GAMES ON GRAPHS, VISIBILITY REPRESENTATIONS, AND GRAPH COLORINGS

BY

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Abstract

In this thesis we study combinatorial games on graphs and some graph parameters whose consideration was inspired by an interest in the symmetry of hypercubes.

A *capacity function* f on a graph G assigns a nonnegative integer to each vertex of $V(G)$. An *f-matching* in G is a set $M \subseteq E(G)$ such that the number of edges of M incident to v is at most $f(v)$ for all $v \in V(G)$. In the *f-matching game* on a graph G, denoted (G, f), players Max and Min alternately choose edges of G to build an f-matching; the game ends when the chosen edges form a maximal f-matching. Max wants the final f-matching to be large; Min wants it to be small. The *game f-matching number* is the size of the final f-matching under optimal play. We extend to the f-matching game a lower bound due to Cranston et al. [14] on the game matching number. We also consider a directed version of the f-matching game on a graph G.

Peg Solitaire is a game on connected graphs introduced by Beeler and Hoilman [5]. In the game, pegs are placed on all but one vertex. If x, y, and z form a 3-vertex path and x and y each have a peg but z does not, then we can remove the pegs at x and y and place a peg at z; this is called a *jump*. Beeler and Rodriguez [6] proposed a variant where we want to maximize the number of pegs remaining when no more jumps can be made. Maximizing over all initial locations of a single hole, the maximum number of pegs left on a graph G when no jumps remain is the Fool's Solitaire number $F(G)$. We determine the Fool's Solitaire number for the join of any graphs G and H. For the cartesian product, we determine $F(G \square K_k)$ when $k \geq 3$ and G is connected. Finally, we give conditions on graphs G and H that imply $F(G \square H) \geq F(G)F(H)$.

A *t-bar visibility representation* of a graph G assigns each vertex a set that is the union of at most t horizontal segments ("bars") in the plane so that vertices are adjacent if and only if there is an unobstructed vertical line of sight (having positive width) joining the sets assigned to them. The visibility number of a graph G, written $b(G)$, is the least t such that G has a t-bar visibility representation. Let Q_n denote the n-dimensional hypercube. A simple application of Euler's Formula yields $b(Q_n) \geq \lceil (n+1)/4 \rceil$. To prove that equality holds, we decompose Q_{4k-1} explicitly into k spanning subgraphs whose components have the form $C_4 \square P_{2^l}$. The *visibility number $b(D)$* of a digraph D is the least t such that D can be represented by assigning

each vertex at most t horizontal bars in the plane so that $uv \in E(D)$ if and only if there is an unobstructed vertical line of sight (with positive width) joining some bar for u to some higher bar for v. It is known that $b(D) \leq 2$ for every outerplanar digraph. We give a characterization of outerplanar digraphs with $b(D) = 1$.

A proper vertex coloring of a graph G is *r-dynamic* if for each $v \in V(G)$, at least $\min\{r, d(v)\}$ colors appear in $N_G(v)$. We investigate r-dynamic versions of coloring and list coloring. We give upper bounds on the minimum number of colors needed for any r in terms of the genus of the graph.

Two vertices of Q_n are *antipodal* if they differ in every coordinate. Two edges uv and xy are *antipodal* if u is antipodal to x and v is antipodal to y. An *antipodal edge-coloring* of Q_n is a 2-coloring of the edges in which antipodal edges have different colors. DeVos and Norine conjectured that for $n \geq 2$, in every antipodal edge-coloring of Q_n there is a pair of antipodal vertices connected by a monochromatic path. Previously this was shown for $n \leq 5$. Here we extend this result to $n = 6$.

Hovey [26] introduced *A-cordial* labelings as a simultaneous generalization of cordial and harmonious labelings. If A is an abelian group, then a labeling $f : V(G) \to A$ of the vertices of a graph G induces an edge-labeling on G; the edge uv receives the label $f(u) + f(v)$. A graph G is *A-cordial* if there is a vertex-labeling such that (1) the vertex label classes differ in size by at most 1, and (2) the induced edge label classes differ in size by at most 1. The smallest non-cyclic group is V_4 (also known as $\mathbb{Z}_2 \times \mathbb{Z}_2$). We investigate V_4-cordiality of many families of graphs, namely complete bipartite graphs, paths, cycles, ladders, prisms, and hypercubes. Finally, we introduce a generalization of A-cordiality involving digraphs and quasigroups, and we show that there are infinitely many Q-cordial digraphs for every quasigroup Q.

To my family and friends.

Acknowledgments

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Chapter 1

Introduction

In this thesis we study combinatorial games on graphs, visibility representations, and several types of graph colorings.

Graph parameters are computed under an ideal situation in which the graph is completely known and the algorithm makes all the decisions. Realistically, information may be unknown, or some choices may become unavailable. Game versions of graph parameters model worst-case interventions by nature as an opponent, though still acting under fairly rigid rules. Players make moves with opposing objectives. Chapter 2 studies a two-player game on graphs in which the players jointly form a subgraph with given vertex degree restrictions; one wants a largest such graph, while the other wants a smallest such graph.

In Chapter 3 we look at Fool's Solitaire, a variation on a popular table game. If in a graph adjacent vertices x and y have pegs, and a vertex z adjacent to y has no peg, then we may *jump* the peg at x over the peg at y and into the "hole" at z. This removes the peg at y so that x and y become holes and z has a peg. A jump can be seen as moving the resources at x using the resources at y to resupply those consumed by the move so that one ends with a full load of resources at z. In this way, Fool's Solitaire can be viewed as an analogue of another resource-transportion model, called pebbling. In pebbling, when two pebbles are at one vertex, they can be replaced by one pebble at a neighboring vertex; the other pebble is consumed along the way. In pebbling, one studies the minimum number of pebbles in an initial allocation on the graph such that moves can result in a pebble on any vertex. A similar question could be asked for jumps. This situation is not monotone, however, since adding pegs can block jumps. In Fool's Solitaire, we look to maximize the amount of resources remaining when no more resources can be moved.

In computational geometry, graphs are used to model visibility relations in the plane. For example, we may say that two vertices of a polygon "see" each other if the line segment joining them lies inside the polygon. In the *visibility graph* on the vertex set, vertices are adjacent if they see each other. Similarly, we can define visibility on a set of line segments; two segments see each other if some segment joining them crosses no other segment in the set. Dozens of papers have been written concerning the computation and the recognition of visibility graphs, with applications to search problems, motion planning, and robotics

(for surveys see [42, 49]). Here we study t-bar visibility representations. In a t-*bar visibility representation* (Chang et al. [12]), each vertex of G is represented by at most t bars, and $uv \in E(G)$ if and only if there is an unobstructed vertical line of sight (having positive width) joining some bar for u to some bar for v. The *bar visibility number* $b(G)$ of a graph G is the least t such that G has a t-bar visibility representation. Chapter 4 studies bar-visibility representations of the hypercube and oriented outerplanar graphs.

In Chapters 5 through 7 we consider a trio of graph coloring problems. Graph coloring can be applied to scheduling problems, in many of which the colors represent possible time slots or meetings given the adjacency restrictions of the graph. In Chapter 5, we look at a variation on graph coloring called r-dynamic coloring. In this variation we want a proper coloring of the vertices in which every vertex $v \in V(G)$ has at least $\min\{r, d(v)\}$ distinct colors on the vertices in its neighborhood. If we view each set consisting of a vertex and its neighbors as a social club where the vertex is the president and its neighbors are members, then r-dynamic coloring can be viewed as a scheduling problem with the following constraints. The colors correspond to meetings, each person goes to only one meeting, and when a person attends a meeting, they can sign in as attending for every club of which they are a member. Also, each club must have members attend at least r meetings (or, if there are fewer than r members, each member must attend a different meeting) and the president cannot attend a meeting attended by any of the other members. We want to hold the fewest number of meetings so that every club can have members attend the required number of meetings.

Two vertices of the hypercube Q_n are *antipodal* if they differ in every coordinate. Two edges uv and xy are *antipodal* if u is antipodal to x and v is antipodal to y. An *antipodal edge-coloring* of Q_n is a 2-coloring of the edges in which antipodal edges have different colors. Chapter 6 considers a conjecture by DeVos and Norine [16] that every antipodal edge-coloring of Q_n contains a monoochromatic path joining some pair of antipodal vertices.

Chapter 7 presents several results on A-cordial labelings of graphs, where A is a group. Graph labelings of diverse types are the subject of much study. In an attempt to provide something of a framework for these results, Hovey introduced A-cordial labelings in [26] as a common generalization of cordial labeling (introduced by Cahit [10]) and harmonious labeling (introduced by Graham and Sloane [23]). If A is an additive abelian group, then a vertex-labeling $f\colon V(G) \to A$ of a graph G induces an edge-labeling on G as well by giving the edge uv the label $f(u) + f(v)$. We say that a graph G is A-*cordial* if there is a vertex-labeling $f\colon V(G) \to A$ such that (1) the vertex sets labeled by any two elements of A differ in size by at most 1, and (2) the induced edge sets labeled by any two elements of A differ in size by at most 1. *Cordial graphs* are simply the $\mathbb{Z}_2$-cordial graphs, while *harmonious graphs* are simply the $\mathbb{Z}_{|E(G)|}$-cordial graphs. Each of

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these concepts is well studied. Almost all other work on A-cordiality has also focused on the case where A is cyclic. This case is indeed very interesting, particularly in light of Hovey's conjecture from [26] that all trees are A-cordial for all cyclic groups A (which he proved for $|A| < 6$). The conjecture does not extend to even the smallest non-cyclic group, V_4 (also known as $\mathbb{Z}_2 \times \mathbb{Z}_2$); the paths P_4 and P_5 are easily seen to be not V_4-cordial. This leads us to consider the V_4-cordiality of some classes of graphs.

Subsequent sections of this chapter give an overview of our results in each chapter. In Section 1.7, we give formal definitions of many of the concepts used in this thesis; most of these concepts can be found in any introductory graph theory textbook.

1.1   Game f-matching

Given graphs F and G, a subgraph H of G is *F-saturated relative to G* if F is not a subgraph of H but is a subgraph of $H + e$ for every $e \in E(G) - E(H)$. In the *F-saturation game* on a graph G, two players, Max and Min, alternately choose edges of G to add to a common subgraph H until H is F-saturated relative to G. Max wants the final graph to be large; Min wants it to be small. The *game F-saturation number* of G is the length of the game under optimal play.

A *capacity function f* on a graph G assigns a nonnegative integer to each vertex of G. An *f-matching* in G is a set M of edges of G such that the number of edges in M incident to any vertex v is at most $f(v)$. When $f(v) = k$ for all $v \in V(G)$, a maximal f-matching in G is simply a $K_{1,k+1}$-saturated subgraph relative to G.

In the *f-matching game* on a graph G, players Max and Min alternately choose edges of G to build an f-matching; the game ends when the chosen edges form a maximal f-matching. Max wants the final f-matching to be large; Min wants it to be small. The *game f-matching number* is the size of the final f-matching under optimal play, meaning the common size that each player can guarantee. We denote this value by $\nu_f(G)$ when Max plays first and by $\hat{\nu}_f(G)$ when Min plays first, calling these two versions of the game the *Max-start* and *Min-start* games *with respect to f*.

The *matching game* is the special case of the f-matching game where $f(v) = 1$ for all $v \in V(G)$ (that is, $f \equiv 1$). The matching game is also the special case of the F-saturation game where $F = P_3$. Max and Min alternately choose edges forming a matching in a graph G, and the game ends when the chosen edges form a maximal matching. With the same objectives as in the general game, the *game matching number* is the size of the final matching under optimal play. We denote this value by $\nu_1(G)$ when Max plays first and by $\hat{\nu}_1(G)$ when Min plays first.

In Section 2.2, we extend to the f-matching game a lower bound due to Cranston et al. [14] on the game matching number of a graph when Max starts and consider some other aspects of the f-matching game.

In Section 2.3, we consider bounds on the game f-matching numbers of the disjoint union of graphs. Specifically, we consider the disjoint union of a graph H with a complete graph. We then consider the disjoint union of any number of complete graphs.

In general, the outcome of the F-saturation game may depend greatly on which player starts. Cranston et al. [14] proved that this does not occur in the special case of the matching game. In particular, for every graph G, we have $|\nu_1(G) - \hat{\nu}_1(G)| \le 1$.

We say that a graph G is *near-fair* for a capacity function f if $|\nu_f(G) - \hat{\nu}_f(G)| \le 1$. Motivated by the result of [14], we ask whether $|\nu_f(G) - \hat{\nu}_f(G)| \le 1$ holds for every graph G and every capacity function f. In Section 2.4 we consider a directed version of the f-matching game on a graph G in which players Max and Min alternately choose edges of G and orient them to build an oriented subgraph H of G in which the outdegree of v in H is bounded above by $f(v)$; the indegree is unconstrained. The game ends when no more edges can be chosen without exceeding some given outdegree capacity. Max wants the final subgraph to be large; Min wants it to be small. The *directed game f-matching number* is the size of the final subgraph under optimal play. We denote this value by $\mu_f(G)$ when Max plays first and by $\hat{\mu}_f(G)$ when Min plays first. We use an auxiliary graph to show that the choice of starting player makes little difference in this variation.

Cranston et al. [14] also proved $3\lfloor \frac{n}{7} \rfloor \le \nu_g(P_n) \le 3\lceil \frac{n}{7} \rceil$ for all n. There is a gap of 3 between the lower and upper bounds, except when n is a multiple of 7. In Section 2.5, in order to determine the exact value, we solve a more general problem and determine the exact value of the game matching number for a linear forest.

This chapter contains work done jointly with Xuding Zhu and (separately) with Douglas West.

1.2 Fool's Solitaire

Peg Solitaire is a table game played with pegs and board that consists of a set of lattice points at which pegs can be placed. The game traditionally begins with pegs at all but one of the points; a point without a peg is called a "hole". If in some row or column two adjacent pegs are next to a hole (as in Fig. 1.1), then the peg at x can jump over the peg at y into the hole at z. The peg in y is then removed. The goal is to remove every peg but one. If this is achieved, then the board is considered *solved*.

Beeler and Hoilman [5] generalized this to a game of Peg Solitaire on graphs. Since the placement of the vertices does not affect a graph, the geometric notions of "rows" and "columns" are modified to 3-vertex
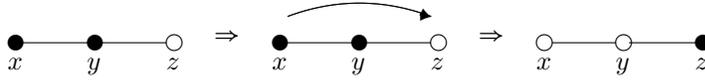
Figure 1.1: A typical jump

paths. If adjacent vertices x and y have pegs, and z adjacent to y is a hole, then we may jump the peg at x over the peg at y and into the hole at z. As in the original version, this removes the peg at y so that x and y become holes and z has a peg.[1] This jump can be seen as moving the resources at x using the resources at y to resupply those consumed by the move so that one ends with a full load of resources at z.

The goal in Peg Solitaire on graphs is again to remove all but one peg by a succession of jumps. If this is achievable starting with one hole, then G is again *solvable*. If G can be solved starting with a single hole at any vertex, then G is *freely solvable*.

Beeler and Rodriguez [6] proposed a variant where we instead want to maximize the number of pegs remaining when no more jumps can be made. Maximizing over all initial locations of a single hole, the maximum number of pegs left on a graph G when no jumps remain is the *Fool's Solitaire number* $F(G)$.

In Section 3.2 we extend a result by Beeler and Rodriguez [6] on complete bipartite graphs by determining the Fool's Solitaire number of all graphs whose complements are disconnected.

Beeler and Rodriguez [6] asked for the behavior of the Fool's Solitaire number under the cartesian product operation. (The *cartesian product* of two graphs G and H, denoted $G \square H$, is the graph with vertex set $V(G) \times V(H)$ such that two vertices are adjacent if and only if they are equal in one coordinate and adjacent in the other.) In this direction, we prove the following in Section 3.3, where $\alpha(H)$ and $\chi(H)$ denote the independence number and chromatic number of a graph H.

Theorem 1.2.1. *Let G be a connected graph. If $k \geq 3$, then $F(G \square K_k) = \alpha(G \square K_k)$. In particular, $F(G \square K_k) = |V(G)|$ when $k \geq \chi(G)$. However, $F(G \square K_2) = \alpha(G \square K_2) - 1$.*

We give sufficient conditions for $F(G \square H) \geq F(G)F(H)$ in Section 3.4. This is a partial answer to the question in [6] asking for the relationship among $F(G), F(H)$, and $F(G \square H)$.

This chapter contains work done jointly with Sarah Loeb and appearing in [35].

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[1]There are several traditional boards marketed commercially, a triangle with 15 positions in the U.S., a portion of a grid in England (marketed as "Hi-Q" in the U.S.), and a European board with more positions than the English board. The significant distinction between these games and the graph version is that they restrict jumps to be made along geometric straight lines.

1.3 Bar visibility representations

In computational geometry, graphs are used to model visibility relations in the plane. For example, we may say that two vertices of a polygon "see" each other if the line segment joining them lies inside the polygon. In the *visibility graph* on the vertex set, vertices are adjacent if they see each other. Similarly, we can define visibility on a set of line segments; two segments see each other if some segment joining them crosses no other segment in the set. Dozens of papers have been written concerning the computation and the recognition of visibility graphs, with applications to search problems, motion planning, and robotics (for surveys see [42, 49]).

A *bar visibility representation* of a graph G assigns the vertices distinct horizontal line segments ("bars") in the plane such that $uv \in E(G)$ if and only if there is an unobstructed vertical line of sight (having positive width) joining the bar assigned to u and the bar assigned to v. A graph is a *bar visibility graph* if it has a bar visibility representation. Bar-visibility graphs must be planar; they have been characterized, applied, and studied from both theoretical and algorithmic viewpoints. The requirement that lines of sight have positive width is important. It permits us to use closed bars so that bars with endpoints having a common x-coordinate cannot see each other but can block vertical visibility between them.

Inspired by an analogous parameter for interval graphs, Chang et al. [12] introduced t-bar visibility representations, where each vertex of G is represented by at most t bars, and $uv \in E(G)$ if and only if there is an unobstructed vertical line of sight (having positive width) joining some bar for u to some bar for v. The *bar visibility number* $b(G)$ of a graph G is the least t such that G has a t-bar visibility representation.

Let Q_n be the n-dimensional hypercube. Using Euler's Formula relating the numbers of vertices, edges, and faces in a planar graph, Axenovich et al. [3] observed that $b(Q_n) \geq \lceil \frac{n+1}{4} \rceil$ and asked whether this trivial lower bound is also an upper bound, yielding $b(Q_n) = \lceil \frac{n+1}{4} \rceil$ for the n-dimensional hypercube. Kleinert [33] decomposed Q_n into $\lceil \frac{n+1}{4} \rceil$ planar graphs. Wismath [55], Tamassia and Tollis [47], and Hutchinson [28] independently proved that 2-connected planar graphs are bar visibility graphs. Thus proving that the components of of the graphs in Kleinert's decomposition are 2-connected answers the question in the affirmative.

In Section 4.2 we give an explicit description of such a structural decomposition of Q_n, which may be useful for additional applications. Our decomposition turns out to be isomorphic to that of Kleinert, but our proof is simpler. The key case is $n \equiv 3 \bmod 4$, where we decompose Q_{4k-1} into k spanning subgraphs whose components are isomorphic cartesian products of 4-cycles with paths whose order is an appropriate power of 2. The visibility result is a corollary. Let C_n denote the n-cycle and P_n denote the path on n vertices.

Theorem 1.3.1. *Q_{4k-1} can be decomposed into k spanning subgraphs $G_1, \ldots, G_k$ such that each component*

of G_i is isomorphic to $C_4 \square P_{2^{i+1}}$ for $i < k$ and to $C_4 \square P_{2^k}$ for $i = k$. Each such G_i can be represented using one bar per vertex, so $b(Q_n) = \lceil \frac{n+1}{4} \rceil$ for $n \in \mathbb{N}$.

The visibility number $b(D)$ of a digraph D (Axenovich et al. [3]) is the least t such that D can be represented by assigning each vertex at most t horizontal bars in the plane so that $uv \in E(D)$ if and only if there is an unobstructed vertical line of sight joining some bar for u to some higher bar for v. Every t-bar representation of an undirected graph G such that each edge is represented in only one direction yields $b(D) \leq t$ for a corresponding orientation D of G. If $b(D) = 1$, then we say that D is a *bar-visibility digraph*. In Section 4.3, we give a characterization of when an outerplanar digraph is a bar-visibility digraph.

This chapter contains work done jointly with Douglas West.

1.4 r-Dynamic coloring of graphs on surfaces

For a graph G and a positive integer r, an *r-dynamic coloring* of G is a proper vertex coloring such that for each $v \in V(G)$, at least $\min\{r, d(v)\}$ distinct colors appear in $N_G(v)$. The *r-dynamic chromatic number*, denoted $\chi_r(G)$, is the minimum k such that G admits an r-dynamic k-coloring. Montgomery [41] introduced 2-dynamic coloring and the generalization to r-dynamic coloring.

List coloring assigns to each vertex v a set $L(v)$ of available colors. An *L-coloring* is a proper coloring c such that the color for each vertex v comes from the list assigned to it. A graph G is *k-choosable* if an L-coloring exists whenever every vertex is assigned at least k colors. The *choice number* of a graph G, denoted $\mathrm{ch}(G)$, is the least k such that G is k-choosable. Since a k-choosable graph must, in particular, have a list coloring when every vertex is assigned the same list of k colors, we always have $\mathrm{ch}(G) \geq \chi(G)$. A graph G is *r-dynamically L-colorable* if an r-dynamic coloring can be chosen from the list assignment L. The *r-dynamic choosability* of G, denoted $\mathrm{ch}_r(G)$, is the least k such that G is *r-dynamically L-colorable* for every list assignment L in which every vertex is assigned at least k colors.

The *square* of a graph G, denoted G^2, is the graph resulting from adding an edge joining every two vertices separated by distance 2 in G. For any graph G, it is clear that

$$\chi(G) = \chi_1(G) \leq \chi_2(G) \leq \cdots \leq \chi_{\Delta(G)}(G) = \cdots = \chi(G^2),$$
$$\mathrm{ch}(G) = \mathrm{ch}_1(G) \leq \mathrm{ch}_2(G) \leq \cdots \leq \mathrm{ch}_{\Delta(G)}(G) = \cdots = \mathrm{ch}(G^2), \qquad (1)$$

and that $\chi_r(G) \leq \mathrm{ch}_r(G)$ for all r. Thus we can think of r-dynamic coloring as bridging the gap between coloring a graph and coloring its square.

Heawood [24] proved that for $g > 0$, graphs embeddable on the orientable surface with genus g are

$(h(g) - 1)$-degenerate and hence $h(g)$-colorable, where

$$h(g) = \left\lfloor \frac{7 + \sqrt{1 + 48g}}{2} \right\rfloor.$$

Chen et al. [13] proved that such a graph is 2-dynamically $h(g)$-choosable.

Let $\gamma(G)$ denote *genus of a graph G*, the minimum genus of a surface on which G embeds. In Section 5.2, we prove the following theorem, giving an upper bound on the r-dynamic choosability of a graph with given genus and thus also an upper bound on the graph's r-dynamic chromatic number.

Theorem 1.4.1. *Let G be a graph, and let $g = \gamma(G)$.*

1. *If $g = 0$ and $r \geq 11$, then $\operatorname{ch}_r(G) \leq 5(r + 1) + 3$*

2. *If $g \geq 1$ and $r \geq 24g - 11$, then $\operatorname{ch}_r(G) \leq (12g - 6)(r + 1) + 3$.*

Even though Theorem 1.4.1 assumes a lower bound on r, by (1) it gives an upper bound on $\operatorname{ch}_r(G)$ for all r. Loeb et al. [34] showed that for planar and toroidal graphs, this bound can be improved to $\operatorname{ch}_r(G) \leq 10$, which is sharp in the toroidal case.

An online version of list coloring (introduced by Zhu [56] and independently by Schauz [46]) can be described using a game with players Lister and Painter. On each round Lister marks a set of vertices allowed to receive a particular color, and Painter chooses an independent subset of the marked vertices to receive that color. Colored vertices will not be marked again; in essence, they are removed from the graph. Each vertex v can tolerate being marked $f(v)$ times before it must be colored, and the graph is *f-paintable* if Painter can guarantee coloring the graph subject to these restrictions. When a graph is f-paintable, we say "Painter has a winning strategy", or more simply, "Painter wins".

Analogously, we say that a graph G is *r-dynamically k-paintable* when Painter has a winning strategy that produces an r-dynamic coloring of G when all vertices can be marked k times. The least k such that Painter can accomplish this is the *r-dynamic paint number*. All of the results in this chapter can be extended to paintability.

This chapter contains work done jointly with Sarah Loeb, Thomas Mahoney, and Benjamin Reiniger, appearing in [34].

1.5 Antipodal edge colorings of hypercubes

Two vertices in the hypercube Q_n are *antipodal* if they differ in every coordinate. Two edges uv and xy are *antipodal* if u is antipodal to x and v is antipodal to y. An *antipodal edge-coloring* of Q_n is a 2-coloring of

the edges of Q_n such that antipodal edges have different colors.

DeVos and Norine [16] conjectured the following

Conjecture 1.5.1. *For $n \geq 2$, in every antipodal edge-coloring of Q_n there is a pair of antipodal vertices connected by a monochromatic path.*

In an antipodal edge-coloring, the graphs formed by the two colors are isomorphic (under complementation of the vertices). Feder and Subi [19] proved a strengthening of Conjecture 1.5.1 for $n \leq 5$. A *geodesic* is a shortest path between the endpoints of the path. In Q_n, a geodesic crosses each dimension of the hypercube at most once. Any geodesic in Q_n between two antipodal vertices has length n. They showed that for $n \leq 5$, in every antipodal edge-coloring of Q_n there are two antipodal vertices joined by a monochromatic geodesic.

Feder and Subi [19] also proved that the conclusion holds for any 2-edge-coloring (not necessarily antipodal) that contains no 4-cycle along which the colors alternate. Feder and Subi [19] further noted that a counterexample for Q_n can be extended to a counterexample for Q_{n+1}. Suppose we have a counterexample of a labelled d-dimensional hypercube Q_n. Making two copies of the labelled Q_n and joining them arbitrarily but antipodally gives a counter example Q_{n+1}, as a monochromatic antipodal path in the resulting coloring on Q_{n+1} would yield such a path in Q_n by copying the portion form one copy of the Q_n to the other.

In this chapter, we give proofs of the strengthening of Conjecture 1.5.1 for $n \in \{4, 5\}$ using a conceptually easier technique than [19] and further prove the strengthening of Conjecture 1.5.1 for $n = 6$ yielding

Theorem 1.5.2. *For $2 \leq n \leq 6$, in every antipodal edge-coloring of Q_n there is a pair of antipodal vertices connected by a monochromatic geodesic.*

It is our hope that this approach can be used to prove the statement for larger n.

This chapter builds upon work done jointly with Oliver Pechenik and Hannah Spinoza and contains work done jointly with Douglas West.

1.6   Generalized graph cordiality

Many types of vertex labelings of graphs have been studied. The state of the field is described in detail in Gallian's dynamic survey [21]. Results obtained so far, while numerous, are mainly piecemeal in nature and lack generality. In an attempt to provide something of a framework for these results, Hovey [26] introduced A-cordial labelings as a simultaneous generalization of cordial labeling (introduced by Cahit [10]) and harmonious labeling (introduced by Graham and Sloane [23]).

If A is an additive abelian group, then a labeling $f : V(G) \to A$ of the vertices of a graph G induces an edge-labeling of G as well by giving the edge uv the label $f(u) + f(v)$. A graph G is A-*cordial* if there is a

vertex labeling of G such that the number of times any two elements of A are used as a vertex label differs by at most 1, and the number of times any two elements of A are used as an edge label differs by at most 1.

In Section 7.2 we consider some necessary conditions for a graph G to be A-cordial for certain A. Research on A-cordiality has focused on the case where A is cyclic. The smallest non-cyclic group is V_4 (also known as $\mathbb{Z}_2 \times \mathbb{Z}_2$). In Section 7.3, we investigate V_4-cordiality of many families of graphs, namely complete bipartite graphs, paths, cycles, ladders, prisms, and hypercubes. We find that all complete bipartite graphs are V_4-cordial except $K_{m,n}$ where $m, n \equiv 2 \bmod 4$. All paths are V_4-cordial except P_4 and P_5. All cycles are V_4-cordial except C_4, C_5, and C_k, where $k \equiv 2 \bmod 4$. All ladders $P_2 \square P_k$ are V_4-cordial except C_4. All prisms $P_2 \square C_k$ are V_4-cordial except when $k \equiv 2 \bmod 4$. All hypercubes are V_4-cordial except C_4.

Further research on V_4-cordiality could address which grids, $P_h \square P_k$, are V_4-cordial. Our results on ladders resolve the case $h = 2$. Also, it is not hard to see that the Petersen graph is V_4-cordial. By one of our necessary conditions in Section 7.2, the Kneser graph $K(n,k)$ is not V_4-cordial, if $\binom{n-k}{k}$ is even and $\frac{1}{2}\binom{n-k}{k}\binom{n}{k} \equiv 2 \bmod 4$. For example, $K(7,3)$ is not V_4-cordial. This leads us to ask which generalized Petersen graphs or Kneser graphs are V_4-cordial.

Finally, we introduce a generalization of A-cordiality involving digraphs and quasigroups, and in Section 7.4, we show that there are infinitely many Q-cordial digraphs for every quasigroup Q. In particular, this gives us the following corollary.

Corollary 1.6.1. *For every abelian group A, there are infinitely many A-cordial cycles and infinitely many A-cordial paths.*

In the case where A is V_4, we obtain much stronger results. All paths with six or more vertices are V_4-cordial. For any particular abelian group A, Corollary 1.6.1 is fairly weak. However, it suggests that, for each abelian group A, the class of A-cordial graphs will be an interesting subject of study. It would be of interest to study how the structure of the abelian group A relates to the set of natural numbers n for which the path P_n is A-cordial. For example, V_4 has the special property that all sufficiently long paths are V_4-cordial. Is it true that, for each abelian group A, there exists N such that P_n is A-cordial whenever $n > N$? If the answer is no, then a characterization of the groups having this property would be very interesting. The only groups known to have this property are the cyclic groups (Theorem 2 in [26]) and V_4.

This chapter contains work done jointly with Oliver Pechenik and appearing in [44].

1.7 Definitions and notation

When k is a nonnegative integer, we use $[k]$ to denote the set consisting of the first k positive integers; note that $[0] = \varnothing$. We use $\mathbb{N}$ to denote the set of positive integers. When X is a set and k is a nonnegative integer, we write $\binom{X}{k}$ for the family of all k-element subsets of X. A *graph* G is a pair $(V(G), E(G))$, where $V(G)$ is an arbitrary set and $E(G) \subseteq \binom{V(G)}{2}$. The elements of $V(G)$ are called the *vertices* of G, and the elements of $E(G)$ are called the *edges* of G. When writing edges, we usually suppress set brackets and commas, writing uv instead of $\{u, v\}$. We are not allowing loops and multiedges. When $uv \in E(G)$, we say that u and v are *adjacent*; we also say that v is a *neighbor* of u. Two edges are *incident* if they share a common vertex. We also say that an edge uv is *incident* to the vertices u and v. The *complement* of a graph G, denoted $\overline{G}$, is the graph with $V(\overline{G}) = V(G)$ such that $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$.

An *isomorphism* from a graph G to a graph H is a map $f: V(G) \to V(H)$ such that, for any $u, v \in V(G)$, we have $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$. Two graphs are *isomorphic* if there is an isomorphism from one to the other. When G and H are isomorphic, we write $G \cong H$. The class of graphs isomorphic to a given graph G is called the *isomorphism class* of G. An *automorphism* of a graph G is an isomorphism from G to G. A graph G is *vertex-transitive* if for any $u, v \in V(G)$ some automorphism maps u to v. A graph G is *edge-transitive* if for any $e, f \in E(G)$ some automorphism maps the vertex set of e to the vertex set of f. If $\overline{G}$ is edge-transitive, then G^+ denotes the graph formed by adding any edge of the complement to G.

The *neighborhood* of a vertex v in a graph G, denoted $N_G(v)$, is the set of vertices adjacent to v in G. The *degree* of a vertex v, denoted $d_G(v)$, is the number of vertices adjacent to v, which is also $|N_G(v)|$. When the graph G is understood, we may omit the subscripts in this notation. The *maximum degree* of a graph G, denoted $\Delta(G)$, is $\max\{d_G(v): v \in V(G)\}$. Similarly, the *minimum degree* of G, denoted $\delta(G)$, is $\min\{d_G(v): v \in V(G)\}$. We say that G is *r-regular* if $d_G(v) = r$ for all $v \in V(G)$.

