is a function of spacetime; we will however not do this presently.

[4] These scale factors are put in for the following reason: recall that the sources admit the quasi-local expansions (6.5), (6.6). These expressions will get modified upon the introduction of the conformal factor z in the metric, as $\delta^{(d)}(x-y) \to z^d\delta^{(d)}(x-y)$. The scale factors introduced above precisely remove this additional z-dependence. The extra z^2 in B_{new} ensures that it transforms tensorially under $CU(L_2)$.

separation, the action becomes

$$S_{Bos.}[\phi, z, B, W_\mu] = S_0 + S_{int} \tag{6.30}$$

$$S_0 = -\frac{1}{z^{d-2}} \int_{x,y,u} \phi^*_m(x) \eta^{\mu\nu} D^{(0)}_\mu(x,u) D^{(0)}_\nu(u,y) \phi^m(y) \tag{6.31}$$

$$S_{int} = \frac{1}{z^{d-2}} \int_{x,y} \phi^*_m(x) \Big(B(x,y) - \Big\{ \widehat{W}^\mu, D^{(0)}_\mu \Big\}_{\cdot} (x,y) - \widehat{W}_\mu {\cdot} \widehat{W}^\mu(x,y) \Big) \phi^m(y) \tag{6.32}$$

where $D^{(0)}_\mu = \partial_\mu + W^{(0)}_\mu$. Since $\widehat{W}_\mu$ is tensorial, it is possible to redefine B to absorb the terms involving $\widehat{W}$

$$\mathcal{B} = B - \Big\{ \widehat{W}^\mu, D^{(0)}_\mu \Big\}_{\cdot} - \widehat{W}_\mu {\cdot} \widehat{W}^\mu \tag{6.33}$$

In other words, from a renormalization group point of view, it is redundant to turn on both B and $\widehat{W}_\mu$. Therefore, one can use the above freedom to set $\widehat{W}_\mu = 0$ in S_{int}; we will henceforth do so, and write the deformation away from the fixed point as

$$S_{int} = \frac{1}{z^{d-2}} \int_{x,y} \phi^*_m(x) \mathcal{B}(x,y) \phi^m(y) \tag{6.34}$$

Note that this was in fact the starting point of Ref. [74], but the geometrical structure has now been made manifest. In our discussion of the exact RG equations to follow, were we not to absorb $\widehat{W}_\mu$, we would find that the exact RG equation cannot unambiguously be separated into independent equations for B and $\widehat{W}_\mu$.

6.2 Infinite jet bundles

We have seen above that the large symmetry of free field theory, which is best elucidated in the path integral formulation, has a naturally geometric flavor. In particular, W – which sources a certain bi-local current operator in the field theory – transforms like a 'connection'. A natural interpretation for W is that it is a connection on the *infinite jet bundle* of the field theory. Said another way, the background $U(L_2)$ and $CU(L_2)$ symmetries of free field theory can be characterized as gauge transformations of its infinite jet bundle, and sourcing all possible single-trace operators is equivalent to picking a connection on (and a section of the endomorphism bundle of) the infinite jet bundle corresponding to the field theory. For completeness, we will end this chapter by briefly describing a few details about infinite jet bundles – a much more detailed account can be found in Appendix C.

While it is true that the $U(L_2)$ and $CU(L_2)$ symmetries we have discussed resemble gauge symmetries, the main problem we must confront in order for such an interpretation to hold, is their non-local nature. The gauge transformations one usually encounters in physics are local – consider for instance a $U(1)$ gauge transformation $\delta\phi(x) = i\alpha(x)\phi(x)$. In this case, ϕ is thought of as a section of a vector bundle associated to a principal $U(1)$ bundle, and the gauge transformation may be thought of as a vertical group action. On the other hand, a $U(L_2)$ transformation

$$\delta\phi^m(x) = \int_y \epsilon(x,y)\phi^m(y) \tag{6.35}$$

depends on the value of ϕ^m not merely at one point, but over the entire common support of ϵ and ϕ^m. In other words, the action in (6.35) depends on the value of the ϕ^m at a point, and its derivatives at that point (at least if we subscribe to a quasi-local gauge transformations of the form (6.17)). In order to interpret this as a gauge transformation then, there is a need to construct a vector bundle whose fibre at each point keeps track of ϕ^m, and its derivatives. In mathematics, this construction is referred to as the infinite jet bundle. Loosely speaking, the infinite jet bundle is a vector bundle whose fibre at a point p consists of all equivalence classes of functions (or more generally sections) which have the same derivatives at p. Schematically, an element Φ of the fibre at p correspondent to the function ϕ looks like

$$\Phi^m[\phi](p) = \left(\phi^m(x), \frac{\partial\phi^m}{\partial x^\mu}(p), \frac{\partial^2\phi^m}{\partial x^\mu \partial x^\nu}(p), \cdots\right) \tag{6.36}$$

and is called the *jet* of ϕ at p. The space of all jets at a point constitutes the fibre of the infinite jet bundle at that point. Going back to equation (6.35), we see the action of ϵ on ϕ^m can be represented in terms of a linear and local action on its jet $\Phi^m[\phi]$. This is why we can think of $U(L_2)$ transformations as gauge transformations acting on the infinite jet bundle, satisfying the $U(L_2)$ condition. For instance, the gauge transformation in equation (6.17) written in terms of jets, takes the *local* matrix-form:

$$\delta\Phi[\phi](x) = \mathcal{E}[\epsilon](x)\cdot\Phi[\phi](x), \quad \mathcal{E}[\epsilon](x) = \begin{pmatrix} i\xi(x) & \xi^\nu(x) & \cdots \\ i\partial_\mu\xi(x) & i\delta^\nu_\mu\xi(x) + \partial_\mu\xi^\nu(x) & \cdots \\ \vdots & \vdots & \end{pmatrix} \tag{6.37}$$

Given this interpretation, the 1-form W_μ is naturally identified as a connection 1-form over the infinite jet bundle, while the 0-form B can be thought of as a section of its endormorphism bundle. Indeed, this interpretation fits nicely with our intuition for quasi-local expansions for our bi-local sources[5]

$$W_\mu(x,y) \simeq \sum_{s=1}^{\infty} W_\mu^{a_1\cdots a_{s-1}}(x)\partial_{a_1}^{(x)}\cdots\partial_{a_{s-1}}^{(x)}\delta^d(x-y) \tag{6.38}$$

$$B(x,y) \simeq \sum_{s=1}^{\infty} B^{a_1\cdots a_{s-1}}(x)\partial_{a_1}^{(x)}\cdots\partial_{a_{s-1}}^{(x)}\delta^d(x-y), \tag{6.39}$$

which can be written in a *local* matrix form similar to equation (6.37), in jet-space. The above quasi-local expansions basically express the fact that both W_μ and B are valued in (a sub-bundle) of the endomorphism bundle of the jet bundle.[6] In this way, a purely field theoretic exercise of sourcing all possible single-trace operators leads us to a beautiful geometric framework. More details about the jet-space formulation described here can be found in Appendix C.

Before we end this chapter, we would like to introduce the notion of a *Wilson line*. Let $w(s;x,y)$ be a one-parameter family of bi-local sources; we will encounter such a family in the next chapter, with s playing the role of the renormalization group flow parameter. We define the Wilson line $\mathscr{K}(t;t_0)$ corresponding to this source as the following path ordered exponential:

$$\begin{aligned} \mathscr{K}(t;t_0) &= \mathscr{P}.\exp\left(\int_{t_0}^{t} ds\, w(s)\right) \\ &= \mathbf{1} + \int_{t_0}^{t} ds\, w(s) + \frac{1}{2}\int_{t_0}^{t} ds_1 \int_{t_0}^{s_1} ds_2\, w(s_1)\boldsymbol{\cdot}\, w(s_2) + \frac{1}{2}\int_{t_0}^{t} ds_2 \int_{t_0}^{s_2} ds_1\, w(s_2)\boldsymbol{\cdot}\, w(s_1) + \cdots \end{aligned} \tag{6.40}$$

Note that the path-ordered exponential above is a "$\boldsymbol{\cdot}$"-exponential, in that all the products involved in defining it are "$\boldsymbol{\cdot}$"-products. As usual, the Wilson line defined above satisfies

$$\frac{d}{dt}\mathscr{K}(t;t_0) = w(t)\boldsymbol{\cdot}\mathscr{K}(t;t_0) \tag{6.41}$$

These Wilson lines will play an important role in the construction of bulk-to-boundary propagators in chapter 8. It is also possible to define such Wilson lines for the $U(L_2)$ connection introduced above, using the jet-bundle formulation; we do this in Appendix C.

• • •

[5] These quasi-local expansions should be regarded as schematic. More precisely, we should think of the bilocal fields as sourcing all possible quasi-primary operators and their descendants, and hence the expansion is in terms of conformal modules.

[6] In most physics literature, the connection is thought of as a 1-form valued in the Lie-algebra of the gauge group, $W = W_\mu^\alpha T^\alpha dx^\mu$. The quasi-local expansions should be thought of in the same spirit, with the differential operators $T^{(s)} \simeq \partial^s_{(x)}\delta^d(x-y)$ playing the role of the Lie-algebra elements.

Chapter 7

The Exact Renormalization Group and Holography

In this chapter, we will construct the renormalization group flow for the free bosonic vector model (described in detail in the previous chapter), perturbed away from the fixed point by the bi-local source $\mathcal{B}$.

7.1 The Renormalization Group

The general principle of Wilsonian renormalization is that the action of a quantum field theory should be thought of as a function of the energy scale at which it is probed. In simple terms, this amounts to having cutoff dependent sources (or couplings) – this is because, in say lowering the ultraviolet (UV) cutoff from M to λM ($\lambda < 1$), one is really integrating over the fast modes in the path integral, which consequently changes the values of the couplings, thus making them cutoff dependent. The remarkable feature of the Wilson-Polchinski exact renormalization group [60] is the description of renormalization of a QFT action in terms of a diffusion-like equation, with the cutoff M being the flow parameter.

Alternatively, it is also possible to think of the conformal scale z of the background metric $g^{(0)} = z^{-2}\eta$ as parameterizing the RG flow, with *fixed* UV cutoff. In this version, one lowers the cutoff $M \mapsto \lambda M$ by integrating out fast modes, but then performs a scale transformation $g^{(0)} \mapsto \lambda^2 g^{(0)}$ (or equivalently $z \mapsto \lambda^{-1} z$) to take the cutoff back to M. Naturally, in this case, the conformal factor z acts as the flow parameter, and the sources may be thought of as z-dependent. From a geometric point of view, this version of RG is more appealing, and we will adopt it in our discussions below. In the notation introduced in the previous section, we will then regard the sources, $\mathcal{B}(z; x, y)$ and $W_\mu^{(0)}(z; x, y)$, as one-parameter families of bilocal sources, with z parametrizing the RG flow.

In order to proceed, we must regulate the path integral – following Polchinski's formalism [60], we will do so by introducing a smooth cutoff function $K(s)$ which has the property that $K(s) \to 1$ for $s < 1$ and $K(s) \to 0$

for $s > 1$. We thus write the new *regulated* action as

$$S^{reg.}_{Bos.} = -\frac{1}{z^{d-2}} \int_{\vec{x},\vec{y}} \phi^*_m(\vec{x}) K^{-1}\left(-z^2 D^2_{(0)}/M^2\right) D^2_{(0)}(\vec{x},\vec{y})\phi^m(\vec{y}) + \frac{1}{z^{d-2}} \int_{\vec{x},\vec{y}} \phi^*_m(\vec{x})\mathcal{B}(\vec{x},\vec{y})\phi^m(\vec{y}) \tag{7.1}$$

where $D^2_{(0)} = \eta^{\mu\nu} D^{(0)}_\mu \cdot D^{(0)}_\nu$, and M is the UV cutoff, which we will regard as fixed.[1] The particular choice of K will not be important in our discussion below – any sufficiently well-behaved cut-off function will do. We emphasize that the cut-off procedure we utilize preserves the global $U(N)$ symmetry, and this is sufficient to ensure that the truncation to $U(N)$ invariant, single-trace operators is a consistent truncation.

The regulated path integral is then given by

$$Z_{CFT}[z; U, \mathcal{B}, W^{(0)}] = \int [d\phi d\phi^*]\, e^{-U - S^{reg.}_{Bos.}} \tag{7.2}$$

where U is the source corresponding to the identity operator. For clarity, we now restate the RG program as a 2-step process:

Step 1: Lower the "cutoff" $M \to \lambda M$ (for $\lambda < 1$), by integrating out a shell of "fast modes" – this changes the sources, and we will use the notation $U \to \widetilde{U}$, $\mathcal{B} \to \widetilde{\mathcal{B}}$ to denote this. The calculation can be efficiently carried out using Polchinski's exact RG formalism (see Appendix C for details).

Step 2: Perform a $CU(L_2)$ transformation $\phi \to \mathcal{L}\cdot\phi$ to bring M back to its original value, but in the process changing $z \to \lambda^{-1} z$ – thus, the RG flow is parametrized by z in our description, and not M (M is an auxiliary parameter in the cut-off function). The $CU(L_2)$ transformation additionally acts on the sources, and as we will see below, leads to a covariantization of the RG equations.

The above two-step process can be succinctly stated in the form of the following equality of partition functions:

$$Z[M, z, \mathcal{B}, W^{(0)}, U] = Z[\lambda M, z, \widetilde{\mathcal{B}}, W^{(0)}, \widetilde{U}] \tag{7.3}$$

$$= Z[M, \lambda^{-1} z, \mathcal{L}^{-1}\cdot\widetilde{\mathcal{B}}\cdot\mathcal{L}, \mathcal{L}^{-1}\cdot W^{(0)}\cdot\mathcal{L} + \mathcal{L}^{-1}\cdot d\mathcal{L}, \widetilde{U}] \tag{7.4}$$

where we have assumed there there is no anomalous contribution to U under the $CU(L_2)$ transformation.

[1]Note that this choice of regulator preserves the $U(L_2)$ symmetry. We used a slightly different regulator in [38, 39]. The present choice is somewhat more convenient – the differences are merely notational, and not physical.

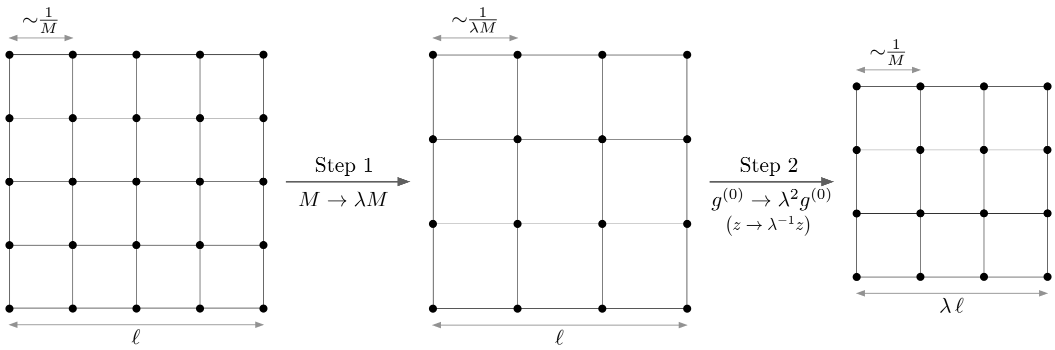

Figure 7.1: In this illustration, we depict the renormalization group flow as a two-step process. In step one, we coarse-grain by lowering the UV cutoff, while in step two we perform a scale transformation to bring the cutoff back to its original value, but changing the background metric.

We can parametrize the infinitesimal RG transformation by writing $\lambda = 1 - \varepsilon$, and

$$\mathcal{L}(x, y) = \delta(x - y) + \varepsilon\, z W_z^{(0)}(x, y) + O(\varepsilon^2) \tag{7.5}$$

where we have suggestively denoted the infinitesimal piece of $\mathcal{L}$ as $W_z^{(0)}$, to indicate that it should be thought of as the z-component of the connection; $W_z^{(0)}$ is merely a book-keeping device which keeps track of the gauge transformations along the RG flow. Equations (7.3) and (7.4) then give us

$$W_\mu^{(0)}(z + \varepsilon z) = W_\mu^{(0)}(z) + \varepsilon z \left[D_\mu^{(0)}, W_z^{(0)}\right]_\bullet + O(\varepsilon^2) \tag{7.6}$$

$$\mathcal{B}(z + \varepsilon z) = \mathcal{B}(z) - \varepsilon z \left[W_z^{(0)}, \mathcal{B}\right]_\bullet + \varepsilon z \beta^{(\mathcal{B})} + O(\varepsilon^2) \tag{7.7}$$

$$U(z + \varepsilon z) = U(z) - i\varepsilon z N\, \mathrm{Tr.}\, (\Delta_B \bullet \mathcal{B}) \tag{7.8}$$

where the *tensorial* beta function $\beta^{(\mathcal{B})}$ is given by (see Appendix C for details)

$$\beta^{(\mathcal{B})} = \mathcal{B} \bullet \Delta_B \bullet \mathcal{B}. \tag{7.9}$$

and we have defined

$$\Delta_B = \frac{2z}{M^2} \dot{K}\Big(- z^2 D_{(0)}^2 / M^2 \Big) \tag{7.10}$$

with $\dot{K}(s) = \partial_s K(s)$. We note that Δ_B defines a *regulated* or *smeared* version of the $\bullet$-product between bi-locals.[2]

[2]A convenient choice of regulator which is useful for computations is the exponential cutoff $K(s) = e^{-s}$. In this case, the kernel Δ_B is proportional to the heat kernel for the operator $D_{(0)}^2$: $\Delta_B = -\frac{2z}{M^2} e^{\frac{z^2}{M^2} D_{(0)}^2}$. In the limit $z \to 0$, $\Delta_B(z; \vec{x}, \vec{y}) \to -\frac{2z}{M^2} \delta^d(\vec{x} - \vec{y})$.

By continuing the renormalization group flow, we can extend $\mathcal{B}$ and $W^{(0)}$ into the entire RG mapping space $\mathbb{R}^3 \times \mathbb{R}^+$, where the half-line $\mathbb{R}^+$ is parametrized by z. We will often refer to this space as the *bulk*, for reasons which will become clear shortly – the RG equations will be interpreted as a dynamical (Hamiltonian) system on this one-higher dimensional space in the next section. We will also henceforth refer to the extended fields as $\mathfrak{B}$ and $\mathcal{W}^{(0)}$, to emphasize that they live in the bulk. Note that $\mathcal{W}^{(0)}$ is a one-form in the bulk; indeed, $\mathcal{W}^{(0)}$ "grows a leg" in the z-direction, with $\mathcal{W}^{(0)}_z$ keeping track of the gauge transformations along the RG flow, as discussed above. Comparing the ε terms on both sides of equations (7.6) and (7.7), we find

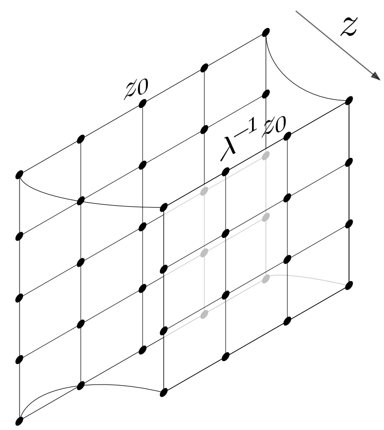

Figure 7.2: The renormalization group equations can be interpreted as a dynamical (Hamiltonian) system on a one-higher dimensional space.

$$\partial_z \mathcal{W}^{(0)}_\mu - \partial_\mu \mathcal{W}^{(0)}_z + \left[\mathcal{W}^{(0)}_z, \mathcal{W}^{(0)}_\mu\right]_{\cdot} = 0 \tag{7.11}$$

$$\partial_z \mathfrak{B} + \left[\mathcal{W}^{(0)}_z, \mathfrak{B}\right]_{\cdot} = \beta^{(\mathfrak{B})} \tag{7.12}$$

Therefore, the renormalization group equations turn out to be gauge-covariant equations in the bulk. Given that $\mathcal{W}^{(0)}_\mu$ is also flat in the transverse directions (by construction; see equation (6.23)), the first of these equations can be promoted to

$$\mathcal{F}^{(0)} \equiv \boldsymbol{d}\mathcal{W}^{(0)} + \mathcal{W}^{(0)} \wedge \mathcal{W}^{(0)} = 0 \tag{7.13}$$

where $\boldsymbol{d} = dx^\mu \left[\partial_\mu, \cdot\right] + dz\partial_z$ is the bulk exterior derivative. Equation (7.12) can similarly be promoted to a full-fledged one-form equation in the bulk

$$\mathcal{D}_{(0)}\mathfrak{B} \equiv \boldsymbol{d}\mathfrak{B} + \left[\mathcal{W}^{(0)}, \mathfrak{B}\right]_{\cdot} = \boldsymbol{\beta}^{(\mathfrak{B})} \tag{7.14}$$

The z-component of the one-form $\boldsymbol{\beta}^{(\mathfrak{B})} = \beta^{(\mathfrak{B})}_\mu dx^\mu + \beta^{(\mathfrak{B})} dz$ is given by equation (7.9); the transverse components on the other hand get determined by the Bianchi identity[3]

$$\mathcal{D}_{(0)}\boldsymbol{\beta}^{(\mathfrak{B})} \equiv \boldsymbol{d}\boldsymbol{\beta}^{(\mathfrak{B})} + \Big[\mathcal{W}^{(0)}, \boldsymbol{\beta}^{(\mathfrak{B})}\Big]_{\cdot} = 0 \tag{7.15}$$

Thus, the renormalization group equations for single-trace perturbations away from the free fixed point organize themselves into covariant equations, with the beta function playing the role of "curvature". In the following section, we will argue that these can be naturally interpreted as (one-half of the) equations of motion describing the holographic dual of free field theory. It might be surprising that the equations we have derived above are remarkably simple, as compared to the Vasiliev higher spin equations. We emphasize that the equations are exact and form a consistent closed system. We have been able to establish these simple equations precisely because we have not insisted on locality. This is an essential aspect of free field theories.

We can similarly write down the Callan-Symanzik equations for $\Pi(x, y)$ following the two step RG prescription outlined above. We find (see Appendix C.3 for details)

$$\Pi(z + \varepsilon z; x, y) = \Pi(z; x, y) - z\varepsilon \Big[W_z^{(0)}, \Pi\Big]_{\cdot} + i\varepsilon z N\Delta_B + \varepsilon z \mathrm{Tr}.\gamma(x, y; u, v){\cdot}\Pi(v, u) \tag{7.16}$$

where we have introduced the notation

$$\gamma(x, y; u, v) = -\frac{\delta\beta^{(\mathcal{B})}(u, v)}{\delta\mathcal{B}(y, x)} \tag{7.17}$$

Note that $\Pi(x, y)$ transforms tensorially under $CU(L_2)$; we may extend it to a bulk adjoint-valued zero form $\mathcal{P}(x, y)$. Comparing ε terms on both sides of equation (7.16), we arrive at

$$\mathcal{D}_z^{(0)}\mathcal{P} \equiv \partial_z\mathcal{P} + \Big[\mathcal{W}_z^{(0)}, \mathcal{P}\Big]_{\cdot} = iN\Delta_B + \mathrm{Tr}\ \gamma(x, y; u, v){\cdot}\mathcal{P}(v, u) \tag{7.18}$$

In the next section, we will see that $\mathfrak{B}$ and $\mathcal{P}$ can be treated as canonically conjugate variables, and the equations (7.14) and (7.18) can be interpreted as Hamiltonian equations of motion.

Notably, equation (7.13) implies that the $\mathcal{W}^{(0)}$ is a *flat* connection in the *bulk* – this is where AdS comes into the picture. In higher-spin theories, geometry is not manifest in the usual sense, i.e. there is no meaningful way to talk about metrics, and the corresponding curvature etc. Instead, the AdS geometry appears in the

[3]The Bianchi identity is derived by acting on equation (7.14) with $\mathcal{D}_{(0)}$, and using the fact that $\mathcal{W}^{(0)}$ is flat.

form of a *flat*, $\mathfrak{g} = \mathfrak{o}(2,d)$-valued connection one form,

$$\mathcal{W}^{(0)}_{AdS} = -\frac{dz}{z}D + \frac{dx^\mu}{z}P_\mu \tag{7.19}$$

which is in fact the Maurer-Cartan form for $O(2,d)$. This is also familiar from, for example, the Chern-Simons formulation of gravity in $2+1$ dimensions, and is in fact a generalization of this story to higher dimensions. By picking out an $\mathfrak{h} = \mathfrak{o}(1,d)$ subalgebra inside $\mathfrak{g}$, one identifies the corresponding $\mathfrak{h}$-valued part of $\mathcal{W}^{(0)}$ as the AdS spin connection, while the remaining $\mathfrak{g}/\mathfrak{h}$-valued piece is identified as the AdS co-frame. The fact that the isometry group of AdS_{d+1} is precisely the conformal group $O(2,d)$ of $\mathbb{R}^{1,d-1}$ is, in this language, manifested in the fact that there exists a subalgebra isomorphic to $\mathfrak{o}(2,d)$ inside the set of all gauge transformations which *preserve* $\mathcal{W}^{(0)}_{AdS}$. It is therefore natural to interpret the flat $U(L_2)$ connection which emerges in the bulk at the free fixed point, as the AdS connection.

Finally, we state the infinitesimal version of the RG "Ward identities" (7.3) and (7.4) explicitly

$$-\frac{\partial}{\partial z}Z = \mathrm{Tr}\left\{\left(\left[\mathcal{B}, W^{(0)}_z\right]_\cdot + \beta^{(\mathcal{B})}\right)\cdot\frac{\delta}{\delta\mathcal{B}} + \left[D^{(0)}_\mu, W^{(0)}_z\right]_\cdot\cdot\frac{\delta}{\delta W^{(0)}_\mu}\right\}Z + N\mathrm{Tr}\,\left(\Delta_B\cdot\mathcal{B}\right)Z \tag{7.20}$$

where by $\frac{\partial}{\partial z}Z$ we mean the partial derivative with respect to z keeping all the sources fixed. As we will see in the next section, this identity can be interpreted as the *Hamilton-Jacobi* equation (z being the parameter for "radial evolution"), and plays a very crucial role in making contact with holography.

7.2 Holography via Hamilton-Jacobi theory

In the previous section, we have seen how the renormalization group organizes field theory data in the one-higher dimensional RG mapping space. In this setup, the sources and the corresponding vacuum expectation values for single-trace deformations away from the fixed point turn into fields living in the bulk, with their dynamics governed by renormalization group equations. However, in order to ascribe a holographic interpretation to this, we must go further and show that all the correlation functions of the field theory can be reproduced from the bulk theory. The first step towards this, of course, is to construct the bulk action.

The defining property of holography is contained in the following equation (which is equivalent to equation

(5.7) adapted to the present case)

$$Z[\epsilon, w_\mu^{(0)}, b^{(0)}] = e^{-S_{HJ}[\epsilon, w_\mu^{(0)}, b^{(0)}]} \tag{7.21}$$

where S_{HJ} is the *Hamilton-Jacobi* functional for the bulk theory; i.e., the bulk action evaluated on-shell, with the boundary conditions $\mathfrak{B}(\epsilon) = b^{(0)}$ and $\mathcal{W}^{(0)}(\epsilon) = w^{(0)}$. Said another way, the generating functional of the CFT is a wavefunctional (defined on a constant $z = \epsilon$ hypersurface) from the bulk point of view in radial quantization. Therefore, while we might not have access directly to the bulk action, the field theory gives us the Hamilton-Jacobi functional instead. As is well-known from Hamilton-Jacobi theory, the (connected) boundary expectation value

$$\Pi = \frac{\delta S_{HJ}}{\delta b^{(0)}} \tag{7.22}$$

can be thought of as the boundary value of the momentum conjugate to $\mathfrak{B}$ in the bulk. Thus, we see a bulk phase space picture emerging, with $\mathfrak{B}$ and $\mathcal{P}$ forming a canonical pair. The canonical 1-form (of which the symplectic 2-form is the exterior derivative) is given by

$$\theta = \text{Tr. } \mathcal{P} \cdot \delta \mathfrak{B} \tag{7.23}$$

The crucial observation is that the RG Ward identity (7.20) takes the form of the Hamilton-Jacobi equation

$$\frac{\partial}{\partial z} S_{HJ} = -\mathcal{H}_{bulk} \tag{7.24}$$

with the bulk Hamiltonian given by

$$\mathcal{H}_{bulk} = \text{Tr.} \left\{ \left(\left[\mathfrak{B}, \mathcal{W}_z^{(0)} \right]_{\cdot} + \boldsymbol{\beta}_z^{(\mathfrak{B})} \right) \cdot \mathcal{P} + \left[\mathcal{D}_\mu^{(0)}, \mathcal{W}_z^{(0)} \right]_{\cdot} \cdot \mathcal{P}^\mu \right\} - N \text{ Tr.} \left(\Delta_B \cdot \mathfrak{B} \right). \tag{7.25}$$

It is straightforward to check that the Hamilton equations of motion which follow from the above are precisely the RG equations (7.11), (7.12) and the Callan-Symanzik equations (7.18). Furthermore, we note that $\mathcal{W}_z^{(0)}$ is a Lagrange multiplier, which enforces the $U(L_2)$ Ward identity. In addition to the above "dynamical" terms in the Hamiltonian, we may also introduce constraint terms, which enforce the transverse components (i.e., the dx^μ components) of equations (7.13), (7.14)

$$\mathcal{H}_{constr.} = \text{Tr} \left\{ \left(\mathcal{D}_\mu^{(0)} \mathfrak{B} - \boldsymbol{\beta}_\mu^{(\mathfrak{B})} \right) \cdot \mathcal{Q}^\mu + \mathcal{F}_{\mu\nu}^{(0)} \cdot \mathcal{Q}^{\mu\nu} \right\} \tag{7.26}$$

where $\mathcal{Q}^\mu$ and $\mathcal{Q}^{\mu\nu}$ are Lagrange multipliers. Note that the Hamiltonian is linear in momenta, and as such there is no distinction between phase space and configuration space formalisms. Nevertheless, we may construct a "phase space action" (of the type "$p\dot{q} - H$", reminiscent of BF theories in condensed matter physics) given by

$$S_{bulk} = \int_{\infty}^{\epsilon} dz \ \mathrm{Tr.} \left\{ \mathcal{P}^I \boldsymbol{\cdot} \left(\mathcal{D}_I^{(0)} \mathfrak{B} - \boldsymbol{\beta}_I^{(\mathfrak{B})} \right) + \mathcal{P}^{IJ} \boldsymbol{\cdot} \mathcal{F}_{IJ}^{(0)} + N \ \Delta_B \boldsymbol{\cdot} \mathfrak{B} \right\} \tag{7.27}$$

where we have collected together $\mathcal{P}, \mathcal{Q}^\mu$ into $\mathcal{P}^I$, etc. It is worthwhile noting that the first variation of this action reproduces all the RG and Callan-Symanzik equations. In the gauge where the Lagrange multipliers $\mathcal{Q}^\mu$ and $\mathcal{Q}^{\mu\nu}$ are set to zero, we obtain the action for $\mathfrak{B}$:

$$S_{bulk}[\mathfrak{B}, \mathcal{P}] = \int_{\infty}^{\epsilon} dz \ \mathrm{Tr.} \left\{ \mathcal{P} \boldsymbol{\cdot} \left(\mathcal{D}_z^{(0)} \mathfrak{B} - \mathfrak{B} \boldsymbol{\cdot} \Delta_B \boldsymbol{\cdot} \mathfrak{B} \right) - N \ \Delta_B \boldsymbol{\cdot} \mathfrak{B} \right\} \tag{7.28}$$

As we will see in this next chapter, this action reproduces all the correlation functions of the boundary theory, in the form of Witten diagrams, thus making the holographic correspondence further manifest. We will conclude this chapter by noting that the above action is strikingly reminiscent of Chern Simons theory; in fact we can make this more manifest if we define the new field $\widetilde{\mathfrak{B}}$ as $\mathcal{P} = \Delta_B \boldsymbol{\cdot} \widetilde{\mathfrak{B}} \boldsymbol{\cdot} \Delta_B$, and the new bilocal product $*$ and trace Tr_* as

$$(f * g) = f \boldsymbol{\cdot} \Delta_B \boldsymbol{\cdot} g, \quad \mathrm{Tr}_* f = \mathrm{Tr.} \left(\Delta_B \boldsymbol{\cdot} f \right) \tag{7.29}$$

In terms of these quantities, the action takes the form

$$S_{bulk}[\mathfrak{B}, \widetilde{\mathfrak{B}}] = \int_{\infty}^{\epsilon} dz \ \mathrm{Tr}_* \left(\widetilde{\mathfrak{B}} * \mathcal{D}_z^{(0)} \mathfrak{B} - \widetilde{\mathfrak{B}} * \mathfrak{B} * \mathfrak{B} \right) + N \int_{\infty}^{\epsilon} dz \ \mathrm{Tr}_* \left(\mathfrak{B} \right) \tag{7.30}$$

which as we noted above, is reminiscent of a non-commutative version of Chern-Simons theory, which appears in Witten's construction of open string field theory. It is indeed an interesting question whether this can be made more precise. Additionally, our discussion of higher-spin equations of motion is similar in spirit to the Vasiliev higher-spin theories, although the precise details appear very different (see Appendix C for some further comments) – we leave these questions for future exploration.

• • •

Chapter 8

Correlation functions and Witten diagrams

In this chapter, we will show that the bulk action derived from the exact renormalization group in the previous chapter, precisely reproduces all the correlation functions of the free vector model in terms of Witten diagrams. In order to compute the field theory correlation functions, we follow the standard prescription, i.e., we compute the bulk action on-shell, and extract the boundary generating functional from it, as per equation (7.21). Note that on-shell, the only non-trivial contribution comes from the last term

$$S_{bulk,o.s} = -N \int_{\epsilon}^{\infty} dz \, \mathrm{Tr}_* \mathfrak{B} = -N \int_{\epsilon}^{\infty} dz \; \mathrm{Tr.} \, (\Delta_B \bullet \mathfrak{B}) \tag{8.1}$$

where the additional minus sign comes from flipping the limits of integration. We remark that this term may be traced back to the RG flow of the source corresponding to the identity operator (7.8). Since the field $\mathfrak{B}$ above is a solution to the bulk equation of motion $\mathcal{D}^{(0)}\mathfrak{B} = \boldsymbol{\beta}^{(\mathfrak{B})}$, what we should do is solve this equation (along with the Callan-Symanzik equation), and substitute back into the action. But before we do that, we need to set up boundary conditions. Since we have two equations at hand, we need two boundary conditions. In the present context, one boundary condition presents itself naturally – we fix the value of $\mathfrak{B}$ at the boundary $z = \epsilon$:

$$\mathfrak{B}(\epsilon; x, y) = b^{(0)}(x, y) \tag{8.2}$$

From the field theory point of view, $b^{(0)}$ clearly has the interpretation of fixing the source at the ultraviolet cutoff. For the other boundary condition, we fix $\mathcal{P}$ in the infra-red:

$$\lim_{z\to\infty} \mathcal{P}(z; x, y) = 0 \tag{8.3}$$

This condition is of course consistent with the Hamilton-Jacobi structure (and the canonical 1-form (7.23)), and is akin to the interior boundary condition one encounters regularly in holography.[1]

[1]For the variational principle to be well defined, we must either fix $\mathfrak{B}$ on the boundary, or set $\mathcal{P} = 0$ on the boundary, and thus the boundary conditions we have chosen are consistent with the variational principle without any additional boundary terms.

