DIFFERENTIAL GAMES AND APPLICATIONS TO COUNTER-TERRORISM

BY

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THESIS

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Abstract

The application of dynamic games to the study of terrorism has the potential to provide insight for policymakers on the behavior of terrorists and inform them how best response to terror threats. We studied several applications of differential games to counter-terrorism and analyzed the usefulness of each approach. We also discussed possible future research topics.

Acknowledgements

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Chapter 1

Introduction

Dynamic or differential game theory is the study of strategic decision making between more than one player where actions are contigent on the information about the opponents' actions. First introduced by Isaacs [1] in the 1960s, differential games concern the decision making of more than one player, each trying to maximize an objective function, subjected to differential equations.

The study of differential games is very closely related to the theory of optimal control in that deriving the best strategies (solutions) for each player in a game is equivalent to deriving control laws for multiple controllers for a given system. However, unlike optimal control, what constitutes a "solution" is ambiguous. There exists many different types of solutions, such as Nash, Stackelberg, Pareto, minimax, that are derivable within certain contexts and with certain assumptions. As such, the applicability of differential games is sparse, with subject areas such as aircraft control [2], mobile robotics [3] and unmanned vehicles [4], having different types of approaches.

In this thesis, we will study one recent area of application of differential games. Given the increase of worldwide terror incidents, particularly major events such as 9/11, there have been considerable attention to the study of counter-terrorism. Among the many issues being studied is the interaction between governments and terrorists and how both sides anticipate each other's action and develop strategic responses. A natural tool to model the conflict between the government and terrorists is differential game theory. Some applications include examining the terrorist's choice (e.g., terrorists' objective(s)), government's choice (e.g., deterrence policies and security programs) and government-terrorist concessionary strategies (e.g., hostage negotiations).

1.1 Outline

The thesis is organized into four chapters. In Chapter 1, we will review the literature on the dynamics of terrorism. Chapter 2 introduces differential games by defining the proper notations and preliminaries beginning with optimal controls and how it relates to the formulation of a differential game. We will define different solutions approaches in analyzing differential games. We will study several applicable games to studying counter-terrorism in Chapter 3. We will conclude with future possible research opportunities in the final chapter.

1.2 Literature Review of Dynamic Models

The application of differential game theory to studying terrorism is scarcer than the classical game theory literature. The reason is the trade-off between the characterization of the dynamics of a terrorist organization, the government and affecting population and the ability to derive solutions. As games are formulated as optimal control problems, we would need to solve Hamilton-Jacobi-Bellman equation or use Pontryagin's maximum principle in deriving optimal strategies. Depending on how we model the dynamics of the game, deriving solutions is oftentimes very difficult if not impossible.

There are many ways to model the dynamics of a terrorist organization, such as its operations, political ideology or its methods of recruitment. Like most mathematical models, these models are imperfect because they must assume large simplification of the underlying phenomena and their results oftentimes cannot be validated. This, however, does not mean we cannot gain practical insights. For example, if we can model and predict recruitment levels of a terrorist organization on a given time horizon, we can determine when it is best to deploy counter-terror policies.

The study of the dynamics of conflict was developed well before differential games was established. One of the first model of conflict was developed by Lewis Fry Richardson in the 1940s and published posthumously in 1963. Richardson Arms Race Model is a system of differential equations describing the weaponry available to multiple countries. The system of differential equations yield four possible outcomes for this conflict: (1) all trajectory approach an equilibrium point, (2) all trajectories go to infinity (continuous arms race), (3) all trajectories go to zero (disarmament) and (4) the trajectory path depends on the initial conditions. Several works in differential games have since incorporated Richardson's model [5, 6].

Uwadia *et al.* [7] provided a system of three nonlinear differential equations that describe the behavior of terrorists and people who are susceptible and non-susceptible to terrorists' influence. They found the conditions under which non-violent interventions and law enforcement interventions are useful to fight terrorism. Faria and Arce [8] used a system of difference equations to describe the time evolution of public support, terrorist recruitment and terrorist attacks. Their model found that four counter-terrorist policies (gather information, political participation, terrorist opportunity cost and law enforcement) are effective to reducing terrorism. Faria [9] also used a system of nonlinear ODEs to describe the dynamics of public support, the government's counter-terrorism actions and terrorist attacks. He showed that public support for an incumbent politician in time of terror attacks is high when the government overreacts to terror attacks since the unitary cost of counter-terrorist actions is higher than the marginal impact of terrorist attacks.