We say that a graph H is a *subgraph* of G, written $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A graph G *contains* a graph H if H is a subgraph of G. We abuse notation by writing $F \subseteq G$ when G contains a subgraph belonging to the isomorphism class F, and indeed in this case we also abuse terminology by saying G contains F. A *spanning subgraph* of G is a subgraph H such that $V(H) = V(G)$. An *induced subgraph* of G, induced by vertex set $S \subseteq V(G)$, is the subgraph with vertex set S and edge set $E(G) \cap \binom{S}{2}$; we denote this subgraph by $G[S]$. When S is a subset of $V(G)$, we use $G - S$ to denote the subgraph of G induced by $V(G) - S$. When T is a subset of $E(G)$, we use $G - T$ to denote the subgraph of G with vertex set $V(G)$ and edge set $E(G) - T$. When removing a single vertex v from G, we write $G - v$ instead of $G - \{v\}$, and similarly, we use $G - uv$ to denote the removal of a single edge uv. We use k^--, k-, and k^+-*vertex* to refer to a vertex with degree at most k, exactly k, and at least k, respectively. A graph is *d-degenerate* if every

subgraph contains a d^--vertex.

A *clique* is a set of pairwise adjacent vertices. An *independent set* is a set of pairwise nonadjacent vertices. An independent set S is *maximal* if no other independent set contains S as a proper subset. The *independence number* of G is the maximum size of an independent set in G and is denoted by $\alpha(G)$. A *matching* in a graph G is a set of pairwise non-adjacent edges of G. A matching M is *maximal* if no other matching contains M as a proper subset. The *matching number* of G is the maximum size of a matching in G and is denoted by $\alpha'(G)$. A vertex v is *covered* by a matching M if v lies in some edge of M; a matching *covers* a set X if it covers every vertex in X.

An n-vertex graph in which any two vertices are adjacent is a *complete graph*. A graph with no edges is *empty* or *trivial*. A graph whose vertex set can be expressed as the union of sets X and Y so that $G[X]$ and $G[Y]$ are empty graphs is a *bipartite graph*; the sets X and Y are the *partite sets* or *parts*. A bipartite graph such that every vertex in one part is adjacent to every vertex in the other is a *complete bipartite graph*. The isomorphism classs of n-vertex complete and empty graphs are denoted by K_n and $\overline{K_n}$ respectively. The isomorphism class of complete bipartite graphs with m vertices in X and n vertices in Y is denoted by $K_{m,n}$.

A graph G is *connected* if for any two vertices u and v, there exists a u,v-path in G. When k is a positive integer, we say that G is *k-connected* if $|V(G)| > k$ and $G-S$ is connected for any $S \subseteq V(G)$ such that $|S| < k$. A *cut-vertex* of a graph G is a vertex v such that $G-v$ has more components thatn G. A *component* of G is a maximal connected subgraph of G.

An n-vertex *path* is a graph whose vertices can be named $v_1, \ldots, v_n$ so that the edge set is $\{v_i v_{i+1} : i \in [n-1]\}$; v_1 and v_n are the *endpoints* of the path. An n-vertex *cycle* is a graph whose vertices can be named $v_1, \ldots, v_n$ so that the edge set is $\{v_i v_{i+1} : i \in [n]\}$, where $v_{n+1} = v_1$. The *length* of a path or cycle is the number of edges. A cycle is *even* if its length is even and *odd* if its length is odd. If G contains no cycles, then G is a *forest*. A connected forest is a *tree*. A *leaf* is a 1-vertex. An n-vertex *star* is an n-vertex tree with $n-1$ leaves. The isomorphism classes of n-vertex paths and cycles are denoted by P_n and C_n.

Two graphs are *disjoint* if they have no common vertices. If P is a path with endpoints u and v and $P \subseteq G$, then we say that P is a u,v-path in G. The *distance* between u and v in G, denoted $d_G(u,v)$, is the minimum length of a u,v-path in G. The *diameter* of a graph G is $\max\{d(u,v) : u,v \in V(G)\}$. A *trail* is alternating sequence of vertices and edges, starting and ending at a vertex, in which each edge is consecutive in the sequence with its two endpoints and in which no edge is repeated. A *circuit* is an equivalence class of closed trails with the same consecutivity of edges. An *Eulerian trail* is a trail in a graph that visits every edge exactly once. Similarly, an *Eulerian circuit* is an Eulerian trail that starts and ends at the same vertex.

An *Eulerian graph* is a graph having an Eulerian circuit. A *Hamiltonian cycle* in a graph G is a cycle that is a spanning subgraph of G.

We denote by Q_n the *n-dimensional hypercube*, defined by $V(Q_n) = \{0,1\}^n$ and $xy \in E(Q_n)$ if and only if x and y differ in exactly one coordinate. The *Kneser graph*, denoted $KG(n,k)$ is the graph whose vertices correspond to the k-element subsets of a set of n elements, in which two vertices are adjacent if and only if the two corresponding sets are disjoint. The *Petersen graph* is $KG(5,2)$.

Given graphs G and H with disjoint vertex sets, the *disjoint union* $G+H$ has vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. A *linear forest* is a disjoint union of paths. The *join* of graphs G and H, denoted[2] $G \oplus H$, is obtained by adding to the disjoint union of G and H the edges $\{uv \colon u \in V(G), v \in V(H)\}$. The *cartesian product* of two graphs G and H, denoted $G \square H$, is the graph with vertex set $V(G) \times V(H)$ such that two vertices are adjacent if and only if they are equal in one coordinate and adjacent in the other. Note that the cartesian product is associative. The *dth cartesian power* of a graph, denoted G^d, is the iterated cartesian product of d copies of G. Given a graph G and an edge $e \in \overline{G}$, we denote by $G+e$ the graph formed by adding e to the edge set of G.

A *digraph G* is a pair $(V(G), E(G))$ where $V(G)$ is a vertex set and the edge set $E(G)$ is a set of ordered pairs of vertices. In each edge, the first vertex is the *tail* and the second vertex is the *head*. We write uv for the edge (u,v) and describe it as an edge *from u to v*; in figures this is denoted by an arrow on the edge pointing from u to v. A *consistent* path or cycle is one in which the head of every edge is the tail of the next edge (except for the last edge in a path).

A *coloring* of a graph G is an assignment of colors to the vertices of G. When k is a nonnegative integer and at most k colors are used in a coloring, we call this a *k-coloring*. A *color class* for a given coloring is the set of all vertices receiving some fixed color. Given a set C of colors, a coloring $f \colon V(G) \to C$ is *proper* if $f(u) \neq f(v)$ whenever $uv \in E(G)$. Thus each color class in a proper coloring induces an empty graph. When G has a proper coloring using at most k colors, we say that G is *k-colorable*. The *chromatic number* of G, denoted $\chi(G)$, is the smallest integer k such that G is k-colorable. Given a set C of colors, an edge-coloring $f \colon E(G) \to C$ is *proper* if $f(d) \neq f(e)$ whenever the edges d and e are incident.

An *edge-coloring* of a graph G is an assignment of colors to the edges of G. When k is a nonnegative integer and at most k colors are used in an edge-coloring, we call this a *k-edge-coloring*. A *color class* for a given edge-coloring is the set of all edges receiving some fixed color.

The *list coloring* model assigns to each vertex v a set $L(v)$ of available colors. An *L-coloring* is a proper

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[2]Popularized by Douglas West, this notation is consistent with the "Czech notation" introduced by Nešetřil in which the notation displays the result of the operation on K_2 and K_2, evokes the additivity of the vertex sets, and avoids conicting with the proper use of "+" for disjoint union

coloring c such that the color for each vertex v comes from the list assigned to it. A graph G is *k-choosable* if an L-coloring exists whenever every vertex is assigned at least k colors. The *choice number* of a graph G, denoted ch(G), is the least k such that G is k-choosable. Since a k-choosable graph must have a list coloring when every vertex is assigned the same list of k colors, we always have ch$(G) \geq \chi(G)$.

An *orientable surface* is a surface without boundary that has an inside and an outside. A *handle* is added to a surface by cutting two holes and adding a tube connecting them. The *genus* of an orientable surface is the number of handles added to form it from a surface homeomorphic to a sphere. A *drawing* of a graph on a surface is a mapping of the vertices into points and the edges into continuous curves on the surface that preserves the incidence relation so that for every edge the images of the endpoints are the endpoints of the image of the edge. Since the incidence relation is preserved, we may view these points and curves as the vertices and edges. By moving edges slightly, we may restrict a drawing to a mapping in which no three edges share a single internal point, that no edge has a vertex as an internal point, that no two edges are tangent, and that no two incident edges cross. In a drawing, a *crossing* of two edges is a common internal point. A graph *embeds* on a surface if there is a drawing of the graph on that surface containing no crossing; a drawing without crossings is an *embedding*. The *genus* of G, denoted $\gamma(G)$, is the minimum number of handles of an orientable surface on which G embeds. A graph is *planar* if it embeds on the plane. A planar graph is *outerplanar* if it has a planar embedding in which all vertices lie on the unbounded face. with k edges. A *2-cell* on a surface is a region in which any closed curve in the interior can be continuously contracted to a single point. A *2-cell embedding* of a graph on a surface is an embedding whose regions are all 2-cells.

A *group* is a set on which an associative binary operation is defined with a two-sided identity element and such that every element has an inverse element. An *abelian group* is a group in which the binary operation is commutative. A *cyclic group* is an abelian group which contains an element g such that every other element of the group may be obtained by applying the group operation repeatedly to multiple copies of g. For every positive integer n, the set of integers modulo n under the operation of addition forms a finite cyclic group denoted $\mathbb{Z}_n$.

Chapter 2

Game f-Matching

2.1 Background

Let $\mathcal{F}$ be a family of graphs. A graph G is $\mathcal{F}$-*saturated* if no subgraph of G is in $\mathcal{F}$, but for any nonadjacent vertices u and v in G the graph obtained by adding uv to G does contain some member of $\mathcal{F}$. When $\mathcal{F} = \{F\}$, we write F-*saturated* instead of $\{F\}$-saturated. A spanning subgraph H of G is $\mathcal{F}$-*saturated relative to* G if no member of $\mathcal{F}$ is a subgraph of H but for every $e \in E(G) - E(H)$ some member of $\mathcal{F}$ is a subgraph of $H + e$.

Fix a graph G and a family $\mathcal{F}$. The $\mathcal{F}$-*saturation game* on G, introduced by Füredi, Reimer, and Seress [20] (see also [11, 43]), begins with the spanning subgraph of G having no edges. Players Max and Min take turns adding edges of G until the subgraph becomes $\mathcal{F}$-saturated relative to G. Max wants the final graph to be large; Min wants it to be small. The *game $\mathcal{F}$-saturation number* of G is the length of the game under optimal play; that is, each player can guarantee doing no worse than this length. We denote the game $\mathcal{F}$-saturation number by $\mathrm{sat}_g(\mathcal{F}; G)$ when Max plays first and by $\widehat{\mathrm{sat}}_g(\mathcal{F}; G)$ when Min plays first. This value is well-defined; it follows by induction on the number of edges that the maximum that Max can guarantee equals the minimum that Min can guarantee. Letting $\mathrm{sat}(\mathcal{F}; G)$ and $\mathrm{ex}(\mathcal{F}; G)$ be the minimum and maximum sizes of $\mathcal{F}$-saturated subgraphs relative to G, we have $\mathrm{sat}(\mathcal{F}; G) \leq \mathrm{sat}_g(\mathcal{F}; G) \leq \mathrm{ex}(\mathcal{F}; G)$. The notation $\mathrm{ex}(\mathcal{F}; G)$ arises from the Turán problem, where $\mathrm{ex}(\mathcal{F}; G)$ is the maximum number of edges in a subgraph of G containing no graph in $\mathcal{F}$.

A *capacity function* f on a graph G assigns a nonnegative integer to each vertex of $V(G)$. An f-*matching* in G is a set $M \subseteq E(G)$ such that the number of edges of M incident to v is at most $f(v)$ for all $v \in V(G)$. When $f(v) = k$ for all $v \in V(G)$, a maximal f-matching in G is simply a $K_{1,k+1}$-saturated subgraph relative to G. The optimization problem of finding a maximum f-matching is a classical problem studied in [17] and more than a hundred other papers, mostly under the name b-*matching*, where the capacity function is named b. We use "f-matching" due to the relationship with f-factors and other uses of f as a capacity function.

Here we consider a competitive version of f-matching. In the *f-matching game* on a graph G, denoted (G, f), players Max and Min alternately choose edges of G to build an f-matching; the game ends when the chosen edges form a maximal f-matching. Max wants the final f-matching to be large; Min wants it to be small. The *game f-matching number* is the size of the final f-matching under optimal play, meaning the common size that each player can guarantee. We denote this value by $\nu_f(G)$ when Max plays first and by $\hat{\nu}_f(G)$ when Min plays first, calling these two versions of the game the *Max-start* and *Min-start* games *with respect to f*.

The *matching game* is the special case of the f-matching game where $f(v) = 1$ for all $v \in V(G)$ (that is, $f \equiv 1$). The matching game is also the special case of the $\mathcal{F}$-saturation game where $\mathcal{F} = \{P_3\}$. Max and Min alternately choose edges forming a matching in a graph G, and the game ends when the chosen edges form a maximal matching. With the same objectives as in the general game, the *game matching number* is the size of the final matching under optimal play. We denote this value by $\nu_1(G)$ when Max plays first and by $\hat{\nu}_1(G)$ when Min plays first. The matching game was studied in [14] and in [53]. When $f \equiv k$ on G, we denote the game matching number by $\nu_k(G)$ when Max plays first and by $\hat{\nu}_k(G)$ when Min plays first.

In general, the outcome of the F-saturation game may depend greatly on which player starts. Let G be obtained from a star with r-edges by subdividing one edge, and let $F = 2K_2$. This is a standard example showing that $\mathrm{sat}_g(F; G)$ can be arbitrarily larger than $\widehat{\mathrm{sat}}_g(F; G)$, with $\mathrm{sat}_g(F; G) = r$ and $\widehat{\mathrm{sat}}_g(F; G) = 2$. Optimal first moves for Max and Min are shown by the bold edges in Figure 2.1. A very simple example showing that $\widehat{\mathrm{sat}}_g(F; G)$ can be arbitrarily larger than $\mathrm{sat}_g(F; G)$ is $G = rP_3$ and $F = P_2 + P_3$. Here $\mathrm{sat}_g(F; G) = 2$ and $\widehat{\mathrm{sat}}_g(F; G) = r$. Optimal second moves for Min and Max are shown by the bold edges in Figure 2.2 where the dashed edge represents up to symmetry the only possible first move.
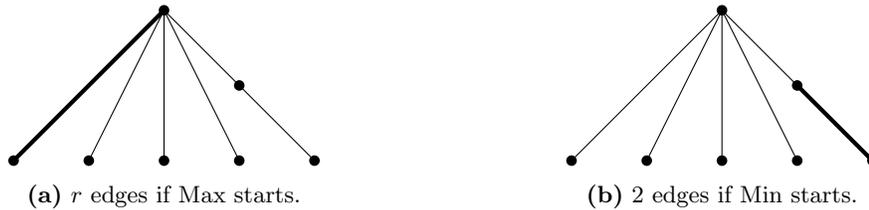

(a) r edges if Max starts.   (b) 2 edges if Min starts.

Figure 2.1: $(2K_2)$-saturation in G.

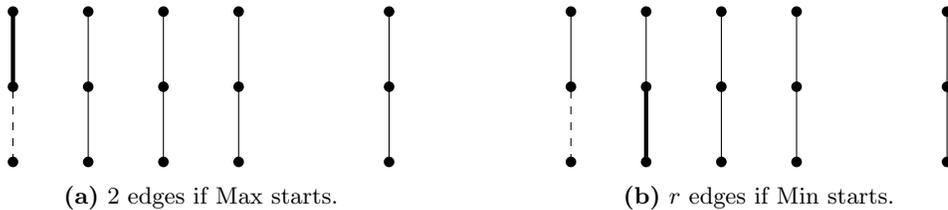(a) 2 edges if Max starts.   (b) r edges if Min starts.

Figure 2.2: $(P_2 + P_3)$-saturation in rP_3.

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Cranston et al. [14] proved that large differences depending on who starts do not occur in the special case of the matching game.

Theorem 2.1.1. *[14] For every graph G, we have $|\nu_1(G) - \hat{\nu}_1(G)| \leq 1$.*

We say that a graph G is *near-fair* for a capacity function f if $|\nu_f(G) - \hat{\nu}_f(G)| \leq 1$. Motivated by the result of [14], we ask whether $|\nu_f(G) - \hat{\nu}_f(G)| \leq 1$ holds for every graph G and every capacity function f. Lacking an answer, we ask the following.

Question 2.1.2. *If G_1 and G_2 are both near-fair for capacity functions f_1 and f_2, then is the disjoint union $G + H$ near fair for the capacity function f that restricts to f_1 and f_2 on G_1 and G_2?*

In Section 2.2, we extend to the f-matching game a lower bound due to Cranston et al. [14] on the game matching number of a graph when Max starts and consider some other aspects of the f-matching game.

In Section 2.3, we consider bounds on the game f-matching numbers of disjoint unions of graphs. Specifically, we consider the disjoint union of a graph H with a complete graph when one player wins (or ties) on H and the other player wins (or ties) on the complete graph. We also consider the disjoint union of any number of complete graphs.

In Section 2.4 we consider a directed version of the f-matching game on a graph G in which players Max and Min alternately choose edges of G and orient them to build an oriented subgraph H of G in which the outdegree of v in H is bounded above by $f(v)$; the indegree is unconstrained. The game ends when no more edges can be chosen without exceeding some given outdegree capacity. Max wants the final subgraph to be large; Min wants it to be small. The *directed game f-matching number* is the size of the final subgraph under optimal play. We denote this value by $\mu_f(G)$ when Max plays first and by $\hat{\mu}_f(G)$ when Min plays first. We say that a digraph G is *near-fair* for a capacity function f if $|\mu_f(G) - \hat{\mu}_f(G)| \leq 1$. We use an auxiliary graph to show that this variation is near-fair.

Cranston et al. [14] also proved that $3\lfloor \frac{n}{7} \rfloor \leq \nu_g(P_n) \leq 3\lceil \frac{n}{7} \rceil$ for all n. There is a gap of 3 between the lower and upper bounds, unless n is a multiple of 7. In Section 2.5, we determine the exact value by solving a more general problem. We determine the exact value of the game matching number for all linear forests.

2.2   Properties of Game f-matching

Given a capacity function f on $V(G)$ and $S \subset V(G)$ such that $f(w) \geq 1$ for $w \in S$, the *S-reduction* of f is the capacity function f' on $V(G)$ defined by $f'(w) = f(w) - 1$ for $u \in S$ and otherwise $f'(w) = f(w)$. When we speak of an *optimal first move* for Max or Min in (G, f), we mean an optimal first move in the Max-start

or Min-start game on G with respect to f, respectively. Note that if $f(v) \geq d(v)$ for a vertex $v \in V(G)$, then there is effectively no restriction on the number of edges played at v.

Our results hold also for multigraphs via the same proofs in the sense that xy indicates one designated edge with endpoints $\{x, y\}$, and similarly $G - xy$ indicates the deletion of one edge with these endpoints. Other copies of the edge remain. This next proposition was stated for the special case $f \equiv 1$ in [14].

Proposition 2.2.1. *Let f be a capacity function on a graph G. If $uv \in E(G)$ with $f(u), f(v) \geq 1$, and f' is the $\{u, v\}$-reduction of f, then $\nu_f(G) \geq 1 + \hat{\nu}_{f'}(G - uv)$, with equality if and only if uv is an optimal first move for Max in (G, f). Similarly, $\hat{\nu}_f(G) \leq 1 + \nu_{f'}(G - uv)$, with equality if and only if uv is an optimal first move for Min in (G, f).*

Proof. The right side of each claimed inequality is the result under optimal play after uv is played as the first move. The optimal move for the first player does at least as well as playing uv first, with equality (by definition) if and only if uv is an optimal first move. $\qquad\square$

When f is a capacity function on a graph G, we also let f denote the capacity function on any induced subgraph of G obtained by restricting f to that vertex set. In [14], Theorem 2.1.1 was proven by proving in the same induction that the removal of any one vertex could not increase ν_1 or $\hat{\nu}_1$. This then allowed for an intermediate step between the game matching number after removing an edge (and thus its endpoints) and the game matching number of the original graph. When $f \equiv 1$, reducing capacity by 1 at the endpoints of a chosen edge has the effect of deleting those endpoints from the graph. When capacities are larger, reducing them not only does not have the effect of deleting the vertices, but also would allow the edge to be picked again if neither endpoint has its remaining capacity reduced to 0. This leads us to ask the following question.

Question 2.2.2. *When f is a capacity function on a graph G, can reducing the capacity on one vertex by 1 increase $\nu_f(G)$ or $\hat{\nu}_f(G)$?*

A possible alternative monotonicity statement for an induction argument comes from the definition of the game f-matching number. Note that $\nu_f(G) = \max_{uv \in E(G)}(1 + \hat{\nu}_{f'}(G - uv))$, where f' denotes the $\{u, v\}$-reduction of f. If $\hat{\nu}_{f'}(G') \leq \hat{\nu}_f(G)$, where $G' = G - uv$ and f' is the $\{u, v\}$-reduction of f, then we obtain $\nu_f(G) \leq 1 + \hat{\nu}_f(G)$. Similarly, $\hat{\nu}_f(G) = \min_{uv \in E(G)}(1 + \nu_{f'}(G - uv))$, where f' denotes the $\{u, v\}$-reduction of f. If $\nu_{f'}(G') \leq \nu_f(G)$, where $G' = G - uv$ and f' is the $\{u, v\}$-reduction of f, then we obtain $\hat{\nu}_f(G) \leq 1 + \nu_f(G)$. If true, then together these inequalities imply the following conjecture.

Conjecture 2.2.3. *If f is a capacity function on a graph G, then $|\nu_f(G) - \hat{\nu}_f(G)| \leq 1$.*

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Let $m_f(G)$ denote the maximum size of an f-matching in G and $\underline{m}_f(G)$ denote the minimum size of a maximal f-matching in G. We write $m_k(G)$ and $\underline{m}_k(G)$ when $f \equiv k$. We begin with a basic result on f-matchings that gives us a relationship between $m_f(G)$ and $\underline{m}_f(G)$ for any graph G and capacity function f on G.

Given an f-matching M, an M-*alternating trail* is a trail that alternates between edges in M and edges not in M. An M-alternating trail whose endpoints have capacity greater than their number of incident edges in M is an M-*augmenting trail*. When f is identically 1, the concept of an M-alternating trail reduces to that of an M-alternating path, where M is a matching. Berge [8] proved that a matching M has maximum size in G if and only if G contains no M-augmenting path. This result has been generalized to f-matchings.

Given a graph G and a capacity function f, let $e(v) = d(v) - f(v)$. We construct the graph G', sometimes known as *Tutte's f-blowup graph* as follows. First replace each vertex $v \in V(G)$ with a copy of $K_{d(v),e(v)}$, where the part of size $d(v)$ is denoted $A(v)$ and the part of size $e(v)$ is denoted $B(v)$. For each edge $uv \in E(G)$, add an edge joining one vertex of $A(v)$ to one vertex of $A(u)$; each vertex of $A(v)$ is incident to exactly one such edge.

Gondran and Minoux [22] generalized Berge's characterization of maximum matchings to maximum f-matchings by applying Berge's result to G'. The key to this proof is that every matching in G' that covers all the vertices of $\bigcup_{v \in V(G)} B(v)$ induces an f-matching in G and that the converse is also true. We give an alternative proof that does not use Tutte's f-blowup graph but instead generalizes Berge's result directly.

Theorem 2.2.4. *For a capacity function f on a graph G, an f-matching M in G is a maximum f-matching in G if and only if G has no M-augmenting trail.*

Proof. If P is an M-augmenting trail, then replacing $M \cap E(P)$ with $E(P) - M$ produces a new f-matching M' with one more edge than M. Thus a maximum f-matching M admits no M-augmenting trail.

We prove the converse by induction on $m_f(G)$. It holds by inspection when $m_f(G) = 0$, meaning that no adjacent vertices have positive capacity. Suppose $m_f(G) > 0$.

Let M' be an f-matching larger than M; we construct an M-augmenting trail. Let F be the spanning subgraph of G whose edge set is the symmetric difference of M and M'. Since M and M' are f-matchings, every vertex v has at most $f(v)$ incident edges from each of M and M'. Hence $\Delta(F) \leq 2 \max_{v \in V(G)} f(v)$.

Since $|M'| > |M|$, some component of F has more edges of M' than M. Start at a vertex v_1 with more incident edges of M' than M and pick an edge $e_1 \in M'$ incident with v_1. If the second endpoint, v_2, of e_1 also has more incident edges of M' than M, then both endpoints have excess capacity relative to M, and e_1 itself forms an M-augmenting path. Otherwise, there is an edge $e_2 \in M$ incident with v_2 and v_3. Add e_2 to the trail, reaching v_3 and ending stage 1.

We continue growing a trail. For $i \geq 2$, at the start of stage i, we have reached vertex v_{2i-1} along the trail via an edge e_{2i-2} of M, and the trail contains the same number of edges from M' and M. If some edge $e_{2i-1} \in M'$ incident with v_{2i-1} is not yet on the trail, then extend the trail along e_{2i-1} (to v_{2i}) and proceed as before; either v_{2i} has more incident edges of M' than M and the M-augmenting trail is complete with excess capacity under M at v_{2i}, or the trail can extend from v_{2i} along an edge $e_{2i} \in M$ not already on the trail. Although we may repeat vertices, we never repeat edges. Since the graph is finite, the process must end.

If the process does not find an M-augmenting trail, then it ends when all edges of M' incident with v_{2i-1} are already on the trail. Let T be the set of edges of the trail. Since every edge of M' on the trail is followed by an edge of M, deleting the edges of T removes the same number of edges from both M and M'. Let f' be the capacity function defined by subtracting from $f(v)$ capacity equal to the number of times T visits v for all $v \in V(G)$. For every vertex $v \in V(T)$, except the endpoints, this means subtracting capacity equal to the number of edges of M (and also of M') in T at v.

We chose the starting vertex, v_1, of T to be a vertex with more incident edges of M' than of M, so T cannot end at v_1, and $f'(v_1)$ is $f(v_1)$ minus one more than the number of edges of M in T at v_1. Since M left excess capacity at v_1, f' still gives v_1 enough capacity for all the edges in $M - T$ at v_1 to be in an f'-matching. Similarly, if v_ℓ is the final vertex of T, then $f'(v_\ell)$ is $f(v_\ell)$ minus the number of edges of M in T at v_ℓ, which is nonnegative ($M' - T$ has fewer edges at v_ℓ, and none not in T).

Thus, since M' and M are f-matchings in G, also $M' - T$ and $M - T$ are f'-matchings in $G - T$. By the induction hypothesis, since $M - T$ is not a maximum f'-matching ($M' - T$ is larger), $G - T$ has an $(M - T)$-augmenting trail; this is also an M-augmenting trail in G. $\qquad\square$

Corollary 2.2.5. $\underline{m}_f(G) \geq \frac{1}{2} m_f(G)$.

Proof. We proceed by induction on $|E(G)|$. If $|E(G)| = 1$, then the statement is true, so let G be such that $|E(G)| > 1$. Let M and M' be maximal f-matchings in G. Let T be an M-alternating trail. Consider the graph $G' = G - E(T)$. Let f' be defined by $f'(v) = f(v)$ if $v \notin V(T)$, $f'(v) = f(v) - 1$ if v is an endpoint of T, and $f'(v) = f(v) - 2$ otherwise. Note that $M - T$ and $M' - T$ are maximal f'-matchings in G'. Since G' has fewer edges, the induction hypothesis says that $\underline{m}_f(G') \geq \frac{1}{2} m_f(G')$. Thus

$$\frac{1}{2} \leq \frac{|M - T|}{|M' - T|} = \frac{|M| - |M \cap T|}{|M'| - |M' \cap T|}.$$

The maximality of M implies that $|M \cap T| \geq \frac{1}{2}|M' \cap T|$. Thus $|M| \geq \frac{1}{2}|M'|$ for all maximal f-matchings in G and $\underline{m}_f(G) \geq \frac{1}{2} m_f(G)$. $\qquad\square$

The special case of the next result for game matching ($f \equiv 1$) was proved in [14].

Theorem 2.2.6. *If G is a graph and f is a capacity function on G, then $\nu_f(G) \geq \frac{2}{3}m_f(G)$.*

Proof. Let M be a largest f-matching on G. As long as there is an unplayed edge of M with positive remaining capacity at both endpoints, Max plays such an edge uv. This reduces the capacities on u and v by 1, yielding f'. Also let $G' = G - uv$. Note that $m_{f'}(G') \geq m_f(G) - 1$, since $M - \{uv\}$ is an f'-matching in G'.

When $m_f(G) \geq 3$, the edge played by Min in response to Max playing uv reduces the capacities of each of its endpoints by 1, yielding a new capacity function f'' and a new graph G'' with one fewer edge than G'. Note that $m_{f''}(G'') \geq m_{f'}(G') - 2$. Hence a round reduces the maximum available size by at most 3 while adding 2 to the number of edges played. When $m_f(G) = j \leq 2$ and Max starts, j edges will be played. Thus if $m_f(G) = j$ before the last move by Max, with $0 \leq j \leq 2$, then $\nu_f(G) \geq \frac{2}{3}(m_f(G) - j) + j \geq \frac{2}{3}m_f(G)$. $\square$

The following is a sharpness example for Theorem 2.2.6 for the case where every vertex has either capacity 1 or capacity k.

Example 2.2.7. Let $G = K_n \oplus \overline{K}_n$. Let T be the clique of dominating vertices, and let S be the remaining independent n-set. Let $f(v) = k$ for all $v \in S$ and $f(v) = 1$ for all $v \in T$. Note that $m_f(G) = n$ and $\underline{m}_f(G) = \lceil \frac{1}{2}n \rceil$. We will show that $\nu_k(G) = \lceil \frac{2}{3}n \rceil$.

The total capacity on vertices of T is $|T|$. Min's strategy on each turn is to play an edge joining two vertices of T if possible and any legal move otherwise. Due to the choice of S and T, every edge in G has an endpoint in T. Thus every move by Max reduces the capacity of at least one vertex of T to 0, and each move by Min (until Min can no longer play an edge between vertices of T) reduces the capacity of two vertices of T to 0. Thus each round increases the size of the f-matching by 2 and decreases the sum of available capacity on T by at least 3, until at most two vertices in T have positive remaining capacity. Thus at most $\lfloor \frac{|T|}{3} \rfloor$ rounds are played, during which at most $\lfloor \frac{2|T|}{3} \rfloor$ edges are added to the f-matching. After these rounds, there is capacity at most 2 left on T, and so at most two more edges can be added to the final f-matching. Thus $\nu_f(G) \leq 3 \lfloor \frac{|T|}{3} \rfloor + 2 \leq \lceil \frac{2}{3}n \rceil$.

As an upper bound (since $|\nu_1(G) - \hat{\nu}_1(G)| \leq 1$), Cranston et al. [11] showed that for the 1-matching game on a graph G, $\hat{\nu}_1(G) \leq \frac{3}{2}\underline{m}_1(G)$, and this is sharp. We believe that this statement also generalizes to the f-matching game.

Conjecture 2.2.8. *If G is a graph and f is a capacity function on G, then $\nu_f(G) - 1 \leq \hat{\nu}_f(G) \leq \frac{3}{2}\underline{m}_f(G)$.*

Note that Corollary 2.2.5 yields $\frac{2}{3}m_f(G) \leq \frac{3}{2}\underline{m}_f(G)$ when $m_f(G) \geq 1$ so Conjecture 2.2.8 does not contradict Theorem 2.2.6. If Conjecture 2.2.8 holds for general f, then the following examples give sharpness for the case of constant capacity and for a case where all capacities are 1 except for one special vertex with a different capacity.