The equations at hand are non-linear; it is convenient (and perhaps physically more instructive) to solve them iteratively. Consequently, we introduce a formal organizing parameter λ, writing

$$\mathfrak{B} = \lambda \mathfrak{B}_{(1)} + \lambda^2 \mathfrak{B}_{(2)} + \cdots \tag{8.4}$$

$$\mathcal{P} = \mathcal{P}_{(0)} + \lambda \mathcal{P}_{(1)} + \lambda \mathcal{P}_{(2)} + \cdots \tag{8.5}$$

and we will solve the equations of motion order by order in λ; the reader may regard the parameter $\lambda \sim \frac{1}{N}$ in a suitable normalization. Let us focus on the $\mathfrak{B}$ equation first. In fact, it suffices to focus on the z-component of the equation of motion, as the remaining components are automatically enforced by the Bianchi identity (7.15). Then, we have

$$\begin{aligned} \left[\mathcal{D}_z^{(0)}, \mathfrak{B}_{(1)}\right]_{\boldsymbol{\cdot}} &= 0 && (8.6)\\ \left[\mathcal{D}_z^{(0)}, \mathfrak{B}_{(2)}\right]_{\boldsymbol{\cdot}} &= \mathfrak{B}_{(1)} \boldsymbol{\cdot} \Delta_B \boldsymbol{\cdot} \mathfrak{B}_{(1)} && (8.7)\\ \left[\mathcal{D}_z^{(0)}, \mathfrak{B}_{(3)}\right]_{\boldsymbol{\cdot}} &= \mathfrak{B}_{(2)} \boldsymbol{\cdot} \Delta_B \boldsymbol{\cdot} \mathfrak{B}_{(1)} + \mathfrak{B}_{(1)} \boldsymbol{\cdot} \Delta_B \boldsymbol{\cdot} \mathfrak{B}_{(2)} && (8.8)\\ &\vdots \end{aligned}$$

We immediately see that the system of equations can be solved sequentially, with the solution of one equation (and all before it) determining the right-hand side of the next. The first equation (8.6) is homogeneous and has the solution

$$\mathfrak{B}_{(1)}(z; x, y) = \int_{x', y'} K(z; x, x') b_{(0)}(x', y') K^{-1}(z; y', y) \tag{8.9}$$

where we have defined the *boundary-to-bulk Wilson line*

$$K(z) = \mathscr{P}. \exp\Big(- \int_{\epsilon}^{z} dz' \, \mathcal{W}_z^{(0)}(z') \Big) \tag{8.10}$$

satisfying the equation

$$\partial_z K(z) + \mathcal{W}_z^{(0)}(z) \boldsymbol{\cdot} K(z) = 0 \tag{8.11}$$

This Wilson line should be interpreted in the terms we described in Section 6.2 above. As usual, we will surreptitiously write equation (8.9) as

$$\mathfrak{B}_{(1)}(z) = K(z) \boldsymbol{\cdot} b_{(0)} \boldsymbol{\cdot} K^{-1}(z) \tag{8.12}$$

in favor of compact notation. What we have done above, is to recognize that conjugating by K (i.e., pulling back from a bulk point to the boundary) effectively converts the covariant derivative in (8.6) to ∂_z. Since $W^{(0)}$ is flat by its equation of motion, K is independent of the path connecting the endpoints. At this order, the on-shell action is simply

$$S^{(1)}_{bulk,o.s.} = -N \int_\epsilon^\infty dz \; \mathrm{Tr} \; \Delta_B \cdot \mathfrak{B}_{(1)} = -N \int_\epsilon^\infty dz \; \mathrm{Tr} \; \left(K^{-1} \cdot \Delta_B \cdot K \cdot b_{(0)}\right) \tag{8.13}$$

It is convenient at this point to define the *Wilsonian Green function* for the boundary field theory

$$g(z; x, y) = \int_\epsilon^z dz' \; H(z'; x, y) = \int_\epsilon^z dz' \; \left(K^{-1} \cdot \Delta_B \cdot K\right)(z'; x, y) \tag{8.14}$$

where

$$H(z) \equiv K^{-1}(z) \cdot \Delta_B(z) \cdot K(z) = \partial_z g(z) \tag{8.15}$$

and furthermore we will denote

$$g_{(0)}(x, y) \equiv g(\infty; x, y), \tag{8.16}$$

which is in fact closely related to the free elementary field propagator of the boundary theory. To see this, note from the result (8.13) (or equivalently by solving the Callan-Symanzik equation at the zeroth order $\mathcal{D}^{(0)} \mathcal{P}_{(0)} = iN\Delta_B$, subject to the boundary condition eq. (8.3)), that

$$\mathcal{P}_{(0)}(\epsilon; x, y) \equiv \langle \phi^*_m(y) \phi^m(x) \rangle_{CFT} = \frac{\delta S_{bulk,o.s.}}{\delta b_{(0)}(y, x)}\Big|_{b_{(0)}=0} = -N \int_\epsilon^\infty dz \; H(z; x, y) = -N \; g_{(0)}(x, y) \tag{8.17}$$

where the subscript CFT means the correlation function at the free-fixed point. Thus we find that the linear term in the on-shell action can be written entirely in terms of boundary quantities

$$S^{(1)}_{bulk,o.s.} = -N \; \mathrm{Tr} \; g_{(0)} \cdot b_{(0)}. \tag{8.18}$$

The above computation can be represented in terms of a Witten diagram as in Fig. 8.1.

Proceeding to second order, we solve equation (8.7) with $\mathfrak{B}_{(1)}$ given by equation (8.12)

$$\left[\mathcal{D}^{(0)}_z, \mathfrak{B}_{(2)}\right]_\cdot = \Phi_{(2)}(z) \quad \equiv \quad K(z) \cdot b_{(0)} \cdot K^{-1}(z) \cdot \Delta_B(z) \cdot K(z) \cdot b_{(0)} \cdot K^{-1}(z) \tag{8.19}$$

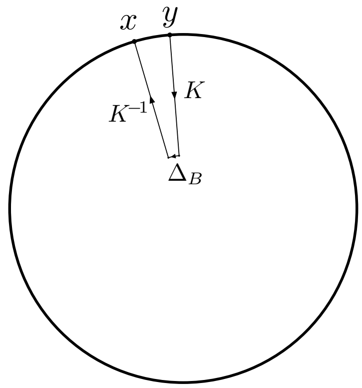

Figure 8.1: The Witten diagram representation for the boundary one-point function $\mathcal{P}_{(0)}(x, y)$. The arrows indicate radial orientation, while the turn-around in the bulk represents an insertion of Δ_B.

More generally, at any given order, we can always write

$$\left[\mathcal{D}_z^{(0)}, \mathfrak{B}_{(k)}\right]_{\bullet} = \Phi_{(k)}(z) \tag{8.20}$$

where $\Phi_{(k)}$ is the inhomogenous term at the corresponding order. To solve this, we first conjugate by K to reduce the covariant derivative to an ordinary derivative

$$K^{-1}(z)\bullet \left[\mathcal{D}_z^{(0)}, \mathfrak{B}_{(k)}\right]_{\bullet}(z)\bullet K(z) = \partial_z \Big(K^{-1}(z)\bullet \mathfrak{B}_{(k)}(z)\bullet K(z)\Big) \tag{8.21}$$

and so we obtain

$$\partial_z \Big(K^{-1}(z)\bullet \mathfrak{B}_{(k)}(z)\bullet K(z)\Big) = K^{-1}(z)\bullet \Phi_{(k)}(z)\bullet K(z) \tag{8.22}$$

Taking without loss of generality the boundary condition to be $\mathfrak{B}_{(k)}(\epsilon) = 0,\ \forall k \geq 2$ (since eq. (8.2) has been satisfied at first order in α), the above equation can be easily solved

$$\mathfrak{B}_{(k)}(z) = K(z)\bullet \left[\int_\epsilon^\infty dz'\ \Theta(z - z')\ K^{-1}(z')\bullet \Phi_{(k)}(z')\bullet K(z')\right]\bullet K^{-1}(z) \tag{8.23}$$

We can recognize here the *ingoing bulk-to-bulk* Wilson line

$$G(z; z') = \Theta(z - z')\ K(z)\bullet K^{-1}(z') = \Theta(z - z')\ \mathscr{P}_{\bullet} \exp\Big(- \int_{z'}^{z} du\ \mathcal{W}_z^{(0)}(u)\Big) \tag{8.24}$$

and the *outgoing bulk-to-bulk* Wilson line

$$G^{-1}(z'; z) = \Theta(z - z')\ K(z')\bullet K^{-1}(z) = \Theta(z - z')\ \mathscr{P}_{\bullet} \exp\Big(- \int_{z}^{z'} du\ \mathcal{W}_z^{(0)}(u)\Big) \tag{8.25}$$

viz

$$\mathfrak{B}_{(k)}(z) = \int_\epsilon^\infty dz'\ G(z;z')\cdot\Phi_{(k)}(z')\cdot G^{-1}(z';z) \tag{8.26}$$

Collecting everything together, we get the integral equation

$$\mathfrak{B}(z) = K(z)\cdot b_{(0)}\cdot K^{-1}(z) + \int_\epsilon^\infty dz'\ G(z;z')\cdot\beta^{(\mathfrak{B})}[\mathfrak{B}](z')\cdot G^{-1}(z';z) \tag{8.27}$$

Returning to the second order calculation, we have

$$\mathfrak{B}_{(2)} = \int_\epsilon^z dz'\ K(z)\cdot b_{(0)}\cdot K^{-1}(z')\cdot\Delta_B(z')\cdot K(z')\cdot b_{(0)}\cdot K^{-1}(z) \tag{8.28}$$

and thus, the on-shell action at second order is given by

$$\begin{aligned} S^{(2)}_{bulk,o.s} &= -N\int_\epsilon^\infty dz \int_\epsilon^z dz'\ \mathrm{Tr}\left(K^{-1}(z)\cdot\Delta_B(z)\cdot K(z)\cdot b_{(0)}\cdot K^{-1}(z')\cdot\Delta_B(z')\cdot K(z')\cdot b_{(0)}\right) && (8.29)\\ &= -N\int_\epsilon^\infty dz \int_\epsilon^z dz'\ \mathrm{Tr}\left(H(z)\cdot b_{(0)}\cdot H(z')\cdot b_{(0)}\right) && (8.30)\end{aligned}$$

We can once again represent this in terms of a Witten diagram as in Figure 2. Using equation (8.14), the

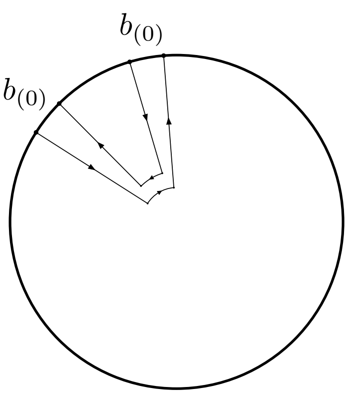

Figure 8.2: The Witten diagram representing the second order term $S^{(2)}_{o.s}$ in the bulk on-shell action. The $b_{(0)}$s are boundary insertions of the ultraviolet bi-local source $b_{(0)}$.

z-integrations can be straightforwardly performed

$$\begin{aligned} S^{(2)}_{bulk,o.s} &= -N\int_\epsilon^\infty dz \int_\epsilon^z dz'\ \mathrm{Tr}\left(H(z)\cdot b_{(0)}\cdot\partial_{z'}g(z')\cdot b_{(0)}\right) && (8.31)\\ &= -N\int_\epsilon^\infty dz\mathrm{Tr}\left(\partial_z g(z)\cdot b_{(0)}\cdot g(z)\cdot b_{(0)}\right) && (8.32)\\ &= -\frac{N}{2}\int_\epsilon^\infty dz\ \partial_z\mathrm{Tr}\left(g(z)\cdot b_{(0)}\cdot g(z)\cdot b_{(0)}\right) && (8.33)\end{aligned}$$

which integrates to

$$S^{(2)}_{bulk,o.s} = -\frac{N}{2}\ \mathrm{Tr}\left(g_{(0)}\cdot b_{(0)}\cdot g_{(0)}\cdot b_{(0)}\right). \tag{8.34}$$

This result reproduces the correct two-point functions of the free field theory.

This procedure can be followed to arbitrary order. One finds the k^{th}-order term has the form

$$S^{(k)}_{bulk,o.s.} = -N\int_{\epsilon}^{\infty} dz_1 \int_{\epsilon}^{z_1} dz_2 ... \int_{\epsilon}^{z_{k-1}} dz_k\ \mathrm{Tr}\left(H(z_1)\cdot b_{(0)}\cdot H(z_2)\cdot b_{(0)}\cdot ...\cdot H(z_k)\cdot b_{(0)} + \text{permutations}\right) \tag{8.35}$$

The permutations include all of the distinct orderings of $\{H(z_2), ..., H(z_k)\}$. Proceeding with the z-integrals as before, we find the on-shell action at this order is given by

$$S^{(k)}_{bulk,o.s.} = -\frac{N}{k}\ \mathrm{Tr}\left(g_{(0)}\cdot b_{(0)}\right)^k \tag{8.36}$$

As an example, the Witten diagram for the three point function is shown in Fig. 3.

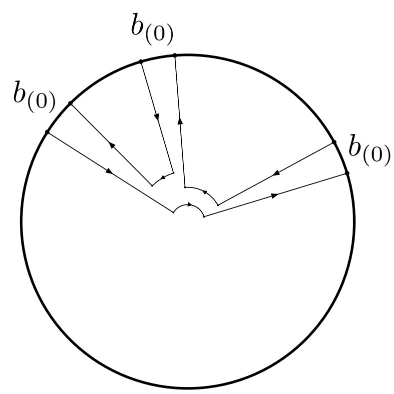

Figure 8.3: The Witten diagram for the bulk on-shell action at third order.

Collecting equations (8.18), (8.34), (8.36), we note that the on-shell action

$$S_{bulk,o.s.} = -N\Big(\mathrm{Tr}\left(g_{(0)}\cdot b_{(0)}\right) + \frac{1}{2}\ \mathrm{Tr}\left(g_{(0)}\cdot b_{(0)}\cdot g_{(0)}\cdot b_{(0)}\right) + \frac{1}{3}\ \mathrm{Tr}\left(g_{(0)}\cdot b_{(0)}\cdot g_{(0)}\cdot b_{(0)}\cdot g_{(0)}\cdot b_{(0)}\right) + \cdots\Big) \tag{8.37}$$

precisely reproduces the boundary generating functional

$$Z[b_{(0)}]/Z[0] = e^{-S_{bulk,o.s.}} = \det{}^{-N}\left(1 - g_{(0)}\cdot b_{(0)}\right) \tag{8.38}$$

Thus we conclude that the holographic formulation correctly reproduces all of the correlation functions of the boundary field theory.

Several comments are in order at this point. First, we have seen that a 'double-line notation' naturally emerges for the Witten diagrams, essentially due to the bi-locality of the bulk field $\mathfrak{B}$. However, because the connection $\mathcal{W}^{(0)}$ is flat, the corresponding Wilson lines can follow any path.[2] Second, the 'bulk vertex' is non-local. Each of these properties is a manifestation of unbroken higher spin symmetry at the free fixed point. Third, the above computation strengthens our claim that the action (7.27) describes the holographic dual to the free bosonic vector model. It is only because the field theory in this case is completely under control, that we could construct the bulk holographic description by hand, and then check that we can go back and forth between the bulk and boundary descriptions. Finally, note from (8.38) that our holographic description reproduced the ratio of partition functions $Z[b_{(0)}]/Z[0]$. $Z[0]$ is the domain of holographic renormalization. The divergences as $\epsilon \to 0$ contained in $Z[0]$ can be cancelled by local boundary counterterms.

•••

[2]If the region between Wilson lines were filled in (as it would be in the presence of a dynamical $U(N)$ gauge field in the field theory) to obtain 'open string worldsheets', the string tension would be zero.

Chapter 9

Higher Spin Fronsdal equations

So far we have constructed the bulk dynamics from the renormalization group, and shown that it reproduces all the correlation functions of the boundary CFT within the framework of the AdS/CFT correspondence. However, our bulk theory is expressed abstractly in terms of connections on certain infinite jet-bundles; it is not clear how the traditional fields such as the photon, graviton etc., presumably propagating on an emergent AdS spacetime, are described by these equations. In this chapter, we will show how these fields are contained in our equations, to linear order in the $1/N$ expansion. Since we expect our theory to contain an infinite tower of massless gauge fields, we will see not merely the spin-one photon and the spin-two graviton, but also massless higher-spin gauge fields of all integer spins. Since we're working at leading order in $1/N$, the bulk fields should be non-interacting, free fields – the corresponding equations of motion are known as *Fronsdal equations*. Interestingly, Vasiliev theories of higher-spin gravity also reduce to Fronsdal equations at the linearized order – therefore our discussion in this chapter will prove the equivalence between our equations of motion and Vasiliev theory at this order.

We first begin with a brief review of the Fronsdal equations of motion in AdS, and then proceed to show that the bulk equations of motion derived previously contain these dynamical equations within them.

9.1 The Fronsdal Equations

The Fronsdal higher spin theory in AdS space is described by symmetric tensors $h_{I_1 \ldots I_n}$ which satisfy the double-tracelessness conditions ${\varphi''}_{I_5 \ldots I_n} \equiv g^{I_1 I_2} g^{I_3 I_4} \varphi_{I_1 \ldots I_n} = 0$. Here the bulk coordinate indices run over the boundary coordinate indices $\mu = 0, 1, ..., d-1$ and the radial direction z, i.e., $I = (\mu, z)$. The equations of motion are explicitly

$$\nabla_I \nabla^I \varphi_{I_1 \ldots I_n} - n \nabla_I \nabla_{(I_1} \varphi_{I_2 \ldots I_n)}{}^I + \frac{1}{2} n(n-1) \nabla_{(I_1} \nabla_{I_2} \varphi_{I_3 \ldots I_n) I}{}^I - 2(n-1)(n+d-2) \varphi_{I_1 \ldots I_n} = 0 \quad (9.1)$$

where the indices $I_1, \cdots I_n$ should be taken to be symmetrized as indicated by parentheses. These equations are invariant under the gauge transformations

$$\delta_\Lambda \varphi_{I_1 \dots I_n} = \nabla_{(I_1} \Lambda_{I_2 \dots I_n)} \tag{9.2}$$

where ∇ is the AdS_{d+1} covariant derivative and the symmetric gauge parameters $\Lambda_{I_2 \dots I_n}$ satisfy the single-tracelessness conditions $g^{I_2 I_3} \Lambda_{I_2 \dots I_n} \equiv \Lambda'_{I_4 \dots I_n} = 0$. For $n = 1$, equation (9.1) is the familiar Maxwell's equation, while for $n = 2$ it is the linearized Einstein's equation.

Such a presentation of the higher spin equations is inconvenient in the present context. We wish to isolate specific (lowest weight) representations of $O(2, d)$; such representations are given by irreducible spin-s representations of $SO(1, d-1)$. We can accomplish this by appropriately fixing the gauge invariance. Many different choices of gauge have been considered in the literature, but the appropriate one here is the "Coulomb gauge"[1]

$$\varphi_{\underbrace{z \dots z}_{m} \mu_1 \dots \mu_s} = 0, \qquad \partial^\mu \varphi_{\mu \mu_1 \dots \mu_s} = 0 \qquad \forall m > 0, \forall s \tag{9.3}$$

In addition, in order to have an irreducible $SO(1, d-1)$ representation, we require $\varphi^{\mu}{}_{\mu \mu_1 \dots \mu_{s-2}} = 0$. In this gauge, the equations of motion reduce to

$$\left[z^2 \partial_z^2 + (2s - d + 1) z \partial_z + s(s-d) + (2-s)(s+d-2) + z^2 \Box_{(\vec{x})} \right] \varphi_{\mu_1 \dots \mu_s}(z, \vec{x}) = 0 \tag{9.4}$$

where $\Box_{(\vec{x})} = \eta^{\mu\nu} \vec{\partial}_\mu \vec{\partial}_\nu$.[2] It is illuminating to obtain equation (9.4) directly from the AdS/CFT point of view, as a statement about the matching of quadratic Casimirs between the bulk and boundary representations [75]. Starting from the CFT, consider a local, symmetric, traceless, spin s, quasi-primary operator $\hat{\mathcal{O}}_{a_1 \dots a_s}(0)$ of dimension Δ in the boundary CFT (where $a_k = 0, \cdots d-1$ are boundary indices). Such an operator satisfies (by definition)

$$\left[K_a, \hat{\mathcal{O}}_{a_1 \dots a_s}(0) \right] = 0 \tag{9.5}$$

$$\left[M_{ab}, \hat{\mathcal{O}}_{a_1 \dots a_s}(0) \right] = \Sigma_{ab}(\hat{\mathcal{O}}_{a_1 \dots a_s}(0)) = -is\eta_{a(a_1} \hat{\mathcal{O}}_{a_2 \dots a_s)b}(0) + is\eta_{b(a_1} \hat{\mathcal{O}}_{a_2 \dots a_s)a}(0) \tag{9.6}$$

$$\left[D, \hat{\mathcal{O}}_{a_1 \dots a_s}(0) \right] = -i\Delta \hat{\mathcal{O}}_{a_1 \dots a_s}(0) \tag{9.7}$$

[1] The terminology "gauge" is somewhat incorrect in this context – what is being said really, is that the fields with z-indices $\varphi_{z \cdots z \mu_1 \cdots \mu_k}$ are non-dynamical, in the sense that they do not contribute to the symplectic structure.

[2] In this section, we will often use the notation $\vec{x}$ to denote coordinates in the boundary spacetime directions.

where Σ_{ab} is the appropriate spin matrix. The quadratic Casimir of the conformal group is given by

$$C_2^{O(2,d)} = -D^2 + \frac{1}{2} M_{ab} M^{ab} - \frac{1}{2}\{P_a, K^a\} \tag{9.8}$$

From equations (9.5) and (9.8), we find straightforwardly[3]

$$\left[C_2^{O(2,d)}, \hat{\mathcal{O}}_{a_1 \ldots a_s}(\vec{x})\right] = \Big(- \Delta(d-\Delta) + s(s+d-2)\Big) \hat{\mathcal{O}}_{a_1 \ldots a_s}(\vec{x}) \tag{9.9}$$

Now the corresponding bulk field $\varphi_{a_1 \cdots a_s}$ of course must have the same value for the Casimir, as it transforms in the corresponding dual AdS representation. We note that $O(2,d)$ is represented as the isometry algebra on AdS:

$$\begin{aligned}
[D, \varphi_{a_1 \ldots a_s}(z,\vec{x})] &= i\vec{x}^a\, [P_a, \varphi_{a_1 \ldots a_s}(z,\vec{x})] + iz\partial_z \varphi_{a_1 \ldots a_s}(z,\vec{x}) \\
[M_{ab}, \varphi_{a_1 \ldots a_s}(z,\vec{x})] &= i\vec{x}_a\, [P_b, \varphi_{a_1 \ldots a_s}(z,\vec{x})] - i\vec{x}_b\, [P_a, \varphi_{a_1 \ldots a_s}(z,\vec{x})] + \Sigma_{ab}(\varphi_{a_1 \ldots a_s})(z,\vec{x}) \\
[K_a, \varphi_{a_1 \ldots a_s}(z,\vec{x})] &= -i(2\vec{x}_a \vec{x}^b - (\vec{x}^2 + z^2)\delta_a^b)\, [P_b, \varphi_{a_1 \ldots a_s}(z,\vec{x})] - i2\vec{x}_a z \partial_z \varphi_{a_1 \ldots a_s}(z,\vec{x}) \\
&- 2\vec{x}^b \Sigma_{ab}(\varphi_{a_1 \ldots a_s})(z,\vec{x}) \\
[P_a, \varphi_{a_1 \ldots a_s}(z,\vec{x})] &= i\vec{\partial}_a \varphi_{a_1 \ldots a_s}(z,\vec{x})
\end{aligned}$$

In this bulk representation, we then have

$$\begin{aligned}
\left[C_2^{O(2,d)}, \varphi_{a_1 \ldots a_s}(z,\vec{x})\right] &= z^2\partial_z^2 \varphi_{a_1 \ldots a_s}(z,\vec{x})) + (-d+1) z\partial_z \varphi_{a_1 \ldots a_s}(z,\vec{x}) \\
&\quad + s(s+d-2)\varphi_{a_1 \ldots a_s}(z,\vec{x}) - z^2\, [P^a, [P_a, \varphi_{a_1 \ldots a_s}(z,\vec{x})]]
\end{aligned} \tag{9.10}$$

But from the CFT calculation, we know that $C_2^{O(2,d)} = -\Delta(d-\Delta) + s(s+d-2)$; therefore, requiring that the two Casimirs agree gives us

$$z^2\partial_z^2 \varphi_{a_1 \ldots a_s}(z,\vec{x}) + (-d+1) z\partial_z \varphi_{a_1 \ldots a_s}(z,\vec{x}) + \Delta(d-\Delta)\varphi_{a_1 \ldots a_s}(z,\vec{x}) + z^2\, [P^a, [P_a, \varphi_{a_1 \ldots a_s}(z,\vec{x})]] = 0 \tag{9.11}$$

To compare this with equation (9.4), we simply note that in the bulk representation, the $a, b, \ldots$ indices must be interpreted as those corresponding to a local frame, as it is in that case that $O(1, d-1)$ acts in the simple fashion stated. Converting to coordinate indices, $\varphi_{a_1 \ldots a_s}(z,\vec{x})$ becomes $z^s \varphi_{\mu_1 \ldots \mu_s}(z,\vec{x})$. Inserting this in the

[3] This result is independent of the spacetime location $\vec{x}$ of the operator, because the quadratic Casimir commutes with translations. Equivalently, every element of the conformal module of course shares the same value of the Casimir.

above equation gives

$$\left[z^2\partial_z^2+(2s-d+1)z\partial_z+s(s-d)+\Delta(d-\Delta)+z^2\Box_{(\vec{x})}\right]\varphi_{\mu_1\ldots\mu_s}(z,\vec{x})=0 \tag{9.12}$$

In the case when the boundary operator is in a short representation, i.e., $\hat{\mathcal{O}}_{a_1\cdots a_s}$ is a conserved current, we have $\Delta=s+d-2$, and so this becomes

$$\left[z^2\partial_z^2+(2s-d+1)z\partial_z+s(s-d)+(2-s)(s+d-2)+z^2\Box_{(\vec{x})}\right]\varphi_{\mu_1\ldots\mu_s}(z,\vec{x})=0 \tag{9.13}$$

in agreement with (9.4). So we conclude that indeed the linearized higher spin equations simply state the value of the Casimir of the appropriate conformal module. Consequently, it must be that the bulk equations of motion derived from RG previously contain the Fronsdal equations. In the rest of the chapter, we proceed to show this explicitly.

9.2 From Wilson-Polchinski to Fronsdal

Let us now embark on our main goal in this chapter, that of reproducing the AdS-Fronsdal equations from the Wilson-Polchinksi exact renormalization group equations:

$$\mathcal{D}_z^{(0)}\mathfrak{B}=\mathfrak{B}\cdot\Delta_B\cdot\mathfrak{B} \tag{9.14}$$

$$\mathcal{D}_z^{(0)}\mathcal{P}=iN\Delta_B-\mathcal{P}\cdot\mathfrak{B}\cdot\Delta_B-\Delta_B\cdot\mathfrak{B}\cdot\mathcal{P} \tag{9.15}$$

In particular, we want to study the above equations upon linearizing about the background

$$\mathfrak{B}=0,\quad \mathcal{P}=\mathcal{P}^{(0)} \tag{9.16}$$

where $\mathcal{P}^{(0)}$ satisfies $\mathcal{D}_z^{(0)}\mathcal{P}^{(0)}=iN\Delta_B$. Clearly, this background is a solution of the equations (9.14) and (9.15), albeit the trivial one which corresponds to the unperturbed boundary CFT. We introduce an auxiliary expansion parameter λ and write

$$\mathfrak{B}(z;\vec{x},\vec{y})=\lambda\,\mathfrak{b}_1(z;\vec{x},\vec{y})+O(\lambda^2),\ \ \mathcal{P}(z;\vec{x},\vec{y})=\mathcal{P}^{(0)}(z;\vec{x},\vec{y})+\lambda\,\mathfrak{p}_1(z;\vec{x},\vec{y})+O(\lambda^2) \tag{9.17}$$

This is where large N plays an important role because such an expansion exists in practice only at large N, with $1/N$ providing the expansion parameter. At linear order in λ, we thus obtain the equations

$$\mathcal{D}_z^{(0)}\mathfrak{b}_1 \quad = \quad 0 \tag{9.18}$$

$$\mathcal{D}_z^{(0)}\mathfrak{p}_1 \quad = \quad -\mathcal{P}^{(0)}\boldsymbol{\cdot}\mathfrak{b}_1\boldsymbol{\cdot}\Delta_B - \Delta_B\boldsymbol{\cdot}\mathfrak{b}_1\boldsymbol{\cdot}\mathcal{P}^{(0)} \tag{9.19}$$

Also recall, that these equations were written for the "*new*" fields defined below equation (6.26). We now revert back to the "*old*" fields by restoring the appropriate powers of z:

$$\mathfrak{b}_1^{new} = \frac{1}{z^{d+2}}\mathfrak{b}_1^{old}, \quad \mathfrak{p}_1^{new} = \frac{1}{z^{d-2}}\mathfrak{p}_1^{old}$$

With this replacement, we get

$$\mathcal{D}_z^{(0)}\mathfrak{b}_1^{old} \quad = \quad \frac{(d+2)}{z}\ \mathfrak{b}_1^{old} \tag{9.20}$$

$$\mathcal{D}_z^{(0)}\mathfrak{p}_1^{old} \quad = \quad \frac{d-2}{z}\mathfrak{p}_1^{old} - \frac{1}{z^4}\Big(\mathcal{P}^{(0)}\boldsymbol{\cdot}\mathfrak{b}_1^{old}\boldsymbol{\cdot}\Delta_B + \Delta_B\boldsymbol{\cdot}\mathfrak{b}_1^{old}\boldsymbol{\cdot}\mathcal{P}^{(0)}\Big) \tag{9.21}$$

In the rest of the section, we will restrict our attention to the case of odd boundary dimension d, with brief comments about even d towards the end.

Spin-zero

For simplicity, let us practice with the spin $s = 0$ case first, before moving on to the arbitrary spin case. In other words, we turn on bulk fields which are dual to the $s = 0$ operator $J^{(0)}(\vec{x}) =: \phi_m^*\phi^m : (\vec{x})$ in the boundary field theory. To that effect, we take[4]

$$\mathfrak{b}_1^{old}(z;\vec{x},\vec{y}) = \phi(z,\vec{x})\Big(z^d\delta^d(\vec{x}-\vec{y})\Big) \tag{9.22}$$

$$\pi(z,\vec{x}) = \frac{1}{N}\lim_{\vec{x}\to\vec{y}}\mathfrak{p}_1^{old}(z;\vec{x},\vec{y}) = \frac{1}{N}\langle J^{(0)}\rangle_1(z,\vec{x}) \tag{9.23}$$

The above projection onto local fields is consistent only because we are working at the linearized level, where the different spins are decoupled in the bulk (as we will see explicitly below). Note that the operator $J^{(0)}(\vec{x})$

[4]Here the bulk field $\phi(z,\vec{x})$ should not be confused with the elementary scalar $\phi^m(\vec{x})$ of the boundary field theory.

above is "normal ordered" with respect to the free CFT, meaning

$$J^{(0)}(\vec{x}) = \lim_{\vec{y}\to\vec{x}} \Big(\phi^*_m(\vec{x})\phi^m(\vec{y}) - \langle\phi^*_m(\vec{x})\phi^m(\vec{y})\rangle_{CFT}\Big) \tag{9.24}$$

and the subscript $\langle J^{(0)}\rangle_1$ in equation (9.23) stands for linearized order in α. The linearized equations of motion (9.20), (9.21) become

$$z\partial_z\phi(z,\vec{x}) = \Delta_-\ \phi(z,\vec{x}) \tag{9.25}$$

$$z\partial_z\pi(z,\vec{x}) = \Delta_+\pi(z,\vec{x}) - z^{2\nu+1}\int_{\vec{u}} \frac{1}{N}\Big(\mathcal{P}^{(0)}(z;\vec{x},\vec{u})\Delta_B(z;\vec{u},\vec{x}) + \Delta_B(z;\vec{x},\vec{u})\mathcal{P}^{(0)}(z;\vec{u},\vec{x})\Big)\phi(z,\vec{u}) \tag{9.26}$$

where we have defined

$$\Delta_+ = d-2, \qquad \Delta_- = 2, \qquad \Delta_+ - \Delta_- = 2\nu \tag{9.27}$$

To simplify the notation somewhat, we rewrite the above equations in the compact form

$$z\partial_z\phi(z,\vec{x}) \ = \ \Delta_-\ \phi(z,\vec{x}) \tag{9.28}$$

$$z\partial_z\pi(z,\vec{x}) \ = \ \Delta_+\pi(z,\vec{x}) + \frac{z^{2\nu}}{2}\int d^d\vec{u}\ \dot{G}_{(0,0)}(z;\vec{x},\vec{u})\phi(z,\vec{u}) \tag{9.29}$$

where the meaning of $\dot{G}_{(0,0)}$ will become clear shortly. These equations of motion come from the linearized action

$$S^{(2)}_{bulk} = \int_\epsilon^\infty \frac{dzd^d\vec{x}}{z^{d+1}}\ \left(\pi(z,\vec{x})z\partial_z\phi(z,\vec{x}) - \Delta_-\pi(z,\vec{x})\phi(z,\vec{x}) + \int d^d\vec{y}\ \frac{z^{2\nu}}{4}\phi(z,\vec{x})\dot{G}_{(0,0)}(z;\vec{x},\vec{y})\phi(z,\vec{y})\right) \tag{9.30}$$

A convenient way to keep track of the boundary condition on $\phi(z,\vec{x})$ at $z=\epsilon$ is to add the boundary term

$$S_{bdry} = \frac{1}{\epsilon^d}\int d^d\vec{x}\ \pi(\epsilon,\vec{x})\Big(\phi(\epsilon,\vec{x}) - \epsilon^{\Delta_-}\phi^{(0)}(\vec{x})\Big) \tag{9.31}$$

to the action. Our aim now is to show that equations (9.28), (9.29) are completely equivalent to the Fronsdal equation for spin $s=0$.