An extension of [9] was presented in Faria and Arce [10] with a discrete optimal control model in which terrorist support and recruitment is a dynamic constraint to the government's optimization problem. A discrete optimal control approach was also taken by Barros *et al.* [11]. They formulated a discrete optimal control problem in which the terrorist organization maximizes attacks subjected to limited resources are that negatively affected by the law enforcement. Their conclusion is that terror attacks follow a random walk so the the number of terrorst attacks are not predictable.

A continuous dynamic optimization approach was used in Caulkins *et al.* [12]. Their optimal control model consists of the government minimizing the cost of counter-terrorist operations through the control of operations that provoke and do not provoke terrorist recruitment, called fire and water strategies. The constraint of the model is the time evolution of the terrorist organization. The authors found that at levels below a certain threshold, it is optimal to use fire strategies. An extension of this model was presented in [13] with the addition of a dynamic constraint describing the dynamics of public support for counterterrorism. They found that DNSS (Dechert-Nishimura-Sethi-Skiba) points may exist.

Although there has been much research effort in the developing and studying the dynamics of terrorism, there have been little work in formulating them as optimal control problems or incorporating them into differential games. We will examine several differential games for the terrorist problem that have been proposed in the literature.

Chapter 2

Differential Games

Let us first establish the mathematical foundation for understanding differential games. We will begin with optimal control and how it relates to the formulation of games.

2.0.1 Optimal Control

The standard model in optimal control theory states that the state of a system is represented by $x \in \mathbb{R}^n$ which evolves in time according to some ordinary differential equation

$$\dot{x}(t) = f(t, x(t), u(t)) \qquad t \in [0, T] \tag{2.1}$$

where $u(\cdot)$ is the *control function* within a set of *admissible control laws* U. We assume $f : \mathbb{R}^n \times U \to \mathbb{R}^n$ to be smooth and bounded so the solution will be defined. For some initial condition $x(0) = x_0$, we wish to find a control function which maximizes the *payoff function*

$$J(u(\cdot)) = \phi(x(T)) - \int_0^T r(t, x(t), u(t))dt \tag{2.2}$$

where $r(\cdot)$ is the *running cost* and $\phi(\cdot)$ is the *terminal payoff*.

There are two fundamental approaches in optimal control in obtaining a solution: (1) Bellman's principle of optimality and (2) Pontryagin's maximum principle.

According to Bellman [14], "An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision." That is, if the state-action sequence is optimal then the remaining sequence is optimal regardless the initial state and action. The solution to HJB, if it exists, is a *value function* which gives the minimum cost. Formally, the value function, defined by,

$$V(x,t) = \min_{u} J(u(\cdot)) \tag{2.3}$$

solves the HJB equation

$$-V_t = \min_{u} \{r(t,x,u) + V_x \cdot f(t,x,u)\} \tag{2.4}$$

with boundary condition given by the terminal payoff:

$$V(x,T) = \phi(x(T)) \tag{2.5}$$

When solved locally, HJB gives a sufficient condition for optimality. When solved in the entire state space, HJB is a necessary and sufficient condition for optimality. As it is a partial differential equation, deriving the value function is difficult or often not possible.

The other approach is Potryagin's minimum principle which seeks to find an admissable control input that minimizes a Hamiltonian function. It provides a necessary condition for optimality. In the context of differential games, either approach can be used to derive optimal strategies though it is more common to use the minimum principle. The formulation of HJB for differential games is straightforward (i.e., it is essentially of the form (2.4) so let us discuss the minimum principle in detail.

2.0.2 Two-Player Differential Game

The two-player differential game extends this optimal control model in which the state $x \in \mathbb{R}^n$ evolves according to the system

$$\dot{x}(t) = f(t, x(t), u_1(t), u_2(t)) \qquad t \in [0, T]. \tag{2.6}$$

where u_i is a measurable mapping or strategy for player i within a set of admissible control laws U_i for $i = 1, 2$.

The *payoff or objective function* of a *zero-sum game* is

$$J(u_1(\cdot), u_2(\cdot)) = \phi(x(T)) - \int_0^T r(t, x(t), u_1(t), u_2(t))dt. \tag{2.7}$$

which player 1 wants to maximize and player 2 wants to minimize. That is, in a zero-sum game, a gain for player 1 represents a loss for player 2.

In a *non-zero-sum game*, we represent the payoff as follows

$$J_i(u_1(\cdot), u_2(\cdot)) = \phi_i(x(T)) - \int_0^T r_i(t, x(t), u_1(t), u_2(t))dt. \tag{2.8}$$

Both players want to maximize this payoff with controls $u_1(\cdot)$ and $u_2(\cdot)$, respectively.