Example 2.2.9. Let G consist of r copies of P_4 with edge-multiplicity k (with kr even). Let $f(v) = k$ for all $v \in (G)$. Note that $\underline{m}_k(G) = kr$. Since the proof that $\nu_k(G) \geq \frac{2}{3}kr$ depends on the game matching number of the linear forest where all components are isomorphic to P_4, we postpone the proof of this example to the end of this chapter.

Example 2.2.10. Let G be the wheel $K_1 \oplus C_r$ with r spokes (and thus r rim vertices), and let the capacity function f give all the rim vertices capacity 1 and the center of the wheel capacity r. Note that $m_f(G) = r$ and $\underline{m}_f(G) = \lceil r/2 \rceil$.

To show that $\nu_f(G) \geq \left\lfloor \frac{3r}{4} \right\rfloor$ and $\hat{\nu}_f(G) \geq \left\lfloor \frac{3r}{4} \right\rfloor$, we give a strategy for Max. Let v_0 be the center vertex and $v_1, \ldots, v_r$ be the rim vertices in clockwise order. It suffices to consider the capacity of the rim vertices, since the capacity on the center of the wheel is large enough to accommodate all incident edges. Let c denote the sum of the capacities remaining on the rim vertices. Note that $c = r$ before any edges are chosen and that the final value of c is 0; if a rim vertex has positive capacity and no incident edge on the rim can be chosen, then the incident spoke can always be picked. Every move decreases c by at least 1 and at most 2.

Let v_i and v_{i+1} be the first consecutive pair of rim vertices with positive capacity going clockwise around the rim from a rim vertex used in Min's latest move. Max picks the spoke edge $v_0 v_{i+1}$. As long as it is not the first move of the game, this effectively adds two edges to the final f-matching and effectively reduces c by 2 since Min cannot block the edge $v_0 v_i$ from being chosen at the end of the game. Note that playing $v_0 v_{i+1}$ might also force the play of $v_0 v_{i+2}$ if v_{i+3} has no remaining capacity. However, since this is not certain, and only helps Max, we may leave $v_0 v_{i+2}$ in play and count it at the end. Min can pick an edge that takes capacity from at most two rim vertices.

If there is no consecutive pair of rim vertices with unused capacity, then every remaining available edge is a spoke, and all remaining spokes whose rim vertices have unused capacity will be included in the final f-matching. Thus, in any given round (until there are only spokes whose rim vertices are isolated on the rim), Max can ensure adding at least three edges to the final f-matching while decreasing c by at most 4. When only spokes remain, all rounds that do not choose edges that Max saved in an earlier round add 2 edges to the final f-matching while decreasing c by 2.

Thus until no consecutive pair of rim vertices has unused capacity, each round reduces c by at most 4 while adding at least three edges to the final f-matching. Each move after no consecutive pair of rim vertices

has unused capacity either reduces c by 1 and adds one edge to the final f-matching, or picks an edge that was already forced to be in the final f-matching and has thus been accounted for already. Therefore, since c is initially r, we have at least $\lfloor \frac{3r}{4} \rfloor$ edges in the final matching.

Thus, the bounds are as stated and, if Conjecture 2.2.8 is true, then it is also sharp.

2.3   Disjoint Unions of Graphs

Note that trying to form a $K_{1,3}$-saturated graph in as few, or as many, moves as possible is the same as trying to create a maximal 2-matching in as few, or as many, moves as possible. Theorem 2.3 of Carraher et al. [11] thus concerns game 2-matching in complete graphs. Here we give corrected version of their posted proof. In analyzing game f-matching inductively, it is helpful to have terminology for the comparison of ν_f and $\hat{\nu}_f$. If $\nu_f(G) > \hat{\nu}_f(G)$, then we say that G is a *first player wins* game. If $\nu_f(G) < \hat{\nu}_f(G)$, then we say that G is a *second player wins* game.

Theorem 2.3.1. *Let f give all vertices of a graph capacity 2.*

For $n \in \{1, 2\}$, $\nu_f(K_n) = \hat{\nu}_f(K_n) = n - 1$.

For $n \in \{3, 4, 7\}$, $\nu_f(K_n) = \hat{\nu}_f(K_n) = n$.

For all odd $n \in \mathbb{N} - \{1, 3, 7\}$, $\nu_f(K_n) = n - 1$ and $\hat{\nu}_f(K_n) = n$.

For all even $n \in \mathbb{N} - \{2, 4\}$, $\nu_f(K_n) = n$ and $\hat{\nu}_f(K_n) = n - 1$.

Proof. Note that $\nu_f(K_n)$ and $\hat{\nu}_f(K_n)$ are both $n - 1$ or n since a maximal 2-matching on K_n consists of disjoint cycles plus possibly a single vertex or a single edge. We can thus say that *Max wins* if there are n edges in the final 2-matching and that *Min wins* if there are only $n - 1$. For odd $n \in \mathbb{N} - \{1, 3, 7\}$ we prove that the second player wins and for even $n \in \mathbb{N} - \{2, 4\}$ we prove that the first player wins. For $n \leq 9$, ad hoc case analysis is needed, but for $n \geq 10$ the argument depends only on the parity of n, inductively.

We proceed by induction on n. For $n \in \{1, 2\}$, no cycle is possible so Min wins no matter who starts. When $n = 3$, no matter who starts, the final 2-matching will be all of K_3 and Max wins.

If $n = 4$ and Max starts, then Max forms P_4 on his second turn. If Min starts, then Max forms $2K_2$ on his first turn. In either case the final 2-matching is a 4-cycle, so Max wins no matter who starts.

If $n = 5$ and Max starts, then Min forms $2K_2$ on the first round and completes either a 3-cycle or a 4-cycle on the second round, leaving an isolated edge or vertex. If $n = 5$ and Min starts, then Max forms $2K_2$ on the first round and P_5 on the second, forcing Min to complete a 5-cycle. Thus $\nu_f(K_5) = 4$ and $\hat{\nu}_f(K_5) = 5$.

If $n = 6$ and Min starts, then Min forms P_4 on the second round and next completes a 4-cycle leaving an isolated edge or a 5-cycle leaving an isolated vertex. Suppose instead that Max starts. If Min forms P_3 on

the first round, then Max completes the 3-cycle, which reduces the problem to Min starting on K_3, where Max wins. If Min forms $2K_2$ on the first round, then Max makes $3K_2$ and next P_6, forcing Min to complete a 6-cycle. Thus $\nu_f(K_6) = 6$ and $\hat{\nu}_f(K_6) = 5$.

Let $n = 7$. If Min starts, then Max first makes P_3 and next completes a 3-cycle or a 4-cycle on the second round, leaving a game on four or three vertices, which Max wins. Suppose instead that Max starts. If Min forms P_3, then Max completes the 3-cycle, reducing the problem to Min starting on K_4, where Max wins. If Min forms $2K_2$ on the first round, then Max makes $3K_2$. On the third round, Max completes either a 3-cycle or a 4-cycle and wins.

For K_8 we first consider the Max-start game. Max forms P_4 on his second turn. If Min extends the path to P_5, then Max completes the 5-cycle and wins on the remaining K_3, ending with the final 2-matching being $C_5 + C_3$. If Min completes the 4-cycle or plays an isolated edge, then Max does the other of these two. In either case, the game reduces to Max starting on K_4, and the final 2-matching is $C_4 + C_4$. Thus $\nu_f(K_8) = 8$.

We now consider Min-start game on K_8. In the first round Max can form P_3 or $2K_2$. If Max forms P_3, then Min completes the 3-cycle, leaving the Max-start game on K_5, which Min wins. If Max forms $2K_2$, then Min forms $P_2 + P_3$. Max again has two choices: pick an edge incident to an already picked edge or pick an independent edge.

If Max picks an independent edge forming $2P_2 + P_3$, then Min completes the 3-cycle, leaving a position equivalent to $2K_2$ in K_5 with Max to move, which we have shown is a winning position for Min in the Max-start game on K_5.

If Max picks an edge adjacent to an already picked edge then one of the following is true: Max has just closed a 3-cycle, or the chosen edges form two paths, or the chosen edges form a 5-vertex path. In the first case, the single edge not on the 3-cycle acts as the first move in the Max-start game on K_5, which Min wins. In the last two cases, Min forms a 6-vertex path, and then Min's next move completes a 7-cycle leaving an isolated vertex or a 6-cycle leaving an isolated edge. Thus $\hat{\nu}_f(K_8) = 7$.

For K_9 we first consider the Max-start game. In the first round Min forms P_3. If Max completes the 3-cycle or plays an isolated edge, then Min does the opposite, leaving a position equivalent to Min-start on K_6, so Min wins. If Max extends the P_3 to P_4, then Min completes the 4-cycle and Max starts on the remaining graph K_5, so Min wins. Thus $\nu_f(K_9) = 8$.

We now consider the Min-start game. In the first round Max forms P_3. If Min completes the 3-cycle, then Max starts on K_6 and wins. Otherwise, Max forms P_5. If Min extends the P_5 to P_6, then Max closes the 6-cycle and wins on the remaining K_3, ending with the final 2-matching being $C_6 + C_3$. If Min completes the 5-cycle or plays an isolated edge, then Max does the other of these two. In either case, the remaining

position is equivalent to Max starting on K_4, and the final 2-matching is $C_5 + C_4$. Thus $\hat{\nu}_f(K_9) = 9$.

We now consider K_n for $n \geq 10$. We first consider the case where n is even. We want to show that Player 1 wins. The first move creates one nontrivial path. As long as there is exactly one nontrivial path, there are three types of edge that Player 2 can choose: an edge extending the path, an isolated edge, or an edge completing the cycle. If there remain more than six isolated vertices, then Player 1 extends the path again in the first case, joins the isolated edge to the existing path in the second case, or picks an isolated edge to start a new path in the last case. Since these are the only moves Player 1 makes, there is always exactly one nontrivial path after Player 1's turn as long as there remain at least than six isolated vertices. Also, there are always an even number of isolated vertices when Player 1 completes his turn, so it will be Player 2's turn when there are exactly six isolated vertices remaining, with exactly one nontrivial path in the $f-$matching. If this nontrivial path consists of exactly one edge, then this is equivalent to Player 1 starting on K_8 where Player 1 wins. If there is a longer path, then Player 2 has three options. If Player 2 ccompletes the cycle or plays an isolated edge, then Player 1 does the opposite and the game reduces to Player 1 starting on K_6, where Player 1 wins. If Player 2 extends the path, then Player 1 completes the cycle and becomes the second player in a game on K_5, thereby winning.

We now consider the case where n is odd. We show that the second player can win. Player 2 forms P_3 on the first round. If Player 1 completes the 3-cycle, then Player 2 becomes the starting player on a smaller even complete graph and wins. If Player 1 does not close the cycle, then Player 2 plays as the first player in a smaller even case, treating the 3-vertex path as P_2. Since the first player's strategy when n is even and at least 10 is to form P_4 on the second round, when n is odd Player 2 forms P_5 (which is then treated as P_4 on $n-1$ vertices) on the second round. Player 1 can no longer complete a 3-cycle using the original P_3, so play can continue as in the even case. Since we have shown that the first player wins on the smaller even case, Player 2 wins here. $\qquad \square$

A common strategy when playing on the disjoint union of graphs is for a player, say Player A, to reply to Player B in the same component of G where Player B just played whenever possible, using an optimal strategy for that component. We call this the *follower strategy*. The following proposition gives bounds on the game f-matching number of $H + K_n$ when one player wins (or ties) on H and the other player wins (or ties) on K_n.

Proposition 2.3.2. *Let G be the disjoint union of two graphs H and K_i for $i \in \mathbb{N}$. Let f be a capacity function on G that gives capacity 2 to all the vertices of the component K_i and restricts to a general capacity function on H.*

If $\nu_f(H) \leq \hat{\nu}_f(H)$, then $\nu_f(G) \geq \hat{\nu}_f(H) + \hat{\nu}_f(K_i)$ and $\hat{\nu}_f(G) \leq \nu_f(H) + \hat{\nu}_f(K_i)$ for even $i \in \mathbb{N} - \{4\}$ or $i \in \{3, 7\}$.

If $\nu_f(H) \geq \hat{\nu}_f(H)$, then $\hat{\nu}_f(G) \leq \hat{\nu}_f(H) + \nu_f(K_i)$ and $\nu_f(G) \geq \nu_f(H) + \nu_f(K_i)$ for odd $i \in \mathbb{N} - \{3, 7\}$ or $i = 4$.

Proof. For even $i \in \mathbb{N} - \{4\}$ or $i \in \{3, 7\}$ and H such that $\nu_f(H) \leq \hat{\nu}_f(H)$, since going second on H makes Player 1 at least as happy as going first would, Player 1 plays on the copy of K_i, forcing Player 2 to go first on H by employing Min's strategy to force only $i - 1$ edges on K_i if $i \notin \{3, 7\}$ and i edges on K_i if $i \in \{3, 7\}$. Whenever Player 2 plays (second) on the K_i, Player 1 also plays on the K_i, keeping play on H alternating with Player 2 starting. The result is then $\nu_f(K_i)$ plus the outcome on H when the other player starts.

For odd $i \in \mathbb{N} - \{3, 7\}$ or $i = 4$ and H such that $\nu_f(H) \geq \hat{\nu}_f(H)$, Player 1 starts on H. Player 1 then employs the follower strategy. For $i \neq 4$, the second player to play on K_i can force there to be only $i - 1$ edges played on K_i if play alternates on K_i, and $i - 1$ is even. For $i = 4$, the second player can force i edges to be played on K_i, and i is even. Whenever Player 2 plays on the K_i, Player 1 will also play next on K_i, keeping play on H alternating with Player 1 starting. If Max is Player 1, then he can force more edges to be played on the K_i, but by doing so will allow Min to play two consecutive moves on H which may allow Min to force fewer edges to be played on H. $\qquad\square$

Proving equality in these bounds, and in general for disjoint unions, requires knowing what will happen if a player goes twice consecutively on one component. This has the effect of changing the starting player in the remaining game. If Conjecture 2.2.3 is true, then going twice in a row makes a difference of at most 1 in the number of edges in the final f-matching. The example on K_n shows that going twice in a row may be beneficial or detrimental to the player in changing the outcome by 1.

Theorem 2.3.3. *Let f be a capacity function on a graph G that gives all vertices capacity 2.*

If $G = \sum K_{n_i}$ for odd $n_i \in \mathbb{N} - \{1, 3, 7\}$, then $\nu_f(G) = \sum(n_i - 1)$ and $\hat{\nu}_f(G) = 1 + \sum(n_i - 1)$.

If $G = \sum K_{n_i}$ for an odd number of even $n_i \in \mathbb{N} - \{2, 4\}$, then $\nu_f(G) = 1 + \sum(n_i - 1)$ and $\hat{\nu}_f(G) = \sum(n_i - 1)$.

If $G = \sum K_{n_i}$ for an even number of even $n_i \in \mathbb{N} - \{2, 4\}$, then $\nu_f(G) = \sum(n_i - 1)$ and $\hat{\nu}_f(G) = 1 + \sum(n_i - 1)$.

Proof. We first note that in all the graphs $\underline{m}_f(G) = \sum(n_i - 1)$.

Suppose first that all the n_i are odd. Note that $\nu_f(K_{n_i}) = n_i - 1$ and $\hat{\nu}_f(K_{n_i}) = n_i$ for each component K_{n_i}. In the Max-start or Min-start game, Min can force at most the claimed final number of edges by employing the follower strategy. Since $n_i - 1$ is even and n_i is odd, in each component of G the last move will be played by Min, and Max will start every component except (possibly) the first. Note that if Max starts then he cannot achieve in any component a final f-matching with more edges than his optimal play.

Thus Max cannot attempt to give away an edge in one component in order to gain more edges in another component. Since Min is employing the follower strategy, there is never a first component that Max finishes.

If Min starts, then Max can play less than optimally in one component to force Min to start the next component, but it will not increase the number of edges in the final f-matching since either Max will do this for every complete graph yielding only $\sum(n_i - 1)$ edges, or Max will have to play optimally on at least one component to gain the final desired edge at which point Max will start the remaining complete graphs as in the Max-start case. This strategy for Min establishes the upperbounds. Since always $\nu_f(G) \geq \underline{m}_f(G)$, the matching lower bound is trivial in the Max-start game. For the Min-start game, we show that Max can force at least one edge more than $\underline{m}_f(G)$. To do this Max employs the follower strategy. Thus in any K_{n_i} that Min starts, which will be at least the first one, Max forces n_i edges giving the extra edge as desired. Therefore, the number of edges in the final f-matching on G will be $\sum \nu(K_{n_i})$ if Max starts and $1 + \sum \nu(K_{n_i})$ if Min starts.

If instead all n_i are even, then $\nu_f(K_{n_i}) = n_i$ and $\hat{\nu}(K_{n_i}) = n_i - 1$ for each component K_{n_i}. In the Max-start or Min-start game, Min can force at most the claimed final number of edges by employing the follower strategy, except that Min never follows Max to pick the second edge in any component unless all first edges have been chosen. When Max plays second in K_{n_i} with n_i even, as follower Min acts as first player, achieving $n_i - 1$ edges. Since $n_i - 1$ is odd and Min acts as first player, Min plays the last move in this component, and Max will have to move to another component. Thus Max will go second in every component except, possibly, one. If Min is Player 1, then he will have to go second once if there are an even number of components. If Min is Player 2, then he will have to go second once if there are an odd number of components. In the other cases, Max will go second in every component. Acting as second player in a component, Max cannot achieve a final f-matching with fewer edges than his optimal play there. Thus Max cannot attempt to give away an edge in one component in order to gain more edges in another component. Since always $\hat{\nu}_f(G) \geq \underline{m}_f(G)$, this is enough to compute the value in the Min-start game on an odd number of components and in the Max-start game on an even number of components.

To solve the Max-start game with an odd number of components or the Min-start game with an even number of components, we show that Max can force at least one edge more than $\underline{m}_f(G)$. To do this Max employs the follower strategy, except that Max never follows Min to pick the second edge in any component unless all first edges have been chosen; if Min plays the first edge in a component, then Max plays the first edge in a new component, if a component exists that has not been played on. Thus Max only picks the second edge in a component once at least one edge of each component has been played. If Min is Player 1, then he will have to go second once if there are an even number of components. If Min is Player 2, then

(a) A graph G with capacity function f.　　**(b)** The auxiliary graph G'.
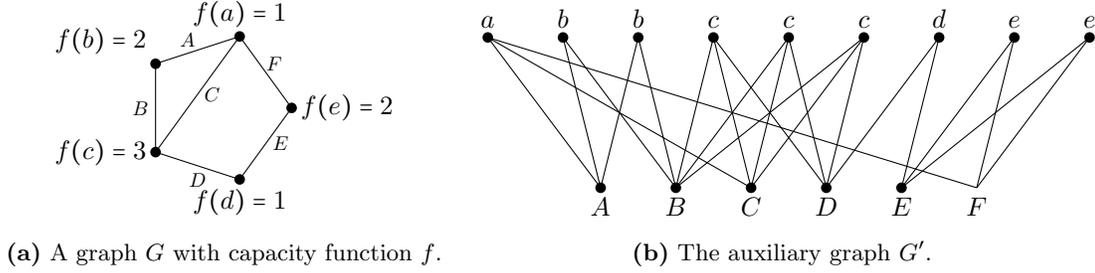
Figure 2.3: A graph and its auxiliary.

he will have to go second once if there are an odd number of components. When Min plays second in K_{n_i} with n_i even, as follower Max acts as first player. If Max plays optimally, he can force n_i edges. On other components, the outcome cannot be less than the number of vertices minus 1, so the lower bound holds. □

2.4 Directed Game f-matching

Motivated by Conjecture 2.2.3, we now consider a directed version of the f-matching game on a graph G in which players Max and Min alternately choose edges of G and orient them to build an oriented subgraph H of G in which the outdegree of v in H is bounded above by $f(v)$; the indegree is unconstrained. The game ends when no more edges can be chosen without exceeding some given outdegree capacity. Max wants the final subgraph to be large; Min wants it to be small. The *directed game f-matching number* is the size of the final subgraph under optimal play. We denote this value by $\mu_f(G)$ when Max plays first and by $\hat{\mu}_f(G)$ when Min plays first, calling these two versions of the game the *Max-start* and *Min-start* games *with respect to f*. We use an auxiliary graph to show that the choice of starting player makes little difference in this variation.

Theorem 2.4.1. *If G is a graph and f is a capacity function on G, then $|\mu_f(G) - \hat{\mu}_f(G)| \le 1$.*

Proof. We begin by building an auxiliary bipartite graph G'. One part consists of the edges of G, and the other consists of $f(v)$ copies of v for each $v \in V(G)$. For $v \in V(G)$, we make each copy of v adjacent in G' to each vertex representing an edge incident to v. Since the choice of starting player in the 1-matching game never makes a difference of more than 1, it suffices to prove $\mu_f(G) = \nu_1(G')$ and $\hat{\mu}_f(G) = \hat{\nu}_1(G')$. An example of a graph G with capacity function f and the corresponding graph G' is shown in Figure 2.3.

We claim that the matching game on G' models the directed f-matching game on G. Picking the edge e oriented away from v in the directed f-matching game on G corresponds to picking an edge ev in the matching game on G'. Each edge of G can be chosen at most once and each vertex in G' representing an

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edge of G can be matched at most once. No vertex $v \in V(G)$ can be matched more than $f(v)$ times in the matching game on G' or have outdegree greater than $f(v)$ in the directed f-matching game on G. □

2.5 Game Matching Numbers of Linear Forests

Let P_n denote the path with n vertices. The following upper and lower bounds for $\nu_1(P_n)$ are obtained in [14].

Theorem 2.5.1. *[14] For all n, we have $3\left\lfloor\frac{n}{7}\right\rfloor \leq \nu_1(P_n) \leq 3\left\lceil\frac{n}{7}\right\rceil$.*

There is a gap of 3 between the lower and upper bounds, unless n is a multiple of 7. Theorem 2.5.1 can be applied to obtain lower and upper bounds for $\nu_1(F)$ when F is a linear forest. However, the gap between the lower and upper bounds will become large when G has many components.

In this section, we determine the exact value of $\nu_1(F)$ and $\hat{\nu}_1(F)$ for every linear forest F. In particular, we determine $\nu_1(P_n)$ and $\hat{\nu}_1(P_n)$ for all n.

We call a component P of a linear forest *critical* if $|V(P)|$ is congruent to 4 or 6 modulo 7. Let $\sigma(P_n) = \left\lfloor\frac{3n+3}{7}\right\rfloor$. Given a linear forest F, let β_F be the number of critical components in F, and let

$$\sigma(F) = \sum\{\sigma(P) : P \text{ is a component of } F\}.$$

Our main result is the computation of $\nu_1(F)$ and $\hat{\nu}_1(F)$ whenever F is a linear forest. We will prove that $\nu_1(F) = \sigma(F) - \left\lfloor\frac{\beta_F}{2}\right\rfloor$ and $\hat{\nu}_1(F) = \sigma(F) - \left\lceil\frac{\beta_F}{2}\right\rceil$. Note that the values for the Max-start and Min-start games differ by at most 1, illustrating Theorem 2.1.1.

For $xy \in E(F)$, let $F_{xy} = F - \{x, y\}$. Note that playing xy leaves the remaining game with P replaced by paths Q_1 and Q_2, where P is the component of containing xy and $P - \{x, y\} = Q_1 + Q_2$ (possibly one is empty). Our proof rests on comparing $\sigma(P)$ and $\sigma(Q_1) + \sigma(Q_2)$ with $3|V(P)|/7$.

Note that

$$\sigma(F) - \sigma(F_{xy}) = \sigma(P) - (\sigma(Q_1) + \sigma(Q_2)). \tag{1}$$

The following lemma follows from the definition of $\sigma(P)$ and the fact that $|V(P)| = |V(Q_1)| + |V(Q_2)| + 2$.

Lemma 2.5.2. *Let $q = |V(P)|$, $q_1 = |V(Q_1)|$, and $q_2 = |V(Q_2)|$. The following hold:*

(a) If P is critical, then

$$\frac{2}{7} \leq \sigma(P) - \frac{3q}{7} \leq \frac{3}{7}.$$

(b) If P is not critical, then

$$-\frac{3}{7} \le \sigma(P) - \frac{3q}{7} \le \frac{1}{7}.$$

(c) If Q_1 and Q_2 are both critical, then

$$-\frac{2}{7} \le \sigma(Q_1) + \sigma(Q_2) - \frac{3q}{7} \le 0.$$

(d) If exactly one of Q_1 and Q_2 is critical, then

$$-1 \le \sigma(Q_1) + \sigma(Q_2) - \frac{3q}{7} \le -\frac{2}{7}.$$

(e) If neither Q_1 nor Q_2 is critical, then

$$-\frac{12}{7} \le \sigma(Q_1) + \sigma(Q_2) - \frac{3q}{7} \le -\frac{4}{7}.$$

Proof. We first compute $\sigma(P_q) - \frac{3q}{7} = \left\lfloor \frac{3q+3}{7} \right\rfloor - \frac{3q}{7}$. Note that the denominator in every case will be 7, so we need only consider the numerator.

$q \bmod 7$	0	1	2	3	5	4	6
$7\left(\left\lfloor \frac{3q+3}{7} \right\rfloor - \frac{3q}{7}\right)$	0	-3	1	-2	-1	2	3

Figure 2.4: $7\left(\sigma(P) - \frac{3q}{7}\right)$

(a) If P is critical, then $q \equiv 4,6 \bmod 7$. The last two columns of Table 2.4 yield $\frac{2}{7} \le \sigma(P) - \frac{3q}{7} \le \frac{3}{7}$.

(b) If P is not critical, then $q \not\equiv 4,6 \bmod 7$. The first five columns of Table 2.4 yield $-\frac{3}{7} \le \sigma(P) - \frac{3q}{7} \le \frac{1}{7}$.

To compute $\sigma(Q_1) + \sigma(Q_2) - \frac{3q}{7}$, let $d(r) = 7\left(\left\lfloor \frac{3s+3}{7} \right\rfloor - \frac{3s}{7}\right)$ for $s \equiv r \bmod 7$. Note that $q = q_1 + q_2 + 2$. Thus, if $q_1 \equiv i \bmod 7$ and $q_2 \equiv j \bmod 7$, then $\sigma(Q_1) + \sigma(Q_2) - 3q/7 = [d(i) + d(j) - 6]/7$. We present a table giving the numerator of $\sigma(Q_1) + \sigma(Q_2) - \frac{3q}{7}$ to assist in finding our bounds.

(c) If Q_1 and Q_2 are both critical, then q_1 and q_2 are congruent to 4 or 6 modulo 7. The bottom right 2-by-2 portion of Table 2.5 yields $-\frac{2}{7} \le \sigma(Q_1) + \sigma(Q_2) - \frac{3q}{7} \le 0$.

(d) If exactly one of Q_1 and Q_2 is critical, then we may assume $q_1 \equiv 4,6 \bmod 7$ and $q_2 \not\equiv 4,6 \bmod 7$. The bottom left 5-by-2 and top right 2-by-5 portions of Table 2.5 yield $-1 \le \sigma(Q_1) + \sigma(Q_2) - \frac{3q}{7} \le -\frac{2}{7}$.

(e) If neither Q_1 nor Q_2 is critical, then $q_1 \not\equiv 4,6 \bmod 7$ and $q_2 \not\equiv 4,6 \bmod 7$. The top left 5-by-5 portion of Table 2.5 yields $-\frac{12}{7} \le \sigma(Q_1) + \sigma(Q_2) - \frac{3q}{7} \le -\frac{4}{7}$. $\qquad\square$

		$q_1 \bmod 7$						
		0	1	2	3	5	4	6
	0	-6	-9	-5	-8	-7	-4	-3
	1	-9	-12	-8	-11	-10	-7	-6
	2	-5	-8	-4	-7	-6	-3	-2
$q_2 \bmod 7$	3	-8	-11	-7	-10	-9	-6	-5
	5	-7	-10	-6	-9	-8	-5	-4
	4	-4	-7	-3	-6	-5	-2	-1
	6	-3	-6	-2	-5	-4	-1	0

Figure 2.5: $7\left(\sigma(Q_1) + \sigma(Q_2) - \frac{3q}{7}\right)$

We prove our theorem by induction on the number of vertices of F. The claim is easy to check if each component of F has at most three vertices. There are then no critical components, and the outcome will always equal the number of nontrivial components, each of which contributes 1 to $\sigma(F)$.

Now assume that F is a linear forest having at least one component with more than three vertices, and assume that the claim holds for linear forests with fewer vertices. It follows from the definition that

$$
\begin{aligned}
\nu_1(F) &= 1 + \max_{e \in F} \hat{\nu}_1(F_e), \\
\hat{\nu}_1(F) &= 1 + \min_{e \in F} \nu_1(F_e).
\end{aligned}
$$

By the induction hypothesis, $\nu_1(F_e) = \sigma(F_e) - \left\lfloor \frac{\beta_{F_e}}{2} \right\rfloor$ and $\hat{\nu}_1(F_e) = \sigma(F_e) - \left\lceil \frac{\beta_{F_e}}{2} \right\rceil$. To prove the claim it thus suffices to prove the following proposition. It reduces the problem to an inductive computation, eliminating the complexities of games and strategies.

Proposition 2.5.3.

$$
\sigma(F) - \left\lfloor \frac{\beta_F}{2} \right\rfloor = 1 + \max_{e \in F} \left(\sigma(F_e) - \left\lceil \frac{\beta_{F_e}}{2} \right\rceil \right), \tag{2}
$$

$$
\sigma(F) - \left\lceil \frac{\beta_F}{2} \right\rceil = 1 + \min_{e \in F} \left(\sigma(F_e) - \left\lfloor \frac{\beta_{F_e}}{2} \right\rfloor \right). \tag{3}
$$

is equivalent to the statement that for any edge e of F,

$$
\left\lfloor \frac{\beta_F}{2} \right\rfloor - \left\lceil \frac{\beta_{F_e}}{2} \right\rceil \le \sigma(F) - \sigma(F_e) - 1 \le \left\lceil \frac{\beta_F}{2} \right\rceil - \left\lfloor \frac{\beta_{F_e}}{2} \right\rfloor \tag{4}
$$

and moreover, the upper and lower bounds are each achieved for some edge (not necessarily the same edge).

Proof. We can rewrite Equation (2) as "$\sigma(F) - \sigma(F_e) - 1 \ge \left\lfloor \frac{\beta_F}{2} \right\rfloor - \left\lceil \frac{\beta_{F_e}}{2} \right\rceil$ for any edge e of F, with equality for some edge e" and Equation (3) as "$\sigma(F) - \sigma(F_e) - 1 \le \left\lceil \frac{\beta_F}{2} \right\rceil - \left\lfloor \frac{\beta_{F_e}}{2} \right\rfloor$ for any edge e of F, with equality

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for some edge e." Together these are precisely the desired statement. $\qquad\square$

Note that the edges achieving equality in Proposition 2.5.3 are optimal moves for the appropriate player. We now prove our main result.

Theorem 2.5.4. *For any linear forest F,*

$$
\begin{aligned}
\nu_1(F) &= \sigma(F) - \left\lfloor \frac{\beta_F}{2} \right\rfloor, \\
\hat{\nu}_1(F) &= \sigma(F) - \left\lceil \frac{\beta_F}{2} \right\rceil.
\end{aligned}
$$

Proof. By Proposition 2.5.3, the proof is reduced to showing that (4) holds for any edge e of F, with equality in each bound for some edge. Let P be the component of F containing e. Recall that $\sigma(F) - \sigma(F_e) - 1 = \sigma(P) - (\sigma(Q_1) - \sigma(Q_2)) - 1$

Case 1: P is critical. In this case, Q_1 and Q_2 are not both critical.

If Q_1 or Q_2 is critical, then $\beta_F = \beta_{F_e}$. By (a) and (d) of Lemma 2.5.2,

$$
\frac{2}{7} + \frac{2}{7} \le \sigma(P) - (\sigma(Q_1) + \sigma(Q_2)) \le \frac{3}{7} + 1.
$$

As $\sigma(P) - (\sigma(Q_1) + \sigma(Q_2))$ is an integer, we conclude by (1) that $\sigma(P) - (\sigma(Q_1) + \sigma(Q_2)) - 1 = 0$. Since $\beta_F = \beta_{F_e}$, (4) holds, with equality throughout.

