GOODWILLIE CALCULUS AND $\mathbb{I}$

BY

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DISSERTATION

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Abstract

We study the implications of using the indexing category of finite sets and injective maps in Goodwillie's calculus of homotopy functors. By careful analysis of the cross-effects of a reduced endofunctor of based spaces, this point of view leads to a monoidal model for the derivatives. Such structure induces operad and module structures for derivatives of monads and their modules, leading to a chain rule for higher derivatives. We also define a category through which n-excisive finitary functors to spectra factor, up to homotopy, and give a classification of such functors as modules over a certain spectral monoid.

For Jim Veneri, for vitamins, pens, umbrellas, and excellent lighting.

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Chapter 1

Introduction

In an effort to understand homotopy types, Goodwillie developed a theory of calculus for homotopy invariant functors from pointed topological spaces $\mathcal{T}$ to the categories of spaces $\mathcal{T}$ or spectra $\mathcal{S}\mathrm{p}$ in a series of landmark papers [Goo90, Goo92, Goo03]. He described a way to canonically assign to a functor F a sequence of "polynomial" (called *n-excisive*) functors P_nF approximating F, which fit into a tower of fibrations analogous to a Taylor series expanded at the zero object.

Goodwillie's theory has been extended to more abstract homotopy theoretical settings [BR14, Kuh07, Per13], and the methods involved in Goodwillie's calculus have provided new insights in various areas of topology, including chromatic homotopy theory [AM99, Beh12, Kuh07], algebraic K-theory [DGM13], and geometric topology [BCKS14, Mal15, Wei99]. For example, analysis of the Taylor tower of the identity functor of spaces (a surprisingly interesting nonlinear functor) has led to calculations of the periodic homotopy of odd dimensional spheres [AM99]. Another triumph of Goodwillie calculus is the identification of the trace map from algebraic K-theory to topological cyclic homology as an isomorphism on differentials, showing that the difference between the computationally difficult K and the more tractable TC is locally constant [DGM13].

Goodwillie defines the n-excisive approximation P_nF of a homotopy invariant functor $F : \mathcal{C} \to \mathcal{D}$ as the homotopy colimit of an infinite iteration of intermediate functors T_nF:

$$F(X) \xrightarrow{t_nF(X)} (T_nF)(X) \xrightarrow{(t_n(T_nF))(X)} T_n^2F(X) \xrightarrow{(t_nT_n^2F)(X)} \cdots$$

Goodwillie shows that P_nF is an n-excisive homotopy functor and there is a natural map $p_nF : F \to P_nF$. These functors fit into a tower of fibrations, called the Taylor tower.

$$F(X) \longrightarrow \quad \cdots \longrightarrow P_nF(X) \longrightarrow P_{n-1}F(X) \longrightarrow \cdots \longrightarrow P_1F(X) \longrightarrow P_0F(X) \simeq F(*)$$

As in function calculus, one wishes to study the functor F by studying its Taylor tower, and this is a good approximation when F is *analytic*, which implies $F(X) \simeq \mathrm{holim}_n P_nF(X)$ for sufficiently connected X.

Many functors are analytic; for example, the identity functor of spaces is analytic. Our work will mainly focus on analytic functors because of their nice stability properties.

The n-excisive approximations of F are difficult to compute in general, so attention shifts to the homotopy fibers $D_nF = \text{fiber}(P_nF \to P_{n-1}F)$, or *layers*, of the Taylor tower, with the hopes that the polynomial parts can be reconstructed once the layers are known. Goodwillie showed in [Goo03] that the layers of the Taylor tower for a finitary functor take the form of infinite loop spaces.

$$D_nF(X) \simeq \Omega^\infty(\partial_nF \wedge X^{\wedge n})_{h\Sigma_n}$$

The Σ_n-spectrum ∂_nF is called the *nth derivative of* F. The symmetric group Σ_n acts on the smash product by permuting the factors and $(-)_{h\Sigma_n}$ denotes the homotopy orbits. Taken together, the derivatives form a symmetric sequence in the category of spectra, which is just a sliver of the interesting structure they have been shown to possess [Joh95, AM99, Chi05]. Utilizing operadic duality, Arone and Ching have a significant body of work [AC11, AC15, AC] developing which properties permit the derivatives to reconstruct the Taylor tower of a functor.

This thesis explores the implications of using the indexing category $\mathbb{I}$ of finite sets and injective maps in functor calculus. The use of the category $\mathbb{I}$ has already found great success in the areas of algebraic K-theory [SS13] and representation stability [CEF15], and we've found that some unresolved questions in Goodwillie's theory have straightforward answers by amending definitions to include the inherent symmetry. By indexing homotopy colimits over $\mathbb{I}$, we prove our main result in theorem 4.2.3, giving a monoidal model for the derivatives of reduced endofunctors of spaces, and thus a positive answer to a conjecture posed by Arone and Ching which streamlines the work in [AC11]. The strategy of including more maps in the homotopy colimit, inspired by Bökstedt's definition of topological Hochschild homology, has also led us to a classification of n-excisive functors from simplicial model categories with a cofibrant generator to spectra, a reformulation of the results of [JM03a, JM03b] in the topological setting.

This thesis is organized as follows. In Chapter 2, we review our conventions and the definitions of Goodwillie calculus. In Chapter 3, we define a category $\mathbb{P}_n\mathcal{C}$ through which finitary n-excisive functors from certain simplicial model categories $\mathcal{C}$ to spectra factor, up to homotopy. We prove that such functors correspond precisely, in the sense of an equivalence of homotopy categories, to modules over the spectrum of endomorphisms of $n+1$ points in the category $\mathbb{P}_n\mathcal{C}$. This differs from the classification of [AC15] by considering the polynomial approximations themselves, instead of reconstructing them from the homogeneous layers of the Taylor tower.

In Chapter 4, we give a monoidal model for the derivatives of analytic, reduced endofunctors of spaces. Such structure sidesteps the technical aspects of Arone and Ching's work and automatically produces a natural operad structure on the derivatives of the identity functor of spaces, a result which required a foray into operadic Koszul duality in [Chi05]. The derivatives of a functor also necessarily inherit the structure of a module over the derivatives of the identity, a hard-earned theorem in [AC11]. We also prove a chain rule in this setting and indicate ways in which to extend this to functors of other categories.

In appendix A, we give some background on the category $\mathbb{I}$, proving that Goodwillie's T_nF's fit into an $\mathbb{I}$ diagram, and giving a proof of Bökstedt's approximation lemma which gives conditions for when homotopy colimits over $\mathbb{I}$ agree with homotopy colimits over $\mathbb{N}$. In appendix B, we give results analogous to those in Chapter 3 for functors to spaces and indicate why they do not assemble to form a classification. Finally, in the last appendix, we give the technical details of the associativity of the monoidal derivative map defined in section 4.2.

Chapter 2

Background and conventions

We will start with basic definitions necessary for Goodwillie's calculus of functors and the use of the category $\mathbb{I}$ of finite sets and injective maps.

2.1 Simplicial objects

Let $\mathcal{T}$ denote the category of based topological spaces and let $\mathcal{S}\mathrm{p}$ be a good category of spectra, for example, symmetric spectra.

Definition 2.1.1. We write Δ for the category whose objects are the totally ordered sets $\mathbf{n} = \{0, 1, \ldots, n\}$ for $n \geq 0$ and whose morphisms are the order-preserving functions. A *simplicial object* in a category $\mathcal{C}$ is a functor $X_\bullet : \Delta^{op} \to \mathcal{C}$. More explicitly, a simplicial object in $\mathcal{C}$ consists of a sequence of objects $X_k \in \mathcal{C}$ for $k \geq 0$, along with face maps, $d_i : X_k \to X_{k-1}$ for $0 \leq i \leq k$, and degeneracy maps, $s_i : X_k \to X_{k+1}$ for $0 \leq i \leq k$, satisfying the simplicial identities.

Definition 2.1.2. If $\mathcal{C}$ is the category of sets, spaces, or spectra, then a simplicial object $X_\bullet$ has a homotopy invariant *geometric realization*, denoted $\|X_\bullet\|$. This is defined by $X_\bullet \otimes_{\Delta_{inj}} \Delta^\bullet$, and is sometimes called the *fat* realization of $X_\bullet$, because the coend is taken over only the injective maps of Δ.

Definition 2.1.3. A *forward contracting homotopy* for an augmented simplicial set $X_\bullet \to X_{-1}$ is a collection of maps $s_{-1} : X_n \to X_{n+1}$ for $n \geq -1$ such that for each $x \in X_n$, one has $s_{-1}s_i x = s_i s_{-1} x$ for $0 \leq i \leq n$ and

$$d_i s_{-1} x = \begin{cases} s_{-1} d_i x & \text{if } 0 \leq i < n \\ x & \text{if } i = n \end{cases}$$

For homotopy limits and colimits, we will use the definitions of Bousfield and Kan in [BK72].

Definition 2.1.4. The *homotopy limit* of a diagram $\mathcal{X} : \mathcal{J} \to \mathcal{T}$, $\operatorname{holim}_{\mathcal{J}} \mathcal{X}$, is given by the totalization of the cosimplicial replacement $crep\ \mathcal{X}$, which has nth term $(crep\ \mathcal{X})_n = \prod_{j_0 \to \cdots \to j_n} \mathcal{X}(j_n)$.

Dually, the *homotopy colimit* of a diagram $\mathcal{X} : \mathcal{J} \to \mathcal{T}$, $\operatorname{hocolim}_{\mathcal{J}} \mathcal{X}$, is given by the realization of the simplicial replacement $srep\ \mathcal{X}$, which has nth term $(srep\ \mathcal{X})_n = \bigvee_{j_0 \to \cdots \to j_n} \mathcal{X}(j_0)$.

2.2 Calculus

Now we will review relevant definitions of the homotopy calculus of functors. In [Goo03], Goodwillie constructs the Taylor tower of a functor from topological spaces to spaces or spectra, and Kuhn shows that Goodwillie's work extends to functors between model categories [Kuh07].

Definition 2.2.1. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor where $\mathcal{C}$ and $\mathcal{D}$ are each either $\mathcal{S}$p or Top. Then we say

- F is *reduced* if $F(*) \simeq *$.
- F is *continuous* if the natural map $\mathcal{C}(X, Y) \to \mathcal{D}(F(X), F(Y))$ is a continuous homomorphism.
- F is a *homotopy functor* if it preserves weak equivalences.
- F is *finitary* if it preserves filtered homotopy colimits, i.e., for any filtered category $\mathcal{I}$ and diagram $X : \mathcal{I} \to \mathcal{C}$, $\operatorname{hocolim}_{\mathcal{I}} F(X_\alpha) \xrightarrow{\simeq} F(\operatorname{hocolim}_{\mathcal{I}} X_\alpha)$. We will also say that such functors satisfy the *colimit axiom.*

If the objects of $\mathcal{C}$ are equivalent to homotopy colimits of filtered diagrams of finite subobjects, then the colimit axiom allows us to evaluate a finitary functor on an object of $\mathcal{C}$ by restricting to the subcategory of finite objects. For example, in the category of spaces, every object is weakly equivalent to a CW complex, which is equivalent to a colimit of its finite dimensional subcomplexes.

Lemma 2.2.2. *If F is a continuous functor, then F has* assembly*, a binatural tranformation*

$$\alpha_F : Z \wedge F(X) \longrightarrow F(Z \wedge X).$$

Proof. The assembly map is given by pushing the identity through the following

$$\begin{aligned} \operatorname{Hom}(Z \wedge X, Z \wedge X) &\cong \operatorname{Hom}(Z, \operatorname{Hom}(X, Z \wedge X)) \\ &\xrightarrow{F} \operatorname{Hom}(Z, \operatorname{Hom}(F(X), F(Z \wedge X))) \\ &\cong \operatorname{Hom}(Z \wedge F(X), F(Z \wedge X)) \end{aligned}$$

□

Definition 2.2.3. A *cubical diagram* is a functor $\mathcal{X} : \mathcal{P}(S) \to \mathcal{C}$, where S is a finite set and $\mathcal{P}(S)$ is the poset of all subsets of S. An n-cube will be a functor $\mathcal{X}$ where the cardinality of S is n, so 0-cubes are objects of $\mathcal{C}$, 1-cubes are morphisms of $\mathcal{C}$, 2-cubes are commutative squares, etc.

Definition 2.2.4. Let $\mathcal{P}_0(\mathbf{n})$ denote the poset of all nonempty subsets of $\mathbf{n} = \{0, 1, \ldots, n\}$. An n-cube is called *homotopy cartesian* if the map $a(\mathcal{X}) : \mathcal{X}(\varnothing) \to \operatorname{holim}_{\mathcal{P}_0(S)} \mathcal{X}$ is a weak equivalence (note that $a(\mathcal{X})$ factors through $\lim_{\mathcal{P}_0(S)} \mathcal{X}$). An n-cube is (homotopy) *k-cartesian* if $a(\mathcal{X})$ is k-connected. An n-cube $\mathcal{X}$ is called *strongly homotopy (co-)cartesian* if each face of dimension ≥ 2 is (co-)cartesian. If every two-dimensional face of $\mathcal{X}$ is (co-)cartesian, then $\mathcal{X}$ is strongly (co-)cartesian.

We will often omit the word "homotopy" from our pushouts, pullbacks, and (co)cartesian cubes, but it is always intended, unless noted otherwise.

Definition 2.2.5. A homotopy functor $F : \mathcal{C} \to \mathcal{D}$ is *n-excisive* if for every strongly cocartesian $(n+1)$-cubical diagram $\mathcal{X} : \mathcal{P}(S) \to \mathcal{C}$, the diagram $F(\mathcal{X}) : \mathcal{P}(S) \to \mathcal{D}$ is cartesian.

Definition 2.2.6. Let X be an object in a symmetric monoidal category $\mathcal{C}$ and let U be a finite set, so $X \times U = \coprod^{|U|} X$. The *join* of X and U is given by $X * U = \operatorname{hocolim}(X \leftarrow X \times U \to U)$.

If X is cofibrant, then this model for the join produces a cofibrant object. This defines an associative join which agrees with the usual join when $\mathcal{C}$ is the category of topological spaces. That is, $(X * U) * V \cong X * (U * V)$ for $X \in \mathcal{C}$ and $U, V \in \mathcal{P}(\mathbf{n})$.

Goodwillie defines the n-excisive approximation $P_n F$ of a homotopy functor F as the homotopy colimit of an infinite iteration of intermediate functors $T_n F$. These functors are defined as follows.

Any object $X \in \mathcal{C}$ defines an $(n+1)$-cubical diagram in $\mathcal{C}$ by $U \mapsto X * U$, for $U \subset \mathbf{n}$. We may then apply F to this diagram to get an $(n+1)$-cube in $\mathcal{D}$. Define a homotopy functor $T_n F : \mathcal{C} \to \mathcal{D}$ by $T_n F(X) = \operatorname*{holim}_{U \in \mathcal{P}_0(\mathbf{n})} F(X * U)$. There is a natural transformation $t_n : F \to T_n F$ because there is a map from the initial corner $F(X * \varnothing) = F(X)$ to the homotopy limit of the rest of the cube. The process may be iterated to produce $T_n^k F(X) \simeq \operatorname{holim}_{U_i \in \mathcal{P}_0(n)^{\times k}} F(X * \bigast_{i=1}^{k} U_i)$, where $\bigast_{i=1}^{k} U_i$ denotes the join of the sets $U_1, \ldots, U_k$.

Definition 2.2.7. The functor $P_n F(X)$ is defined to be the homotopy colimit of the diagram

$$F(X) \xrightarrow{t_n F(X)} (T_n F)(X) \xrightarrow{(t_n (T_n F))(X)} T_n^2 F(X) \xrightarrow{(t_n T_n^2 F)(X)} \cdots$$

Example 2.2.8. When $n = 1$, $\mathcal{P}(\mathbf{1})$ forms the pushout square

$$\begin{array}{ccc} \varnothing & \longrightarrow & \{0\} \\ \downarrow & & \downarrow \\ \{1\} & \longrightarrow & \{0,1\} \end{array}$$

Then $U \mapsto F(X * U)$ gives the square

$$\begin{array}{ccc} F(X * \varnothing) & \longrightarrow & F(X * \{0\}) \\ \downarrow & & \downarrow \\ F(X * \{1\}) & \longrightarrow & F(X * \{0,1\}) \end{array} \quad \simeq \quad \begin{array}{ccc} F(X) & \longrightarrow & F(CX) \\ \downarrow & & \downarrow \\ F(CX) & \longrightarrow & F(\Sigma X) \end{array}$$

where CX is the cone on X and ΣX is the suspension of X. So $T_1F(X)$ is given by the homotopy limit of the diagram resulting from removing the initial corner (where $\varnothing$ sits in the indexing category). If F is a reduced functor, then $F(*) \simeq *$ and

$$T_1F(X) \simeq \operatorname{holim} \begin{pmatrix} & & * \\ & & \downarrow \\ * & \to & F(\Sigma X) \end{pmatrix} \simeq \Omega F(\Sigma X)$$

Repeating the process, we see that $T_1^2F(X) \simeq \Omega^2F(\Sigma^2X)$ and so $P_1F(X) \simeq \operatorname{hocolim}_n \Omega^nF(\Sigma^nX) = \Omega^\infty F(\Sigma^\infty X)$.

When $n = 2$, $\mathcal{P}(\mathbf{2})$ is a 3-dimensional cube; identifying T_nF and P_nF is much harder to do in practice when $n > 1$.

In Theorem 1.8 of [Goo03], Goodwillie shows that P_nF is an n-excisive homotopy functor and the natural map $p_nF : F \to P_nF$ is the universal map from F to an n-excisive functor, up to homotopy.

Definition 2.2.9. The *layers* of the Taylor tower of $F : \mathcal{C} \to \mathcal{D}$ are the functors $D_nF : \mathcal{C} \to \mathcal{D}$ for $n \geq 1$ given by

$$D_nF = \operatorname{hofib}(P_nF \to P_{n-1}F)$$

The functor D_nF is *n-homogeneous*, that is, both n-excisive and n-reduced ($P_{n-1}(D_nF) \simeq *$.)

Definition 2.2.10. Let $F : \mathcal{C} \to \mathcal{D}$ be a homotopy functor. F is *stably n-excisive* or satisfies stable nth order excision, if the following condition holds for some numbers c and κ:

$E_n(c, \kappa)$: If $\mathcal{X} : \mathcal{P}(S) \to \mathcal{C}$ is any strongly co-cartesian $(n+1)$-cube such that for all $s \in S$, the map

$\mathcal{X}(\varnothing) \to \mathcal{X}(\{s\})$ is k_s-connected and $k_s \geq \kappa$, then the diagram $F(\mathcal{X})$ is $(-c + \Sigma k_s)$-Cartesian.

Definition 2.2.11. The functor F is *ρ-analytic* if there is some number q such that F satisfies $E_n(n\rho - q, \rho + 1)$ for all $n \geq 1$. (Note that it is the same q for all n.)

Example 2.2.12 ([Goo92] 4.3, 4.5)**.** An analytic functor is one whose deviation from being n-excisive is bounded in a certain way for all n. The identity functor of spaces is 1-analytic by the higher Blakers-Massey theorem. The functor $\mathrm{Hom}(K, -)$ is k-analytic, where $k = dim(K)$.

Theorem 2.2.13 ([Goo03] 1.13)**.** *If F is ρ-analytic and X is (at least) ρ-connected, then the connectivity of the map $F(X) \to P_n F(X)$ tends to infinity with n, so that $F(X)$ is equivalent to the homotopy limit $P_\infty F(X)$ of the tower. Thus, the number ρ gives a sort of radius of convergence for the Taylor tower.*

The following definition will be useful in Lemma 2.3.4. The notation O stands for 'osculating.'

Definition 2.2.14. A map $\alpha : F \to G$ between two functors from $\mathcal{C}$ to $\mathcal{D}$ satisfies $O_n(c, \kappa)$ if, for every $k \geq \kappa$, for every object X of $\mathcal{C}$ such that $X \to *$ is k-connected, the map $\alpha_X : F(X) \to G(X)$ is $(-c + (n+1)k)$-connected.

2.3 $\mathbb{I}$ and symmetric sequences

We will exploit the properties of a particular indexing category used by Bökstedt to define topological Hochschild homology. He attributes the idea to Illusie. More facts about $\mathbb{I}$ are given in the appendix.

Definition 2.3.1. Let $\mathbb{I}$ denote the (skeleton of the) category of finite sets and injective maps. Let $\mathbb{N}$ denote the category of finite sets with only the standard inclusions (those induced by subset inclusion). Let Σ denote the category of finite sets with only bijections.

Bökstedt showed that under certain conditions on a functor $G : \mathbb{I} \to \mathcal{T}$, $\mathrm{hocolim}_{\mathbb{N}} G \to \mathrm{hocolim}_{\mathbb{I}} G$ is an equivalence. Essentially, the condition is that maps further in the diagram become more and more connected.

Lemma 2.3.2. *([Bök85]) Let $G : \mathbb{I} \to \mathcal{T}$ be a functor, $\mathbf{x} \in ob\ \mathbb{I}$, and let $\mathbf{x} \downarrow \mathbb{I}$ be the full subcategory of $\mathbb{I}$ of objects supporting maps from $\mathbf{x}$. If G sends maps in $\mathbf{x} \downarrow \mathbb{I}$ to $n_{|\mathbf{x}|}$-connected maps and $n_{|\mathbf{x}|} \to \infty$ as $|\mathbf{x}| \to \infty$, then* $\mathrm{hocolim}_{\mathbb{N}} G \to \mathrm{hocolim}_{\mathbb{I}} G$ *is an equivalence.*

A published proof can be found in [DGM13] (Lemma 2.2.2.2), and we provide a version in the appendix. We also show in the appendix that Goodwillie's T_n^k's fit into an $\mathbb{I}$ diagram, so we can make the following definition.

Definition 2.3.3. Let $\mathbb{P}_n F = \operatorname{hocolim}_{k\in\mathbb{I}} T_n^k F$.

Lemma 2.3.4. *When F is stably n-excisive, $P_n F \to \mathbb{P}_n F$ is an equivalence.*

Proof. We will show that the functor $\Theta : \mathbb{I} \to Fun(\mathcal{T},\mathcal{T})$ defined by $\Theta(\mathbf{k}) = T_n^{|\mathbf{k}|}F$ satisfies the hypotheses of Bökstedt's lemma (2.3.2) when F satisfies $E_n(c,\kappa)$. By Proposition 1.4 of [Goo03], if F satisfies $E_n(c,\kappa)$, then T_nF satisfies $E_n(c-1,\kappa-1)$ and $t_nF : F \to T_nF$ satisfies $O_n(c,\kappa)$. By induction on i, T_n^iF satisfies $E_n(c-i,\kappa-i)$, and $T_n^iF \to T_n^{i+1}F$ satisfies $O_n(c-i,\kappa-i)$. By the definition of O_n, all the maps $T_n^iF(X) \to T_n^{i+1}F(X)$ are $(i-c+(n+1)\ell)$-connected for $\ell \geq \kappa$, where ℓ is the connectivity of $X \to *$. Since $(i-c+(n+1)\ell)$ increases as i increases, Θ satisfies the condition of Bökstedt's lemma. □

Definition 2.3.5. Let $\mathcal{C}$ be a category. A *symmetric sequence* in $\mathcal{C}$ is a functor $A : \Sigma \to \mathcal{C}$. This is a sequence $\{A(n)\}$ of objects of $\mathcal{C}$ with a Σ_n-action on $A(n)$ for each $n \geq 1$. A morphism of symmetric sequences $f : A \to B$ is a natural tranformation of functors or, explicitly, a sequence of Σ_n-equivariant morphisms $f(n) : A(n) \to B(n)$.

Definition 2.3.6. If $\mathcal{C}$ is a cocomplete closed symmetric monoidal category with monoidal product denoted $\wedge$ and if A, B are symmetric sequences in $\mathcal{C}$, then the *composition product* or $\circ$-*product* of A and B is the symmetric sequence $A \circ B$ defined by

$$(A\circ B)(n) = \bigvee_{\text{unordered partitions of } \{1,\dots,n\}} A(k)\wedge B(n_1)\wedge\cdots\wedge B(n_k).$$

The Σ_n action on $(A\circ B)(n)$ is not immediately obvious. We give a quick description using the definition of symmetric sequences on the category of all finite sets and isomorphisms. For a finite set T, $(A\circ B)(T) = \bigvee_{T=\coprod_{i\in I} T_i} A(I)\wedge(\wedge_{i\in I}B(T_i))$ for nonempty subsets T_i. A bijection $\sigma : T \to T'$ induces a bijection on partitions $\coprod_{i\in I} T_i \to \coprod_{i\in I'} T'_i$ and there are induced maps $\sigma_* : I \to I'$ so $T_i \to T'_{\sigma_*(i)}$. Following this through for the finite sets $T = \{1,2\}$ and $T' = \{a,b\}$ with the map $\sigma(1) = b, s(2) = a$ shows that there are two types of action maps.

$$\begin{array}{c} A(\{i\})\wedge B(\{1,2\}) \vee A(\{j,k\})\wedge B(\{1\})\wedge B(\{2\}) \\ \downarrow \\ A(\{i'\})\wedge B(\{a,b\}) \vee A(\{j',k'\})\wedge B(\{a\})\wedge B(\{b\}) \end{array}$$

Essentially, the map on the first summand is $id\wedge\Sigma_2$ and the map on the second summand is $\Sigma_2\wedge$ block permute. So we need to account for both of these types of actions when we consider equivariance of maps of symmetric sequences.