2.0.3 Information Structure and Equilibrium

It is important to consider the information available to the players as it may yield different type of equilibrium solutions to the problem. A player can have information regarding either the current state of the system or the control of the other player. If player i can observe the current state of the system then he can adopt a *feedback loop* of the form $u_i = u_i(x, t)$. Otherwise, he can implement an *open-loop* of the form $u_i = u_i(t)$.

In a non-cooperative game involving two or more players, the *Nash equilibrium* is an optimal outcome in which players have no incentive to change his or her strategy after considering an opponent's choice. Note, the Nash solution may not be unique. Formally, we define a Nash equilibrium of our differential game as follows.

Definition 1 *(Open-loop Nash Equilibrium) The pair of controls* $(u_1^*(t), u_2^*(t))$ *is a open-loop Nash equilibrium provided*

$$J_1(u_1^*(t), u_2^*(t)) \geq J_1(u_1(t), u_2^*(t))$$

and

$$J_2(u_1^*(t), u_2^*(t)) \geq J_1(u_1^*(t), u_2(t))$$

To obtain a Nash equilibrium, we solve the following optimal control problem simultaneously for both players.

1. $u_1^*(t)$ is a solution to the dynamic optimization problem for Player 1:

$$\text{maximize} \quad J_1(u_1, u_2^*)(x) = \phi_1(x(T)) - \int_0^T r_1(t, x(t), u_1(t), u_2^*(t))dt$$
$$\text{subject to} \quad \begin{cases} \dot{x}(t) = f(t, x(t), u_1(t), u_2^*(t)) & t \in [0, T], u_1 \in U_1 \\ x(0) = x_0 \in \mathbb{R}^n \end{cases} \tag{2.9}$$

2. $u_2^*(t)$ is a solution to the dynamic optimization problem for Player 2:

$$\text{maximize} \quad J_2(u_1^*, u_2)(x) = \phi_2(x(T)) - \int_0^T r_2(t, x(t), u_1^*(t), u_2(t))dt$$
$$\text{subject to} \quad \begin{cases} \dot{x}(t) = f(t, x(t), u_1^*(t), u_2(t)) & t \in [0, T], u_2 \in U_2 \\ x(0) = x_0 \in \mathbb{R}^n \end{cases} \tag{2.10}$$

The following are necessary conditions to determine a Nash solution. Define the Hamiltonian

$$H_i = r_i(x, u_1, u_2, t) + \lambda_i f(x, u_1, u_2, t) \tag{2.11}$$

where λ is a vector containing the co-states of the system which satisfies the adjoint equation $\dot{\lambda}_i = -H_i^x$ and $\lambda^i(T) = \phi_i^x(x(T))$.

The optimal control pair (u_1^*, u_2^*) must satisfy the following conditions

$$H_i(x^*, u_1^*, u_2^*, \lambda, t) \geq H_i(x^*, u_1, u_2^*, \lambda, t) \tag{2.12}$$

$$H_i(x^*, u_1^*, u_2^*, \lambda, t) \geq H_i(x^*, u_1^*, u_2, \lambda, t) \tag{2.13}$$

The Nash equilibrium for a game with feedback is defined as follows.

Definition 2 *(Feedback Nash Equilibrium) The pair of controls* $(u_1^*(x,t), u_2^*(x,t))$ *is a open-loop Nash equilibrium provided*

$$J_1(u_1^*(x,t), u_2^*(x,t)) \geq J_1(u_1(x,t), u_2^*(x,t))$$

and

$$J_2(u_1^*(x,t), u_2^*(x,t)) \geq J_1(u_1^*(x,t), u_2(x,t))$$

Now the controls are dependent on the state variable x, the adjoint equation is now

$$\dot{\lambda}_i = -H_i^x - [(H_1^{u_1} u_1^*) + (H_1^{u_2} u_2^*)] \tag{2.14}$$

The second term makes deriving the feedback Nash solution complex. Note that in the two-player zero-sum game the second term in (2.14) is 0 since $H_1 = -H_2$. Note that very rarely do we have games which result in a unique Nash solution.

If we assume that each player chooses their strategy in a leader-follower way (i.e., player 1 picks his strategy then player 2 follows), then we have a Stackelberg model. For a two-player game, Player 1 first chooses some strategy

$\hat{u_1}(t)$. Player 2 observes this and chooses a strategy $u_2^* \in \mathcal{U}_2^*(u_1)$, where $\mathcal{U}_2^*(u_1)$ is the set of best responses, to maximizes his own payoff relative to $\hat{u}_1(t)$.