If neither Q_1 nor Q_2 is critical, then $\beta_{F_e} = \beta_F - 1$. Hence $\left\lfloor \frac{\beta_F}{2} \right\rfloor - \left\lceil \frac{\beta_{F_e}}{2} \right\rceil = 0$ and $\left\lceil \frac{\beta_F}{2} \right\rceil - \left\lfloor \frac{\beta_{F_e}}{2} \right\rfloor = 1$, and we need to show $0 \le \sigma(P) - (\sigma(Q_1) + \sigma(Q_2)) - 1 \le 1$. By (a) and (e) of Lemma 2.5.2,

$$
\frac{2}{7} + \frac{4}{7} \le \sigma(P) - (\sigma(Q_1) + \sigma(Q_2)) \le \frac{3}{7} + \frac{12}{7}. \tag{5}
$$

As $\sigma(P) - (\sigma(Q_1) + \sigma(Q_2))$ is an integer, $0 \le \sigma(P) - (\sigma(Q_1) + \sigma(Q_2)) - 1 \le 1$.

Case 2: P is not critical. We consider the three subcases by how many of $\{Q_1, Q_2\}$ are critical, obtaining (4) in each case.

If both are critical, then $\beta_{F_e} = \beta_F + 2$. Hence $\left\lfloor \frac{\beta_F}{2} \right\rfloor - \left\lceil \frac{\beta_{F_e}}{2} \right\rceil \le -1$ and $\left\lceil \frac{\beta_F}{2} \right\rceil - \left\lfloor \frac{\beta_{F_e}}{2} \right\rfloor \ge -1$. By (b) and (c) of Lemma 2.5.2,

$$
-\frac{3}{7} \le \sigma(P) - (\sigma(Q_1) + \sigma(Q_2)) \le \frac{1}{7} + \frac{2}{7}.
$$

As $\sigma(P) - (\sigma(Q_1) + \sigma(Q_2))$ is an integer, $\sigma(P) - (\sigma(Q_1) + \sigma(Q_2)) - 1 = -1$.

If exactly one of Q_1 and Q_2 is critical, then $\beta_{F_e} = \beta_F + 1$. Hence $\left\lfloor \frac{\beta_F}{2} \right\rfloor - \left\lceil \frac{\beta_{F_e}}{2} \right\rceil = -1$ and $\left\lceil \frac{\beta_F}{2} \right\rceil - \left\lfloor \frac{\beta_{F_e}}{2} \right\rfloor = 0$.

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By (b) and (d) of Lemma 2.5.2,

$$-\frac{3}{7} + \frac{2}{7} \le \sigma(P) - (\sigma(Q_1) + \sigma(Q_2)) \le \frac{1}{7} + 1. \tag{6}$$

As $\sigma(P) - (\sigma(Q_1) + \sigma(Q_2))$ is an integer, $-1 \le \sigma(P) - (\sigma(Q_1) + \sigma(Q_2)) - 1 \le 0$.

If neither Q_1 nor Q_2 is critical, then $\beta_{F_e} = \beta_F$. By (b) and (e) of Lemma 2.5.2,

$$-\frac{3}{7} + \frac{4}{7} \le \sigma(P) - (\sigma(Q_1) + \sigma(Q_2)) \le \frac{1}{7} + \frac{12}{7}. \tag{7}$$

As $\sigma(P) - (\sigma(Q_1) + \sigma(Q_2))$ is an integer, $\sigma(P) - (\sigma(Q_1) + \sigma(Q_2)) - 1 = 0$, establishing (4) with equality throughout.

We have shown that (4) holds with equality throughout if we play an edge on a critical component and leave a critical component and a non-critical component. We also showed that (4) holds with equality throughout if we play an edge on a non-critical component and do not create a critical component. If $q \notin \{4, 6\}$, then there is such an edge for every congruence class modulo 7. The following table gives one option for the values of q_1 and q_2 modulo 7 for each congruence class of q; some congruence classes have another option.

$q \bmod 7$	0	1	2	3	5	4	6
$q_1 \bmod 7$	2	3	0	0	1	4	0
$q_2 \bmod 7$	3	3	0	1	2	5	4

Figure 2.6: Choosing an edge to attain equality in (4).

We end by considering $n \in \{4, 6\}$. If $n = 4$, then since there is one less critical component in F_e than in F, we have $\beta_{F_e} = \beta_F - 1$. Hence $\left\lfloor \frac{\beta_F}{2} \right\rfloor - \left\lceil \frac{\beta_{F_e}}{2} \right\rceil = 0$ and $\left\lceil \frac{\beta_F}{2} \right\rceil - \left\lfloor \frac{\beta_{F_e}}{2} \right\rfloor = 1$. If e is an end edge of P, then $q_1 = 2$ and $q_2 = 0$ so $\sigma(P) - (\sigma(Q_1) + \sigma(Q_2)) - 1 = 0$. If e is the central edge of P, then $q_1 = q_2 = 1$ so $\sigma(P) - (\sigma(Q_1) + \sigma(Q_2)) - 1 = 1$.

Let $n = 6$. If there is one less critical component in F_e than in F, we have $\beta_{F_e} = \beta_F - 1$. Hence $\left\lfloor \frac{\beta_F}{2} \right\rfloor - \left\lceil \frac{\beta_{F_e}}{2} \right\rceil = 0$ and $\left\lceil \frac{\beta_F}{2} \right\rceil - \left\lfloor \frac{\beta_{F_e}}{2} \right\rfloor = 1$. If e is the second or fourth edge of P, then $q_1 = 1$ and $q_2 = 3$ so $\sigma(P) - (\sigma(Q_1) + \sigma(Q_2)) - 1 = 1$. If e is the central edge of P, then $q_2 = q_2 = 2$ so $\sigma(P) - (\sigma(Q_1) + \sigma(Q_2)) - 1 = 0$.

Therefore (4) holds for any edge e of F, with equality in each bound for some edge. □

We can now prove that Example 2.2.9 is really a sharpness example for Conjecture 2.2.8.

Proposition 2.5.5. *If G consists of r copies of P_4 with edge-multiplicity k (with kr even), and $f(v) = k$ for all $v \in (G)$, then $\nu_k(G) \ge \frac{2}{3}kr$.*

Proof. Note that $\underline{m}_k(G) = kr$. Let G' consis of kr components, all copies of P_4. We use the matching game on G' to give a strategy for Max. Number the edges of P_4 with 1, 2, 3 in order. We denote P_4 with edge-multiplicity k by $P_4^{(k)}$. Group the kr copies of P_4 in G' into r copies of kP_4 and index these copies from 1 to r.

Whenever Max's strategy in G' is to play an edge numbered j in one of the copies of P_4 in the ith set in G', then Max plays a copy of an edge numbered j in the ith copy of $P_4^{(k)}$ in G. If Min responds with an edge numbered j' in the i'th P_4' in G, then Max plays in G' as if Min played an edge numbered j' in the lowest indexed of the copies of P_4 in the i'th set in G' where an edge numbered j' is allowed to be played.

Note that no more than k edges will be played incident to the ℓth vertex in the ith copy of P_4' in G; otherwise more than k edges incident to copies of the ℓth vertex in the ith group of P_4s in G' were played. This cannot happen since there are only k copies of the ℓth vertex in the ith group, and none of them can have more than one incident edge played.

Since Max plays optimally on G', and as many edges are played on G as on G', by Theorem 2.5.4, $\nu_1(G) \geq \nu_k(G') \geq \frac{2}{3}kr$. $\qquad\qquad\square$

Chapter 3

Fool's Solitaire

3.1 Background

Peg Solitaire is a game generalized to graphs by Beeler and Hoilman [5]. In the Peg Solitaire game on graphs, each vertex except one starts with a peg. Vertices without pegs are said to be *holes*. If adjacent vertices x and y have pegs, and z adjacent to y is a hole, then we may *jump* the peg at x over the peg at y and into the hole at z. This removes the peg at y so that x and y become holes and z has a peg. We denote this jump, shown in Figure 3.1, by xyz.
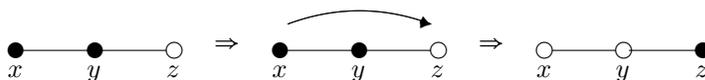
Figure 3.1: A typical jump

In general, if we start with some configuration of pegs and holes, and some succession of jumps reduces the number of pegs to 1, then the configuration is *solvable*. In the Peg Solitaire game on a graph G, if some configuration with a hole at one vertex and pegs at all other vertices is solvable, then we say G is *solvable*. If G can be solved starting with a single hole at any vertex, then G is *freely solvable*. Note that solvability requires G to be connected.

Beeler and Hoilman [5] determined which graphs are solvable and freely solvable among stars, paths, cycles, complete graphs, and complete bipartite graphs. They also proved that cartesian products of solvable graphs are solvable and gave additional sufficient conditions for the solvability of cartesian products of graphs. Walvoort [52] determined which trees of diameter 4 are solvable.

An alternate goal for the Peg Solitaire game was proposed by Beeler and Rodriguez [6]. In the *Fool's Solitaire* game, we instead try to maximize the number of pegs at the end of the process (when there are no remaining available moves). A *terminal state* is a set of vertices that are the final locations of pegs when the game is played starting with a single hole, and played until no more jumps are possible. Because no more

jumps are possible from a terminal state and any two adjacent pegs allow for a jump (when the graph is connected and has a hole), all terminal states are independent sets of vertices. The *Fool's Solitaire number* of a graph G is the maximum number of pegs in a terminal state and is denoted $F(G)$. A fundamental observation follows from the fact that moves from a configuration are the reverse of moves that reach the complementary configuration. If we have a configuration Q reached by playing Peg or Fool's Solitaire from the configuration starting with a hole in vertex v, then we can switch the holes and the pegs in Q and play the game where we reverse the order and direction of the jumps to end with a single peg in vertex v and vice versa. This gives us the following proposition.

Proposition 3.1.1. *[6] A set of vertices T is a terminal state of some Peg/Fool's Solitaire game on G if and only if a starting configuration with holes at vertices of T and pegs at vertices of $V(G) - T$ can be reduced to a single peg.*

Proposition 3.1.1 is used in our proofs of lower bounds on the Fool's Solitaire number. Beeler and Rodriguez [6] also proved

Proposition 3.1.2. *[6] Let G be a graph. Because terminal states are independent sets, $F(G) \leq \alpha(G)$. Also, if $\alpha(G) \leq |V(G)| - 2$ and $V(G) - A$ is independent whenever A is a maximum independent set, then $F(G) \leq \alpha(G) - 1$.*

The proposition holds since if the complement of every maximum independent set is independent and has at least two vertices, then by Proposition 3.1.1 no maximum independent set can be the terminal state of a solitaire game.

The Fool's Solitaire numbers for complete graphs, stars, complete bipartite graphs, paths, cycles, and hypercubes were found in [6]. The Fool's Solitaire number of trees with diameter 4 was computed by Walvoort [52]. In particular, there is a class of trees with diameter 4 for which $\alpha(G) - F(G)$ approaches $\alpha(G)/6$, disproving an earlier conjecture that $\alpha(G) - F(G) \leq 1$ [6]. It remains open how small $F(G)$ can be in terms of $\alpha(G)$.

Beeler and Rodriguez [6] proved $F(K_{n,m}) = \alpha(K_{n,m}) - 1$, and thus Proposition 3.1.2 (which yields the upper bound) is sharp. In Section 3.2, we extend their result on complete bipartite graphs by determining the Fool's Solitaire number of all graphs whose complements are disconnected.

Beeler and Rodriguez [6] also asked for the behavior of the Fool's Solitaire number under the cartesian product operation. In Section 3.3, we show $F(G \square K_k) = \alpha(G \square K_k)$ for $k \geq 3$ when G is any connected graph. However, this behavior does not hold when $k = 2$. In particular, we show that if G is a bipartite graph with a Hamiltonian path, then $F(G \square K_2) = \alpha(G \square K_2) - 1$.

In Section 3.4, we give sufficient conditions for $F(G \square H) \geq F(G)F(H)$. This is a partial answer to the question in [6] asking for the relationship among $F(G), F(H)$, and $F(G \square H)$.

3.2 Joins

Note that the join of any two graphs is connected, and joins are precisely the graphs whose complements are disconnected. For the complete bipartite graph $K_{n,m}$ with $n \geq m > 1$, Beeler and Rodriguez [6] showed $F(K_{n,m}) = n - 1$. By viewing $K_{n,m}$ as $\overline{K}_n \oplus \overline{K}_m$, we expand their method to find the Fool's Solitaire number of all graph joins, starting with the case of joins with K_1.

Lemma 3.2.1. *If G is a graph, then $F(G \oplus K_1) = \alpha(G \oplus K_1)$.*

Proof. Always $F(G \oplus K_1) \leq \alpha(G \oplus K_1) = \alpha(G)$, so we must show $F(G \oplus K_1) \geq \alpha(G \oplus K_1)$. If $G = \overline{K}_n$, then $G \oplus K_1$ is a star and $F(G \oplus K_1) = \alpha(G \oplus K_1)$, because there are no available moves if we place the starting hole at the center of the star. Otherwise, let S be a largest independent set of G, and let z be the vertex outside G. We wish to show that S is a terminal state; by Proposition 3.1.1 it suffices to solve the game where S gives the locations of the starting holes. Since S is a maximum independent set, every peg is adjacent to a hole in G. Start by jumping any peg in G over the peg at z to land in a hole adjacent to another peg in G. We now have two adjacent pegs and we next jump one over the other to land in the hole at z. By repeating this two-jump process, the number of pegs is reduced to 1. $\square$

The remaining case is when $G \oplus H$ is not a complete bipartite graph and has no dominating vertex.

Theorem 3.2.2. *If G and H are graphs with $|V(G)|, |V(H)| \geq 2$ and $|E(G)| + |E(H)| \geq 1$, then $F(G \oplus H) = \alpha(G \oplus H)$.*

Proof. Always $F(G \oplus H) \leq \alpha(G \oplus H)$, so we must show $F(G \oplus H) \geq \alpha(G \oplus H)$. Without loss of generality, $\alpha(G) \geq \alpha(H)$, so $\alpha(G \oplus H) = \alpha(G)$. Motivated by Proposition 3.1.1, consider the complementary problem, where the set of holes is some maximum independent set S in G.

If G has an edge, then G has a vertex with a peg and a vertex without a peg. Jump a peg from G over a peg in H and land in G. Because all vertices in H start with pegs and $|V(H)| \geq 2$, there remains a peg in H, so H now has a peg and a hole. Jump a peg from H over a peg in G, landing in a hole in H. Continue jumping from H over G to H until there is exactly one peg left in G. Because $|V(G)| \geq 2$, there is a hole in G. Now we can jump the peg in G over a peg in H and land in G until all pegs in H are removed. This leaves a single peg.

If G has no edge, then H has an edge e. Also $S = V(G)$, so every vertex of G is a hole and every vertex of H has a peg. In particular, both endpoints of e have pegs. Use one to jump over the other and into a hole in G. Now because $|V(G)| \geq 2$, we may repeatedly jump the peg in G over a peg in H to land in a hole in G until all pegs in H are gone. This leaves a single peg. $\square$

The proof Beeler and Rodriguez [6] give for $F(K_{n,m})$ uses Proposition 3.1.2 for the upper bound and Proposition 3.1.1 for the lower bound. Assuming that $n \geq m \geq 2$, they start with holes at $n-1$ of the vertices in the larger part and jump the single peg in this part over pegs in the other part until the configuration is reduced to a single peg. Our proof above extends this concept to general graphs. Combining our results with theirs, we have $F(G \oplus H) = \alpha(G \oplus H)$ unless G and H are both independent and have at least two vertices, in which case $F(G \oplus H) = \alpha(G \oplus H) - 1$. This gives the Fool's Solitaire number of all graphs whose complements are disconnected.

3.3   Cartesian Products

In this section we find $F(G \square K_k)$ for $k \geq 3$ when G is a connected graph. A cartesian product is connected if and only if both factors are connected.

We start with three lemmas that aid in finding $F(G \square K_k)$. The first two discuss the location of the final peg when solving a complete graph. Note that since $\alpha(K_n) = 1$, the Fool's Solitaire game on the complete graph is the same as the Peg Solitaire game on the complete graph.

Lemma 3.3.1. *For $k > 4$, the Peg/Fool's Solitaire game on K_k with initial hole at a specified vertex may end with the final peg at any vertex.*

Proof. Let v be the vertex required to be occupied at the end of the game. Since it takes $k-2$ jumps to end the game, at least three jumps occur.

If v starts with the hole, then the first jump ends with a peg at v. With the second jump, we can jump a peg over the peg at v to one of the new holes. Now $K_k - v$ has at least one hole and we can play on $K_k - v$ until two pegs remain. Finally, jump one peg over the other to leave the last peg at v.

If v starts with a peg, then we can first jump it over another peg and land in the hole. Now proceed on $K_k - v$ as in the previous case. $\square$

Lemma 3.3.2. *The Peg/Fool's Solitaire game on K_4 may end with the peg in any location except the location of the starting hole.*

Proof. Let u be the location of the initial hole and v be the vertex required to be occupied at the end of the game. Two jumps will end the game. Because the first jump must end with a peg at u, the second jump must end with no peg at u. Hence we cannot have $v = u$. If $v \neq u$, then the first jump can remove the peg at v, and the second jump can land there. □

In contrast, in the Peg/Fool's Solitaire game on K_3 there is a single jump. Therefore, the final peg must be at the location of the starting hole. Lacking the flexibility guaranteed by Lemma 3.3.1 and 3.3.2, when studying $G \square K_3$ we use a property of the game on $P_2 \square K_3$.

Figure 3.2: Cases for peg placement in the proof of Lemma 3.3.3.

Lemma 3.3.3. *For the peg game on $P_2 \square K_3$ such that each triangle has at least one peg and at least one hole and the locations of the starting pegs do not form an independent set, a succession of jumps can lead to no pegs on one triangle and at least one peg and one hole on the other triangle. If there is only one peg at the end, then there are two possible locations for that peg in the specified triangle.*

Proof. Let T_1 and T_2 be the copies of K_3, where T_1 is the copy we wish to clear. In each subfigure of Figure 3.2, T_1 is on the left and T_2 is on the right. The arrows give the second edge involved in the jump(s) and are numbered to indicate order of moves. If a vertex in a Figure is solid, then there is a peg there; if a vertex is an empty circle, then there is a hole.

We first consider the case where there are two pegs on T_1. If there are two pegs on T_2, then we make two jumps. The first is within T_2 leaving two holes in T_2 and allowing us to jump one peg in T_1 over the other into T_2. An example of this is shown in Figure 3.2a. If there is one peg in T_2, then we start with two holes in T_2 and we jump one peg in T_1 over the other into T_2. An example of this configuration is shown in Figure 3.2b. Either way, we end with two pegs on T_2 and no pegs on T_1.

We next consider the case when there is one peg in T_1. First suppose that there are two pegs on T_2. Either the peg on T_1 is adjacent to a peg on T_2 or it is not. These cases are illustrated in Figures 3.2c and 3.2d.

If possible, we jump the peg on T_1 over a peg on T_2 into T_2. This leaves T_1 with no pegs and and T_2 with two pegs. If this jump is unavailable, we instead make a jump within T_2. This leaves a peg in T_2 adjacent to the peg in T_1. We can then jump the peg in T_1 over the peg in T_2 into either hole in T_2. Suppose instead that T_2 starts with one peg. By our assumption that the locations of the pegs do not form an independent set, the peg on T_2 is adjacent to the peg on T_1. Jump the peg on T_1 over the peg in T_2 into either hole in T_2. $\qquad\square$

We can now use the lemmas to find the Fool's Solitaire number of $G \square K_k$ when $k \geq 3$. Berge [8] proved that $\alpha(G \square K_k) = |V(G)|$ if and only if $k \geq \chi(G)$, where $\chi(G)$ is the chromatic number of G.

Theorem 3.3.4. *Let G be a connected graph. If $k \geq 3$, then $F(G \square K_k) = \alpha(G \square K_k)$. In particular, $F(G \square K_k) = |V(G)|$ when $k \geq \chi(G)$.*

Proof. We will denote the copy of K_k that contains all copies of a vertex $v \in V(G)$ by $K(v)$. The vertices of $K(v)$ will be $\{v_1, \ldots, v_k\}$, where v_i plays the role of v in the ith copy of G. By Proposition 3.1.1, it suffices to show that some configuration with holes at a maximum independent set can be reduced to a single peg. Start with holes at a maximum independent set S in $G \square K_k$. Note that S has at most one vertex in each copy of K_k. We perform jumps in two phases.

Phase 1 achieves a configuration in which each copy of K_k has exactly one hole. Since S is a maximum independent set, for any copy $K(v)$ of K_k having no hole, there is an edge $uv \in E(G)$ such that $K(u)$ has one hole (otherwise a hole can be added in $K(v)$ to enlarge S). Jump a peg from $K(v)$ over a peg in $K(u)$ into the hole in $K(u)$. Now both $K(v)$ and $K(u)$ have one hole. Making such a jump for every copy of K_k having no hole independently completes Phase 1.

For Phase 2, let T be a spanning tree of G.

Case 1: $k \geq 4$. Let v be a leaf of T, and let u be the neighbor of v in T. Let u_i be the vertex of $K(u)$ that has a hole. Since $K(v)$ has a single hole, we may solve $K(v)$. Because $k \geq 4$, by Lemmas 3.3.1 and 3.3.2, we may choose the location of the final peg on $K(v)$ to be v_j with $j \neq i$. We can then jump $v_j u_j u_i$ (see Figure 3.3). Now $K(u)$ has a hole only at u_j. Remove v from T and repeat this process with a new leaf. Continue until the remaining pegs lie in a single complete subgraph, which is solvable.
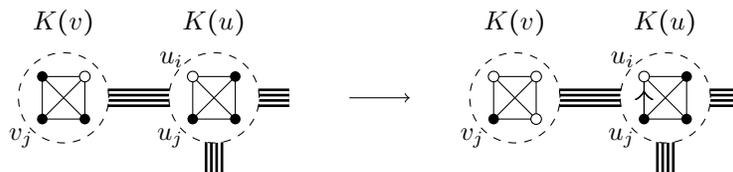

Figure 3.3: Case 1 of Phase 2 in the proof of Theorem 3.3.4.

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Case 2: $k = 3$. Because we cannot control the location of the final peg in each copy of K_3, the previous strategy does not work, and we instead use Lemma 3.3.3. Again let v be a leaf of T and u be the neighbor of v in T. When $K(v)$ has one or two pegs, Lemma 3.3.3 allows us to remove all pegs from $K(v)$ and leave one or two pegs on $K(u)$. Remove v from T and repeat with a new leaf. Continue until the remaining pegs lie in a single copy of K_3, which is solvable. The only possible problem with this strategy is that Lemma 3.3.3 does not apply when each of $K(v)$ and $K(u)$ has only one peg and they sit at nonadjacent vertices. Since each copy of K_3 starts with at least two pegs, this situation arises only for adjacent vertices of T from which neighbors have been eliminated.

Suppose that the neighbor of $K(v)$ most recently losing all pegs is $K(x)$ and the neighbor of $K(u)$ most recently losing all pegs is $K(y)$ (see Figure 3.4). In the applications of Lemma 3.3.3 to the pair $K(x)$ and $K(v)$ and the pair $K(y)$ and $K(u)$, there were two choices for the location of the remaining peg on $K(v)$ and on $K(u)$. Since two element subsets of a set of three indices have a common element, we may choose the moves in the application of Lemma 3.3.3 to $K(x)$ and $K(v)$ and to $K(y)$ and $K(u)$ so that the pegs on $K(v)$ and $K(u)$ are adjacent. Also, choosing these moves does not affect future applications of Lemma 3.3.3 involving $K(u)$, since we have two choices for the location of the peg resulting from the application of Lemma 3.3.3 to $K(v)$ and $K(u)$. □

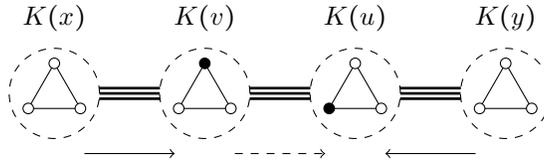Figure 3.4: Possible problem in Case 2 of Phase 2 in the proof of Theorem 3.3.4.

The methods above do not work when $k = 2$. Since $K_2 = P_2$, it follows from Theorem 3.3.6 that $k \geq 3$ is required to guarantee that $F(G \square K_k) = \alpha(G \square K_k)$ for every graph G.

Lemma 3.3.5. *If H is a connected, n-vertex, bipartite graph having a Hamiltonian path and at least four vertices, then $F(H) \geq \lceil \frac{n}{2} \rceil - 1 = \alpha(H) - 1$.*

Proof. Let $v_1, \ldots, v_n$ in order form a Hamiltonian path in H. Because H is bipartite and has a Hamiltonian path, $\alpha(H) = \lceil \frac{n}{2} \rceil$. To show $F(H) \geq \alpha(H) - 1$, we claim that the set of odd-indexed vertices other than v_1 is a terminal state. Consider the game that starts with pegs at the even-indexed vertices and at v_1. To solve this configuration, jump the peg at v_1 over the pegs at the even-indexed vertices from smallest index to largest index. If n is odd, then the process ends with this peg at v_n and no other pegs. If n is even, then the process ends with this peg at v_{n-1} and a peg at v_n. Performing the jump $v_n v_{n-1} v_{n-2}$ then leaves a single

peg. □

Theorem 3.3.6. *If G is a connected, bipartite graph having a Hamiltonian path and at least two vertices, then $F(G \square P_k) = \alpha(G \square P_k) - 1$ for $k \geq 2$.*

Proof. Let $X \cup Y$ be the bipartition of G. Without loss of generality, we may assume $|X| = \lceil \frac{n}{2} \rceil$, and $|Y| = \lfloor \frac{n}{2} \rfloor$, where $n = |V(G)|$. In $G \square P_k$ we have k copies of G, say $G_1, \ldots, G_k$, corresponding to the vertices of P_k. Let S be the set of vertices of X in G_i for odd i and vertices of Y in G_i for even i; S is an independent set of size $\lceil \frac{k}{2} \rceil |X| + \lfloor \frac{k}{2} \rfloor |Y|$ in $G \square P_k$. This forms a maximum independent set because an independent set can contain a copy of $v \in V(G)$ in at most one of G_i and G_{i+1}. Note that $V(G \square P_k) - S$ is also an independent set.

Let S' be an independent set of size $\lceil \frac{k}{2} \rceil \lceil \frac{n}{2} \rceil + \lfloor \frac{k}{2} \rfloor \lfloor \frac{n}{2} \rfloor$. We claim that $S' = S$ or $S' = V(G \square P_k) - S$. Let Q be a Hamiltonian path in G; let Q_i denote the copy of Q in G_i. Consider the grid $Q \square P_k$. Since G_i corresponds to the ith vertex of P_k, each Q_i corresponds to a row. Since every independent set in G is independent in $Q \square P_k$, it suffices that the only independent sets of size $\alpha(G)$ in $Q \square P_k$ are S and its complement. To show that $S' = S$ (or $S' = V(Q \square P_k) - S$ for even k), we use induction on k. For $k \in \{1, 2\}$, the statement follows by inspection. For $k \geq 3$, consider Q', where Q' consists of $Q \square P_k$ minus the last two rows (Q_{k-1} and Q_k). Because the last two rows contribute at most n vertices to the independent set, reaching the desired size requires the restriction to the first $k - 2$ rows to be a largest independent set in that subgraph (regardless of the parity of k). If k is odd, then the last row of Q' is an odd-indexed copy of Q and, by the induction hypothesis, we use the vertices of $|X|$ in our independent set. Similarly, if k is even, then applying the induction hypothesis yields that the last row contributes all its vertices of Y or all its vertices of X. Hence in order to achieve the desired number of vertices in our maximum independent set in the larger graph, we must use the desired vertices from the last two rows of $Q \square P_k$.

Since G has at least two vertices and $k \geq 2$, each of S and $V(G \square P_k) - S$ has at least two vertices. Thus by Proposition 3.1.2, $F(G \square P_k) \leq \alpha(G \square P_k) - 1$. Furthermore, $G \square P_k$ is a connected, bipartite graph with a Hamiltonian path and at least four vertices, so by Lemma 3.3.5, $F(G \square P_k) = \alpha(G \square P_k) - 1$. □

The special case of Theorem 3.3.6 where $k = 2$ tells us that unlike for the larger complete graphs in Theorem 3.3.4, $F(G \square K_2) = \alpha(G \square K_2) - 1$. This leads us to ask

Question 3.3.7. *What is $F(G \square K_2)$ when G is a graph other than a bipartite graph having a spanning path?*

3.4 A Product Lower Bound

Beeler and Rodriguez [6] asked what can be said about the value of $F(G \square H)$ in terms of $F(G)$ and $F(H)$. We obtain a sufficient condition for $F(G \square H) \geq F(G)F(H)$. Let $N[v]$ denote the closed neighborhood $N(v) \cup \{v\}$ of a vertex v. We say that a graph G is *freely neighborhood-solvable* if, for every $v \in V(G)$, G is solvable from the position with a single hole at v so that the final peg is in $N[v]$. Graphs previously known to be freely solvable that have this stronger property include complete graphs, even cycles of length up to 10 (C_{12} does not), the platonic solids, and the Petersen graph. Beeler and Gray [4] found that 103 of the 112 six-vertex graphs and 820 of the 853 seven-vertex graphs are freely solvable. Computer search shows that 95 of these 103 and 796 of these 820 are freely neighborhood-solvable. Furthermore, over 98% of eight-vertex and nine-vertex graphs are freely neighborhood-solvable.

Theorem 3.4.1. *If G is freely solvable and H is freely neighborhood-solvable, then $F(G \square H) \geq F(G)F(H)$.*

Proof. For $v \in V(G)$, let $H(v)$ be the copy of H with first coordinate v. Similarly, for $w \in V(H)$, let $G(w)$ be the copy of G with second coordinate w. Let S_G be a maximum-sized terminal state for G and let S_H be a maximum-sized terminal state for H. By Proposition 3.1.1, to show that $S_G \times S_H$ is a terminal state in $G \square H$, it suffices to solve the configuration with holes at $S_G \times S_H$.

For each $x \in S_H$, we know that $G(x)$ is solvable from this configuration of holes, by Proposition 3.1.1. Solve these so that all copies of G corresponding to S_H leave their final peg in the same location, say the copy of v. Now all vertices of $H(v)$ have pegs, and every copy of H except $H(v)$ has holes at the vertices of S_H. Now by Proposition 3.1.1 we may solve every copy of H except $H(v)$ so that the final pegs all end up at a copy of the same vertex in H; call it w. This initial portion of the procedure does not use the properties assumed for G and H.

We now prepare to solve $H(v)$ from a hole at w with the peg ending in $N[w]$. Suppose first that the final peg in the solution of $H(v)$ from a hole at w ends at w. Let w' be any neighbor of w in H, and let v' be any neighbor of v in G. Now $\{(v,w),(v',w),(v',w'),(v,w')\}$ induce a 4-cycle in $G \square H$. Make the jumps in Figure 3.5: jump $(v,w)(v,w')(v',w')$ and then $(v',w)(v',w')(v,w')$. Now $H(v)$ has a hole at (v,w) and at no other location. Solve $H(v)$ so that the final peg ends at (v,w). Now the remaining pegs occur on $V(G(w)) - \{(v',w)\}$. Because G is freely solvable, we may solve $G(w)$ and thus solve $G \square H$.