There are two main obstacles we must confront: (i) The π equation of motion seems non-local, due to the presence of the bilocal kernel $\dot{G}_{(0,0)}$, and it is not clear how to go from our non-local equations of motion to the local Fronsdal equations. (ii) A second confusing property of the above action (and the corresponding Hamiltonian) is the absence of a π^2 term. Naively, this gives the impression of a lack of any interesting dynamics. Another manifestation of this problem is that the field ϕ seems to satisfy an ultra-local first order equation, which is obviously not true of the usual bulk fields in AdS/CFT.

To resolve these issues, we must remember that we're in a phase space formulation – ϕ and π are coordinates on the bulk phase space, with the symplectic structure[5]

$$\boldsymbol{\Omega}(z) = \int \frac{d^d\vec{x}}{z^d}\, \boldsymbol{\delta\phi}(z,\vec{x}) \wedge \boldsymbol{\delta\pi}(z,\vec{x}) \tag{9.32}$$

In the specific symplectic frame coordinatized by ϕ and π, $\phi(z,\vec{x})$ is fixed through (9.28) by its boundary value, and $\pi(z,\vec{x})$ contains all of the information about the renormalized 2-point function of the current. Indeed, it is straightforward to see from equations (9.26) and (9.29) that if we define

$$G_{(0,0)}(z;\vec{x},\vec{y}) = \frac{2i}{N}\left\langle J^{(0)}(\vec{x})J^{(0)}(\vec{y})\right\rangle_{CFT,Mink}(z) \tag{9.33}$$

then

$$\dot{G}_{(0,0)}(z;\vec{x},\vec{y}) = \frac{2i}{N} z\partial_z \left\langle J^{(0)}(\vec{x})J^{(0)}(\vec{y})\right\rangle_{CFT,Mink}(z), \tag{9.34}$$

where the correlator is defined in the *regulated* CFT on Minkowski space, with the cut-off procedure described in section 2 (see appendix C for more details).

An essential feature of the phase space formulation is that we have the freedom to perform *canonical (symplectic) transformations*, which are field redefinitions (i.e., coordinate transformations on phase space) which leave the symplectic 2-form unchanged. Consider for instance, a general linear transformation on phase space

$$\begin{aligned} \phi &= A{\cdot}\varphi + B{\cdot}\varpi \\ \pi &= C{\cdot}\varphi + D{\cdot}\varpi \end{aligned} \tag{9.35}$$

for general bi-local kernels A, B, C, D. The requirement that the symplectic 2-form be preserved, namely

$$\int \frac{d^d\vec{x}}{z^d}\, \boldsymbol{\delta\phi}(z,\vec{x}) \wedge \boldsymbol{\delta\pi}(z,\vec{x}) = \int \frac{d^d\vec{x}}{z^d}\, \boldsymbol{\delta\varphi}(z,\vec{x}) \wedge \boldsymbol{\delta\varpi}(z,\vec{x}) \tag{9.36}$$

leads to the constraints

$$A^T{\cdot}C = C^T{\cdot}A, \quad D^T{\cdot}B = B^T{\cdot}D \tag{9.37}$$

$$A^T{\cdot}D - C^T{\cdot}B = \mathbf{1} \tag{9.38}$$

[5]We use bold symbols $\boldsymbol{\delta\phi}$, $\boldsymbol{\delta\pi}$ etc. to denote differential 1-forms on the phase space.

For simplicity (and because this suffices for our purpose), we will restrict our attention to the case where A, B, C, D are symmetric, and translationally and rotationally invariant. In this case, the constraints (9.37) are automatically satisfied, and we only have to satisfy the constraint (9.38).

To avoid unnecessary complications, we begin by choosing a simpler canonical transformation[6]

$$\begin{aligned} \phi(z,\vec{x}) &= \varphi(z,\vec{x}) + \frac{2\delta}{z^{2\nu}} \int_{\vec{y}} \dot{G}^{-1}_{(0,0)}(z;\vec{x},\vec{y})\varpi(z,\vec{y}) \\ \pi(z,\vec{x}) &= \varpi(z,\vec{x}) \end{aligned} \tag{9.39}$$

for some constant δ to be fixed later. This ansatz clearly satisfies all of the constraints (because $\dot{G}^{-1}_{(0,0)}$ is a symmetric kernel), and is therefore a canonical transformation. We will presently show that for a specific choice of δ, the field φ satisfies the spin-zero AdS_{d+1} Fronsdal equation, up to higher-derivative corrections (i.e., up to $O(z^4\vec{\partial}^4)$ terms). We will later show that these higher derivative terms can in fact be systematically eliminated by a more sophisticated choice of the canonical transformation, but we postpone that discussion to section 9.2.

Substituting equation (9.39) into (9.30), the action in terms of the new fields becomes

$$\begin{aligned} S^{(2)}_{bulk} &= \int \frac{dz}{z} \left(\frac{1}{z^d}\varpi\cdot (z\partial_z\varphi - (\Delta_- - \delta)\varphi) + \frac{2\delta}{z^d}\varpi\cdot z\partial_z \left(z^{-2\nu}\dot{G}^{-1}_{(0,0)}\cdot\varpi\right) \right. \\ &- \left. \frac{(2\Delta_- - \delta)\delta}{z^{d+2\nu}}\varpi\cdot\dot{G}^{-1}_{(0,0)}\cdot\varpi + \frac{1}{4z^{d-2\nu}}\varphi\cdot\dot{G}_{(0,0)}\cdot\varphi \right) \end{aligned} \tag{9.40}$$

where we have switched to the $\cdot$-product notation for convenience. Let us focus on the second term above:

$$\begin{aligned} 2nd\text{ term} &= -4\nu\delta \int \frac{dz}{z}\frac{1}{z^{d+2\nu}}\varpi\cdot\dot{G}^{-1}_{(0,0)}\cdot\varpi + 2\delta\int \frac{dz}{z}\frac{1}{z^{d+2\nu}}\varpi\cdot z\partial_z\left(\dot{G}^{-1}_{(0,0)}\cdot\varpi\right) \\ &= -4\nu\delta \int \frac{dz}{z}\frac{1}{z^{d+2\nu}}\varpi\cdot\dot{G}^{-1}_{(0,0)}\cdot\varpi + 2\delta\int dz\frac{1}{z^{d+2\nu}}\varpi\cdot\left(\partial_z(\dot{G}^{-1}_{(0,0)})\cdot\varpi + \dot{G}^{-1}_{(0,0)}\cdot\partial_z\varpi\right) \\ &= \delta(d-2\nu)\int \frac{dz}{z}\frac{1}{z^{d+2\nu}}\varpi\cdot\dot{G}^{-1}_{(0,0)}\cdot\varpi + \delta\int \frac{dz}{z}\frac{1}{z^{d+2\nu}}\varpi\cdot z\partial_z(\dot{G}^{-1}_{(0,0)})\cdot\varpi \\ &- \frac{\delta}{\epsilon^{d+2\nu}}\,\varpi\cdot\dot{G}^{-1}_{(0,0)}\cdot\varpi\Big|_{z=\epsilon} \end{aligned} \tag{9.41}$$

where in the last line we have integrated by parts with respect to z. Putting everything together, we get the

[6] A similar transformation also appeared in [58], although higher-derivative corrections were not under control in that case. We also note that in the quantum RG formulation of [58], canonical transformations are simply changes of integration variables in the bulk path-integral, which leave the measure invariant.

bulk action

$$S^{(2)}_{bulk} = \int \frac{dz}{z^{d+1}} \left(\varpi{\cdot}z\partial_z\varphi - (\Delta_- - \delta)\varpi{\cdot}\varphi + \frac{1}{z^{2\nu}}\varpi{\cdot}\left[\delta^2 \dot{G}^{-1}_{(0,0)} + \delta z \partial_z(\dot{G}^{-1}_{(0,0)})\right]{\cdot}\varpi + \frac{1}{4z^{-2\nu}}\varphi{\cdot}\dot{G}_{(0,0)}{\cdot}\varphi \right) \quad (9.42)$$

Evidently, the new action has a ϖ^2 term in it, as opposed to the previous version. Of course, the integration by parts we have performed above also produces a new boundary term

$$\delta S_{bdry} = -\frac{\delta}{z^{d+2\nu}}\ \varpi{\cdot}\dot{G}^{-1}_{(0,0)}{\cdot}\varpi\Big|_{z=\epsilon} \quad (9.43)$$

This boundary term has a clear interpretation from the bulk point of view – it is the *generating function* for the canonical transformation. From the boundary point of view, it appears to be a multi-trace deformation. We will return to the boundary terms shortly.

In order to proceed, we need to examine the various bi-local kernels appearing in the above equations. The kernel $\dot{G}_{(0,0)}$ admits an asymptotic expansion of the form (see Appendix C)

$$\dot{G}_{(0,0)}(z;\vec{x},\vec{y}) = -\frac{C}{z^{2\nu}}\left(1 + \alpha z^2 \Box_{(\vec{x})} + \cdots\right)\delta^d(\vec{x}-\vec{y}) \quad (9.44)$$

$$\dot{G}^{-1}_{(0,0)}(z;\vec{x},\vec{y}) = -\frac{z^{2\nu}}{C}\left(1 - \alpha z^2 \Box_{(\vec{x})} + \cdots\right)\delta^d(\vec{x}-\vec{y}) \quad (9.45)$$

where $\alpha > 0$ and C are (dimensionful) constants, which are evaluated in the Appendix. While the numerical values of these constants are irrelevant, the positivity of α is important in the present discussion for the bulk metric to have the correct signature. The ellipsis above indicate higher-derivative terms, which we will address in Section 9.2, because presently our aim is to obtain a two-derivative action. An intuitive way to understand the above expansions is as follows: in any CFT, the two point function of a given operator is universally determined by conformal invariance. Ambiguities which arise upon introducing a regulator come in the form of *local counterterms* – equations (9.44) and (9.45) parametrize precisely such counterterms.

The ϖ^2 term in the action simplifies to

$$\frac{1}{C}\int \frac{dz d^d\vec{x}}{z^{d+1}}\varpi(z,\vec{x})\Big(-\delta(\delta+2\nu) + \alpha\delta(\delta+2\nu+2)z^2\Box_{(\vec{x})} + \cdots\Big)\varpi(z,\vec{x}) \quad (9.46)$$

where again the ellipsis indicates higher-derivative terms. To see that the action (9.42) gives rise to the spin-zero AdS_{d+1}-Fronsdal equation, we write down the equations of motion:

$$
\begin{aligned}
z\partial_z\varphi - (\Delta_- - \delta)\varphi &= \frac{2\delta}{C}\Big((2\nu+\delta)\varpi - \alpha(2\nu+\delta+2)z^2\Box_{\vec{x}}\varpi + ...\Big) && (9.47)\\
-z\partial_z\varpi + (\Delta_+ + \delta)\varpi &= \frac{C}{2}\Big(\varphi + \alpha z^2\Box_{(\vec{x})}\varphi + \cdots\Big) && (9.48)
\end{aligned}
$$

Combining these two equations into a second order differential equation, we get (up to $O(z^4\vec{\partial}^4)$ terms)

$$
z\partial_z(z\partial_z\varphi) - dz\partial_z\varphi + \Delta_-\Delta_+\varphi - 2\alpha\delta z^2\Box_{(\vec{x})}\varphi = -\frac{4\alpha\delta(2\nu+\delta+2)}{C}z^2\Box_{\vec{x}}\varpi + \cdots \tag{9.49}
$$

We see that the right-hand side of (9.49) can be removed (and thus is of order $z^4\vec{\partial}^4$) with the choice $\delta = -(2\nu+2)$. (Equivalently, the second term on the right hand side of (9.47) drops out with this choice of δ.) Further, by rescaling the $\vec{x}$ coordinates, we can set the coefficient $-2\alpha\delta = 2\alpha(2\nu+2) > 0$ of the $\Box_{(\vec{x})}$ term to one. We thus recognize the above equation as the Fronsdal equation for spin $s = 0$

$$
z\partial_z(z\partial_z\varphi) - dz\partial_z\varphi + \Delta_-\Delta_+\varphi + z^2\Box_{(\vec{x})}\varphi = 0 \tag{9.50}
$$

up to higher order corrections. As expected, the scalar mass is given by

$$
(mL)^2 = -\Delta_-\Delta_+
$$

Note that the particular value for δ is picked out by the requirement that the spurious term on the right hand side of equation (9.49) cancels out. Since δ was the parameter in the symplectic transformation (9.39), we see here the first indication that a symplectic transformation is capable of removing spurious higher order terms, and we will see shortly that this can be done systematically to all orders.

At the level of the action, we obtain

$$
S^{(2)}_{bulk} = \int \frac{dzd^d\vec{x}}{z^{d+1}}\Big(\varpi z\partial_z\varphi - d\varpi\varphi - \frac{2(2+2\nu)}{C}\varpi^2 - \frac{C}{4}\left(\varphi^2 + \alpha z^2\varphi\Box_{(\vec{x})}\varphi\right) + \cdots\Big) \tag{9.51}
$$

Solving for the ϖ equation of motion, and plugging it back into the action straightforwardly gives the action (once again up to higher derivative terms)[7]

$$
S^{(2)}_{bulk} = k\int \frac{dzd^d\vec{x}}{z^{d+1}}\left(z\partial_z\varphi\, z\partial_z\varphi - z^2\varphi\,\Box_{(\vec{x})}\varphi + (mL)^2\varphi\varphi\right) + \cdots \tag{9.52}
$$

[7] We also generate an extra boundary term which can be removed by a boundary counterterm, as a part of holographic renormalization.

where k is some dimensionful constant.

Having established the bulk action and equations of motion, now let us turn our attention to the boundary terms. Combining equations (9.31) and (9.43), we find that the boundary action is given by

$$S_{bdry} = \frac{1}{\epsilon^d} \int d^d\vec{x}\; \varpi(\epsilon, \vec{x}) \left(\varphi(\epsilon, \vec{x}) - \epsilon^{\Delta_-} \phi^{(0)}(\vec{x})\right) - \frac{\delta}{C\epsilon^d} \int d^d\vec{x}\; \varpi(\epsilon, \vec{x}) \left(1 + O(\epsilon^2)\right) \varpi(\epsilon, \vec{x}) \tag{9.53}$$

This gives rise to the boundary condition

$$\varphi - \frac{2\delta}{C}\varpi = \epsilon^{\Delta_-} \phi^{(0)} \left(1 + O(\epsilon^2)\right) \tag{9.54}$$

which upon using the ϖ equation of motion gives

$$(z\partial_z - \Delta_+)\varphi = 2\epsilon^{\Delta_-} \phi^{(0)} \left(1 + O(\epsilon^2)\right) \tag{9.55}$$

As usual, as a consequence of the equation of motion (9.49), φ behaves asymptotically as

$$\varphi(z, \vec{x}) = z^{\Delta_+} \varphi^{(+)}(\vec{x}) \left(1 + O(z^2)\right) + z^{\Delta_-} \varphi^{(-)}(\vec{x}) \left(1 + O(z^2)\right) \tag{9.56}$$

and the above boundary condition then becomes

$$\varphi^{(-)}(\vec{x}) = -\frac{1}{\nu} \phi^{(0)}(\vec{x}) \tag{9.57}$$

which is the appropriate boundary condition up to a trivial rescaling. For instance in $d = 2 + 1$, we have thus correctly found that the bulk field comes with the "alternate quantization" as expected.

Having warmed up with the spin-zero case, we now generalize the discussion at two levels − in the next section, we repeat the above exercise for general spin, which will allow us to reproduce the spin-s Fronsdal equation in AdS_{d+1} (once again up to $O(z^4\vec{\partial}^4)$ corrections). Then in section 9.2, we revisit the higher derivative corrections we have been neglecting, and show how to eliminate them systematically. This will complete our argument that the bulk equations obtained from RG are canonically equivalent to AdS Fronsdal equations.

Higher spins

Moving onto the higher-spin case, we now want to recover the Fronsdal equation for arbitrary spin. As we will show, the computation proceeds in essentially the same way as the $s = 0$ case. Going back to the RG equations (9.20) and (9.21), we now wish to turn on bulk fields which are related to the conserved, symmetric and traceless spin-s current in the boundary field theory schematically denoted

$$J^{(s)}_{\mu_1\cdots\mu_s}(\vec{x}) = :\phi^*_m\, f_{\mu_1\cdots\mu_s}(\overleftarrow{\partial}, \overrightarrow{\partial})\phi^m : (\vec{x})$$

where $f_{\mu_1\cdots\mu_s}(\vec{u},\vec{v})$ is a homogenous, symmetric polynomial of order s in $\vec{u}$ and $\vec{v}$, which is symmetric and traceless in all of its indices. To this end, we choose

$$\mathfrak{b}^{old}_1(z;\vec{x},\vec{y}) = z^s\phi_{\mu_1\cdots\mu_s}(z,\vec{x})f^{\mu_1\cdots\mu_s}(\vec{\partial}_{(x)},\vec{\partial}_{(y)})\Big(z^d\delta^d(\vec{x}-\vec{y})\Big) \tag{9.58}$$

$$\pi^{\mu_1\cdots\mu_s}(z,\vec{x}) = \frac{1}{N}\lim_{\vec{x}\to\vec{y}} z^{-s}\, f^{\mu_1\cdots\mu_s}(\vec{\partial}_{(x)},\vec{\partial}_{(y)})\mathfrak{p}^{old}_1(z;\vec{x},\vec{y}) = \frac{1}{N}\langle J^{\mu_1\cdots\mu_s}_{(s)}\rangle_1(z,\vec{x}) \tag{9.59}$$

When the current $J^{\mu_1\cdots\mu_s}_{(s)}$ is conserved in the boundary theory, it is clear that the boundary value $\phi^{(0)}_{\mu_1\cdots\mu_s}$ of the source $\phi_{\mu_1\cdots\mu_s}$ is defined only modulo the gauge transformation

$$\delta\phi^{(0)}_{\mu_1\cdots\mu_s}(\vec{x}) = \vec{\partial}_{(\mu_1}\epsilon^{(0)}_{\mu_2\cdots\mu_s)}(\vec{x}) \tag{9.60}$$

This is of course a manifestation of the $U(L_2)$ gauge symmetry at the linearized level. Furthermore, since $J_{(s)}$ is traceless, only the traceless part of the boundary source is relevant. We can use these considerations to our advantage by making the gauge choice

$$\vec{\partial}^{\mu}\phi^{(0)}_{\mu\mu_2\cdots\mu_s} = 0, \quad \eta^{\mu_1\mu_2}\phi^{(0)}_{\mu_1\cdots\mu_s} = 0 \tag{9.61}$$

For brevity, we introduce the notation $\underline{\mu}_s \equiv \mu_1\cdots\mu_s$. The equations of motion (9.20), (9.21) in the present case are given by

$$\begin{aligned} z\partial_z\phi_{\underline{\mu}_s}(\vec{x}) &= \Delta_-\,\phi_{\underline{\mu}_s}(\vec{x}) &(9.62)\\ z\partial_z\pi^{\underline{\mu}_s}(\vec{x}) &= \Delta_+\pi^{\underline{\mu}_s}(\vec{x}) + \frac{z^{2\nu}}{2}\int d^d\vec{u}\;\dot{G}^{\underline{\mu}_s,\underline{\nu}_s}_{(s,s)}(z,\vec{x},\vec{u})\phi_{\underline{\nu}_s}(\vec{u}) &(9.63)\end{aligned}$$

where

$$\Delta_+ = d - 2 + s, \qquad \Delta_- = 2 - s, \qquad 2\nu = \Delta_+ - \Delta_- = d - 4 + 2s \tag{9.64}$$

The kernel in eq. (9.63) can be identified with

$$\begin{aligned} G^{\underline{\mu}_s,\underline{\nu}_s}_{(s,s)}(z;\vec{x},\vec{y}) &= \frac{2i}{N}\left\langle J^{\mu_1\cdots\mu_s}_{(s)}(\vec{x})J^{\nu_1\cdots\nu_s}_{(s)}(\vec{y})\right\rangle_{CFT,Mink}(z) & (9.65)\\ \dot{G}^{\underline{\mu}_s,\underline{\nu}_s}_{(s,s)}(z;\vec{x},\vec{y}) &= z\partial_z G^{\underline{\mu}_s,\underline{\nu}_s}_{(s,s)}(z;\vec{x},\vec{y}) & (9.66)\end{aligned}$$

where the correlator is defined in the regulated CFT on Minkowski space. To avoid cluttering the notation, we will drop the subscript (s,s) on these kernels henceforth.

Remarkably, the equations of motion are compatible with the gauge choice on the boundary, which implies that we can take the bulk fields (or more precisely, on-shell bulk fields) to satisfy the same gauge conditions

$$\vec{\partial}^\mu\phi_{\mu\mu_2\cdots\mu_s} = 0 = \vec{\partial}_\mu\pi^{\mu\mu_2\cdots\mu_s}, \quad \eta^{\mu_1\mu_2}\phi_{\mu_1\cdots\mu_s} = 0 = \eta_{\mu_1\mu_2}\pi^{\mu_1\cdots\mu_s} \tag{9.67}$$

This choice of (on-shell) gauge is once again the higher-spin *Coulomb gauge* (described previously) at the level of RG. The above equations of motion come from the action

$$S^{(2)}_{bulk} = \int\frac{dzd^d\vec{x}}{z^{d+1}}\left(\pi^{\underline{\mu}_s}(z,\vec{x})z\partial_z\phi_{\underline{\mu}_s}(z,\vec{x}) - \Delta_-\pi^{\underline{\mu}_s}(z,\vec{x})\phi_{\underline{\mu}_s}(z,\vec{x}) + \frac{z^{2\nu}}{4}\phi_{\underline{\mu}_s}(z,\vec{x})\dot{G}^{\underline{\mu}_s,\underline{\nu}_s}(z;\vec{x},\vec{y})\phi_{\underline{\nu}_s}(z,\vec{y})\right) \tag{9.68}$$

along with the boundary action

$$S_{bdry} = \frac{1}{\epsilon^d}\int d^d\vec{x}\;\pi^{\underline{\mu}_s}(\epsilon,\vec{x})\left(\phi_{\underline{\mu}_s}(\epsilon,\vec{x}) - \epsilon^{\Delta_-}\phi^{(0)}_{\underline{\mu}_s}(\vec{x})\right) \tag{9.69}$$

Let us pause briefly to explain why the higher-spin Coulomb gauge simplifies the analysis significantly. As before, the kernel $\dot{G}^{\underline{\mu}_s,\underline{\nu}_s}$ admits an asymptotic expansion, which in general is complicated because of the index structure. But precisely in this gauge (9.67), we see from the action above that the index structures become irrelevant; the only part of the kernels which survive in the action take the generic form

$$\begin{aligned} \dot{G}^{\underline{\mu}_s,\underline{\nu}_s}(\vec{x},\vec{y}) &= -C_s z^{-2\nu}\left(1 + \alpha_s z^2\Box_{(\vec{x})} + \cdots\right)\eta^{\langle\mu_1\langle\nu_1}\cdots\eta^{\mu_s\rangle\nu_s\rangle}\;\delta^d(\vec{x}-\vec{y}) & (9.70)\\ \dot{G}^{-1}_{\underline{\mu}_s,\underline{\nu}_s}(\vec{x},\vec{y}) &= -\frac{z^{2\nu}}{C_s}\left(1 - \alpha_s z^2\Box_{(\vec{x})} + \cdots\right)\eta_{\langle\mu_1\langle\nu_1}\cdots\eta_{\mu_s\rangle\nu_s\rangle}\;\delta^d(\vec{x}-\vec{y}) & (9.71)\end{aligned}$$

where $\alpha_s > 0$ and C_s are (dimensionful) constants (see Appendix C).[8] The notation $\langle\mu_1 \cdots \mu_s\rangle$ denotes the symmetrized traceless combination, and the ellipsis above indicate higher-derivative terms.

Moving on, we now perform the canonical transformation

$$\begin{aligned}
\phi_{\underline{\mu}_s}(z,\vec{x}) &= \varphi_{\underline{\mu}_s}(z,\vec{x}) + \frac{2\delta}{z^{2\nu}} \int_{\vec{y}} \dot{G}^{-1}_{\underline{\mu}_s,\underline{\nu}_s}(z;\vec{x},\vec{y}) \varpi^{\underline{\nu}_s}(z,\vec{y}) \\
\pi^{\underline{\mu}_s}(z,\vec{x}) &= \varpi^{\underline{\mu}_s}(z,\vec{x})
\end{aligned} \tag{9.72}$$

for some constant δ to be fixed later. This canonical transformation preserves the higher-spin Coulomb gauge condition $\vec{\partial}^\mu \varphi_{\mu\mu_2\cdots\mu_s} = 0$, $\eta^{\mu_1\mu_2}\varphi_{\mu_1\cdots\mu_s} = 0$, as can be easily checked.

In terms of the new fields, the action becomes

$$\begin{aligned}
S^{(2)}_{bulk} &= \int \frac{dz}{z^{d+1}} \Big(\varpi^{\underline{\mu}_s} \cdot z\partial_z \varphi_{\underline{\mu}_s} - (\Delta_- - \delta)\varpi^{\underline{\mu}_s} \cdot \varphi_{\underline{\mu}_s} + \frac{1}{z^{2\nu}} \varpi^{\underline{\mu}_s} \cdot \Big[\delta^2 \dot{G}^{-1}_{\underline{\mu}_s,\underline{\nu}_s} + \delta z \partial_z \dot{G}^{-1}_{\underline{\mu}_s,\underline{\nu}_s}\Big] \cdot \varpi^{\underline{\nu}_s} \\
&\quad + \frac{1}{4z^{-2\nu}} \varphi_{\underline{\mu}_s} \cdot \dot{G}^{\underline{\mu}_s,\underline{\nu}_s} \cdot \varphi_{\underline{\nu}_s} \Big) \qquad (9.73) \\
S_{bdry} &= \frac{1}{\epsilon^d} \int d^d\vec{x}\, \varpi^{\underline{\mu}_s}(x) \left(\varphi_{\underline{\mu}_s}(\vec{x}) - \epsilon^{\Delta_-} \phi^{(0)}_{\underline{\mu}_s}(\vec{x}) \right) - \frac{\delta}{\epsilon^{d+2\nu}}\, \varpi^{\underline{\mu}_s} \cdot \dot{G}^{-1}_{\underline{\mu}_s,\underline{\nu}_s} \cdot \varpi^{\underline{\nu}_s} \Big|_{z=\epsilon} \qquad (9.74)
\end{aligned}$$

Substituting equations (9.70) and (9.71) into the above action, we find that the ϖ^2 term in the action becomes

$$\frac{1}{C_s} \int \frac{dz d^d\vec{x}}{z^{d+1}} \varpi^{\underline{\mu}_s}(z,\vec{x}) \Big(-\delta(\delta + 2\nu) + \alpha_s \delta(\delta + 2\nu + 2) z^2 \Box_{(\vec{x})} + \cdots \Big) \varpi_{\underline{\mu}_s}(z,\vec{x}) \tag{9.75}$$

As in the $s = 0$ case above, choosing $\delta = -(2\nu + 2)$ will ensure that the $\varpi\Box_{(\vec{x})}\varpi$ term drops out, and the full bulk action then becomes

$$\begin{aligned}
S^{(2)}_{bulk} &= \int \frac{dz d^d\vec{x}}{z^{d+1}} \Big(\varpi^{\underline{\mu}_s} z\partial_z \varphi_{\underline{\mu}_s} - (d+s)\, \varpi^{\underline{\mu}_s} \varphi_{\underline{\mu}_s} - \frac{1}{C_s} 2(\Delta_+ + s)\, \varpi^{\underline{\mu}_s} \varpi_{\underline{\mu}_s} \\
&\quad - \frac{C_s}{4} \varphi_{\underline{\mu}_s} \left(1 + \alpha_s z^2 \Box_{(\vec{x})}\right) \varphi^{\underline{\mu}_s} \Big) + \cdots
\end{aligned} \tag{9.76}$$

The equations of motion for this action are now

$$z\partial_z \varphi_{\underline{\mu}_s} - (d+s)\, \varphi_{\underline{\mu}_s} = \frac{2}{C_s} 2(\Delta_+ + s) \varpi_{\underline{\mu}_s} + \cdots \tag{9.77}$$

$$-z\partial_z \varpi^{\underline{\mu}_s} - s\varpi^{\underline{\mu}_s} = \frac{C_s}{2} \left(1 + \alpha_s \frac{z^2}{M^2} \Box_{(\vec{x})}\right) \varphi^{\underline{\mu}_s} + \cdots \tag{9.78}$$

[8]While in the present discussion $\alpha_s > 0$ is required for the bulk metric to have the correct signature, one could imagine having a cut-off function where this condition is not satisfied. The more general argument of the next subsection will show that this condition (namely $\alpha_s > 0$) is not actually necessary – it is merely an artifact of the simple-minded canonical transformation we have chosen here.

Combining these two equations into a second order differential equation, we get (up to higher derivative terms)

$$z\partial_z\left(z\partial_z\varphi_{\underline{\mu}_s}\right) - d\, z\partial_z\varphi_{\underline{\mu}_s} + \ell_s^2 z^2\Box_{(\vec{x})}\varphi_{\underline{\mu}_s} - s(s+d)\varphi_{\underline{\mu}_s} + 2(\Delta_+ + s)\varphi_{\underline{\mu}_s} = 0 \tag{9.79}$$

where $\ell_s^2 = \frac{2(\Delta_+ + s)\alpha_s}{M^2}$ is a positive constant. As before, ℓ_s can be set equal to one, by rescaling the boundary coordinate $\vec{x}$. Finally, in order to put the above equation in the standard Fronsdal form, we redefine

$$\varphi_{\underline{\mu}_s} = z^s\widehat{\varphi}_{\underline{\mu}_s} \tag{9.80}$$

We note that this is not an arbitrary redefinition, but corresponds to going from frame indices to coordinate indices. Having done so, the above equation in terms of $\widehat{\varphi}_{\underline{\mu}_s}$ becomes

$$z\partial_z\left(z\partial_z\widehat{\varphi}_{\underline{\mu}_s}\right) + (2s-d)\, z\partial_z\widehat{\varphi}_{\underline{\mu}_s} + z^2\Box_{(\vec{x})}\widehat{\varphi}_{\underline{\mu}_s} + \left[s(s-d) + \Delta_+\Delta_-\right]\widehat{\varphi}_{\underline{\mu}_s} = 0 \tag{9.81}$$

which is precisely the Fronsdal equation in the higher-spin Coulomb gauge (see equation (9.4)). It is worth pointing out that in the special case $s = 1$ this is the familiar Maxwell's equation in AdS space written in Coulomb gauge, and the Hamiltonian obtained from equation (9.76) can be cast in the form $\vec{E}^2 + \vec{B}^2$. Similarly, in the case $s = 2$ the above equation is the Einstein's equation linearized about AdS space, in the $s = 2$ Coulomb gauge.

Finally, we revisit the boundary action

$$S_{bdry} = \frac{1}{\epsilon^d}\int d^d\vec{x}\;\varpi^{\underline{\mu}_s}(\epsilon,\vec{x})\left(\varphi_{\underline{\mu}_s}(\epsilon,\vec{x}) - \epsilon^{\Delta_-}\phi^{(0)}_{\underline{\mu}_s}(\vec{x})\right) - \frac{\delta}{C_s\epsilon^d}\int d^d\vec{x}\;\varpi^{\underline{\mu}_s}(\epsilon,\vec{x})\left(1 + O(\epsilon^2)\right)\varpi_{\underline{\mu}_s}(\epsilon,\vec{x}) \tag{9.82}$$

which gives us the boundary condition

$$\varphi_{\underline{\mu}_s} - \frac{2\delta}{C_s}\varpi_{\underline{\mu}_s} = \epsilon^{\Delta_-}\phi^{(0)}_{\underline{\mu}_s}\left(1 + O(\epsilon^2)\right) \tag{9.83}$$

Using $\delta = -2\nu - 2 = -(\Delta_+ + s)$ and the equation of motion (9.77), we get

$$z\partial_z\varphi_{\underline{\mu}_s} - \Delta_+\varphi_{\underline{\mu}_s} = 2\epsilon^{\Delta_-}\phi^{(0)}_{\underline{\mu}_s}\left(1 + O(\epsilon^2)\right) \tag{9.84}$$

Equation (9.79) implies the asymptotics

$$\lim_{z\to 0}\varphi_{\underline{\mu}_s}(z,\vec{x}) \sim \varphi^{(+)}_{\underline{\mu}_s}(\vec{x})z^{\Delta_+}\left(1 + O(z^2)\right) + \varphi^{(-)}_{\underline{\mu}_s}(\vec{x})z^{\Delta_-}\left(1 + O(z^2)\right)$$

Therefore, the boundary condition becomes

$$\varphi^{(-)}_{\underline{\mu}_s} = -\frac{1}{\nu}\phi^{(0)}_{\underline{\mu}_s} \tag{9.85}$$

or equivalently $\widehat{\varphi}_{\underline{\mu}_s} \sim -\frac{z^{2-2s}}{\nu}\phi^{(0)}_{\underline{\mu}_s}$, which is indeed the correct boundary condition up to a trivial rescaling.