The solution approach to the Stackelberg problem is backward induction in that we first solve the Player 2's optimal control problem given the control chosen by Player 1. We then substitute Player 2's response to Player 1's problem and solve. The Stackelberg equilibrium is defined as follows.

Definition 3 *(Open-Loop Stackelberg Equilibrium) The pair of controls* $(u_1^*(t), u_2^*(t))$ *is a open-loop Stackelberg equilibrium, if it exists, provided the following*

1. $u_2^* \in \mathcal{U}_2^*(u_1)$ *when* $u_1 = u_1^*$

2. $J_1(u_1^*(t), u_2^*(t)) \geq J_1(u_1(t), u_2(t))$

Given Player 1 plays $\hat{u}_1$, Player 2's optimal control problem is as follows

$$\begin{aligned} &\text{maximize} \quad J_2(\hat{u}_1, u_2)(x) = \phi_2(x(T)) - \int_0^T r_2(t, x(t), \hat{u}_1(t), u_2(t))dt \\ &\text{subject to} \quad \begin{cases} \dot{x}(t) = f(t, x(t), u_1(t), u_2^*(t)) & t \in [0, T], u_1 \in U_1 \\ x(0) = x_0 \in \mathbb{R}^n \end{cases} \end{aligned} \tag{2.15}$$

Let us derive the necessary conditions using a maximum principle argument. Player 2's Hamiltonian is

$$H_2(t, x, \lambda, u_1, u_2) = r_2(t, x, u_1, u_2) + \lambda_2 f(t, x, u_1, u_2) \tag{2.16}$$

where λ_2, again, is the vector satisfying the adjoint equation

$$\begin{aligned}
\dot{\lambda}_2 &= \frac{\partial H_2^T}{\partial x}(t, x, \hat{u}_1, u_2, \lambda_2) \\
\lambda_2(T) &= \phi_2^x(x(T))
\end{aligned} \tag{2.17}$$

We wish to obtain an optimal response function for Player 2 as $u_2^* = \text{argmax}_{u_2}\{H_2\} = \hat{u}_2(t, x, \lambda_2, u_1)$, such that if u_2^* is an interior solution, we have the following first order condition

$$\frac{\partial H_2(t, x, \lambda_2, u_1, \hat{u}_2)}{\partial u_2} = 0 \tag{2.18}$$

Given the optimal response function for Player 2, $\hat{u}_2$, we can derive the problem for Player 1:

$$\begin{aligned}
&\text{maximize} \quad J_1(u_1, u_2^*)(x) = \phi_1(x(T)) - \int_0^T r_1(t, x(t), u_1(t), \hat{u}_2(t, x, \lambda_2, u_1))dt \\
&\text{subject to} \quad \begin{cases}
\dot{x}(t) = f(t, x(t), u_1(t), \hat{u}_2(t, x, \lambda_2, u_1)) \qquad t \in [0, T], u_1 \in U_1 \\
\dot{\lambda}_2 = \frac{\partial H_2^T}{\partial x}(x, \hat{u}_1, \hat{u}_2, \lambda, t) \\
\lambda_2(T) = \phi_2^x(x(T)) \\
x(0) = x_0 \in \mathbb{R}^n
\end{cases}
\end{aligned} \tag{2.19}$$

Assumming that all functions are continuously differentiable, we can apply the maximum principle. Player 1's Hamiltonian is given as

$$
\begin{aligned}
H_1(t,x,\lambda_2,\lambda_1,\gamma_1,u_1) =& r_1(t,x,u_1,\hat{u}_2(t,x,\lambda_2,u_1)) \\
&+ \lambda_1 f(t,x,u_1,\hat{u}_2(t,x,\lambda_2,u_1)) \\
&+ \gamma_1\left(\frac{\partial H_2^T}{\partial x}(x,\hat{u}_1,\hat{u}_2,\lambda,t)\right)
\end{aligned}
\tag{2.20}
$$

where the vectors must satisfy the following adjoint equations:

$$
\begin{aligned}
\dot{\lambda}_1 &= \frac{\partial H_1^T}{\partial x}(t,x,u_1,\hat{u}_2(t,x,\lambda_2,u_1),\lambda_2,\lambda_1,\gamma_1) \\
\dot{\gamma}_1 &= \frac{\partial H_1^T}{\partial \lambda_2}(t,x,u_1,\hat{u}_2(t,x,\lambda_2,u_1),\lambda_2,\lambda_1,\gamma_1) \\
\lambda_1(T) &= \phi_1^x(x(T))
\end{aligned}
\tag{2.21}
$$

The necessary condition for the Stackelberg game is obtained when

$$
u_1^* = \underset{u_1}{\operatorname{argmax}}\{H_1(t,x,\lambda_2,\lambda_1,\gamma_1,u_1)\} \tag{2.22}
$$

2.1 Special Case: Linear Quadratic Games

A special case of differential games are games in which the state equations are linear and the payoff function is quadratic. Linear quadratic differential games have been well-studied and are very popular in application due to the fact that analytical solutions are more easily attained. Let us examine a two-player zero-sum game.