Suppose instead that when we solve H from a hole at w that we end with a peg at w', a neighbor of w in H. Let v' be any neighbor of v in G. Again $\{(v,w),(v',w),(v',w'),(v,w')\}$ induces a 4-cycle in $G \square H$. Starting from the state with pegs only on $H(v)$ and $G(w)$, make the jumps in Figure 3.6: start with jump $(v,w)(v',w)(v',w')$. Now $H(v)$ has a hole at (v,w) and at no other location. Solve $H(v)$ so that the final
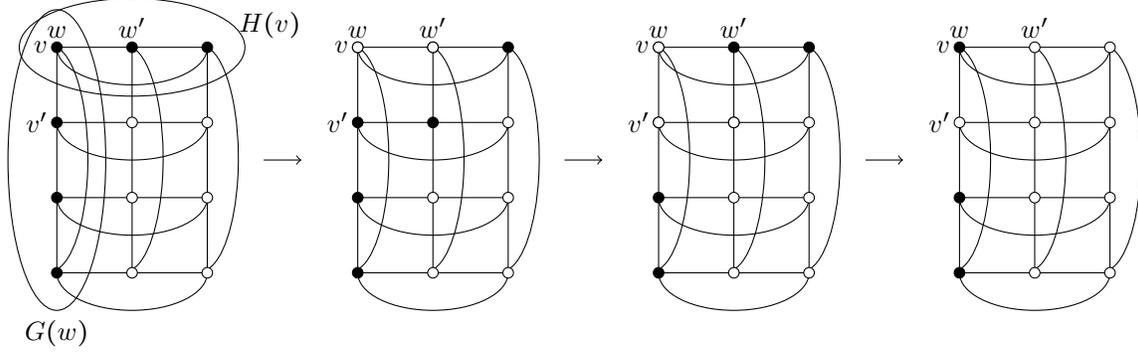
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Figure 3.5: Completion of Theorem 3.4.1 when solving $H(v)$ ends at (v, w).

peg ends at (v, w'). Then jump $(v', w')(v, w')(v, w)$. Now the remaining pegs are on $V(G(w)) - (v', w)$. Because G is freely solvable, we may solve $G(w)$ and thus solve $G \square H$. $\qquad \square$
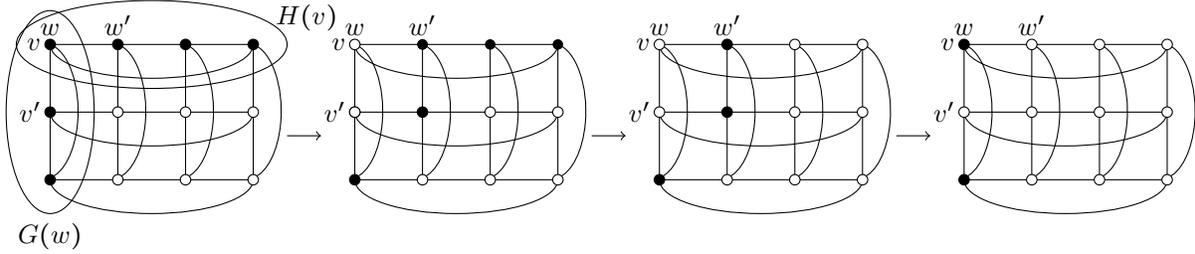

Figure 3.6: Completion of Theorem 3.4.1 when solving $H(v)$ ends at (v, w').

We can strengthen Theorem 3.4.1 by weakening the hypothesis on G. As indicated in our proof, it suffices for the configuration having a hole only at some neighbor of v to be solvable, where v is a vertex to which the configuration with holes at S_G is solvable.

Computer testing shows that equality holds in the bound of Theorem 3.4.1 for $P_2 \square P_2$ and $(K_4 - e) \square (K_4 - e)$ but not for $(K_4 - e) \square C_4$, $C_4 \square C_4$ or $P_2 \square C_n$ when $n \in \{4, 6, 8\}$. However, this bound does not hold for every graph: the Fool's Solitaire number of the cartesian product of $K_{1,3}$ with either P_3 or the paw graph $K_{1,3}^+$ is less than $F(K_{1,3})F(P_3)$ or $F(K_{1,3})F(K_{1,3}^+)$ respectively. In considering these examples and the sharpness of our bound, we ask

Question 3.4.2. *By how much can $F(G \square H)$ exceed $F(G)F(H)$? When does $F(G)F(H)$ exceed $F(G \square H)$?*

Lemma 3.3.1 shows that one can solve a complete graph with more than four vertices with the hole starting at any given vertex and the final peg ending at any specified vertex. We ask where other graphs satisfy this more restricted notion of freely solvable graphs.

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Question 3.4.3. *On what graphs, other than K_k for $k > 4$, can the game start with one hole in any specified vertex and end with one peg at any specified vertex?*

Bell [7] determined that several geometrically defined boards have this property.

Chapter 4

Bar visibility representations

4.1 Introduction

A *bar visibility representation* of a graph G assigns the vertices distinct horizontal line segments ("bars") in the plane such that $uv \in E(G)$ if and only if there is an unobstructed vertical line of sight (having positive width) joining the bar assigned to u and the bar assigned to v. A graph is a *bar visibility graph* if it has a bar visibility representation. The requirement that lines of sight have positive width is important. It permits us to use closed bars so that bars with endpoints having a common x-coordinate cannot see each other but can block vertical visibility between them. Early work used the name "visibility representation" for a slightly different model using zero-width lines of sight [2, 37, 36, 47].

The model we study here was introduced by Melnikov [40] under the name ϵ-visibility graphs, but it is now commonly called bar visibility representation. Wismath [55] and Tamassia and Tollis [47] independently gave a simple characterization of bar visibility graphs under this definition. Hutchinson [28] later gave a simpler proof.

Theorem 4.1.1. *[28, 47, 55] A graph G is a bar visibility graph if and only if G has an embedding in the plane in which all cut-vertices lie on a single face.*

In a *t-bar visibility representation* (Chang et al. [12]), each vertex of G is represented by at most t bars, and $uv \in E(G)$ if and only if there is an unobstructed vertical line of sight (having positive width) joining some bar for u to some bar for v. The *bar visibility number* $b(G)$ of a graph G is the least t such that G has a t-bar visibility representation. (This is analogous to the relationship between interval graphs and the interval number of a graph.)

Let Q_n be the n-dimensional hypercube, defined by $V(Q_n) = \{0, 1\}^n$ and $xy \in E(Q_n)$ if and only if x and y differ in exactly one coordinate. Given a t-bar representation of Q_n with minimum t, let $\hat{Q}_n$ be the planar graph formed by letting the bars be vertices and the lines of sight corresponding to $E(Q_n)$ be edges. Note that $\hat{Q}_n$ is triangle-free, since any triangle in $\hat{Q}_n$ would yield a triangle in Q_n. For a triangle-free planar

graph G, Euler's Formula yields $|E(G)| \leq 2|V(G)| - 4$. Note also that $t \geq |V(\hat{Q}_n)|/2^n$. Thus

$$b(Q_n) = t \geq \left\lceil \frac{|E(\hat{Q}_n)| + 4}{2 \cdot 2^n} \right\rceil = \left\lceil \frac{n \cdot 2^{n-1} + 4}{2 \cdot 2^n} \right\rceil = \left\lceil \frac{n+1}{4} \right\rceil.$$

Axenovich et al. [3] asked whether this trivial lower bound suffices, yielding $b(Q_n) = \left\lceil \frac{n+1}{4} \right\rceil$. Kleinert [33] decomposed Q_n into $\left\lceil \frac{n+1}{4} \right\rceil$ planar graphs. In light of Theorem 4.1.1, proving that his graphs are 2-connected answers the question in the affirmative.

Theorem 4.1.2. $b(Q_n) = \left\lceil \frac{n+1}{4} \right\rceil$ *for $n \in \mathbb{N}$.*

We explicitly describe such a decomposition; the description is of independent interest. Let C_n denote the n-vertex cycle and P_n denote the n-vertex path. The cartesian product $G \square H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$ in which two vertices are adjacent if in one coordinate they are equal and in the other they are adjacent. For the case $n = 4k - 1$, our decomposition consists of spanning subgraphs $G_1, \ldots, G_k$ such that the components of G_i for $1 \leq i < k$ are isomorphic to $C_4 \square P_{2^{i+1}}$, and $G_k \cong G_{k-1}$.

The product $C_4 \square P_{2^l}$ can be drawn using 2^l concentric 4-cycles joined by matchings, as shown in Figure 4.1. The labels will be used later to explain how the component is assembled as a subgraph of Q_n. Clearly $C_4 \square P_{2^l}$ is planar and 2-connected. Our graphs are isomorphic to those used by Kleinert, but our presentation is more intuitive and somewhat simpler.

The analogue of visibility representation for digraphs establishes an edge uv only when there is a positive-width line of sight from some bar for u to some higher bar for v. The visibility number $b(D)$ of a digraph D (Axenovich et al. [3]) is the least number of bars per vertex that permits such a representation. A t-bar representation of a graph G with each edge represented only once defines an orientation D of G such that $b(D) \leq t$. The visibility representation of $C_4 \square P_{2^l}$ has this property. Hence our decomposition answers another question of [3], showing that Q_n has an orientation D with $b(D) = \left\lceil \frac{n+1}{4} \right\rceil$.

Nevertheless, for Q_2 and Q_3 there are other orientations D_2' and D_3' such that $b(D_2') = b(D_3') = 2$. Hence it is natural to ask how large $b(D_n') - b(Q_n)$ can be when D_n' is an orientation of Q_n.

4.2 Upper Bound Construction

We first reduce the problem to $n \equiv 3 \bmod 4$, where there is no slack in the counting argument.

Lemma 4.2.1. *To prove Theorem 4.1.2, it suffices to decompose Q_{4k-1} into k planar subgraphs $G_1, \ldots, G_k$ whose components are 2-connected, for $k \in \mathbb{N}$.*
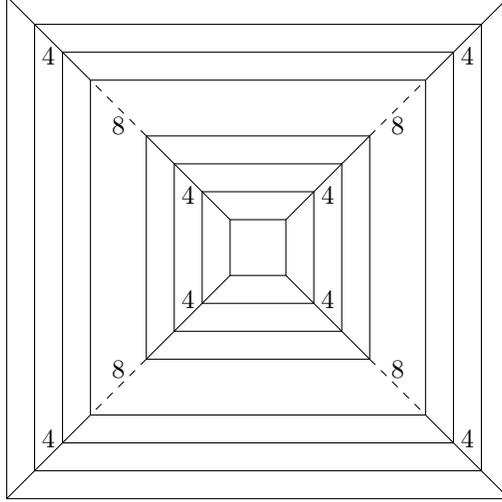
Figure 4.1

Proof. Let $n = 4k - 1 + s$, where $1 \leq s \leq 3$. Given such a decomposition of Q_{4k-1}, decompose Q_n into 2^s copies of Q_{4k-1} and 2^{4k-1} copies of Q_s. Construct a $(k+1)$-bar visibility representation of Q_{4k-1+s} by placing 2^s copies of the representation of Q_{4k-1} in disjoint vertical strips and 2^{4k-1} copies of the bar visibility representation of Q_s in other disjoint vertical strips. Since $b(Q_s) = 1$ for $s \in \{1, 2, 3\}$, the extra bar allowed per vertex suffices for these representations. $\qquad\square$

Our decomposition of Q_{4k-1} is described by allocating edges to subgraphs, based on the coordinates where the endpoints of edges differ and the constant values in other coordinates.

Definition 4.2.2. An edge $e \in E(Q_{4k-1})$ is of *type* r if its endpoints differ in coordinate r. Let E_r be the set of edges of type r. The edges of type $4j$ for $1 \leq j \leq k-1$ are *special edges*. For an edge e of type r and $s \neq r$, let e_s denote the common value that the endpoints of e have in coordinate s. For $e \in E_r$ with $r \leq 4k - 4$, let e' denote the edge in Q_{4k-5} obtained by deleting the last four coordinates of the endpoints of e; the edge e' is the *truncation* of e, and e is an *extension* of e'.

We will decompose Q_{4k-1} into spanning subgraphs $G_1, \ldots, G_k$. Example 4.2.3 shows the allocation of edges to subgraphs for $k \leq 5$. We will prove several properties inductively.

For $1 \leq i < k$, the spanning subgraph G_i has 2^n vertices in $2^{4(k-1)-i}$ components, each isomorphic to $C_4 \square P_{2^{i+1}}$; also $G_k \cong G_{k-1}$. The subgraph G_i contains $E_{4(k-i)+1} \cup E_{4(k-i)+2} \cup E_{4(k-i)+3}$. The resulting 3-dimensional subcubes are linked into larger components using special edges. For example, for $k > 2$ the components of G_2 are copies of $C_4 \square P_8$. As shown in Figure 4.1 for $k = 3$, these arise by combining four copies of Q_3 using eight edges of type $4k - 8$ and four edges of type $4k - 4$ (when $k = 3$ these are types 4 and

8, respectively).

To discuss $G_1, \ldots, G_k$ for Q_{4k-1}, let $G'_1, \ldots, G'_{k-1}$ be the specified decomposition of Q_{4k-5}. The key idea, illustrated in Example 4.2.3, is that for $i > 1$ the subgraph G_i for Q_{4k-1} is obtained from G'_{i-1}. Indeed, G_i begins with 16 copies of G'_{i-1}, extending its vertices by fixed choices in the four new coordinates. These copies will be linked in pairs using the new special type $4k - 4$. It thus follows inductively that edges of all types other than $4k - 4$ are used exactly once in the decomposition. We will need to prove this also for type $4k - 4$ and allocate its edges to combine components in pairs.

Example 4.2.3. *Decompositions of Q_3, Q_7, Q_{11}, Q_{15}, and Q_{19}:*

$k = 1$

G_1	E_1
	E_2
	E_3

$\rightarrow$

$k = 2$

		$e \in E_4$
G_1	E_5	$e_1 = e_5$
	E_6	
	E_7	
G_2	E_1	$e_1 \neq e_5$
	E_2	
	E_3	

$\searrow$

$k = 3$

		$e \in E_4$	$e \in E_8$
G_1	E_9		$e_4 = e_9$
	E_{10}		
	E_{11}		
G_2	E_5	$e_1 = e_5$	$e_1 \neq e_5$
	E_6		$e_4 \neq e_9$
	E_7		
G_3	E_1	$e_1 \neq e_5$	$e_1 = e_5$
	E_2		$e_4 \neq e_9$
	E_3		

$\searrow$

$k = 4$

		$e \in E_4$	$e \in E_8$	$e \in E_{12}$
G_1	E_{13}			$e_8 = e_{13}$
	E_{14}			
	E_{15}			
G_2	E_9		$e_4 = e_9$	$e_4 \neq e_9$
	E_{10}			$e_8 \neq e_{13}$
	E_{11}			
G_3	E_5	$e_1 = e_5$	$e_1 \neq e_5$	$e_1 \neq e_5$
	E_6		$e_4 \neq e_9$	$e_4 = e_9$
	E_7			$e_8 \neq e_{13}$
G_4	E_1	$e_1 \neq e_5$	$e_1 = e_5$	$e_1 = e_5$
	E_2		$e_4 \neq e_9$	$e_4 = e_9$
	E_3			$e_8 \neq e_{13}$

$\searrow$

$k = 5$

		$e \in E_4$	$e \in E_8$	$e \in E_{12}$	E_{16}
G_1	E_{17}				$e_{12} = e_{17}$
	E_{18}				
	E_{19}				
G_2	E_{13}			$e_8 = e_{13}$	$e_8 \neq e_{13}$
	E_{14}				$e_{12} \neq e_{17}$
	E_{15}				
G_3	E_9		$e_4 = e_9$	$e_4 \neq e_9$	$e_4 \neq e_9$
	E_{10}			$e_8 \neq e_{13}$	$e_8 = e_{13}$
	E_{11}				$e_{12} \neq e_{17}$
G_4	E_5	$e_1 = e_5$	$e_1 \neq e_5$	$e_1 \neq e_5$	$e_1 \neq e_5$
	E_6		$e_4 \neq e_9$	$e_4 = e_9$	$e_4 = e_9$
	E_7			$e_8 \neq e_{13}$	$e_8 = e_{13}$
					$e_{12} \neq e_{17}$
G_5	E_1	$e_1 \neq e_5$	$e_1 = e_5$	$e_1 = e_5$	$e_1 = e_5$
	E_2		$e_4 \neq e_9$	$e_4 = e_9$	$e_4 = e_9$
	E_3			$e_8 \neq e_{13}$	$e_8 = e_{13}$
					$e_{12} \neq e_{17}$

The constraints used to allocate E_{4k-4} to subgraphs will ensure the desired properties. Before allocating type $4k - 4$ edges, G_1 consists of 2^{4k-4} copies of Q_3, which is $C_4 \square P_{2^1}$. To combine into copies of $C_4 \square P_{2^2}$, we need to use four edges 2^{4k-5} times, for a total of 2^{4k-3} edges. Since there are 2^{4k-2} edges of each type, we use half the edges of E_{4k-4}, which is accomplished by imposing one constraint on the coordinates.

With each increase in i, the size of the components doubles, and the number of components halves. Hence the number of special edges needed also halves. To obtain the desired number of edges of E_{4k-4}, we add one constraint at each step, and the constraints are satisfied by half of the remaining edges. The last

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step has the same number of constraints as the step before it, using the remaining edges of E_{4k-4}. Hence we allocate each edge of E_{4k-4} once, but we still must show that this produces the claimed subgraphs.

The inductive specification facilitates proof.

Construction 4.2.4. *Decomposition of Q_{4k-1}* We define spanning subgraphs $G_1, \ldots, G_k$ by specifying the edge sets, letting $F_i = E(G_i)$. For $k = 1$, let $F_1 = E_1 \cup E_2 \cup E_3 = E(Q_3)$. For $k > 1$, let $G'_1, \ldots, G'_{k-1}$ be the decomposition of Q_{4k-5}, with $F'_i = E(G'_i)$.

1. For $e \in E_r$ with $r < 4k - 4$, put $e \in F_i$ if $e' \in F'_{i-1}$. Also put $E_{4k-1} \cup E_{4k-2} \cup E_{4k-3} \in F_1$.

2. For $k = 2$ and $e \in E_4$, put e in F_1 if $e_1 = e_5$, in F_2 if $e_1 \neq e_5$.

3. For $k > 2$ and $e \in E_{4k-4}$ with $e_{4k-8} = e_{4k-3}$, put $e \in F_1$.

4. For $k > 2$ and $e \in E_{4k-4}$ with $e_{4k-8} \neq e_{4k-3}$,

   (a) If $e_{4j-4} = e_{4j+1}$ for $i' < j \leq k - 2$ and $e_{4i'-4} \neq e_{4i'+1}$, then put $e \in F_{k-i'}$ (here $i' \geq 2$).

   (b) If $e_{4j-4} = e_{4j+1}$ for $2 \leq j \leq k - 2$, then put $e \in F_{k-|e_1-e_5|}$.

Theorem 4.2.5. *Construction 4.2.4 decomposes Q_{4k-1} into spanning subgraphs $G_1, \ldots, G_k$ such that each component of G_i is isomorphic to $C_4 \square P_{2^{i+1}}$ for $i < k$ and to $C_4 \square P_{2^k}$ for $i = k$.*

Proof. The proof is by induction on k. We first check that $G_1, \ldots, G_k$ is a decomposition. For $k = 1$, $Q_3 = G_1$. For $k \geq 2$, let $G'_1, \ldots, G'_{k-1}$ be the specified decomposition of Q_{4k-5}. Rule (1) allocates all types other than E_{4k-4}, putting edges into G_i for $1 \leq i \leq k$ forming a spanning subgraph whose components are isomorphic to Q_3 for $i = 1$ and to G'_{i-1} for $i > 1$.

To allocate E_{4k-4}, in Rules 3 and 4a of Construction 4.2.4 we impose i constraints on the edges to be used in F_i for $i < k - 1$. Furthermore, the constraints for edges put in F_i are not satisfied by those put in $F_1, \ldots, F_{i-1}$. In Rule 4b, we allocate all the remaining edges, half to F_{k-1} and half to F_k. Thus each edge of E_{4k-4} is allocated exactly once, and $G_1, \ldots, G_k$ is a decomposition of Q_{4k-1}.

We must show that for $1 \leq i < k$, the edges of E_{4k-4} in G_i combine copies of $C_4 \square P_{2^i}$ into copies of $C_4 \square P_{2^{i+1}}$, and that in G_k they combine copies of $C_4 \square P_{2^{k-1}}$ into copies of $C_4 \square P_{2^k}$.

Let $H_i = C_4 \square P_{2^i}$. The vertices in the two copies of H_i that will be linked by four edges of E_{4k-4} are obtained by adding four coordinates to extend the names of the vertices in H_i. These two extensions will differ only in coordinate $4k - 4$, which is one of the four new coordinates. This allows us to link them using edges of E_{4k-4}.

In H_i, we call the four copies of P_{2^i} the *diagonals*. A copy of H_i can be embedded in the plane with a specified end of the diagonals on the unbounded face or on the central face. The two copies we want to link have extensions differing only in coordinate $4k - 4$. We embed them with one inside the other, so that the

outer face of the inner copy has extensions of the same vertices whose extensions are on the inner face of the outer copy. Hence the vertices incident to the region bounded by the two embeddings are matched in Q_{4k-1} by edges of E_{4k-4}. Adding those edges creates copies of H_{i+1} in which the diagonals have twice as many vertices as in H_i, and the central edge of each diagonal is in E_{4k-4}.

It remains to show that the edges of type $4k-4$ we use to link these pairs are in fact the ones we have specified for G_i. Each pairing uses the following discussion. When each diagonal has exactly one edge of type r, the vertices on the outer face and those on the central face in an embedding of H_i have opposite values in coordinate r. If also the copy of H_i has no edge of type s, then an embedding of H_i has coordinates r and s agreeing in the vertices of one extreme face and disagreeing in the vertices of the other extreme face. Furthermore, either property can be chosen for the outer face.

For $k = 2$, the components of the graph G_1' are copies of H_1 (which is Q_3) using types 1, 2, and 3. To form G_2, draw the copies of G_1' with edges of type 1 on the diagonal. Because coordinate 5 is constant on the two copies and the diagonal edge in each is type 1, we may embed the inside and outside copies of H_1 so that $e_1 \neq e_5$ for the endpoints of each edge e of type 4 used to link the two copies across the region between them.

We also combine copies of Q_3 to form G_1 for $k = 2$. The copies of Q_3 use edges of types 5, 6, and 7, with type 5 along the diagonal. We link two copies whose extensions differ only in coordinate 4. Since coordinate 1 is constant on the two copies of Q_3 being linked and the diagonal edges are type 5, we may choose to embed the inside and outside copies of Q_3 so that $e_1 = e_5$ for the endpoints of each edge e of type 4 used to link the two copies. Hence we have the claimed allocation for $k = 2$, as specified by Rule 2.

For $k \geq 3$, we form G_1 in almost the same manner as for $k = 2$. We take copies of Q_3 using edges of types $4k-1$, $4k-2$, and $4k-3$, with type $4k-3$ on the diagonals. Since coordinate $4k-8$ is constant on the copies of Q_3 being linked and the diagonal edges are type $4k-3$, we may embed the copies of Q_3 so that coordinates $4k-8$ and $4k-3$ are equal at vertices on the outer face of the inner copy of Q_3 and on the inner face of the outer copy of Q_3 linked to it. Thus we link the copies by edges of E_{4k-4} satisfying $e_{4k-8} = e_{4k-3}$, as specified in Rule 3. (The case $k = 2$ differs from this in using e_1, since $1 \neq 4 \cdot 2 - 8$.)

In forming G_1 we have used every edge $e \in E_{4k-4}$ such that $e_{4k-8} = e_{4k-3}$; there are 2^{4k-3} of them in 2^{4k-5} components. To allocate the edges satisfying $e_{4k-8} \neq e_{4k-3}$, we want to agree with Rule 4. By the induction hypothesis, for $2 \leq i \leq k$ the central edge of the diagonal in each component of G_{i-1}' has type $4k-8$. No other edges of G_{i-1}' have type $4k-8$, and coordinate $4k-3$ is constant on the vertices in components being paired. Hence by the usual discussion we embed the paired components so that the outer vertices of the inner copy and the inner vertices of the outer copy are matched via edges in E_{4k-4} satisfying $e_{4k-8} \neq e_{4k-3}$,

agreeing with Rule 4.

Finally, to show that each edge of E_{4k-4} is used only once, we check that the remaining specified constraints on which subgraph contains which edges of E_{4k-4} are satisfied. We are checking Rule 4, involving only the edges of E_{4k-4} satisfying $e_{4k-8} \neq e_{4k-3}$. The condition in Rule 4b is vacuous when $k = 3$, so Rule 4b always puts edges into F_{k-1} and F_k, while Rule 4a puts edges into $F_2, \ldots, F_{k-2}$ for $k \geq 4$. For easier understanding, we recommend that the reader compare the arguments with Example 4.2.3.

A component of G_i is formed by combining extensions of two copies of G'_{i-1}. By the inductive construction, the central edge on each diagonal in G'_{i-1} is type $4k - 8$. When $i = 2$, the edges of type $4k - 8$ in G'_1 satisfy $e_{4k-12} = e_{4k-7}$. The other diagonal edges have type $4k - 7$, and one of them is traversed to reach the edge of type $4k - 4$. Thus the edges of type $4k - 4$ in G_2 satisfy $e_{4k-12} \neq e_{4k-7}$, as specified by Rule 4a with $i' = k - 2$.

To understand the constraints on the other edges, it is helpful to describe the list of types along each diagonal in a component of G_i; let this list be $L_k(i)$. By construction, $L_k(1) = 4k-3, 4k-4, 4k-3$. For $i \geq 2$, we have a recursive concatenation: $L_k(i) = L_{k-1}(i-1), 4k-4, L_{k-1}(i-1)$. Note that the special types on the diagonal are types $4k - 4$ down to $4i'$, where $i' = k - i$. The key point follows inductively: for $2 < i \leq k - 2$ and $i' < j \leq k - 2$, between the central edge e of type $4k - 4$ and the nearest edge of type $4j$ in either direction, there is exactly one edge of type $4j - 4$. Note also that type $4i'$ is the lowest special type in G_i, and the special edges of type $4i'$ nearest to the central edge e of type $4k - 4$ on the diagonal are separated from e only by an edge of type $4i' + 1$.

By Rule 1, the edges of special type $4j$ in G_i were created when forming the $(k-2-j)$th subgraph in the decomposition of $Q_{4(j+1)-1}$. If $j = i'$, then the $(k - 2 - j)$th subgraph is the first. By Rule 3, the endpoints of these edges have the same value in coordinates $4i' - 4$ and $4i' + 1$. Since we follow an edge of type $4i' + 1$ to reach the new edge e of type $4k - 4$, we have $e_{4i'-4} \neq e_{4i'+1}$, as specified in Rule 4a. If $i' < j \leq k - 2$, then the edges of type $4j$ are not introduced into the first subgraph in decomposing $Q_{4(j+1)-1}$, so their endpoints have different values in coordinates $4j - 4$ and $4j + 1$. Since we follow one edge of type $4j - 4$ and no edges of type $4j + 1$ in reaching the new edge e of type $4k-4$ on the diagonal, we have $e_{4j-4} = e_{4j+1}$, again as specified in Rule 4a.

Finally, we consider the edges of type $4k - 4$ placed in F_{k-1} and F_k. In these graphs all special types occur. For $2 \leq j \leq k - 2$, the reasoning is as above: the edges of type $4j$ are not introduced into a subgraph other than the first when decomposing $Q_{4(j+1)-1}$, so their endpoints have different values in coordinates $4j - 4$ and $4j + 1$. We follow one edge of type $4j - 4$ and none of type $4j + 1$ to reach the new edge e, and hence $e_{4j-4} = e_{4j+1}$.

To determine the last constraint on edge e for G_{k-1} and G_k, we consider the edges of type 4 nearest to e along the diagonal. Inductively, these edges were originally created in copies of the first or second subgraph when decomposing Q_7 (that is, $k = 2$). In G_{k-1}, the types around e along the diagonal are $5, 4, 5, 4k-4, 5, 4, 5$ in order. In G_k, they are $1, 4, 1, 4k-4, 1, 4, 1$. The endpoints of the edges of type 4 have coordinates 1 and 5 the same in G_{k-1}, different in G_k. Since the edge e is separated from these edges by one edge of type 1 or 5, we have $e_1 \neq e_5$ in G_{k-1} and $e_1 = e_5$ in G_k, as specified in Rule 4b. $\qquad \square$

It is easy to describe an explicit bar visibility representation for $C_4 \square P_{2^i}$. We indicate the resulting representations for Q_4 and Q_{11} in Figures 4.2 and 4.3, respectively. Vertical cuts in horizontal segments indicate shared endpoints of bars.
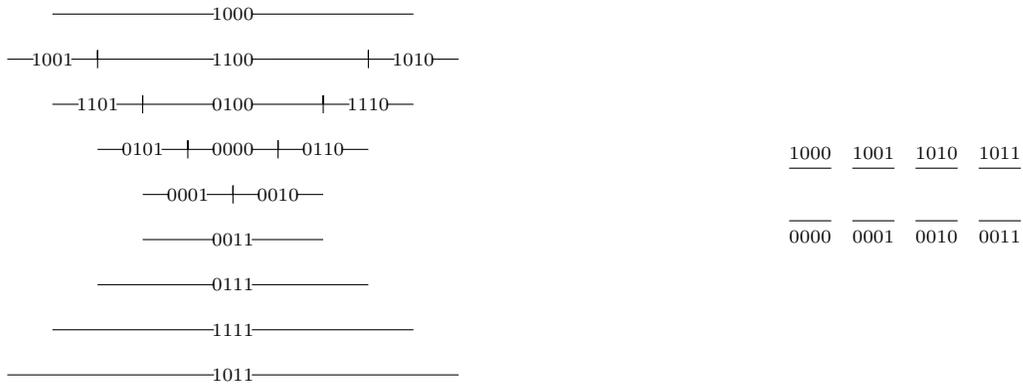
Figure 4.2: 2-Bar visibility representation for Q_4
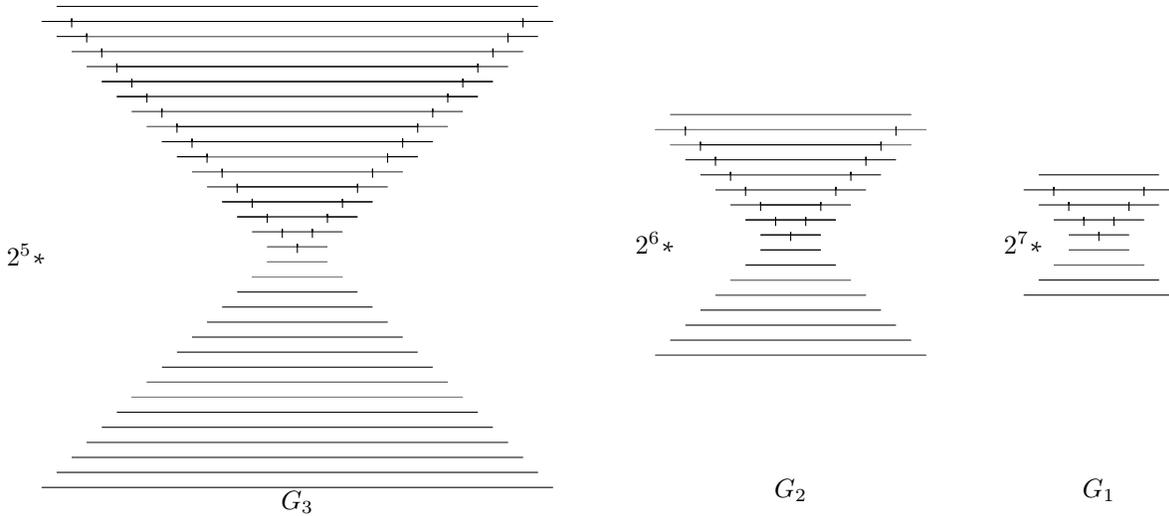
Figure 4.3: 3-Bar visibility representation for Q_{11}

4.3 Outerplanar digraphs

In this section, we look closer at the bar visibility representations of digraphs.

When $b(D) = 1$, we say that D is a *bar-visibility digraph*. Axenovich et al. [3] characterized the oriented trees that are bar-visibility digraphs and showed that every simple outerplanar digraph has visibility number at most 2. In this section, we give a forbidden subgraph characterization of the outerplanar digraphs that are bar-visibility digraphs.

In a digraph, a vertex with indegree 0 or outdegree 0 is a *source* or a *sink*, respectively. We use *consistent cycle* to mean an orientation of a cycle having no source or sink.

Shrinking bars to vertices converts a bar visibility representation of a digraph G into a planar embedding of G; hence every bar visibility digraph is planar. Thomassia and Tollis [47] and Wismath [54] characterized the planar digraphs that are bar visibility digraphs.

Definition 4.3.1. Given a digraph G, the *auxiliary digraph* G' is formed by adding two vertices s and t, a directed edge sv for every source vertex v in G, a directed edge wt for every sink vertex w, and the directed edge st.

Theorem 4.3.2. [47, 54] *A planar digraph G is a bar-visibility digraph if and only if G has no consistent cycle and its auxiliary digraph G' is planar.*

Axenovich et al. [3] proved $b(G) \le 2$ for every outerplanar digraph G. Theorem 4.3.2 thus computes $b(G)$ when G is an outerplanar digraph, via a check for acyclicity and a planarity test of the auxiliary digraph. In essence, our result analyzes how nonplanar subgraphs can arise in the auxiliary digraph of an outerplanar digraph.