Higher order terms

So far we have demonstrated that the linearized bulk equations obtained from RG are canonically equivalent to AdS_{d+1} Fronsdal equations, up to $O(z^4\vec{\partial}^4)$ terms. These higher derivative terms are only an artifact of choosing a simple canonical transformation. Indeed, it is possible to construct a more general canonical transformation such that the higher derivative terms are completely eliminated, as we will now show. For notational simplicity, we revert back to the spin zero case; all the arguments carry through straightforwardly in the general spin case. So consider once again a general linear canonical transformation

$$\phi = A{\cdot}\varphi + B{\cdot}\varpi \tag{9.86}$$

$$\pi = C{\cdot}\varphi + D{\cdot}\varpi \tag{9.87}$$

where we take all the matrices A, B, C, D to be symmetric as well as translationally and rotationally invariant. The requirement that this be a canonical transformation gives us one constraint

$$A{\cdot}D - C{\cdot}B = \mathbf{1} \tag{9.88}$$

where $\mathbf{1}$ of course is the delta function $\delta^d(\vec{x}-\vec{y})$. The original bulk action (9.30) in terms of the new variables is given by

$$\begin{aligned} S^{(2)}_{bulk} &= \int \frac{dz}{z^{d+1}}\Big\{\varpi{\cdot}z\partial_z\varphi - \varpi{\cdot}\left((\dot{C}-\Delta_+ C){\cdot}B - D{\cdot}(\dot{A}-\Delta_- A) - \frac{z^{2\nu}}{2}B{\cdot}\dot{G}{\cdot}A\right){\cdot}\varphi \\ &- \frac{1}{2}\varphi{\cdot}\left(\dot{C}{\cdot}A - C{\cdot}\dot{A} - 2\nu(C{\cdot}A) - \frac{z^{2\nu}}{2}A{\cdot}\dot{G}{\cdot}A\right){\cdot}\varphi \\ &- \frac{1}{2}\varpi{\cdot}\left(\dot{D}{\cdot}B - D{\cdot}\dot{B} - 2\nu(D{\cdot}B) - \frac{z^{2\nu}}{2}B{\cdot}\dot{G}{\cdot}B\right){\cdot}\varpi\Big\} \end{aligned} \tag{9.89}$$

with additional boundary terms coming from the integrations by parts we have performed above

$$\delta S_{bdry} = \frac{1}{2\epsilon^d}\Big(\varphi\boldsymbol{\cdot}(C\boldsymbol{\cdot}A)\boldsymbol{\cdot}\varphi + \varpi\boldsymbol{\cdot}(D\boldsymbol{\cdot}B)\boldsymbol{\cdot}\varpi + 2\varphi\boldsymbol{\cdot}(C\boldsymbol{\cdot}B)\varpi\Big) \tag{9.90}$$

Here $\dot{A} = z\partial_z A$, and recall the definitions

$$\Delta_+ = d-2, \quad \Delta_- = 2, \quad 2\nu = \Delta_+ - \Delta_-$$

relevant to $s = 0$. Remember that our aim here is to map this action on to the Klein-Gordon action in (9.51), with no higher-derivative corrections surviving. So this gives us three more constraints:

$$(\dot{C} - \Delta_+ C)\boldsymbol{\cdot}B - D\boldsymbol{\cdot}(\dot{A} - \Delta_- A) - \frac{z^{2\nu}}{2}B\boldsymbol{\cdot}\dot{G}\boldsymbol{\cdot}A = d\,\mathbf{1} \tag{9.91}$$

$$\dot{C}\boldsymbol{\cdot}A - C\boldsymbol{\cdot}\dot{A} - 2\nu(C\boldsymbol{\cdot}A) - \frac{z^{2\nu}}{2}A\boldsymbol{\cdot}\dot{G}\boldsymbol{\cdot}A = \frac{C}{2}\left(1 + \alpha z^2\Box_{(\vec{x})}\right)\mathbf{1} \tag{9.92}$$

$$\dot{D}\boldsymbol{\cdot}B - D\boldsymbol{\cdot}\dot{B} - 2\nu(D\boldsymbol{\cdot}B) - \frac{z^{2\nu}}{2}B\boldsymbol{\cdot}\dot{G}\boldsymbol{\cdot}B = \frac{4(2+2\nu)}{C}\mathbf{1} \tag{9.93}$$

Together with the symplectic constraint $A\boldsymbol{\cdot}D - C\boldsymbol{\cdot}B = \mathbf{1}$, we now have four constraints and four unknown kernels — so we can try to solve for them order by order in an asymptotic expansion in powers of $z^2\Box_{(\vec{x})}$. Of course, we have already found the solution to these constraints up to second order in derivatives previously, so we might as well retain the previous solution up to two derivatives. We parametrize the higher derivatives as follows:

$$A = \delta^d(\vec{x}-\vec{y}) + \Big(\alpha_2^A\ z^4\Box^2_{(\vec{x})} + \alpha_3^A\ z^6\Box^3_{(\vec{x})} + \cdots\Big)\delta^d(\vec{x}-\vec{y}) \tag{9.94}$$

$$B = -2(2+2\nu)\left(\frac{1}{\alpha_0^G} - \frac{\alpha_1^G}{(\alpha_0^G)^2}z^2\Box_{(\vec{x})}\right)\delta^d(\vec{x}-\vec{y}) + \Big(\alpha_2^B\ z^4\Box^2_{(\vec{x})} + \alpha_3^B\ z^6\Box^3_{(\vec{x})} + \cdots\Big)\delta^d(\vec{x}-\vec{y}) \tag{9.95}$$

$$C = \Big(\alpha_2^C\ z^4\Box^2_{(\vec{x})} + \alpha_3^C\ z^6\Box^3_{(\vec{x})} + \cdots\Big)\delta^d(\vec{x}-\vec{y}) \tag{9.96}$$

$$D = \delta^d(\vec{x}-\vec{y}) + \Big(\alpha_2^D\ z^4\Box^2_{(\vec{x})} + \alpha_3^D\ z^6\Box^3_{(\vec{x})} + \cdots\Big)\delta^d(\vec{x}-\vec{y}) \tag{9.97}$$

where $\boldsymbol{\alpha}^{(i)} = (\alpha_i^A, \alpha_i^B, \alpha_i^C, \alpha_i^D)$ for $i \geq 2$ are coefficients to be determined from the constraints. We have also introduced the convenient notation

$$\dot{G}_{(0,0)}(z;\vec{x},\vec{y}) = z^{-2\nu}\Big(\alpha_0^G + \alpha_1^G\ z^2\Box_{(\vec{x})} + \cdots\Big)\delta^d(\vec{x}-\vec{y}) \tag{9.98}$$

with $\alpha_0^G \neq 0$. Note that we have taken the expansions for A, B, C, D to be polynomial in $z^2\Box_{(\vec{x})}$. While this is correct in odd dimensions, in general one needs to include logarithmic terms in even dimensions. In order to avoid such complications, we have restricted our attention to odd dimensions in this chapter; the same arguments should go through in even dimensions with logarithmic terms properly taken into account.

The game now is to determine the coefficients $\boldsymbol{\alpha}^{(i)}$. Let us describe this process in general. Let's say we have determined the coefficients to the $(r-1)$th order in the above expansion. At the rth order ($r \geq 2$), we now have four variables $\alpha_r^A, \cdots \alpha_r^D$ to determine, from the four constraints (9.88, 9.91–9.93) listed above. Plugging our expansions (9.94–9.98) into the constraints, we get four constraint equations on the coefficients $\boldsymbol{\alpha}^{(r)} = (\alpha_r^A, \alpha_r^B, \alpha_r^C, \alpha_r^D)$:

1. Symplectic contraint:
$$\alpha_r^A + \alpha_r^D + \frac{2(2+2\nu)}{\alpha_0^G}\alpha_r^C = f_1^{(r)} \tag{9.99}$$

2. $\varpi\varphi$ constraint:
$$\frac{2(2+2\nu)}{\alpha_0^G}(2r-\Delta_+)\alpha_r^C + (2r-2-\Delta_+)\alpha_r^A - \Delta_-\alpha_r^D + \frac{\alpha_0^G}{2}\alpha_r^B = f_2^{(r)} \tag{9.100}$$

3. φ^2 constraint:
$$(2r-2\nu)\alpha_r^C - \alpha_0^G\alpha_r^A = f_3^{(r)} \tag{9.101}$$

4. ϖ^2 constraint:
$$-\frac{2(2+2\nu)}{\alpha_0^G}(2r-2\nu)\alpha_r^D - (2r-2\nu-4)\alpha_r^B = f_4^{(r)} \tag{9.102}$$

where on the right hand side we have functions $\boldsymbol{f}^{(r)} = (f_1^{(r)}, \cdots, f_4^{(r)})$ of all the previously determined coefficients and $\{\alpha_j^G\}$, i.e., $\boldsymbol{f}^{(r)} = \boldsymbol{f}^{(r)}(\boldsymbol{\alpha}^{(0)}, \cdots, \boldsymbol{\alpha}^{(r-1)}; \{\alpha_j^G\})$. So the general structure of these equations for any r is given by
$$\boldsymbol{M}^{(r)} \cdot \boldsymbol{\alpha}^{(r)} = \boldsymbol{f}^{(r)} \tag{9.103}$$

where

$$\boldsymbol{M}^{(r)} = \begin{pmatrix} 1 & 0 & \frac{2(2+2\nu)}{\alpha_0^G} & 1 \\ (2r-2-\Delta_+) & \frac{\alpha_0^G}{2} & \frac{2(2+2\nu)}{\alpha_0^G}(2r-\Delta_+) & -\Delta_- \\ -\alpha_0^G & 0 & (2r-2\nu) & 0 \\ 0 & -(2r-2\nu-4) & 0 & -\frac{2(2+2\nu)}{\alpha_0^G}(2r-2\nu) \end{pmatrix} \tag{9.104}$$

and $\boldsymbol{\alpha}^{(r)} = (\alpha_r^A, \alpha_r^B, \alpha_r^C, \alpha_r^D)$, $\boldsymbol{f}^{(r)} = (f_1^{(r)}, \cdots, f_4^{(r)})$ are defined above. The above matrix has the determinant

$$\det \boldsymbol{M}^{(r)} = -8r(r-\nu)(r+\nu) \tag{9.105}$$

We see that the determinant is non-zero for generic $r > 0$, except at the pathological levels $r = |\nu|$, where the determinant vanishes. However, r is an integer, while for d odd, ν is always half-integral – hence there are no pathologies for any $r > 0$ when d is odd. Consequently, det $\boldsymbol{M}^{(r)} \neq 0$ for any $r > 0$, which means that we can solve equation (9.103) to obtain $\boldsymbol{\alpha}^{(r)}$. By induction on r, we can thus determine all the coefficients of the kernels A, B, C, D uniquely, and determine the canonical transformation at any desired order in the asymptotic expansion. While we demonstrated this in the case of $s = 0$ above, the same calculation generalizes straightforwardly for general spin with the same conclusion. This completes our proof of the statement that in all odd dimensions, the RG equations are canonically equivalent to the bulk Fronsdal equations.

A few comments are in order: firstly, if we naively carry over all the above expressions to d even, then it might seem that the program fails at $r = |\nu|$. This indicates that the asymptotic form of the expansions for A, B, C, D we have considered above is incomplete for d even – we must also include terms logarithmic in $z^2\vec{\partial}^2$. Having done so, the arguments we have presented above will go through for even dimensions as well, but we will not repeat the details here. Secondly, our discussion does not crucially depend on the choice of the cut-off function $K(s)$ – as long as $\dot{G}$ has an expansion of the form (9.98), all the arguments go through. Of course, the detailed form of the canonical transformation would depend on the choice of the cut-off function. From this point of view, we conclude that the various different choices of cut-off functions in the boundary correspond to different choices of a canonical-frame in the bulk. Finally, we note that although we have shown the existence of the canonical transformation to all orders in the expansion in powers of $z^2\vec{\partial}^2$, these expansions are still somewhat formal, i.e., we do not have any handle on the convergence of the series we have found for A, B, C, D.

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Chapter 10

Multi-trace Interactions

In our discussion so far, we have focussed on the holographic dual of the free fixed point deformed by single-trace operators. In this chapter, we will include multi-trace deformations on the CFT side. In the large N limit, we will show that the effect of multi-trace deformations simply amounts to a change in the boundary conditions. Since this discussion is easiest to formulate in the fermionic vector model, we will take this opportunity to first recall our construction in the fermion case.

10.1 Fermionic $U(N)$ vector model

To be concrete, we will work with the free fermionic $U(N)$ vector model on $d = 2+1$ Minkowski spacetime, but we expect our discussion to generalize to higher dimensions (and to the Bosonic case as well). The free Dirac theory has a path-integral definition, in terms of elementary Dirac fields $\psi^m(\vec{x}), \ \ \vec{x} \in \mathbb{R}^{1,2}$

$$S_{Dirac} = \int d^3\vec{x}\ \bar{\psi}_m(\vec{x})\gamma^\mu \vec{\partial}_\mu \psi^m(\vec{x}) \tag{10.1}$$

All the single-trace operators in this theory can be conveniently packaged into the two bi-local operators

$$\mathcal{O}(\vec{x},\vec{y}) = \bar{\psi}_m(\vec{x})\psi^m(\vec{y}), \qquad \mathcal{O}^\mu(\vec{x},\vec{y}) = \bar{\psi}_m(\vec{x})\gamma^\mu\psi^m(\vec{y}) \tag{10.2}$$

Corresponding to these operators, we may introduce in the action, the following source terms:

$$V_{free}[\mathcal{O},\mathcal{O}^\mu] = U + \int_{\vec{x},\vec{y}} \Big(B(\vec{x},\vec{y})\mathcal{O}(\vec{y},\vec{x}) + W_\mu(\vec{x},\vec{y})\mathcal{O}^\mu(\vec{y},\vec{x}) \Big) \tag{10.3}$$

Here we have also introduced a field U which sources the identity operator to keep track of the overall normalization. We note that any ambiguity (such as normal ordering) in the definition of the operators

$\mathcal{O}(\vec{x}, \vec{y}), \mathcal{O}^\mu(\vec{x}, \vec{y})$ that are being sourced by A and W_μ can be absorbed into U. We write the corresponding (unregulated) generating function (or partition function) as

$$Z_{free}[B, W] = e^{-S_{free}[B, W_\mu]} = \int [D\psi D\bar{\psi}]\; e^{-S_{Dirac} - V_{free}[\mathcal{O}, \mathcal{O}^\mu]} \tag{10.4}$$

where S_{free} is the generating function for connected correlators. The subscript $free$ is meant to distinguish this generating function from the one involving multi-trace deformations (to be introduced shortly). Note that the source W_μ combines with $\vec{\partial}_\mu$ in the kinetic term to form the "covariant" derivative

$$D_\mu(\vec{x}, \vec{y}) = \vec{\partial}_\mu^{(x)} \delta^3(\vec{x} - \vec{y}) + W_\mu(\vec{x}, \vec{y}) \tag{10.5}$$

The term covariant here is again used in the context of $U(L_2)$,which we will discuss shortly. Given the bi-local nature of our sources and operators, it is convenient to introduce the following "$\bullet$-product" (following the Bosonic version)

$$(f \bullet g)(\vec{x}, \vec{y}) = \int d^3\vec{u}\; f(\vec{x}, \vec{u}) g(\vec{u}, \vec{y}) \tag{10.6}$$

and the trace

$$\mathrm{Tr}\;(f) = \int d^3\vec{x}\; f(\vec{x}, \vec{x}) \tag{10.7}$$

Background symmetries

As the reader may have anticipated by now, the free Dirac theory has a $U(L_2)$ background symmetry [38, 39]

$$\psi'^m(\vec{x}) = \int_{\vec{y}} \mathcal{L}(\vec{x}, \vec{y}) \psi^m(\vec{y}), \qquad \mathcal{L}^\dagger \bullet \mathcal{L}(\vec{x}, \vec{y}) = \delta^d(\vec{x} - \vec{y}) \tag{10.8}$$

under which the path-integral measure is formally invariant, and the sources W_μ and B transform like a connection and an adjoint-valued field respectively:

$$W'_\mu = \mathcal{L}^{-1} \bullet W_\mu \bullet \mathcal{L} + \mathcal{L}^{-1} \bullet [\partial_\mu, \mathcal{L}]_\bullet\,, \qquad B' = \mathcal{L}^{-1} \bullet B \bullet \mathcal{L} \tag{10.9}$$

Given these transformation properties, the derivative operator D_μ introduced in equation (10.5) transforms covariantly under $U(L_2)$. The above discussion can be naturally formulated in terms of the geometry of infinite jet bundles (see Appendix C for details), but we will not have the need for this formalism in the present chapter.

An important consequence of the above symmetry is that the free fixed point can be reached by setting $B = 0$ and $W_\mu = W_\mu^{(0)}$, with $W_\mu^{(0)}$ any flat connection

$$dW^{(0)} + W^{(0)} \wedge W^{(0)} = 0 \tag{10.10}$$

where $d = dx^\mu \left[\partial_\mu, \cdot\right]_.$ is the exterior derivative. For this reason, we will henceforth pull out a flat piece from the full source W and write it as (see also equations (10.13) and (10.14) below)

$$W = W^{(0)} + \widehat{W} \tag{10.11}$$

Indeed, it is $\widehat{W}$ and B which parametrize arbitrary single-trace, *tensorial* deformations away from the free-fixed point, and thus single-trace RG flows away from the free CFT.

In addition to $U(L_2)$, we also have a background Weyl symmetry, which together with $U(L_2)$ we refer to as $CU(L_2)$. In order to make this explicit, we introduce a conformal factor z in the background metric $\eta_{\mu\nu} \mapsto z^{-2}\eta_{\mu\nu}$, and redefine the sources by rescaling them:

$$B_{old} = z^{d+1} B_{new}, \qquad W_{old} = z^d W_{new} \tag{10.12}$$

where of course $d = 3$. For simplicity, we will drop the subscripts *new* presently, and resurrect them when required. With these changes, the kinetic term and the source terms take the form

$$S_{Dirac}[\psi] = \frac{1}{z^{d-1}} \int_{\vec{x},\vec{u},\vec{y}} \bar{\psi}_m(\vec{x})\gamma^\mu D_\mu^{(0)}(\vec{x},\vec{y})\psi^m(\vec{y}) \tag{10.13}$$

$$V_{free}[\mathcal{O},\mathcal{O}^\mu] = \frac{1}{z^{d-1}} \int_{\vec{x},\vec{y}} B(\vec{x},\vec{y})\mathcal{O}(\vec{y},\vec{x}) + \frac{1}{z^{d-1}} \int_{\vec{x},\vec{y}} \widehat{W}_\mu(\vec{x},\vec{y})\mathcal{O}^\mu(\vec{y},\vec{x}) \tag{10.14}$$

where we have now absorbed the flat piece $W^{(0)}$ of the full connection into the kinetic term, by defining

$$D_\mu^{(0)} = \vec{\partial}_\mu^{(x)} \delta^d(\vec{x}-\vec{y}) + W_\mu^{(0)}(\vec{x},\vec{y})$$

It is clear now that the action is invariant under

$$\psi^m(\vec{x}) \mapsto \lambda^{\frac{d-1}{2}}\ \psi^m(\vec{x}), \qquad z \to \lambda\, z \tag{10.15}$$

where λ is a constant (i.e. spacetime independent) scale factor. More generally, we could consider arbitrary

Weyl transformations by making λ $\vec{x}$-dependent, but we will not do so here. As before, we note that $\mu \propto 1/z$ will end up being the effective "renormalization scale".

Renormalization group & holography

Of course, equation (10.4) needs regularization in order to render the path integral convergent. Following our discussion in the Bosonic case, we will once again regulate the kinetic term in the action by introducing a smooth cutoff function $K(s)$ which has the property that $K(s) \to 1$ for $s < 1$ and $K(s) \to 0$ for $s > 1$. We thus write the new regulated kinetic term as

$$S_{Dirac}^{reg.} = -\frac{1}{z^{d-1}} \int_{\vec{x},\vec{y}} \bar{\psi}_m(\vec{x}) K^{-1}\left(-z^2 D_{(0)}^2/M^2\right) \gamma^\mu D_\mu^{(0)}(\vec{x},\vec{y}) \psi^m(\vec{y}) \tag{10.16}$$

where $D_{(0)}^2 = \eta^{\mu\nu} D_\mu^{(0)} \cdot D_\nu^{(0)}$, and M is an auxiliary cutoff scale. Note that this choice of regulator preserves the $U(L_2)$ symmetry, while ostensibly breaking the dilatation symmetry. The regulated path integral becomes

$$Z_{free}[z; B, \widehat{W}] = e^{-S_{free}[z;B,\widehat{W}]} = \int \left[D\bar{\psi} D\psi\right] \; e^{-S_{Dirac}^{reg.} - V_{free}[\mathcal{O},\mathcal{O}^\mu]} \tag{10.17}$$

It is clear from the path integral that as we tune z from ϵ to ∞, the effective cutoff for the field theory decreases from $\Lambda_{UV} = \frac{M}{\epsilon}$ to zero. What we're interested in computing, of course, is the path integral at $z = \epsilon$, in the limit $\epsilon \to 0$. From the Wilsonian point of view, this process of lowering the cutoff is interpreted as progressively integrating out fast modes. The partition function Z_{free} must therefore remain unchanged under this process, and the effect of integrating out modes can be accounted for by making the source B and $\widehat{W}$ z-dependent. We will label the resulting fields $\mathfrak{B}(z; \vec{x}, \vec{y})$ and $\widehat{\mathcal{W}}(z; \vec{x}, \vec{y})$ respectively, to indicate that they live in the one-higher dimensional bulk space. The boundary values of these fields at $z = \epsilon$ (or in RG terms, the bare values) will be denoted by $b^{(0)}$ and $\widehat{w}_\mu^{(0)}$ respectively. Similarly, the vevs

$$\langle \mathcal{O} \rangle = \frac{\delta S_{free}}{\delta B}, \qquad \langle \mathcal{O}^\mu \rangle = \frac{\delta S_{free}}{\delta W_\mu} \tag{10.18}$$

also evolve into bulk fields which we denote as $\mathcal{P}(z; \vec{x}, \vec{y})$ and $\mathcal{P}^\mu(z; \vec{x}, \vec{y})$ respectively. In fact, $(\mathfrak{B}, \mathcal{P})$, and $(\widehat{\mathcal{W}}_\mu, \mathcal{P}^\mu)$ form canonically conjugate coordinates on the bulk phase space. Finally, along the RG trajectory, we also have the freedom to perform arbitrary $U(L_2)$ gauge transformations, and as a result, the connection $W^{(0)}$ also evolves into a flat connection in the bulk, which we label $\mathcal{W}^{(0)}$ (the z-component of which keeps track of the gauge transformations along RG). The RG evolution equations for the above fields are most conveniently obtained using Polchinki's formulation of the exact renormalization group [60]. These are most compactly written by introducing the fields

$$\boldsymbol{A} = \mathfrak{B} + \gamma^{\mu}\widehat{\mathcal{W}}_{\mu}, \qquad \boldsymbol{P} = \mathcal{P} + \gamma_{\mu}\mathcal{P}^{\mu} \tag{10.19}$$

In these terms, the renormalization group equations are given by (the details are identical to the Bosonic case, which can be found in Appendix C)

$$\mathcal{F}^{(0)} = \boldsymbol{d}\mathcal{W}^{(0)} + \mathcal{W}^{(0)} \wedge_{\cdot} \mathcal{W}^{(0)} = 0 \tag{10.20}$$

$$\mathcal{D}_z^{(0)}\boldsymbol{A} = \partial_z\boldsymbol{A} + \Big[\mathcal{W}_z^{(0)}, \boldsymbol{A}\Big]_{\cdot} = \boldsymbol{A}\boldsymbol{\cdot}\Delta_F\boldsymbol{\cdot}\boldsymbol{A} \tag{10.21}$$

$$\mathcal{D}_z^{(0)}\boldsymbol{P} = \partial_z\boldsymbol{P} + \Big[\mathcal{W}_z^{(0)}, \boldsymbol{P}\Big]_{\cdot} = iN\Delta_F - \boldsymbol{P}\boldsymbol{\cdot}\boldsymbol{A}\boldsymbol{\cdot}\Delta_F - \Delta_F\boldsymbol{\cdot}\boldsymbol{A}\boldsymbol{\cdot}\boldsymbol{P} \tag{10.22}$$

where $\boldsymbol{d} = dz\partial_z + d\vec{x}^{\mu}\left[\partial_{\mu}, \cdot\right]_{\cdot}$ is the bulk exterior derivative, and we have used the convenient notation

$$\Delta_F = -\frac{2z}{M^2}\gamma^{\mu}D_{\mu}^{(0)}\boldsymbol{\cdot}\dot{K}\Big(z^2 D_{(0)}^2/M^2\Big) \tag{10.23}$$

with $\dot{K}(s) = \partial_s K(s)$. Most importantly, the above equations are in fact the Hamilton's equations of motion for the bulk Hamiltonian

$$\mathcal{H}_{free}[\boldsymbol{A}, \boldsymbol{P}] = \mathrm{Tr}\ \boldsymbol{P}\boldsymbol{\cdot}\Big(\Big[\boldsymbol{A}, \mathcal{W}_z^{(0)}\Big]_{\cdot} + \boldsymbol{A}\boldsymbol{\cdot}\Delta_F\boldsymbol{\cdot}\boldsymbol{A}\Big) - N\mathrm{Tr}\ \Delta_F\boldsymbol{\cdot}\boldsymbol{A} \tag{10.24}$$

which in fact, satisfies the Hamilton-Jacobi relation

$$\mathcal{H}_{free} = -\frac{\partial S_{free}}{\partial z} \tag{10.25}$$

Note that the trace Tr now includes trace over the Clifford indices, and an additional factor of half for convenience. This is the central observation which leads to a holographic interpretation of the renormalization group equations. We can use the above Hamiltonian, to construct a bulk "action"

$$S_{bulk} = \int_{\infty}^{\epsilon} dz\ \mathrm{Tr}\ \Big(\boldsymbol{P}\boldsymbol{\cdot}\Big(\mathcal{D}_z^{(0)}\boldsymbol{A} - \boldsymbol{A}\boldsymbol{\cdot}\Delta_F\boldsymbol{\cdot}\boldsymbol{A}\Big) + N\Delta_F\boldsymbol{\cdot}\boldsymbol{A}\Big) \tag{10.26}$$

The boundary conditions

$$\boldsymbol{A}(\epsilon; \vec{x}, \vec{y}) = b^{(0)}(\vec{x}, \vec{y}) + \gamma^{\mu}\widehat{w}_{\mu}^{(0)}(\vec{x}, \vec{y}), \tag{10.27}$$

can be implemented by adding to the bulk action, boundary terms at $z = \epsilon$:

$$S_{bdry} = \Big(\text{Tr}\,(\boldsymbol{P}\boldsymbol{\cdot}\boldsymbol{A}) - V_{free}\Big)_{z=\epsilon} = \text{Tr}\,\Big(\boldsymbol{P}\boldsymbol{\cdot}\Big(\boldsymbol{A} - b^{(0)} - \gamma^\mu \widehat{w}_\mu^{(0)}\Big)\Big)_{z=\epsilon} \tag{10.28}$$

Solving the bulk equations of motion with respect to the boundary conditions (10.27) and

$$\lim_{z\to\infty} \boldsymbol{P}(z;\vec{x},\vec{y}) = 0 \tag{10.29}$$

one obtains the bulk action on-shell. Once again, it turns out that the on-shell action organizes itself in terms of a Witten-diagram expansion, and indeed, precisely reproduces the generating function of connected correlators in the boundary field theory

$$S_{bulk,o.s}[\epsilon; b^{(0)}, \widehat{w}_\mu^{(0)}] = S_{free}[\epsilon; b^{(0)}, \widehat{w}_\mu^{(0)}] \tag{10.30}$$

which is precisely the statement of holographic duality. The detailed derivation for these statements in the Bosonic case has already appeared in previous chapters, so we do not repeat it here. Note that S_{bdry} drops out of the above equation, because it is zero on-shell; this will change however in the interacting version.

This concludes our introduction to the holographic dual of the free, Dirac $U(N)$ vector model. Our main aim in the following sections is to deduce the holographic dual to the above theory with generic multi-trace deformations turned on. In particular, this will allow us to understand the holography of the critical vector model.

10.2 Multi-trace Interactions in Vector Models

In the previous section, we considered the holographic representation of the generating functional of free vector models with single-trace operators turned on. Vector models perturbed by local double-trace deformations (particularly those involving spin zero or one currents) are well understood at large N, and are known to possess non-trivial critical points. Here, we are interested in the more general case of vector models with arbitrary multi-trace interactions, i.e. interaction terms constructed as products of the single-trace, bi-local operators $\mathcal{O}(\vec{x},\vec{y}) = \bar{\psi}_m(\vec{x})\psi^m(\vec{y})$ and $\mathcal{O}^\mu(\vec{x},\vec{y}) = \bar{\psi}_m(\vec{x})\gamma^\mu\psi^m(\vec{y})$:

$$\begin{aligned} V_{int}[\left\{A^{(k)}_{\mu_1\cdots\mu_s}\right\}, \frac{1}{N}\mathcal{O}, \frac{1}{N}\mathcal{O}^\mu] &= \sum_{k=0}^{\infty}\sum_{s=0}^{k} \frac{1}{k!\epsilon^{k(d-1)}}\int_{\{\vec{x}_k,\vec{y}_k\}} A^{(k)}_{\mu_1\cdots\mu_s}(\vec{x}_1,\vec{y}_1;\cdots;\vec{x}_k,\vec{y}_k) \\ &\times \quad \frac{1}{N}\mathcal{O}(\vec{x}_1,\vec{y}_1)\cdots\frac{1}{N}\mathcal{O}(\vec{x}_{k-s},\vec{y}_{k-s})\frac{1}{N}\mathcal{O}^{\mu_1}(\vec{x}_{k-s+1},\vec{y}_{k-s+1})\cdots\frac{1}{N}\mathcal{O}^{\mu_s}(\vec{x}_k,\vec{y}_k) \end{aligned} \tag{10.31}$$

where the powers of N have been appropriately chosen for the large-N limit to exist. Although such deformations go well beyond what is known from field theory analyses, within our formalism there is no particular advantage to be gained by simplifying further. The corresponding path integral, which we denote with the subscript int, is given by

$$Z_{int}[\epsilon;\left\{A^{(k)}_{\mu_1\cdots\mu_s}\right\}] = e^{-S_{int}[\epsilon;\{A^{(k)}_{\mu_1\cdots\mu_s}\}]} = \int[D\psi D\bar{\psi}]e^{-S^{reg.}_{Dirac}-NV_{int}[\{A^{(k)}_{\mu_1\cdots\mu_s}\},\mathcal{O},\mathcal{O}^\mu]} \tag{10.32}$$

Note that we specify the multi-trace interactions above at $z=\epsilon$: from the field theory point of view, this has the interpretation of specifying the bare values of the various couplings. Of course, $A^{(0)}=U$, while $A^{(1)}=b^{(0)}$ and $A^{(1)}_\mu = \widehat{w}^{(0)}_\mu$ source single-trace operators, and the rest are multi-trace sources. The interactions (10.31) might seem like a drastic generalization from free field theory. But from the bulk point of view (as we will see from various different angles), they merely lead, at leading order in $1/N$, to a different choice of boundary conditions, which of course, is known to be the correct picture [76].

In order to proceed, we use the *Hubbard-Stratanovich* trick — we introduce four auxiliary bi-local fields

$$\lambda(\vec{x},\vec{y}), \qquad \rho(\vec{x},\vec{y}), \qquad \lambda_\mu(\vec{x},\vec{y}), \qquad \rho^\mu(\vec{x},\vec{y}) \tag{10.33}$$

and write the above path integral as

$$Z_{int} = \int[D\bar{\psi}D\psi D\lambda D\rho D\lambda_\mu D\rho^\mu]e^{-S^{reg}_{Dirac}-\mathrm{Tr}\ \lambda\cdot(\mathcal{O}-N\rho)-\mathrm{Tr}\ \lambda_\mu\cdot(\mathcal{O}^\mu-N\rho^\mu)-NV_{int}[\{A^{(k)}_{\mu_1\cdots\mu_s}\},\rho,\rho^\mu]} \tag{10.34}$$

where (schematically)

$$V_{int}[\{A^{(k)}_{\mu_1\cdots\mu_s}\},\rho,\rho^\mu] = \sum_{k=0}^{\infty}\sum_{s=0}^{k}\frac{1}{k!\epsilon^{k(d-1)}}\int_{\{x_k,y_k\}} A^{(k)}_{\mu_1\cdots\mu_s}\underbrace{\rho\cdots\rho}_{k-s}\ \rho^{\mu_1}\cdots\rho^{\mu_s} \tag{10.35}$$

Since path integrals over bi-local fields are perhaps not familiar, we pause momentarily to define them here. To do so, we introduce an orthonormal basis of functions $\chi_\alpha(\vec{x})$, a basis for $L_2(\mathbb{R}^{1,d-1})$. Then, any bi-local field $\lambda(\vec{x},\vec{y})$ can be written in terms of these as

$$\lambda(\vec{x},\vec{y}) = \sum_{\alpha,\beta} \lambda_{\alpha\beta}\ \chi_\alpha(\vec{x})\chi^*_\beta(\vec{y})$$

where $\lambda_{\alpha\beta}$ are $\mathbb{C}$-valued. Thus, we can regard the path integral over $\lambda(\vec{x},\vec{y})$ to be defined as

$$\int [D\lambda] = \prod_{\alpha,\beta}\int_{\mathbb{C}} d\mathrm{Re}\ \lambda_{\alpha\beta}\ d\mathrm{Im}\ \lambda_{\alpha\beta}.$$

With this definition, it is a straightforward exercise to check the validity of equation (10.34). Moving on, we can now perform the $\psi,\bar{\psi}$ integrations in (10.34) and rewrite this path integral in terms of the generating functional for the original free CFT (defined in equation (10.17))

$$Z_{int}[\epsilon,\{A^{(k)}_{\mu_1\cdots\mu_s}\}] = \int [D\lambda D\rho D\lambda_\mu D\rho^\mu]\ e^{-S_{free}[\epsilon;\lambda,\lambda_\mu]+N\mathrm{Tr}\ \lambda\cdot\rho+N\mathrm{Tr}\ \lambda_\mu\cdot\rho^\mu-NV_{int}[\{A^{(k)}_{\mu_1\cdots\mu_s}\},\rho,\rho^\mu]}. \qquad (10.36)$$

We will now try to deduce and analyze the holographic representation of Z_{int} given in the above equation. We will do this in two steps – we first deal with the case $N\to\infty$, where the answer is easiest to state. Then in step two we connect this with the Wilsonian RG flow for the multi-trace couplings $\{A^{(k)}_{\mu_1\cdots\mu_s}\}$ (at $N=\infty$).

1. Saddle point evaluation at $N=\infty$: Since all the terms in the exponential in the above path-integral are proportional to N, in the $N\to\infty$ limit, the path integral localizes on the solutions to the Euler-Lagrange equations with respect to $\lambda,\lambda_\mu,\rho,\rho^\mu$, the saddle point equations (gap equations):

$$\lambda_* = \frac{\delta V_{int}}{\delta\rho_*}, \qquad \lambda_{*,\mu} = \frac{\delta V_{int}}{\delta\rho^\mu_*} \qquad (10.37)$$

$$\rho_* = \frac{1}{N}\frac{\delta S_{free}}{\delta\lambda_*}, \qquad \rho^\mu_* = \frac{1}{N}\frac{\delta S_{free}}{\delta\lambda_{*,\mu}} \qquad (10.38)$$

Of course, from the bulk point of view, $\lambda_*,\lambda_{*;\mu},\rho_*,\rho^\mu_*$ are boundary values of bulk fields, and so the above saddle point equations lead to a generalized set of boundary conditions. Substitution into (10.36) gives

$$S_{int}[\epsilon;\{A^{(k)}_{\mu_1\cdots\mu_s}\}] = S^{o.s.}_{bulk}[\epsilon;\lambda_*,\lambda_{*,\mu}] - N\mathrm{Tr}\ \lambda_*\cdot\rho_* - N\mathrm{Tr}\ \lambda_{*,\mu}\cdot\rho^\mu_* + NV_{int}[\{A^{(k)}_{\mu_1\cdots\mu_s}\},\rho_*,\rho^\mu_*] \qquad (10.39)$$

where we have used equation (10.30). From equation (10.39), it is obvious that the difference between the holographic dual of the free vector model and the interacting vector model, at large N, boils down to boundary terms. Indeed, all the additional terms in (10.39), as well as the modified boundary conditions, can be accounted for simply, by replacing the boundary action in (10.28) by

$$\widetilde{S}_{bdry} = \Big(\mathrm{Tr}\ \boldsymbol{P}\boldsymbol{\cdot}\boldsymbol{A} - NV_{int}[\{A^{(k)}_{\mu_1\cdots\mu_s}\}, \mathcal{P}, \mathcal{P}^{\mu}]\Big)_{z=\epsilon} \tag{10.40}$$

The role of these generalized boundary terms, is to implement the generalized boundary conditions[1]

$$\boldsymbol{A}(\epsilon) = N\frac{\delta V_{int}}{\delta \boldsymbol{P}(\epsilon)} \tag{10.41}$$

which from the field theory point of view correspond to the saddle point equations (10.37). Indeed, these are precisely the multi-trace boundary conditions described in [76]. Thus, we conclude that *in the $N \to \infty$ limit, the bulk equations of motion and the bulk action dual to the interacting vector model are precisely the same as that of the free vector model, the only difference being that we must use the modified boundary conditions* (10.41).