Suppose we have state equation given by

$$\dot{x}(t) = Ax(t) + Bu_1(t) + Cu_2(t) \tag{2.23}$$

with $x(0) = x_0$ and a quadratic payoff function given by

$$J(u_1, u_2) = \int_0^T \left(x^T Q x + u_1^T R u_1 - \gamma u_2^T u_2 \right) dt + x^T(T) M x(T) \tag{2.24}$$

where $A, B, C, Q, Q_T, R, \gamma$ are constant matrices/values and $Q, Q_T, R, \gamma > 0$. Assume also that each player have perfect information. Let us derive the open-loop Nash equilibria for this game.

By definition, the Hamiltonian is given as

$$H = x^T Q x + u_1^T R u_1 - \gamma u_2^T u_2 + 2x^T P(Ax + Bu_1 + Cu_2) \tag{2.25}$$

We can compute the controls from the Hamiltonian to be

$$\begin{aligned} \frac{\partial H}{\partial u_1} &= 2Ru_1 + 2B^T P x = 0 \\ \implies \hat{u}_1 &= -R^{-1} B^T P x \end{aligned} \tag{2.26}$$

and

$$\begin{aligned} \frac{\partial H}{\partial u_2} &= 2\gamma u_2 - 2C^T P x = 0 \\ \implies \hat{u}_2 &= \gamma^{-1} C^T P x \end{aligned} \tag{2.27}$$

Substituting these controls back into the Hamiltonian, we obtain the Riccati equation

$$\dot{P} = Q - PBR^{-1}B^T P + PA + A^T P + \frac{1}{\gamma} PCC^T P \tag{2.28}$$

If there exists a solution $P(t) > 0$ to the Riccati equation with boundary condition $P(T) = M$, then $(\hat{u}_1, \hat{u}_2)$ is the saddle-point solution.

Note, there is an alternative approach in deriving the open-loop Nash equilibrium similar to what we had discussed in the general case of the previous section. Instead of solving the Riccati equation, we can characterize the open-loop Nash equilibrium for the linear quadratic game by explicitly solving the linear canonical system in the state and co-state variables. That is, instead of substituting (2.26) and (2.27) into the Hamiltonian, we substitute the controls into the state equation (2.23) and formulate the adjoint equations for both players. The solution for this canonical system of linear differential equations will yield the open-loop Nash equilibrium.

Chapter 3

Applications to Counter-Terrorism

In this chapter we will discuss several applicable differential games that is applicable to counter-terrorism in detail. We will formulate several different games within the context of studying terrorism and will derive solution strategies either numerically or analytically.

3.1 Resource-Allocation Games

The modeling of terrorism as a resource or stock is quite common in the economics and policy literature [15, 16, 17]. Such modeling approach allows for the application of game theory in the form of resource-allocation games where the government's and terrorist organization's (TO) decisions affect how much capital stock is available for the terrorist. We refer to [18] for the application of the resource-allocation game to the terrorist problem with the classical game theory approach. A differential game approach was studied in [19]. The authors modeled terror stock as a state variable and analyzed a two-player non-zero sum differential game. Under the assumption of state-separability, they explicitly derived Nash and Stackelberg solutions.

Let us consider a resource-allocation game which exhibits a Nash solution using a simple single-state model. Assume that TO is seeking to maximize its members through recruitment and the government is seeking to how to best response. TO has a cap for how many members it can sustain given its resources.

Example 3.1.1 (Nash Game) *Let the state variable x represent the number of terrorists which changes according to*

$$\dot{x} = \alpha x(t)\left(1 - \frac{x(t)}{M}\right) + u_1 x(t) - u_2 x(t) \tag{3.1}$$

where α is the intrinsic growth rate, M is the maximum number of terrorists TO can support, u_1 is the intensity of recruitment and u_2 is the intensity of government action. Assume the initial condition is $x(0) = x_0$. This is an example of the Verhulst growth model. Let us assume the payoff (3.2) for the government and (3.3) for the terrorist. Compute the open-loop Nash equilibrium.