We define several substructures that must be forbidden from bar-visibility digraphs. An *oriented cycle* is an orientation of a cycle, not necessarily a consistent cycle. A *claw* is a copy of the star $K_{1,3}$.

Definition 4.3.3. A *flower* in a digraph D consists of an oriented cycle C such that from three distinct vertices on C there are paths in the underlying graph to a sink and a source of D, and all six paths are disjoint except for the initial three vertices on C, (see Figure 4.4a).

A *gear* in D consists of an oriented cycle C such that from four distinct vertices on C (in order) there are paths in the underlying graph to a source, a sink, a source, and a sink of D, (see Figure 4.4b). These paths may have length 0.

A *tripod* in a digraph D consists of a claw such that the underlying graph has from each leaf of the claw a path to a source of D and a path to a sink of D; the two paths from one leaf need not be disjoint, but the paths from one leaf are disjoint from the paths from the other leaves and avoid the center, (see Figure 4.4c).

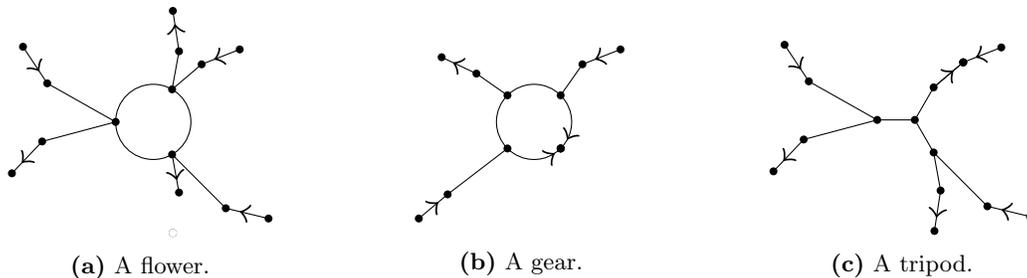(a) A flower.　　　(b) A gear.　　　(c) A tripod.

Figure 4.4: Examples of forbidden substructures in an outerplanar bar-visibility digraph.

Axenovich et al. [3] also characterized the oriented trees T that are bar visibility graphs. This computes $b(T)$ for every oriented tree T, since $b(G) \leq 2$ when G is outerplanar.

Theorem 4.3.4. [3] *An oriented tree is a bar-visibility graph if and only if it contains three internally disjoint inconsistent paths from a single vertex.*

In a tree, the forbidden condition in Theorem 4.3.4 is equivalent to the existence of a tripod.

Theorem 4.3.5. *An oriented tree is a bar-visibility graph if and only if it contains no tripod.*

Proof. We show that an oriented tree T contains three internally disjoint inconsistent paths from a single vertex if and only if it contains a tripod.

Given a tripod in T whose claw has center w, let v be a leaf of the claw. If the edge is oriented from v to w, then appending the tripod path from v to a sink yields an inconsistent path starting at w. If it is oriented from w to v, then appending the tripod path from v to a source yields an inconsistent path starting at w. Doing this for each leaf of the claw yields the three desired paths.

Given three inconsistent paths from w, from each we obtain one leaf in the claw for a tripod. By symmetry, suppose that such a path begins with the directed edge wv. Following a consistent path from v eventually reaches a sink in T, since T has no cycle. The given inconsistent path from w contains a first edge yx oriented toward w. Following a consistent path from y (in reverse) eventually reaches a source in T. Hence we obtain two paths from v in the underlying graph (which may share an initial portion) to a sink and a source in T. □

Axenovich et al. [3] also noted that a bar visibility digraph cannot contain a consistent cycle. Our main result in this section is that also forbidding the configurations of Definition 4.3.3 characterizes outerplanar digraphs that are bar visibility digraphs.

Theorem 4.3.6. *If G is an outerplanar digraph, then $b(G) = 1$ if and only if G contains no flower, gear, tripod, or consistent cycle.*

Proof. We first prove necessity.

If G contains a consistent cycle, then a bar visibility representation of G must put every bar in the cycle above the previous bar, which is impossible.

If G contains a flower, then in the auxiliary digraph G' of Theorem 4.3.2 the three given vertices on the cycle C plus s and t are the branch vertices of a K_5-subdivision. The paths in the subdivison are three on C, the edge st, and the six paths from the specified vertices C to a source or sink and then to s or t.

If G contains a gear, then in G' the four vertices on the cycle C plus s and t are the branch vertices of a $K_{3,3}$-subdivision. The branch vertices for one part of $K_{3,3}$ consist of s and the two vertices on the cycle whose specified paths lead to sinks; the other part consists of t and the two specified vertices whose paths lead to sources. We add four paths on C, the four paths leading to sinks or sources and then to s or t, and the edge st itself.

If G contains a tripod, then in G' there is a $K_{3,3}$-subdivision whose branch vertices are s, t, and the center of the claw in one part, and the last common vertex on the specified paths to a source and a sink from each of the three leaves in the other part. The tripod provides internally disjoint paths from each vertex of one part to each vertex of the other.

For sufficiency, we suppose that G is not a bar visibility digraph and show that G contains a forbidden substructure. We may assume that G has no consistent cycle. Thus the auxiliary digraph G' must be nonplanar, containing in its underlying graph a subdivision of $K_{3,3}$ or K_5. Since G is outerplanar, G contains no subdivision of $K_{2,3}$ or K_4. Thus s and t must both be branch vertices in the subdivision of $K_{3,3}$ or K_5.

Suppose first that G' contains a $K_{3,3}$-subdivision H. If s and t are branch vertices in the same part, then let w be the third branch vertex in that part, and let Y be the set of branch vertices in the other part. The three edges incident to w in H form a claw that extends on to Y. From each vertex of Y, in H there is a path to a source followed by an edge from s and a path to a sink followed by an edge to t. Thus H contains a tripod (the two paths from a neighbor of w to a source and a sink run together until they reach Y).

If s and t are branch vertices in opposite parts, then let s_1 and s_2 be the other two branch vertices in the same part as s, and let t_1 and t_2 be the other two branch vertices in the same part as t. Note that $s_1, t_1, s_2,$ and t_2 lie in order on a cycle C. In H there must be paths from s_1 and s_2 to t, and paths to t reach a sink just before t. Similarly, in H there are paths from t_1 and t_2 to s, reaching a source just before s. Hence we obtain a gear in G.

Finally, if G' contains a K_5-subdivision H, then let Z be the set of the three branch vertices other than s and t. Note that H contains a cycle C through Z. Leaving C, in H there are paths to s and t from each

vertex of Z. These paths reach s and t via a source or sink vertex of G, respectively. Hence we obtain a flower in G. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Chapter 5

r-Dynamic coloring of graphs on surfaces

5.1 Background

For a graph G and a positive integer r, an r-*dynamic coloring* of G is a proper vertex coloring such that for each $v \in V(G)$, at least $\min\{r, d(v)\}$ distinct colors appear in $N_G(v)$. The r-*dynamic chromatic number*, denoted $\chi_r(G)$, is the minimum k such that G admits an r-dynamic k-coloring. Montgomery [41] introduced 2-dynamic coloring and the generalization to r-dynamic coloring.

List coloring was introduced independently by Vizing [50] and by Erdős, Rubin, and Taylor [18]. A *list assignment* L for G assigns to each vertex v a list $L(v)$ of permissible colors. Given a list assignment L for a graph G, if a proper coloring ϕ can be chosen so that $\phi(v) \in L(v)$ for all $v \in V(G)$, then G is *L-colorable*. The *choosability* of G is the least k such that G is L-colorable for every list assignment L satisfying $|L(v)| \geq k$ for all $v \in V(G)$. We consider the r-dynamic version of this parameter. A graph G is r-*dynamically L-colorable* if an r-dynamic coloring can be chosen from the list assignment L. The r-*dynamic choosability* of G, denoted $\mathrm{ch}_r(G)$, is the least k such that G is r-*dynamically L-colorable* for every list assignment L satisfying $|L(v)| \geq k$ for all $v \in V(G)$.

Much of the previous work in on this topic has focused on $\chi_2(G)$ and $\mathrm{ch}_2(G)$. Akbari et. al [1] proved that if G is a graph with $\Delta(G) \geq 3$ not having C_5 as a component, then $\mathrm{ch}_2(G) \leq \Delta(G) + 1$. They determined $\mathrm{ch}_2(C_n)$ for every natural number n. Jahanbekam et. al [30] considered bounds on the r-dynamic number under maximum degree and diameter conditions and gave bounds on the the r-dynamic number of the m-by-n grid. Kang et. al [31] completed the determination of the r-dynamic chromatic number of the m-by-n grid for all r, m, n. For further work, see [13, 32].

The *square* of a graph G, denoted G^2, is the graph resulting from adding an edge uv whenever the distance between u and v in G is 2. For any graph G, it is clear that

$$\chi(G) = \chi_1(G) \leq \chi_2(G) \leq \cdots \leq \chi_{\Delta(G)}(G) = \cdots = \chi(G^2),$$
$$\mathrm{ch}(G) = \mathrm{ch}_1(G) \leq \mathrm{ch}_2(G) \leq \cdots \leq \mathrm{ch}_{\Delta(G)}(G) = \cdots = \mathrm{ch}(G^2), \qquad (1)$$

and that $\chi_r(G) \le \mathrm{ch}_r(G)$ for all r. Thus we can think of r-dynamic coloring as bridging the gap between coloring a graph and coloring its square, and similarly for r-dynamic choosability.

Heawood [24] proved that for $g > 0$, graphs of (orientable) genus g are $(h(g) - 1)$-degenerate and hence $h(g)$-colorable, where

$$h(g) = \left\lfloor \frac{7 + \sqrt{1 + 48g}}{2} \right\rfloor.$$

Chen et al. [13] proved that such a graph is 2-dynamically $h(g)$-choosable.

Loeb et al. [34] proved that if G is a toroidal graph, then $\chi_3(G) \le \mathrm{ch}_3(G) \le 10$. This is sharp: the Petersen graph P has maximum degree 3 and diameter 2, so $\chi_3(P) = \chi(P^2) = \chi(K_{10}) = 10$. As an immediate corollary, $\chi_3(G) \le \mathrm{ch}_3(G) \le 10$ when G is a planar graph.

In this chapter, we consider the r-dynamic choosability of graphs with arbitrary genus. Let $\gamma(G)$ denote the minimum genus among the surfaces on which G embeds. We prove the following theorem, giving an upper bound on the r-dynamic choosability of a graph with given genus, and thus also an upper bound on the graph's r-dynamic chromatic number.

Theorem 1.4.1. *Let G be a graph, and let $g = \gamma(G)$.*

1. *If $g = 0$ and $r \ge 11$, then $\mathrm{ch}_r(G) \le 5(r + 1) + 3$*

2. *If $g \ge 1$ and $r \ge 24g - 11$, then $\mathrm{ch}_r(G) \le (12g - 6)(r + 1) + 3$.*

Even though Theorem 1.4.1 assumes a lower bound on r, by (1) it gives an upper bound on $\mathrm{ch}_r(G)$ for all r.

Our proof uses the Discharging Method. A *configuration* in a graph is a set of vertices that satisfies some specified condition, for example, a condition on the degrees or adjacencies of the vertices in the configuration. We say that a configuration in a graph is *reducible* for a graph property if it cannot occur in a minimal graph failing that property. In the Discharging Method, one shows that various configurations are reducible for the desired property. One then assigns charge to the vertices of a graph and redistributes charge, under the hypothesis that the reducible configurations do not occur, to violate some global bound on the total charge. The contradiction shows that these reducible configurations form a set that is unavoidable by a minimal graph failing to to have the desired property. Hence the desired property always holds. For more on the Discharging Method see [15].

5.2 General bounds

In this section we consider general genus and large r. We prove Theorem 5.2.5, giving bounds on the r-dynamic choosability of graphs in terms of their genus.

We begin by showing that there are no vertices of degree at most 2 in a smallest counterexample to our bound on r-dynamic choosability. We then show that for some edge the endpoints have a relatively small degree sum. We then use discharging to show that these reducible configurations form an unavoidable set for minimal graphs failing to be r-dynamically $\omega(g, r)$-choosable, where $\omega(g, r)$ is the bound on $\mathrm{ch}_r(G)$ in the statement of the theorem.

We say that a vertex w is r-*dynamic* if at least $\min\{r, d(w)\}$ colors appear on $N(w)$. If G' is an induced subgraph of G, we say that an r-dynamic coloring f of G' *extends* to G if for any lists of size $\omega(g, r)$ on $V(G) - V(G')$ there is a way to color the remaining vertices of G from their lists such that the coloring of G is r-dynamic and restricts to f on G'.

Suppose that G is a graph with the fewest vertices violating Theorem 5.2.5 and let L be a list assignment for G with $|L(v)| \geq \omega(g, r)$ for all $v \in V(G)$ such that G is not r-dynamically L-colorable. Using an r-dynamic L-coloring of a graph with fewer vertices than G we produce an r-dynamic L-coloring of G, giving a contradiction.

Lemma 5.2.1. *A vertex of degree at most* 2 *is reducible for r-dynamic* $(2r + 1)$*-choosability.*

Proof. Consider a list assignment L with lists of size at least $2r + 1$. Suppose that G has a 1-vertex v with neighbor u (Figure 5.1(i)). Let $G' = G - v$. Since G is a smallest counterexample, there is an r-dynamic L-coloring of G'. We construct an r-dynamic L-coloring of G by appropriately coloring v. If $N(u)$ already contains r colors, then v only needs to avoid the color on u. Otherwise, for u to be r-dynamic, v must avoid at most $r - 1$ colors that appear in the neighborhood of u and the color on u. Thus v only needs to avoid at most r colors, which is easy with a list of size at least $2r + 1$.

Suppose that G has a 2-vertex v with neighbors y and z (Figure 5.1(ii)). Let $G' = (G \cup yz) - v$. Because y and z are on the same face, we may add the edge yz to G' without increasing the genus. Select an r-dynamic L-coloring f of G'. We now construct an r-dynamic L-coloring of G by appropriately coloring v. Since yz is in $E(G')$ even though it may not be in $E(G)$, we have ensured that the colors on y and z are distinct, so v is r-dynamic. To ensure that y and z are r-dynamic, v only needs to avoid at most $r - 1$ colors that appear in the neighborhood of y, at most $r - 1$ colors that appear in the neighborhood of z, and the colors on y and z. Thus v need only avoid at most $2r$ colors, which is easy with a list of size at least $2r + 1$. $\qquad\square$

Given a graph G and edge $uv \in E(G)$, the *weight* of uv, denoted $w(uv)$, is $d(u) + d(v)$.

Figure 5.1: Configuration for Lemma 5.2.1: A 1-vertex v with neighbor u or a 2-vertex v with neighbors y and z.

Lemma 5.2.2. *[9] Planar graphs with minimum degree at least 3 have an edge of weight at most 13.*

Ivančo extended this to a sharp bound for every orientable surface.

Lemma 5.2.3 (Ivančo [29])**.** *If G is a simple graph with genus g such that $\delta(G) \geq 3$, then G has an edge of weight at most*

$$\begin{cases} 2g + 13 & \text{if } 0 \leq g \leq 2, \\ 4g + 7 & \text{otherwise.} \end{cases}$$

When $g = 1$ the graph obtained from a 6-regular triangulation of the torus by adding a vertex inside each face and making it adjacent to each vertex of that face has edges of weight 15 and 24. For $g \geq 3$, the graph formed from the complete bipartite graph $K_{3,4g+2}$ by adding three edges to the part with three vertices has genus g has edges of weight $4g + 7$. In particular, all the edges other than the added triangle have weight $4g + 7$. Therefore Lemma 5.2.3 is sharp for all g.

Here we give an easy proof of a weaker bound. For $g = 1$ this bound equals the bound given by Lemma 5.2.3. For all other g, Lemma 5.2.3 is a stronger bound.

Lemma 5.2.4. *If G is a simple graph satisfying $0 < \gamma(G) \leq g$ and $\delta(G) \geq 3$, then there exists $uv \in E(G)$ with $w(uv) \leq 24g - 9$.*

Proof. Consider a 2-cell embedding of G on a surface of genus $\gamma(G)$. Since we may add edges without lowering the degree sum of any existing edges, we may assume that each face of the embedding is a triangle. Suppose to the contrary that $w(uv) > 24g - 9$ for all $uv \in E(G)$. To each vertex v, we assign charge $d(v) - 6$. From Euler's formula the total charge assigned to G is $12g - 12$. We now move charge according to the following rules:

- If $d(v) = 3$, then v takes charge 1 from each neighbor.

- If $d(v) = 4$, then v takes charge $\frac{1}{2}$ from each neighbor.

- If $d(v) = 5$, then v takes charge $\frac{1}{5}$ from each neighbor.

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The final charge of a vertex v is denoted by $c(v)$.

Claim 1: $c(v) \geq 0$ for all $v \in V(G)$.

Since $g \geq 1$, any edge uv satisfies $w(uv) > 24g - 9 \geq 15$. Therefore vertices with degree at most 10 lose no charge, and $c(v) = 0$ when $d(v) \leq 6$. Also, $c(v) > 0$ when $6 < d(v) < 11$. Because all edges have weight at least 15, a vertex v cannot have adjacent 5^--vertices. Since also G is a triangulation, v loses charge to at most $d(v)/2$ neighbors. If $11 \leq d(v) < 13$, then v has no 3-neighbors, and $c(v) \geq d(v) - 6 - \frac{1}{2} \cdot \frac{d(v)}{2} \geq 0$. If $d(v) \geq 13$, then $c(v) \geq d(v) - 6 - 1 \cdot \frac{d(v)}{2} \geq 0$.

Claim 2: There exists $v \in V(G)$ such that $c(v) > 12g - 12$.

If v has a 3-neighbor u, then $d(v) > 24g - 12$. Since v cannot have consecutive 3-neighbors, $c(v) \geq d(v) - 6 - \frac{d(v)}{2} > 12g - 12$. If v has no 3-neighbors, but has a neighbor u with $d(u) \in \{4, 5\}$, then $d(v) > 24g - 14$. Again v cannot have consecutive such neighbors, so $c(v) \geq d(v) - 6 - \frac{d(v)}{4} > 18g - \frac{66}{4} > 12g - 6$. We may now assume $\delta(G) \geq 6$, implying that no vertex loses any charge. Since $w(uv) > 24g - 9$, at least one endpoint has degree at least $12g - 5$.

Claims 1 and 2 imply $\sum_{v \in V(G)} c(v) > 12g - 12$, contradicting that the total charge is $12g - 12$. $\qquad\square$

Theorem 1.4.1. *Let G be a graph, and let $g = \gamma(G)$.*

1. *If $g = 0$ and $r \geq 11$, then $\mathrm{ch}_r(G) \leq 5(r + 1) + 3$*

2. *If $g \geq 1$ and $r \geq 24g - 11$, then $\mathrm{ch}_r(G) \leq (12g - 6)(r + 1) + 3$.*

Proof. We proceed by induction on $|V(G)|$. If $|V(G)| \leq 4$, then G is planar and $\mathrm{ch}_r(G) \leq |V(G)| < 5(r+1)+3$.

Let

$$\ell = \begin{cases} 5(r + 1) + 3 & \text{if } g = 0, \\ (12g - 6)(r + 1) + 3 & \text{otherwise,} \end{cases}$$

$$\omega = \begin{cases} 13 & \text{if } g = 0, \\ 24g - 9 & \text{otherwise.} \end{cases}$$

Let L be a list assignment to G with $|L(v)| \geq \ell$ for all $v \in V(G)$. Suppose G has a 2^--vertex v. Since $\ell > 2r$, by Lemma 5.2.1 v is reducible and applying induction to $G - v$ completes the proof. Therefore we may assume $\delta(G) \geq 3$. By Lemmas 5.2.2 and 5.2.4, there exists an edge $uv \in E(G)$ with weight at most ω. Suppose $d(u) \leq d(v)$, and let G' be obtained by contracting uv. View v as "absorbing" the edges incident to u, deleting multiedges. Let the list assignment L' be the restriction of L to $V(G) - u$.

Because G' was formed through edge contraction, we satisfy $\gamma(G') \leq \gamma(G)$. Since the bound for $g = 0$ is less than the bound for $g = 1$, the induction hypothesis implies that G' has a r-dynamic L'-coloring ϕ. Note that $r \geq \omega - 2$, so all neighbors of v in G' receive distinct colors. We extend ϕ to an L-coloring of G by appropriately coloring u. To ensure that u is r-dynamic, we must avoid the colors on $N_G(u)$; there are at most $d_G(u)$ of these. To ensure that v is r-dynamic, we also avoid the colors on $N_G(v) - u$; there are at most $d_G(v) - 1$ of these. To ensure that each vertex in $N_G(u) - v$ is r-dynamic, we must avoid at most $r - 1$ colors for each vertex in $N_G(u) - v$. Thus we must avoid at most $d_G(u) + d_G(v) - 1 + (r-1)(d_G(u)-1)$ colors when coloring u. Since $d_G(u) + d_G(v) \leq \omega$ and $d_G(u) \leq (\omega - 1)/2$ (since ω is odd), simplifying yields

$$d(u) + d(v) - 1 + (r-1)(d(u)-1) = (d(u) + d(v)) + (r-1)d(u) - r$$

$$\leq \omega + (r-1)\frac{\omega - 1}{2} - r$$

$$= \frac{(r+1)\omega}{2} - \frac{r-1}{2} - r$$

$$= \frac{(r+1)\omega}{2} - \frac{3r+3}{2} + 2$$

$$= \frac{(\omega - 3)(r+1)}{2} + 2$$

$$= \begin{cases} 5(r+1) + 2 & \text{if } g \leq 2, \\ (12g - 6)(r+1) + 2 & \text{else} \end{cases}$$

$$= \ell - 1.$$

Thus $L(u)$ has more colors than u needs to avoid, and we can obtain a proper r-dynamic L-coloring of G. □

Lemma 5.2.3 proves the existence of an edge with smaller edge weight than the Lemma 5.2.4. This allows us to strengthen Theorem 1.4.1 since we do not need as many colors when we extend an r-dynamic L'-coloring of the graph formed by contracting an edge if the edge has smaller weight. Asymptotically Theorem 5.2.5 improves Theorem 1.4.1 by a factor of 6 for large g. However, for $g \in \{0, 1\}$ the two bounds are equal.

Theorem 5.2.5. *Let G be a graph, and let $g = \gamma(G)$.*

1. *If $g \leq 2$ and $r \geq 2g + 11$, then $\text{ch}_r(G) \leq (g + 5)(r + 1) + 3$*

2. *If $g \geq 3$ and $r \geq 4g + 5$, then $\text{ch}_r(G) \leq (2g + 2)(r + 1) + 3$.*

The proof of Theorem 5.2.5 follows from Lemma 5.2.3 in much the same way that the proof of Theo-

rem 1.4.1 follows from Lemma 5.2.4. The only differences lie in the bounds on ℓ and ω and for which g the bounds switch.

Even the tighter Theorem 5.2.5 is unlikely to be sharp even for the plane. Hell and Seyffarth [25] found examples of planar graphs with diameter 2, maximum degree r, and $\lfloor \frac{3r}{2} \rfloor + 1$ vertices. For such planar graphs, $\chi_r(G) = \lfloor \frac{3r}{2} \rfloor + 1$.

Zhu [56] and Schauz [46] independently introduced an online version of choosability, which is modeled by the following game. Suppose that G is a graph and that each vertex $v \in V(G)$ is assigned a positive number $f(v)$ of *tokens*. The *f-paintability game* is played by two players: Lister and Painter. On the ith round, Lister *marks* a nonempty set of uncolored vertices; each marked vertex loses one token. Painter responds by choosing a subset of the marked set that forms an independent set in the graph and assigning color i to each vertex in that subset. Lister wins the game by marking a vertex with no tokens, and Painter wins by coloring all vertices.

We say that G is *f-paintable* when Painter has a winning strategy in the f-paintability game. When G is f-paintable and $f(v) = k$ for all $v \in V(G)$, we say that G is *k-paintable*. The least k such that G is k-paintable is the *paint number* (or *online choice number*) of G, denoted by $\mathrm{\mathring{c}h}(G)$.

In the f-paintability game, Painter's goal is to generate a proper coloring of the graph. We say that a graph G is *r-dynamically k-paintable* if Painter has a winning strategy that produces an r-dynamic coloring of G when all vertices have k tokens. The least k such that Painter can accomplish this is the *r-dynamic paint number*, denoted by $\mathrm{\mathring{c}h}_r(G)$.

As with r-dynamic colorability and r-dynamic choosability, we have the following string of inequalities for r-dynamic paintability:

$$\mathrm{\mathring{c}h}(G) = \mathrm{\mathring{c}h}_1(G) \le \mathrm{\mathring{c}h}_2(G) \le \cdots \le \mathrm{\mathring{c}h}_{\Delta(G)}(G) = \cdots = \mathrm{\mathring{c}h}(G^2).$$

Furthermore, $\chi_r(G) \le \mathrm{ch}_r(G) \le \mathrm{\mathring{c}h}_r(G)$ for all r.

Because $(k-1)$-degenerate graphs are k-paintable, this also shows that graphs with genus g are $h(g)$-paintable, where $h(g)$ is the Heawood bound. Mahoney [38] strenthened the result of Chen et al. [13] to prove that such a graph is 2-dynamically $h(g)$-paintable.

Let $\mathcal{G}_g$ be the family of graphs embeddable on a surface of genus g. Bounds on the r-dynamic coloring parameters for graphs of given genus are well known for $r = 1$: for $G \in \mathcal{G}_0$, $\chi_1(G) \le 4$, while $\mathrm{ch}_1(G) \le 5$ and $\mathrm{\mathring{c}h}_1(G) \le 5$ with equality achievable for each bound. Loeb et al. [34] showed that, for $G \in \mathcal{G}_1$, $\chi_3(G) \le \mathrm{ch}_3(G) \le \mathrm{\mathring{c}h}_3(G) \le 10$. On the torus, equality is achieved by the Petersen graph, but equality is not known

in the plane.

Question 5.2.6. *Over $G \in \mathcal{G}_0$, what are $\max \chi_3(G)$, $\max \mathrm{ch}_3(G)$, and $\max \check{\mathrm{ch}}_3(G)$?*

An example of a planar graph G with $\chi_3(G) = 7$ is the graph obtained from K_4 by subdividing the three edges incident to one vertex, shown in Figure 5.2. Note that G has maximum degree 3 and diameter 2, so $\chi_3(G) = \chi(G^2) = \chi(K_7) = 7$. Hence the maximum of $\chi_3(G)$ over planar graphs is at least 7.
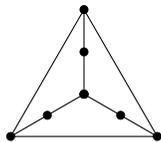

Figure 5.2: Example of planar graph with r-dynamic chromatic number 7.

Thomassen [48] proved that planar graphs are 5-choosable, and Voigt [51] proved sharpness. Schauz [46] further proved that planar graphs are 5-paintable. Kim, Lee, and Park [32] proved that planar graphs are actually 2-dynamically 5-choosable by invoking Thomassen's result. By using Schauz's result that planar graphs are 5-paintable instead, the result of Kim, Lee, and Park was strengthened to the following corollary.

Corollary 5.2.7. *[38] If G is a planar graph, then $\check{\mathrm{ch}}_2(G) \le 5$.*

Thus over $G \in \mathcal{G}_0$, we have $\max \chi(G) = 4 < 5 = \max \mathrm{ch}(G) = \max \check{\mathrm{ch}}(G)$, but $\max \chi_2(G) = \max \mathrm{ch}_2(G) = \max \check{\mathrm{ch}}_2(G) = 5$. Determining tight upper bounds for $\max_{G \in \mathcal{G}_g} \mathrm{ch}_r(G)$ for $r = 3$ and $g > 1$ and for all $r > 3$ is also of interest.

Question 5.2.8. *Over graphs $G \in \mathcal{G}_g$ for $r \ge 3$ what are $\max \chi_r(G)$, $\max \mathrm{ch}_r(G)$, and $\max \check{\mathrm{ch}}_r(G)$?*

In addition, recall that for graphs $G \in \mathcal{G}_g$ with $g > 0$, Chen et al. [13] showed that $\max \chi(G) = \max \chi_2(G) = \max \mathrm{ch}(G) = \max \mathrm{ch}_2(G)$, which motivates the questions of equality for other fixed r. Note that every nonplanar graph G is 2-dynamically $h(\gamma(G))$-paintable [38], which implies equality for $r = 2$.

Question 5.2.9. *Except when $r = 1$ and $g = 0$, is it true that $\max_{G \in \mathcal{G}_g} \chi_r(G) = \max_{G \in \mathcal{G}_g} \check{\mathrm{ch}}_r(G)$?*

Theorem 5.2.5 gives upper bounds on these maximum values.

Chapter 6

Antipodal edge-colorings of hypercubes

6.1 Background

Two vertices in the hypercube Q_n are *antipodal* if they differ in every coordinate. Two edges uv and xy are *antipodal* if u is antipodal to x and v is antipodal to y. An *antipodal edge-coloring* of Q_n is a 2-coloring of the edges of Q_n such that antipodal edges have different colors.

DeVos and Norine [16] conjectured the following

Conjecture 1.5.1. *For $n \geq 2$, in every antipodal edge-coloring of Q_n there is a pair of antipodal vertices connected by a monochromatic path.*

In an antipodal edge-coloring, the graphs formed by the two colors are isomorphic (under complementation of the vertices). Feder and Subi [19] proved a strengthening of Conjecture 1.5.1 for $n \leq 5$. A *geodesic* is a shortest path between the endpoints of the path. In Q_n, a geodesic crosses each dimension of the hypercube at most once. Any geodesic in Q_n between two antipodal vertices has length n. They showed that for $n \leq 5$, in every antipodal edge-coloring of Q_n there are two antipodal vertices joined by a monochromatic geodesic.

Feder and Subi [19] also proved that the conclusion holds for any 2-edge-coloring (not necessarily antipodal) that contains no 4-cycle along which the colors alternate. Despite the theorem of Feder and Subi not requiring the coloring be antipodal, weakening the hypothesis in the conjecture to require only giving each color to exactly half the edges in each dimension permits a counterexample, as shown in Figure 6.1 for Q_4, where the two color subgraphs are also isomorphic (the half-half condition suffices in Q_3). Thus antipodal edge-colorings are more special than splitting the edges in half.

Feder and Subi [19] further noted that a counterexample for Q_n can be extended to a counterexample for Q_{n+1}. Suppose we have a counterexample of a labelled n-dimensional hypercube Q_n. Making two copies of the labelled Q_n and joining them arbitrarily but antipodally gives a counter example Q_{n+1}, as a monochromatic antipodal path in the resulting coloring on Q_{n+1} would yield such a path in Q_n by copying the portion form one copy of the Q_n to the other.
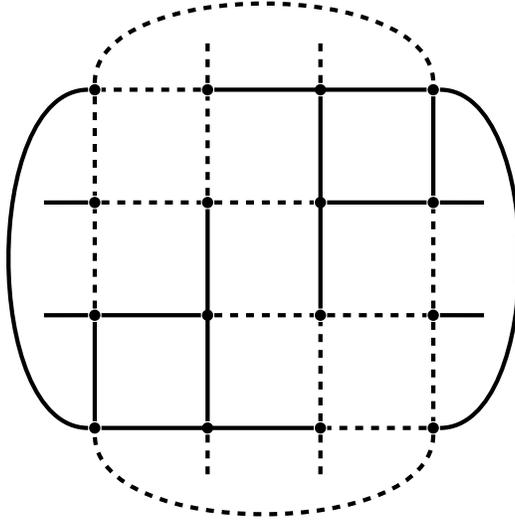
Figure 6.1: 2-edge coloring of Q_4 with exactly half the edges in every dimension red.

In this chapter, we give proofs of the strengthening of Conjecture 1.5.1 for $n \in \{4, 5\}$ using a conceptually easier technique than [19] and further prove the strengthening of Conjecture 1.5.1 for $n = 6$ yielding

Theorem 1.5.2. *For $2 \le n \le 6$, in every antipodal edge-coloring of Q_n there is a pair of antipodal vertices connected by a monochromatic geodesic.*

It is our hope that this approach can be used to prove the statement for larger n.

6.2 The cubes

The vertex antipodal to a vertex v will be denoted $\overline{v}$. In all figures in this section, colored edges are either bold (red) or dashed (blue). Gray edges have color unspecified. Some edges of the hypercube are omitted from the figures for clarity. An *alternating 4-cycle* is a 4-cycle $[a, b, c, d]$ such that the edges ab and cd are red, while the edges bc and ad are blue. Theorem 1.5.2 is obvious for $n = 2$. We prove cases $\le d \le 6$ as individual lemmas.