2. Renormalization group at $N = \infty$: As we lower the cut-off from $z = \epsilon$ to $z = \infty$, the multi-trace sources $\{A^{(k)}_{\mu_1\cdots\mu_s}\}$ flow with z. Indeed, the *Polchinski exact RG equation*

$$\frac{1}{N}z\partial_z V_{int} = \frac{\delta V_{int}}{\delta\bar{\psi}_m}\boldsymbol{\cdot}\Delta_F\boldsymbol{\cdot}\frac{\delta V_{int}}{\delta\psi^m} - \frac{i}{N}\mathrm{Tr}\ \Delta_F\boldsymbol{\cdot}\frac{\delta^2 V_{int}}{\delta\bar{\psi}_m\delta\psi^m} \tag{10.42}$$

in this case gives us an infinite set of flow equations for $\{A^{(k)}_{\mu_1\cdots\mu_s}\}$. This might seem to imply that all the multi-trace sources $A^{(k)}_{\mu_1\cdots\mu_s}$ become dynamical fields in the bulk holographic description. However, as should be clear by now, this is incorrect. A simple way to see this in the $N \to \infty$ limit is to realize that the vevs of all the multi-trace operators factorize into products of vevs of single-trace operators. Consequently, if we regard multi-trace sources to be dynamical, then the purported symplectic structure would be degenerate. So, at least in this limit, the multi-trace sources do not become dynamical in the bulk – it is only the single-trace sources which become dynamical bulk fields.

However, there is a puzzle here: if the multi-trace sources do not become dynamical in the bulk, then is it possible to deduce their renormalization group flow (namely equation (10.42)) from our bulk holographic equations? In particular, there are infinitely many multi-trace sources, while there are only four bulk equations for single-trace sources and vevs. The answer to this puzzle turns out to be in the affirmative, and remarkably simple. If $\boldsymbol{A}(z)$ and $\boldsymbol{P}(z)$ satisfy the bulk equations of motion with boundary conditions

[1] Here we have defined $\frac{\delta}{\delta \boldsymbol{P}} = \frac{\delta}{\delta \mathcal{P}} + \gamma^{\mu}\frac{\delta}{\delta \mathcal{P}^{\mu}}$.

(10.41), then define a one-parameter family of functions $V_{int}(z)$ which satisfies

$$\boldsymbol{A}(z) = N\frac{\delta V_{int}(z)}{\delta \boldsymbol{P}(z)} \tag{10.43}$$

We claim that the running sources $\{A^{(k)}_{\mu_1\cdots\mu_s}(z)\}$ can be read off from $V_{int}(z)$ by simply expanding it in a power series in $\boldsymbol{P}$ and reading off the coefficients, i.e.

$$V_{int}(z) = \sum_{k=0}^{\infty}\sum_{s=0}^{k}\frac{1}{N^k k! z^{k(d-1)}}\int_{\{x_k,y_k\}} A^{(k)}_{\mu_1\cdots\mu_s}(z)\,\mathcal{P}(z)\cdots\mathcal{P}(z)\,\mathcal{P}^{\mu_1}(z)\cdots\mathcal{P}^{\mu_s}(z) \tag{10.44}$$

Let us now show that this is indeed the case. Differentiating equation (10.43) with respect to z, we obtain (we use minimal notation to avoid cluttering up the equations)

$$z\frac{d\boldsymbol{A}}{dz} = N\frac{\delta}{\delta \boldsymbol{P}}z\partial_z V_{int} + Nz\frac{d\boldsymbol{P}}{dz}\,\frac{\delta^2 V_{int}}{\delta \boldsymbol{P}^2} \tag{10.45}$$

Using the Hamiltonian equations of motion, we get

$$\frac{\delta\mathcal{H}_{free}}{\delta \boldsymbol{P}} = N\frac{\delta}{\delta \boldsymbol{P}}z\partial_z V_{int} - N\frac{\delta\mathcal{H}_{free}}{\delta \boldsymbol{A}}\,\frac{\delta^2 V_{int}}{\delta \boldsymbol{P}^2} \tag{10.46}$$

Rearranging the above equation gives the following flow equation for V_{int}:

$$z\partial_z V_{int} = \frac{1}{N}\mathcal{H}_{free}[N\frac{\delta V_{int}}{\delta \boldsymbol{P}}, \boldsymbol{P}] + V_0 \tag{10.47}$$

where the $\boldsymbol{P}$-independent term V_0 is basically the overall normalization of the path integral, and might be ignored. In order to obtain the RG flow of multi-trace couplings, we need only solve this equation. As a check, note that using the explicit form of the Hamiltonian from equation (10.24), we get

$$\frac{1}{N}z\partial_z V_{int} = -\frac{1}{N}\mathrm{Tr}\;\boldsymbol{P}\boldsymbol{\cdot}\left[\mathcal{W}^{(0)}_z, \frac{\delta V_{int}}{\delta \boldsymbol{P}}\right]_{\boldsymbol{\cdot}} + \mathrm{Tr}\;\boldsymbol{P}\boldsymbol{\cdot}\frac{\delta V_{int}}{\delta \boldsymbol{P}}\boldsymbol{\cdot}\Delta_F\boldsymbol{\cdot}\frac{\delta V_{int}}{\delta \boldsymbol{P}} - i\mathrm{Tr}\;\Delta_F\boldsymbol{\cdot}\frac{\delta V_{int}}{\delta \boldsymbol{P}} \tag{10.48}$$

This equation is precisely the covariantized version of the Polchinski exact RG equation (10.42) for the Dirac theory *in the large-N limit.*[2] In this way, the bulk theory reproduces all the infinitely many RG equations for the multi-trace couplings from the bulk equations of motion. Interestingly, although we have derived equation (10.47) in the specific example of the free fermion CFT, the above arguments go through more

[2]This is because, while acting on V_{int}, we have

$$\frac{\delta}{\delta\psi^m(\vec{x})} = \int_{\vec{y}}\bar{\psi}_m(\vec{y})\frac{\delta}{\delta\boldsymbol{P}(\vec{x},\vec{y})}, \qquad \frac{\delta}{\delta\bar{\psi}_m(\vec{x})} = \int_{\vec{y}}\psi^m(\vec{y})\frac{\delta}{\delta\boldsymbol{P}(\vec{y},\vec{x})}$$

generally for any CFT with a holographic dual. Consider a CFT with a set of "single-trace operators", which we collectively label $\mathcal{O}$, and let $\boldsymbol{A}$ and $\boldsymbol{P}$ be the corresponding bulk sources and their conjugate momenta. Let $\mathcal{H}[\boldsymbol{A}, \boldsymbol{P}]$ be the bulk Hamiltonian. Then the Wilsonian effective action for multi-trace sources satisfies the flow equation [56]

$$z\partial_z V_{int} = \mathcal{H}\left[\frac{\delta V_{int}}{\delta \boldsymbol{P}}, \boldsymbol{P}\right]$$

In this context, we might regard the above equation as a *generalized Polchinski equation.*

This concludes our discussion of the exact renormalization group and its interpretation as a dual holographic theory (in the Hamiltonian framework). To summarize, we considered the Bosonic and Fermionic $U(N)$ vector models close to their free fixed points, with single-trace deformations turned on. We formulated these theories by recasting the single-trace deformations in terms of a background $U(L_2)$ connection and an adjoint-valued 0-form, which are closely related to the geometry of infinite jet bundles. The renormalization group equations for these sources were written as $U(L_2)$ covariant, geometric equations of motion. Including the RG flow of the vevs of the corresponding operators, RG was formulated in terms of a Hamiltonian system of equations on a one-higher dimensional *bulk* spacetime. The bulk on-shell action was evaluated explicitly, and shown to reproduce all the correlation functions of the boundary CFT, consistent with the holographic dictionary. Furthermore, the linearized bulk equations of motion were shown to contain the Fronsdal equations of motion on AdS space, thus proving the equivalence of our equations with Vasiliev higher spin equations at the linearized level. Finally, it was argued that turning on multi-trace deformations in the CFT corresponds to a different choice of boundary conditions in the limit $N \to \infty$, which is again consistent with the known AdS/CFT dictionary. Our construction provides a concrete boundary to bulk realization of the AdS/CFT correspondence as a geometrization of the renormalization group, in addition to a new (potentially more accessible) formulation of higher spin theories on AdS.

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Appendix A

Supersymmetric Quantum Mechanics

In this appendix, we present a detailed review of Supersymmetric Quantum Mechanics (SQM) on manifolds with torsion and curvature, and use it to compute asymptotic expansions involving the heat kernel of the Dirac operator on such manifolds.

A.1 Classical $N = 1$ SQM

We start with the Lagrangian

$$L_{min} = \frac{1}{2} g_{ij}(x) \frac{\mathrm{d}x^i}{\mathrm{d}t} \frac{\mathrm{d}x^j}{\mathrm{d}t} + \frac{i}{2} g_{ij}(x) \psi^i \left(\frac{\mathrm{d}\psi^i}{\mathrm{d}t} + \frac{\mathrm{d}x^k}{\mathrm{d}t} \Gamma^j{}_{kl} \psi^l \right) \tag{A.1}$$

where the quantity $\left(\frac{\mathrm{d}\psi^i}{\mathrm{d}t} + \frac{\mathrm{d}x^k}{\mathrm{d}t} \Gamma^j{}_{kl} \psi^l \right)$ is simply the covariant derivative of ψ^j along $\dot{x}^k$, and will henceforth be denoted by $\nabla_t \psi^j$. It is worthwhile noting that the Lagrangian can also be written equivalently as

$$L_{min} = \frac{1}{2} g_{ij}(x) \dot{x}^i \dot{x}^j + \frac{i}{2} \eta_{ab} \psi^a \left(\dot{\psi}^b + \dot{x}^k \omega_k{}^b{}_c \psi^c \right) \tag{A.2}$$

where we have introduced the new variables $\psi^a = e^a_i \psi^i$, and ω is the connection for local Lorentz transformations. The ψ^a's are to be treated as the canonical variables, and therefore the correct canonical momenta are

$$p_i = g_{ij} \dot{x}^j + \frac{i}{2} \eta_{ab} \psi^a \omega_i{}^b{}_c \psi^c, \qquad \pi_a = -\frac{i}{2} \psi_a \tag{A.3}$$

We are interested in constructing a torsionful Lagrangian L invariant under the following supersymmetry transformation:

$$\delta x^i = i\eta\psi^i, \quad \delta\psi^i = -\eta\dot{x}^i \tag{A.4}$$

or in terms of the ψ^a's

$$\delta x^i = i\eta e^i_a \psi^a, \qquad \delta\psi^a = -\eta e^a_i \dot{x}^i - i\eta e^i_c \omega_i{}^a{}_d \psi^c\psi^d + \frac{i}{2}\eta T^a_{bc}\psi^b\psi^c \tag{A.5}$$

$N = 1$ Superspace

We construct the superfield corresponding to (x^i, ψ^i) as $X^i = x^i + \theta\psi^i$. The differential representation of the supersymmetry generator Q on superfields is given by $Q = (-\partial_\theta + i\theta\partial_t)$ with $Q^2 = -i\partial_t$, and $\delta X = -i\eta Q\, X$. It is easy to check that a supersymmetry covariant derivative is given by $D = (\partial_\theta + i\theta\partial_t)$, with $\{Q, D\} = 0$, and $D^2 = i\partial_t$. It is evident from the above constructions, that for any function $f(X) = f_t(x^i, \psi^i) + \theta f_\theta(x^i, \psi^i)$, one can obtain a supersymmetry invariant by constructing

$$L = \int \mathrm{d}\theta f(X), \qquad \delta L = -\eta\partial_t f_t \tag{A.6}$$

For example, if we take $f(X) = -\frac{1}{2}\eta_{ij} DX^i(i\partial_t X^j) = -\frac{1}{2}\eta_{ij} DX^i D^2 X^j$, we get the supersymmetric lagrangian in flat space, $L_{flat} = \frac{1}{2}\dot{x}^i\dot{x}_i + \frac{i}{2}\psi^i\dot{\psi}_i$. More generally, it is easy to see that for

$$f(X) = -\frac{1}{2}g_{ij}(X) DX^i D^2 X^j, \quad L^{(0)} = \frac{1}{2}g_{ij}\dot{x}^i\dot{x}^j + \frac{i}{2}\psi^i\hat{\nabla}_t\psi_j \tag{A.7}$$

which is the supersymmetric Lagrangian in curved space without torsion. Here,$\hat{\nabla}$ stands for Levi-Civita connection. From equation (A.6), we get

$$\delta L^{(0)} = -\eta\partial_t f_t = \frac{i\eta}{2}\frac{\mathrm{d}}{\mathrm{d}t}(g_{ij}\psi^i\dot{x}^j). \tag{A.8}$$

It is also worth noting that if we consider the superfield $Df(X) = f_\theta + i\theta\partial_t f_t$, and construct a similar supersymmetry-invariant out of it, we find that $i\delta f_t = -\eta f_\theta + C$, where C is some time-independent constant. Thus in most cases, we find that supersymmetry closed constructions are also supersymmetry exact. For instance, in the example of curved torsionless spacetime considered above, we conclude that

$$\delta\left(\frac{1}{2}g_{ij}\psi^i\dot{x}^j\right) = -\eta\left(\frac{1}{2}g_{ij}\dot{x}^i\dot{x}^j + \frac{i}{2}\psi^i\hat{\nabla}_t\psi_j\right) \tag{A.9}$$

Torsion terms

Let $C^i{}_{jk}$ be the torsion tensor, while $\Omega_{ijk} = \frac{1}{2}(C_{ijk} - C_{jki} + C_{kij})$ is the contorsion tensor. We now attempt to add torsion terms:

$$\begin{aligned} L^{(1A)} = \frac{1}{2}\Omega_{ijk}(X)DX^iDX^jDX^k|_\theta &= \frac{i}{2}\Omega_{ijk}\left(\dot{x}^i\psi^j\psi^k - \dot{x}^j\psi^i\psi^k + \dot{x}^k\psi^i\psi^j\right) + \frac{1}{2}\partial_m\Omega_{ijk}\psi^m\psi^i\psi^j\psi^k \\ &= \frac{i}{2}\Omega_{ijk}\left(2\dot{x}^i\psi^j\psi^k - \dot{x}^j\psi^i\psi^k\right) + \frac{1}{4}\partial_m C_{ijk}\psi^m\psi^i\psi^j\psi^k \\ &= \frac{i}{2}\Omega_{ijk}\left(2\dot{x}^i\psi^j\psi^k - \dot{x}^j\psi^i\psi^k\right) + \frac{1}{2}N_{mijk}\psi^m\psi^i\psi^j\psi^k \end{aligned} \tag{A.10}$$

where N_{mijk} is defined as $(T^a \wedge T_a - R_{ab} \wedge e^a \wedge e^b) = N_{mijk}\mathrm{d}x^m \wedge \mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k$. In the last line we have used $\partial_m C_{ijk} = g_{il}(\partial_m C^l{}_{jk} + \Gamma^l{}_{mp}C^p{}_{jk}) + \Gamma_{lmi}C^l{}_{jk}$, along with the Bianchi identity for torsion $DT^a = R^a{}_b \wedge e^b$. Using the fact that $\Omega_{ijk}\dot{x}^i\psi^j\psi^k = \Omega_{ijk}\dot{x}^j\psi^i\psi^k + C_{ijk}\dot{x}^k\psi^i\psi^j$ we can write $L^{(1)}$ as

$$L^{(1A)} = \frac{i}{2}\Omega_{ijk}\dot{x}^j\psi^i\psi^k + iC_{ijk}\dot{x}^k\psi^i\psi^j + \frac{1}{2}N_{mijk}\psi^m\psi^i\psi^j\psi^k \tag{A.11}$$

It is curious to note that there is one more way of writing the above action, which is

$$\begin{aligned} L^{(1A)} &= \frac{i}{2}\Omega_{ijk}\left(\dot{x}^i\psi^j\psi^k - \dot{x}^j\psi^i\psi^k + \dot{x}^k\psi^i\psi^j\right) + \frac{1}{2}N_{mijk}\psi^m\psi^i\psi^j\psi^k \\ &= -\frac{i}{2}\Omega_{ijk}\dot{x}^j\psi^i\psi^k + \frac{i}{2}(\Omega_{ijk} - \Omega_{ikj})\dot{x}^i\psi^j\psi^k + \frac{1}{2}N_{mijk}\psi^m\psi^i\psi^j\psi^k \\ &= -\frac{i}{2}\Omega_{ijk}\dot{x}^j\psi^i\psi^k + \frac{i}{2}C_{ijk}\dot{x}^i\psi^j\psi^k + \frac{1}{2}N_{mijk}\psi^m\psi^i\psi^j\psi^k \end{aligned} \tag{A.12}$$

The different ways of writing this term are simply different ways of packaging Ω_{ijk} - the equivalence can be easily checked. The supersymmetry variation is given by

$$\begin{aligned} \delta L^{(1A)} &= -\eta\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\Omega_{ijk}(X)DX^iDX^jDX^k|_t\right) \\ &= -\frac{\eta}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\Omega_{ijk}\psi^i\psi^j\psi^k) \end{aligned} \tag{A.13}$$

We notice that the first term in $L^{(1A)}$ is the minimal coupling term. We will henceforth stick to the second version of $L^{(1A)}$ in equation $A.12$, because the contractions are more natural[1] If we take the full Lagrangian to be $L = L^{(0)} - L^{(1A)}$, we get

$$L = \frac{1}{2}g_{ij}\dot{x}^i\dot{x}^j + \frac{i}{2}\psi^i\nabla_t\psi_j - \frac{i}{2}C_{ijk}\dot{x}^i\psi^j\psi^k - \frac{1}{2}N_{mijk}\psi^m\psi^i\psi^j\psi^k \tag{A.14}$$

[1]In fact, it is this version which gives the correct Dirac operator as we will see in a short while.

where now ∇ stands for the full connection. The supersymmetry variation is given by

$$\delta L = \eta \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{i}{2}g_{ij}\psi^i\dot{x}^j + \frac{1}{2}\Omega_{ijk}\psi^i\psi^j\psi^k\right) \tag{A.15}$$

We also remind ourselves from our discussion of SUSY closed implies SUSY exact, that we have

$$\delta\left(\frac{i}{2}g_{ij}\psi^i\dot{x}^j + \frac{1}{2}\Omega_{ijk}\psi^i\psi^j\psi^k\right) = -i\eta L \tag{A.16}$$

Hamiltonian

We now compute the classical Hamiltonian for this theory. We have the bosonic momentum $p_i = \dot{x}_i + \frac{i}{2}\omega_{i,ab}\psi^{ab} - \frac{i}{2}C_{i,ab}\psi^{ab}$, and the fermionic momentum $\rho_a = -\frac{i}{2}\psi_a$. The Hamiltonian is given by

$$\begin{aligned} H &= g^{ij}p_i\dot{x}_j + \eta^{ab}\dot{\psi}_a\rho_b - L \\ &= \frac{1}{2}g^{ij}\dot{x}_i\dot{x}_j + \frac{1}{2}N_{abcd}\psi^a\psi^b\psi^c\psi^d \\ &= \frac{1}{2}g^{ij}\pi_i\pi_j + \frac{i}{2}\psi^a\psi^b C^k_{ab}\pi_i - \frac{1}{4}R_{ab,cd}\psi^a\psi^b\psi^c\psi^d \end{aligned} \tag{A.17}$$

where $\pi_i = p_i - \frac{i}{2}\omega_{i,ab}\psi^a\psi^b$.

Yang-Mills

In order to add Yang-Mills (internal) gauge degrees to the theory, we introduce new superfields $N^M = \eta^M + \theta D^M$ and $\bar{N}_M = \bar{\eta}_M + \theta\bar{D}_M$, with the superspace action $S_{YM} = -\int \mathrm{d}t\mathrm{d}\theta\ \bar{N}_M(DN^M + iDX^i A^\alpha_i(X)T^{\alpha M}{}_N N^N)$. In component language, we get (we suppress the vector index)

$$S_{YM} = \int \mathrm{d}t\ i\bar{\eta}(\dot{\eta} + i\dot{x}^k A^\alpha_k T^\alpha\eta) - i\bar{\eta}\psi^k A^\alpha_k T^\alpha D + i\bar{\eta}\psi^j\psi^k\partial_j A^\alpha_k T^\alpha\eta - (\bar{D}D + i\bar{D}\psi^i A^\alpha_i T^\alpha\eta) \tag{A.18}$$

We can eliminate the auxiliary field D by using the equations of motion $D = -i\psi^i A^\alpha_i T^\alpha\eta$ and $\bar{D} = -i\bar{\eta}\psi^i A^\alpha_i T^\alpha$ to get

$$S_{YM} = \int \mathrm{d}t\ i\bar{\eta}(\frac{\mathrm{d}}{\mathrm{d}t}\eta + \dot{x}^k A^\alpha_k T^\alpha\eta) + \frac{i}{2}\bar{\eta}\psi^j\psi^k F^\alpha_{jk}T^\alpha\eta \tag{A.19}$$

We can add S_{YM} to the action derived before. Now the full hamiltonian gets modified to

$$H = \frac{1}{2}g^{ij}\pi_i\pi_j + \frac{i}{2}\psi^a\psi^b C^k_{ab}\pi_i - \frac{1}{4}R_{ab,cd}\psi^a\psi^b\psi^c\psi^d - \frac{i}{2}\bar{\eta}\psi^j\psi^k F^\alpha_{jk}T^\alpha\eta \tag{A.20}$$

where now $\pi_i = p_i - \frac{i}{2}\omega_{i,ab}\psi^a\psi^b + \bar{\eta}A^\alpha_i T^\alpha\eta$.

Supercharge

We can calculate the supercharge from here as

$$\begin{aligned} \eta Q &= \delta x^i p_i + \delta\psi^a\pi_a - \left(\frac{i}{2}\eta g_{ij}\psi^j\dot{x}^i + \frac{\eta}{2}\Omega_{ijk}\psi^i\psi^j\psi^k\right) \\ &= i\eta\psi^a e^i_a p_i + \frac{1}{2}\eta e^i_c\psi^c\hat{\omega}_{i,ad}\psi^a\psi^d - \frac{\eta}{2}\Omega_{abc}\psi^a\psi^b\psi^c \end{aligned} \tag{A.21}$$

Upon quantization, we must replace $p_i \to -i\partial_i$, and $\psi^a \to \frac{1}{\sqrt{2}}\gamma^a$. The first two terms give us the Dirac operator with the Levi-Civita connection, while the last term can be simplified as

$$\begin{aligned} \frac{\eta}{2}\Omega_{abc}\gamma^a\gamma^b\gamma^c &= -\frac{\eta}{2}\Omega_{abc}\gamma^b\gamma^a\gamma^c + \eta\Omega_{aac}\gamma^c \\ &= -\frac{\eta}{2}\Omega_{abc}\gamma^b\gamma^{ac} + \eta\Omega_{aac}\gamma^c \\ &= -\frac{\eta}{2}\Omega_{abc}\gamma^{bac} \end{aligned}$$

Thus the supercharge upon quantization becomes

$$\eta Q = \frac{\eta}{\sqrt{2}}\left(\not{D}_{\mathrm{LC}} + \frac{1}{4}\Omega_{abc}\gamma^{bac}\right) = \frac{\eta}{\sqrt{2}}\not{D} \tag{A.22}$$

where the subscript LC stands for Levi-Civita. So we learn that ignoring ordering ambiguities of quantum mechanics, the torsional Dirac operator can be realized as the supersymmetry charge corresponding to this model. We now deal with the operator ordering issues.

A.2 Quantum $N = 1$ SQM

In order to calculate anomalies, we need to work with $\mathcal{R} = -\frac{1}{2}e^{1/2}\not{\mathcal{D}}^2 e^{-1/2}$ as our regulator, where $\not{\mathcal{D}} = \gamma^a e^i_a(\partial_i + \frac{1}{4}\omega_{i,bc}\gamma^{bc} + B_i + iA_i)$ and $e = \det(e^a_i)$. A standard Weitzenbock calculation yields

$$\not{\mathcal{D}}^2 = g^{ij}\mathcal{D}^{(\Gamma)}_i\mathcal{D}_j - \frac{1}{2}\gamma^{ij}C^k{}_{ij}\mathcal{D}_k + \frac{1}{2}\gamma^{ij}B_{ij} + \frac{1}{8}\gamma^{ij}\gamma^{ab}R_{ab,ij} + \frac{i}{2}\gamma^{ab}F_{ab} \tag{A.23}$$

Using this, we write our regulator in the form

$$\begin{aligned}
\mathcal{R} &= -\frac{1}{2}e^{1/2}\left(g^{ij}\mathcal{D}^{(\Gamma)}_i\mathcal{D}_j - \frac{1}{2}\gamma^{ij}C^k{}_{ij}\mathcal{D}_k + \frac{1}{2}\gamma^{ij}B_{ij} + \frac{1}{8}\gamma^{ij}\gamma^{ab}R_{ab,ij} + \frac{i}{2}\gamma^{ab}F_{ab}\right)e^{-1/2} \\
&= -\frac{1}{2}e^{-1/2}\left(\mathcal{D}_i e g^{ij}\mathcal{D}_j - 2eB^j\mathcal{D}_j\right)e^{-1/2} - \frac{1}{2}e^{1/2}\left(-\frac{1}{2}\gamma^{ij}C^k{}_{ij}\mathcal{D}_k + \frac{1}{2}\gamma^{ij}B_{ij} + \frac{1}{8}\gamma^{ij}\gamma^{ab}R_{ab,ij} + \frac{i}{2}\gamma^{ab}F_{ab}\right)e^{-1/2} \\
&= -\frac{1}{2}e^{-1/2}\left(D_i e g^{ij}D_j + (\partial_i(eB^i) - eB_iB^i)\right)e^{-1/2} - \frac{1}{2}e^{1/2}\left(-\frac{1}{2}\gamma^{ij}C^k{}_{ij}\mathcal{D}_k + \frac{1}{2}\gamma^{ij}B_{ij}\right. \\
&+ \left.\frac{1}{8}\gamma^{ij}\gamma^{ab}R_{ab,ij} + \frac{i}{2}\gamma^{ab}F_{ab}\right)e^{-1/2}
\end{aligned} \tag{A.24}$$

where $D_i = \partial_i + \frac{1}{4}\omega_{i,ab}\gamma^{ab} + iA^\alpha_i T^\alpha$. Now in the quantum theory, we can make the replacements $\hat{p}_i = -i\hbar\partial_i$, $\hat{\psi}^a = \frac{1}{\sqrt{2}}\gamma^a$ and $T^\alpha = \hat{c}^*_M(T_\alpha)^M{}_N\hat{c}^N$. We denote $\hat{\pi}_i = \hat{p}_i - \frac{i\hbar}{2}\omega_{i,ab}\hat{\psi}^a\hat{\psi}^b + \hbar A^\alpha_i\hat{c}^*T^\alpha\hat{c}$. From now on, we will drop the hats on the operators, with the understanding that until we Weyl order the expression and put it inside a path integral, the p_i's, ψ^a's and c's are operators. The regulator becomes

$$\begin{aligned}
\hbar^2\mathcal{R} &= \frac{1}{2}e^{-1/2}\left(\pi_i e g^{ij}\pi_j\right)e^{-1/2} + \frac{i\hbar}{2}e^{1/2}\psi^a\psi^b C^k{}_{ab}\pi_k e^{-1/2} - \frac{\hbar^2}{4}\psi^c\psi^d\psi^a\psi^b R_{ab,cd} - \frac{i\hbar^2}{2}\psi^a\psi^b F^\alpha_{ab}c^*T^\alpha c \\
&- \frac{1}{2}\left(\hbar^2\psi^a\psi^b B_{ab} + \hbar^2(e^{-1}\partial_i(eB^i) - B_iB^i) - \hbar^2\psi^a\psi^b C^k{}_{ab}B_k\right)
\end{aligned} \tag{A.25}$$

Next, we Weyl order the terms in the first line; terms in the second line are already Weyl ordered. Note that the internal fields will not be Weyl ordered. Let us first look at $\frac{1}{2}e^{-1/2}\pi_i e g^{ij}\pi_j e^{-1/2}$. We have

$$\begin{aligned}
\frac{1}{2}e^{-1/2}p_i e g^{ij}p_j e^{-1/2} &= \frac{1}{2}p_i g^{ij}p_j + \frac{\hbar^2}{4}\partial_i(g^{ij}\partial_j ln(e)) + \frac{\hbar^2}{8}g^{ij}\partial_i ln(e)\partial_j ln(e) \\
&= \frac{1}{2}(p_i g^{ij}p_j)_S + \frac{\hbar^2}{8}\partial_i\partial_j g^{ij} + \frac{\hbar^2}{4}\partial_i(g^{ij}\partial_j ln(e)) + \frac{\hbar^2}{8}g^{ij}\partial_i ln(e)\partial_j ln(e)
\end{aligned} \tag{A.26}$$

A quick calculation gives us

$$\frac{1}{2}e^{-1/2}p_i e g^{ij}p_j e^{-1/2} = \frac{1}{2}(p_i g^{ij}p_j)_S - \frac{\hbar^2}{8}\hat{R} + \frac{\hbar^2}{8}\hat{\Gamma}^i_{jk}\hat{\Gamma}^j_{il}g^{kl} \tag{A.27}$$

where the hats indicate Levi-Civita connection. Similarly, Weyl ordering the four fermion term gives us

$$\frac{1}{2}e^{-1/2}(\frac{i\hbar}{2}\omega_{i,ab}\psi^a\psi^b)eg^{ij}(\frac{i\hbar}{2}\omega_{j,cd}\psi^c\psi^d)e^{-1/2} = -\frac{\hbar^2}{8}g^{ij}\omega_{i,cd}\omega_{i,ab}(\psi^c\psi^d\psi^a\psi^b)_A + \frac{\hbar^2}{16}g^{ij}\omega_{i,cd}\omega_j^{cd} \quad \text{(A.28)}$$

The two-fermion term gives no additional counterterms and can be written in the Weyl ordered form as $-\frac{i\hbar}{4}\{p_i, g^{ij}\omega_{j,ab}\psi^{ab}\}$. Now we move on to the second term - $\frac{i\hbar}{2}e^{1/2}\psi^a\psi^b C^k{}_{ab}\pi_k e^{-1/2}$. We observe that

$$\frac{i\hbar}{2}e^{1/2}\psi^a\psi^b C^k{}_{ab}p_k e^{-1/2} = \frac{i\hbar}{2}(\psi^{ab}C^k{}_{ab}p_k)_S - \frac{\hbar^2}{4}\psi^{ab}(\partial_k C^k_{ab} + \Gamma^m{}_{mk}C^k{}_{ab} + 2B_k C^k{}_{ab}) \quad \text{(A.29)}$$

$$\frac{\hbar^2}{4}\psi^{ab}C^k{}_{ab}\omega_{k,cd}\psi^{cd} = \frac{\hbar^2}{4}C^k{}_{ab}\omega_{k,cd}(\psi^a\psi^b\psi^c\psi^d)_A + \frac{\hbar^2}{8}\omega_{k,ab}C^k{}_{ba} + \frac{\hbar^2}{2}C^k{}_{ab}\omega_{k,bc}\psi^{ac} \quad \text{(A.30)}$$

The curvature term can be written as

$$-\frac{\hbar^2}{4}R_{ab,cd}\psi^c\psi^d\psi^a\psi^b = -\frac{\hbar^2}{4}R_{ab,cd}(\psi^c\psi^d\psi^a\psi^b)_A + \frac{\hbar^2}{8}R - \frac{\hbar^2}{2}R_{ab,ad}\psi^{bd} \quad \text{(A.31)}$$

Thus the regulator after Weyl ordering becomes

$$\hbar^2\mathcal{R} \quad = \quad \left(\frac{1}{2}g^{ij}\pi_i\pi_j + \frac{i\hbar}{2}\psi^a\psi^b C^k{}_{ab}\pi_k - \frac{\hbar^2}{4}R_{ab,cd}\psi^a\psi^b\psi^c\psi^d - \frac{i\hbar^2}{2}\psi^a\psi^b F^\alpha_{ab}c^*T^\alpha c\right)_W + \text{counterterms} \quad \text{(A.32)}$$

We observe that the Weyl ordered piece of the regulator is essentially the Hamiltonian for N=1 Supersymmetric quantum mechanics derived previously. We now focus on the counterterms.