We wish to determine the pair of controls (u_1^*, u_2^*) such that $J_1(u_1^*, u_2^*) \geq J_1(u_1, u_2^*)$ and $J_2(u_1^*, u_2^*) \geq J_2(u_1^*, u_2)$. This will require solving the TO's and the government's problems simultaneously. Recall from the previous section that the optimal controls are determined in terms of the adjoints.

$$\begin{aligned} u_1^*(t) &= \underset{u_1}{\operatorname{argmax}}\{\lambda_1 u_1 + \beta_1 ln(u_1 x)\} = -\frac{\beta_1}{\lambda_1} \\ u_2^*(t) &= \underset{u_2}{\operatorname{argmax}}\{-\lambda_2 u_2 + \beta_2 ln(u_2)\} = \frac{\beta_2}{\lambda_2} \end{aligned} \tag{3.2}$$

Noticed that we do not have to solve one player's problem and then use it to determine the other's optimal strategies like the Stackelberg problem. The state and the adjoints λ_1 and λ_2 are determined by the following boundary value problem:

$$\begin{aligned}
\dot{x} &= \alpha x(t)\left(1-\frac{x(t)}{M}\right)+u_1^*x(t)-u_2^*x(t)\\
\dot{\lambda}_1 &= -\lambda_1\left(-\alpha\left(1-\frac{2x}{M}\right)\right)+(u_1^*-u_2^*)+\frac{\beta_1}{x}\\
\dot{\lambda}_2 &= -\lambda_2\left(-\alpha\left(1-\frac{2x}{M}\right)\right)+(u_1^*-u_2^*)\\
x(0) &= x_0\\
\lambda_1(T) &= \alpha\\
\lambda_2(T) &= 1
\end{aligned} \tag{3.3}$$

Substituting in the values for (u_1^*, u_2^*) into equation (3.3), we obtain the following:

$$\begin{aligned}
\dot{x} &= \alpha x(t)\left(1-\frac{x(t)}{M}\right)+\left(-\frac{\beta_1}{\lambda_1}-\frac{\beta_2}{\lambda_2}\right)x(t)\\
\dot{\lambda}_1 &= \alpha\lambda_1\left(1-\frac{2x}{M}\right)+\left(-\frac{\beta_1}{\lambda_1}-\frac{\beta_2}{\lambda_2}\right)+\frac{\beta_1}{x}\\
\dot{\lambda}_2 &= \alpha\lambda_2\left(1-\frac{2x}{M}\right)+\left(-\frac{\beta_1}{\lambda_1}-\frac{\beta_2}{\lambda_2}\right)\\
x(0) &= x_0\\
\lambda_1(T) &= \alpha\\
\lambda_2(T) &= 1
\end{aligned} \tag{3.4}$$

In Figure 3.1, we provide a numerical solution for the nonlinear system (3.4) with parameters $x_0 = 1, \alpha = 0.25, M = 200, \beta_1 = 3$, and $\beta_2 = 5$. We can determine the optimal controls for the government and TO as both depends on the adjoint variables, λ_1 and λ_2, respectively. We have determined the rate of which TO would want to recruit and how the government's response. Neither players have the incentive to deviate from this strategy.