The composition product defines a monoidal product on the category of symmetric sequences in $\mathcal{C}$. If the unit of $\mathcal{C}$ is S, the unit object of $[\Sigma, \mathcal{C}]$ is given by

$$1(n) = \begin{cases} S & \text{if } n = 1 \\ * & \text{else} \end{cases}$$

Definition 2.3.7. An *operad* in $\mathcal{C}$ is a monoid under the composition product; that is, an operad is a symmetric sequence $\mathcal{O}$ with a composition map $\gamma : \mathcal{O} \circ \mathcal{O} \to \mathcal{O}$ and a unit map $\eta : 1 \to \mathcal{O}$ satisfying associativity and unitality diagrams.

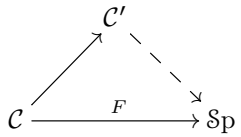
Definition 2.3.8. Let $\mathcal{O}$ be an operad in $\mathcal{C}$. A *right $\mathcal{O}$-module* is a symmetric sequence M with an action map $M \circ \mathcal{O} \to M$ satisfying associativity and unitality diagrams. A *left $\mathcal{O}$-module* is a symmetric sequence M with map $\mathcal{O} \circ M \to M$ again satisfying associativity and unit.

Definition 2.3.9. A functor $F : \mathcal{C} \to \mathcal{D}$ between monoidal categories $(\mathcal{C}, \otimes_{\mathcal{C}}, 1_{\mathcal{C}})$ and $(\mathcal{D}, \otimes_{\mathcal{D}}, 1_{\mathcal{D}})$ is *monoidal* if there is a morphism $\epsilon : 1_{\mathcal{D}} \to F(1_{\mathcal{C}})$ and a natural tranformation $\mu_{X,Y} : F(X) \otimes_{\mathcal{D}} F(Y) \to F(X \otimes_{\mathcal{C}} Y)$ satifying associativity and unitality diagrams.

Chapter 3

A classification of n-excisive functors to spectra

In [JM04], Brenda Johnson and Randy McCarthy developed an algebraic version of calculus for functors to chain complexes that produces "n-additive" approximations, and in [JM03a, JM03b], they defined a category $P_n\mathcal{C}$ classifying degree n functors. In this chapter, we mimic these results in the topological setting by constructing a category $\mathcal{C}'$ through which n-excisive functors F factor and use this to classify n-excisive functors:



To define $P_n\mathcal{C}$, Johnson and McCarthy consider the composition rule for morphisms in $\mathcal{C}$:

$$\mathrm{Hom}_{\mathcal{C}}(X,Y)\times\mathrm{Hom}_{\mathcal{C}}(Y,Z)\to\mathrm{Hom}_{\mathcal{C}}(X,Z)$$

Applying their algebraic intermediate functors to $\mathrm{Hom}(X,-)$ and taking colimits yields a map

$$P_n\,\mathrm{Hom}_{\mathcal{C}}(X,Y)\times P_n\,\mathrm{Hom}_{\mathcal{C}}(Y,Z)\to P_nP_n\,\mathrm{Hom}_{\mathcal{C}}(X,Z)$$

In their algebraic setting, there is a map $P_nP_n\,\mathrm{Hom}_{\mathcal{C}}(X,Z)\to P_n\,\mathrm{Hom}_{\mathcal{C}}(X,Z)$, so defining the morphism set $P_n\mathcal{C}(X,Y)$ as $P_n\,\mathrm{Hom}(X,-)(Y)$ gives the category $P_n\mathcal{C}$ an associative composition.

If we try to mimic this with Goodwillie's definitions in the topological setting, we get a map

$$\operatorname*{hocolim}_{k\in\mathbb{N}} T_n^k\,\mathrm{Hom}(X,-)(Y)\times\operatorname*{hocolim}_{\ell\in\mathbb{N}} T_n^\ell\,\mathrm{Hom}(Y,-)(Z)\to\operatorname*{hocolim}_{(k,\ell)\in\mathbb{N}\times\mathbb{N}} T_n^kT_n^\ell\,\mathrm{Hom}(X,-)(Z)$$

and we would be forced to find a map $\mathbb{N}\times\mathbb{N}\to\mathbb{N}$ that induces a map $T_n^kT_n^\ell\to T_n^m$ giving an associative composition rule. This is not possible. One could choose a path as in Adams' handicrafted smash product, but this will not be associative on the nose.

Our solution changes the indexing category of the homotopy colimit to $\mathbb{I}$, the category of finite sets

with injective maps. This technique was used by Bökstedt to define topological Hochschild homology before spectra were known to have a strictly associative smash product, because it fixes this problem of combining homotopy colimits in an associative way. The map $\mathrm{hocolim}_{\mathbb{I}\times\mathbb{I}} \to \mathrm{hocolim}_{\mathbb{I}}$ is induced by the disjoint union of sets $(U,V) \mapsto U \coprod V$. This is essentially the reason that symmetric spectra have an associative smash product; the extra symmetry provides more room. In some sense, indexing over $\mathbb{I}$ is choosing all paths at once, instead of picking one.

Let $\mathcal{C}$ be a simplicial model category with a cofibrant generator c such that all objects of $\mathcal{C}$ are equivalent to the realization of the simplicial object built from a resolution by the cotriple $\mathrm{Hom}(c,-)\otimes c$. For example, $\mathcal{C}$ could be the category of based spaces with generator S^0 or a category of spectra with generator $\mathbb{S}$. In section 3.1, we define a category $\mathbb{P}_n\mathcal{C}$ through which n-excisive functors $F : \mathcal{C} \to \mathcal{S}\mathrm{p}$ factor up to homotopy and define an evaluation map $\mathbb{P}_n\mathcal{C}(X,Y) \wedge \mathbb{P}_n F(X) \to \mathbb{P}_n F(Y)$. We show in section 3.2 that n-excisive functors to spectra are determined by their value on $n+1$ points or the n-fold coproduct of the generator of $\mathcal{C}$. In section 3.3, we prove that the derived evaluation map is an equivalence, thus giving the desired factorization of n-excisive functors through $\mathbb{P}_n\mathcal{C}$. Finally, in section 3.4, we show that n-excisive functors to spectra which preserve filtered homotopy colimits correspond precisely, in the sense of an equivalence of homotopy categories to modules over the spectrum of endomorphisms of $n+1$ points in the category $\mathbb{P}_n\mathcal{C}$.

3.1 The category

In this section, we define a category, $\mathbb{P}_n\mathcal{C}$, through which n-excisive functors with domain $\mathcal{C}$ will be shown to factor up to homotopy and define other maps which will allow for the classification.

Proposition 3.1.1. *For a simplicial category $\mathcal{C}$, there is a well-defined category, $\mathbb{P}_n\mathcal{C}$, whose objects are the objects of $\mathcal{C}$ and whose morphisms are given by*

$$\mathbb{P}_n\mathcal{C}(X,Y) = \mathbb{P}_n\Sigma^\infty \mathrm{Hom}_{\mathcal{C}}(X,-)(Y) = \mathrm{hocolim}_{U\in\mathbb{I}}\, T_n^U \Sigma^\infty \mathrm{Hom}_{\mathcal{C}}(X,-)(Y)$$

for objects X and Y in $\mathcal{C}$. That is, the morphisms are given by applying the construction from Definition 2.3.3 to the simplicial Hom functor.

Because $\mathcal{C}$ is simplicial, $\mathrm{Hom}_{\mathcal{C}}(X,-) : \mathcal{C} \to \mathcal{T}$ is functor to spaces thus $\mathbb{P}_n\Sigma^\infty \mathrm{Hom}_{\mathcal{C}}(X,-); \mathcal{C} \to \mathcal{S}\mathrm{p}$ lands in the category of spectra, so $\mathbb{P}_n\mathcal{C}$ is enriched in $\mathcal{S}\mathrm{p}$. There is a natural functor $\mathbb{p}_n : \mathcal{C} \to \mathbb{P}_n\mathcal{C}$, given by the identity on objects and by the natural transformations $F \to \Sigma^\infty \circ F \to \mathbb{P}_n(\Sigma^\infty \circ F)$ on morphisms.

This construction is analogous to Theorem 4.8 of [JM03b].

Proof. The categorical composition is a map $\mathbb{P}_n\mathcal{C}(X,Y)\otimes\mathbb{P}_n\mathcal{C}(Y,Z)\to\mathbb{P}_n\mathcal{C}(X,Z)$ given by the following composition:

$$
\begin{aligned}
\mathbb{P}_n\mathcal{C}(X,Y)\otimes\mathbb{P}_n\mathcal{C}(Y,Z) &= \operatorname*{hocolim}_{U\in\mathbb{I}} T_n^U\Sigma^\infty\operatorname{Hom}(X,Y)\otimes\operatorname*{hocolim}_{V\in\mathbb{I}} T_n^V\Sigma^\infty\operatorname{Hom}(Y,Z)\\
&\xrightarrow{\otimes\leftrightarrow\text{hocolim}} \operatorname*{hocolim}_{U\in\mathbb{I}}\operatorname*{hocolim}_{V\in\mathbb{I}} T_n^U\Sigma^\infty\operatorname{Hom}(X,Y)\otimes T_n^V\Sigma^\infty\operatorname{Hom}(Y,Z)\\
&\xrightarrow{(\text{holim}\,\mathcal{X})\otimes Y\to\text{holim}(\mathcal{X}\otimes Y)} \operatorname*{hocolim}_{U\in\mathbb{I}}\operatorname*{hocolim}_{V\in\mathbb{I}}\operatorname*{holim}_{U_i\in\mathcal{P}_0(n)^k}\operatorname*{holim}_{V_j\in\mathcal{P}_0(n)^\ell}\Sigma^\infty\operatorname{Hom}(X,Y*\overset{k}{\divideontimes}U_i)\otimes\Sigma^\infty\operatorname{Hom}(Y,Z*\overset{\ell}{\divideontimes}V_j)\\
&\xrightarrow{\text{fubini for ho(co)lims}} \operatorname*{hocolim}_{(U,V)\in\mathbb{I}\times\mathbb{I}}\operatorname*{holim}_{(U_i,V_j)\in\mathcal{P}_0(n)^{k+\ell}}\Sigma^\infty\operatorname{Hom}(X,Y*\overset{k}{\divideontimes}U_i)\otimes\Sigma^\infty\operatorname{Hom}(Y,Z*\overset{\ell}{\divideontimes}V_j)\\
&\xrightarrow{\Sigma^\infty\leftrightarrow\otimes} \operatorname*{hocolim}_{(U,V)\in\mathbb{I}\times\mathbb{I}}\operatorname*{holim}_{(U_i,V_j)\in\mathcal{P}_0(n)^{k+\ell}}\Sigma^\infty\big[\operatorname{Hom}(X,Y*\overset{k}{\divideontimes}U_i)\otimes\operatorname{Hom}(Y,Z*\overset{\ell}{\divideontimes}V_j)\big]\\
&\xrightarrow{id\otimes(\divideontimes U_i)} \operatorname*{hocolim}_{(U,V)\in\mathbb{I}\times\mathbb{I}}\operatorname*{holim}_{(U_i,V_j)\in\mathcal{P}_0(n)^{k+\ell}}\Sigma^\infty\big[\operatorname{Hom}(X,Y*\overset{k}{\divideontimes}U_i)\otimes\operatorname{Hom}(Y*\overset{k}{\divideontimes}U_i,Z*\overset{\ell}{\divideontimes}V_j*\overset{k}{\divideontimes}U_i)\big]\\
&\xrightarrow{\text{composition in }\mathcal{C}} \operatorname*{hocolim}_{(U,V)\in\mathbb{I}\times\mathbb{I}}\operatorname*{holim}_{(U_i,V_j)\in\mathcal{P}_0(n)^{k+\ell}}\Sigma^\infty\big[\operatorname{Hom}(X,Z*\overset{\ell}{\divideontimes}V_j*\overset{k}{\divideontimes}U_i)\big]\\
&\xrightarrow{\text{defn of }T_n} \operatorname*{hocolim}_{(U,V)\in\mathbb{I}\times\mathbb{I}} T_n^{U\amalg V}\Sigma^\infty\operatorname{Hom}(X,Z)\\
&\xrightarrow{\text{induced by }\amalg} \operatorname*{hocolim}_{\mathbb{I}} T_n^{U\amalg V}\Sigma^\infty\operatorname{Hom}(X,Z)\\
&= \mathbb{P}_n\mathcal{C}(X,Z)
\end{aligned}
$$

This composition is strictly associative by the associativity of the disjoint union of sets. The identity of this composition is the image of the identity under the composite natural transformation $F\to\Sigma^\infty F\to\mathbb{P}_n\Sigma^\infty F$, which can be verified using naturality. □

The spectrum of endomorphisms of any object in the category $\mathbb{P}_n\mathcal{C}$ has a strict monoidal structure induced by composition. That is, if Y is an object of $\mathcal{C}$, the endomorphisms of Y in $\mathbb{P}_n\mathcal{C}$ form a monoid,

$$\mathbb{P}_n\mathcal{C}(Y,Y)=\mathbb{P}_n\Sigma^\infty\operatorname{Hom}(Y,-)(Y).$$

The other hom spectra inherit module structures over these monoids using the composition of $\mathbb{P}_n\mathcal{C}$. For example, the map

$$\mathbb{P}_n\mathcal{C}(Y,Y)\otimes\mathbb{P}_n\mathcal{C}(Y,Z)\to\mathbb{P}_n\mathcal{C}(Y,Z)$$

exhibits a left action of $\mathbb{P}_n\mathcal{C}(Y,Y)$ on $\mathbb{P}_n\mathcal{C}(Y,Z)$.

We will now show that the n-excisive approximation of an analytic functor is a module over one of these monoids. For functors landing in a simplicial category of spectra, $\mathcal{S}\mathrm{p}$, there is a continuous map $\operatorname{Hom}(X,Y)\to\operatorname{Hom}(F(X),F(Y))$ whose adjoint, $F(X)\wedge\operatorname{Hom}(X,Y)\to F(Y)$, we call *evaluation*. This

evaluation structure descends to the category $\mathbb{P}_n\mathcal{C}$, producing natural maps

$$\mathbb{P}_n\mathcal{C}(X,Y)\otimes\mathbb{P}_nF(X)\to\mathbb{P}_nF(Y)\quad\text{ and }\quad\mathbb{P}_nF(X)\otimes\mathbb{P}_n\mathcal{C}(X,Y)\to\mathbb{P}_nF(Y)$$

which exhibit $\mathbb{P}_nF(X)$ as both a left and a right module over $\mathbb{P}_n\mathcal{C}(X,X)$.

Note that given a space X and a spectrum A, the monoidal product of spectra, $(\Sigma^\infty X)\otimes_{\mathbb{S}\mathrm{p}}A$, is equivalent to the smash product $X\wedge A$, which is the levelwise smashing of X with A, $(X\wedge A)_n = X\wedge A_n$. This can be seen using adjunctions and the Yoneda lemma:

$$\mathrm{Hom}((\Sigma^\infty X)\wedge A,B)\cong\mathrm{Hom}(\Sigma^\infty X,\mathrm{Hom}_{\mathbb{S}\mathrm{p}}(A,B))\cong\mathrm{Hom}(X,\Omega^\infty\mathrm{Hom}(A,B))\cong\mathrm{Hom}(X,\mathrm{Hom}(A,B))\cong\mathrm{Hom}(X\wedge A,B)$$

Due to the difficulty in showing that $\mathbb{P}_nF$ defines a functor $\mathbb{P}_n\mathcal{C}\to\mathbb{S}\mathrm{p}$, we will describe the evaluation map explicitly. The left module evaluation map is given by the following composition, and the right module map is defined similarly. We refer to the composition map $\mathbb{P}_n\mathcal{C}(X,Y)\otimes\mathbb{P}_n\mathcal{C}(Y,Z)\to\mathbb{P}_n\mathcal{C}(X,Z)$ from Proposition 3.1.1 as $\triangledown$, and the isomorphism $\mathrm{Hom}((\Sigma^\infty X)\wedge A,B)\cong\mathrm{Hom}(X\wedge A,B)$ as equation $\diamond$.

$$\begin{aligned}
\mathbb{P}_n\mathcal{C}(X,-)\otimes\mathbb{P}_nF(X) &= \operatorname*{hocolim}_{U\in\mathbb{I}} T_n^U\Sigma^\infty\mathrm{Hom}(X,-)\otimes_{\mathbb{S}\mathrm{p}}\operatorname*{hocolim}_{V\in\mathbb{I}}T_n^VF(X)\\
&\xrightarrow{\text{as in }\triangledown}\operatorname*{hocolim}_{U\in\mathbb{I}}\operatorname*{hocolim}_{V\in\mathbb{I}}\operatorname*{holim}_{U_i\in\mathcal{P}_0(n)^k}\operatorname*{holim}_{V_j\in\mathcal{P}_0(n)^\ell}\Sigma^\infty\mathrm{Hom}(X,-*\overset{k}{\divideontimes}U_i)\otimes_{\mathbb{S}\mathrm{p}}F(X*\overset{\ell}{\divideontimes}V_j)\\
&\xrightarrow{\text{ho(co)lims}}\operatorname*{hocolim}_{(U,V)\in\mathbb{I}\times\mathbb{I}}\operatorname*{holim}_{(U_i,V_j)\in\mathcal{P}_0(n)^{k+\ell}}\Sigma^\infty\mathrm{Hom}(X,-*\overset{k}{\divideontimes}U_i)\otimes_{\mathbb{S}\mathrm{p}}F(X*\overset{\ell}{\divideontimes}V_j)\\
&\xrightarrow{\text{by eqn }\diamond}\operatorname*{hocolim}_{(U,V)\in\mathbb{I}\times\mathbb{I}}\operatorname*{holim}_{(U_i,V_j)\in\mathcal{P}_0(n)^{k+\ell}}\mathrm{Hom}(X,-*\overset{k}{\divideontimes}U_i)\otimes F(X*\overset{\ell}{\divideontimes}V_j)\\
&\xrightarrow{\text{as in }\triangledown}\operatorname*{hocolim}_{(U,V)\in\mathbb{I}\times\mathbb{I}}\operatorname*{holim}_{(U_i,V_j)\in\mathcal{P}_0(n)^{k+\ell}}\mathrm{Hom}(X*\overset{\ell}{\divideontimes}V_j,-*\overset{k}{\divideontimes}U_i*\overset{\ell}{\divideontimes}V_j)\otimes F(X*\overset{\ell}{\divideontimes}V_j)\\
&\xrightarrow{\text{evaluation}}\operatorname*{hocolim}_{(U,V)\in\mathbb{I}\times\mathbb{I}}\operatorname*{holim}_{(U_i,V_j)\in\mathcal{P}_0(n)^{k+\ell}}F(-*\overset{k}{\divideontimes}U_i*\overset{\ell}{\divideontimes}V_j)\\
&\xrightarrow{\text{defn of }T_n}\operatorname*{hocolim}_{(U,V)\in\mathbb{I}\times\mathbb{I}}T_n^{U\amalg V}F(-)\\
&\xrightarrow{U\amalg V\to W}\operatorname*{hocolim}_{W\in\mathbb{I}}T_n^WF(-)\\
&= \mathbb{P}_nF(-).
\end{aligned}$$

In section 3.3, we will show that if $F:\mathcal{C}\to\mathbb{S}\mathrm{p}$ is n-excisive, then F factors through $\mathbb{P}_n\mathcal{C}$ up to homotopy.

$$\begin{array}{ccc}\mathcal{C} & \xrightarrow{\quad F\quad} & \mathbb{S}\mathrm{p}\\ & {\scriptstyle\mathbb{p}_n}\searrow & \\ & & \mathbb{P}_n\mathcal{C}\end{array}$$

In the situation above, the homotopy left Kan extension of F along $\mathbb{p}_n$ is the realization of the simplicial object defined by

$$[m] \mapsto \coprod_{c_0, c_1, \ldots, c_m \in \mathcal{C}} (\mathbb{P}_n \Sigma^\infty \operatorname{Hom}(c_m, -) \times \mathcal{C}(c_{m-1}, c_m) \times \cdots \times \mathcal{C}(c_0, c_1)) \otimes F(c_0).$$

Later, in our classification, we will restrict our attention to a subcategory of $\mathbb{P}_n \mathcal{C}$ and consider the left Kan extension of $\mathbb{P}_n F$ along the inclusion of this subcategory. Since we have not shown that $\mathbb{P}_n F$ is a functor on $\mathbb{P}_n \mathcal{C}$, we need to use the module maps described above to make sense of this construction. This will be described further in section 3.3.

3.2 An n-excisive functor is determined by its value on points

In this section, we classify n-excisive functors which satisfy the colimit axiom as functors determined by their value on $n+1$ points. This is in direct analogy with the fact that real-valued degree n polynomial functions are determined by their value on $n+1$ points.

Let $\mathcal{C}$ be a simplicial model category. We will assume that $\mathcal{C}$ has a cofibrant generator c, and that all objects of $\mathcal{C}$ are equivalent to the realization of the simplicial object built from a resolution by the cotriple $\operatorname{Hom}(c, -) \otimes c$. For $\mathcal{C} = \mathcal{T}$, the generator is S^0 and a space X is equivalent to the realization of the singularization of X, $\| Sing_\bullet X \|$, where $Sing_\bullet X = \operatorname{Hom}(S^0, X)$ as a simplicial set. These ideas are spelled out in more detail in section 2 of [JM03a] and section 6 of [McC].

Definition 3.2.1. We say that c is a *generator* if every object $X \in \mathcal{C}$ is equivalent to the homotopy colimit of a filtration X_n

$$X \simeq \operatorname{hocolim}(X_0 \to X_1 \to X_2 \to \cdots)$$

such that the cofibers X_n / X_{n-1} are equivalent to $\coprod S^n \otimes c$.

Note that for pointed spaces, $c = S^0$, for unpointed spaces, $c = *$, and for spectra, $c = \mathbb{S}$.

Definition 3.2.2. Let $\varnothing$ denote the initial object of $\mathcal{C}$ and $*$ the final object; when $\mathcal{C}$ is based, these agree and we use $*$ for the initial object. If $\mathcal{C}$ is based, define $\mathbf{n}_c = \coprod_*^n c$, with $\mathbf{0}_c = \varnothing$. That is,

$$\mathbf{n}_c = \operatorname{colim}\left(c \leftarrow c \leftarrow * \rightarrow \cdots \rightarrow c \rightarrow c \right).$$

If $\mathcal{C}$ is unbased, define $\mathbf{n}_c = \coprod_{\varnothing}^{n+1} c$

For pointed spaces, $\mathbf{n}_c = \bigvee^n S^0$, the space of $n+1$ points, for unpointed spaces, $\mathbf{n}_c = \coprod^{n+1} *$, the space of $n+1$ points, and for spectra, $\mathbf{n}_c = \bigvee^n \mathbb{S}$, the wedge sum of n sphere spectra (which levelwise is the n-fold wedge sum of k-spheres.)

The main result of this section is the following theorem, which we will prove for $\mathcal{C} = \mathcal{T}$ first, then indicate the generalization to simplicial model categories with cofibrant generator c at the end of this section.

Theorem 3.2.3. *Let $F, G : \mathcal{C} \to \mathcal{S}\mathrm{p}$ be two finitary, n-excisive functors; that is, F and G preserve filtered homotopy colimits and take strongly cocartesian $n+1$-cubes in $\mathcal{C}$ to cartesian cubes of spectra. If a natural transformation $\eta : F \to G$ is an equivalence on $\mathbf{n}_c$, then it is an equivalence on all objects of $\mathcal{C}$.*

Example 3.2.4. Here is an example demonstrating that this theorem does not hold for functors to spaces, although a modified version of this theorem holds (see appendix B). Let $F = \Omega\Omega^\infty(H\mathbb{Z} \wedge -)$ and $G = \Omega\Omega^\infty(H\mathbb{Z}/2 \wedge -)$. There is a natural transformation $\eta : F \to G$. Both F and G are linear, reduced, finitary functors, and η is an equivalence on all discrete sets, but $F(S^1) \not\simeq G(S^1)$ since $\pi_0 F(S^1) = \mathbb{Z}$ and $\pi_0 G(S^1) = \mathbb{Z}/2$.

We will prove Theorem 3.2.3 through a series of lemmas. The first says that the value of a finitary, n-excisive functor on discrete sets is determined by its value on the space $\mathbf{n}_c = \bigvee^n S^0$.