Counterterms

Let us consider

$$R_{ab,cd} = \hat{R}_{ab,cd} + (D_c\Omega_{d,ab} - D_d\Omega_{c,ab}) + \Omega_{e,ab}T^e{}_{cd} - (\Omega_{c,ae}\Omega_{d,eb} - \Omega_{d,ae}\Omega_{c,eb}) \quad \text{(A.33)}$$

We can use this to see that

$$\begin{aligned}
R_{ab,ad}\psi^{bd} &= \hat{R}_{ab,ad}\psi^{bd} + (D_a\Omega_{d,ab} - D_d\Omega_{a,ab})\psi^{bd} + \Omega_{e,ab}T^e{}_{ad}\psi^{bd} - (\Omega_{a,ae}\Omega_{d,eb} - \Omega_{d,ae}\Omega_{a,eb})\psi^{bd} \\
&= \frac{1}{2}D_aT_{a,db}\psi^{bd} + (2D_dB_b\psi^{bd} + B_eT_{e,db})\psi^{bd} + \Omega_{e,ab}T^e{}_{ad}\psi^{bd} + \Omega_{d,ae}\Omega_{a,eb}\psi^{bd} \\
&= \frac{1}{2}D_aT_{a,db}\psi^{bd} + B_{db}\psi^{bd} + \frac{1}{2}(T_{a,eb} + T_{b,ae})T^e{}_{ad}\psi^{bd} - \frac{1}{4}(T_{a,de} + T_{e,ad})T_{b,ae}\psi^{bd} \\
&= \frac{1}{2}D_aT_{a,db}\psi^{bd} + B_{db}\psi^{bd} + \frac{1}{2}T_{a,eb}T^e{}_{ad}\psi^{bd} \qquad (A.34)
\end{aligned}$$

This yields

$$-\frac{\hbar^2}{2}R_{ab,ad}\psi^{bd} = \frac{\hbar^2}{2}\left(\frac{1}{2}D_aT_{a,bd} + B_{bd} - \frac{1}{2}T_{a,eb}T^e{}_{ad}\right)\psi^{bd} \qquad (A.35)$$

Thus the ψ^{ab} counterterms all cancel out. In order to compute the remaining counterterms, we notice that

$$\begin{aligned}
\frac{\hbar^2}{8}R &= \frac{\hbar^2}{8}\hat{R} + \frac{\hbar^2}{2}(D_aB^a + B_aB^a) + \frac{\hbar^2}{8}\Omega_{e,ab}T^e{}_{ab} + \frac{\hbar^2}{8}\Omega_{b,ae}\Omega_{a,eb} \\
&= \frac{\hbar^2}{8}\hat{R} + \frac{\hbar^2}{2}(e^{-1}\partial_i(eB^i) - B_iB^i) + \frac{\hbar^2}{16}T_{a,eb}T_{e,ab} + \frac{\hbar^2}{32}T_{e,ab}T_{e,ab} \qquad (A.36)
\end{aligned}$$

The ψ independent counterterms reduce to

$$\frac{\hbar^2}{8}\hat{\Gamma}^i_{jk}\hat{\Gamma}^j_{il}g^{kl} + \frac{\hbar^2}{16}g^{ij}\omega_{i,cd}\omega^{cd}_j + \frac{\hbar^2}{8}\Omega_{b,ae}\Omega_{a,eb} - \frac{\hbar^2}{8}\hat{\omega}_{k,ab}C^k{}_{ab} \qquad (A.37)$$

We henceforth denote all the counterterms by L_{ct}. It should be noted that this can be rearranged to put it in the form

$$\begin{aligned}
L_{ct} &= \frac{\hbar^2}{8}\hat{\Gamma}^i_{jk}\hat{\Gamma}^j_{il}g^{kl} + \frac{\hbar^2}{16}\hat{\omega}_{i,cd}\hat{\omega}^{cd}_i + \frac{\hbar^2}{16}\hat{\omega}_{e,ab}(T_{a,eb} + T_{e,ba} + T_{b,ae}) - \frac{\hbar^2}{32}(T_{b,ae} + T_{a,eb} + T_{e,ba})\Omega_{b,ae} \\
&= \frac{\hbar^2}{8}\hat{\Gamma}^i_{jk}\hat{\Gamma}^j_{il}g^{kl} + \frac{\hbar^2}{16}\hat{\omega}_{i,cd}\hat{\omega}^{cd}_i + \frac{3\hbar^2}{16}\hat{\omega}_{e,ab}T_{[a,eb]} - \frac{3\hbar^2}{32}T_{[b,ae]}\Omega_{[b,ae]} \\
&= \frac{\hbar^2}{8}\hat{\Gamma}^i_{jk}\hat{\Gamma}^j_{il}g^{kl} + \frac{\hbar^2}{16}W_{i,cd}W^{cd}_i - \frac{3\hbar^2}{8}\Omega_{[b,ae]}\Omega_{[b,ae]} \qquad (A.38)
\end{aligned}$$

where $W_{i,ab} = \hat{\omega}_{i,ab} + 3\Omega_{[i,ab]}$. Notice that the counterterms only see the totally anti-symmetric part of torsion. In fact, we could have kept only the totally antisymmetric part of torsion to begin with, and we would've gotten the same answer much faster - this is again a consequence of the fact that the Dirac operator only sees the totally anti-symmetric part of torsion. Finally, we conclude that the regulator can be written in the form

$$\hbar^2\mathcal{R} = \left(\frac{1}{2}g^{ij}\pi_i\pi_j + \frac{i\hbar}{2}\psi^a\psi^bC^k{}_{ab}\pi_k - \frac{\hbar^2}{4}R_{ab,cd}\psi^a\psi^b\psi^c\psi^d\right)_W + L_{ct} \qquad (A.39)$$

where the counterterms are given in two equivalent forms in Eqs A.37, A.38.

A.3 Dirac index and the chiral anomaly

Our first aim is to compute the chiral anomaly in a torsional background using supersymmetric quantum mechanics. We will mostly follow the treatment of Boer et. al. in this calculation. Using standard arguments which we don't repeat here, we have

$$\text{ind } \not{D} = \text{Tr } \gamma^5 e^{\frac{1}{2}\beta\hbar e^{1/2}\not{D}^2 e^{-1/2}} = \text{Tr } \gamma^5 e^{-\beta\hbar\mathcal{R}} \tag{A.40}$$

where $2(\beta\hbar)^{-1} = \Lambda^2$, Λ being the ultraviolet cutoff. From our previous discussion on SQM, we can write this as

$$\text{ind } \not{D} = \text{Tr } (-1)^F e^{-\frac{\beta}{\hbar}(H+L_{ct})} \tag{A.41}$$

where H is the classical Hamiltonian for $N = 1$ SQM. Again, from standard arguments this can be written as

$$\text{ind } \not{D} = \int_{\text{PBC}} [dx^i(\tau) d\psi^j(\tau)] e^{-\frac{1}{\hbar}\int_{-\beta}^{0} \, \mathrm{d}\tau L_E} \tag{A.42}$$

where

$$\begin{aligned} L_E &= \frac{1}{2} g_{ij}\dot{x}^i\dot{x}^j + \frac{\hbar}{2} g_{ij}\psi^i\nabla_\tau\psi^j - \frac{\hbar}{2} C_{ijk}\dot{x}^i\psi^j\psi^k + \frac{\hbar^2}{2} N_{mijk}\psi^m\psi^i\psi^j\psi^k \\ &+ \hbar\bar{\eta}(\dot{\eta} + \dot{x}^k A_k^\alpha T^\alpha \eta) - \frac{i\hbar^2}{2}\bar{\eta}\psi^j\psi^k F_{jk}^\alpha T^\alpha \eta + L_{ct} \end{aligned} \tag{A.43}$$

is the Euclidean version of Eq A.14 (now with the counterterms added), obtained by the rotation $\tau = it$ - the dots in the above equation now obviously refer to derivatives with respect to τ. The symbol PBC stands for periodic boundary conditions [2]. We also have to add to the Lagrangian certain ghost fields which enter as a consequence of a careful time-slicing analysis of the path integral. The ghost Lagrangian is $L_g = \frac{1}{2} g_{ij} b^i c^j + \frac{1}{2} g_{ij} a^i a^j$, where b_i, c_i are real anti-commuting ghosts while a_i is a real commuting ghost. Finally, we redefine the time coordinate as $s = \beta^{-1}\tau$, and rescale the fermion fields $\psi = \frac{1}{\sqrt{\beta\hbar}}\psi'$, and dropping

[2] In Euclidean superspace, we have $Q = -\partial_\theta - \theta\partial_\tau$, $Q^2 = \partial_\tau$, $D = \partial_\theta - \theta\partial_\tau$, $D^2 = -\partial_\tau$ and $DX^i = \psi^i - \theta\dot{x}^i$. The classical part of the above Lagrangian can be written as $L_E = \int \mathrm{d}\theta \left(\frac{1}{2} g_{ij}(X) DX^i D^2 X^j + \frac{1}{2}\Omega_{ijk}(X) DX^i DX^j DX^k\right)$

the primes for convenience, we have the action

$$-\frac{1}{\hbar}S = -\frac{1}{\hbar\beta}\int_{-1}^{0}\mathrm{d}s\,\left(\frac{1}{2}g_{ij}(\dot{x}^i\dot{x}^j + b^ic^j + a^ia^j) + \frac{1}{2}\eta_{ab}\psi^a\nabla_s\psi^b - \frac{1}{2}C_{iab}\dot{x}^i\psi^a\psi^b + \frac{1}{2}N_{abcd}\psi^a\psi^b\psi^c\psi^d\right)$$
$$- \int_{-1}^{0}\mathrm{d}s\,\left(\bar{\eta}(\dot{\eta} + \dot{x}^kA_k^\alpha T^\alpha\eta) - \frac{i}{2}\bar{\eta}\psi^j\psi^kF_{jk}^\alpha T^\alpha\eta\right) - \hbar\beta\int_{-1}^{0}\mathrm{d}s\,L_{ct} \tag{A.44}$$

where $L_{ct} = \frac{1}{8}\hat{\Gamma}^i_{jk}\hat{\Gamma}^j_{il}g^{kl} + \frac{1}{16}W_{i,cd}W_i^{cd} - \frac{3}{8}\Omega_{[b,ae]}\Omega_{[b,ae]}$. We now expand the fields as $x^i(s) = x_0^i + \sqrt{\beta\hbar}q^i(s)$ and $\psi^a(s) = \psi_0^a + \sqrt{\beta\hbar}\zeta^a(s)$[3] about the classical solutions of the free action, which we take to be $S_0 = -\int_{-1}^0 \mathrm{d}s\,\left(\frac{1}{2}g_{ij}(x_0)(\dot{q}^i\dot{q}^j + b^ic^j + a^ia^j) + \frac{1}{2}\eta_{ab}\zeta^a\dot{\zeta}^b\right)$. We have also scaled the ghost fields by a factor of $\sqrt{\beta\hbar}$.

Let us now define $H_{i,ab} = \Omega_{[i,ab]}$. We will find it convenient to work with the modified connection one-form $W_{i,ab} = \hat{\omega}_{i,ab} + 3H_{i,ab}$, so all our expressions will involve these new quantities henceforth. The interaction terms are

$$\begin{aligned}
-\frac{1}{\hbar}S_{int} &= \int_{-1}^{0}\mathrm{d}s\,(-\frac{\beta\hbar}{4}\partial_k\partial_l g_{ij}(x_0)(x_0)q^kq^l + \cdots)(\dot{q}^i\dot{q}^j + b^ic^j + a^ia^j) - \beta\hbar\int_{-1}^{0}L_{ct} \\
&- \int_{-1}^{0}\mathrm{d}s\,\frac{1}{2\sqrt{\beta\hbar}}\dot{q}^i\left(W_{i,ab}(x_0) + \sqrt{\beta\hbar}q^k\partial_kW_{i,ab}(x_0) + \frac{1}{2}\beta\hbar q^kq^m\partial_k\partial_mW_{i,ab}(x_0) + \cdots\right) \\
&\times \left(\psi_0^a\psi_0^b + 2\sqrt{\beta\hbar}\zeta^a\psi_0^b + \beta\hbar\zeta^a\zeta^b\right) \\
&- \int_{-1}^{0}\mathrm{d}s\,\frac{1}{2\beta\hbar}\left(\tilde{N}_{abcd}(x_0) + \sqrt{\beta\hbar}q^i\partial_i\tilde{N}_{abcd}(x_0) + \frac{1}{2}\beta\hbar q^iq^j\partial_i\partial_j\tilde{N}_{abcd}(x_0) + \cdots\right) \\
&\times \left(\psi_0^a\psi_0^b\psi_0^c\psi_0^d + 4\sqrt{\beta\hbar}\zeta^a\psi_0^b\psi_0^c\psi_0^d + 6\beta\hbar\zeta^a\zeta^b\psi_0^c\psi_0^d + 4(\beta\hbar)^{3/2}\zeta^a\zeta^b\zeta^c\psi_0^d + (\beta\hbar)^2\zeta^a\zeta^b\zeta^c\zeta^d\right) \\
&- \sqrt{\beta\hbar}\int_{-1}^{0}\mathrm{d}s\,\dot{q}^k\bar{\eta}(A_k^\alpha + \sqrt{\beta\hbar}\partial_jA_kq^j + \cdots)T^\alpha\eta \\
&+ \int_{-1}^{0}\mathrm{d}s\,\frac{i}{2}\bar{\eta}(\psi_0^a\psi_0^b + 2\sqrt{\beta\hbar}\zeta^a\psi_0^b + \beta\hbar\zeta^a\zeta^b)(F_{ab}^\alpha + \sqrt{\beta\hbar}\partial_kF_{ab}^\alpha q^k + \beta\hbar\partial_j\partial_kF_{ab}^\alpha q^jq^k)T^\alpha\eta
\end{aligned} \tag{A.45}$$

where $\tilde{N}_{abcd} = N_{[abcd]}$. In order to simplify the interaction terms, we will chose Riemann normal coordinates about the point x_0 and chose a frame such that the modified connection $W_{i,ab}(x_0) = 0$.

Perturbative evaluation

We now evaluate the Dirac index in perturbation theory in $d = 4$, with $\beta\hbar$ as the coupling constant (in the weak coupling limit). The same computation can be repeated in higher dimensions, but we do not present all the details here.

[3]The factor of $\sqrt{\beta\hbar}$ makes it easier to keep track of the Feynman diagrams we need to compute.

We first state all the propagators

$$\langle q^i(t)q^j(s)\rangle = -g^{ij}(x_0)\Delta(t-s) \tag{A.46a}$$

$$\langle a^i(t)a^j(s)\rangle = +g^{ij}(x_0)\delta(t-s) \tag{A.46b}$$

$$\langle b^i(t)c^j(s)\rangle = -2g^{ij}(x_0)\delta(t-s) \tag{A.46c}$$

$$\langle \zeta^a(t)\zeta^b(s)\rangle = \frac{1}{2}\delta^{ab}\epsilon(t-s) - (t-s) \tag{A.46d}$$

where $\Delta(t-s) = t(s+1)\theta(t-s) + s(t+1)\theta(s-t)$.

Let us perform the computation order by order - we'll mainly be interested in the results at $\beta\hbar^{-1}$ and $\beta\hbar^0$ order. At $\beta\hbar^{-1}$, we have only one non-trivial diagram

$$-\frac{1}{2\beta\hbar}\tilde{N}_{abcd}\psi_0^a\psi_0^b\psi_0^c\psi^d\langle 1\rangle \tag{A.47}$$

There are many terms at $\beta\hbar^0$, and we will list the terms (ignoring gauge internal gauge fields for the moment) below:

$$\frac{1}{8}\int_{-1}^{0}\mathrm{d}t\mathrm{d}s\ \tilde{N}_{abcd}(x_0)\partial_k\partial_l g_{ij}(x_0)\langle q^k q^l(\dot{q}^i\dot{q}^j + b^i c^j + a^i a^j)\rangle\psi_0^{abcd} = \frac{1}{48}\hat{R}(x_0)\tilde{N}_{abcd}(x_0)\psi_0^{abcd} \tag{A.48}$$

In the above, we have used the normal coordinates formula $\partial^2_{(kl)}g_{ij} = \frac{2}{3}R_{i(k,l)j}$.[4]

$$-\frac{1}{4}\int_{-1}^{0}\mathrm{d}t\ \partial_i\partial_j\tilde{N}_{abcd}\psi_0^{abcd}\langle q^i(t)q^j(t)\rangle = -\frac{1}{24}g^{ij}(x_0)\partial_i\partial_j N_{abcd}(x_0)\psi_0^{abcd} \tag{A.49}$$

$$\int_{-1}^{0}\mathrm{d}s\mathrm{d}t\ 4\tilde{N}_{abcd}\tilde{N}_{efgh}\psi_0^{hbcd}\langle\zeta^a(t)\zeta^e(s)\zeta^f(s)\zeta^g(s)\rangle = 0 \tag{A.50}$$

$$\int_{-1}^{0}\mathrm{d}t\mathrm{d}s\ 2\tilde{N}_{abcd}\partial_k W_{i,ef}\langle\dot{q}^i(s)q^k(s)\zeta^a(t)\zeta^e(s)\rangle\psi_0^{fbcd} = \frac{1}{6}\delta^{ae}g^{ik}\tilde{N}_{abcd}\partial_k W_{i,ef}\psi_0^{fbcd} \tag{A.51}$$

$$\frac{9}{2}\int\mathrm{d}t\mathrm{d}s\ \tilde{N}_{abcd}\tilde{N}_{efgh}\langle\zeta^a(t)\zeta^b(t)\zeta^e(s)\zeta^f(s)\rangle\psi_0^{cdgh} = -\frac{3}{4}\tilde{N}_{abcd}\tilde{N}_{abgh}\psi_0^{cdgh} \tag{A.52}$$

$$\frac{1}{8}\int_{-1}^{0}\mathrm{d}s\mathrm{d}t\ \partial_k W_{i,ab}\partial_l W_{j,cd}\langle\dot{q}^i(t)\dot{q}^j(s)q^k(t)q^l(s)\rangle\psi_0^{abcd} = \frac{1}{96}(g^{kl}g^{ij} - g^{kj}g^{il})\partial_k W_{i,ab}\partial_l W_{j,cd}\psi_0^{abcd} \tag{A.53}$$

$$-\frac{1}{4}\int_{-1}^{0}\mathrm{d}t\mathrm{d}s\ F_{ab}F_{cd}\langle\bar{\eta}(t)\eta(t)\bar{\eta}(s)\eta(s)\rangle\psi_0^{abcd} \tag{A.54}$$

[4]The more general formula (see [?]) is $g_{ij}(x_0+y) = g_{ij}(x_0) + \frac{1}{3}R_{i(k,l)j}y^k y^l + \frac{1}{6}\nabla_m R_{i(k,l)j}y^k y^l y^m + \cdots$.

There is only one counterterm diagram

$$-\frac{3}{16}\int_{-1}^{0}\mathrm{d}t\mathrm{d}s\;\tilde{N}_{abcd}H_{e,fg}H_{e,fg}\psi_0^{abcd}=-\frac{3}{16}\tilde{N}_{abcd}H_{e,fg}H_{e,fg}\psi_0^{abcd} \tag{A.55}$$

Putting everything together, and adding the appropriate normalization factor we get

$$\begin{aligned}\mathrm{ind}(\not{D}) \quad &= \quad \frac{1}{8\pi^2\beta\hbar}\epsilon^{abcd}N_{abcd}+\frac{1}{4\pi^2}\epsilon^{abcd}\left(\frac{1}{48}\hat{R}(x_0)\tilde{N}_{abcd}(x_0)-\frac{1}{24}g^{ij}(x_0)\partial_i\partial_j N_{abcd}(x_0)\right.\\ &+\quad \left.\frac{1}{96}(g^{kl}g^{ij}-g^{kj}g^{il})\partial_k W_{i,ab}\partial_l W_{j,cd}-\frac{3}{16}\tilde{N}_{abcd}H_{e,fg}H_{e,fg}-\frac{3}{4}N_{mnab}N_{mncd}\right)\end{aligned} \tag{A.56}$$

Let us now make an important observation - let $\mathbb{R}^{(c)}_{\mu\nu,\lambda\sigma}$ be the curvature corresponding to the connection $\hat{\omega}_{i,ab}+cH_{i,ab}$, written in coordinate indices. We have (it is understood that all of the quantities in the below equation are antisymmetric in μ,ν and λ,σ)

$$\mathbb{R}^{(c)}_{\mu\nu\lambda\sigma}-\mathbb{R}^{(-c)}_{\lambda\sigma,\mu\nu}=2c\left(\hat{D}_\lambda H_{\sigma,\mu\nu}+\hat{D}_\mu H_{\nu,\lambda\sigma}\right)=2c(\partial_\lambda H_{\sigma,\mu\nu}-\partial_\mu H_{\sigma,\lambda\nu}) \tag{A.57}$$

We see that this combination is totally anti-symmetric in all it's indices, and is in fact proportional to the Nieh Yan tensor. We thus conclude that

$$\mathbb{R}^{(c)}_{\mu\nu\lambda\sigma}-\mathbb{R}^{(-c)}_{\lambda\sigma,\mu\nu}=-4cN_{[\mu\nu\lambda\sigma]} \tag{A.58}$$

Further, since all our calculations are in a special choice of frame, we are allowed to replace $\partial_i W_{j,ab}$ with $\frac{1}{2}\mathbb{R}^{(3)}_{ab,ij}$. Therefore, we will covariantize our answers by making the replacement $\partial_i W_{j,ab}\rightarrow\frac{1}{2}(\mathbb{R}^{(-3)}_{ij,ab}-12N_{abij})$. This immdietly gives

$$\mathrm{ind}(\not{D})=\int_M\frac{1}{8\pi^2\beta\hbar}(T^a\wedge T_a-R_{ab}\wedge e^a\wedge e^b)+\frac{1}{192\pi^2}\left(\mathbb{R}^{(-3)}_{ab}\wedge\mathbb{R}^{(-3)}_{ba}+2\mathrm{d}\mathrm{d}^\dagger(T^a\wedge T_a-R_{ab}\wedge e^a\wedge e^b)\right) \tag{A.59}$$

where we have used $\mathbb{R}^{(-3)}=\hat{R}-9H_{a,bc}H_{a,bc}$, and $\mathrm{d}^\dagger={}^*\mathrm{d}^*$. Finally, we notice that the last term is the derivative of an invariantly defined form; we will drop such terms. Also, we haven't considered the Yang-Mills gauge fields in the above computation, but one can include them easily. Adding back a $U(1)$ gauge field gives us

$$\mathrm{ind}(\not{D})=\int_M\frac{1}{8\pi^2\beta\hbar}(T^a\wedge T_a-R_{ab}\wedge e^a\wedge e^b)+\frac{1}{192\pi^2}\mathbb{R}^{(-3)}_{ab}\wedge\mathbb{R}^{(-3)}_{ba}+\frac{1}{8\pi^2}F\wedge F \tag{A.60}$$

In the general non-Abelian case, the result is simply modified by the Chern class $\frac{1}{8\pi^2}\mathrm{Tr}\,F\wedge F$.

A.4 Diffeomorphism anomaly

The diffeomorphism anomaly due to coupling of chiral fermions to gravity is given by

$$\mathcal{A}[v^i] = \mathrm{Tr}\left(\gamma^5 v^i D_i e^{-\beta\hbar\mathcal{R}}\right) \tag{A.61}$$

Similar to the case of the chiral anomaly, we can write this in terms of a Euclidean path integral as

$$\mathcal{A}[v^i] = \int_{\mathrm{PBC}} [dx^i(\tau) d\psi^j(\tau)] v_i(-\dot{x}^i + \frac{1}{2}C^i{}_{jk}\psi^j\psi^k) e^{-\int_0^\beta \mathrm{d}\tau L_E} \tag{A.62}$$

Since the operator insertions are already Weyl ordered, there are no additional Weyl ordering contributions. We simply exponentiate the insertions, because we are interested in terms first order in v. The additional terms are

$$\begin{aligned} -\frac{1}{\hbar}S_{diff} &= -\frac{1}{\beta\hbar}\int_{-1}^0 \mathrm{d}s\ v_i\left(\dot{x}^i - \frac{\beta\hbar}{2}C^i{}_{ab}\psi^a\psi^b\right) \\ &= \int_{-1}^0 \mathrm{d}s\ \left\{-\dot{q}^i\left(q^k\partial_k v_i + \frac{1}{2!}\sqrt{\beta\hbar}\partial^2_{kl}v_i q^k q^l + \frac{1}{3!}\beta\hbar\partial^3_{klm}v_i q^k q^l q^m + \cdots\right)\right. \\ &+ \left.\frac{1}{2\beta\hbar}(v_{ab} + \sqrt{\beta\hbar}q^k\partial_k v_{ab} + \frac{1}{2}\beta\hbar q^k q^l \partial^2_{kl}v_{ab} + \cdots)(\psi_0^a\psi_0^b + 2\sqrt{\beta\hbar}\zeta^a\psi_0^b + \beta\hbar\zeta^a\zeta^b)\right\} \end{aligned} \tag{A.63}$$

Perturbative evaluation

Once again, we present a sample calculation of the diffeomorphism anomaly in $d = 2$, although we expect the technique to go through in higher dimensions as well. In two dimensions, the computation is greatly simplified by the fact that $H_{i,ab} = 0$. So the Lagrangian is effectively torsion-free. The only contribution to torsion comes from the operator insertions. Once again, let us perturbatively compute the diffeomorphism anomaly - this time, we chose Riemann normal coordinates, and the frame is chosen such that $\hat{\omega}_{i,ab}(x_0) = 0$.

At order $(\beta\hbar)^{-1}$, we have only one diagram

$$\frac{1}{2\beta\hbar}v_i C^i{}_{ab}\psi_0^a\psi_0^b \tag{A.64}$$

Let us now list the diagrams at order $(\beta\hbar)^0$:

$$\int_{-1}^{0}\int \mathrm{d}t\mathrm{d}s\ \frac{1}{2}\langle \dot{q}^i(t)q^k(t)\dot{q}^j(s)q^l(s)\rangle \partial_k\hat{\omega}_{i,ab}\partial_l v_j\psi_0^{ab} = \frac{1}{24}(g^{ij}g^{kl}-g^{il}g^{jk})\partial_k\hat{\omega}_{i,ab}\partial_l v_j\psi_0^{ab} \tag{A.65}$$

$$-\frac{1}{8}\int_{-1}^{0}\mathrm{d}t\mathrm{d}s\ v_iC^i{}_{ab}\partial_k\partial_l g_{ij}\langle q^kq^l(\dot{q}^i\dot{q}^j+a^ia^j+b^ic^j)\rangle\psi_0^{ab} = -\frac{1}{48}\hat{R}v_iC^i{}_{ab}\psi_0^{ab} \tag{A.66}$$

$$-\frac{1}{4}\int_{-1}^{0}\mathrm{d}t\ \partial_k\partial_m(v_iC^i{}_{ab})\langle q^k(t)q^m(t)\rangle\psi_0^{ab} = -\frac{1}{24}g^{km}\partial_k\partial_m(v_iC^i{}_{ab})\psi_0^{ab} \tag{A.67}$$

$$\int_{-1}^{0}\mathrm{d}t\mathrm{d}s\ v_iC^i{}_{ab}\partial_k W_{j,cd}\langle\zeta^a(t)\zeta^c(s)\dot{q}^j(s)q^k(s)\rangle = 0 \tag{A.68}$$

Putting all the terms together and covariantizing, we get

$$\mathcal{A}[v] = \int_M \mathrm{d}^2x\ \sqrt{g}\frac{-i\epsilon^{ab}}{2\pi}\left(\frac{1}{2\beta\hbar}v_iC^i{}_{ab}+\frac{1}{24}\hat{R}_{ij,ab}(\partial_i v_j)-\frac{1}{48}\hat{R}v_iC^i{}_{ab}\right) \tag{A.69}$$

Using the special properties of Riemann curvature in 2 dimensions, we write this in differential form notation as

$$\mathcal{A}[v] = \int_M -iv_a\left(\frac{1}{2\pi\beta\hbar}T^a+\frac{1}{48\pi}e^a\wedge\mathrm{d}\hat{R}-\frac{1}{48\pi}\hat{R}T^a\right) \tag{A.70}$$

We have ignored the term $\mathrm{d}\mathrm{d}^\dagger(v_aT^a)$ in the above expression, because it is a total derivative.

A.5 Trace anomaly

Here the trace that we are interested in is

$$\mathcal{A}[\sigma] = \mathrm{Tr}\ \sigma e^{-\beta\hbar\mathcal{R}} \tag{A.71}$$

Note that the absence of $\gamma^5=(-1)^F$ means that the fermions have anti-periodic boundary conditions. This makes computations harder at higher orders in $\beta\hbar$. We will restrict ourselves to two-loop calculations, which are the relevant ones in two dimensions. In fact, the same computation carries over to higher dimensions (unlike in the case of the previous calculations), and so we will compute the two-loop coefficient in arbitrary dimension.

We will expand $x^i(t) = x_0^i + \sqrt{\beta\hbar}q^i(t)$ and $\psi^i(t) = \sqrt{\beta\hbar}\xi^i(t)$. As before, the free action is

$$S_0 = -\int_{-1}^{0} \mathrm{d}s \; \left(\frac{1}{2}g_{ij}(x_0)(\dot{q}^i\dot{q}^j + b^i c^j + a^i a^j) + \frac{1}{2}\eta_{ab}\zeta^a\dot{\zeta}^b\right). \tag{A.72}$$

The interaction terms are now given by

$$\begin{aligned}
-\frac{1}{\hbar}S_{int} \;\; &= \;\; \int_{-1}^{0}\mathrm{d}s\;(\frac{-\beta\hbar}{4}\partial_k\partial_l g_{ij}(x_0)(x_0)q^k q^l + \cdots)(\dot{q}^i\dot{q}^j + b^i c^j + a^i a^j) - \beta\hbar\int_{-1}^{0} L_{ct} \\
&- \;\; \int_{-1}^{0}\mathrm{d}s\;\frac{1}{2\sqrt{\beta\hbar}}\dot{q}^i\left(\sqrt{\beta\hbar}q^k\partial_k W_{i,ab}(x_0) + \frac{1}{2}\beta\hbar q^k q^m\partial_k\partial_m W_{i,ab}(x_0) + \cdots\right)\beta\hbar\zeta^a\zeta^b \\
&- \;\; \int_{-1}^{0}\mathrm{d}s\;\frac{1}{2\beta\hbar}\left(\tilde{N}_{abcd}(x_0) + \sqrt{\beta\hbar}q^i\partial_i\tilde{N}_{abcd}(x_0) + \frac{1}{2}\beta\hbar q^i q^j\partial_i\partial_j\tilde{N}_{abcd}(x_0) + \cdots\right)(\beta\hbar)^2\zeta^a\zeta^b\zeta^c\zeta^d \\
&- \;\; \sqrt{\beta\hbar}\int_{-1}^{0}\mathrm{d}s\;\dot{q}^k\bar{\eta}(A_k^\alpha + \sqrt{\beta\hbar}\partial_j A_k q^j + \cdots)T^\alpha\eta \\
&+ \;\; \int_{-1}^{0}\mathrm{d}s\;\frac{i}{2}\bar{\eta}(\beta\hbar\zeta^a\zeta^b)(F_{ab}^\alpha + \sqrt{\beta\hbar}\partial_k F_{ab}^\alpha q^k + \beta\hbar\partial_j\partial_k F_{ab}^\alpha q^j q^k + \cdots)T^\alpha\eta
\end{aligned} \tag{A.73}$$

At zero-loops, the trace is given by

$$\frac{2^{[d/2]}}{(2\pi\beta\hbar)^{d/2}}\int d^d x_0\;\sqrt{g(x_0)}\sigma(x_0) \tag{A.74}$$

At the next order in $\beta\hbar$ we have the diagram

$$-\frac{\beta\hbar}{4}\int_{-1}^{0} ds\;\partial_k\partial_l g_{ij}\langle q^k(s)q^l(s)(\dot{q}^i(s)\dot{q}^j(s) + \cdots)\rangle = -\beta\hbar\frac{\hat{R}}{24} \tag{A.75}$$

where $\cdots$ represents ghost terms. Additionally, we have the counterterm diagram$\beta\hbar\frac{3}{8}H_{a,bc}H_{a,bc}$. These two combine to give us

$$\beta\hbar\left(-\frac{\hat{R}}{24} + \frac{3}{8}H_{a,bc}H_{a,bc}\right) = -\beta\hbar\frac{1}{24}R^{(-3)} \tag{A.76}$$

So, the trace anomaly becomes

$$\mathcal{A}[\sigma] = \frac{2^{[d/2]}}{(2\pi\beta\hbar)^{d/2}}\int d^d x_0\;\sqrt{g(x_0)}\sigma(x_0)\left(1 - \frac{\beta\hbar}{24}R^{(-3)} + O((\beta\hbar)^2)\right) \tag{A.77}$$

For $d = 2$, this reduces to

$$\mathrm{Tr}\;e^{t\not{D}^2} = \int d^2 x_0\;\sqrt{g(x_0)}\left(\frac{1}{2\pi t} - \frac{\hat{R}}{24\pi} + \cdots\right) \tag{A.78}$$

and for $d = 3$ we get

$$\mathrm{Tr}\;e^{t\not{D}^2} = \int d^3 x_0\;\sqrt{g(x_0)}\frac{2}{(4\pi t)^{3/2}}\left(1 - \frac{R^{(-3)}}{12}t + \cdots\right) \tag{A.79}$$

This concludes our review of SQM and its applications.

Appendix B

Supplement to Part I

In this appendix, we will present some more supplementary material relevant to the discussion in part I of this thesis.