Lemma 6.2.1. *If an antipodally edge-colored n-cube contains a geodesic joining antipodal vertices that is the union of two monochromatic paths, then it contains a monochromatic geodesic from the common vertex of these paths to its antipode. In particular, if Q_n contains a monochromatic geodesic of length $n - 1$, then Q_n contains a monochromatic geodesic of length n.*

Proof. Let v and $\overline{v}$ be the endpoints of the given geodesic, with u the common vertex of its monochromatic path. We obtain the desired geodesic by concatenating the monochromatic geodesic from u to v with the

monochromatic geodesic from v to $\overline{u}$ that is antipodal to the monochromatic geodesic from u to $\overline{v}$. The resulting path is a monochromatic geodesic of length n from u to $\overline{u}$. □

Proposition 6.2.2. *Every antipodal edge-coloring of Q_4 that has an alternating 4-cycle contains a monochromatic geodesic joining some two antipodal vertices.*

Proof. An alternating 4-cycle is connected by a geodesic P of length 2 to its antipodal alternating 4-cycle (see Figure 6.2). If P is monochromatic, then extending by the incident edge of that color in the two alternating 4-cycles yields a monochromatic geodesic of length 4.

Therefore, we may assume that P is not monochromatic and changes color at the vertex v as in Figure 6.2. Let e be another edge incident to v. Regardless of the color of e, it forms a monochromatic geodesic of length 3 with one edge from P and one from the adjacent alternating 4-cycle. By Lemma 6.2.1, Q_4 contains a monochromatic geodesic of length 4. □
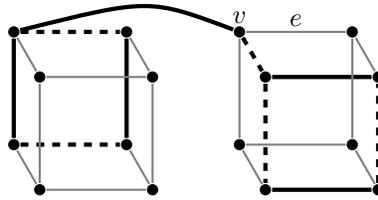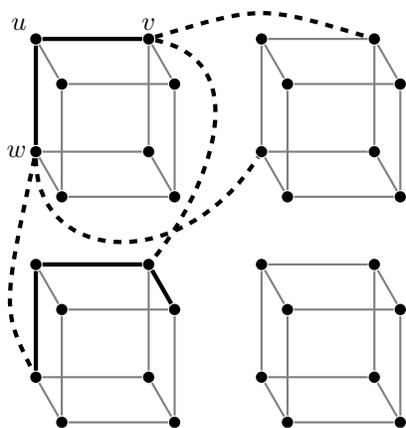

Figure 6.2: Finding a monochromatic geodesic of length 3 in Q_4.

Lemma 6.2.3. *Every antipodal edge-coloring of Q_4 contains a monochromatic geodesic joining some two antipodal vertices.*

Proof. By Proposition 6.2.2, we may assume that Q_4 does not contain an alternating 4-cycle.

If there exists a 4-cycle with two edges of each color, then let v and u be the vertices of the 4-cycle where the color changes. Let e be another edge incident to v. Regardless of the color of e, it forms a monochromatic geodesic of length 3 with two edges from the 4-cycle. By Lemma 6.2.1, Q_4 contains a monochromatic geodesic of length 4.

Thus, we may assume that every 4-cycle contains three edges of one color and one of the opposite color. Every edge leaving a 4-cycle C with three red edges must be blue, or it forms a monochromatic geodesic of length 3. The antipodal 4-cycle C' contains three blue edges and one red. Every edge leaving this antipodal 4-cycle must thus be red. Consider two vertices at distance 2, one on C and one on C'. The two geodesics of length 2 joining these two vertices form a 4-cycle with two edges of each color, a contradiction. □

Proposition 6.2.4. *For $n \geq 5$, any 2-edge coloring of Q_n contains a monochromatic geodesic of length 3.*

Proof. Since $\delta(Q_n) \geq 3$, Q_n contains a monochromatic geodesic P of length 2, say v to u to w in red. Figure 6.3 shows a 5-dimensional subcube of Q_n.

If any edge incident to v or w and passing into a third dimension is red, then we have the desired monochromatic geodesic of length 3. Hence assume that these edges are not red, but rather blue as shown in Figure 6.3. These blue edges give more monochromatic geodesics, each of length 2. If any edge incident to an endpoint of any of these new geodesics of length 2 and crossing a third dimension not already crossed by a geodesic ending there is blue, then we are done. If none of these edges are blue, then since the endpoints of the blue geodesic centered at v and w include vertices at distance 2, the red edges incident to them include geodesics of length 3, such as that shown in the lower left of Figure 6.3. □

Figure 6.3: Finding a monochromatic geodesic of length 3 in Q_n.

Lemma 6.2.5. *Every antipodal edge-coloring of Q_5 contains a monochromatic geodesic joining some two antipodal vertices.*

Proof. By Proposition 6.2.4, every antipodal edge-coloring of Q_5 has a monochromatic geodesic P of length 3, as shown in red in the upper left copy of Q_3 in Figure 6.4. The corresponding antipodal geodesic $\overline{P}$ is shown in blue the lower right subcube. It suffices to produce a monochromatic geodesic of length 4, because then Proposition 6.2.1 guarantees a monochromatic geodesic of length 5. Consider the edges of Figure 6.4 that join two of the four indicated copies of Q_3, forming a 4-cycle using any corresponding (not antipodal) endpoints of P and $\overline{P}$. If any of these are colored other than as shown, then they form a monochromatic geodesic of length 4 with P or $\overline{P}$ and we are done.

Each of these edges has an endpoint in either the upper right or the lower left copy of Q_3 at s or t, respectively. Consider the edges incident to s and t in the given copies of Q_3. First suppose that two edges incident to t in that copy of Q_3 are different colors, as shown in the lower left of Figure 6.4. In the top right
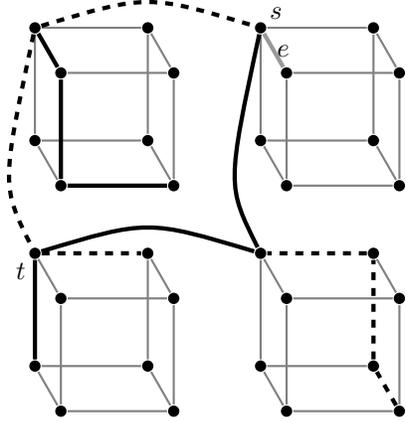
Figure 6.4: A partial antipodal edge-coloring of Q_5. Two edges incident to t in its Q_3 are different colors.

copy of Q_3, e completes a monochromatic geodesic of length 4, regardless of its color. Hence, we may assume that the situation shown in the lower left copy of Q_3 of Figure 6.4 is not the case, i.e. all edges incident at t in that copy of Q_3 are the same color. Without loss of generality, they are all blue as in Figure 6.5.
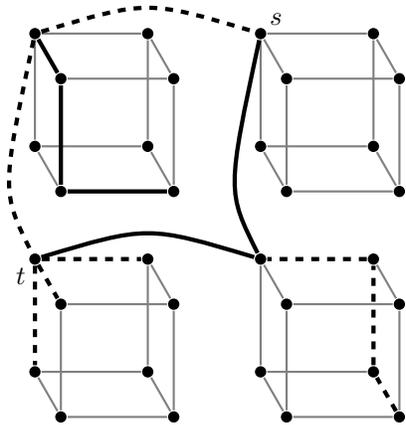

Figure 6.5: A partial antipodal edge-coloring of Q_5. All edges incident to t in its Q_3 are blue.

Now consider the upper right copy of Q_3 in Figure 6.5. The three edges adjacent to s must all be the same color, by the same reasoning. If they are blue, then we have a monochromatic geodesic of length 4. Hence they must all be red, as shown in Figure 6.6. The edges in the bottom left copy of Q_3 antipodal to these must then be blue. If any of the remaining edges in the bottom left copy of Q_3 are blue, then we have a monochromatic geodesic of length 5 in blue, so they must form a red 6-cycle as shown. Finally, consider the two edges a and b of Figure 6.6. If either of them is red, then we have a monochromatic geodesic of length at least 4. However, if they are both blue, then we again have a monochromatic geodesic of length 4.

Thus, the proof is complete and every antipodal edge-coloring of Q_5 contains a monochromatic geodesic joining some two antipodal vertices. □
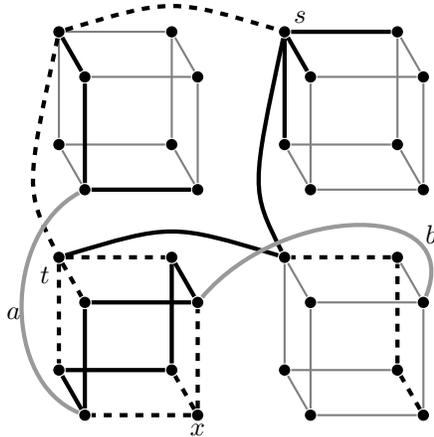
Figure 6.6: A partial antipodal edge-coloring of Q_5. Color of gray edges is undetermined.

For clarity in our next proof we write the vertex names by collapsing six bits to two octal digits, with each digit representing the binary triple given by its binary expansion as in Figure 6.7. Note that $\overline{ij} = (7-i)(7-j)$.
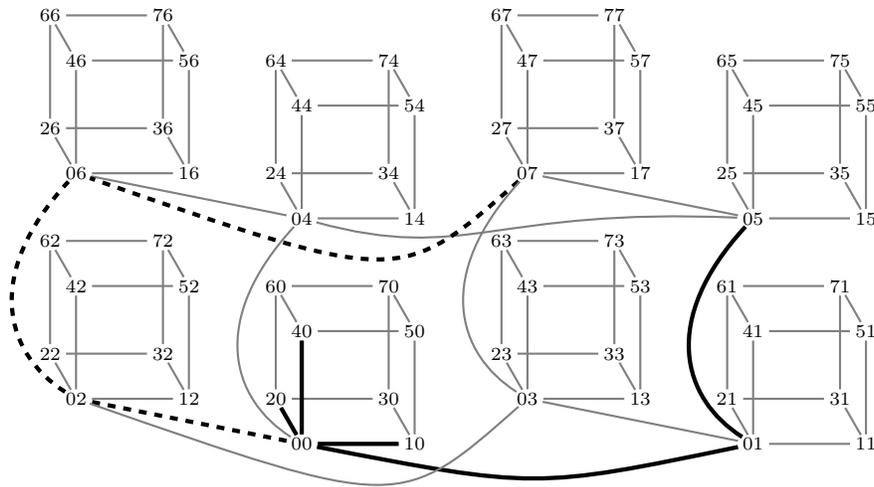

Figure 6.7: A partial antipodal edge-coloring of Q_6. Color of gray edges is undetermined.

Lemma 6.2.6. *Every antipodal edge-coloring of Q_6 contains a monochromatic geodesic joining some two antipodal vertices.*

Proof. We first show by contradiction that every antipodal edge-coloring of Q_6 has a monochromatic geodesic R of length 4. Consider an antipodal coloring c of Q_6 with no monochromatic geodesic of length 4. By Proposition 6.2.4, c has a monochromatic geodesic P of length 3. Without loss of generality let P have endpoints 06 and 76 (crossing the first three dimensions) and be colored red. Since there is no monochromatic geodesic of length 4, the other edges incident to these endpoints must be blue; in particular 06 : 02 and 06 : 07

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are blue as in Figure 6.7. Since c is an antipodal coloring, $\overline{P}$ is blue and the edges incident to its endpoints that cross the last three dimensions must be red; in particular $01:05$ and $01:00$ are red. Consider the edge $02:00$. Since the coloring is symmetric whether this edge is blue or red, without loss of generality, we may assume this edge is blue. Now $07:06:02:00$ is a blue geodesic of length 3. Thus the edges incident with 00 that cross the first three dimensions must be red. Now $20:00:01:05$ a red geodesic of length 3, and the edges $05:07$ and $05:45$ must be blue. This yields $45:05:07:06:02$ as a blue geodesic of length 4.

We now show by contradiction that every antipodal edge-coloring of Q_6 has a monochromatic geodesic of length 5. We have just seen that there is a monochromatic geodesic R of length 4. Without loss of generality let R have endpoints 76 and 02 and be colored red. Assume that there is no monochromatic geodesic of length 5. The edges incident to these endpoints and crossing the fifth and sixth dimensions must be blue. Since this is an antipodal coloring, $\overline{R}$ is blue and the edges incident to its endpoints and crossing the fifth and sixth dimensions must be red, as shown in Figure 6.8.
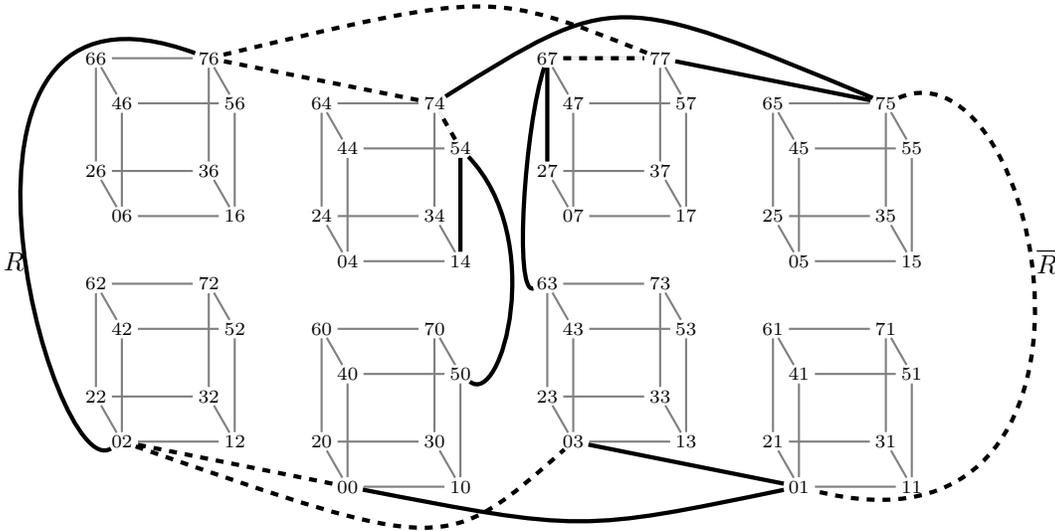

Figure 6.8: A partial antipodal edge-coloring of Q_6. Case 1. Color of gray edges is undetermined.

Case 1: 77 *and* 74 *are incident to edges of the same color in distinct dimensions among the first four dimensions.*

By symmetry, we may assume that $74:54$ and $77:67$ are blue. These two edges together with $74:76:77$ form a monochromatic geodesic of length 4 from 54 to 67. Thus $14:54:50$ and $27:67:63$ must be red.

Case 1.1: *Another edge incident to* 74 *crossing one of the first four dimensions, other than* $64:74$ *parallel*

to $67 : 77$, *is also blue.*

Without loss of generality, let the new edge be $74 : 34$. Since $34 : 74 : 76 : 77 : 67$ is a blue geodesic, $14 : 34 : 30$ and $47 : 67$ must be red and the antipodal edge $30 : 10$ must be blue.

Case 1.1.1: $34 : 24$ *or* $54 : 44$ *is blue.* The edges are symmetric at this point since no choice has distinguished between the first two dimensions, so we may assume $34 : 24$ is blue. Since $50 : 10$ is antipodal to $27 : 67$, it is blue. Since $24 : 34 : 74 : 76 : 77$ is a blue geodesic, $04 : 24 : 20$ and $57 : 77 : 73$ must be red and the antipodal geodesics, $73 : 53 : 57$ and $20 : 00 : 04$, must be blue as in Figure 6.9.
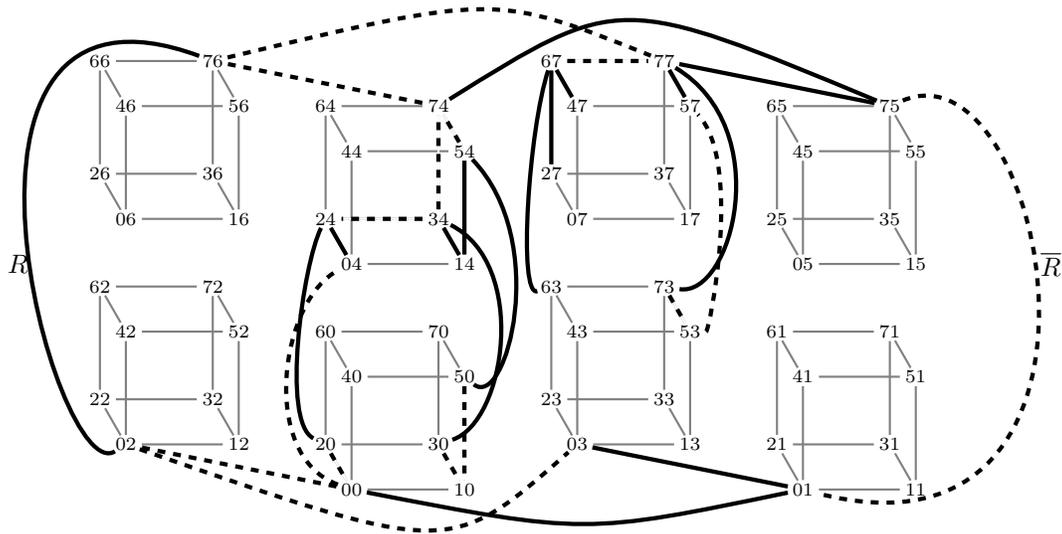
Figure 6.9: A partial antipodal edge-coloring of Q_6. Case 1.1.1. Color of thin edges is undetermined.

Case 1.1.1.1: $56 : 54$ *is blue.*

Geodesic	forces
$56:54:74:34:24$ is blue	$52:56:57$ and $20:24:25$ red, $25:21:20$ and $57:53:52$ blue
$25:21:20:30:10:50$ not blue	$20:30$ red
$20:30:34:14:54$ red	$22:20$ and $54:55$ blue
$25:24:04:14:54:50$ not red	$04:14$ blue
$14:04:00:20:22$ blue	$14:15$ red
$15:14:34:30:20$ red	$20:60$ and $55:15:17$ blue
$17:15:55:54:74$ blue	$74:70$ red
$57:77:75:74:70$ red	$70:60$ blue
$25:21:20:60:70$ blue	$70:50$ red
$70:50:54:14:15$ red	$70:72$ blue
$04:00:20:60:70:72$ blue	

Case 1.1.1.2: $56:54$ *is red.*

Geodesic	forces
$56:54:14:34:30$ red	$46:56:57$ blue
$46:56:57:53:73$ blue	$46:44$ and $73:33$ red, $04:44$ blue
$03:02:00:04:44$ blue	$54:44$ red
$46:44:54:14:34:30$ red	

Case 1.1.2: $34:24$ *and* $54:44$ *are both red. See Figure 6.10*

Geodesic	forces
$24:34:14:54:50$ red	$52:50:51$ blue
$44:54:14:34:30$ red	$32:30:31$ blue
$32:30:10:50:51$ blue	$22:32:36$ and $41:51:55$ red, $36:26:22$ blue
$24:34:14:54$ red	$53:43:63:23$ blue
$53:43:63:23:22:26$ not blue	$22:23$ red
$36:32:22:23:03:43$ red	

Case 1.2: $77:67$, $77:57$, $74:64$, $74:54$ *are all blue. Antipodally,* $77:73$, $77:37$, $74:70$, $74:34$ *are red.* Since Case 1.1 is complete, symmetry allows us to assume that $74:34$, $74:70$, $77:37$, $77:73$ are all red. If $74:64$ or $77:57$ is also red, then we have a case symmetric to Case 1.1, switching blue and red. Hence we may assume that $77:67$, $77:57$, $74:64$, $74:54$ are all blue. Antipodally, $77:73$, $77:37$, $74:70$, $74:34$ are red.
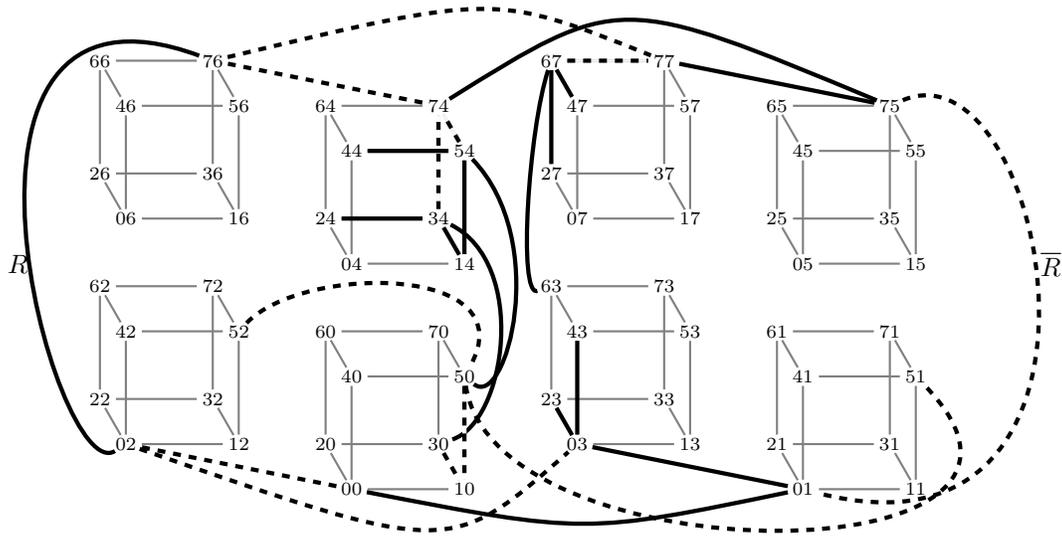
Figure 6.10: A partial antipodal edge-coloring of Q_6. Case 1.1.2. Color of thin edges is undetermined.

Geodesic	forces
$54:74:76:77:67$ blue	$50:54:14$ and $63:67:27$ red, $27:23:63$ and $14:10:50$ blue
$70:74:75:77:37$ red	$50:70:60$ and $17:37:27$ blue, $27:07:17$ and $60:40:50$ red
$14:04:24:64:60$ red	$60:62$ blue
$14:34:24:20:60:62$ blue	

Case 2: *The edges incident to* 77 *that cross the first four dimensions are all blue, and the edges incident to* 74 *that cross the first four dimensions are all red.* To see that this is the remaining case, consider the edges incident to 77 crossing the first four dimensions. Without loss of generality there are at least two blue edges. If 74 is incident to some blue edge crossing one of the first four dimensions, then we have Case 1. Hence all edges incident to 74 that cross one of the first four dimensions are red. By the symmetric argument, all edges incident to 77 that cross one of the first four dimensions are blue.

Case 2.1: $73:63$ *is blue. Antipodally* $04:14$ *is red.* The geodesic $74:76:77:73:63$ is blue, so $43:63:23$ must be red, and antipodally $34:14:54$ is blue as in Figure 6.11..

Since Case 1 is done, every geodesic of length 4 must behave as specified here in Case 2. Thus each of $43, 23, 34, 54$ has five incident edges of one color and one incident edge of the other color. Either 43 and 54 each have five incident red edges or 23 and 34 each have five incient red edges. The other pair each have five incident blue edges. The choice is symmetric since at this point there is no distinction between the first two
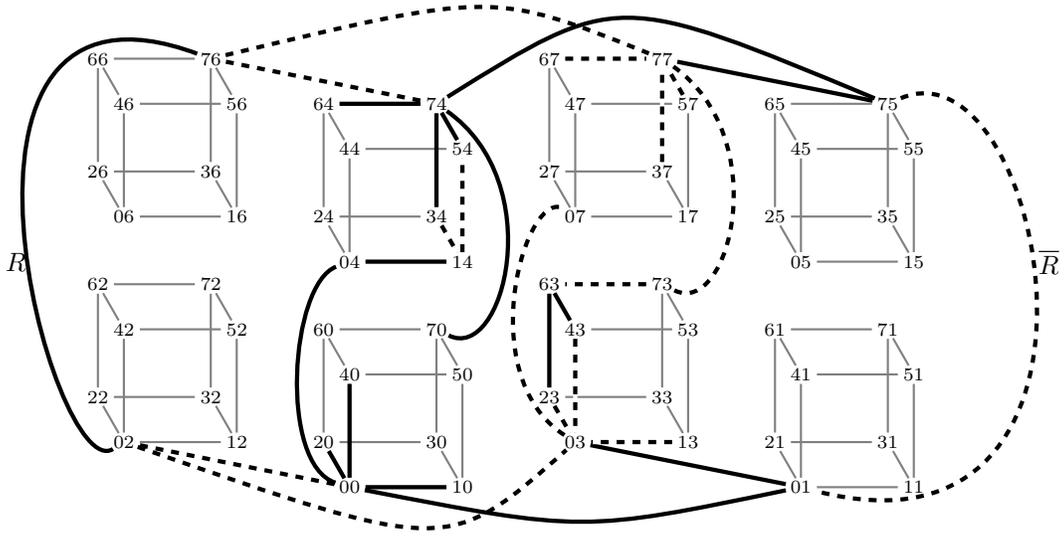
Figure 6.11: A partial antipodal edge-coloring of Q_6. Case 2.1. Color of thin edges is undetermined.

dimensions. Hence we may assume that 43 and 54 each have five incident blue edges and 23 and 34 each have five incident red edges.

The geodesic $00 : 02 : 03 : 43 : 47$ is blue, so $67 : 47 : 57$ must be red. The geodesic $00 : 02 : 03 : 43 : 53$ is blue, so $73 : 53 : 57$ must be red and antipodally, $20 : 24 : 04$ is blue.

Case 2.1.1: $51 : 53$ *is red.*

Geodesic	forces
$67 : 47 : 57 : 53 : 51$ red	$11 : 51 : 50$ blue
$56 : 54 : 50 : 51 : 11$ blue	$01 : 11 : 31$ red
$31 : 11 : 01 : 00 : 04$ red	$04 : 44$ blue
$20 : 24 : 04 : 44 : 54 : 56$ blue	

Case 2.1.2: $51 : 53$ *is blue, and antipodally* $26 : 24$ *is red.*

Geodesic	forces
$51 : 53 : 43 : 03 : 02$ blue	$71 : 51 : 55$ and $22 : 02 : 06$ red, $06 : 26 : 22$ and $55 : 75 : 71$ blue
$51 : 53 : 43 : 03 : 07$ blue	$50 : 51$ and $27 : 07 : 06$ red, $27 : 26$ and $50 : 70 : 71$ blue

Let R' be the red geodesic $26 : 24 : 34 : 74 : 75$. By Case 1, 06 and 22 each have five incident edges of the same color, as reasoned at the beginning of Case 2, and these colors must be different at 06 and 22. However, we have $23 : 22 : 02$ and $08 : 06 : 02$ all red. Hence the repeated colors at 06 and 22 must both be red, a contradiction.

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Case 2.2: *All edges incident to any endpoint of an edge in T other than 74 and 00 are blue, where T is the set of edges incident to 74 or 00 crossing one of the first four dimensions.*

In this case, all edges of the copy of Q_4 on vertices with second coordinate 0 or 4 that are not incident to 74 or 00 are blue, as shown in Figure 6.12. Thus, to avoid a geodesic of length 5, all edges leaving this copy of Q_4 are red except those incident to 74 or 00.

Similarly, all edges of the copy of Q_4 on vertices with second coordinate 3 or 7 that are not incident to 03 or 77 are red. Thus, to avoid a geodesic of length 5, all edges leaving this copy of Q_4 are blue except those incident to 03 or 77.
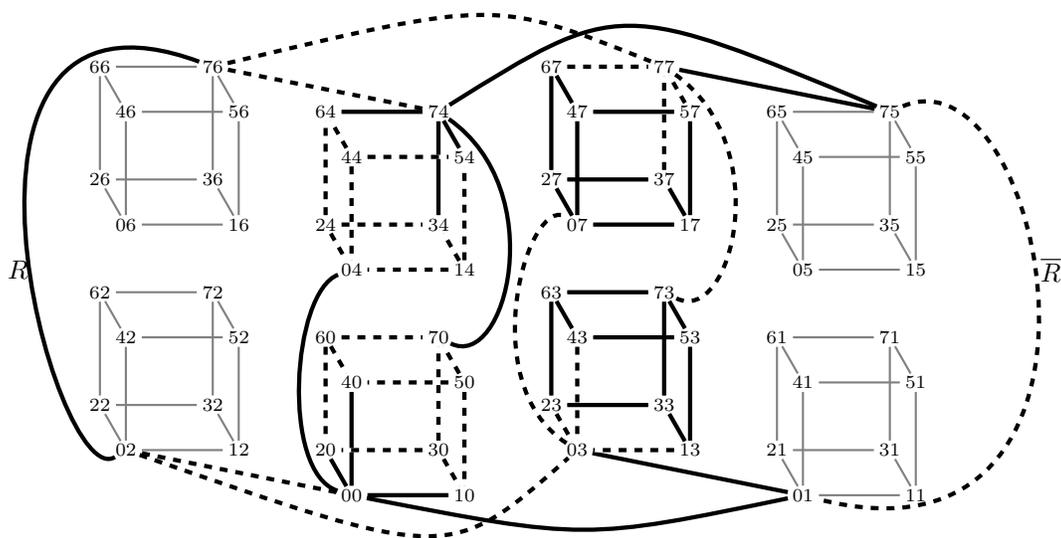

Figure 6.12: A partial antipodal edge-coloring of Q_6. Case 2.2. Color of thin edges is undetermined.

Note that $17 : 16$ leaves the Q_4 on vertices with second coordinate 3 or 7 and is not incident to either 03 or 77, so must be blue.

Geodesic	forces
$55 : 54 : 74 : 64 : 66$ red	$62 : 66 : 26$ blue
$71 : 70 : 74 : 34 : 36$ red	$16 : 36 : 26$ blue
$62 : 66 : 26 : 36 : 16 : 17$ blue	

This completes the proof. □

Chapter 7

Generalized graph cordiality

7.1 Background

Graph labelings of diverse types are the subject of much study. The state of the field is described in detail in Gallian's dynamic survey [21]. Results obtained so far, while numerous, are mainly piecemeal in nature and lack generality. In an attempt to provide something of a framework for these results, Hovey introduced A-cordial labelings in [26] as a common generalization of cordial labeling (introduced by Cahit [10]) and harmonious labeling (introduced by Graham and Sloane [23]).

If A is an additive abelian group, then a vertex-labeling $f: V(G) \to A$ of the vertices of a graph G induces an edge-labeling on G as well by giving the edge uv the label $f(u) + f(v)$.

Definition 7.1.1. Let A be an abelian group. We say that a graph G is A-*cordial* if there is a vertex-labeling $f: V(G) \to A$ such that (1) the vertex sets labeled by any two elements of A differ in size by at most 1, and (2) the induced edge sets labeled by any two elements of A differ in size by at most 1.

Such a labeling is *balanced*. If the sizes of the vertex label classes are exactly equal in a balanced labeling, then that vertex labeling is *perfectly balanced*. Similarly, if the sizes of the edge label classes are exactly equal in a balanced labeling, then that edge labeling is *perfectly balanced*.

Cordial graphs are simply the $\mathbb{Z}_2$-cordial graphs, while *harmonious graphs* are simply the $\mathbb{Z}_{|E(G)|}$-cordial graphs. Each of these concepts is well studied. Almost all other work on A-cordiality has also focused on the case where A is cyclic. This case is indeed very interesting, particularly in light of Hovey's conjecture from [26] that all trees are A-cordial for all cyclic groups A (which he proved for $|A| < 6$). The conjecture does not extend to even the smallest non-cyclic group, V_4 (also known as $\mathbb{Z}_2 \times \mathbb{Z}_2$); the paths P_4 and P_5 are easily seen to be not V_4-cordial, as we show in the next example.

The group V_4 has four elements; in this thesis we call them $0, a, b, c$. In the operation table, 0 is the identity, every element is its own inverse, and the remaining operations have the form $u + v = w$ where u, v, w are distinct elements of $\{a, b, c\}$.

Example 7.1.2. In order to be V_4-cordial, the vertices in P_4 must receive distinct labels and no label can be used on more than one edge. However, since the vertices have distinct labels, the outer two edges must have the same label d as follows. If 0 is on a leaf vertex, then d is the label on the vertex adjacent to the leaf labeled 0; otherwise d is the label on the leaf adjacent to the vertex labeled 0.