Lemma 3.2.5. *Let $F, G : \mathcal{T} \to \mathcal{S}\mathrm{p}$ be two finitary, n-excisive functors such that the natural transformation $\eta : F \to G$ is an equivalence on the space $\mathbf{n}_c$. Then η is an equivalence on all discrete sets.*

Proof. Since the functors are finitary and uncountable sets are filtered colimits of their finite subsets, we need only check that $F \simeq G$ on finite sets. For all $n > 0$, $\mathbf{n-1}_c$ is a retract of $\mathbf{n}_c$, because the maps

$$\bigvee^{n-1} S^0 \hookrightarrow \bigvee^{n} S^0 \longrightarrow \bigvee^{n-1} S^0$$

compose to the identity. The first map is inclusion into the first $n-1$ summands while the second map is the identity on the first $n-1$ summands and folds the nth summand with the $n-1$st.

Then we can apply F and G to this sequence to get:

$$\begin{array}{ccccc} F(\bigvee^{n-1} S^0) & \longrightarrow & F(\bigvee^{n} S^0) & \longrightarrow & F(\bigvee^{n-1} S^0) \\ \downarrow{\scriptstyle \eta_{\mathbf{n-1}_c}} & & \downarrow{\scriptstyle \simeq\ \eta_{\mathbf{n}_c}} & & \downarrow{\scriptstyle \eta_{\mathbf{n-1}_c}} \\ G(\bigvee^{n-1} S^0) & \longrightarrow & G(\bigvee^{n} S^0) & \longrightarrow & G(\bigvee^{n-1} S^0) \end{array}$$

Now $\eta_{\mathbf{n-1}_c}$ is a retract of $\eta_{\mathbf{n}_c}$ because both horizontal composites are the identity ($F(id) = id = G(id)$).

Since weak equivalences are preserved under retracts and $\eta_{\mathbf{n}_c}$ is an equivalence, $\eta_{\mathbf{n-1}_c}$ is also a weak equivalence.

So inductively η is an equivalence on the space $\mathbf{k}_c$ where $0 \leq k \leq n$. To show that η is an equivalence on $\mathbf{k}_c$ for $k > n$, we will use the excisiveness of F and G.

Let $n = 1$. Consider the cocartesian diagram

$$\begin{array}{ccc} * & \longrightarrow & S^0 \\ \downarrow & & \downarrow \\ S^0 & \longrightarrow & S^0 \vee S^0 \end{array}$$

Since F and G are 1-excisive, the front and back squares of the following cube of spectra are cartesian and thus also cocartesian:

$$\begin{array}{ccccc} F(*) & \longrightarrow & F(S^0) & & \\ \downarrow & \searrow^{\simeq} & \downarrow & \searrow^{\simeq} & \\ & G(*) & \longrightarrow & & G(S^0) \\ F(S^0) & \longrightarrow & F(S^0 \vee S^0) & & \downarrow \\ & \searrow^{\simeq} & \downarrow & \searrow & \\ & & G(S^0) & \longrightarrow & G(S^0 \vee S^0) \end{array}$$

The three labelled diagonal maps are equivalences by the above argument, and so the map $\eta_{\mathbf{2}_c}$ on the final corners is also an equivalence by the homotopy invariance of homotopy colimits.

Similarly for higher dimensions, we can form the strongly cocartesian $(n+1)$-cube $U \mapsto \bigvee^{|U|} S^0$. Applying F and G yield cartesian (and thus cocartesian) cubes which are equivalent on all U except the final corner, but these are also equivalent by homotopy invariance of the homotopy colimit. Thus η is an equivalence on $\mathbf{k}_c$ for all finite k. The colimit axiom assures us that F and G agree on all collections of points (infinite or finite). □

The generalization of Lemma 3.2.5 to a simplicial category $\mathcal{C}$ with cofibrant generator is straightforward, but we'd like to point out that for $\mathcal{C}$ unbased, we can recover the initial object $\varnothing$ (the empty coproduct of c's) as the homotopy limit of the cosimplicial object $\coprod_{\varnothing}^{n+1} c$. Since η is an equivalence on every object of the diagram, $F(\varnothing) \simeq G(\varnothing)$.

Definition 3.2.6. If $X_\bullet$ is a simplicial object in $\mathcal{C}$, we say that F *commutes with realization* if

$$\|F(X_\bullet)\| \xrightarrow{\simeq} F(\|X_\bullet\|).$$

As long as the category $s\mathcal{C}$ of simplicial objects in $\mathcal{C}$ has a decent notion of realization, we can extend a

functor $F : \mathcal{C} \to \mathcal{S}\mathrm{p}$ by applying it levelwise to an object in $s\mathcal{C}$. This yields an object in $s\,\mathcal{S}\mathrm{p}$ which can be realized in $\mathcal{S}\mathrm{p}$. We could also apply F to the realization of the object of $s\mathcal{C}$, and compare the results. That is, a functor $F : \mathcal{C} \to \mathcal{S}\mathrm{p}$ commutes with realization if the following diagram commutes

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{S}\mathrm{p} \\ {\scriptstyle \|-\|}\uparrow & & \uparrow{\scriptstyle \|-\|} \\ s\mathcal{C} & \xrightarrow{F^{\Delta^{op}}} & s\,\mathcal{S}\mathrm{p} \end{array}$$

Lemma 3.2.7. *If two functors $F, G : \mathcal{T} \to \mathcal{S}\mathrm{p}$ commute with realization, then a natural transformation $\eta : F \to G$ which is an equivalence on discrete sets is an equivalence on all spaces.*

Proof. By the singularization/realization adjunction of spaces and simplicial sets, there is a weak equivalence $\|Sing(X)_\bullet\| \xrightarrow{\simeq} X$, which the homotopy functor F preserves. For all k, $Sing(X)_k$ is a simplicial (discrete) space, and since F commutes with realization, for all spaces X,

$$\|F(Sing(X)_\bullet)\| \xrightarrow{\simeq} F(\|Sing(X)_\bullet\|) \xrightarrow{\simeq} F(X).$$

We have assumed that $F \xrightarrow{\eta} G$ is an equivalence on discrete sets, so $\eta_\bullet : F(Sing(X)_k) \xrightarrow{\simeq} G(Sing(X)_k)$ for all k. By Segal's realization lemma ([Seg74] Lemma A.1.ii), simplicial spaces which are levelwise equivalent by a natural transformation have equivalent realization; Proposition X.1.2 of [EKMM97] gives the analogous result for simplicial spectra. Thus, for all X, η is an equivalence.

$$\begin{array}{ccc} \|F(Sing(X)_\bullet)\| & \xrightarrow{\simeq} & F(X) \\ {\scriptstyle \eta_\bullet}\downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \eta} \\ \|G(Sing(X)_\bullet)\| & \xrightarrow{\simeq} & G(X) \end{array}$$

□

McCarthy's argument in Corollary 6.5 of [McC] for endofunctors of spectra generalizes to functors $F : \mathcal{C} \to \mathcal{S}\mathrm{p}$ from a simplicial model category $\mathcal{C}$ with cofibrant generator c by building the c-cellular replacement of an object $X \in \mathcal{C}$. To prove lemma 3.2.7 for such categories, one needs to note that objects of $\mathcal{C}$ are equivalent to the realization of a simplicial object of $\mathcal{C}$, and levelwise, these are sets tensored with the generator c, so are discrete c-elements, $\mathbf{k}_c$ for some k.

The next lemma shows that n-excisive functors to spectra commute with realization. Together with the previous two lemmas, this completes the proof of Theorem 3.2.3 for functors $F : \mathcal{T} \to \mathcal{S}\mathrm{p}$.

Lemma 3.2.8. *An n-excisive functor to spectra commutes with realization.*

This is proven as Corollary 6.4 of [McC] for functors with domain spectra and as Corollary 5.11 of [MO02] for domain spaces. We offer a proof similar to McCarthy's.

Proof. The proof is by cases and induction. Let $F : \mathcal{T} \to \mathcal{S}\mathrm{p}$ be an n-excisive functor.

Case 0: Suppose F is 0-excisive, so F is constant. Clearly constant functors commute with realization.

Case 1: F is n-homogeneous, so $F \simeq D_nF$. By Goodwillie's classification in [Goo03], $D_nF(X) \simeq (\partial_nF(*) \wedge X^{\wedge n})_{h\Sigma_n}$. The realization is a homotopy colimit construction, so commutes with homotopy orbits, smashing with a fixed spectrum, and smash products. Thus D_nF commutes with realization.

Case 2: Suppose F is n-excisive, but not necessarily n-reduced, so $F \simeq P_nF$. The fiber sequence $D_nF \to P_nF \to P_{n-1}F$ is also a cofiber sequence in spectra, and applying realization (a homotopy colimit) preserves this, so $\|D_nF(X_\bullet)\| \to \|P_nF(X_\bullet)\| \to \|P_{n-1}F(X_\bullet)\|$ is a fibration. By induction $\|P_{n-1}F(X_\bullet)\| \simeq P_{n-1}F(\|X_\bullet\|)$, and by case 1, $\|D_nF(X_\bullet)\| \simeq D_nF(\|X_\bullet\|)$, so the result follows.

For functors with simplicial domain $\mathcal{C}$, one can use the results of [Kuh07] to write D_nF as composition of constructions which commute with homotopy colimits for Case 1, and Case 2 goes through as written. □

3.3 Evaluation is an equivalence

In this section, we will show that the natural evaluation map (defined in section 3.1) is an equivalence when F is n-excisive:

$$\mathbb{P}_nF(\mathbf{n}_c) \otimes^{\mathbb{L}}_{\mathbb{P}_n\mathcal{C}(\mathbf{n}_c,\mathbf{n}_c)} \mathbb{P}_n\mathcal{C}(\mathbf{n}_c, -) \xrightarrow{evaluation} \mathbb{P}_nF(-).$$

We use the superscript $\mathbb{L}$ to denote that this is a derived tensor product, given by the bar construction. That is,

Definition 3.3.1. Let

$$\begin{aligned} L_nF &= \mathbb{P}_nF(\mathbf{n}_c) \otimes^{\mathbb{L}}_{\mathbb{P}_n\mathcal{C}(\mathbf{n}_c,\mathbf{n}_c)} \mathbb{P}_n\mathcal{C}(\mathbf{n}_c, -) \\ &= \left\| [\mathbf{m}] \mapsto \mathbb{P}_nF(\mathbf{n}_c) \otimes (\overbrace{\mathbb{P}_n\mathcal{C}(\mathbf{n}_c,\mathbf{n}_c) \times \cdots \times \mathbb{P}_n\mathcal{C}(\mathbf{n}_c,\mathbf{n}_c)}^{m \text{ times}}) \times \mathbb{P}_n\Sigma^\infty \operatorname{Hom}(\mathbf{n}_c, -)) \right\|. \end{aligned}$$

where the $\mathbb{P}_n\mathcal{C}$ hom sets are defined as in section 3.1:

$$\mathbb{P}_n\mathcal{C}(\mathbf{n}_c,\mathbf{n}_c) = \operatorname{hocolim}_{\mathbb{I}} T_n^k\Sigma^\infty \operatorname{Hom}(\mathbf{n}_c, -)(\mathbf{n}_c).$$

We will show that the natural evaluation $L_nF \to \mathbb{P}_nF$ is an equivalence on all objects of $\mathcal{C}$. We'll focus on functors $F : \mathcal{C} \to \mathcal{S}\mathrm{p}$ with $\mathcal{C}$ a simplicial model category with cofibrant generator c. In particular, $\mathcal{C}$ can be the category of pointed topological spaces with generator S^0.

We will start by showing that each level of the simplicial functor L_nF is n-excisive, then that levelwise n-excisive functors realize to n-excisive functors. By applying Theorem 3.2.3, we will obtain the desired equivalence $L_nF \xrightarrow{\simeq} \mathbb{P}_nF$.

We first show that $(L_nF)_m = \mathbb{P}_nF(\mathbf{n}_c) \otimes \mathbb{P}_n\mathcal{C}(\mathbf{n}_c, \mathbf{n}_c)^{\otimes m} \otimes \mathbb{P}_n\mathcal{C}(\mathbf{n}_c, -)$ is n-excisive for $m \geq 0$.

Lemma 3.3.2. *$\mathbb{P}_n\Sigma^\infty \operatorname{Hom}(X, -) : \mathcal{T} \to \mathcal{S}\mathrm{p}$ is n-excisive when X is cofibrant and finite dimensional.*

Proof. If $\mathcal{C}$ is the category of based spaces, then Goodwillie shows (in Example 4.5 of [Goo92]) that $\Sigma^\infty \operatorname{Hom}(X, -)$ is ρ-analytic when X is finite of dimension ρ. Thus the $\mathbb{I}$-functor $[k] \mapsto T_n^{k+1}\Sigma^\infty \operatorname{Hom}(X, -)$ satisfies the conditions of Bökstedt's lemma (2.3.2)), because the connectivity increases as k increases. Then there is an equivalence

$$P_n\mathcal{C}(X, -) = \operatorname{hocolim}_{\mathbb{N}} T_n^k\Sigma^\infty \operatorname{Hom}(X, -) \xrightarrow{\simeq} \operatorname{hocolim}_{\mathbb{I}} T_n^k\Sigma^\infty \operatorname{Hom}(X, -) = \mathbb{P}_n\mathcal{C}(X, -).$$

Since $P_n\mathcal{C}(X, -)$ is n-excisive, so is $\mathbb{P}_n\Sigma^\infty \operatorname{Hom}(X, -)$. □

Note that $\mathbf{n}_c$ is cofibrant when c is a cofibrant generator of $\mathcal{C}$.

Lemma 3.3.3. *If G is an n-excisive functor to spectra, then for any spectrum Y, $Y \wedge G(-)$ is also n-excisive.*

Proof. Let $G : \mathcal{C} \to \mathcal{S}\mathrm{p}$ be an n-excisive functor. For any strongly cocartesian $(n+1)$-cube $\mathcal{X}$, the cube $G(\mathcal{X})$ of spectra is cartesian, thus also cocartesian. When $\mathcal{C}$ has functorial cofibrant replacement, smashing means applying the derived tensor, so $Y \wedge -$ is a homotopy left adjoint and $Y \wedge G(\mathcal{X})$ is also cocartesian, thus cartesian, and so $Y \wedge G(-)$ is n-excisive for any spectrum Y. □

Levelwise, L_nF is the product of an n-excisive functor with finitely many spectra, so we have shown that L_nF is levelwise n-excisive.

Remark 3.3.4. Note that these lemmas are exactly why we have chosen to enrich $\mathbb{P}_n\mathcal{C}$ in spectra. If we had used the simplicial enrichment of $\mathcal{C}$ as the enrichment of $\mathbb{P}_n\mathcal{C}$, we would need a map $\mathcal{S}\mathrm{p} \otimes \mathcal{T} \to \mathcal{S}\mathrm{p}$ that preserves n-excision, but the following example shows that this does not always exist.

Example 3.3.5. The functor $\Omega^\infty\Sigma^\infty : \mathcal{T} \to \mathcal{T}$ is linear, but upon tensoring with the spectrum $\mathbb{S}$, one gets

the functor $\Sigma^\infty\Omega^\infty\Sigma^\infty$, which is not linear by the Snaith splitting

$$\Sigma^\infty\Omega^\infty\Sigma^\infty(X) \simeq \bigvee_{j\geq 1} \Sigma^\infty(X^{\wedge j})_{h\Sigma_j}.$$

Given a simplicial functor $[k] \mapsto F_k$ to spectra such that each F_k is an n-excisive functor, the next lemma shows that the realization $\|F_\bullet\|$ is n-excisive. That is, given a strongly cocartesian $(n+1)$-cube $\mathcal{X}$, the map $\|F_\bullet(\mathcal{X}_\varnothing)\| \to \operatorname{holim}_{U\in\mathcal{P}_0(\mathbf{n})} \|F_\bullet(\mathcal{X}_U)\|$ is an equivalence.

Lemma 3.3.6. *([MO02] 5.4) The realization of a levelwise n-excisive functor to spectra is n-excisive.*

Proof. If $[k] \mapsto F_k$ is a simplicial functor to spectra such that F_k is n-excisive for all k, then $F_k \simeq P_nF_k$ for all $k \geq 0$. Let $\mathcal{X}$ be a strongly cocartesian $(n+1)$-cube, then $P_nF_k(\mathcal{X})$ is cartesian for all k. In the category of spectra, this is also cocartesian. Since fat realization commutes with homotopy colimits, $\|P_nF_k(\mathcal{X})\|$ is also cocartesian and thus cartesian. Then $\|P_nF_k\|$ is n-excisive. □

Thus, L_nF is an n-excisive functor, because it is levelwise n-excisive.

Finally, we can show that the evaulation map is an equivalence when F is n-excisive and finitary.

Corollary 3.3.7. *The natural map* $\mathbb{P}_nF(\mathbf{n}_c) \otimes^{\mathbb{L}}_{\mathbb{P}_n \operatorname{Hom}(\mathbf{n}_c,\mathbf{n}_c)} \mathbb{P}_n \operatorname{Hom}(\mathbf{n}_c, -) \xrightarrow{evaluation} \mathbb{P}_nF(-)$ *defined in section 3.1 is an equivalence when F is n-excisive and finitary.*

Proof. We have shown in the previous section that L_nF is n-excisive. When F is n-excisive, $\mathbb{P}_nF \simeq P_nF \simeq F$, where the first equivalence is by Bökstedt's lemma ([Bök85]) and the second is due to Goodwillie ([Goo03], Prop 1.5). Then $\mathbb{P}_nF$ is also n-excisive and satisfies the colimit axiom. We see that L_nF satisfies the colimit axiom, since realization, finite monoidal products, and $\operatorname{Hom}(X, -)$ commute with filtered colimits when X is compact.

To apply Theorem 3.2.3, we must show that the natural transformation $L_nF \to \mathbb{P}_nF$ is an equivalence on $\mathbf{n}_c$, i.e., on $n+1$ points. At $\mathbf{n}_c$, there is a map $s : \mathbb{P}_nF(\mathbf{n}_c) \to \mathbb{P}_nF(\mathbf{n}_c) \otimes \mathbb{P}_n \operatorname{Hom}(\mathbf{n}_c, \mathbf{n}_c)$ defined by $x \mapsto (x, [id])$, where $[id]$ is the image of $id \in \operatorname{Hom}(\mathbf{n}_c, \mathbf{n}_c)$ in $\mathbb{P}_n\Sigma^\infty \operatorname{Hom}(\mathbf{n}_c, \mathbf{n}_c)$. Clearly, $ev \circ s = \mathbf{1}_{\mathbb{P}_nF(\mathbf{n}_c)}$.

The other composition $s \circ ev$ is simplicially homotopic to the identity. That is, the map s allows us to build a contracting simplicial homotopy (Definition 2.1.3) by defining an extra degeneracy map at each stage which inserts the identity $[id]$ in the last spot.

Explicitly, let $\mathbb{P}_nF(\mathbf{n}_c)$ be denoted by A. Note that the map $d_0 : A \otimes P \to A : (a, p) \mapsto a \cdot p$ coequalizes $d_0, d_1 : A \otimes P \otimes P \to A \otimes P$, so $A \otimes^{\mathbb{L}}_{\mathbb{P}_n\mathcal{C}(\mathbf{n}_c,\mathbf{n}_c)} \mathbb{P}_n\mathcal{C}(\mathbf{n}_c, -)(\mathbf{n}_c)$ is augmented by A. We will consider A as the -1 object of the simplicial object L_nF. There is a map $s_{-1} : A \to A \otimes P : a \to (a, id)$.

We'll denote the simplicial complex $A \otimes^{\mathbb{L}}_{\mathbb{P}_n\mathcal{C}(\mathbf{n}_c,\mathbf{n}_c)} \mathbb{P}_n\mathcal{C}(\mathbf{n}_c,\mathbf{n}_c)$ by $X_\bullet$, so

$$X_n = A \otimes \underbrace{\mathbb{P}_n\mathcal{C}(\mathbf{n}_c,\mathbf{n}_c) \otimes \cdots \otimes \mathbb{P}_n\mathcal{C}(\mathbf{n}_c,\mathbf{n}_c)}_{n+1}$$

with s_i including an identity in the ith spot and

$$d_i(a, p_0, \ldots, p_n) = \begin{cases} (a \cdot p_0, p_1, \ldots, p_n) & \text{if } i = 0 \\ (a, p_0, \ldots, p_{i-1} \cdot p_i, \ldots, p_n) & \text{if } 1 \leq i < n \end{cases}$$

Let $s_{-1} : X_n \to X_{n+1}$ be defined by $(a, p_0, \ldots, p_n) \mapsto (a, p_0, \ldots, p_n, id)$, and note that s_{-1} satisfies the identities of a contracting homotopy (definition 2.1.3). Thus $|X| \to A$ is a homotopy equivalence, and $L_nF \to \mathbb{P}_nF$ is an equivalence on all spaces. □

3.4 Classification

Let $F : \mathcal{C} \to \mathcal{S}\mathrm{p}$ be a functor where $\mathcal{C}$ is a simplicial model category with cofibrant generator c. In a series of papers in the 90's, Kuhn classifies degree n functors of vector spaces by modules over matrix rings, which he calls generic representations. For reasons explained in remark 3.4.3, we adopt Kuhn's terminology from [Kuh00].

Definition 3.4.1. A rank n generic representation, A, is a spectrum equipped with a continuous monoid map of spectra $\mathbb{P}_n\mathcal{C}(\mathbf{n}_c,\mathbf{n}_c) \to \mathrm{Hom}(A, A)$.

By adjunction, this is the data of a right module over $\mathbb{P}_n\mathcal{C}(\mathbf{n}_c,\mathbf{n}_c)$, i.e., an object $A \in \mathcal{S}\mathrm{p}$ with a unital, associative, and continuous action map: $A \otimes \mathbb{P}_n\mathcal{C}(\mathbf{n}_c,\mathbf{n}_c) \to A$, and so a morphism of rank n generic representations $f : A \to B$ is a map respecting this module structure, i.e., a morphism of spectra such that the following commutes:

$$\begin{array}{ccc} A \otimes \mathbb{P}_n\mathcal{C}(\mathbf{n}_c,\mathbf{n}_c) & \longrightarrow & A \\ \downarrow{\scriptstyle f\otimes id} & & \downarrow{\scriptstyle f} \\ B \otimes \mathbb{P}_n\mathcal{C}(\mathbf{n}_c,\mathbf{n}_c) & \longrightarrow & B \end{array}$$

Two rank n generic representations, A and B, are called equivalent if there is a map of representations $A \to B$ which is an equivalence of spectra.

Recall that two functors $F, G : \mathcal{C} \to \mathcal{S}\mathrm{p}$ are equivalent if there is a natural transformation $\eta : F \to G$ that is an equivalence on each object of $\mathcal{C}$. We have shown that n-excisive functors which satisfy the colimit axiom (i.e., finitary functors) are equivalent if η is an equivalence on $\mathbf{n}_c$.

Theorem 3.4.2. *The homotopy category of rank n generic representations is equivalent to the homotopy category of finitary n-excisive functors $F : \mathcal{C} \to \mathcal{S}\mathrm{p}$.*

Proof. Consider the functors

$$\mathrm{Fun}_{n\text{-}exc}(\mathcal{C}, \mathcal{S}\mathrm{p}) \underset{\mathbb{P}_n - (\mathbf{n}_c)}{\overset{-\otimes^{\mathbb{L}}_{\mathbb{P}_n\mathcal{C}(\mathbf{n}_c,\mathbf{n}_c)} \mathbb{P}_n\mathcal{C}(\mathbf{n}_c,\star)}{\longleftarrow}} \mathrm{Mod}\text{-}\mathbb{P}_n\mathcal{C}(\mathbf{n}_c, \mathbf{n}_c)$$

Let F be a finitary, n-excisive functor and let A be a rank n generic representaiton. The bottom functor takes F to $\mathbb{P}_n F(\mathbf{n}_c)$, and the top functor sends A to the realization of the simplicial functor $A \otimes^{\mathbb{L}}_{\mathbb{P}_n\mathcal{C}(\mathbf{n}_c,\mathbf{n}_c)} \mathbb{P}_n\mathcal{C}(\mathbf{n}_c, -) : \mathcal{C} \to \mathcal{S}\mathrm{p}$, which is levelwise n-excisive. We have shown in section 3.1 that $\mathbb{P}_n F(\mathbf{n}_c)$ is a right $\mathbb{P}_n\mathcal{C}(\mathbf{n}_c, \mathbf{n}_c)$-module, so the bottom functor is well-defined; by lemma 3.3.6, the top functor produces an n-excisive functor so is also well-defined.

We will show that the functors preserve weak equivalences. Given two equivalent n-excisive functors $\eta : F \xrightarrow{\simeq} G$, we have

$$\begin{array}{ccc} F & \xrightarrow{\simeq} & G \\ \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} \\ \mathbb{P}_n F & \longrightarrow & \mathbb{P}_n G \end{array}$$

Thus $\mathbb{P}_n F(\mathbf{n}_c) \simeq \mathbb{P}_n G(\mathbf{n}_c)$ so the functor $\mathbb{P}_n - (\mathbf{n}_c)$ is a homotopy functor.