B.1 Divergences in higher dimensions

In this section, we discuss the torsional divergences in anomaly polynomials in arbitrary dimensions, and their Pauli-Villar's regularization. As we noted in Section 2, divergences of the anomaly polynomials in $d = 4n$ and $d = 4n + 2$ are the same. Therefore, to study the cancellation of divergences, it suffices to focus on the anomaly polynomials in $d = 4n$. We have dealt with the case of $n = 1$ explicitly in section 2, so we now take $n > 1$. Now in $d = 4n$, we have the asymptotic expansion

$$\mathrm{Tr}_{4n}\Gamma^{4n+1}e^{s\not{\mathcal{D}}_{4n}^2} \simeq \frac{1}{s^n}\sum_{k=0}^{\infty} b_k s^k = \frac{1}{s^n}\sum_{k=0}^{n} b_k s^k + O(s) \tag{B.1}$$

where the b_k are $4n$-form polynomials made out of curvature, torsion, and their covariant derivatives (see Eqs. (2.16) and (2.49)). For instance, in $d = 4n$ we have $b_0 \propto \int_{M_{4n}}(dH)^n$, while in $d = 4n + 2$ we have $b_0 \propto \int_{M_{4n+2}} F \wedge (dH)^n$.[1] As before, we will not consider $O(s)$ terms because these lead to $1/m$ corrections in the anomaly polynomial. The un-regulated anomaly polynomial thus takes the form

$$\mathcal{P}^{(0)}(m) = \lim_{\epsilon\to 0} i\sqrt{\pi}m\sum_{k=0}^{n}\Gamma_\epsilon(-n + \frac{1}{2} + k, m^2)\ b_k \tag{B.2}$$

where

$$\Gamma_\epsilon(\alpha, m^2) = \int_\epsilon^\infty s^{\alpha-1}e^{-sm^2} \tag{B.3}$$

[1] The explicit form of b_k is difficult to compute in arbitrary dimension in the presence of torsion.

with $\epsilon = \frac{1}{\Lambda^2}$. Therefore, the UV divergences of the anomaly polynomial in $d = 4n$ are contained in

$$\left\{ m\Gamma_\epsilon(-n + \frac{1}{2} + k, m^2) \right\}, \quad 0 \le k < n \tag{B.4}$$

where $\epsilon = \frac{1}{\Lambda^2}$. Let us examine these integrals schematically:

$$m\Gamma_\epsilon(-n + \frac{1}{2} + k, m^2) = a_0^{(k)} m\Lambda^{2n-2k-1} + a_1^{(k)} m^3 \Lambda^{2n-3-2k} + \cdots a_{n-k-1}^{(k)} m^{2n-1-2k}\Lambda + a_{n-k}^{(k)} \mathrm{sign}(m) m^{2n-2k} \tag{B.5}$$

where the $a_\ell^{(k)}$ are finite numerical coefficients. As before, we introduce Pauli-Villar's regulator fermions with masses M_I and parities C_I, where $I = 1, 2 \cdots N$. For convenience, we label the original low-energy fermion as $I = 0$ with $M_0 = m$ and $C_0 = 1$. From equation (B.5), it is amply clear that to cancel all the UV divergences, we must require

$$\sum_{I=0}^{N} C_I M_I = 0, \ \sum_{I=0}^{N} C_I M_I^3 = 0, \cdots, \sum_{I=0}^{N} C_I M_I^{2n-1} = 0. \tag{B.6}$$

Additionally, we must also check the finiteness of the remaining Λ-independent coefficients

$$\alpha_0 = \sum_{I=0}^{N} a_n^{(0)} C_I \mathrm{sign}(M_I) M_I^{2n}, \ \alpha_1 = \sum_{I=0}^{N} a_{n-1}^{(1)} C_I \mathrm{sign}(M_I) M_I^{2n-2}, \cdots, \ \alpha_n = \sum_{I=0}^{N} a_0^{(n)} C_I \mathrm{sign}(M_I) \tag{B.7}$$

in both the topological and trivial phases, where we note that $a_{n-k}^{(k)} = \widetilde{\Gamma}(-n + k + \frac{1}{2})$, where $\widetilde{\Gamma}$ stands for analytic continuation of the Gamma function. Having done so, the regulated anomaly polynomial is

$$\mathcal{P}(m) = \sum_{k=0}^{n} \alpha_k(m)\, b_k. \tag{B.8}$$

In order to see that the constraints in (B.6) can be satisfied, and that the coefficients $\{\alpha_k\}$ are finite, we go back to the lattice Dirac model in $d = 4n - 1$. We will work with the lattice Hamiltonian

$$H = \sum_{\vec{k}} c_{\vec{k}}^\dagger \left\{ m + \mu_{bw} \left(4n - 2 - \sum_{\mu=1}^{4n-2} \cos(k_\mu) \right) \gamma^{4n-1} + v_F \sum_{\mu=1}^{4n-2} \sin(k_\mu) \gamma^\mu \right\} c_{\vec{k}}. \tag{B.9}$$

The Hamiltonian has 2^{4n-2} Dirac points - the one at $\vec{k} = (0, 0, \cdots, 0)$ will be labelled by $I = 0$ and interpreted as the low-energy Dirac fermion, while the other fermions will be labelled by I from 1 to $4n - 2$ and interpreted as Pauli-Villar's regulator fermions. The fermions have a degenracy of $N_I = \binom{4n-2}{I}$, parities $C_I = (-1)^I$, and masses $M_I = (m + 2I\mu_{bw})$. Now in this model, all of the UV constraints (B.6) translate to

$$\sum_{I=0}^{4n-2} C_I N_I = 0,\ \sum_{I=0}^{4n-2} C_I N_I I = 0,\ \sum_{I=0}^{4n-2} C_I N_I I^2 = 0 \cdots,\ \sum_{I=0}^{4n-2} C_I N_I I^{2n-1} = 0. \tag{B.10}$$

These constraints are obviously satisfied on account of the following identity

$$\sum_{I=0}^{4n-2} \binom{4n-2}{I} (-1)^I I^k = \left(x \frac{\partial}{\partial x} \right)^k (1-x)^{4n-2} \Bigg|_{x=1} = 0, \qquad \forall\, 0 \leq k \leq 2n-1. \tag{B.11}$$

Moving on to the finiteness of the coefficients (B.7), we have to deal with these separately for $m < 0$ and $m > 0$. For $m > 0$, these are all zero (for $n > 1$) as a result of identity (B.11). On the other hand for $m < 0$, we get

$$\alpha_k = -2m^{2n-2k} \widetilde{\Gamma} \left(-n + k + \frac{1}{2} \right). \tag{B.12}$$

This proves that the parity-odd fermion effective action for the lattice Dirac model is finite in arbitrary dimension even in presence of torsion, provided we take into account the contributions from spectator fermions.

B.2 Hall viscosity for Lattice Dirac fermions from Berry connection

The Hamiltonian for lattice Dirac fermions is usually written as

$$\begin{aligned} H_{LD} &= \sum_{m_x, m_y} \Big\{ \left(c^\dagger_{m_x+1,m_y} \sigma_x c_{m_x,m_y} - c^\dagger_{m_x,m_y} \sigma_x c_{m_x+1,m_y} \right) + \left(c^\dagger_{m_x,m_y+1} \sigma_y c_{m_x,m_y} - c^\dagger_{m_x,m_y} \sigma_y c_{m_x,m_y+1} \right) \\ &- \left(c^\dagger_{m_x+1,m_y} \sigma_z c_{m_x,m_y} + c^\dagger_{m_x,m_y} \sigma_z c_{m_x+1,m_y} \right) - \left(c^\dagger_{m_x,m_y+1} \sigma_z c_{m_x,m_y} + c^\dagger_{m_x,m_y} \sigma_z c_{m_x,m_y+1} \right) \\ &+ (2-m) c^\dagger_{m_x,m_y} \sigma_z c_{m_x,m_y} \Big\} \end{aligned} \tag{B.13}$$

We will interprete m_x, m_y as the lattice coordinates, with a fixed integral spacing (i.e. we set the lattice constant to one). Now usually, the matrices $\sigma^\mu = (\sigma_x, \sigma_y, \sigma_z)$ are taken to be the Pauli matrices. In the more general case, we replace these with

$$\sigma^\mu = \sum_{a=1,2} e^\mu_a \sigma^a \tag{B.14}$$

where the σ^a are Pauli matrices. We will take the coefficients e^μ_a to be constant, but importantly, we will take $e^z_a = \delta^z_3$– this means no mixing between time and space directions. This is why the sum in the above equation runs over the spatial directions only. With this, the lattice Hamiltonian becomes

$$\begin{aligned} H_{LD} &= \sum_{m_x,m_y} \Big\{ \Big(c^\dagger_{m_x+1,m_y} e^x_a \sigma^a c_{m_x,m_y} - c^\dagger_{m_x,m_y} e^x_a \sigma^a c_{m_x+1,m_y} \Big) + \Big(c^\dagger_{m_x,m_y+1} e^y_a \sigma^a c_{m_x,m_y} - c^\dagger_{m_x,m_y} e^y_a \sigma^a c_{m_x,m_y+1} \Big) \\ &- \Big(c^\dagger_{m_x+1,m_y} \sigma^3 c_{m_x,m_y} + c^\dagger_{m_x,m_y} \sigma^3 c_{m_x+1,m_y} \Big) - \Big(c^\dagger_{m_x,m_y+1} \sigma^3 c_{m_x,m_y} + c^\dagger_{m_x,m_y} \sigma^3 c_{m_x,m_y+1} \Big) \\ &+ (2-m) c^\dagger_{m_x,m_y} \sigma^3 c_{m_x,m_y} \Big\} \end{aligned} \tag{B.15}$$

In the continuum limit, if we regard

$$c_{m_x+1,m_y} = c_{m_x,m_y} + \partial_x c_{m_x,m_y} + \cdots \tag{B.16}$$

$$c_{m_x,m_y+1} = c_{m_x,m_y} + \partial_y c_{m_x,m_y} + \cdots \tag{B.17}$$

then the Hamiltonian becomes (up to total derivatives)

$$H_{cont.} = \sum_{i=1,2} \int d^2x \Big(\partial_i c^\dagger(x,y) e^i_a \sigma^a c(x,y) - c^\dagger(x,y) e^i_a \sigma^a \partial_i c(x,y) - m c^\dagger(x,y) \sigma^3 c(x,y) \Big) \tag{B.18}$$

which is the correct continuum limit, provided we interpret e^i_a as the spatial frame. More generally, we could take e^i_a to be functions of the lattice coordinates, but we will not do so here.

We now switch to momentum space

$$c_{m_x,m_y} = \sum_{p_x,p_y} e^{ip_x m_x + ip_y m_y} c_{p_x,p_y} \tag{B.19}$$

The Hamiltonian in momentum space becomes

$$H_{LD} = \sum_{p_x,p_y} c^\dagger_{p_x,p_y} \Big(\sin(p_x) e^x_a \sigma^a + \sin(p_y) e^y_a \sigma^a + (2 - m - \cos(p_x) - \cos(p_y))\, \sigma^3 \Big) c_{p_x,p_y} \tag{B.20}$$

Note that p_x, p_y are momentum coordinates, dual to the lattice coordinates; consequently their periodicity is fixed to be 2π once and for all, irrespective of any lattice deformations. Any effect of lattice deformations is encoded only in e^i_a. Clearly, the above Hamiltonian approaches the continuum Dirac Hamiltonian as $p_x, p_y \to 0$.

We now specialize to the case

$$e_1^x = \sqrt{\tau_2}, \quad e_2^x = -\frac{\tau_1}{\sqrt{\tau_2}}, \quad e_2^y = \frac{1}{\sqrt{\tau_2}} \tag{B.21}$$

In this set up, the parameter space is the τ-space, with a torus (coordinatized by p_x, p_y) sitting at each τ, with the corresponding τ dependent metric:

$$g^{-1} = \frac{1}{\tau_2}\left(|\tau|^2 dp_x^2 - 2\tau_1 dp_x dp_y + dp_y^2\right). \tag{B.22}$$

The Hamiltonian becomes

$$H_{LD} = \sum_{p_x,p_y} c^\dagger_{p_x,p_y}\Big\{\sqrt{\tau_2}\sigma^1 \sin(p_x) + \frac{1}{\sqrt{\tau_2}}\sigma_2\Big(\sin(p_y) - \tau_1\sin(p_x)\Big) + \Big(2 - m - \cos(p_x) - \cos(p_y)\Big)\sigma^3\Big\}c_{p_x,p_y}. \tag{B.23}$$

Clearly, this Hamiltonian takes the familiar form[2]

$$H = \sum_{a=1}^{3} c_a(R)\sigma^a \tag{B.24}$$

We read off the coefficients c_a

$$c_1(p_i,\tau) = \sqrt{\tau_2}\sin(p_x) \tag{B.25}$$

$$c_2(p_i,\tau) = \frac{1}{\sqrt{\tau_2}}\Big(\sin(p_y) - \tau_1\sin(p_x)\Big) \tag{B.26}$$

$$c_3(p_i,\tau) = 2 - m - \cos(p_x) - \cos(p_y) \tag{B.27}$$

Hamiltonians of this form are known to have the Berry curvature

$$F = \frac{1}{(c_1^2+c_2^2+c_3^2)^{3/2}}\Big(c_1\, dc_2\wedge dc_3 + c_2\, dc_3\wedge dc_1 + c_3\, dc_1\wedge dc_2\Big) \tag{B.28}$$

We can now straightforwardly pull back the above 2-form on to our parameter space:

$$\begin{aligned} F &= \frac{1}{|c|^3}\Big\{-\Big(\cos(p_x)+\cos(p_y)-(2-m)\cos(p_x)\cos(p_y)\Big)dp_x\wedge dp_y \\ &+ \frac{1}{\tau_2}\Big(\,(2-m-\cos(p_x)-\cos(p_y))\sin^2(p_x)\Big)d\tau_1\wedge d\tau_2 \\ &+ \sin(p_x)\Big(1+\cos(p_x)\,(-2+m+\cos(p_y))\,\Big)d\tau_1\wedge dp_x + \cdots \end{aligned} \tag{B.29}$$

where we do not write the remaining terms because they're quite complicated, and not required in our

[2] The common notation is $d_a\sigma^a$, but we use c_a so that there is no confusion between exterior derivatives and the coefficients d_a.

discussion. In the $p_x, p_y \to 0$ limit (continuum limit), F reduces to

$$\lim_{p\to 0} F|_{\tau=i} = \frac{m}{(m^2+p_x^2+p_y^2)^{3/2}} dp_x \wedge dp_y - \frac{m p_x^2}{(m^2+p_x^2+p_y^2)^{3/2}} d\tau_1 \wedge d\tau_2 + \cdots \tag{B.30}$$

We only show the result at $\tau = i$, but it is not very hard to compute the curvature at generic τ. It is now a straightforward exercise to show that the first term above, when integrated over the Brillouin zone gives the Hall conductivity (which needs further regularization using Pauli-Villar's regulators). Importantly, the second term gives the Berry curvature in the τ-space at $\tau = i$; upon averaging over the entire Brillouin zone (and suitable regularization) one finds precisely the coefficient ζ_H computed in chapter 2:

$$\zeta_h^{(0)} = \int_0^{2\pi}\int_0^{2\pi} dp_x dp_y \sqrt{g}\, F_{\tau\bar{\tau}} \tag{B.31}$$

In fact, the above integral can be performed numerically (without going to the continuum limit), and we show the result in the following figure. We note however, that ζ_H as computed using (B.31) is not a topological

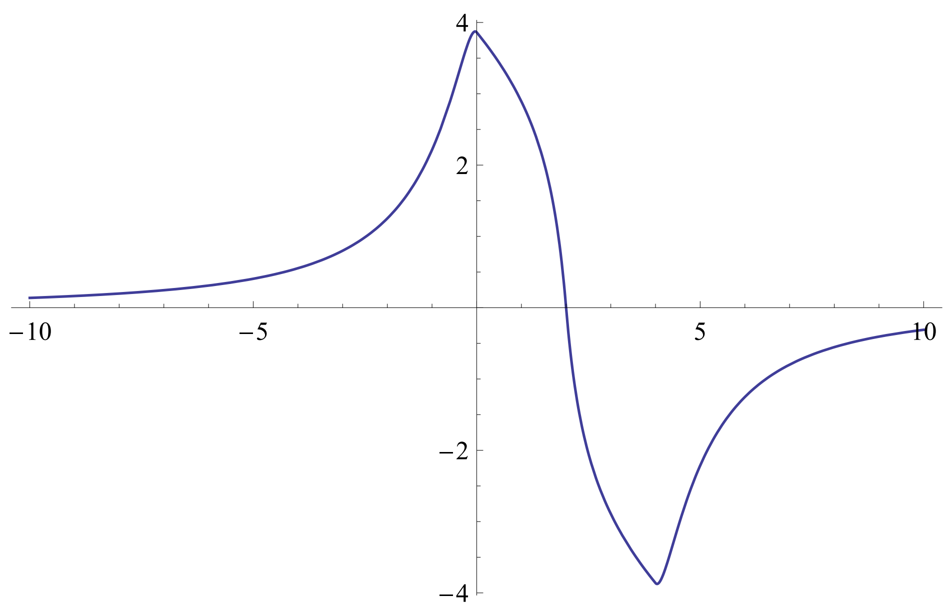

Figure B.1: The Hall viscosity as a function of m at $\tau = i$.

invariant. One can define a topological invariant by integrating F over the fundamental domain in τ-space, but we will leave this to future investigation.

B.3 Energy Spectra for 3+1-d Weyl Fermions

$U(1)$ Magnetic Field

Let us consider the energy spectra of isolated Weyl fermions in the presence of a uniform $U(1)$ magnetic field. This result is well-known but we recount it here to compare it with the case of the torsional magnetic field. We take the spatial geometry to be $\Sigma_3 = \mathbb{R} \times S^1 \times S^1$, parametrized by $x^i = (x^1, x^2, x^3)$ respectively. The $U(1)$ gauge field is taken to be $A = f(x)dy$. We chose the Weyl basis for gamma matrices

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix}, \; \gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{B.32}$$

With this, the Dirac equation for the left and right modes $\psi_L = \frac{1-\gamma^5}{2}\psi_L$, $\psi_R = \frac{1+\gamma^5}{2}\psi_R$ becomes

$$i\left(\partial_0 - \sigma^i(\partial_i + iqA_i)\right)\psi_R = 0, \; i\left(\partial_0 + \sigma^i(\partial_i + iqA_i)\right)\psi_L = 0. \tag{B.33}$$

Let us now concentrate on the left handed modes, and we will drop the L subscript from here on. If ψ is a zero mode of $\partial_0 + \sigma^i(\partial_i + iqA_i)$, then so is $(\partial_0 - \sigma^i(\partial_i + iqA_i))\psi$ (because the A_i are time independent), and hence we try to solve the second order equation[3]

$$\left(\partial_0^2 - \sigma^i(\partial_i + iqA_i)\sigma^j(\partial_j + iqA_j)\right)\psi \quad = \quad 0. \tag{B.34}$$

Using $\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k$ and the fact that p_2, p_3 are good quantum numbers, we find that energy eigenfunctions must satisfy

$$\left(-\partial_1^2 + (p_2 + qA_2)^2 + p_3^2 + \frac{q}{2}\epsilon_{ijk}F_{ij}\sigma^k\right)\psi = E^2\psi. \tag{B.35}$$

Now let us consider the special case of a uniform magnetic field. Choose $A = Bx^1dx^2$ corresponding to a uniform magnetic field B parallel to x^3. Substituting into Eq. B.35 we find

$$\left(-\partial_1^2 + (qB)^2\left(x^1 + \frac{p_2}{qB}\right)^2 + p_3^2 + qB\sigma^3\right)\psi = E^2\psi \tag{B.36}$$

[3]Eventually, we should be careful to discard solutions of $(\partial_0 - \sigma^i(\partial_i + iqA_i))\psi = 0$

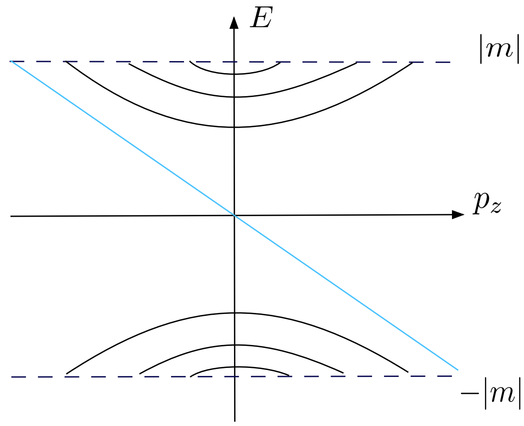

Figure B.2: An illustration of the energy spectrum for a left-handed Weyl fermion in the presence of a uniform background $U(1)$ magnetic field. The linear dispersing mode is the zeroth Landau level and the gapped modes are higher Landau levels (or bulk states). We have drawn a mass cut-off $\pm|m|$ to represent the energy at which the low-energy chiral modes begin to couple with the bulk modes in the gapped topological insulator and lose their chirality and boundary localization properties.

which is the simple harmonic oscillator equation with frequency $|qB|$. The dispersion relations are

$$E(\ell, p_3, \sigma_3) = \pm\left(p_3^2 + 2|qB|(\ell + \frac{1}{2}) + qB\sigma_3\right)^{1/2}, \qquad \ell = 0, 1, 2, \cdots, \sigma_3 = \pm 1 \tag{B.37}$$

and the wavefunctions are

$$\psi(\ell, p_3, \sigma_3) = A_\ell e^{ip_3x^3 + ip_2x^2} e^{-|qB|x_1^2/2} H_\ell\left(\sqrt{|qB|}(x^1 + \frac{p_2}{qB})\right)|\sigma_3\rangle \tag{B.38}$$

with $A_\ell = \frac{1}{2^\ell \ell!}(|qB|)^{1/4}$ being the normalization.

The solutions corresponding to $\ell = 0, \sigma_3 = -\text{sign}(qB)$ are the gapless modes $E(p_3) = \pm p_3$. But note that we still need to eliminate the spurious solutions which satisfy $(i\partial_0 - i\sigma^i(\partial_i + iqA_i))\psi = 0$, i.e.

$$\begin{pmatrix} E + p_3 & (p_1 - ieB(x^1 + p_2/qB)) \\ p_1 + iqB(x^1 + p_2/qB) & E - p_3 \end{pmatrix}\psi(\ell, p_3, \sigma) = 0. \tag{B.39}$$

Thus, the $E = \text{sign}(qB)p_3$ mode gets eliminated, and we are left with only one gapless branch

$$E = -\text{sign}(qB)p_3. \tag{B.40}$$

The number of states for each p_3 is given by $\frac{|q\Phi_B|}{2\pi}$, which comes from demanding $-\frac{L_1}{2} < \frac{p_2}{qB} < \frac{L_1}{2}$; here Φ_B is the magnetic flux. If we had chosen to study the right-handed chirality then $-\text{sign}(qB)p_3$ would have been eliminated and the remaining mode would be $E = +\text{sign}(qB)p_3$.

Torsion Magnetic Field

Now set the $U(1)$ magnetic field to zero, and consider the following co-frame and its dual frame

$$e^0 = dt,\ e^1 = dx^1,\ e^2 = dx^2,\ e^3 = dx^3 + f(x^1)dx^2, \tag{B.41}$$

$$\underline{e}_0 = \partial_0,\ \underline{e}_1 = \partial_1,\ \underline{e}_2 = \partial_2 - f(x^1)\partial_3,\ \underline{e}_3 = \partial_3.$$

We will set the spin connection to zero for simplicity. In this case, the above co-frame is torsional with $T^3 = de^3 = \partial_1 f(x^1)dx_1 \wedge dx_2$. The Dirac operator becomes

$$i\not{D} = i\left(\gamma^0\partial_0 + \gamma^1\partial_1 + \gamma^2(\partial_2 - f(x^1)\partial_3) + \gamma^3\partial_3\right). \tag{B.42}$$

For the left-handed Weyl fermions, the Dirac equation reduces to

$$i\left(\partial_0 + \sigma^1\partial_1 + \sigma^2(\partial_2 - f(x^1)\partial_3) + \sigma^3\partial_3\right)\psi_L = 0, \tag{B.43}$$

and since p_2, p_3 are good quantum numbers, we can write the above as

$$\left(i\partial_0 + i\sigma^1\partial_1 - \sigma^2(p_2 - f(x^1)p_3) - \sigma^3 p_3\right)\psi_L = 0. \tag{B.44}$$

We notice that this looks exactly like the Dirac equation with a $U(1)$ gauge field $A = -\frac{p_3}{q}f(x^1)dx_2 = -\frac{p_3}{q}\delta e^3$ and field strength $F = -\frac{p_3}{q}T^3$. Thus (B.35) becomes

$$\left(-\partial_1^2 + (p_2 - p_3\delta e^3_2)^2 + p_3^2 - \frac{p_3}{2}\epsilon_{ijk}T^3_{ij}\sigma^k\right)\psi = E^2\psi. \tag{B.45}$$

To understand the spectrum, we first notice that for $p_3 = 0$, the spectrum is just $E(p_1, p_2, p_3 = 0) = \pm(p_1^2 + p_2^2)^{1/2}$. This must be the case because the $p_3 = 0$ mode is not sensitive to translations/torsion. In order to proceed, we choose $f(x^1) = Cx^1$, this leads to a uniform torsion magnetic field $T^3 = Cdx^1 \wedge dx^2$. The spectrum for $p_3 \neq 0$ is similar to the case of the uniform magnetic field

$$E(\ell, p_3, \sigma_3) = \pm\left(p_3^2 + 2|Cp_3|(\ell + \frac{1}{2}) - Cp_3\sigma^3\right)^{1/2} \qquad \ell = 0, 1, 2\cdots, \sigma_3 = \pm 1. \tag{B.46}$$

Notice that for $\ell = 0, \sigma_3 = \mathrm{sign}(Cp_3)$, the spectrum is simply given by $E = \pm p_3$. But once again we have

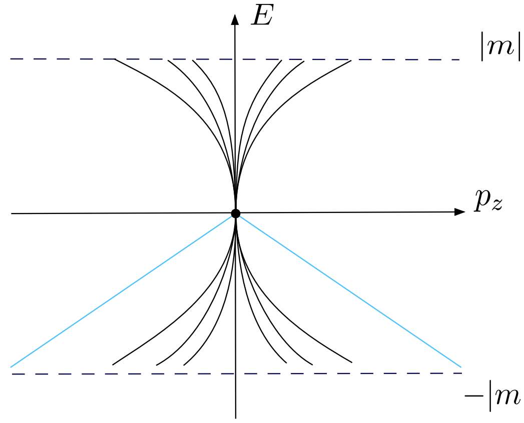

Figure B.3: An illustration of the energy spectrum for a 3+1-d left-handed Weyl fermion in the presence of a uniform background torsion magnetic field. The downward dispersing (blue) curve represents the zeroth Landau level while the non-linear (black) curves represent higher Landau levels as given in Eq. B.46. This should be compared with the result for a $U(1)$ magnetic field shown in Fig. B.2.

to be careful to eliminate the spurious zero mode. This is delicate, so let us work this out explicitly; the spurious mode satisfies

$$\begin{pmatrix} E+p_3 & p_1 - i(p_2 - Cp_3x^1) \\ p_1 + i(p_2 - Cp_3x^1) & E-p_3 \end{pmatrix} \psi = 0. \tag{B.47}$$

We find that $E = -\mathrm{sign}(Cp_3)p_3$ should be eliminated. Thus the remaining gapless ($p_3 \neq 0$) mode is

$$E = \mathrm{sign}(Cp_3)p_3,\ \sigma_3 = \mathrm{sign}(Cp_3). \tag{B.48}$$

The opposite chirality mode will have $E = -\mathrm{sign}(Cp_3)p_3,\ \sigma_3 = -\mathrm{sign}(Cp_3)$. This is different from the case of the $U(1)$ magnetic field in two important ways. First, the number of states for each $p_3 \neq 0$ is now given by $\frac{|p_3\Phi_T|}{2\pi}$, where $\Phi_T = CL_1L_2$ is the torsion magnetic flux. Second the right-handed and left-handed fermions do not give rise to 1+1-d fermion branches with a constant group velocity. In fact, one chirality disperses upward and the other chirality disperses downward. The fact that the association between the different 1+1-d branches and the chirality is modified is exactly what gives rise to the torsional contribution to the chiral anomaly.

Appendix C

Supplement to Part II

In this appendix, we will present some supplementary details relevant to our discussion in part II of this thesis.

C.1 Vasiliev Higher spin gravity

In this section, we will present a short review of the non-linear Vasiliev higher spin equations in general dimension $d+1$ in terms of vector oscillators.[1] Of course, this is not meant to be pedagogical by any means, as the details are not relevant to our discussion in this thesis – our aim here is to merely present the Vasiliev equations so as to facilitate comparison with our RG equations. For more details on the Vasiliev theory, we refer the reader to Refs. [64, 65, 66, 67, 68].

Let $\{Y_i^A\}$ and $\{Z_j^A\}$ be $Sp(2)\times O(2,d)$ variables, where upper-case latin indices $A, B\cdots$ stand for $O(2,d)$ vector indices, while $i, j, \cdots$ stand for $Sp(2)$ indices. The $Sp(2)$ invariant product is defined by $Y^{A\,i}Y_i^B \equiv \epsilon^{ij}Y_i^AY_j^B$. We define the *star-product* between two functions $f(Y,Z)$ and $g(Y,Z)$ as

$$f(Y,Z)\star g(Y,Z) = N^{2D}\int d^{2D}Ud^{2D}V\; e^{-2U_i^AV_A^i}f(Y+U,Z+U)g(Y+V,Z-V) \tag{C.1}$$

where $D=d+2$ and N^{2D} is an appropriate normalization constant chosen such that $f\star 1 = f$. It is easy to check that this implies the relations

$$\begin{aligned}
Y_i^A\star Y_j^B &= Y_i^AY_j^B+\frac{1}{2}\eta^{AB}\epsilon_{ij},\;\; Z_i^A\star Z_j^B = Z_i^AZ_j^B-\frac{1}{2}\eta^{AB}\epsilon_{ij}\\
Y_i^A\star Z_j^B &= Y_i^AZ_j^B-\frac{1}{2}\eta^{AB}\epsilon_{ij},\;\; Z_i^A\star Y_j^B = Z_i^AY_j^B+\frac{1}{2}\eta^{AB}\epsilon_{ij}
\end{aligned} \tag{C.2}$$

[1] We note that the case $d=3$ is special, in that the Vasiliev equations can be formulated in terms of twistor variables, and admit the two versions referred to as A type and B type. In particular, it is not known how to construct the B type theory in terms of vector oscillators.

We introduce the function $\mathcal{K}(t) = e^{-2tz^i y_i}$, where $y_i = Y_i^{-1}$ and $z_i = Z_i^{-1}$. For $t = 1$ this is called the *Kleinian*, and will be denoted by $\mathcal{K}$. It has the important property that

$$\mathcal{K} \star \mathcal{K} = 1, \;\; \mathcal{K} \star f(Y, Z) \star \mathcal{K} = \tilde{f}(Y, Z) \tag{C.3}$$

where $\tilde{f}(Y, Z) = f(Y^A - 2Y^{-1}\delta^A_{-1}, Z^A - 2Z^{-1}\delta^A_{-1})$.

The Vasiliev system is described by two one forms $\mathcal{W}(x|Y, Z) = \mathcal{W}_I(x|Y, Z)dx^I$ and $\mathcal{S}(x|Y, Z) = S^i_A(x|Y, Z)dZ^A_i$, and a zero-form $B(x|Y, Z)$. The Vasiliev equations are given by

$$\begin{aligned}
&d_x\mathcal{W} + \mathcal{W} \star \mathcal{W} = 0 \\
&d_x B + \mathcal{W} \star B - B \star \widetilde{\mathcal{W}} = 0 \\
&d_Z\mathcal{W} + d_x\mathcal{S} + \mathcal{W} \star \mathcal{S} + \mathcal{S} \star \mathcal{W} = 0 \\
&d_Z B + \mathcal{S} \star B - B \star \widetilde{\mathcal{S}} = 0 \\
&d_Z\mathcal{S} + \mathcal{S} \star \mathcal{S} = \frac{2}{3} dZ_i^{-1} dZ^i_{-1}\; B \star \mathcal{K}
\end{aligned} \tag{C.4}$$

In addition, one must impose the appropriate $Sp(2)$ invariance constraints on the above fields, in order for them to describe physical higher spin fields. Note that B transforms in the *twisted* adjoint representation, and in particular the covariant derivatives for B feature the twisted commutators $(\mathcal{W} \star B - B \star \widetilde{\mathcal{W}})$ and $(\mathcal{S} \star B - B \star \widetilde{\mathcal{S}})$. By redefining the 0-form as

$$\mathfrak{B} = B \star \mathcal{K} \tag{C.5}$$

the new 0-form $\mathfrak{B}$ transforms in the adjoint representation, and the twisting can be partially removed from the Vasiliev equations.

C.2 Infinite Jet bundles

In this section, we want to give a general discussion on the language of infinite jet bundles introduced in the main text. The elementary fields could be scalars, or fermions (or more general fields), denoted by $\psi^m(x)$; more formally these are sections of a vector bundle $E \to M_d = \mathbb{R}^d$, which could be a complex line bundle, or the spin bundle (or a more general bundle depending on the spin), tensored with itself N times. We will label by $\Gamma(E)$ the space of all C^∞ sections of E. Corresponding to E, there exists the *infinite jet bundle* over M_d

$$\pi_\infty : J^\infty(E) \mapsto M_d \tag{C.6}$$

which is defined as follows: two sections $\psi^m(x)$ and $\chi^m(x)$ of E are said to have the same rth jet at a point $x \in M_d$ if

$$\frac{\partial^k}{\partial x^{a_1} \cdots \partial x^{a_k}} \psi^m \bigg|_x = \frac{\partial^k}{\partial x^{a_1} \cdots \partial x^{a_k}} \chi^m \bigg|_x , \quad 0 \leq k \leq r \tag{C.7}$$

For any given section $\psi^m(x)$ of E, the rth jet of ψ^m at x, denoted by $j^r_x\psi$, is the equivalence class of all sections which have the same rth jet at x as ψ^m. The rth *jet bundle* $\pi_r : J^r(E) \mapsto M_d$ of E over M_d is then defined by

$$J^r(E) = \{ j^r_x\psi \ : \ \forall x \in M_d, \psi \in \Gamma(E) \} \tag{C.8}$$

with the natural projection $\pi_r : j^r_x\psi \mapsto x$. The *infinite jet bundle* $J^\infty(E)$ of E is defined as above, with $r \to \infty$. Given a section $\psi^m(x)$ of E, we can naturally construct a section $j^\infty\psi^m(x)$ of $J^\infty(E)$ by taking its infinite jet at every point x. This is called the *prolongation* map

$$j^\infty : \Gamma(E) \mapsto \Gamma(J^\infty(E)) \tag{C.9}$$

In simple terms, the prolongation map sends

$$\Gamma(E) \ni \psi^m(x) \mapsto \Psi^m[\psi](x) = \left(\psi^m(x), \frac{\partial \psi^m}{\partial x^{a_1}}(x), \frac{\partial^2 \psi^m}{\partial x^{a_1} \partial x^{a_2}}(x), \cdots \right) \in \Gamma(J^\infty(E)). \tag{C.10}$$

The important point is that a differential operator can be thought of as a section of the *Endormorphism bundle* $\mathrm{End}(J^\infty(E))$ of $J^\infty(E)$, i.e. it is simply a local linear transformation when thought of as acting on sections of the jet bundle. For instance, the derivative operator $\frac{\partial}{\partial x^\mu}$ can be thought of as the matrix

$$\mathbb{P}_\mu = \begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \cdots \\ \vdots & \vdots & \vdots & & \end{pmatrix} \tag{C.11}$$

acting on the jet prolongation, with each entry corresponding to a map between tensors of different ranks. In more precise notation, $\mathbb{P}_\mu$ is a section of $\mathrm{End}(J^\infty(E))$. Acting on a vector $j^\infty_x\psi^m$ at x, it may be defined as the push-forward of the derivative operator:

$$\left(\mathbb{P}_\mu \cdot j_x^\infty \psi^m\right)(x) = j_x^\infty \left(\partial_\mu \psi^m\right)(x) \tag{C.12}$$

or in terms of a commuting diagram

$$\begin{tikzcd} \Gamma(E) \arrow[r, "\partial_\mu"] \arrow[d, "j_x^\infty"'] & \Gamma(E) \arrow[d, "j_x^\infty"] \\ J_x^\infty(E) \arrow[r, "\mathbb{P}_\mu"'] & J_x^\infty(E) \end{tikzcd}$$

where, by $J_x^\infty(E)$ we mean the fiber of the infinite jet bundle over x. Similarly, we may also translate a more general differential operator

$$\epsilon(x,y) = i\xi(x)\delta^d(x-y) + \xi^\mu(x)\partial_\mu^{(x)}\delta^d(x-y) + \cdots \tag{C.13}$$

to the language of jets, as a section of $\mathrm{End}(J^\infty(E))$ acting on the vector $j_x^\infty \psi^m$ at x:

$$\left(\mathcal{E}[\epsilon]{\cdot}j_x^\infty \psi^m\right)(x) = j_x^\infty \left(\epsilon{\cdot}\psi^m\right)(x), \quad \mathcal{E}[\epsilon](x) = \begin{pmatrix} i\xi(x) & \xi^\nu(x) & \cdots \\ i\partial_\mu \xi(x) & i\delta_\mu^\nu \xi(x) + \partial_\mu \xi^\nu(x) & \cdots \\ \vdots & \vdots & \end{pmatrix} \tag{C.14}$$

In terms of commuting diagrams:

$$\begin{tikzcd} \Gamma(E) \arrow[r, "\epsilon"] \arrow[d, "j_x^\infty"'] & \Gamma(E) \arrow[d, "j_x^\infty"] \\ J_x^\infty(E) \arrow[r, "\mathcal{E}[\epsilon]"'] & J_x^\infty(E) \end{tikzcd}$$

There is another slightly more convenient notation which makes the discussion of infinite jets potentially more tractable. Let us introduce the auxiliary variable $\mathrm{Y}^a \in \mathbb{R}^d$; we emphasize that this is completely independent of the spacetime coordinates. Now we write the jet prolongation as

$$\Phi^m[\psi](x) = j_x^\infty \psi^m(x) = \psi^m(x) + \mathrm{Y}^a \partial_a \psi^m(x) + \frac{1}{2!}\mathrm{Y}^a \mathrm{Y}^b \partial_a \partial_b \psi^m(x) + \cdots \tag{C.15}$$

In this notation, we can write the jet-prolongation of differential operators as

$$\mathcal{E}[\epsilon](x) = \sum_{r,s=0}^{\infty} \frac{1}{r!}\epsilon_{a_1\cdots a_s;b_1\cdots b_r}(x)\mathrm{Y}^{b_1}\cdots \mathrm{Y}^{b_r}\frac{\partial}{\partial \mathrm{Y}_{a_1}}\cdots\frac{\partial}{\partial \mathrm{Y}_{a_s}}, \quad \epsilon_{a_1\cdots a_s;b_1\cdots b_r}(x) = \partial_{b_1}\cdots\partial_{b_r}\epsilon_{a_1\cdots a_s}(x) \tag{C.16}$$

Note that Y and ∂_{Y} generate the Weyl algebra, so in this language we can think of our bilocal sources as well as vevs as being Weyl-algebra valued. It would be an interesting exercise to try and rewrite the non-local renormalization group equations in terms of local equations using this formalism. We leave this to future work.