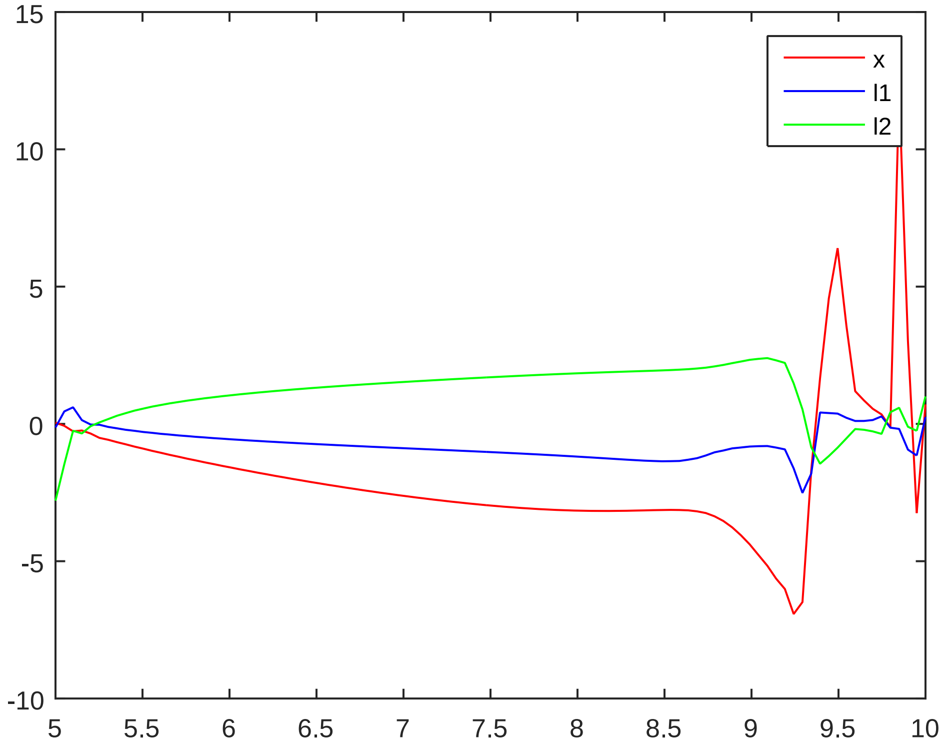

Figure 3.1: Numerical solution for Example 3.1.1 with $x_0 = 1, \alpha = 0.25, M = 200, \beta_1 = 3$, and $\beta_2 = 5$

Now, let us examine a Stackelberg game. In this game, the terrorist organization will choose how much of it's resource to use in an attack. The government will observe this action and response. As there is no longer a resource cap, we will use a different growth model than the previous example.

Example 3.1.2 (Stackelberg Game) *Consider a resource game in which TO utilizes its resource (e.g., financial capital, weaponry, etc.) to carry out* *an attack. Let $x(t)$ describe the total resource stock at time t. Let us assume that this quantity evolves according to*

$$\dot{x}(t) = rx(t) - u_1 x(t) - u_2 \tag{3.5}$$

where r is some constant growth rate, u_1 is a terror attack and u_2 is a government counter-terror response. This can be thought of as a simple economic growth model in which we can think of u_1 as a tax which depletes the terror stock and u_2 as instantaneous consumption. The payoffs for the government and terrorist, respectively, are:

$$J_1 = \alpha x(T) + \int_0^T \beta_1 ln(u_1(t)x(t))dt \tag{3.6}$$

$$J_2 = x(T) + \int_0^T \beta_2 ln(u_2(t))dt \tag{3.7}$$

Let us assume an open-loop information structure where both players cannot observe the current resource stock and TO is the leader. TO will announce an attack $\hat{u}_1$ and the government, as followers, must respond appropriately. Derive the Stackelberg solution for this game.

The Hamiltonian for the government is given as

$$H_2(u_1, u_2, x, t, \lambda_2) = \beta_2 ln(u_2) + \lambda_2(rx - u_1 x - u_2) \tag{3.8}$$

where λ_2 is the adjoint variable satisfying the equation

$$\begin{aligned} \dot{\lambda_2} &= -\frac{\partial H_2(u_1, u_2, x, t, \lambda_2)}{\partial x} = -\lambda_2(r - u_1) \\ \lambda_2(T) &= \frac{\partial (x(T))}{\partial x} = 1 \end{aligned} \tag{3.9}$$

Also, the necessary condition for the government's optimal control is

$$u_2^*(t) = \underset{u_2}{\text{argmax}}\{-\lambda_2 u_2 + \beta_2 ln(u_2)\} = \frac{\beta_2}{\lambda_2} \tag{3.10}$$

Now, we can formulate TO's optimal control problem as follows:

$$\begin{aligned} &\text{maximize} \quad \alpha x(T) + \int_0^T \beta_1 ln(u_1(t)x(t))dt \\ &\text{subject to} \quad \begin{cases} \dot{x}(t) = rx - u_1 x - \frac{\beta_2}{\lambda_2} \\ \dot{\lambda}_2 = -\lambda_2(a - u_1) \\ \lambda_2(T) = 1 \\ x(0) = x_0 \end{cases} \end{aligned} \tag{3.11}$$

Using the maximum principle, we will derive a Stackelberg solution. Solving $\frac{\partial H_1}{\partial u_1} = -\gamma_2 u_1 + \gamma_1 \lambda_2 + \gamma_0 \beta_1 \frac{1}{x} = 0$, we obtain the solution for TO:

$$u_1^*(t) = \underset{u_1}{\text{argmax}}\{\gamma_2(-u_1 x) + \gamma_1 \lambda_2 u_1 + \gamma_0 \beta_1 ln(u_1 x)\} = \frac{\gamma_0 \beta_1}{\gamma_1 x - \gamma_2 \lambda_2} \tag{3.12}$$