Given two equivalent rank n generic representations $A \xrightarrow{\simeq} B$, it is clear (using the homotopy invariant tensor product) that the functors agree objectwise

$$A \otimes \mathbb{P}_n\mathcal{C}(\mathbf{n}_c, \mathbf{n}_c) \otimes \cdots \otimes \mathbb{P}_n\mathcal{C}(\mathbf{n}_c, -) \xrightarrow{\simeq} B \otimes \mathbb{P}_n\mathcal{C}(\mathbf{n}_c, \mathbf{n}_c) \otimes \cdots \otimes \mathbb{P}_n\mathcal{C}(\mathbf{n}_c, -).$$

By homotopy invariance of the homotopy colimit, the functors are equivalent. Thus the top map also preserves weak equivalences.

Finally, we will show that the two compositions are equivalent to the identity.

Starting on the left with n-excisive functor F, the composition yields

$$- \otimes^{\mathbb{L}}_{\mathbb{P}_n\mathcal{C}(\mathbf{n}_c,\mathbf{n}_c)} \mathbb{P}_n\mathcal{C}(\mathbf{n}_c, \star) \circ \mathbb{P}_n - (\mathbf{n}_c)(F) = \mathbb{P}_n F(\mathbf{n}_c) \otimes^{\mathbb{L}}_{\mathbb{P}_n\mathcal{C}(\mathbf{n}_c,\mathbf{n}_c)} \mathbb{P}_n\mathcal{C}(\mathbf{n}_c, -) = L_n F.$$

By Corollary 3.3.7, $L_n F \simeq \mathbb{P}_n F$ and $\mathbb{P}_n F \simeq F$ by n-excision. Thus the composition is equivalent to the identity.

Starting on the right with generic representation A, the composition yields

$$\begin{aligned}
&\mathbb{P}_n - (\mathbf{n}_c) \circ (- \otimes^{\mathbb{L}}_{\mathbb{P}_n\mathcal{C}(\mathbf{n}_c,\mathbf{n}_c)} \mathbb{P}_n\mathcal{C}(\mathbf{n}_c, \star))(A) \\
&= \mathbb{P}_n \left(A \otimes^{\mathbb{L}}_{\mathbb{P}_n\mathcal{C}(\mathbf{n}_c,\mathbf{n}_c)} \mathbb{P}_n\mathcal{C}(\mathbf{n}_c, -) \right) (\mathbf{n}_c) \\
&\simeq A \otimes^{\mathbb{L}}_{\mathbb{P}_n\mathcal{C}(\mathbf{n}_c,\mathbf{n}_c)} \mathbb{P}_n\mathcal{C}(\mathbf{n}_c, -)(\mathbf{n}_c)
\end{aligned}$$

where the last equivalence is by n-excision. Now $A \otimes^{\mathbb{L}}_{\mathbb{P}_n\mathcal{C}(\mathbf{n}_c,\mathbf{n}_c)} \mathbb{P}_n\mathcal{C}(\mathbf{n}_c, \mathbf{n}_c) \xrightarrow{\simeq} A$ by a (forward) contracting simplicial homotopy (as in corollary 3.3.7). So the composition is equivalent to the identity.

Thus we have an equivalence of homotopy categories. □

Remark 3.4.3. When $\mathcal{C} = \mathcal{S}\mathrm{p}$, n-excisive functors are equivalent to modules over the monoid $\mathbb{P}_n\,\mathcal{S}\mathrm{p}(\mathbf{n}_c, \mathbf{n}_c) \simeq \mathbb{P}_n\Sigma^\infty\Omega^\infty M_n(\mathbb{S})$, where $M_n(R)$ is the matrix ring spectrum on R, defined by $M_n(R) = \mathrm{Hom}(\mathbf{n}_+, \mathbf{n}_+ \wedge R)$ with $\mathbf{n}_+ = \{0, 1, \ldots, n\}$ ([Bök85, Sch07]). This is why we have chosen the terminology of Kuhn to describe n-excisive functors as generic representations. This is a reformulation of the case of endofunctors of spectra which was considered in [McC].

Chapter 4

Monoidal Derivatives

The n-excisive approximations of a functor are hard to compute, and so we turn to the n-homogeneous layers of the Taylor tower

$$D_nF = \text{hofib}(P_nF \to P_{n-1}F)$$

One can consider the fiber as a difference of the nth polynomial approximation from the $n-1$st. Indeed, this analogy is justified by Goodwillie's classification of the layers, which look like the n-homogeneous pieces of the Taylor series, $\frac{f^{(n)}(*)\cdot x^n}{n!}$.

Theorem 4.0.4 ([Goo03])**.**

$$D_nF(X) \simeq \Omega^\infty \left(\partial_n F \wedge X^{\wedge n}\right)_{h\Sigma_n}$$

where $\partial_n F$ is a spectrum with Σ_n-action called the n-th derivative of F*.*

Thus the layers of the Taylor tower correspond to Σ_n-spectra associated to F, and Goodwillie went further to identify the homotopy type of these derivatives.

Theorem 4.0.5 ([Goo03])**.** *The n-th derivative of F is equivalent to the multilinearization of the nth cross effect.*

$$(\Omega^\infty)\partial_n^G F \simeq \operatorname*{hocolim}_{k_1,\ldots,k_n\to\infty} \Omega^{k_1}\cdots\Omega^{k_n} cr_nF(\Sigma^{k_1}S^0,\ldots,\Sigma^{k_n}S^0)$$

The Σ_n-action is induced by permuting the variables of cr_nF; in the multilinearization, this also permutes the loops. The nth cross effect is a functor of n variables which can be thought of as a measurement of the failure of F to be degree $n-1$ (in an additive sense). For example, $cr_1F(X) = \text{hofib}(F(X) \to F(*))$, so if F is degree 0 (or constant), cr_1F is trivial.

If we consider all the derivatives of a functor together, we see a symmetric sequence in spectra. Thus we may think of the derivatives as a functor

$$\partial_* : [Top_*, Top_*] \to [\Sigma, \mathcal{S}\text{p}]$$

This point of view leads to a question posed by Arone and Ching in the introduction of [AC11].

Question 1. *Is ∂_* (lax) monoidal?*

Specifically, this asks for a natural transformation $\partial_* F \circ \partial_* G \to \partial_*(F \circ G)$ and a map $\mathbb{S} \to \partial_1 Id$, where the first $\circ$ is the composition product of symmetric sequences and the second is composition of functors.

It is easy to construct a composition map which is associative and unital up to homotopy, but strict associativity requires a different model for the derivatives.

If ∂_* is monoidal, some immediate consequences would be that if F is a monad and G is a module over F, then $\partial_* F$ is naturally an operad and $\partial_* G$ is a $\partial_* F$-module. For the monad $F = Id$, these consequences have been proven. The first is due to work of Johnson [Joh95], Arone-Mahowald [AM99], and Ching [Chi05], and the second is work of Arone-Ching [AC11]. Arone and Ching went on to show that the derivatives have an even nicer property, a chain rule.

Theorem 4.0.6 ([AC11])**.** *For reduced, finitary functors $F \circ G : \mathcal{C} \to \mathcal{D} \to \mathcal{E}$, where $\mathcal{C}, \mathcal{D}, \mathcal{E}$ are either $\mathcal{T}$ or* $\mathcal{S}\mathrm{p}$*,*

$$\partial_* F \circ_{\partial_* Id_{\mathcal{D}}} \partial_* G \simeq \partial_*(F \circ G).$$

Taking the composition product over $\partial_* Id$ is a derived product, i.e., the left hand side is a two-sided bar construction. The equivalence is given as a zigzag of equivalences, and there is no direct map for the chain rule.

In this chapter, we give a monoidal model for the derivatives of a reduced endofunctor of spaces, using the indexing category $\mathbb{I}$ of finite sets and injective maps. In the first section, we define our models for the cross effects of a functor and show that they form something like a functor operad. In section 4.2, we define the new model for the derivatives and show that it is monoidal. Finally, in section 4.3, we prove a chain rule in this setting.

4.1 Key properties of cross effects

We will start with the definition of the cross effects of an endofunctor F of spaces. It is important to choose our model for the homotopy fiber carefully so that the desired maps exist.

Definition 4.1.1. ([May99] 8.6) The homotopy fiber of a map $f : X \to Y$ is given by the strict limit

$$\mathrm{hofib}\, f = \lim \left(\begin{array}{ccc} & & X \\ & & \downarrow f \\ Y^I & \xrightarrow{ev_1} & Y \end{array} \right)$$

where Y^I is the pointed path space of Y. This diagram is a fibrant replacement for the diagram with the terminal object $*$ in the place of Y^I, so the homotopy limit agrees with the strict limit.

Definition 4.1.2. Let $\bigsqcup_n : \mathcal{C}^n \to \mathcal{C}$ be defined by $\bigsqcup_n(X_1, \ldots, X_n) = X_1 \vee \cdots \vee X_n$.

Definition 4.1.3. The *1st cross effect of* F on the space X is given by

$$cr_1F(X) = \text{hofib}[F(X) \to F(*)]$$

The *nth cross effect of* F on the spaces $X_1, \ldots, X_n$ is given by

$$cr_nF(X_1, \ldots, X_n) = cr_1^{(n)} \cdots cr_1^{(1)}(F \circ \bigsqcup_n)(X_1, \ldots, X_n)$$

where $cr_1^{(i)}G$ denotes the first cross effect applied to the ith variable of the multifunctor G.

Traditionally, cr_nF is defined as the total fiber of a cube constructed from coproducts of the inputs [Goo03], but we will now demonstrate that the given model is equivalent to the usual cubical model, cr_n^G.

The second cross effect is given by

$$cr_2^G F(X,Y) = \text{tothofib}\begin{pmatrix} F(X \vee Y) & \longrightarrow & F(Y) \\ \downarrow & & \downarrow \\ F(X) & \longrightarrow & F(*) \end{pmatrix}$$

It is well known that the total homotopy fiber of a cube is equivalent to any of the iterated homotopy fibers (see [BJM15] for detailed definitions) so we can start with taking horizontal fibers to get

$$\begin{array}{ccc} cr_1^{(1)}(F \circ \bigsqcup_2)(X,Y) & = & \text{hofib}\,[F(X \vee Y) \to F(* \vee Y)] \\ \downarrow & & \downarrow \\ cr_1^{(1)}(F \circ \bigsqcup_2)(X,*) & = & \text{hofib}\,[F(X \vee *) \to F(* \vee *)] \end{array}$$

Then the vertical fiber $\text{hofib}[cr_1^{(1)}(F \circ \bigsqcup_2)(X,Y) \to cr_1^{(1)}(F \circ \bigsqcup_2)(X,*)]$ is equivalent to $cr_1^{(2)}cr_1^{(1)}(F \circ \bigsqcup_2)(X,Y)$, so our model of iterated first cross effects is equivalent to the standard cubical model, $cr_2^G F(X,Y) \simeq cr_2F(X,Y)$. The generalization to higher cubes is straightforward.

Note that if F is reduced, cr_1F is also reduced, and the choice of model for the homotopy fiber yields isomorphisms

$$cr_1^{(1)}cr_1^{(2)}(F \circ \bigsqcup_2)(X,Y) \cong cr_1^{(2)}cr_1^{(1)}(F \circ \bigsqcup_2)(X,Y).$$

Lemma 4.1.4. *cr_nF has assembly maps in each variable.*

Proof. Since cr_nF is continuous and reduced in each variable, the assembly map is given by following the identity through the maps

$$\begin{aligned}\mathrm{Hom}(Z\wedge X, Z\wedge X) &\cong \mathrm{Hom}(Z, \mathrm{Hom}(X, Z\wedge X))\\ &\xrightarrow{cr_2F(-,Y)} \mathrm{Hom}(Z, \mathrm{Hom}(cr_2F(X,Y), cr_2F(Z\wedge X, Y)))\\ &\cong \mathrm{Hom}(Z\wedge cr_2F(X,Y), cr_2F(Z\wedge X, Y))\end{aligned}$$

□

Lemma 4.1.5. *If F is stably n-excisive, then cr_kF is stably n-excisive in each variable.*

Proof. If F is stably n-excisive, then F satisfies $E_n(c,\kappa)$ for some c and κ. That is, for any strongly cocartesian $(n+1)$-cube $\mathcal{X}$ such that $\mathcal{X}(\varnothing)\to\mathcal{X}(\{s\})$ is k_s-connected with $k_s\geq\kappa$, the diagram $F(\mathcal{X})$ is $(\Sigma k_s - c)$-cartesian.

We will show that cr_k^GF is stably n-excisive in each variable by showing that $cr_k^GF(-,Y_2,\ldots,Y_k)$ is. Let $\mathcal{X}$ be a strongly cocartesian $(n+1)$-cube such that $\mathcal{X}(\varnothing)\to\mathcal{X}(\{s\})$ is k_s-connected with $k_s\geq\kappa$. Then $cr_k^GF(\mathcal{X},Y_2,\ldots,Y_k)$ is given by the homotopy fiber of cubes (so is of the form to apply Proposition 1.18 of [Goo92])

$$V\mapsto \mathrm{tfib}(F\circ\bigsqcup\nolimits_k)(\mathcal{X}(V),Y_2,\ldots,Y_k) = \mathrm{hofib}\left[F(\mathcal{X}(V)\vee Y_2\vee\cdots\vee Y_k)\to \operatorname*{holim}_{U\in\mathcal{P}_0(\mathbf{k-1})} F\left(\mathcal{X}(V)^{\delta_U}\vee\bigvee_{j\notin U}Y_j\right)\right]$$

where $\delta_U=\begin{cases}0 & \text{if } 1\in U\\ 1 & \text{if } 1\notin U\end{cases}$, i.e. $\mathcal{X}(V)$ is in the last sum if $1\notin U$. This last term can be viewed as an $(n+1)$-cube of homotopy limits of punctured k-cubes. By Proposition 1.22 of [Goo92], this cube is ℓ-cartesian where $\ell = min\{1-|U|+\ell_U\}$, and ℓ_U is the cartesianness of the $(n+1)$-cube at U. If $1\notin U$, this cube is given by $(F\circ\bigsqcup_2)\left(\mathcal{X},\bigvee_{j\notin U}Y_j\right)$, which gives $\ell_U=\Sigma k_s-c$. If $1\in U$, the $(n+1)$-cube is constant, so cartesian. Then the largest that U can be (where the cube at U is not cartesian) is $k-1$, so $\ell=\Sigma k_s-c-k+2$.

The $(n+1)$-cube $(F\circ\bigsqcup_k)(\mathcal{X},Y_2,\ldots,Y_k)$ is (Σk_s-c)-cartesian, so by Proposition 1.6 (and 1.18) of [Goo92], the cube of fibers $cr_k^GF(\mathcal{X},Y_2,\ldots,Y_k)$ is $(\Sigma k_s-c-k+1)$-cartesian. Thus cr_k^GF satisfies $E_n(c+k-1,\kappa)$, and is stably n-excisive. By equivalence, cr_kF is also stably n-excisive in each variable. □

The following proposition shows that the cross effects form a sort of functor operad. We prove it by induction after a series of lemmas.

Proposition 4.1.6. *Let F,$G : \mathcal{T} \to \mathcal{T}$ be functors. For natural numbers $k, j_1, \dots j_k$ and spaces $\{X_{i,\ell}\}_{1 \le \ell \le j_i}^{1 \le i \le k}$, there are natural associative maps*

$$\gamma_{cr_*} : cr_k F(cr_{j_1} G(X_{1,1}, \dots, X_{1,j_1}), \dots, cr_{j_k} G(X_{k,1}, \dots, X_{k,j_k})) \to cr_{j_1+\dots+j_k}(F \circ G)(X_{1,1}, \dots, X_{k,j_k}).$$

Lemma 4.1.7. *There is a map $F \circ cr_1 G \to cr_1(F \circ G)$.*

Proof. $F \circ cr_1 G(X)$ fits into the following commuting diagram

$$\begin{array}{ccc} F \circ cr_1 G(X) & \longrightarrow & FG(X) \\ \downarrow & & \downarrow \\ F(G(*)^I) & \longrightarrow & FG(*) \end{array}$$

There is a map from $F \circ cr_1 G(X)$ to the strict limit of the rest of the diagram. A map from this limit to $cr_1(F \circ G)$ is induced by the map of diagrams

$$\begin{array}{ccccc} & & & & FG(X) \\ & & & & \downarrow \quad = \quad \\ & & & & FG(X) \\ & & & & \downarrow \\ F(G(*)^I) & \xrightarrow{F(ev_1)} & FG(*) & = & \\ & \searrow^{\alpha_I} & & & \\ & & FG(*)^I & \xrightarrow{ev_1} & FG(*) \end{array}$$

where α_Z is the natural transformation given by the adjoint of the composite

$$Z \wedge F(\mathrm{Hom}(Z, G(*))) \xrightarrow{assembly} F(Z \wedge \mathrm{Hom}(Z, G(*))) \xrightarrow{F(evaluation)} FG(*)$$

with the evaluation map $Z \wedge \mathrm{Hom}(Z, G(*)) \to G(*)$ given by the adjoint of the identity $\mathrm{Hom}(Z, G(*)) \to \mathrm{Hom}(Z, G(*))$.

The map of diagrams commutes by viewing the evaluation at 1 map as $\mathrm{Hom}(I, Y) \to \mathrm{Hom}(S^0, Y) \cong Y$, and using naturality. That is, we use the commutativity of

$$\begin{array}{ccc} F(\mathrm{Hom}(I, G(*))) & \xrightarrow{F(ev_1)} & F(\mathrm{Hom}(S^0, G(*))) \\ \downarrow{\scriptstyle \alpha_I} & & \downarrow{\scriptstyle \alpha_{S^0}} \\ \mathrm{Hom}(I, FG(*)) & \xrightarrow{ev_1} & \mathrm{Hom}(S^0, FG(*)) \end{array}$$

Thus, there is a map $F \circ cr_1 G \to cr_1(F \circ G)$. □

Lemma 4.1.8. *The composite $cr_1F \circ cr_1G \to F \circ cr_1G \to cr_1(F \circ G)$ is associative.*

Proof. Consider the diagram

$$\begin{array}{ccccc}
cr_1F \circ cr_1G \circ cr_1H & \longrightarrow & cr_1F \circ G \circ cr_1H & \longrightarrow & cr_1F \circ cr_1(G \circ H) \\
\downarrow & & \downarrow & & \downarrow \\
F \circ cr_1G \circ cr_1H & \longrightarrow & F \circ G \circ cr_1H & \longrightarrow & F \circ cr_1(G \circ H) \\
\downarrow & & \| & & \downarrow \\
cr_1(F \circ G) \circ cr_1H & \longrightarrow & F \circ G \circ cr_1H & \longrightarrow & cr_1(F \circ G \circ H)
\end{array}$$

The top two squares commute by naturality of the map $cr_1F \to F$, and the bottom two commute by naturality of β, which is given by the composite

$$F\left(\lim\left(\begin{array}{cc} & G(X) \\ & \downarrow \\ G(Y)^Z & \longrightarrow G(Y) \end{array}\right)\right) \to \lim\left(\begin{array}{cc} & FG(X) \\ & \downarrow \\ F(G(Y)^Z) & \longrightarrow FG(Y) \end{array}\right) \to \lim\left(\begin{array}{cc} & FG(X) \\ & \downarrow \\ FG(Y)^Z & \longrightarrow FG(Y) \end{array}\right)$$

Letting $(Y, Z) = (*, I)$ or (X, S^0) shows that the vertical maps of the bottom row are instances of β. $\square$

Now we will use induction to show the existence of the maps of proposition 4.1.6.

Proof of proposition 4.1.6. To keep notation at bay, we'll prove the case $k = 2$ and note that the general case follows easily; that is, we will show existence of natural maps

$$cr_2F(cr_{j_1}G(X_1, \ldots, X_{j_1}), cr_{j_2}G(Y_1, \ldots Y_{j_2})) \to cr_{j_1+j_2}(F \circ G)(X_1, \ldots, X_{j_1}, Y_1, \ldots Y_{j_2}).$$

The map is given by the following composition.

$$cr_2F(cr_{j_1}G(X_1, \ldots, X_{j_1}), cr_{j_2}G(Y_1, \ldots Y_{j_2}))$$

$$\stackrel{defn}{\cong} cr_2F(cr_1^{(1)} \cdots cr_1^{(j_1)}(G \circ \bigsqcup_{j_1})(X_1, \ldots, X_{j_1}), cr_{j_2}G(Y_1, \ldots Y_{j_2}))$$

$$= cr_2F(cr_1^{(1)} \cdots cr_1^{(j_1-1)} -, cr_{j_2}G(Y_1, \ldots Y_{j_2})) \circ cr_1^{(j_1)}(G \circ \bigsqcup_{j_1})(X_1, \ldots, X_{j_1})$$

$$\xrightarrow{4.1.7} cr_1^{(j_1)} cr_2 F(cr_1^{(1)} \cdots cr_1^{(j_1-1)} (G \circ \bigsqcup_{j_1})(X_1, \ldots, X_{j_1}), cr_{j_2} G(Y_1, \ldots Y_{j_2}))$$

$$\xrightarrow{4.1.7\cdots} cr_1^{(j_1)} \cdots cr_1^{(1)} cr_2 F((G \circ \bigsqcup_{j_1})(X_1, \ldots, X_{j_1}), cr_{j_2} G(Y_1, \ldots Y_{j_2}))$$

$$\overset{defn}{=} cr_1^{(j_1)} \cdots cr_1^{(1)} cr_2 F((G \circ \bigsqcup_{j_1})(X_1, \ldots, X_{j_1}), cr_1^{(1)} \cdots cr_1^{(j_2)} (G \circ \bigsqcup_{j_2})(Y_1, \ldots Y_{j_2}))$$

$$\xrightarrow{4.1.7\cdots} cr_1^{(j_2+j_1)} \cdots cr_1^{(1+j_1)} cr_1^{(j_1)} \cdots cr_1^{(1)} cr_2 F((G \circ \bigsqcup_{j_1})(X_1, \ldots, X_{j_1}), (G \circ \bigsqcup_{j_2})(Y_1, \ldots Y_{j_2}))$$

$$\xrightarrow{univ} cr_1^{(j_2+j_1)} \cdots cr_1^{(1+j_1)} cr_1^{(j_1)} \cdots cr_1^{(1)} F((G \circ \bigsqcup_{j_1})(X_1, \ldots, X_{j_1}) \vee (G \circ \bigsqcup_{j_2})(Y_1, \ldots Y_{j_2}))$$

$$\xrightarrow{univ} cr_1^{(j_2+j_1)} \cdots cr_1^{(1+j_1)} cr_1^{(j_1)} \cdots cr_1^{(1)} (F \circ G \circ \bigsqcup_{(j_1+j_2)})(X_1, \ldots, X_{j_1}, Y_1, \ldots Y_{j_2})$$

$$\overset{defn}{=} cr_{j_1+j_2}(F \circ G)(X_1, \ldots, X_{j_1}, Y_1, \ldots Y_{j_2})$$

□

4.2 Main theorem

Definition 4.2.1. Let

$$\partial_n F = \operatorname*{hocolim}_{U_1, \ldots U_n \in \mathbb{I}} \Omega^{\coprod U_i} cr_n F(\Sigma^{U_1} S^0, \ldots \Sigma^{U_n} S^0).$$

The Σ_n-action is induced by permuting the n inputs of $cr_n F$, which also permutes the loops in the multilinearization. For example, the Σ_2-action on $\partial_2 F$ is the conjugate action which block swaps the sphere coordinates of the loops and variables, given by sending $f \in \Omega^U \Omega^V cr_2 F(S^U, S^V)$ to the composite

$$S^V \wedge S^U \xrightarrow{\chi_{U,V}^{-1}} S^U \wedge S^V \xrightarrow{f} cr_2 F(S^U, S^V) \xrightarrow{cr_2 F(\tau)} cr_2 F(S^V, S^U)$$

Lemma 4.2.2. *If F is stably 1-excisive, then the natural map $\partial_n^G F \to \partial_n F$ is an equivalence.*

Proof. By lemma 4.1.5, if F satisfies $E_k(c, \kappa)$, then $cr_n F$ satisfies $E_k(c + n - 1, \kappa)$ in each variable. By Proposition 1.4 of [Goo03], $T_1 F$ satisfies $E_1(c - 1, \kappa - 1)$ and $T_1^j F \to T_1^{j+1} F$ satisfies $O_1(c - j, \kappa - j)$. Thus

the natural transformation $T_1^j cr_n F \to T_1^{j+1} cr_n F$ satisfies $O_1(c+n-j, \kappa - j)$ in each variable, so evaluated on S^0 in one variable the map is $(j-c-n+1)$-connected. Thinking of the cross-effect as a functor in variable i, for each $(U_1, \dots, U_n)$, the maps

$$\Omega^{\coprod_{\ell \neq i} U_\ell} T_1^j cr_n F(S^{U_1}, \dots, -, \cdots, S^{U_n})(S^0) \to \Omega^{\coprod_{\ell \neq i} U_\ell} T_1^{j+1} cr_n F(S^{U_1}, \dots, -, \cdots, S^{U_n})(S^0)$$

are $(j - \Sigma_{\ell \neq i} |U_\ell| - c - n + 1)$-connected. As j goes to infinity, so does the connectivity of this map. Thus for each $(U_1, \dots, U_n)$, the map of homotopy colimits induced by the inclusion $\mathbb{N} \hookrightarrow \mathbb{I}$ is an equivalence.