Similarly, for P_5 every element of V_4 must be used at least once on a vertex and no label can be used on two edges. Let x be the vertex label used twice. To have an edge labeled 0, the two vertices labeled x must be adjacent. Up to isomorphism, there are two options for the placement of these two vertices. We first consider the case where the first two vertices of P_5 are labeled x. If the third vertex is labeled 0, then the second and last edge must both have label x. If the third vertex is labeled y, then the second and last edge both have label z. We now consider the case where the second and third vertices of P_5 are labeled x. If the first vertex is labeled 0, then the first and last edge must both have label x. If the first vertex is labeled y, then the first and last edge both have label z.

Throughout this chapter, all our graphs are finite and simple, and all our quasigroups are finite. Section 7.2 considers some necessary conditions for a graph G to be A-cordial for certain A. In Section 7.3 we compare V_4-cordiality with A-cordiality for cyclic A. Finally, Section 7.4 introduces a generalization of A-cordiality involving digraphs and quasigroups, showing that there are infinitely many Q-cordial digraphs for every quasigroup Q.

7.2   Necessary Conditions for A-Cordiality

The following propositions will be used in the next section. The *exponent* of an additive abelian group A is the least $n \in \mathbb{Z}^+$ such that $na = 0$ for all $a \in A$.

Definition 7.2.1. Let $G_1, \ldots, G_n$ be groups. The cartesian product of the sets is a group with respect to componentwise addition. This group is the *(external) direct product* of the groups $G_1, \ldots, G_n$.

Theorem 7.2.2 ([27])**.** *Every finitely generated abelian group is isomorphic to a finite direct product of cyclic groups, each of which is either infinite or of order a power of a prime.*

Lemma 7.2.3. *If A is an abelian group of exponent 2, then $|A|$ is even. If further $|A| > 2$, then $\sum\limits_{a \in A} a = 0$.*

Proof. By Theorem 7.2.2, an abelian group is a finite direct product of cyclic groups. If A has exponent 2 and any one of these cyclic groups is not a copy of $\mathbb{Z}_2$, then there is an element a of for which $a + a \neq 0$. Since the direct product uses componentwise multiplication, any element $x \in A$ which has a as a component

cannot satisfy $a + a \ne 0$. Thus $|A|$ is even. If $|A| > 2$ and $|A|$ is even, each component is raised to an even power when all the elements of A are multiplied together and thus $\sum_{a \in A} a = 0$. $\qquad\square$

Proposition 7.2.4. *Let A be an abelian group of exponent 2, with $N = |A| > 2$. If G is an Eulerian graph with $m = |E(G)| \equiv \pm 2 \mod N$, then G is not A-cordial.*

Proof. Take an Eulerian circuit through G, and label the vertices along it $g_1, \ldots, g_m$ in order. For all i, let $h_i = g_i + g_{i+1}$ (taking the indices modulo m); these are precisely the labels assigned to the edges. In particular, $\sum_{i=1}^m h_i$ is the sum of all the edge labels. Clearly,

$$\sum_{i=1}^{\frac{m}{2}} h_{2i-1} = \sum_{i=1}^{m} g_i = \sum_{i=1}^{\frac{m}{2}} h_{2i}.$$

Since any element added to itself equals 0, we conclude that

$$\sum_{i=1}^{m} h_i = \sum_{i=1}^{\frac{m}{2}} h_{2i-1} + \sum_{i=1}^{\frac{m}{2}} h_{2i} = 2 \sum_{i=1}^{m} g_i = 0.$$

If the edge label classes were balanced, then all but two edge labels would appear equally often. By Lemma 7.2.3, the sum of all the elements of A is 0. Canceling sets of N distinct summands implies that there are two distinct elements of A that sum to 0, which is impossible, since every element of A is its own inverse. Hence, the edge label classes cannot be balanced, and G is not A-cordial. $\qquad\square$

Definition 7.2.5. A graph G is *1-factorable* if the edges of G can be partioned into disjoint perfect matchings.

Proposition 7.2.6. *Let A be an abelian group of exponent 2, with $N = |A| > 2$. If G is a 1-factorable graph with kN vertices and $\ell N \pm 2$ edges, where $k, \ell \in \mathbb{N}$, then G is not A-cordial.*

Proof. In an A-cordial labeling of G, the vertices must be perfectly balanced, since the number of vertices is divisible by N. Partition the edges of G into edge-disjoint perfect matchings. In each perfect matching, the sum of the vertex labels must be equal to the sum of the edge labels. Thus by Lemma 7.2.3, the sum of the labels on the edges in each of these matchings is 0. Thus, the sum of all the edge labels of G is 0. However, G has $\ell N \pm 2$ edges. Canceling sets of N edges with distinct labels leaves two distinct elements of A with sum 0, which is impossible, since every element of A is its own inverse. Thus, G is not A-cordial. $\qquad\square$

7.3 V_4-Cordiality for Some Families of Graphs

We denote the elements of V_4 by $0, a, b, c$; the sum of any two of $\{a, b, c\}$ is the third, and $g + g = 0$ for any $g \in V_4$. The study of V_4-cordiality was initiated by Riskin [45], who claimed the following results.

Claim 7.3.1 (Riskin, [45]). *The complete graph K_n is V_4-cordial if and only if $n < 4$.*

Claim 7.3.2 (Riskin, [45]). *All complete bipartite graphs $K_{m,n}$ are V_4-cordial except $K_{2,2}$.*

Riskin's proof of Claim 7.3.1 is essentially correct, except for some arithmetical errors. However, Claim 7.3.2 is not true.[1] We provide a corrected version of it.

Theorem 7.3.3. *The complete bipartite graph $K_{m,n}$ is V_4-cordial if and only if m and n are not both congruent to $2 \bmod 4$.*

Proof. Let X and Y be the partite sets, with $|X| = m$ and $|Y| = n$. Suppose that $\max\{m, n\} \geq 4$ and suppose that we have a V_4-cordial labeling of $K_{m,n}$. Note that in V_4, for distinct s, t, u, w, we have $s + t = u + w$. We claim that one of the partite sets has four vertices with distinct labels. If not, then some label u appears only in X and some other label w appears only in Y. Since this is a balanced labeling and u and w are on approximately $1/4$ of the vertices, this implies that the number of edges joining u-vertices to w-vertices is at least $\left(\frac{m+n}{4} - 1\right)^2$. We will derive a contradiction by showing that there are more than $\lceil \frac{mn}{4} \rceil$ edges assigned label $u + w$, including edges joining vertices labeled s to vertices labeled t.

By the Arithmetic mean-Geometric mean Inequality, $\frac{(m+n)^2}{16} \geq \frac{mn}{4}$. It remains to show that there are more than $\frac{m+n}{2} - 1$ other uw-edges, which we do by counting the edges joining s-vertices to t-vertices. Let s_X and t_X be the number of vertices in X labeled s or t respectively. Let $s_\ell = 1$ if the label s appears on fewer than $\frac{m+n}{4}$ vertices and let $s_\ell = 0$ if the label s appears on at least $\frac{m+n}{4}$ vertices. Similarly, let $t_\ell = 1$ if the label t appears on fewer than $\frac{m+n}{4}$ vertices and let $t_\ell = 0$ if the label t appears on at least $\frac{m+n}{4}$ vertices.

We see that the number of st-edges is at least

$$s_X \left(\frac{m+n}{4} - t_\ell - t_X \right) + t_X \left(\frac{m+n}{4} - s_\ell - s_X \right) = (s_X + t_X) \frac{m+n}{4} - 2 s_X t_X - t_\ell s_X - s_\ell t_X.$$

Since there must be at least $\lfloor \frac{mn}{4} \rfloor$ edges labeled 0, we have

$$s_X \left(\frac{m+n}{4} - s_\ell - s_X \right) + t_X \left(\frac{m+n}{4} - t_\ell - t_X \right) = (s_X + t_X) \frac{m+n}{4} - s_X^2 - t_X^2 - s_\ell s_X - t_\ell t_X \geq \frac{mn}{4}.$$

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[1] An anonymous reviewer informed us that some of these mistakes were also identified in an unpublished undergraduate thesis by McAlexander [39]. That thesis may also anticipate some of our other results; we were unable to obtain a copy.

Without loss of generality, we may assume $s_X \geq t_X$, in which case

$$(s_X + t_X)\frac{m+n}{4} - 2s_X t_X - t_\ell s_X - s_\ell t_X - (s_X + t_X)\frac{m+n}{4} - s_X^2 - t_X^2 - s_\ell s_X - t_\ell t_X$$

$$= -2s_X t_X - t_\ell s_X - s_\ell t_X - (-s_X^2 - t_X^2 - s_\ell s_X - t_\ell t_X)$$

$$= (s_X - t_X)^2 - t_\ell s_X - s_\ell t_X + s_\ell s_X + t_\ell t_X$$

$$= (s_X - t_X)^2 + (t_\ell - s_\ell)t_X + (s_\ell - t_\ell)s_X$$

$$\geq (s_X - t_X)^2 - (s_X - t_X) \geq 0.$$

Thus, except when $m, n = 1$, we have

$$(s_X + t_X)\frac{m+n}{4} - 2s_X t_X - t_\ell s_X - s_\ell t_X \geq (s_X + t_X)\frac{m+n}{4} - s_X^2 - t_X^2 - s_\ell s_X - t_\ell t_X \geq \frac{mn}{4} \geq \frac{m+n}{2} - 1.$$

Hence in general, the number of edges labeled uw is strictly greater than $\lceil \frac{mn}{4} \rceil$, a contradiction. Thus one of the partite sets has four vertices with distinct labels. Deleting these four vertices yields a V_4-cordial labeling of $K_{m-4,n}$ or $K_{m,n-4}$.

Thus it suffices to consider $m, n < 4$ and the small number of cases for which $\frac{mn}{4} < \frac{m+n}{2} - 1$, namely $m = 1$ or $n = 1$ and the other is at least 4.

Without loss of generality, we may assume $n = 1$. We then have then we have a star with m leaves. In this case, label the center 0 and label the leaves in order, $a, b, c, 0, \ldots$ This will give a V_4-cordial labeling of a star. If $m = n = 2$, then $K_{2,2} = C_4$. Because every label must be used exactly once in a balanced V_4-labeling of the vertices C_4, we must have that C_4 is not V_4-cordial since the label 0 cannot be induced on any edge. If $n = 2$ and $m = 3$, then label the vertices of Y with 0 and a and the vertices of X with $b, c, 0$. The remaining cases where $n = 2$ and $m = 4$, or where $n = 3$ and $m \in \{3, 4\}$, or where $m = n = 4$ are shown in Figure 7.1

Thus we have that $K_{m,n}$ is V_4-cordial if and only if m and n are not both equal to 2. $\square$

As noted above, the paths P_4 and P_5 are not V_4-cordial. However they are exceptional in this regard.

Theorem 7.3.4. *The path P_n is V_4-cordial unless $n \in \{4, 5\}$.*

Proof. If $n < 4$, then the path P_n is V_4-cordial, by inspection.

The path P_6 has a V_4-cordial labeling with vertices labeled $(c, c, 0, b, 0, a)$ in order. The path P_8 has a V_4-cordial labeling with vertices labeled $(a, c, a, b, b, c, 0, 0)$ in order. The path P_{12} has a V_4-cordial labeling with vertices labeled $(a, 0, b, 0, c, c, c, a, b, b, a, 0)$ in order.
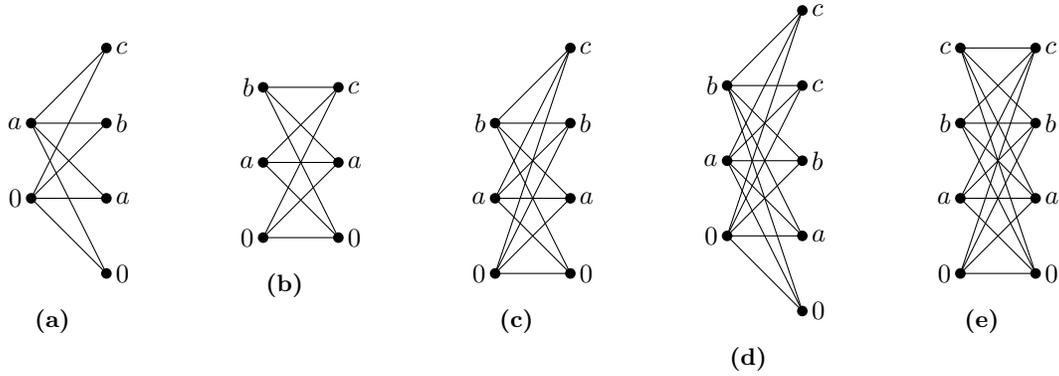
The following two claims complete the proof by induction.

Figure 7.1: Small cases for Theorem 7.3.3.

Claim 1. *If P_n is V_4-cordial and $n \not\equiv 3 \bmod 4$, then P_{n+1} is V_4-cordial.*

Claim 2. *For all $n \in \mathbb{N}$, if P_n is V_4-cordial, then P_{n+8} is V_4-cordial.*

We begin by proving Claim 1. Given a V_4-cordial labeling of P_n, we will append a vertex v to one end and extend the labeling to v, while maintaining V_4-cordiality, in three cases for n modulo 4. Let w be the neighbor of v.

When $n = 4k$, there are exactly k vertices with each label, so the vertex label classes will be balanced in P_{n+1} regardless of how we label v. One edge label appears $k-1$ times, the others k times. Label v so that the edge vw receives the label that was deficient.

When $n = 4k + 1$, there are exactly k edges with each label, so the edge label classes will be balanced in P_{n+1} regardless of how we label v. Label v so that the vertex label classes remain balanced.

When $n = 4k + 2$, there are two labels we could use on v to keep the vertex label classes balanced. Only one label on vw would cause an imbalance in the edge label classes, so at least one of the two potential labels for v avoids this label on vw.

We now prove Claim 2. If P_n has a V_4-cordial labeling with an endvertex labeled 0, extend by eight edges at that vertex and label the new vertices $a, c, a, b, b, c, 0, 0$ in order. Otherwise, without loss of generality, P_n has an endvertex labeled a. In this case, extend by eight edges at that vertex and label the new vertices $0, 0, c, b, b, a, c, a$ in order. $\square$

We now determine which cycles C_n are V_4-cordial.

Theorem 7.3.5. *The cycle C_n is V_4-cordial if and only if $n \notin \{4, 5\}$ and $n \not\equiv 2 \bmod 4$.*

Proof. Note that C_3 is V_4-cordial; label the vertices a, b, c, which induces the labeling c, b, a on the edges. We have already discussed C_4. To see that C_5 is not V_4-cordial we note that in a V_4-cordial labeling of C_5

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exactly one label is used twice on the vertices. Its two uses must be adjacent, since the label 0 must be induced on one of the edges. If 0 is used twice on the vertices, then whichever labels are used on the two vertices adjacent to a vertex labeled 0 will be used twice on the edges, as shown in Figure 7.2a.

Finally suppose that a nonzero element, which without loss of generality we may say is a, is used twice. If a neighboring vertex is labeled 0, then the labels a and whichever label is not on a vertex adjacent to one labeled a will be used twice on the edges, as shown in Figure 7.2b. On the other hand, if neither of the vertices adjacent to the vertices labeled a are labeled 0, then labels b and c will be used twice on the edges, as shown in Figure 7.2c.
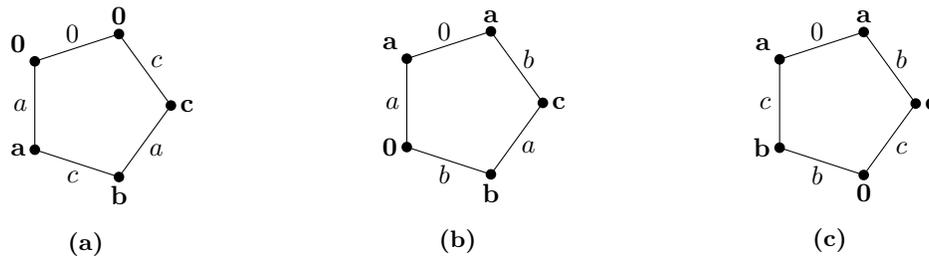

Figure 7.2: Cases for potential V_4-cordial labelings of C_5.

It follows from Proposition 7.2.4 that C_n is not V_4-cordial when $n \equiv 2 \bmod 4$, since V_4 has exponent 2 and order 4.

We now prove that C_n is V_4-cordial whenever n is a nontrivial multiple of 4. We proceed by induction with base cases C_8 and C_{12}. The vertex labels $(a, c, a, b, b, c, 0, 0)$ in order show C_8 is V_4-cordial. The vertex labels $(0, a, b, b, a, c, c, c, 0, b, 0, a)$ in order show C_{12} is V_4-cordial.

For the induction step, consider a V_4-cordial labeling of C_n, where $n \neq 3$. There is an edge labeled 0; its endpoints have the same label. Without loss of generality, assume the endpoints are either both labeled 0 or both labeled a. In either case, insert eight vertices into the cycle between the two endpoints and label them $(a, c, a, b, b, c, 0, 0)$ in order to obtain a V_4-cordial labeling of C_{n+8}.

Finally, we show that if C_n is V_4-cordial and n is a multiple of 4, then C_{n-1} and C_{n+1} are also V_4-cordial. Let $n = 4k$. In a V_4-cordial labeling of C_{4k}, there are exactly k vertices with each label and exactly k edges with each label. In particular, there is an edge labeled 0, the endpoints of which must share the same label, say g. Contracting this edge or subdividing it by a new vertex with label g yields V_4-cordial labelings of C_{4k-1} and C_{4k+1}, respectively.   □

We next determine which ladders $P_2 \square P_n$ are V_4-cordial. The copies of P_2 that appear in each ladder will be referred to as *rungs*. A rung whose vertices are labeled g and h will be called a (g, h)-*rung*.

Theorem 7.3.6. *All ladders $P_2 \square P_k$ are V_4-cordial, except $P_2 \square P_2$.*

Proof. We first note that the ladders $P_2 \square P_3$, $P_2 \square P_4$, $P_2 \square P_5$, and $P_2 \square P_6$ are V_4-cordial, as shown in Figure 7.3. In particular, there is a V_4-cordial labeling of these ladders such that one of the end rungs is a $(0,0)$-rung.
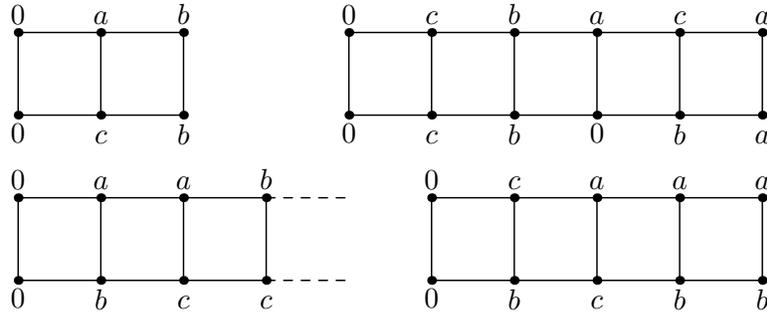

Figure 7.3: V_4-cordial labelings of the ladders $P_2 \square P_3$, $P_2 \square P_4$, $P_2 \square P_5$, and $P_2 \square P_6$.

If the (b,c)-rung of the 4-ladder $P_2 \square P_4$ shown in Figure 7.3 is made adjacent to an end $(0,0)$-rung of any labeled ladder (as suggested in Figure 7.3), then the added vertices and edges are both perfectly balanced. Using this process, we construct a V_4-cordial $P_2 \square P_{k+4}$ with an end $(0,0)$-rung from a V_4-cordial $P_2 \square P_k$ with an end $(0,0)$-rung. With the base cases, we construct V_4-cordial labelings for all ladders except $P_2 \square P_2$. $\square$

We next determine which prisms $P_2 \square C_n$ are V_4-cordial, using "rungs" as above.

Theorem 7.3.7. *The prism $P_2 \square C_k$ is V_4-cordial if and only if $k \not\equiv 2 \bmod 4$.*

Proof. We first note that the prisms $P_2 \square C_3$, $P_2 \square C_4$, and $P_2 \square C_5$ are V_4-cordial, as shown in Figure 7.4. In particular, there is a V_4-cordial labeling of these prisms such that one of the rungs is a $(0,0)$-rung.
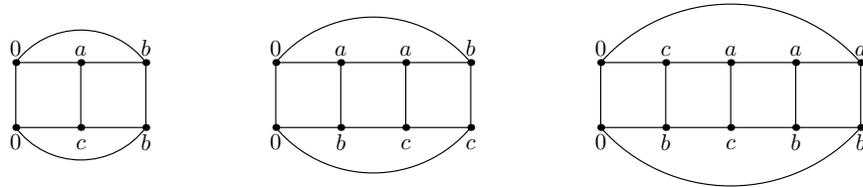

Figure 7.4: V_4-cordial labelings of the prisms $P_2 \square C_3$, $P_2 \square C_4$, and $P_2 \square C_5$.

From a V_4-cordial labeling of $P_2 \square C_k$ with a $(0,0)$-rung, we will construct a V_4-cordially-labeled prism $P_2 \square C_{k+4}$ with a $(0,0)$-rung. Take a V_4-cordially-labeled prism $P_2 \square C_k$ with a $(0,0)$-rung and cut it into a ladder by removing two edges, so that the $(0,0)$-rung becomes an end rung. Now make the (b,c)-rung of the ladder $P_2 \square P_4$ from Figure 7.3 adjacent to this $(0,0)$-rung and add two edges to turn the resulting ladder into a prism. This operation has not changed the balance of the labelings. By induction, all prisms $P_2 \square C_n$ with $n \not\equiv 2 \bmod 4$ are V_4-cordial.

Proposition 7.2.6 shows that $P_2 \square C_{4k+2}$ is not V_4-cordial. $\square$

We next determine which hypercubes Q_d are V_4-cordial. As we saw previously, the square Q_2 is not V_4-cordial.

Theorem 7.3.8. *The n-dimensional hypercube Q_n is V_4-cordial, unless $n = 2$.*

Proof. We prove a stronger statement by induction. We show that if $n > 2$, then Q_n not only has a V_4-cordial labeling, but it has such a labeling with the property that we can cut Q_n into a pair of $(n-1)$-dimensional subcubes each with a perfectly balanced labeling of the vertices by removing a perfectly balanced set of 2^{n-1} edges.

A V_4-cordial-labeling of the cube Q_3 is shown in Figure 7.5. This labeling has the property that the inside square is cut from the outside square by removing a perfectly balanced set of four edges and both the inside square and the outside square use every element of V_4 exactly once.
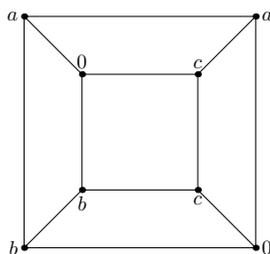

Figure 7.5: A V_4-cordial labeling of the cube Q_3.

Now suppose that Q_n has a V_4-cordial labeling as specified. Let F_1 and F_2 be the two $(n-1)$-dimensional subcubes with perfectly balanced vertex labels obtained by deleting a perfectly balanced cut of size 2^{n-1}. We construct a V_4-cordial-labeling of Q_{n+1} by joining two copies of each of F_1 and F_2 as shown in Figure 7.6. The hypercubes F_1 and F_2 are labeled as in a V_4-cordial-labeling of Q_n. Each of the four sets of 2^{n-1} edges between F_1 and F_2 is perfectly balanced. Furthermore this labeling of Q_{n+1} has the property that it may be cut into two n-dimensional subcubes with perfectly balanced edge labelings by removing a perfectly balanced set of 2^n edges. □
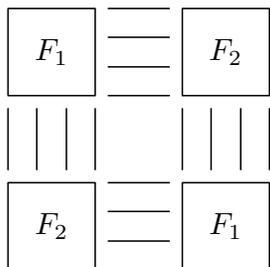

Figure 7.6: A V_4-cordial labeling of the hypercube Q_{n+1}.

Further research on V_4-cordiality could address which grids $P_h \,\square\, P_k$ are V_4-cordial. Our results on ladders resolve the case $h = 2$. The Kneser graph $K(n,k)$ is Eulerian and the number of edges is congruent to 2 modulo 4 when $\binom{n-k}{k}$ is even and $\frac{1}{2}\binom{n-k}{k}\binom{n}{k} \equiv 2 \bmod 4$; in this case $K(n,k)$ is not V_4-cordial by Proposition 7.2.4. For example, $K(7,3)$ is not V_4-cordial. Figure 7.7 shows that the Petersen graph is V_4-cordial. Further research could address which other Kneser graphs are V_4-cordial.
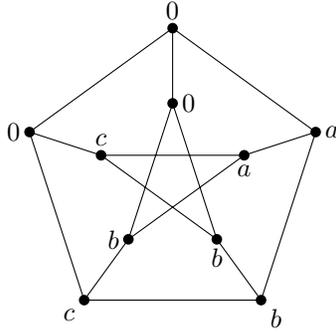

Figure 7.7: V_4-cordial labeling of the Petersen graph.

7.4   Beyond Abelian Groups

We now generalize the idea of A-cordial graphs to labelings from quasigroups. A *quasigroup* Q is a set with a binary operation $\cdot$ such that for all $a, b \in Q$, there exist unique $c, d \in Q$ such that $a \cdot c = b$ and $d \cdot a = b$. In particular, all groups are quasigroups. For a non-abelian group A, lack of commutativity suggests A-cordial labeling of digraphs. We do not delve deeply here into the study of Q-cordial graphs where Q is a quasigroup; our goal is merely to motivate the definition by demonstrating that, for each Q, there is an interesting theory of Q-cordial digraphs.

Definition 7.4.1. Let Q be a quasigroup. A labeling $f\colon V(G) \to Q$ of the vertices of a digraph G induces a labeling of the edges of G in the following way. If (a, b) is a directed edge with head b, then $f(a, b) = f(b)\cdot f(a)$, with the convention that $\sigma \cdot \tau$ means apply τ and then σ. If there is a balanced vertex labeling of G that induces a balanced edge labeling of G, then we say that G is Q-cordial.

Figure 7.8 shows an S_3-cordial labeling of an orientation of $K_{2,3}$.

Lemma 7.4.2. *Let G and H be graphs with $|V(G)| = |V(H)| = 2n$. If each has a Hamiltonian cycle, then $G \square H$ has a Hamiltonian cycle that alternates between an edge in a copy of G and an edge in a copy of H.*

Proof. Let the vertices of the Hamiltonian cycle in G in order be $u_1, \ldots, u_{2n}$ and let the vertices of the Hamiltonian cycle in H in order be $v_1, \ldots, v_{2n}$.
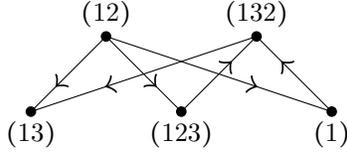
Figure 7.8: An S_3-cordial labeling of an orientation of $K_{2,3}$.

For each pair (G_{2i+1}, G_{2i}), form a cycle that starts with the edge $(u_1, v_{2i})(u_1, v_{2i+1})$ and alternates between the rows formed by G_{2i} and G_{2i+1} in the cartesian product. From each cycle remove the edge $(u_1, v_{2i})(u_1, v_{2i+1})$. Add all edges of the form $(u_1, v_{2i+1})(u_1, v_{2i+2})$. This yields a Hamiltonian cycle of $G \square H$ and since we have replaced edges in copies of H with edges in copies of H, our Hamiltonian cycle alternates between an edge in a copy of G and an edge in a copy of H.

Thus this is a Hamiltonian cycle that alternates between copies of G and copies of H. Figure 7.9 shows the Hamiltonian cycle found in this manner for $n = 3$, showing only edges in the Hamiltonian cycles of G and H. □
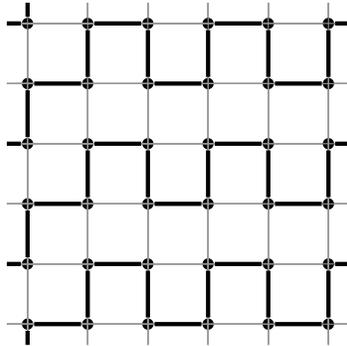

Figure 7.9: Hamiltonian cycle in $G \square H$, $|V(G)| = |V(H)| = 6$, G and H each have a Hamiltonian cycle.

Theorem 7.4.3. *Let Q be an n-element quasigroup. If n is even, then for every positive integer m, there are orientations of C_{mn^2} and P_{mn^2} that are Q-cordial. If n is odd, then for every positive integer m, there are orientations of C_{2mn^2} and P_{2mn^2} that are Q-cordial.*

Proof. Enumerate the elements of Q as $q_1, \ldots, q_n$. Consider the graph $H = C_n \square C_n$, where we name the vertices by elements of $\{1, \ldots, n\} \times \{1, \ldots, n\}$ in the canonical way. We call an edge *horizontal* if its endpoints differ in their first coordinate. Edges that are not horizontal are *vertical*. Addition and subtraction are taken modulo $2n - 1$.

When n is even, consider the Hamiltonian cycle through H that alternates horizontal and vertical edges found in Lemma 7.4.2. Fix a direction along this cycle. Label the vertex (i, j) with the quasigroup element

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q_i if we leave (i,j) by a vertical edge and with q_j if we leave by a horizontal edge. Since in this Hamiltonian cycle we leave every other vertex along each row k by a vertical edge, this gives us $n/2$ vertices labeled q_j. Similarly, since in this Hamiltonian cycle we leave every other vertex along each column k by a horizontal edge, we have an additional $n/2$ vertices labeled q_j. Thus for each of the n elements of Q there are n vertices labeled q_i, and this is a balanced labeling of the vertices of C_{n^2}. Orient each vertical edge of C_{n^2} in the direction that it is traversed, and orient each horizontal edge in the opposite direction to how it is traversed. Each vertex that was labeled q_i because we left (i,j) by a vertical edge is a source in the cycle (and every source is such a vertex) and points to $(i, j+1)$ with label q_{j+1} and to $(i-1, j)$ with label j. Since every other vertex in the ith column has label q_i, we thus have all labels of the form $q_j \cdot q_i$. As there is now one edge labeled with each entry of the multiplication table for Q, we have a balanced labeling of the edges of C_{n^2}, so this orientation of C_{n^2} is Q-cordial.

When n is odd, we may modify the construction by finding an Eulerian circuit through H that alternates vertical and horizontal edges. Form an Eulerian circuit by exiting (i,j) towards $(i+1, j)$ if you entered from $(i, j-1)$ and exiting (i,j) towards $(i, j+1)$ if you entered from $(i-1, j)$. Since every vertex is visited exactly twice, the Eulerian circuit has length $2n^2$. We label the vertices of C_{2n^2} as we did C_{n^2} when n was even to get a labeling that uses every label $2n$ times. Note that every vertex (i,j) of H is left both vertically and horizontally by the Eulerian circuit, so every label gets used $2n$ times on $V(C_{2n^2})$. Again, orient each vertical edge of C_{n^2} in the direction that it is traversed and orient each horizontal edge in the opposite direction to how it is traversed. Each vertex that was labeled q_i because we left (i,j) by a vertical edge is a source in the cycle and points to $(i, j+1)$ with label q_{j+1} and to $(i-1, j)$ with label j. Since every other vertex in the ith column has label q_i, we thus have all labels of the form $q_j \cdot q_i$ twice in our cycle.

For $m > 1$, splice together m copies of the appropriate labeled and oriented cycle.

Deleting any edge, in a label class of maximal size, from any of the labeled and oriented cycles constructed above gives a Q-cordial labeling of an oriented path of the desired length. $\square$

For abelian groups, the orientation of edges is irrelevant, so Theorem 7.4.3 gives results for undirected graphs. In particular, we identify the following easy but important consequence.

Corollary 1.6.1. *For every abelian group A, there are infinitely many A-cordial cycles and infinitely many A-cordial paths.*

For $A = V_4$, we have presented much stronger results. Indeed, by Theorem 7.3.4 all paths with six or more vertices are V_4-cordial. For any particular abelian group A, Corollary 1.6.1 is fairly weak. However, it suggests that, for each abelian group A, the class of A-cordial graphs will be an interesting object. It would

be of interest to study how the structure of the abelian group A relates to the set of natural numbers n for which the path P_n is A-cordial. For example, V_4 has the special property that all sufficiently long paths are V_4-cordial. We can ask the following question:

Question 7.4.4. *Is it true that, for each abelian group A, there exists N such that P_n is A-cordial whenever $n > N$?*

If the answer is no, then a characterization of the groups that have this property would be very interesting. The only groups known to have this property are the cyclic groups (Theorem 2 in [26]) and V_4 (Theorem 7.3.4).

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