It is also convenient to introduce a bilinear form on the fibres of $J^{\infty}(E)$ which, intuitively speaking, we want to look like

$$\langle\cdot,\cdot\rangle = \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \vdots & \vdots & \vdots & & \end{pmatrix} \otimes \epsilon_{\alpha\beta} \otimes \delta_{mn} \tag{C.17}$$

where $\epsilon_{\alpha\beta}$ and δ_{mn} are the metrics for spinor and $O(N)$ indices respectively. More precisely then, we define $\langle\cdot,\cdot\rangle$ as

$$\langle j^{\infty}_x \psi^m, j^{\infty}_x \chi^n\rangle(x) = \delta_{mn}\widetilde{\psi}^m(x)\chi^n(x) \tag{C.18}$$

With this, we naturally get an *inner product* $\langle\cdot,\cdot\rangle_{\Gamma(J^{\infty}(E))}$ on sections of $J^{\infty}(E)$

$$\langle\Phi^m, \Psi^n\rangle_{\Gamma(J^{\infty}(E))} = \int_{M_d} d^d x \sqrt{g^{(0)}(x)}\, \langle\Phi^m(x), \Psi^n(x)\rangle \tag{C.19}$$

where $\Phi, \Psi \in \Gamma(J^{\infty}(E))$, and we have made the (metric) measure on spacetime explicit. The point of choosing this inner product of course, is that *on prolongations*, it agrees with the standard inner product on $\Gamma(E)$, namely

$$\langle j^{\infty}\psi^m, j^{\infty}\chi^n\rangle_{\Gamma(J^{\infty}(E))} = \langle\psi^m, \chi^n\rangle_{\Gamma(E)} = \int_{M_d} d^d x\, \sqrt{g^{(0)}(x)}\, \delta_{mn}\widetilde{\psi}^m(x)\chi^n(x) \tag{C.20}$$

In terms of the auxiliary vectors Y introduced above, this amounts to setting $\mathrm{Y} = 0$ after all the contractions are performed.

C.3 Renormalization group: Details

In this appendix, we spell out the details of the derivation of the exact renormalization group equations. We take the regulated action to be

$$S_{Bos.} = S_0 + S_1 \tag{C.21}$$

$$S_0 = -\frac{1}{z^{d-2}} \int_{x,y,u} \phi^*_m(x) D^{(0)}_\mu(x,y) D^{(0)}_\mu(y,u) \phi^m(u) \tag{C.22}$$

$$S_1 = \frac{1}{z^{d-2}} \int_{x,y} \phi^*_m(x) \mathcal{B}(x,y) \phi^m(y) + U \tag{C.23}$$

Next, we run the two-step RG process:

Step 1: In Step 1 of RG, we want to integrate out a shell of fast modes, and investigate how that changes the sources. In order to perform this integration, we use Polchinski's exact RG formalism. We start by lowering $M \to \lambda M$, where $\lambda = 1 - \varepsilon$. Since this has the interpretation of integrating out fast modes, we can extract the change in the sources $\delta_\varepsilon \mathcal{B} = -\varepsilon M \frac{d}{dM} \mathcal{B}$ and $\delta U = -\varepsilon M \frac{d}{dM} U$ by imposing

$$M \frac{d}{dM} Z = Z_0^{-1} \int [d\phi \, d\phi^*] \left\{ \left(M \frac{d}{dM} e^{iS_0} \right) e^{iS_1} + e^{iS_0} \left(M \frac{d}{dM} e^{iS_1} \right) - Z_0^{-1} e^{iS_0 + iS_1} M \frac{d}{dM} Z_0 \right\} = 0 \tag{C.24}$$

where the last term above is from the normalization of the partition function[2], as in (6.11). Evaluating the first term, we find

$$\begin{aligned} M \frac{d}{dM} e^{iS_0} &= -\frac{i}{z^{d-2}} e^{iS_0} \int \phi^*_m \cdot \left(M \frac{d}{dM} D^2_{(0)} \right) \cdot \phi^m \\ &= \frac{i}{z^{d-2}} e^{iS_0} \int \phi^*_m \cdot D^{(0)2} \cdot \Delta_B \cdot D^{(0)2} \cdot \phi^m \\ &= -i z^{d-2} \int_{x,y} \Delta_B(x,y) \left\{ \frac{\delta^2}{\delta\phi^m(x) \delta\phi^*_m(y)} - i \frac{\delta^2 S_0}{\delta^m \phi(x) \delta_m \phi^*(y)} \right\} e^{iS_0}. \end{aligned} \tag{C.25}$$

where we have defined $\Delta_B = M \frac{d}{dM} (D^{(0)}_\mu D^{(0)}_\mu)^{-1}$. The second term in (C.25) cancels with the contribution from the normalization. Therefore, integrating by parts from equations (C.24) and (C.25), we are left with

$$M \frac{d}{dM} e^{iS_1} - i z^{d-2} \int_{x,y} \Delta_B(x,y) \frac{\delta^2}{\delta\phi^*_m(x) \delta\phi^m(y)} e^{iS_1} = 0 \tag{C.26}$$

[2] We have defined $Z_0 = \int [d\phi \, d\phi^*] e^{iS_0}$.

Evaluating this term by term, we find

$$i\left\langle M\frac{d}{dM}U+\frac{1}{z^{d-2}}\phi_m^*\boldsymbol{\cdot}M\frac{d}{dM}\mathcal{B}\boldsymbol{\cdot}\phi^m\right\rangle=-\left\langle N\mathrm{Tr}\ \Delta_B\boldsymbol{\cdot}\mathcal{B}+\frac{i}{z^{d-2}}\phi_m^*\boldsymbol{\cdot}\mathcal{B}\boldsymbol{\cdot}\Delta_B\boldsymbol{\cdot}\mathcal{B}\boldsymbol{\cdot}\phi^m\right\rangle \tag{C.27}$$

As the notation suggests, the above equations should be regarded as valid inside the path integral. From the above equation, we can now read off the change in the sources (if we treat the 1 and $\phi_m^*(x)\phi^m(y)$ as independent operators)

$$\delta_\varepsilon\mathcal{B}=-\varepsilon M\frac{d}{dM}\mathcal{B}=\varepsilon\ \mathcal{B}\boldsymbol{\cdot}\Delta_B\boldsymbol{\cdot}\mathcal{B} \tag{C.28}$$

$$\delta_\varepsilon U=-\varepsilon M\frac{d}{dM}U=-i\varepsilon N\ \mathrm{Tr}\ \Delta_B\boldsymbol{\cdot}\mathcal{B} \tag{C.29}$$

Step 2: Next in step 2, we perform a $CU(L_2)$ transformation

$$\mathcal{L}(x,y)=\delta^d(x-y)+\varepsilon\ zW_z^{(0)}(x,y) \tag{C.30}$$

to bring the cutoff back while changing the conformal factor of the metric. Having done this, we label the sources $\mathcal{B}(z),\mathcal{B}(z+\varepsilon z)$ and $U(z),U(z+\varepsilon z)$. Together with step 1, we thus conclude

$$\mathcal{B}(z+\varepsilon z)=\mathcal{B}(z)-\varepsilon\left[W_z^{(0)},\mathcal{B}\right]_{\boldsymbol{\cdot}}+\varepsilon\ \mathcal{B}\boldsymbol{\cdot}\Delta_B\boldsymbol{\cdot}\mathcal{B} \tag{C.31}$$

$$U(z+\varepsilon z)=U(z)-i\varepsilon N\ \mathrm{Tr}\ \Delta_B\boldsymbol{\cdot}\mathcal{B} \tag{C.32}$$

In this way, the renormalization group extends the sources defined at a given value of z to all of the bulk RG mapping space. Redefining Δ_B as $\Delta_B=\frac{M}{z}\frac{d}{dM}\left(D_{(0)}^2\right)^{-1}$, we recover equations (7.7) and (7.8).

Callan-Symanzik equations

Similarly, we can derive an expression for the Callan-Symanzik equations of the bi-local operator $\hat{\Pi}(x,y)=\phi_m^*(y)\phi^m(x)$. Again we run the two step RG process:

Step 1: Defining normalized correlation functions by

$$\langle\mathcal{O}\rangle\equiv\frac{\int[d\phi\ d\phi^*]\,\mathcal{O}\,e^{iS}}{\int[d\phi\ d\phi^*]\,e^{iS}} \tag{C.33}$$

it is straightforward to demonstrate the relationship

$$M d_M \Pi \equiv M d_M \left\langle \hat{\Pi} \right\rangle = \mathrm{Tr}\left\{ \Delta_B \boldsymbol{\cdot} \left\langle \frac{\delta S_1}{\delta \phi^*_m} \frac{\delta \hat{\Pi}}{\delta \phi^m} + \frac{\delta \hat{\Pi}}{\delta \phi^*_m} \frac{\delta S_1}{\delta \phi^m} - i \frac{\delta^2 \hat{\Pi}}{\delta \phi^*_m \delta \phi^m} \right\rangle \right\} \tag{C.34}$$

The right hand side can be calculated explicitely. The result is

$$\delta_\varepsilon \Pi = -\varepsilon M d_M \Pi = i\varepsilon \; N \; z \Delta_B - \varepsilon z \; \Delta_B \boldsymbol{\cdot} \mathcal{B} \boldsymbol{\cdot} \Pi - \varepsilon z \; \Pi \boldsymbol{\cdot} \mathcal{B} \boldsymbol{\cdot} \Delta_B \tag{C.35}$$

or more compactly,

$$\delta_\varepsilon \Pi = i\varepsilon z \; N \, \Delta_B + \varepsilon z \; \mathrm{Tr}\left\{ \gamma \boldsymbol{\cdot} \Pi \right\} \tag{C.36}$$

where we define

$$\gamma(x,y;u,v) \equiv -\frac{\delta \beta^{(\mathcal{B})}(u,v)}{\delta \mathcal{B}(y,x)} = -\delta(x-u) \left(\Delta_B \boldsymbol{\cdot} \mathcal{B} \right)(y,v) - \left(\mathcal{B} \boldsymbol{\cdot} \Delta_B \right)(u,x) \, \delta(v-y) \tag{C.37}$$

Step 2: We perform a $CU(L_2)$ transformation as given in (C.30). The result is

$$\Pi(z+\varepsilon z; x,y) = \Pi(z;x,y) - \varepsilon \, z \left[W_z^{(0)}, \Pi \right]_{\boldsymbol{\cdot}} + i\varepsilon z \; N \, \Delta_B + \varepsilon \, z \, \mathrm{Tr}\left\{ \gamma(x,y;u,v) \boldsymbol{\cdot} \Pi(v,u) \right\} \tag{C.38}$$

As with the beta function derived above, this relationship can be extended into the bulk. Denoting the bulk momentum as $\mathcal{P}$, we have

$$\mathcal{D}_z^{(0)} \mathcal{P} \equiv \partial_z \mathcal{P} + \left[\mathcal{W}_z^{(0)}, \mathcal{P} \right]_{\boldsymbol{\cdot}} = iN \, \Delta_B + \mathrm{Tr}\left\{ \boldsymbol{\gamma}(x,y;u,v) \boldsymbol{\cdot} \mathcal{P}(v,u) \right\} \tag{C.39}$$

where $\boldsymbol{\gamma}(z;x,y;u,v) \equiv -\frac{\delta \boldsymbol{\beta}^{(\mathfrak{B})}(z;u,v)}{\delta \mathfrak{B}(z;y,x)}$ is the bulk extension of γ.

C.4 Comments on Exact RG as Vasiliev theory

In this section, we make some observations about the additional structure which we expect to emerge by shifting to the language of principal bundles (see [77] for details), and we make comparison with Vasiliev theory. Let $\mathcal{G} \mapsto P_{\mathcal{G}} \mapsto M_{d+1}$ be a principal bundle over M_{d+1} (with $\mathcal{G}$ being the structure group), of which $J^\infty_{bulk}(E)$ is an associated vector bundle. In particular, we may take $P_{\mathcal{G}}$ to be the frame bundle $\mathrm{Fr}(J^\infty_{bulk}(E))$. Let Z^α be local coordinates on the (infinite-dimensional) fibers of $P_{\mathcal{G}}$. Given a local section $\Sigma : M_{d+1} \mapsto P_{\mathcal{G}}$, we may choose local coordinates[3] (x, Z) on the total space of $P_{\mathcal{G}}$ *adapted* to the section, which is to say the section is given by $Z = 0$ in these coordinates (see figure C.1). Vector fields on $P_{\mathcal{G}}$ of the form $V = V^\alpha \frac{\partial}{\partial Z^\alpha}$ which point along the fiber directions are referred to as *vertical* vector fields.

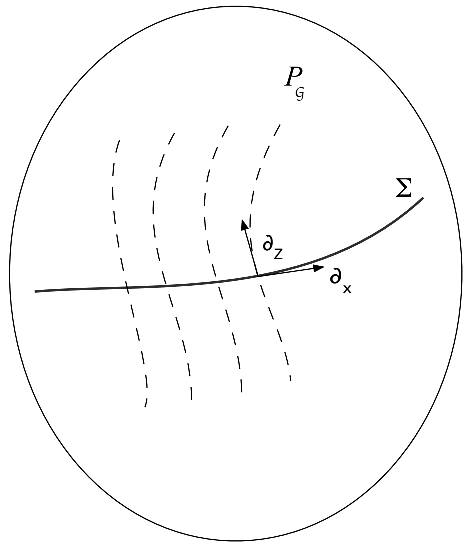

Figure C.1: A pictorial representation of the principal bundle structure.

In order to specify what it means to be *horizontal*, we need to define the notion of a connection on the $\mathcal{G}$-bundle. An *Ehresmann connection* $\boldsymbol{\omega}$ on $P_{\mathcal{G}}$ is a $\mathcal{G}$-equivariant one-form on the total space, valued in the Lie-algebra of $\mathcal{G}$, and may be written locally on $P_{\mathcal{G}}$ as

$$\boldsymbol{\omega} = \boldsymbol{\omega}_I(x, Z|Y)dx^I + \boldsymbol{\omega}_\alpha(x, Z|Y)dZ^\alpha \tag{C.40}$$

Note that both $\boldsymbol{\omega}_I$ and $\boldsymbol{\omega}_\alpha$ are valued in the Lie-algebra of $\mathcal{G}$, which is manifested above by their Y dependence. Having defined the connection, we now refer to vector fields on $P_{\mathcal{G}}$ in the kernel of $\boldsymbol{\omega}$ as *horizontal*. In terms of the local coordinate basis of 1-forms (dx^I, dZ^α), we may think of dx^I as being horizontal because they kill all vertical vector fields, while dZ^α are simply normal to the section Σ. The pull back of the connection by the section, $\Sigma^{-1}\boldsymbol{\omega}$, is a qualified connection 1-form on associated vector bundles, and is what is usually called the connection (or gauge field) in the physics literature. It is this piece which may be identified with what we referred to as the connection over $J^\infty_{bulk}(E)$ in the previous section

$$\mathcal{W}(x|Y) = \boldsymbol{\omega}_I(x, 0|Y)dx^I \tag{C.41}$$

[3] In this section, the symbol x should be taken to stand for $x^I = (z, x^\mu)$.

As was explained in [77], the remaining piece $\boldsymbol{\omega}_\alpha(x,0|Y)dZ^\alpha$ (evaluated on the section) is called the *Faddeev-Popov ghost* in physics, and we suggestively label it as

$$S(x|Y) = \boldsymbol{\omega}_\alpha(x,0|Y)dZ^\alpha \tag{C.42}$$

The fact that S is a 1-form means that it anti-commutes with itself, which is why the ghost is taken to be Grassmann.

The exterior derivative d on the total space $P_{\mathcal{G}}$ can also be separated with respect to our coordinate system into a horizontal and a vertical piece: $d = \boldsymbol{d}_x + d_Z$. The vertical piece d_Z is commonly referred to as the *BRST operator* in physics. The *curvature* 2-form for $\boldsymbol{\omega}$[4]

$$\begin{aligned} \mathcal{F}_{\boldsymbol{\omega}} &= d\boldsymbol{\omega} + \boldsymbol{\omega} \wedge_\star \boldsymbol{\omega} \\ &= \boldsymbol{d}_x\mathcal{W} + \mathcal{W} \wedge_\star \mathcal{W} + d_Z\mathcal{W} + \boldsymbol{d}_x S + \{\mathcal{W}, S\}_\star + d_Z S + S \wedge_\star S \end{aligned} \tag{C.43}$$

consequently splits up into a horizontal, a vertical and a mixed term. A fundamental property of the curvature 2-form is that it is purely horizontal (a quick proof for physicists can be found in [77]). This implies that the curvature 2-form must not have any dZ^α legs, which lead us to conclude that

$$d_Z\mathcal{W} + \boldsymbol{d}_x S + \{\mathcal{W}, S\}_\star = 0 \tag{C.44}$$

$$d_Z S + S \wedge_\star S = 0 \tag{C.45}$$

These relations are referred to as the BRST equations in physics. Of course, the charged 0-form $\mathfrak{B}$ has its own BRST relation as well, which encodes its tensorial transformation property under gauge transformations

$$d_Z\mathfrak{B} + [S, \mathfrak{B}]_\star = 0 \tag{C.46}$$

At this point, putting all of the above BRST equations together with the renormalization group equations

[4]Here $d_Z S$ is to be interpreted appropriately as $d_Z(\boldsymbol{\omega}_\alpha dZ^\alpha)|_{Z=0}$.

(7.3) and (7.4), we obtain the full set of equations satisfied by the various pieces of our Ehresmann connection

$$\begin{aligned}
&\boldsymbol{d}_x\mathcal{W} + \mathcal{W}\wedge_\star \mathcal{W} = \boldsymbol{\beta}_\star^{(\mathcal{W})} \\
&\boldsymbol{d}_x\mathfrak{B} + [\mathcal{W}, \mathfrak{B}]_\star = \boldsymbol{\beta}_\star^{(\mathfrak{B})} \\
&d_Z\mathcal{W} + \boldsymbol{d}_x S + \{\mathcal{W}, S\}_\star = 0 \\
&d_Z\mathfrak{B} + [S, \mathfrak{B}]_\star = 0 \\
&d_Z S + S\wedge_\star S = 0
\end{aligned} \tag{C.47}$$

These equations bear remarkable resemblance with the equations of motion in Vasiliev's higher spin theory, which have been briefly reviewed for completeness in Appendix C.1. Note however, that there are also significant differences:

(i) Firstly, in our construction, Z^α are coordinates on the infinite dimensional fibers of $P_{\mathcal{G}}$. To make contact with Vasiliev, we can introduce a parameterization of these fiber coordinates

$$Z^\alpha = \sum \left(z^\alpha_{A_1 B_1 \ldots} \epsilon^{i_1 j_1} Z^{A_1}_{i_1} \star Z^{B_1}_{j_1} \star \ldots \right) \tag{C.48}$$

That is, by introducing auxiliary $sp(2)\times O(2,d)$ variables Z^A_i, the Z^α can be written as arbitrary $Sp(2)$-invariant $\star$-polynomials. We can then recast

$$S(x|Y) = \boldsymbol{\omega}_\alpha(x,0|Y)dZ^\alpha = \boldsymbol{\omega}^i_A(x|Y,Z)dZ^A_i \tag{C.49}$$

(ii) Secondly, equations (C.47) have been written along the $Z^\alpha = 0$ section, and (C.49) represents some sort of lift to non-zero Z^A_i. While one is eventually supposed to project the non-linear Vasiliev equations to $Z^A_i = 0$ to get the physical variables, such a projection is not straightforward in the Vasiliev theory, and is typically carried out order by order in perturbation theory, thus making a direct comparison non-trivial.

(iii) Finally, in Vasiliev's equations without the projection to $Z^A_i = 0$, the curvature is along vertical (i.e. $dZ^A_i \wedge dZ^i_A$) directions, as opposed to our situation, where the horizontal components of curvature are non-trivial.

It is natural to ask if there is some sort of redefinition of our variables that would render our equations in Vasiliev's form. Such a redefinition was implicit in the construction of Ref. [74], though it is not clear to us if such a redefinition is natural. From our point of view, it seems compelling to think of the RG β functions as the (horizontal) curvature, while the equations for S are interpreted as the analogue of BRST equations. Holographic RG certainly presents us with a notion of a higher spin theory; it is perhaps not obvious that it must agree in all details with Vasiliev's construction, even though the similarities are immense. However, it is our belief that the differences pointed out above conspire to hide the equivalence of our renormalization group equations with the non-linear Vasiliev equations. A better understanding of this equivalence by constructing an explicit map between the two sets of equations will be left to future work. But if the conjectured equivalence is indeed true, then it would shed new light on the auxiliary 1-form S in the Vasiliev system (which has always appeared mysterious, to us anyway), namely, that it is the Faddeev-Popov ghost corresponding to the higher-spin gauge symmetry.

C.5 Calculating $\dot{G}_{(s,s)}$

In this appendix, we want to explicitly compute the kernel $\dot{G}_{(s,s)}$ which appeared in chapter 9, equations (9.44), (9.70). We will first compute the $s=0$ case, and then general s.

$s = 0$

From the definition (9.33), we have

$$G_{(0,0)}(z;\vec{x},\vec{y}) = \int \frac{d^d\vec{p}}{(2\pi)^d}\, G_{(0,0)}(z;\vec{p})e^{i\vec{p}.(\vec{x}-\vec{y})}, \quad G_{(0,0)}(z;\vec{p}) = c\int \frac{d^d\vec{q}}{(2\pi)^d}\frac{K(z^2(\vec{p}-\vec{q})^2/M^2)}{(\vec{p}-\vec{q})^2}\frac{K(z^2\vec{q}^2/M^2)}{\vec{q}^2} \tag{C.50}$$

where c is some constant factor. This is basically the Feynman diagram shown in figure 2.

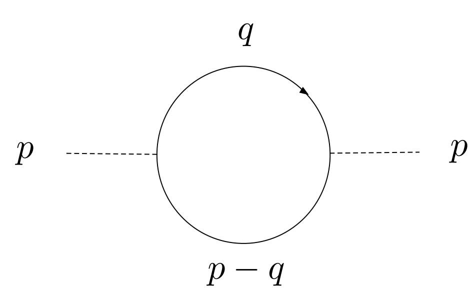

Figure C.2: The Feynman diagram which enters the renormalization group equations at the linearized level. The dotted lines are the external sources, while the solid lines correspond to propagators for elementary scalars.

For concreteness, let us pick a convenient regulator:

$$K(s) = e^{-s}$$

As we have discussed before, the arguments we have presented do not depend on the choice of the cut-off function. Therefore

$$G_{(0,0)}(z;\vec{p}) = c\int \frac{d^d\vec{q}}{(2\pi)^d}\frac{e^{-u^2(\vec{p}-\vec{q})^2}}{(\vec{p}-\vec{q})^2}\frac{e^{-u^2\vec{q}^2}}{\vec{q}^2} \tag{C.51}$$

where we have defined

$$u = z/M$$

We can use Schwinger parameters to rewrite this as

$$G_{(0,0)}(z;\vec{p}) = c\int_{u^2}^{\infty}\int_{u^2}^{\infty} dtds \int \frac{d^d\vec{q}}{(2\pi)^d} e^{-t(\vec{p}-\vec{q})^2-s\vec{q}^2} = c\int_{u^2}^{\infty}\int_{u^2}^{\infty} dtds\ \frac{1}{2^d\pi^{d/2}}\frac{1}{(s+t)^{d/2}}e^{-\frac{ts}{t+s}\vec{p}^2} \tag{C.52}$$

where we have carried out the $\vec{q}$ integration. We can evaluate the $u = 0$ limit straightforwardly

$$G_{(0,0)}(z\to 0;\vec{p}) = c\frac{\Gamma(2-\frac{d}{2})B(\frac{d}{2}-1,\frac{d}{2}-1)}{(4\pi)^{d/2}}\frac{1}{(\vec{p}^2)^{2-\frac{d}{2}}} \tag{C.53}$$

which in position space goes as $|x-y|^{-2\Delta_+}$ – the correct boundary two point function. But what we are interested in is not $G_{(0,0)}$, but $\dot{G}_{(0,0)}$

$$\dot{G}_{(0,0)}(z;\vec{p}) = z\partial_z G_{(0,0)}(z;\vec{p}) = -4cu^2\int_{u^2}^{\infty} dt\ \frac{1}{2^d\pi^{d/2}}\frac{1}{(u^2+t)^{d/2}}e^{-\frac{tu^2}{t+u^2}\vec{p}^2} \tag{C.54}$$

Defining $t = u^2\tau$, we get

$$\dot{G}_{(0,0)}(z;\vec{p}) = -4cu^{4-d}\int_{1}^{\infty} d\tau\ \frac{1}{2^d\pi^{d/2}}\frac{1}{(1+\tau)^{d/2}}e^{-\frac{\tau}{\tau+1}u^2\vec{p}^2} \tag{C.55}$$

For $u^2\vec{p}^2 << 1$, the quantity in the exponential is small, because

$$\frac{1}{2} < \frac{\tau}{1+\tau} < 1$$

Thus, in the limit $u^2\vec{p}^2 \to 0$, the exponential point-wise (in τ) converges to (and is bounded by) 1. This is also the case for all derivatives of the above function with respect to u. Additionally, $\frac{1}{(1+\tau)^{d/2}}$ is integrable on the domain $\tau \in (1, \infty)$. So, using the dominated convergence theorem, we get

$$
\begin{aligned}
\dot{G}_{(0,0)}(z;\vec{p}) &= -4cu^{4-d}\int_1^\infty d\tau\ \frac{1}{2^d\pi^{d/2}}\frac{1}{(1+\tau)^{d/2}}\left(1-\frac{\tau}{\tau+1}u^2\vec{p}^2+\frac{1}{2!}\frac{\tau^2}{(\tau+1)^2}u^4\vec{p}^4+\cdots\right)\\
&= -4cu^{4-d}\frac{1}{2^d\pi^{d/2}}\left(I(d;0)-I(d;1)u^2\vec{p}^2+\frac{1}{2!}I(d;2)u^4\vec{p}^4+\cdots\right)
\end{aligned}
\tag{C.56}
$$

where we have defined

$$
I(d;m) = \int_1^\infty d\tau \frac{\tau^m}{(1+\tau)^{d/2+m}} = \frac{2}{d-2}{}_2F_1\left(\frac{d-2}{2},\frac{d}{2}+m,\frac{d}{2};-1\right) \tag{C.57}
$$

which is well-defined for all m provided $d > 2$. The first few of these integrals are given by

$$
I(d;0) = \frac{2^{2-d/2}}{d-2} \tag{C.58}
$$

$$
I(d;1) = \frac{2^{1-d/2}(d+2)}{d(d-2)} \tag{C.59}
$$

$$
I(d;2) = \frac{2^{-d/2}(d^2+6d+16)}{d(d^2-4)} \tag{C.60}
$$

and so on. So, in position space, we get

$$
\begin{aligned}
\dot{G}_{(0,0)}(z;\vec{x},\vec{y}) &= -\frac{4cu^{-2\nu}}{2^d\pi^{d/2}}\left(I(d;0)+I(d;1)u^2\Box_{(x)}+\frac{1}{2!}I(d;2)u^4\Box_{(x)}^2+\cdots\right)\delta^d(x-y)\\
&= -Cz^{-2\nu}\left(1+\alpha z^2\Box_{(\vec{x})}+\cdots\right)\delta^d(x-y)
\end{aligned}
\tag{C.61}
$$

where

$$
C = \frac{4cI(d;0)}{2^d\pi^{d/2}}M^{2\nu}, \quad \alpha = \frac{d+2}{2dM^2} > 0
$$

are constants, and recall that

$$
2\nu = \Delta_+ - \Delta_- = (d-4)
$$

Higher spins

Now we wish to do the same calculation for generic higher-spin currents. In this case,

$$
G^{\mu_s,\nu_s}_{(s,s)}(z;\vec{x},\vec{y}) = \frac{2i}{N}\langle J^{\mu_1\cdots\mu_s}(\vec{x})J^{\nu_1\cdots\nu_s}(\vec{y})\rangle_{CFT} \tag{C.62}
$$

Using

$$J^{\mu_1\cdots\mu_s}(\vec{x}) = \phi^*_m(\vec{x}) f^{\mu_1\cdots\mu_s}(\overleftarrow{\partial}_{(x)}, \overrightarrow{\partial}_{(x)})\phi^m(\vec{x}) \tag{C.63}$$

we get in momentum space

$$G^{\underline{\mu}_s,\underline{\nu}_s}_{(s,s)}(z;\vec{p}) = c_s \int \frac{d^d\vec{q}}{(2\pi)^d} \frac{K(z^2(\vec{p}-\vec{q})^2/M^2)}{(\vec{p}-\vec{q})^2} f^{\mu_1\cdots\mu_s}(i\vec{q}, i(\vec{p}-\vec{q})) \frac{K(z^2\vec{q}^2/M^2)}{\vec{q}^2} f^{\nu_1\cdots\nu_s}(-i\vec{q}, -i(\vec{p}-\vec{q})) \tag{C.64}$$

Once again, using $K(s) = e^{-s}$ and Schwinger parameters, we get

$$G^{\underline{\mu}_s,\underline{\nu}_s}_{(s,s)}(z;\vec{p}) = c_s \int_{u^2}^{\infty} dt \int_{u^2}^{\infty} ds \int \frac{d^d\vec{q}}{(2\pi)^d} f^{\mu_1\cdots\mu_s}(i\vec{q}, i(\vec{p}-\vec{q})) f^{\nu_1\cdots\nu_s}(-i\vec{q}, -i(\vec{p}-\vec{q})) e^{-s\vec{q}^2 - t(\vec{p}-\vec{q})^2} \tag{C.65}$$

which can be conveniently written as

$$G^{\underline{\mu}_s,\underline{\nu}_s}_{(s,s)}(z;\vec{p}) = \lim_{\vec{j}\to 0} \int_{u^2}^{\infty} dt \int_{u^2}^{\infty} ds\; f^{\mu_1\cdots\mu_s}\left(\frac{\partial}{\partial\vec{j}}, i\vec{p} - \frac{\partial}{\partial\vec{j}}\right) f^{\nu_1\cdots\nu_s}\left(-\frac{\partial}{\partial\vec{j}}, -i\vec{p} + \frac{\partial}{\partial\vec{j}}\right) \int \frac{d^d\vec{q}}{(2\pi)^d} e^{-s\vec{q}^2 - t(\vec{p}-\vec{q})^2 + i\vec{q}\cdot\vec{j}} \tag{C.66}$$

Upon doing the $\vec{q}$ integration, we get

$$\begin{aligned} G^{\underline{\mu}_s,\underline{\nu}_s}_{(s,s)}(z;\vec{p}) &= \frac{c_s}{2^d\pi^{d/2}} \lim_{\vec{j}\to 0} f^{\mu_1\cdots\mu_s}\left(\frac{\partial}{\partial\vec{j}}, i\vec{p} - \frac{\partial}{\partial\vec{j}}\right) f^{\nu_1\cdots\nu_s}\left(-\frac{\partial}{\partial\vec{j}}, -i\vec{p} + \frac{\partial}{\partial\vec{j}}\right) \\ &\times \int_{u^2}^{\infty} dt \int_{u^2}^{\infty} ds\; \frac{1}{(t+s)^{d/2}} e^{-\frac{ts}{t+s}\vec{p}^2 - \frac{1}{4(t+s)}\vec{j}^2 + i\frac{t}{t+s}\vec{p}\cdot\vec{j}} \end{aligned} \tag{C.67}$$

Now taking a u derivative, we see that

$$\begin{aligned} \dot{G}^{\underline{\mu}_s,\underline{\nu}_s}_{(s,s)}(z;\vec{p}) &= -\frac{2c_s u^{4-d-2s}}{2^d\pi^{d/2}} \lim_{\vec{j}'\to 0} f^{\mu_1\cdots\mu_s}\left(\frac{\partial}{\partial\vec{j}'}, iu\vec{p} - \frac{\partial}{\partial\vec{j}'}\right) f^{\nu_1\cdots\nu_s}\left(-\frac{\partial}{\partial\vec{j}'}, -iu\vec{p} + \frac{\partial}{\partial\vec{j}'}\right) \\ &\times \int_1^{\infty} d\tau \frac{1}{(1+\tau)^{d/2}} e^{-\frac{\tau}{\tau+1}u^2\vec{p}^2 - \frac{1}{4(\tau+1)}\vec{j}'^2} \left(e^{\frac{i}{\tau+1}u\vec{p}\cdot\vec{j}'} + e^{\frac{i\tau}{\tau+1}u\vec{p}\cdot\vec{j}'}\right) \end{aligned} \tag{C.68}$$

where $\vec{j} = u\vec{j}'$. In order to proceed, we need to know the explicit form of $f^{\mu_1\cdots\mu_s}$, and the detailed form of the kernel above will depend on this explicitly. However, in the higher-spin Coulomb gauge we choose the higher-spin fields to be divergenceless, and the only piece of interest is the term proportional to $\eta^{<\mu_1<\nu_1}\cdots\eta^{\mu_s>\nu_s>}$, where the angular brackets refer to the traceless, symmetric combination. In this case, it is evident for the same reason as in the $s=0$ case, that we have

$$\dot{G}^{\underline{\mu}_s,\underline{\nu}_s}(\vec{x},\vec{y}) = C_s z^{-2\nu}\left(1 + \alpha_s z^2\Box_{(\vec{x})} + \cdots\right)\delta^d(\vec{x}-\vec{y})\eta^{<\mu_1<\nu_1}\cdots\eta^{\mu_s>\nu_s>} \tag{C.69}$$

with $\alpha_s > 0$.

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