$$\operatorname*{hocolim}_{U_i \in \mathbb{N}} \Omega^{\coprod U_\ell} cr_n F(S^{U_1}, \dots, S^{U_n}) \xrightarrow{\simeq} \operatorname*{hocolim}_{U_i \in \mathbb{I}} \Omega^{\coprod U_\ell} cr_n F(S^{U_1}, \dots, S^{U_n}) \qquad (\heartsuit)$$

We proceed by (reverse) induction on the indexing set. For the base case, let $i = n$ in $(\heartsuit)$, and note that the equivalence is preserved by taking the homotopy colimit over $U_1, \dots, U_{n-1} \in \mathbb{N}$. Let $H = \Omega^{\coprod U_\ell} cr_n F(S^{U_1}, \dots, S^{U_n})$ and assume that

$$\operatorname*{hocolim}_{U_1, \dots, U_{i+1} \in \mathbb{N}} \operatorname*{hocolim}_{U_{i+2}, \dots, U_n \in \mathbb{I}} H \simeq \operatorname*{hocolim}_{U_1, \dots, U_i \in \mathbb{N}} \operatorname*{hocolim}_{U_{i+1}, \dots, U_n \in \mathbb{I}} H.$$

Then

$$\begin{array}{c}
\operatorname*{hocolim}_{U_1, \dots, U_{i-1} \in \mathbb{N}} \operatorname*{hocolim}_{U_i \in \mathbb{N}} \operatorname*{hocolim}_{U_{i+1}, \dots, U_n \in \mathbb{I}} H \\
\simeq \Big\downarrow \text{hocolims commute} \\
\operatorname*{hocolim}_{U_1, \dots, U_{i-1} \in \mathbb{N}} \operatorname*{hocolim}_{U_{i+1}, \dots, U_n \in \mathbb{I}} \operatorname*{hocolim}_{U_i \in \mathbb{N}} H \\
\simeq \Big\downarrow \text{by } \heartsuit \\
\operatorname*{hocolim}_{U_1, \dots, U_{i-1} \in \mathbb{N}} \operatorname*{hocolim}_{U_{i+1}, \dots, U_n \in \mathbb{I}} \operatorname*{hocolim}_{U_i \in \mathbb{I}} H \\
\simeq \Big\downarrow \text{hocolims commute} \\
\operatorname*{hocolim}_{U_1, \dots, U_{i-1} \in \mathbb{N}} \operatorname*{hocolim}_{U_i \in \mathbb{I}} \operatorname*{hocolim}_{U_{i+1}, \dots, U_n \in \mathbb{I}} H
\end{array}$$

Note that analytic functors are, in particular, stably 1-excisive, so this lemma holds for all analytic functors. □

Theorem 4.2.3. *The model for $\partial_* : [\mathcal{T}, \mathcal{T}]_{red} \to [\Sigma, \mathcal{T}]$ given in Definition 4.2.1 is monoidal.*

Proof. Recall from Definition 2.3.9 that we must define a morphism $\epsilon : 1 \to \partial_* Id$ and a natural tranformation $\mu_{F,G} : \partial_* F \circ \partial_* G \to \partial_*(F \circ G)$. First, we define the morphism ϵ. Since the unit of $\mathcal{T}^\Sigma$ is the symmetric sequence with S^0 in level 1 and the trivial space elsewhere, ϵ is determined by the map $S^0 \to \partial_1 Id$, given by

the inclusion of the first object in the homotopy colimit $S^0 \to \operatorname{hocolim}_{k\in\mathbb{I}} \Omega^k cr_1 Id(S^k)$.

The natural transformation $\partial_* F \circ \partial_* G \to \partial_*(F \circ G)$ is a map of symmetric sequences, so a levelwise equivariant map. On level j, this is

$$\bigvee_{\text{partitions of } \{1,\dots,j\}} \partial_k F \wedge \partial_{j_1} G \wedge \cdots \wedge \partial_{j_k} G \longrightarrow \partial_j (F \circ G)$$

which boils down to defining maps

$$\partial_k F \wedge \partial_{j_1} G \wedge \cdots \wedge \partial_{j_k} G \longrightarrow \partial_j (F \circ G) \quad \text{for all } \; j = j_1 + \cdots + j_k.$$

We start by defining the map for the first level $\partial_1 F \wedge \partial_1 G \to \partial_1(F \circ G)$. Recall that the homotopy colimit and loops functors are both continuous, so by lemma 2.2.2 have assembly maps which we denote with α.

$$\begin{array}{c}
\operatorname*{hocolim}_{U\in\mathbb{I}} \Omega^U cr_1 F(S^U) \wedge \operatorname*{hocolim}_{V\in\mathbb{I}} \Omega^V cr_1 G(S^V) \\
\downarrow{\scriptstyle \alpha_{\text{hocolim}},\, \alpha_\Omega} \\
\operatorname*{hocolim}_{U\in\mathbb{I}} \operatorname*{hocolim}_{V\in\mathbb{I}} \Omega^U \Omega^V cr_1 F(S^U) \wedge cr_1 G(S^V) \\
\downarrow{\scriptstyle \alpha_{cr_1}} \\
\operatorname*{hocolim}_{U\in\mathbb{I}} \operatorname*{hocolim}_{V\in\mathbb{I}} \Omega^U \Omega^V cr_1 F(cr_1 G(S^U \wedge S^V)) \\
\downarrow{\scriptstyle 4.1.7} \\
\operatorname*{hocolim}_{(U,V)\in\mathbb{I}\times\mathbb{I}} \Omega^{U \amalg V} cr_1 (F \circ G)(S^{U \amalg V}) \\
\downarrow{\scriptstyle \amalg_*} \\
\operatorname*{hocolim}_{W\in\mathbb{I}} \Omega^W cr_1 (F \circ G)(S^W)
\end{array}$$

Remark 4.2.4. The last step is the key reason for using $\mathbb{I}$; if the homotopy colimit is defined over $\mathbb{N}$, the map can be defined, but it will not be strictly associative on homotopy colimits. This is similar to the reason naive spectra do not have a good smash product, but symmetric spectra have enough extra structure to encode the smash product in an associative way.

To define the composition map in general, we will first define a map

$$\Omega^U F(S^U) \to \Omega^{\amalg_j U} F(S^{\amalg_j U}) \tag{4.1}$$

given by the diagonal inclusion. For example, $\Omega S^1 \to \Omega^2 S^2$ sends f to the perpendicular suspension of f on

the diagonal copy of S^1 in S^2. This is equivariant with respect to the action which permutes the coordinates of S^2. We denote this map with $U \mapsto \coprod_j U$ even though it does not arise from a set map. If a set map is desired, one could consider the indexing category of finite dimensional inner product spaces with linear isometric inclusions and take the actual diagonal map in that setting. The derivatives can be defined in this more general context and perhaps some equivariant information can be gained from this point of view, but we do not require it now.

Remark 4.2.5. The map $U \mapsto \coprod_j U$ is associative. If $U \hookrightarrow V \hookrightarrow W$ are inclusions, then write $V = U \coprod V'$ and $W = U \coprod V' \coprod W'$, where V' and W' represent the orthogonal complements of U and $U \coprod V'$ inside of V and W, respectively. The map $\Omega^U S^U \to \Omega^V S^V$ is given by $f \mapsto id_{V'} \wedge f$ and so the composition $\Omega^U S^U \to \Omega^V S^V \to \Omega^W S^W$ is given by $f \mapsto id_{W'} \wedge id_{V'} \wedge f = id_{W' \coprod V'} \wedge f$ and thus is associative.

To define the appropriate map for the crosseffects, we must follow the map above with an assembly map. That is, f in $\Omega^U F(S^U)$ maps to the composite below, which is associative by the associativity of the assembly maps.

$$S^U_\perp \wedge S^U \xrightarrow{S^U_\perp \wedge f} S^U_\perp \wedge F(S^U) \xrightarrow{\alpha_F} F(S^U_\perp \wedge S^U).$$

We will introduce new notation to save some ink in the definition of the general $\mu_{F,G}$. If $U, V_1, \ldots, V_k$ are finite sets, let $S^{\underline{V}}$ denote the k-tuple of spheres $(S^{V_1}, \ldots, S^{V_k})$ and let $S^{U \coprod \underline{V}} = (S^{U \coprod V_1}, \ldots, S^{U \coprod V_k})$. Then we may define the map

$$\operatorname*{hocolim}_{U_1,\ldots U_k \in \mathbb{I}} \Omega^{\coprod U_i} cr_k F(S^{\underline{U}}) \wedge \operatorname*{hocolim}_{V_1^1,\ldots,V_{j_1}^1 \in \mathbb{I}} \Omega^{\coprod V_i^1} cr_{j_1} G(S^{\underline{V^1}}) \wedge \cdots \wedge \operatorname*{hocolim}_{V_1^k,\ldots V_{j_k}^k \in \mathbb{I}} \Omega^{\coprod V_i^k} cr_{j_k} G(S^{\underline{V^k}})$$

$$\downarrow \alpha_{\text{hocolim}}$$

$$\operatorname*{hocolim}_{U_1,\ldots U_k \in \mathbb{I}} \operatorname*{hocolim}_{V_1^1,\ldots,V_{j_1}^1 \in \mathbb{I}} \cdots \operatorname*{hocolim}_{V_1^k,\ldots V_{j_k}^k \in \mathbb{I}} \Omega^{\coprod U_i} cr_k F(S^{\underline{U}}) \wedge \Omega^{\coprod V_i^1} cr_{j_1} G(S^{\underline{V^1}}) \wedge \cdots \wedge \Omega^{\coprod V_i^k} cr_{j_k} G(S^{\underline{V^k}})$$

$$\downarrow \alpha_\Omega$$

$$\operatorname*{hocolim}_{U_1,\ldots U_k, V_1^1,\ldots,V_{j_k}^k \in \mathbb{I}} \Omega^{\coprod U_i} \Omega^{\coprod V_i^1} \ldots \Omega^{\coprod V_i^k} cr_k F(S^{\underline{U}}) \wedge cr_{j_1} G(S^{\underline{V^1}}) \wedge \cdots \wedge cr_{j_k} G(S^{\underline{V^k}})$$

$$\downarrow U_i \mapsto \coprod_{j_i} U_i$$

$$\operatorname*{hocolim}_{U_i, V_1^1,\ldots,V_{j_k}^k \in \mathbb{I}} \Omega^{\coprod \coprod_{j_i} U_i} \Omega^{\coprod V_i^1} \ldots \Omega^{\coprod V_i^k} cr_k F(S^{\coprod_{j_1} U_1}, \ldots, S^{\coprod_{j_n} U_k}) \wedge cr_{j_1} G(S^{\underline{V^1}}) \wedge \cdots \wedge cr_{j_k} G(S^{\underline{V^k}})$$

$$\downarrow \alpha_{cr_k F}$$

$$\operatorname*{hocolim}_{U_i,V_1^1,\dots,V_{j_k}^k\in\mathbb{I}} \Omega^{\amalg \amalg_{j_i} U_i}\Omega^{\amalg V_i^1}\dots\Omega^{\amalg V_i^k} cr_k F(S^{\amalg_{j_1} U_1}\wedge cr_{j_1}G(S^{\underline{V}^1}),\dots,S^{\amalg_{j_n} U_k}\wedge cr_{j_k}G(S^{\underline{V}^k}))$$

$$\downarrow \alpha_{cr_{j_i}G}$$

$$\operatorname*{hocolim}_{U_i,V_1^1,\dots,V_{j_k}^k\in\mathbb{I}} \Omega^{\amalg(U_1\amalg V_i^1)}\dots\Omega^{\amalg(U_k\amalg V_i^k)} cr_k F(cr_{j_1}G(S^{U_1\amalg\underline{V}^1}),\dots,cr_{j_k}G(S^{U_k\amalg\underline{V}^k}))$$

$$\downarrow \gamma_{cr}$$

$$\operatorname*{hocolim}_{W_1,\dots,W_j\in\mathbb{I}} \Omega^{\amalg W_i} cr_j(F\circ G)(S^{W_1},\dots,S^{W_j}))$$

Note that the assembly maps are equivariant and associative, as is the map described in (4.1), the crosseffect map from proposition 4.1.6 is also equivariant with respect to permuting the variables, so the map defined is equivariant and associative. □

The rest of this section is dedicated to defining a spectrum level description of the derivatives which agrees with Goodwillie's definition and maintains monoidicity.

Definition 4.2.6. Let $\partial_n F$ be the spectrum defined in level ℓ by

$$(\partial_n F)_\ell = \operatorname*{hocolim}_{U_1,\dots U_n\in\mathbb{I}} \Omega^{\amalg U_i}\Sigma^\ell cr_n F(\Sigma^{U_1}S^0,\dots\Sigma^{U_n}S^0).$$

Lemma 4.2.7. *If F is analytic, $\Omega^\infty\,\partial_k F\simeq\partial_k F$.*

Proof. Since $\Omega^\infty\partial_k F = \operatorname{hocolim}_{\mathbb{N}}\Omega^\ell\operatorname{hocolim}_{\mathbb{I}^k}\Omega^{\amalg_i V_i}\Sigma^\ell cr_k F(S^{\underline{V}})$, and F is analytic, we may consider the homotopy colimits over $\mathbb{N}$, so we can interchange loops and directed homotopy colimits. Thus we want to show that the following map is an equivalence

$$\operatorname*{hocolim}_{\mathbb{N}^k}\Omega^{\Sigma_i v_i} cr_k F(S^V)\to\operatorname*{hocolim}_{\mathbb{N}^{k+1}}\Omega^{\Sigma_i v_i}\Omega^\ell\Sigma^\ell cr_k F(S^V)$$

Suppose F is analytic and satisfies $E_{k-1}(c,\kappa)$. The k-cube $\mathcal{X}: U\mapsto\bigvee_{j\notin U}S^{v_j}$ is strongly cocartesian with v_s-connected maps $\mathcal{X}(\varnothing)\to\mathcal{X}(\{s\})$, so $F(\mathcal{X})$ is $\Sigma_i v_i - c$-cartesian, and the total homotopy fiber of $F(\mathcal{X})$ is $\Sigma_i v_i - c - 1$-connected, that is, $cr_k F(S^{v_1},\dots,S^{v_k})$ is $\Sigma_i v_i - c - 1$-connected.

By the Blakers-Massey theorem, the map $cr_k F(S^V)\to\Omega\Sigma cr_k F(S^V)$ is $2(\Sigma_i v_i - c - 1) - 1$-connected, so the map $cr_k F(S^V)\to\Omega^\ell\Sigma^\ell cr_k F(S^V)$ is also $2(\Sigma_i v_i - c - 1) - 1$-connected.

Thus the map $\Omega^{\Sigma_i v_i} cr_k F(S^V)\to\Omega^{\Sigma_i v_i}\Omega^\ell\Sigma^\ell cr_k F(S^V)$ is $\Sigma_i v_i + 2(-c-1) - 1$-connected, so as $v_i\to\infty$, the map on homotopy colimits becomes an equivalence. □

Lemma 4.2.8. *If F is analytic, $\partial_k^G F\simeq\partial_k F$.*

Proof. Recall from [Goo03] that Goodwillie defines the ℓth level of $\partial_k^G F$ to be

$$(\partial_k^G F)_\ell = \Omega^{\overline{V_k}\ell} cr_k F(S^\ell, \dots, S^\ell) \cong \Omega^{(k-1)\ell} cr_k F(S^\ell, \dots, S^\ell)$$

where $\overline{V_k}$ is the reduced standard representation of Σ_k, so has dimension $k-1$. The equivalent associated Ω-spectrum is given in level ℓ by

$$\operatorname*{hocolim}_{u\in\mathbb{N}} \Omega^u \Omega^{(k-1)(\ell+u)} cr_k F(S^{\ell+u}, \dots, S^{\ell+u}) \cong \operatorname*{hocolim}_{u\in\mathbb{N}} \Omega^{k(\ell+u)} \Omega^{-\ell} cr_k F(S^{\ell+u}, \dots, S^{\ell+u})$$

Reindexing by $t = \ell + u$ and using the Blakers-Massey argument of Lemma 4.2.7, this is equivalent to

$$\operatorname*{hocolim}_{t\in\mathbb{N}} \Omega^{kt} \Sigma^\ell cr_k F(S^t, \dots, S^t)$$

Finally, the diagonal map and lemma 4.2.2 show that this is equivalent to $(\partial_n F)_\ell$. □

Theorem 4.2.9. *The model for $\partial_* : [\mathcal{T}, \mathcal{T}] \to [\Sigma, \mathbb{S}\mathrm{p}]$ given in Definition 4.2.6 is monoidal.*

Proof. This is an easy modification of the proof of Theorem 4.2.3. □

We take a moment now to reiterate some consequences of this theorem, compare it with existing work in the area, and discuss its limits. The derivatives of a monad, for example the identity functor, have the structure of an operad and derivatives of other functors (or modules over monads) inherit a module structure over the derivatives of the identity (respectively, the monad). In particular, there is an operad structure on the derivatives of the monad associated to any operad. The operad structure is explicit; this differs from the proof that $\partial_*^G Id_{\mathcal{T}}$ forms an operad in the literature ([Joh95, AM99, Chi05]), which utilizes a cooperad structure on dual spectra.

We hope to generalize theorem 4.2.9 to functors between other categories to get natural operad and module structures in new settings. For example, the derivatives of the identity functor in the category of algebras over an operad are conjectured to be equivalent to the operad itself, but this has not been shown explicitly (see [Per13] for the proof of equivalence as symmetric sequences). It is interesting that we have found a simple solution for endofunctors of spaces, while the easier case in [AC11] was endofunctors of spectra, and the results for spaces were acheived using adjunctions. Because of our heavy use of assembly maps, we do not expect this theorem to transfer as is to endofunctors of spectra; the only spectral functors with assembly are linear. We will also use adjunctions to extend to other settings (see the comments at the end of section 4.3).

4.3 Chain rule

In this section, we will prove a chain rule for reduced, finitary, analytic endofunctors of spaces. This has advantages to the chain rule of [AC11], in that the monoid map $\partial_* F \circ \partial_* G \to \partial_*(F \circ G)$ defines a spectrum level map on the derived composition product $\partial_* F \circ_{\partial_* Id} \partial_* G \to \partial_*(F \circ G)$ instead of having a map only in the homotopy category.

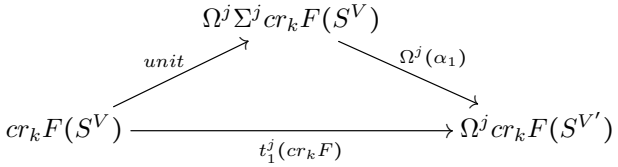
We will need the following technical lemma in the proof of the chain rule.

Lemma 4.3.1. *∂_k preserves fiber sequences of analytic functors.*

Proof. Let $F \to G \to H$ be a fiber sequence of analytic endofunctors of spaces. Then $F(S^n) \to G(S^n) \to H(S^n)$ is a fiber sequence for all S^n, and if $V = (v_1, \dots, v_k) \in \mathbb{N}^k$ and $S^V = (S^{v_1}, \dots, S^{v_k})$, then $(F \circ \bigsqcup_k)(S^V) \to (G \circ \bigsqcup_k)(S^V) \to (H \circ \bigsqcup_k)(S^V)$ is again a fiber sequence. Since cr_k is a total fiber (and a right adjoint), $cr_k F(S^V) \to cr_k G(S^V) \to cr_k H(S^V)$ is still a fiber sequence. It remains a fiber sequence after looping as many times as desired and taking a filtered colimit; that is, the following is a fiber sequence for any $V' \in \mathbb{N}^k$

$$\operatorname*{hocolim}_{V \in \mathbb{N}^k} \Omega^{\Sigma_i v_i} cr_k F(S^{V'}) \to \operatorname*{hocolim}_{V \in \mathbb{N}^k} \Omega^{\Sigma_i v_i} cr_k G(S^{V'}) \to \operatorname*{hocolim}_{V \in \mathbb{N}^k} \Omega^{\Sigma_i v_i} cr_k H(S^{V'})$$

Consider the assembly map for the kth cross effect in the first variable $\Sigma^j cr_k F(S^V) \to cr_k F(S^{V'})$, where $v_1' = v_1 + j$ and $v_i' = v_i$ for $1 < i \leq k$. We will show that this map induces a weak equivalence in the homotopy colimit. The assembly map in the first variable, α_1, is part of a factorization of the map $cr_k F(S^V) \to \Omega^j cr_k F(S^{V'})$. That is,



If F is analytic satisfying $E_{k-1}(c, \kappa)$, then $cr_k F(S^V)$ is $(\Sigma_i v_i - c - 1)$-connected and as above, the unit map is $(2(\Sigma_i v_i - c - 1) - 1)$-connected by Blakers-Massey theorem. By applying $cr_k F(-, S^{v_2}, \dots, S^{v_k})$ to the cocartesian diagram $* \leftarrow S^{v_1} \to *$ and using that $cr_k F$ satisfies $E_n(c + k - 1, \kappa)$ for all n, we get that $t_1(cr_k F)(S^V)$ is $2v_1 - (c+k-1)$-connected. The iterations increase in connectivity, so the map $t_1^j(cr_k F)(S^V)$ is also $2v_1 - (c + k - 1)$-connected. Thus $\Omega^j(\alpha_1)$ is $\min(2\Sigma_i v_i - 2c - 3, 2v_1 - c - k + 1)$-connected, or $2v_1 - C$-connected.

Then $\Omega^{\Sigma_i v_i} \Omega^j(\alpha_1)$ is $v_1 - (v_2 + \cdots + v_k) - C$-connected, and the homotopy colimit over $V_1 \to \infty$ gives an equiv-

alence. Since the homotopy colimit is a homotopy functor, $\operatorname{hocolim}_V \Omega^{\Sigma_i v_i}\Sigma^j cr_k F(S^V) \to \operatorname{hocolim}_V \Omega^{\Sigma_i v_i} cr_k F(S^{V'})$ is an equivalence.

Since F, G, and H are analytic, the filtered colimit is equivalent to the homotopy colimit over $\mathbb{I}$, so we have shown in the following diagram that the top row is a fiber sequence and the vertical maps are equivalences, so the bottom row is also a fiber sequence (where here $v_1' = v_1 + \ell$).

$$\begin{array}{ccccc}
\operatorname*{hocolim}_{V\in\mathbb{N}^k} \Omega^{\Sigma_i v_i} cr_k F(S^{V'}) & \longrightarrow & \operatorname*{hocolim}_{V\in\mathbb{N}^k} \Omega^{\Sigma_i v_i} cr_k G(S^{V'}) & \longrightarrow & \operatorname*{hocolim}_{V\in\mathbb{N}^k} \Omega^{\Sigma_i v_i} cr_k H(S^{V'}) \\
\uparrow\simeq & & \uparrow\simeq & & \uparrow\simeq \\
\operatorname*{hocolim}_{V\in\mathbb{I}} \Omega^{\Sigma_i v_i}\Sigma^\ell cr_k F(S^V) & \longrightarrow & \operatorname*{hocolim}_{V\in\mathbb{I}} \Omega^{\Sigma_i v_i}\Sigma^\ell cr_k G(S^V) & \longrightarrow & \operatorname*{hocolim}_{V\in\mathbb{I}} \Omega^{\Sigma_i v_i}\Sigma^\ell cr_k H(S^V) \\
\uparrow\simeq & & \uparrow\simeq & & \uparrow\simeq \\
(\partial_k F)_\ell & \longrightarrow & (\partial_k G)_\ell & \longrightarrow & (\partial_k H)_\ell
\end{array}$$

□

Note that if ∂ preserves fiber sequences of functors, then it also preserves finite products of endofunctors of spaces, using the fiber sequence $F \to F \times G \to G$.

For the rest of this chapter, we drop the ∂ notation, and let $\partial_* F$ denote the new definition of the derivatives presented here, and we denote Goodwillie's definition by $\partial_*^G F$.

Theorem 4.3.2. *Let $F, G : \mathcal{T} \to \mathcal{T}$ be reduced, analytic, finitary functors. The natural map $\partial_* F \circ_{\partial_* Id} \partial_* G \to \partial_*(F \circ G)$ is an equivalence.*

Proof. First, consider the case when X is a finite CW complex and $F = \operatorname{Hom}(X, -)$. We may filter X by its skeleta $X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n = X$, and $\operatorname{Hom}(X, -) = \operatorname{Hom}(X_n, -)$.

The skeletal filtration of X gives cofiber sequences $X_i \to X_{i+1} \to \bigvee S^{i+1}$, and these yield fiber sequences

$$\operatorname{Hom}(\bigvee S^{i+1}, -) \to \operatorname{Hom}(X_{i+1}, -) \to \operatorname{Hom}(X_i, -).$$

Since ∂_* preserves fiber sequences (by lemma 4.3.1), we have fiber sequences in spectra:

$$\partial_k \operatorname{Hom}(\bigvee S^{i+1}, -) \to \partial_k \operatorname{Hom}(X_{i+1}, -) \to \partial_k \operatorname{Hom}(X_i, -)$$

Taken together (for all k), the derivatives form a fiber sequence of symmetric sequences in spectra. Since the derivatives land in spectra, each level is also a cofiber sequence, so together they also form a cofiber sequence of symmetric sequences in spectra. The bar construction $B(-, \partial_* Id, \partial_* G)$ preserves cofiber

sequences, and thus the following is a fiber sequence of symmetric sequences in spectra:

$$\partial_* \operatorname{Hom}(\bigvee S^{i+1}, -) \circ_{\partial_* Id} \partial_* G \longrightarrow \partial_* \operatorname{Hom}(X_{i+1}, -) \circ_{\partial_* Id} \partial_* G \longrightarrow \partial_* \operatorname{Hom}(X_i, -) \circ_{\partial_* Id} \partial_* G$$

Similarly, since $\operatorname{Hom}(\bigvee S^{i+1}, -) \circ G \to \operatorname{Hom}(X_{i+1}, -) \circ G \to \operatorname{Hom}(X_i, -) \circ G$ is a fiber sequence, we get another fiber sequence of symmetric sequences in spectra and the maps $\mu_{F,G}$ described in Theorem 4.2.3 yield a map of fiber sequences:

$$\begin{array}{ccccc}
\partial_* \operatorname{Hom}(\bigvee S^{i+1}, -) \circ_{\partial_* Id} \partial_* G & \longrightarrow & \partial_* \operatorname{Hom}(X_{i+1}, -) \circ_{\partial_* Id} \partial_* G & \longrightarrow & \partial_* \operatorname{Hom}(X_i, -) \circ_{\partial_* Id} \partial_* G \\
\downarrow{\scriptstyle \mu_{\operatorname{Hom}(\vee S^{i+1}, -), G}} & & \downarrow{\scriptstyle \mu_{\operatorname{Hom}(X_{i+1}, -), G}} & & \downarrow{\scriptstyle \mu_{\operatorname{Hom}(X_i, -), G}} \\
\partial_*(\operatorname{Hom}(\bigvee S^{i+1}, -) \circ G) & \longrightarrow & \partial_*(\operatorname{Hom}(X_{i+1}, -) \circ G) & \longrightarrow & \partial_*(\operatorname{Hom}(X_i, -) \circ G)
\end{array}$$

First, we will show that the left vertical map is an equivalence. Recall that $\bigvee_\beta S^i \to * \to \bigvee_\beta S^{i+1}$ is a cofiber sequence of spaces, so again we get a map of fiber sequences

$$\begin{array}{ccccc}
\partial_* \operatorname{Hom}(\bigvee S^{i+1}, -) \circ_{\partial_* Id} \partial_* G & \longrightarrow & \partial_* \operatorname{Hom}(*, -) \circ_{\partial_* Id} \partial_* G & \longrightarrow & \partial_* \operatorname{Hom}(\bigvee S^{i}, -) \circ_{\partial_* Id} \partial_* G \\
\downarrow & & \downarrow & & \downarrow \\
\partial_*(\operatorname{Hom}(\bigvee S^{i+1}, -) \circ G) & \longrightarrow & \partial_*(\operatorname{Hom}(*, -) \circ G) & \longrightarrow & \partial_*(\bigvee \operatorname{Hom}(S^{i}, -) \circ G)
\end{array}$$

The middle terms are trivial, so the middle map is an equivalence.

We will show that the right map is an equivalence in the base case, $i = 0$. When $i = 0$ and $\beta = 1$, $\operatorname{Hom}(S^0, -) = Id$, and we can build a contracting simplicial homotopy $\partial_* G \xrightarrow{\simeq} \partial_* Id \circ_{\partial_* Id} \partial_* G$. For $\beta > 1$, $\operatorname{Hom}(\bigvee_\beta S^0, -) \cong \prod_\beta Id$, so we want to show an equivalence

$$\partial_*(\prod Id) \circ_{\partial_* Id} \partial_* G \to \partial_*(\prod Id \circ G) = \partial_*(\prod G).$$

The identity functor is analytic, as are products of the identity, so $\partial^G_*(\prod_\beta Id) \simeq \partial_*(\prod_\beta Id)$. The cross-effect functors and Ω commute with products, and filtered homotopy colimits commute with finite limits, so $\partial^G_*(\prod_\beta Id) \simeq \prod_\beta \partial^G_*(Id)$. Similarly, when G is analytic, $\prod G$ is also analytic, so $\partial_*(\prod G) \simeq \partial^G_*(\prod G) \simeq \prod \partial^G_* G \simeq \prod \partial_* G$.

We now show that the bar construction of symmetric sequences in spectra commutes with finite products

$$(\prod \partial_* Id) \circ_{\partial_* Id} \partial_* G \xrightarrow{\simeq} \prod(\partial_* Id \circ_{\partial_* Id} \partial_* G).$$

Since the product of symmetric sequences is defined levelwise, we can consider the product in spectra, so we have the cartesian cocartesian square

$$\begin{array}{ccc} \partial_* Id \times \partial_* Id & \longrightarrow & \partial_* Id \\ \downarrow & & \downarrow \\ \partial_* Id & \longrightarrow & * \end{array}$$

Since the square is cocartesian, it remains cocartesian after applying the bar construction so we have the cartesian square

$$\begin{array}{ccc} (\partial_* Id \times \partial_* Id) \circ_{\partial_* Id} \partial_* G & \longrightarrow & \partial_* Id \circ_{\partial_* Id} \partial_* G \\ \downarrow & & \downarrow \\ \partial_* Id \circ_{\partial_* Id} \partial_* G & \longrightarrow & * \end{array}$$

Thus $(\partial_* Id \times \partial_* Id) \circ_{\partial_* Id} \partial_* G \simeq (\partial_* Id \circ_{\partial_* Id} \partial_* G) \times (\partial_* Id \circ_{\partial_* Id} \partial_* G)$.

Finally, we use the equivalence $\partial_* Id \circ_{\partial_* Id} \partial_* G \xrightarrow{\simeq} \partial_* G$ given by contracting homotopy on each factor. We have shown

$$\partial_*(\prod Id) \circ_{\partial_* Id} \partial_* G \simeq (\prod \partial_* Id) \circ_{\partial_* Id} \partial_* G \simeq \prod(\partial_* Id \circ_{\partial_* Id} \partial_* G) \simeq \prod(\partial_* G) \simeq \partial_*(\prod G).$$

Thus the base case $i = 0$ for the right vertical map is an equivalence. A map of fiber sequences in spectra in which two maps are equivalences yields an equivalence on the third map. Thus induction shows that $\mu_{\mathrm{Hom}(\vee S^i, -), G}$ is an equivalence for all i.

In the original map of fiber sequences, we have that the left map is an equivalence and by induction the right map is an equivalence (the base case is taken care of by the base case for spheres) and so induction yields the chain rule for representable functors

$$\partial_* \mathrm{Hom}(X_{i+1}, -) \circ_{\partial_* Id} \partial_* G \xrightarrow{\simeq} \partial_*(\mathrm{Hom}(X_{i+1}, -) \circ G).$$

Arone and Ching show in [AC15] that a cofibrant model for the derivatives of representable functors can be extended to a model for all functors.

By Proposition 4.23 of [Kel05], any cofibrant functor is equivalent to its left Kan extension along the identity functor, so we may rewrite $F(-) \simeq \mathrm{Hom}(X, -) \wedge_{X \in Top} F(X)$.

Then

$$\partial_* F \circ_{\partial_* Id} \partial_* G \simeq (\partial_* \mathrm{Hom}(X, -) \circ_{\partial_* Id} \partial_* G) \wedge_{X \in Top} F(X) \simeq \partial_* \mathrm{Hom}(X, G(-)) \wedge_{X \in Top} F(X) \simeq \partial_*(F \circ G).$$

Thus the chain rule extends to all (analytic, finitary) functors built out of representable functors. □

We have recovered the results of [AC11], but just for endofunctors of spaces. To extend to functors $F : \mathcal{C} \to \mathcal{D}$ between the categories of spaces and spectra, we can use the adjunction $\Sigma^\infty : \mathcal{T} \rightleftarrows \mathcal{S}\mathrm{p} : \Omega^\infty$. The derivatives of a functor as defined by Goodwillie are equivalent as a symmetric sequence to the derivatives of the associated endofunctor of spaces. For $G : \mathcal{S}\mathrm{p} \to \mathcal{T}$, for example, $\partial_*^G G \simeq \partial_*(G\Sigma^\infty)$ as symmetric sequences. Similarly, for functors $G : \mathcal{T} \to \mathcal{S}\mathrm{p}$, the derivatives $\partial_*^G G \simeq \partial_*(\Omega^\infty G)$ are equivalent as symmetric sequences. For this model of the derivatives to work, we need a category of spectra with all objects fibrant (we could use EKMM spectra), and a category of spaces with all objects cofibrant (we could use simplicial sets). Using simplicial sets would require checking that all maps in chapter 4 are simplicial maps. Once this is done, any simplicial category $\mathcal{C}$ with fibrant objects which has an adjunction with spaces $c \wedge - : \mathcal{T} \rightleftarrows \mathcal{C} : \mathrm{Hom}(c, -)$ for $c \in \mathcal{C}$ will have a monoidal model for derivatives of its endofunctors and a chain rule for functors built from representables.

Appendix A

More on $\mathbb{I}$

A.1 Some lemmas about $\mathbb{I}$

In these appendices, we will give conditions on an endofunctor that ensure that its iterates receive a functor from the category $\mathbb{I}$. We will also show that Goodwillie's string of T_n's actually fits into an $\mathbb{I}$ diagram. Finally in the last section, we give a proof of Bökstedt's lemma 2.3.2.

Let $\mathbb{I}$ be the category with objects given by finite ordered sets $\mathbf{n} = \{0, 1, \ldots, n\}$ and with morphisms given by injective maps. Let $\mathbb{N}$ be the ("wide") subcategory with the same objects but only the standard inclusions $i_{n-1} : \mathbf{n-1} \hookrightarrow \mathbf{n}$. Let Σ denote the wide subcategory of $\mathbb{I}$ consisting of all the isomorphisms. (Σ is the groupoid $\coprod \Sigma_n$.)

We like to visualize the category $\mathbb{I}$ as follows, where the category $\mathbb{N}$ is the subcategory with only the horizontal arrows:

$$\varnothing \longrightarrow \mathbf{0} \xrightarrow{i_0} \mathbf{1} \xrightarrow{i_1} \mathbf{2} \xrightarrow{i_2} \mathbf{3} \xrightarrow{i_3} \cdots$$

with Σ_2, Σ_3, Σ_4 acting on $\mathbf{1}$, $\mathbf{2}$, $\mathbf{3}$ respectively.

Since $\mathbb{I}$ has all inclusions, there are two maps $\mathbf{0} \to \mathbf{1}$ given by $i_0 : 0 \mapsto 0$ and the map $0 \mapsto 1$, but this information is encoded by the Σ_2 action τ on $\mathbf{1} = \{0, 1\}$, because the map $0 \mapsto 1$ is a composition $\tau i_0 : 0 \mapsto 0 \mapsto 1$. (This should give some intuition for why we need to include the empty set for $B\mathbb{I}$ to be contractible.)

Let $F : \mathcal{C} \to \mathcal{C}$ be an endofunctor on a category $\mathcal{C}$. We will give conditions for when there is a well-defined functor from the category $\mathbb{I}$ to the iterates of F, which we denote by $F^\bullet$.

$$Id \longrightarrow F \xrightarrow{i_0} F \circ F \xrightarrow{i_1} F^{\circ 3} \xrightarrow{i_2} F^{\circ 4} \xrightarrow{i_3} \cdots$$

with Σ_2, Σ_3, Σ_4 acting on $F \circ F$, $F^{\circ 3}$, $F^{\circ 4}$ respectively.

First, we would like to define a Σ_n-action on $F^{\circ n}$, or equivalently, give a functor $\Psi : \Sigma \to F^\bullet$ which sends $\mathbf{n}$ to the functor $F^{\circ|\mathbf{n}|} = F^{\circ(n+1)}$ and sends each morphism $s \in \Sigma_{|\mathbf{n}|}$ to a natural transformation $\Psi(s) : F^{n+1} \to F^{n+1}$. The following lemma shows that by functoriality, such a functor is generated by a well

chosen $\Psi((12))$, where (12) is the nontrivial element of Σ_2. This choice, σ, can be used to define images of the other transpositions in Σ_n, for example, $\sigma_F : F^3 \to F^3$ swaps the first two copies of F. Thus σ will also need to satisfy the braiding condition, which essentially says that the images of the two ways of writing (13) as a product of transpositions are equal.

Lemma A.1.1. *Let $F : \mathcal{C} \to \mathcal{C}$ be an endofunctor on $\mathcal{C}$. Let $\sigma : F \circ F \to F \circ F$ be a natural transformation such that $\sigma \circ \sigma = id$ and $F(\sigma) \circ \sigma_F \circ F(\sigma) = \sigma_F \circ F(\sigma) \circ \sigma_F$ (the braiding condition is satisfied). Then there is a well-defined functor $\Psi : \Sigma \to F^\bullet$*

Proof. Recall that the transpositions $\tau_k = (k, k+1)$ generate Σ_n for $0 \le k < n-1$, with relations

$$\begin{aligned} (\tau_i)^2 &= id \\ \tau_i\tau_j &= \tau_j\tau_i && \text{if } j \neq i \pm 1 \\ \tau_i\tau_{i+1}\tau_i &= \tau_{i+1}\tau_i\tau_{i+1} \end{aligned}$$

We will define $\Psi(\tau_k) : F^{\circ n} \to F^{\circ n}$ by using only a Σ_2-action. Let $\sigma : F \circ F \to F \circ F$ be a natural transformation such that $\sigma \circ \sigma = id$, so σ defines an action of Σ_2 on $F \circ F$. We can define symmetric group actions on $F \circ F \circ F$ by $F(\sigma)$ and σ_F where $\sigma_F(X) : F \circ F(F(X)) \to F \circ F(F(X))$ is the natural transformation σ applied to $F(X)$. These are analogous to the transpositions (01) and (12) of Σ_3.

For $n > 1$, define:

$$\Psi(\tau_k) := F^{n-k-2}\sigma_{F^k} \quad \text{for } 0 \le k \le n-2$$

So when $n = 2$, $\Psi(\tau_0) = \sigma$. We will show that Ψ preserves the relations, so Ψ defines a functor. For the rest of the proof, we'll drop Ψ from the notation.

First, $\sigma^2 = id$ so we can see that $(\tau_{n-2})^2 = (\sigma_{F^{n-2}})^2 = id$ for all $n > 1$. Since F is a functor, we have $(\tau_i)^2 = F^{n-i-2}(\sigma_{F^i})F^{n-i-2}(\sigma_{F^i}) = F^{n-i-2}((\sigma_{F^i})^2) = F^{n-i-2}(id) = id$ for $0 \le i < n-2$ and for all $n > 2$.

The proof of the second relation uses functoriality of F and naturality of σ. Let $j > i+1$. Then

$$\tau_i\tau_j = F^{n-i-2}\sigma_{F^i} \circ F^{n-j-2}\sigma_{F^j} = F^{n-j-2}F^{j-i}\sigma_{F^i} \circ F^{n-j-2}\sigma_{F^j} = F^{n-j-2}(F^{j-i}\sigma_{F^i} \circ \sigma_{F^j})$$

$$\overset{*}{=} F^{n-j-2}(\sigma_{F^j} \circ F^{j-i}\sigma_{F^i}) = F^{n-j-2}\sigma_{F^j} \circ F^{n-i-2}\sigma_{F^i} = \tau_j\tau_i$$

The starred equality holds by naturality of σ, displayed in the following diagram:

$$\begin{array}{ccc} F^{j+2} & \xrightarrow{\sigma_{F^j}} & F^{j+2} \\ {\scriptstyle F^{j-i}\sigma_{F^i}}\downarrow & & \downarrow{\scriptstyle F^{j-i}\sigma_{F^i}} \\ F^{j+2} & \xrightarrow{\sigma_{F^j}} & F^{j+2} \end{array}$$

Finally, we need to show the braiding condition, that is, $\tau_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\tau_{i+1}$.

$$\tau_i\tau_{i+1}\tau_i = F^{n-i-2}\sigma_{F^i} \circ F^{n-i-3}\sigma_{F^{i+1}} \circ F^{n-i-2}\sigma_{F^i} = F^{n-i-3}(F\sigma_{F^i} \circ \sigma_{F^{i+1}} \circ F\sigma_{F^i}) = F^{n-i-3}(F(\sigma) \circ \sigma_F \circ F(\sigma))(F^i)$$

$$\overset{*}{=} F^{n-i-3}(\sigma_F \circ F(\sigma) \circ \sigma_F)(F^i) = F^{n-i-3}(\sigma_{F^{i+1}} \circ F\sigma_{F^i} \circ \sigma_{F^{i+1}}) = F^{n-i-3}\sigma_{F^{i+1}} \circ F^{n-i-2}\sigma_{F^i} \circ F^{n-i-3}\sigma_{F^{i+1}} = \tau_{i+1}\tau_i\tau_{i+1}$$

The starred equality is the only thing we could not prove without assumption; that is, $F(\sigma) \circ \sigma_F \circ F(\sigma) = \sigma_F \circ F(\sigma) \circ \sigma_F$ does not come for free. This amounts to saying that the two ways of swapping the first and third functor are equal. □

Let $\mathbf{\Delta}$ be the category of finite ordered sets with order-preserving maps. Let $\mathbf{\Delta}^{inj}$ be the (wide) subcategory of finite ordered sets with order-preserving injections. Any morphism in this category can be uniquely written as a composition $d^{i_1} \cdots d^{i_k}$ such that $i_1 \leq i_2 \leq \cdots \leq i_k$.

Lemma A.1.2. *Let $\eta : Id \to F$ be a natural transformation. Then there is a well defined functor $\Phi : \mathbf{\Delta}^{inj} \to F^\bullet$.*

Proof. Again, since Δ is generated by d^i, we will define the image of Φ on these generators and show that the cosimplicial identity $d^i d^j = d^j d^{i-1}$ for $i > j$ holds.

Define:

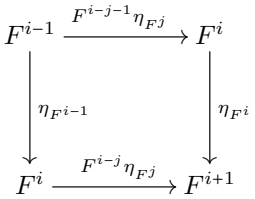
$$\Phi(d^i) = F^{n-i}\eta_{F^i} : F^n \to F^{n+1} \text{ for } 0 \leq i \leq n$$

If $i > j$, then

$$d^i d^j = F^{n+1-i}\eta_{F^i} \circ F^{n-j}\eta_{F^j} = F^{n+1-i}(\eta_{F_i} \circ F^{i-j-1}\eta_{F^j})$$

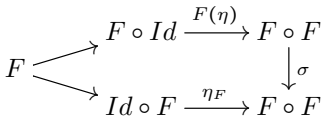
$$\overset{*}{=} F^{n+1-i}(F^{i-j}\eta_{F^j} \circ \eta_{F^{i-1}}) = F^{n+1-j}\eta_{F^j} \circ F^{n-i+1}\eta_{F^i} = d^j d^{i-1}$$

We have the starred equality because the following commutes by naturality for $i > j$:



□

Suppose we have σ and η as above (satisfying the hypotheses of lemmas 1 and 2), and they interact such that the following diagram commutes:



That is, $\sigma \circ F(\eta) = \eta_F$ and so then also $F(\eta) = \sigma \circ \eta_F$; we will call this condition $\star$. The next lemma shows that this is enough to define a functor $\mathbb{I}$ to the iterations of the functor F.

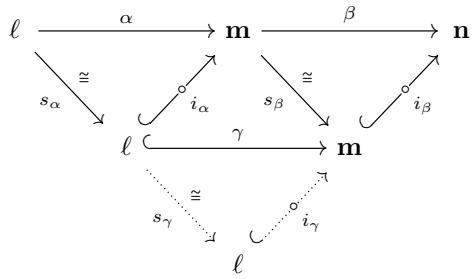
Lemma A.1.3. *If there are natural transformations $\sigma : F^2 \to F^2$ and $\eta : \mathbf{1} \to F$ such that*

- $\sigma^2 = id$
- *the braid condition $F(\sigma) \circ \sigma_F \circ F(\sigma) = \sigma_F \circ F(\sigma) \circ \sigma_F$ holds*
- *the star condition $\sigma \circ F(\eta) = \eta_F$ holds*

then there is a well-defined functor $\Theta : \mathbb{I} \to F^\bullet$.

Proof. Let $\alpha : \mathbf{n} \to \mathbf{m}$ be a morphism of $\mathbb{I}$, so α is an injection. Any set map can be factored as a surjection onto its image followed by an inclusion of the image into the codomain, so there is a canonical factorization of α into a bijection followed by an injection. Since α is injective, the surjection onto its image is also injective, so we may think of the bijection a straightening out of α so that the injection into the codomain is an order-preserving map, that is, a morphism of $\mathbf{\Delta}^{inj}$. We may write $\alpha = i_\alpha \circ s_\alpha$ uniquely, where s_α is a permutation and i_α is an order-preserving injection.

If $\ell \xrightarrow{\alpha} \mathbf{m} \xrightarrow{\beta} \mathbf{n}$ are injections then we may factor each as a permutation followed by an ordered inclusion. Then there is an injection $\gamma : im(s_\alpha) \to im(s_\beta)$ which factors as a permutation followed by an ordered inclusion, as shown with the dotted arrows in the following diagram.



So there is a unique map $\ell \to \mathbf{n}$ which is the composition of a permutation $s_{\beta\alpha} = s_\gamma \circ s_\alpha$ followed by an order-preserving injection $i_{\beta\alpha} = i_\beta \circ i_\gamma$. We will define a functor $\Theta : \mathbb{I} \to F^\bullet$ which takes $\mathbf{n}$ to the functor F^{n+1}. To an injection $\alpha = i_\alpha \circ s_\alpha : \mathbf{m} \to \mathbf{n}$, the functor Θ will assign the composition $\Theta(\alpha) = \Phi(i_\alpha) \circ \Psi(s_\alpha) : F^{m+1} \to F^{n+1}$.

The functor Θ preserves the identity morphism because it is a composition of functors. To show that Θ preserves composition, by functoriality, we must only check the questionable equality below:

$$\Theta(\beta\alpha) = \Phi(i_{\beta\alpha}) \circ \Psi(s_{\beta\alpha}) = \Phi(i_\beta i_\gamma) \circ \Psi(s_\gamma s_\alpha) = \Phi(i_\beta)\Phi(i_\gamma)\Psi(s_\gamma)\Psi(s_\alpha)$$

$$\overset{???}{=} \Phi(i_\beta)\Psi(s_\beta)\Phi(i_\alpha)\Psi(s_\alpha) = \Theta(\beta)\Theta(\alpha)$$

We may write $\Psi(s_\gamma)$ as a composition of $\Psi(\tau_j)$'s and $\Phi(i_\gamma)$ as a composition of $\Phi(d^i)$'s. In $\mathbb{I}$, we have the following relations for moving transpositions past ordered inclusions for $0 \le i \le n,\ 0 \le j \le n-2$:

$$d^i\tau_j = \begin{cases} \tau_j d^i & \text{if} \quad i > j+1 \\ \tau_{j+1} d^i & \text{if} \quad i \le j \\ \tau_j \tau_{j+1} d^j & \text{if} \quad i = j+1 \end{cases}$$

We will show that these hold for $\Psi(\tau_j)$'s and $\Phi(d^i)$'s; that is, we can move the transpositions past the ordered inclusions. Again, we will drop Ψ and Φ from the notation.

If $i > j+1$, then we are proving the commutativity of the following where $\tau_j : F^{n+1} \to F^{n+1}$ is given by $F^{n-j-1}\sigma_{F^j}$.

$$\begin{array}{ccc} F^n & \xrightarrow{\tau_j} & F^n \\ \downarrow{\scriptstyle d^i} & & \downarrow{\scriptstyle d^i} \\ F^{n+1} & \xrightarrow{\tau_j} & F^{n+1} \end{array}$$

$$d^i\tau_j = F^{n-i}\eta_{F^i} \circ F^{n-j-2}\sigma_{F^j} = F^{n-i}(\eta_{F^i} \circ F^{i-j-2}\sigma_{F^j})$$

$$\overset{*}{=} F^{n-i}(F^{i-j-1}\sigma_{F^j} \circ \eta_{F^i}) = F^{n-j-1}\sigma_{F^j} \circ F^{n-i}\eta_{F^i} = \tau_j d^i$$

The starred equality is given by commutativity of the following (because η is a natural transformation and misses the transposition since $i > j+1$.)

$$\begin{array}{ccc} F^i & \xrightarrow{F^{i-j-2}\sigma_{F^j}} & F^i \\ {\scriptstyle \eta_{F^i}}\downarrow & & \downarrow{\scriptstyle \eta_{F^i}} \\ F^{i+1} & \xrightarrow{F^{i-j-1}\sigma_{F^j}} & F^{i+1} \end{array}$$

If $i \leq j$, then

$$d^i\tau_j = F^{n-i}\eta_{F^i} \circ F^{n-j-2}\sigma_{F^j} = F^{n-j-2}(F^{j-i+2}\eta_{F^i} \circ \sigma_{F^j})$$

$$\overset{*}{=} F^{n-j-2}(\sigma_{F^{j+1}} \circ F^{j-i+2}\eta_{F^i}) = F^{n-j-2}\sigma_{F^{j+1}} \circ F^{n-i}\eta_{F^i} = \tau_{j+1}d^i$$

Where again the starred equality is by commutativity of the following (because σ is a natural transformation and misses η since $i < j$):

$$\begin{array}{ccc} F^{j+2} & \xrightarrow{\sigma_{F^j}} & F^{j+2} \\ {\scriptstyle F^{j-i+2}\eta_{F^i}}\downarrow & & \downarrow{\scriptstyle F^{j-i+2}\eta_{F^i}} \\ F^{j+3} & \xrightarrow{\sigma_{F^{j+1}}} & F^{j+3} \end{array}$$

Finally, by naturality of σ, $\sigma_F \circ F^2\eta = F^2\eta \circ \sigma$, then the $\star$ condition implies that $F\sigma \circ F^2\eta = F\eta_F$, so

$$d^{j+1}\tau_j = F^{n-j-1}\eta_{F^{j+1}} \circ F^{n-j-2}\sigma_{F^j} = F^{n-j-2}(F\eta_F \circ \sigma)(F^j)$$

$$\overset{\star}{=} F^{n-j-2}(F\sigma \circ \sigma_F \circ F^2\eta)(F^j) = F^{n-j-1}\sigma_{F^j} \circ F^{n-j-2}\sigma_{F^{j+1}} \circ F^{n-j}\eta_{F^j} = \tau_j\tau_{j+1}d^j$$

Thus $\Phi(i_\gamma)\Psi(s_\gamma) = d^{i_1}\cdots d^{i_k}\tau_{a_1}\cdots\tau_{a_m} = \tau_{a'_1}\cdots\tau_{a'_{m'}}d^{i'_1}\cdots d^{i'_{k'}}$ and the result must agree with $\Psi(s_\beta)\Phi(i_\alpha)$ by the uniqueness of the decompositions. □

A.2 T_n and $\mathbb{I}$

Now, we will show that the iterates of Goodwillie's T_n's fit into an $\mathbb{I}$ diagram.

Proposition A.2.1. *There is a well defined functor $\mathbb{I} \to T_n^\bullet F$.*

Proof. Let the natural tranformation $\sigma : T_nT_n \to T_nT_n$ be defined by the composite:

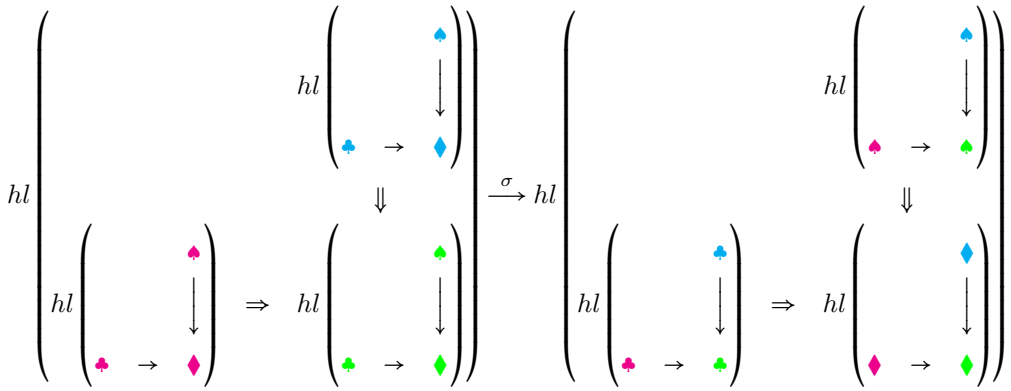
$$
\begin{aligned}
T_nT_nF(X) &= \operatorname*{holim}_{V\in\mathcal{P}_0(\mathbf{n})}\operatorname*{holim}_{U\in\mathcal{P}_0(\mathbf{n})} F((X*V)*U)\\
&\xrightarrow{\phi} \operatorname*{holim}_{V\in\mathcal{P}_0(\mathbf{n})}\operatorname*{holim}_{U\in\mathcal{P}_0(\mathbf{n})} F(X*(V*U))\\
&\xrightarrow{\psi} \operatorname*{holim}_{(V,U)\in\mathcal{P}_0(\mathbf{n})\times\mathcal{P}_0(\mathbf{n})} F(X*(V*U))\\
&\xrightarrow{\overline{\sigma}} \operatorname*{holim}_{(U,V)\in\mathcal{P}_0(\mathbf{n})\times\mathcal{P}_0(\mathbf{n})} F(X*(U*V))\\
&\xrightarrow{\psi^{-1}} \operatorname*{holim}_{U\in\mathcal{P}_0(\mathbf{n})}\operatorname*{holim}_{V\in\mathcal{P}_0(\mathbf{n})} F(X*(U*V))\\
&\xrightarrow{\phi^{-1}} \operatorname*{holim}_{U\in\mathcal{P}_0(\mathbf{n})}\operatorname*{holim}_{V\in\mathcal{P}_0(\mathbf{n})} F((X*U)*V)\\
&= T_nT_nF(X)
\end{aligned}
$$

where ϕ and ψ are the natural isomorphisms given by associativity of the join and consolidation of limits. We will define the map $\overline{\sigma}$ for categories $\mathcal{C}$ and $\mathcal{D}$ for clarity, but we will need $\mathcal{C}=\mathcal{D}=\mathcal{P}_0(\mathbf{n})$ to make sense of the functor.

Recall that given a map on indexing categories $\mathcal{D}\times\mathcal{C}\xrightarrow{\alpha}\mathcal{C}\times\mathcal{D}\xrightarrow{G}\mathcal{E}$, there is an induced map $\operatorname*{holim}_{\mathcal{C}\times\mathcal{D}} G \xrightarrow{\alpha_*} \operatorname*{holim}_{\mathcal{D}\times\mathcal{C}} G\alpha$. If G is a bifunctor which is symmetric in its arguments, then there is a natural transformation $G\alpha\to G$, and thus a map $\operatorname*{holim}_{\mathcal{D}\times\mathcal{C}} G\alpha\to\operatorname*{holim}_{\mathcal{D}\times\mathcal{C}} G$. We will define $\overline{\sigma}$ to be the composition $\operatorname*{holim}_{\mathcal{C}\times\mathcal{D}} G \xrightarrow{\alpha_*} \operatorname*{holim}_{\mathcal{D}\times\mathcal{C}} G\alpha\to\operatorname*{holim}_{\mathcal{D}\times\mathcal{C}} G$. The join of two sets is symmetric, so if $G(U,V)=U*V$, there is a map $\overline{\sigma}:\operatorname*{holim}_{(V,U)\in\mathcal{P}_0(n)^2} F(X*G(V,U))\to\operatorname*{holim}_{(U,V)\in\mathcal{P}_0(n)^2} F(X*G(U,V))$ as desired.

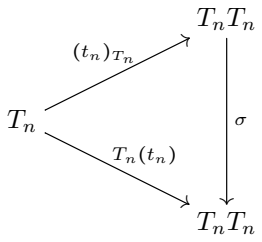
We may see what σ is actually doing with a picture. It swaps the roles of the U and V variable positions. As U changes, the color changes and as V varies, the suit varies. Then the resulting map is

$$
\operatorname*{holim}_{U\in\mathcal{P}_0(\mathbf{1})}\operatorname*{holim}_{V\in\mathcal{P}_0(\mathbf{1})} F((X*V)*U)\to\operatorname*{holim}_{U\in\mathcal{P}_0(\mathbf{1})}\operatorname*{holim}_{V\in\mathcal{P}_0(\mathbf{1})} F((X*U)*V)
$$



Goodwillie has already defined η for us: the natural transformation $t_n : Id \to T_n$. This map exists by the universal map from $F(X * \varnothing) = F(X)$ to $T_n F(X) = \underset{U \in \mathcal{P}_0(n)}{\text{holim}} F(X * U)$.

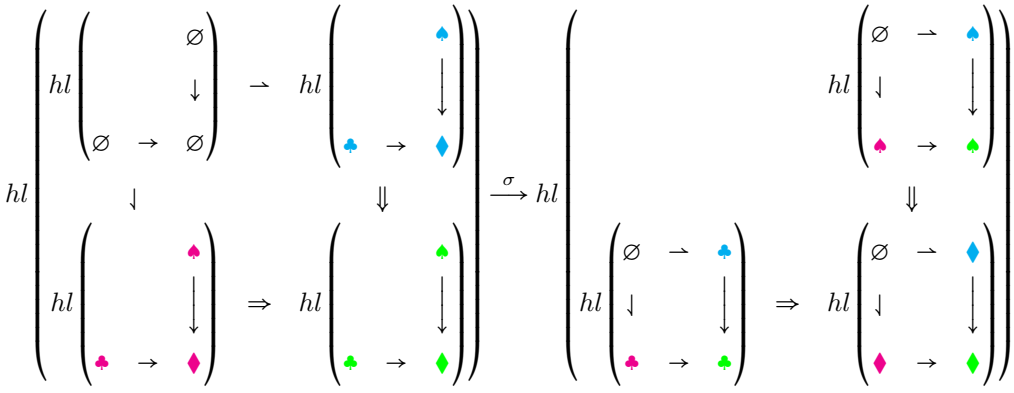
We will show that the star condition of lemma A.1.3 holds. There are two evident maps $T_n \to T_n T_n$ given by $T_n(t_n)$ and $(t_n)_{T_n}$ and we will show that these differ by σ; that is, the following triangle commutes:



We can picture t_n as mapping the empty set into the initial corner, depicted below by placing $\varnothing$ in the initial position with weird arrows (because it should actually be mapping to the homotopy limit as a result of mapping to everything in the diagram):

$$\begin{pmatrix} \varnothing & \rightharpoonup & \bullet \\ \downarrow & & \downarrow \\ \bullet & \to & \bullet \end{pmatrix}$$

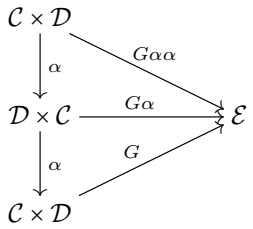
To verify the star condition, one can picture the maps into T_n^2. The two natural transformations are shown by including $\varnothing$ in the diagrams. Notice that including in the U variable gives one large empty set mapping into the upper left which can be written as the homotopy limit of a diagram of empty sets. The two maps clearly differ by σ.



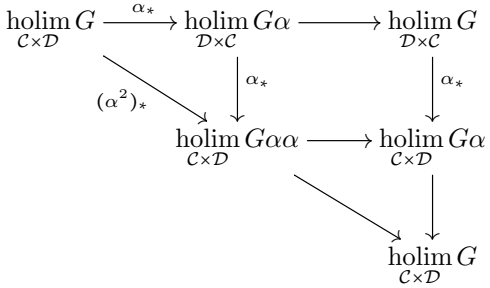
We will show that $\sigma^2 = id$ so we can use the lemma to define transposition maps on the T_n's. If $\overline{\sigma}^2 = id$,

$$\sigma^2 = \phi^{-1}\psi^{-1}\overline{\sigma}\psi\phi\phi^{-1}\psi^{-1}\overline{\sigma}\psi\phi = \phi^{-1}\psi^{-1}\overline{\sigma}^2\psi\phi = id$$

We may see why $\overline{\sigma}^2 = id$ by considering the following diagram.



So on homotopy limits, we get the following:



The maps along the top and right compose to be $\overline{\sigma}^2$ and the maps along the diagonal compose to id. The triangles and square commute by naturality. Then $\sigma^2 = id$.

We showed above that η and σ satisfy the star condition of lemma A.1.3.

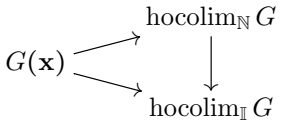
Finally we must verify the braiding condition, that is, that $T_n(\sigma)\sigma_{T_n}T_n(\sigma) = \sigma_{T_n}T_n(\sigma)\sigma_{T_n}$. This could be confused in an enormous diagram, but it is easier to think of the essential quality of $\overline{\sigma}$ that made everything so far work. We knew that $G(U,V) = U * V$ had a natural Σ_2-action because it is symmetric in its variables. Similarly, $G(U,V,W) = U * V * W$ has a natural Σ_3-action, so there is only one map $U * V * W \to W * V * U$ so the two $(0,2)$ shuffles must be the same map. This guarantees that the braid condition will hold. □

A.3 Bökstedt's lemma

A published proof of Bökstedt's approximation lemma may be found as lemma 2.2.2.2 of [DGM13].

Lemma A.3.1. *Let $G : \mathbb{I} \to \mathcal{T}$ be a functor, $\mathbf{x} \in ob\ \mathbb{I}$, and let $\mathbf{x} \downarrow \mathbb{I}$ be the full subcategory of $\mathbb{I}$ of objects supporting maps from $\mathbf{x}$. If G sends maps in $\mathbf{x} \downarrow \mathbb{I}$ to $n_{|\mathbf{x}|}$-connected maps and $n_{|\mathbf{x}|} \to \infty$ as $|\mathbf{x}| \to \infty$, then* $\operatorname{hocolim}_{\mathbb{N}} G \to \operatorname{hocolim}_{\mathbb{I}} G$ *is an equivalence.*

Proof. Fix k. We will show that for some $\mathbf{x}$, the diagonal maps in the following diagram are k-connected, so the natural map $\operatorname{hocolim}_{\mathbb{N}} G \to \operatorname{hocolim}_{\mathbb{I}} G$ is k-connected. Letting $k \to \infty$ recovers the desired result.

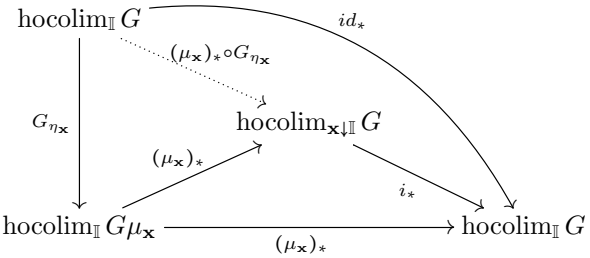


We can find an N such that $|\mathbf{x}| \geq N$ implies $n_{|\mathbf{x}|} \geq k$ so let $|\mathbf{x}| \geq N$ for the rest of the proof. By cofinality, $\mathrm{hocolim}_{\mathbf{x}\downarrow\mathbb{N}} G \to \mathrm{hocolim}_{\mathbb{N}} G$ is an equivalence, so the top map is morally seen to be k-connected. Everything in $G|_{\mathbf{x}\downarrow\mathbb{I}}$ is at least k-connected (that is, isomorphisms on homotopy up to level k) for $|\mathbf{x}| \geq N$, so the standard inclusions $G(\mathbf{y}) \to G(\mathbf{y}+\mathbf{1})$ are all k-connected for $y \geq x$. Thus the map $G(\mathbf{x}) \to \mathrm{hocolim}_{\mathbf{x}\downarrow\mathbb{N}} G$ is k-connected. Similarly, it is morally evident that the map $G(\mathbf{x}) \to \mathrm{hocolim}_{\mathbf{x}\downarrow\mathbb{I}} G$ is at least k-connected by the same argument. The actual proof of these requires a form of Quillen's theorem B which we discuss after the rest of the proof. We have reduced to showing that the map $i_* : \mathrm{hocolim}_{\mathbf{x}\downarrow\mathbb{I}} G \to \mathrm{hocolim}_{\mathbb{I}} G$ induced by the inclusion of the subcategory $i : \mathbf{x}\downarrow\mathbb{I} \to \mathbb{I}$ is a weak equivalence.

Define $\mu_{\mathbf{x}} : \mathbb{I} \to \mathbb{I}$ by $\mu_{\mathbf{x}}(\mathbf{y}) = \mathbf{x} \sqcup \mathbf{y}$, sending a set to its disjoint union with $\mathbf{x}$ so that $\mu_{\mathbf{x}}$ factors through the subcategory $\mathbf{x}\downarrow\mathbb{I}$ of objects supporting maps from $\mathbf{x}$. The inclusion $\mathbf{y} \hookrightarrow \mathbf{x} \sqcup \mathbf{y}$ defines a natural transformation $\eta_{\mathbf{x}}$ from the identity to $\mu_{\mathbf{x}}$, which induces a homotopy from the identity to the composite

$$\mathrm{hocolim}_{\mathbb{I}} G \xrightarrow{G_{\eta_{\mathbf{x}}}} \mathrm{hocolim}_{\mathbb{I}} G\mu_{\mathbf{x}} \xrightarrow{(\mu_{\mathbf{x}})_*} \mathrm{hocolim}_{\mathbb{I}} G.$$

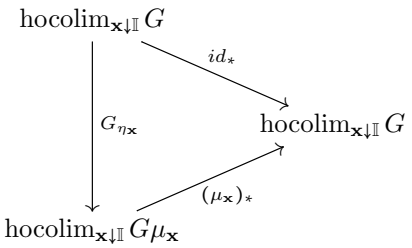
We will first show that i_* is a split surjection in the homotopy category, i.e., that it has a right homotopy inverse. The factorization of $\mu_{\mathbf{x}}$ as the composite $i \circ \mu_{\mathbf{x}} : \mathbb{I} \to \mathbf{x}\downarrow\mathbb{I} \to \mathbb{I}$ gives a factorization on homotopy colimits to yield the diagram:



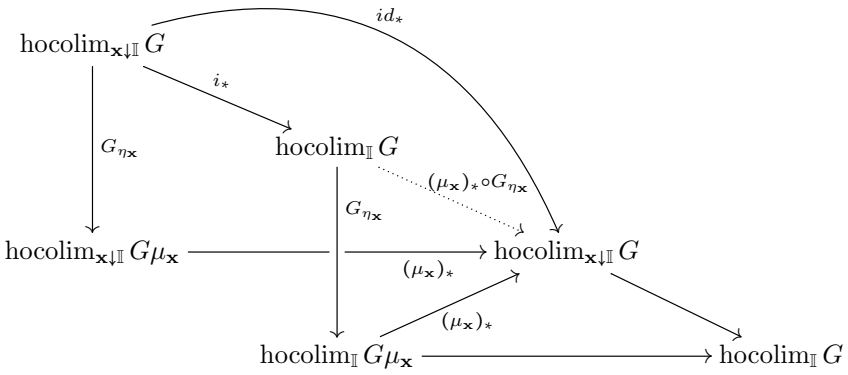
The dotted map in the diagram composed with i_* is homotopic to the identity, and in the homotopy category this is a right inverse.

Following through the same argument for the natural transformation $\eta_{\mathbf{x}}$ restricted to $\mathbf{x}\downarrow\mathbb{I}$ produces a

homotopy commutative diagram:



The following shows that the same map, $(\mu_{\mathbf{x}})_* \circ G_{\eta_{\mathbf{x}}}$, is also a left homotopy inverse of i_*. (We've included the diagram above as the lower triangle for clarity.)



This demonstrates that $i_* : \operatorname{hocolim}_{\mathbf{x}\downarrow\mathbb{I}} G \to \operatorname{hocolim}_{\mathbb{I}} G$ is a weak equivalence, thus proving the lemma (modulo Quillen's theorem B). Below is an explanation of how Quillen's theorem applies. We record the version given in the appendix of [DGM13].

Proposition A.3.2. *Let I be a small category, let λ be a natural number, and let $G : I \to S$ be such that for any $f : i \to j \in I$ the induced map $G(i) \to G(j)$ is λ-connected. Let* $\operatorname{hocolim}_I G \to \operatorname{hocolim}_I * \cong BI$ *be induced by the natural transformation $G(i) \to *$. Given any object i in I, the map from $G(i)$ to the homotopy fiber of* $\operatorname{hocolim}_I G \to BI$ *over the vertex $i \in B_0 I$ is λ-connected.*

The entire argument of the lemma above can be repeated for the constant functor $* : \mathbb{I} \to \mathcal{T}$, so there is a weak equivalence $B(\mathbf{x} \downarrow \mathbb{I}) \cong \operatorname{hocolim}_{\mathbf{x}\downarrow\mathbb{I}} * \to \operatorname{hocolim}_{\mathbb{I}} * \cong B\mathbb{I}$. The latter is contractible ($\mathbb{I}$ has an initial object), so $B(\mathbf{x} \downarrow \mathbb{I})$ is also contractible, although it does not have an initial object (for $\mathbf{x} \neq \varnothing$).

Because $B(\mathbf{x} \downarrow \mathbb{I})$ is contractible, the homotopy fiber of $\operatorname{hocolim}_{\mathbf{x}\downarrow\mathbb{I}} G \to B(\mathbf{x} \downarrow \mathbb{I})$ is equivalent to $\operatorname{hocolim}_{\mathbf{x}\downarrow\mathbb{I}} G$. We also know that all the maps of $G|_{\mathbf{x}\downarrow\mathbb{I}}$ are k-connected, so by the proposition, the map $G(\mathbf{x}) \to \operatorname{hocolim}_{\mathbf{x}\downarrow\mathbb{I}} G$ is k-connected. Similarly, $B(\mathbf{x} \downarrow \mathbb{N})$ is contractible, so $G(\mathbf{x}) \to \operatorname{hocolim}_{\mathbf{x}\downarrow\mathbb{N}} G$ is k-connected. □

Appendix B

Partial results for functors to spaces

Although the results in this appendix do not assemble to give a classification of n-excisive functors to spaces, we offer proofs of results analogous to those that gave the classification for spectra in chapter 3 in the hopes that my failure can be instructive or useful. Let $[n] = \bigvee^n S^{k+1}$.

Definition B.0.3. A space X is k-connected if $\pi_j X = 0$ for $0 \le j \le k$. A map $f : X \to Y$ is k-connected if $\pi_j f : \pi_j X \to \pi_j Y$ is an isomorphism for $0 \le j < k$ and a surjection for $j = k$.

Theorem B.0.4. *Let $F, G : \mathcal{T} \to \mathcal{T}$ be two finitary, n-excisive functors which satisfy $E_n(\rho n - q)$ for all n and $\pi_0 F(*) = \pi_0 G(*) = 0$. If a natural transformation $\eta : F \to G$ is an equivalence on $[n] = \bigvee^n S^{k+1}$, where $k = max(\rho, -q)$, then η is an equivalence on all k-connected spaces.*

That is, n-excisive functors are determined in their radius of convergence by their values on spheres.

Note that we are requiring a *strong* analyticity by asking F to be stably n-excisive for all $n \ge 0$, not just for $n \ge 1$ as in Goodwillie's definition. This requires F to take α-connected maps to $(\alpha - q)$-connected maps. This condition was used by Mauer-Oats in [MO02], and thus to utilize his results, we make the same assumptions. We will prove theorem B.0.4 through a series of lemmas.

Lemma B.0.5. *The value of an n-excisive functor on wedges of k-spheres is determined by its value on the space $\bigvee^n S^k$.*

Proof. Replace $\bigvee S^0$ with $\bigvee S^k$ in the proof of lemma 3.2.5. □

To prove the theorem, we apply the lemma for $k+1$-spheres. The following lemma is an easy adaptation of a result in section 6 of [MO02].

Lemma B.0.6. *Let $F : \mathcal{T} \to \mathcal{T}$ be a finitary functor such that $F(*)$ is connected and F satisfies $E_n(\rho n - q)$ for all n, then F commutes with realizations of simplicial k-connected spaces, where $k \ge max(\rho, -q)$.*

Proof. Let $X_\bullet$ be a simplicial k-connected space with $k \ge max(\rho, -q)$. We will show that all layers in the Taylor tower commute with the realization of $X_\bullet$, and then use induction to show that all $P_n F$ commute with the realization also. We will use analyticity to conclude that it is also true of F.

Recall $D_nF(X) \simeq \Omega^\infty(\partial_n F \wedge X^{\wedge n})_{h\Sigma_n}$. If X_i is k-connected, then smashing with X_i increases connectivity by $k+1$, and smashing with $X_i^{\wedge m+1}$ raises connectivity by $(m+1)(k+1)$. Since $\partial_{m+1}F$ has bottom nonzero homotopy in dimension $q-\rho m$ (thm 6.1 [MO02]), we see that $\partial_{m+1}F \wedge X_i^{\wedge m+1}$ is $k+q+(k-\rho)m+m+1$-connected. Since $k \geq max(\rho, -q)$, both $k+q \geq 0$ and $k-\rho \geq 0$, so this is always connected. Taking homotopy orbits does not lower connectivity, and Ω^∞ commutes with the realization of simplicial connective spectra (thm 6.9 [MO02]), thus $D_{m+1}F$ commutes with realization of $X_\bullet$ for all $m \geq 0$.

Clearly, P_0F commutes with realizations, because it is constant. We have also assumed that $P_0F \simeq F(*)$ is connected. Assume for induction that $P_{n-1}F$ is connected and commutes with the realization of $X_\bullet$. Note that $P_nF(X_i)$ is connected for all i, by analyzing the fiber sequence $D_nF \to P_nF \to P_{n-1}F$. (The fiber is 0-connected if and only if the map is 1-connected, so $\pi_0 P_nF(X_i) \to \pi_0 P_{n-1}F(X_i)$ is an isomorphism.) Then all $P_nF(X_i)$ are connected and we may apply Waldhausen's lemma to the above fiber sequence to get the following map of fiber sequences

$$\begin{array}{ccccc} \|D_nF(X_\bullet)\| & \longrightarrow & \|P_nF(X_\bullet)\| & \longrightarrow & \|P_{n-1}F(X_\bullet)\| \\ \downarrow \simeq & & \downarrow & & \downarrow \simeq \\ D_nF(\|X_\bullet\|) & \longrightarrow & P_nF(\|X_\bullet\|) & \longrightarrow & P_{n-1}F(\|X_\bullet\|) \end{array}$$

By the 5-lemma, we get that $\pi_i\|P_nF(X_\bullet)\| \to \pi_i P_nF(\|X_\bullet\|)$ is an isomorphism for all $i \geq 1$. Since both are connected, we also have an isomorphism when $i = 0$ and so P_nF commutes with the realization of $X_\bullet$.

Finally, the connectivity of the map $F \to P_nF$ grows as n gets larger as long as X is in the radius of convergence of F, that is, when X is at least ρ-connected. We have assumed such, so F also commutes with the realization of $X_\bullet$. □

Lemma B.0.7. *If X is a k-connected space, then $X \simeq \|Y_\bullet\|$ for some simplicial levelwise k-connected space which is levelwise a wedge of spheres.*

This result can be found in section 2 of [Sto90].

Proof. Given a connected pointed CW complex X, define $\mathcal{V}X$ as the following pushout:

$$\begin{array}{ccc} \bigvee_{n\geq 1} \bigvee_{h \in \mathrm{Hom}(D^{n+1}, X)} S^n_{h|S^n} & \longrightarrow & \bigvee_{n\geq 1} \bigvee_{h \in \mathrm{Hom}(D^{n+1}, X)} D^{n+1}_h \\ \downarrow & & \downarrow \\ \bigvee_{n\geq 1} \bigvee_{f \in \mathrm{Hom}(S^n, X)} S^n_f & \longrightarrow & \mathcal{V}X \end{array}$$

where the maps are induced by the inclusions $S^n \hookrightarrow D^{n+1}$. That is, $\mathcal{V}X$ is a wedge of spheres S^n_f indexed

by all maps $f : S^n \to X$ with a disk D^{n+1}_h attached to S^n_f for every null homotopy $h : D^{n+1} \to X$ of f. The construction $\mathcal{V}$ is functorial and forms a cotriple along with the natural maps $\epsilon : \mathcal{V}X \to X$ and $\beta : \mathcal{V}X \to \mathcal{V}^2X$, where ϵ sends S^n_f into X by the indexing map f and sends D^{n+1}_h into X by h, and β takes S^n_f homeomorphically to the copy of S^n in $\mathcal{V}^2X$ that is indexed by the inclusion $S^n_f \to \mathcal{V}X$ and similarly for D^{n+1}_h.

By standard methods, the cotriple $\mathcal{V}$ produces a cellular simplicial space $Y_\bullet$ augmented by X such that $Y_p = \mathcal{V}^{p+1}X$ for all $p \geq 0$ with face and degeneracy maps given by ϵ and β.

We will show that each space Y_p has the homotopy type of a wedge of spheres by showing that there are contractible subcomplexes $C_p \subset Y_p$, $p \geq 0$ such that for all $p \geq 0$, the quotient Y_p/C_p is a bouquet of spheres of positive dimensions, $s_j c_p \in C_{p+1}$ for all degeneracy maps and all points $c_p \in C_p$, and for all $0 \leq j \leq p$, the induced maps $\overline{s}_j : Y_p/C_p \to Y_{p+1}/C_{p+1}$ are inclusions of bouquets.

Let C_0 be the subcomplex of $\mathcal{V}X$ that is obtained by choosing, for each sphere S^n_f whose index map f is null homotopic, exactly one of the disks D^{n+1}_h that are attached to S^n_f. The quotient $\mathcal{V}X/C_0$ is a bouquet of two types of spheres: those of the form $D^{n+1}_h/S^n_{h|S^n}$ where D^{n+1}_h is not in C_0, and those of the form S^n_f where f is not null homotopic. Since $Y_\bullet$ is cellular, induction shows that we can choose $C_1 \subset Y_1$, etc. so that the degeneracy conditions are satisfied.

When X is not connected (-1-connected), we can use the usual singular complex to show that X is equivalent to the realization of a simplicial space which is levelwise a wedge of zero-spheres. Note that when X is k-connected, all maps $S^n \to X$ are nullhomotopic for $n \leq k$, so the spheres S^n_f of the latter type in the wedge $\mathcal{V}X/C_0$ have $n \geq k+1$ and the spheres of type $D^{n+1}_h/S^n_{h|S^n}$ must also have dimension greater than k.

Then we can write $X \simeq \|Y_\bullet\|$, where $Y_p \simeq \bigvee_{n>k} \bigvee^{i_n} S^n$. □

Even better, we can reduce the dimension of these spheres to $k+1$.

Lemma B.0.8. *A k-connected wedge of spheres is equivalent to the realization of a simplicial space which is levelwise a wedge of $k+1$-spheres.*

Proof. Take, for example, the simplicial model $\Delta^1/\partial\Delta^1$ for S^1 given levelwise by $[k] \mapsto \vee^k S^0$ which can be seen as the (coproduct) bar construction applied to S^0. Now similarly,

$$S^2 \simeq S^1 \wedge S^1 \simeq S^1 \wedge B.^\vee S^0 \simeq B.^\vee(S^1 \wedge S^0) \simeq B.^\vee S^1.$$

By induction, we see that $S^n \simeq B.^\vee S^{n-1}$.

Using that the realization of a bisimplicial space is the realization of its diagonal [Qui73], we see that any sphere S^n with $n > k$ is equivalent to the realization of a simplicial space which is levelwise $\vee S^{k+1}$. That

is, $S^n \simeq (B.^\vee)^{n-k-1} S^{k+1}$. Thus the wedge of spheres $Y_p \simeq \vee_{n>k} \vee^{i_n} S^n$ from lemma B.0.7 is equivalent to the realization of a simplicial space $Y_{(p,\bullet)}$ which is levelwise $Y_{(p,q)} \simeq \vee S^{k+1}$. □

Example B.0.9.

$$S^3 \simeq B.^\vee B.^\vee S^1 \simeq \| * \Longleftarrow S^2 \Longleftarrow S^2 \vee S^2 \qquad \cdots \|$$

$$\simeq \text{hocolim} \left(\left\| \begin{array}{c} * \\ \Downarrow \\ * \\ \Downarrow \\ * \end{array} \right\| \Leftarrow \left\| \begin{array}{c} S^1 \vee S^1 \\ \Downarrow \\ S^1 \\ \Downarrow \\ * \end{array} \right\| \Leftarrow \left\| \begin{array}{c} \vee^4 S^1 \\ \Downarrow \\ S^1 \vee S^1 \\ \Downarrow \\ * \end{array} \right\| \right) \simeq \text{hocolim}\, diag \left(\begin{array}{ccccc} * & \Longleftarrow & S^1 \vee S^1 & \Longleftarrow & \bigvee^4 S^1 \\ \Downarrow & & \Downarrow & & \Downarrow \\ * & \Longleftarrow & S^1 & \Longleftarrow & S^1 \vee S^1 \\ \Downarrow & & \Downarrow & & \Downarrow \\ * & \Longleftarrow & * & \Longleftarrow & * \end{array} \right)$$

$$\simeq \| * \Longleftarrow S^1 \Longleftarrow \vee^4 S^1 \qquad \cdots \|$$

Lemma B.0.10. *If F, G commute with realizations of simplicial k-connected spaces, then a natural transformation $\eta : F \to G$ which is an equivalence on $[n] = \vee^n S^{k+1}$ is an equivalence on all k-connected spaces.*

Proof. Let X be a k-connected space. Using that the realization of a bisimplicial space is the realization of its diagonal together with lemmas B.0.7 and B.0.8, we get that $X \simeq \|Z_\bullet\|$ where $Z_m = Y_{(m,m)} \simeq \bigvee^{i_m} S^{k+1}$. F and G are homotopy functors so $F(X) \simeq F(\|Z_\bullet\|)$ and $G(X) \simeq G(\|Z_\bullet\|)$. Consider the diagram

$$\begin{array}{ccccc} \|F(Z_\bullet)\| & \xrightarrow{\simeq} & F(\|Z_\bullet\|) & \xrightarrow{\simeq} & F(X) \\ \downarrow{\scriptstyle \|\eta_\bullet\|} & & \downarrow & & \downarrow{\scriptstyle \eta} \\ \|G(Z_\bullet)\| & \xrightarrow{\simeq} & G(\|Z_\bullet\|) & \xrightarrow{\simeq} & G(X) \end{array}$$

By lemma 3.2.5, η is an equivalence on wedges $\vee S^{k+1}$, so $\eta_\bullet$ is a levelwise equivalence. Then $\|\eta_\bullet\|$ is an equivalence by the realization lemma, so $F(X) \simeq G(X)$. □

This completes the proof of theorem B.0.4, thus we know that n-excisive analytic functors to spaces are determined, in their radius of convergence, by $n+1$ "points" or spheres, but this was only half of the classification argument. We showed that $L_n F$ was n-excisive and thus the map $L_n F \to \mathbb{P}_n F$ was an equivalence, so now we need $L_n F$ to be n-excisive and analytic to feed into this theorem. We showed in 3.3.6 that a levelwise n-excisive functor to spectra realizes to an n-excisive functor, so we now give the analogous results for functors to spaces, which are much harder.

Lemma B.0.11. *Let $[p] \mapsto F_p$ be a simplicial functor such that each $F_p : \mathcal{T} \to \mathcal{T}$ is n-excisive, finitary, satisfies $E_n(\rho n - q)$ for all n, and $F_p(*)$ is connected. Then the realization $\|F_\bullet\|$ is n-excisive on k-connected spaces.*

Proof. Again we will use induction on n as in the proof of lemma B.0.6.

Case 0: Let $n = 0$. Then F_p is 0-excisive and thus constant for each p. If $F_p(Y) = X_p$ for all objects Y of $\mathcal{T}$, then for all Y, $\|F_\bullet(Y)\|$ is also constant with value

$$\|F_\bullet(Y)\| = \left\| X_0 \Longleftarrow X_1 \Longleftarrow X_2 \qquad \cdots \right\|.$$

Case 1: Suppose each F_p is n-homogeneous, that is, n-excisive and n-reduced. Then $F_p \xrightarrow{\simeq} P_n F_p \xleftarrow{\simeq} D_n F_p$ for all $p \geq 0$. By Segal's realization lemma ([Seg74]), $\|F_\bullet\| \simeq \|D_n F_\bullet\|$. So to show that $\|F_\bullet\|$ is n-excisive, we will show $\|D_n F_\bullet\|$ is n-excisive.

Recall from [Goo03] that $D_n F : \mathcal{T} \to \mathcal{T}$ factors through spectra as an n-homogeneous functor $D'_n F$ followed by the right adjoint Ω^∞ back to spaces

$$\mathcal{T} \xrightarrow{D'_n F} \mathcal{S}\mathrm{p} \xrightarrow{\Omega^\infty} \mathcal{T}.$$

Let $\mathcal{X}$ be a strongly cocartesian $(n+1)$-cube of k-connected spaces. Then $D'_n F_p(\mathcal{X})$ is a cartesian cube of spectra by n-excision and thus also cocartesian. The fat realization is a homotopy colimit so preserves cocartesian cubes, thus $\|D'_n F_\bullet(\mathcal{X})\|$ is also homotopy cocartesian. As a cube of spectra, it is also cartesian. As a homotopy right adjoint, Ω^∞ preserves homotopy limits, so $\Omega^\infty \|D'_n F_\bullet(\mathcal{X})\|$ is cartesian.

By an induction argument on skeleta, Mauer-Oats ([MO02] Theorem 6.9) shows that the map $\|\Omega^\infty A_\bullet\| \to \Omega^\infty \|A_\bullet\|$ is an equivalence for simplicial connective spectra $A_\bullet$. By chapter 6 of [MO02], $D'_n F_p$ is connective on k-connected spaces, so $\|D_n F_\bullet(\mathcal{X})\| = \|\Omega^\infty D'_n F_\bullet(\mathcal{X})\|$ is homotopy cartesian when $\mathcal{X}$ is made up of k-connected spaces. Thus if F_p is n-homogeneous for all p, $\|F_\bullet\|$ is n-excisive.

Case 2: If F_p is n-excisive but not n-reduced, then $F_p \xrightarrow{\simeq} P_n F_p$, and we will show that $\|P_n F_\bullet\|$ is n-excisive. Consider the levelwise fibration

$$D_n F_p \to P_n F_p \to P_{n-1} F_p.$$

A lemma of Waldhausen ([Wal78] Lemma 5.2) states that if $X_\bullet \to Y_\bullet \to Z_\bullet$ is a levelwise fibration up to homotopy of simplicial spaces and each Z_n is connected, then $\|X_\bullet\| \to \|Y_\bullet\| \to \|Z_\bullet\|$ is a fibration up to homotopy. By chapter 6 of [MO02], $P_n F_p$ is connected on k-connected spaces, thus the resulting diagram of fat realizations $\|D_n F_\bullet(X)\| \to \|P_n F_\bullet(X)\| \to \|P_{n-1} F_\bullet(X)\|$ is a homotopy fibration, and by proposition 1.7

of [Goo03], T_n preserves fibrations, so there is a map of fiber sequences

$$\begin{array}{ccccc} \|D_nF_\bullet\| & \longrightarrow & \|P_nF_\bullet\| & \longrightarrow & \|P_{n-1}F_\bullet\| \\ \downarrow & & \downarrow & & \downarrow \\ T_n\|D_nF_\bullet\| & \longrightarrow & T_n\|P_nF_\bullet\| & \longrightarrow & T_n\|P_{n-1}F_\bullet\| \end{array}$$

The outside two vertical maps are homotopy equivalences (by the induction hypothesis and case 1, respectively), and since both rows are fibrations, they induce long exact sequences in homotopy:

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & \pi_1 A & \longrightarrow & \pi_1 B & \longrightarrow & \pi_1 C & \longrightarrow & \pi_0 A & \longrightarrow & \pi_0 B & \longrightarrow & \pi_0 C \\ & & \downarrow\cong & & \downarrow & & \downarrow\cong & & \downarrow\cong & & \downarrow & & \downarrow\cong \\ \cdots & \longrightarrow & \pi_1 A & \longrightarrow & \pi_1 T_n B & \longrightarrow & \pi_1 C & \longrightarrow & \pi_0 A & \longrightarrow & \pi_0 T_n B & \longrightarrow & \pi_0 C \end{array}$$

Since A and C are connected, we reduce to the map of exact sequences

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & \pi_1 A & \longrightarrow & \pi_1 B & \longrightarrow & \pi_1 C & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\ & & \downarrow\cong & & \downarrow & & \downarrow\cong & & \downarrow\cong & & \downarrow & & \downarrow\cong \\ \cdots & \longrightarrow & \pi_1 A & \longrightarrow & \pi_1 T_n B & \longrightarrow & \pi_1 C & \longrightarrow & 0 & \longrightarrow & \pi_0 T_n B & \longrightarrow & 0 \end{array}$$

By a diagram chase, we see that $\pi_0 B \cong \pi_0 T_n B = 0$. The five lemma gives the necessary isomorphisms $\pi_k B \to \pi_k T_n B$ for $k \geq 1$, so $T_n\|P_nF_\bullet\| \simeq \|P_nF_\bullet\|$ and thus $\|P_nF_\bullet\|$ is n-excisive on k-connected spaces. This finishes the proof that $\|F_\bullet\|$ is n-excisive. □

To use theorem B.0.4 for a classification, we would also need the realization to be analytic.

Lemma B.0.12. *If $[p] \mapsto F_p$ is a simplicial functor and each $F_p : \mathcal{T} \to \mathcal{T}$ is a strongly analytic functor (satisfying $E_n(\rho n - q)$ for all n) and $F_p(*)$ is connected for all p, then $\|F_\bullet\|$ is analytic on $k+1$-connected spaces, where $k = max(\rho, -q)$.*

Proof. We will prove this by induction on n. First, recall that the fiber of a k-connected map is a $k-1$-connected space.

Suppose F_p is levelwise analytic satisfying $E_n(n\rho - q)$ and that $F_p(*)$ is connected for all p. For a strongly cocartesian $(n+1)$-cube $\mathcal{X}$ of $(k+1)$-connected spaces with initial maps of connectivities k_s, the map $F_p(\mathcal{X}_\varnothing) \to \operatorname{holim}_{U \in \mathcal{P}_0(n)} F_p(\mathcal{X}_U)$ is $(\Sigma k_s - (\rho n - q))$-connected. Realization preserves connectivity, so if $\|\operatorname{holim}_{U \in \mathcal{P}_0(n)} F_p(\mathcal{X}_U)\| \to \operatorname{holim}_{U \in \mathcal{P}_0(n)} \|F_p(\mathcal{X}_U)\|$ is an equivalence, $\|F_p(\mathcal{X})\|$ will be $(\Sigma k_s - (\rho n - q))$-cartesian, so $\|F_p\|$ will be analytic.

Let $n = 1$, and let the following be a strongly cocartesian square of k-connected spaces.

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ W & \longrightarrow & Z \end{array}$$

Then consider the following levelwise cartesian square

$$\begin{array}{ccc} \text{holim}_p & \longrightarrow & F_p(Y) \\ \downarrow & & \downarrow \\ F_p(W) & \longrightarrow & F_p(Z) \end{array}$$

Since each F_p satisfies $E_n(\rho n - q)$ for all n, each satisfies $E_0(-q)$. This means that F_p takes α-connected maps to $(\alpha + q)$-connected maps. Since Y is k-connected, the map $Y \to *$ is $(k+1)$-connected, so the map $F_p(Y) \to F_p(*)$ is $(k+1+q)$-connected. As long as the map is at least 1-connected and $F_p(*)$ is 0-connected for all p, $F_p(Y)$ will be connected. Since $k + q \geq 0$, we have the desired connectivity on the map. We have assumed that $F_p(*)$ is connected for all p. Similarly, $F_p(Z)$ is connected, so by the Bousfield-Friedlander theorem, the realization square below is cartesian.

$$\begin{array}{ccc} \| \text{holim}_p \| & \longrightarrow & \|F_p(Y)\| \\ \downarrow & & \downarrow \\ \|F_p(W)\| & \longrightarrow & \|F_p(Z)\| \end{array}$$

Let $n = 2$ and let $\mathcal{X} : \mathcal{P}(2) \to \text{Top}_*$ be a 3-cube of $(k+1)$-connected spaces, so all the maps are $(k+1)$-connected. Suppose $F_p(*)$ is connected for all p, so $\pi_0 F_p(X_U) = 0$ (for $U = 0, 2, 02, 01, 12, 012$ in particular). Replace $F_p(\mathcal{X}_\varnothing)$ with $\text{holim}_{U \in \mathcal{P}_0(2)} F_p(\mathcal{X}_U)$ and call this cartesian cube $\mathcal{Y}_p$. Now since $\mathcal{X}_2 \to \mathcal{X}_{02}$ is $(k+1)$-connected, $F_p(X_2) \to F_p(X_{02})$ is $(k+1+q)$-connected and since $k+q \geq 0$, the fiber of this map is 0-connected. Similarly for $X_{12} \to X_{012}$. Then we may consider the fiber of the cube $\mathcal{Y}_p$, that is, the square below

$$\begin{array}{ccc} fib(\text{holim} \to F(\mathcal{X}_0)) & & \\ \downarrow & \searrow & \\ & & fib(F(\mathcal{X}_2) \to F(\mathcal{X}_{02})) \\ fib(F(\mathcal{X}_1) \to F(\mathcal{X}_{01})) & & \downarrow \\ & \searrow & \\ & & fib(F(\mathcal{X}_{12}) \to F(\mathcal{X}_{012})) \end{array}$$

$$\begin{array}{ccccc} \text{holim} & \longrightarrow & F(\mathcal{X}_0) & & \\ \downarrow & \searrow & \downarrow & \searrow & \\ & F(\mathcal{X}_2) & \longrightarrow & F(\mathcal{X}_{02}) & \\ F(\mathcal{X}_1) & \longrightarrow & F(\mathcal{X}_{01}) & & \downarrow \\ & \searrow & \downarrow & \searrow & \\ & & F(\mathcal{X}_{12}) & \longrightarrow & F(\mathcal{X}_{012}) \end{array}$$

We have denoted connected spaces in blue and 1-connected maps in green.

Since the cube is cartesian, the fiber square is also cartesian, and since both (blue) fibers are connected, this square satisfies the conditions of the Bousfield-Friedlander theorem, so the realization square is also cartesian. By Waldhausen's lemma, the realized fibers are equivalent to the fibers of the realizations, so the realized cube $\|\mathcal{Y}\|$ is also cartesian. That is, $\|\operatorname{holim}_{U\in\mathcal{P}_0(2)} F_\bullet(\mathcal{X}_U)\| \simeq \operatorname{holim}_{U\in\mathcal{P}_0(2)} \|F_\bullet(\mathcal{X}_U)\|$, so the realization satisfies $E_n(c)$.

Let $n=3$ and let $\mathcal{X}:\mathcal{P}(3)\to \mathrm{Top}_*$ be a 4-cube of $(k+1)$-connected spaces. Again, let $F_p(*)$ be connected for all p, so that $F_p(\mathcal{X}_U)$ is connected. Again, morphisms are $(k+1)$-connected, so the maps in $\mathcal{Y}$ are 1-connected. The fiber cube of $\mathcal{Y}$ will need to satisfy the conditions necessary of the $n=2$ case, so we need two maps of fibers to be 1-connected. This is equivalent to asking that the squares be 1-cartesian. Consider

$$\begin{array}{ccc} fib(X\to X') & & X \longrightarrow X' \\ \downarrow{\scriptstyle f} & & \downarrow \quad \downarrow \\ fib(Y\to Y') & & Y \longrightarrow Y' \end{array}$$

Note that the fiber of f is the total fiber of the square S. The map f is 1-connected if and only if the total fiber is 0-connected. The total fiber is also the fiber of the map $X\to \mathrm{holim}$, so this is also 1-connected, and S is 1-cartesian. Now, F_p is analytic, satisfying $E_1(\rho-q)$, and the initial maps of S are $(k+1)$-connected, so $F_p(S)$ is $2(k+1)-\rho+q$-cartesian, and this is ≥ 2. (We only need ≥ 1.)

This raises the question of what is necessary for a simplicial cartesian n-cube to still be cartesian upon realization, i.e., is there a higher Bousfield Friedlander theorem? We have outlined why it would be sufficient to have 2 1-cartesian $(n-2)$-cubes, 6 1-cartesian $(n-3)$-cubes, 12 1-cartesian $(n-4)$-cubes, etc., in the appropriate places.

That is, if every level of the simplicial cartesian n-cube has $2^{i-2}3$ 1-cartesian $(n-i)$-faces, for $2\leq i<n$, and $2^{n-2}3$ 0-connected objects in the appropriate spots, then the realization is cartesian. An $(n-i)$-face is a cube $\mathcal{X}$ which we can apply the E_{n-i-1} condition to. That is, F_p satisfies $E_{n-i-1}(\rho(n-i-1)-q)$ and since all initial maps of the $(n-i)$-cube $\mathcal{X}$ are $(k+1)$-connected, $F_p(\mathcal{X})$ is $((n-i)(k+1)-\rho(n-i-1)+q)$-cartesian. Since $k-\rho\geq 0$ and $k+q\geq 0$, this quantity is $\geq n-i$. Perhaps one could generalize Rezk's generalization of the π_*-Kan condition to give weaker conditions, but we leave this pursuit to the interested reader. □

Herein lies the problem with the classification. The analyticity of the realization requires $k+1$-connected spaces, but we are feeding in k-spheres, leaving us with the wrong analyticity for theorem B.0.4. We were unable to rectify this, so we have not found a classification for functors to spaces.

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