Given the pair of controls (u_1^*, u_2^*), we need to solve the system of ODEs consisting of the state and adjoints

$$
\begin{aligned}
&\dot{x} = rx - \tfrac{\gamma_0\beta_1}{\gamma_1 x - \gamma_2\lambda_2} x - \tfrac{\beta_2}{\lambda_2} \\
&\dot{\lambda}_2 = -\lambda_2 \left(r - \tfrac{\gamma_0\beta_1}{\gamma_1 x - \gamma_2\lambda_2} \right) \\
&\dot{\gamma}_1 = -\gamma_0 \tfrac{\beta_1}{x} - \gamma_1 \left(\tfrac{\gamma_0\beta_1}{\gamma_1 x - \gamma_2\lambda_2} \right) \\
&\dot{\gamma}_2 = -\gamma_1 \tfrac{\beta_2}{\lambda_2} + \gamma_2 \left(\tfrac{\gamma_0\beta_1}{\gamma_1 x - \gamma_2\lambda_2} \right) \\
&x(0) = x_0 \\
&\lambda_2(T) = 1 \\
&\gamma_1(T) = \gamma_0 b \\
&\gamma_2(0) = 0
\end{aligned}
\tag{3.13}
$$

to obtain the Stackelberg solution. Given the terminal conditions, it is very difficult to solve this system of ODEs, even numerically. However, we can still gather some insights about the game based on the optimal control for the players without explicitly deriving the solution. Novak, Feichtinger & Leitmann [19] found that both players are more cautious in the Stackelberg game in comparison to the Nash game which leads to a lower resource utilization. In addition, in a Stackelberg game, the follower have a lower value while the leader has a higher value compared to their respective values in the Nash game.

3.2 Information Accumulation Game

Linear quadratic games are widely used in the macroeconomics literature. The common application is that the state and/or control variables describe some type of deviation from economic indicators. These deviation are penalized by quadratic cost functions and it is the goal of the decision makers to minimize these quadratic deviations from a target value. Often, it is very difficult to model the underlying economic system and so the assumption of linearity in quite common. We refer the reader to the text [20] for a survey of differential games in the economic literature.

A common game used in macroeconomics is the capital accumulation type of game where players invest in a public stock such as knowledge. In the context of the counter-terrorism, [18] discussed the mechanisms to foster international counter-terrorism cooperation between governments and law enforcement agencies. The author also discuss the sharing of information between terrorist organizations. Let us look at a game of information accumulation.

Let the state variable, $x(t)$, denote the stock of knowledge which accumulates according to:

$$\dot{x}(t) = u_1(t) + u_2(t) - \alpha x(t) \tag{3.14}$$

Here, u_1 and u_2 can be considered either governments or terrorist organizations who wish to increase information stock. As information can become obsolete over time, let us define α as some constant depreciation rate.

Let us assume that each player's utility function is linear from the consumption of information and so the payoff is given by:

$$J_i = \int_0^T e^{-rt}[x(t) - C_i(u_i)]dt + e^{-rT}Q_i x(T) \tag{3.15}$$

where r is a discount rate, Q_i is the terminal value of information for player i and C_i is the cost of getting that information for player i. Let us assume that the cost is quadratic, given by $C_i(u_i) = \frac{1}{2}u_i^2$. Given this linear quadratic game, let us now derive the open-loop Nash equilibrium.

It is possible to solve accumulation games using an HJB approach (see [20] and [21]) by assuming the general form of the value function to be:

$$V_i(x,t) = a_i(t)x + b_i(t) \tag{3.16}$$

Therefore, let us attempt to derive the Nash equilibrium using by solving the HJB equation. The HJB equations for this game is given by:

$$r(a_i(t)x+b_i(t))-(\dot{a}_i(t)x+\dot{b}_i(t)) = \max_{u_i}\left\{x - C_i(u_i) + \frac{\partial V_i(x,t)}{\partial x}\left(\sum u_i - \alpha x\right)\right\} \tag{3.17}$$

with the boundary condition given by the terminal payoff, $V_i(x,T) = Q_i x(T)$.

Assuming the general form of the value function to be (3.16), the right-hand side of (3.17) is given by

$$\frac{\partial C_i(u_i)}{\partial u_i} = u_i = a_i(t) \tag{3.18}$$

We substitute (3.18) into (3.17),

$$r(a_i(t)x + b_i(t)) - (\dot{a}_i(t)x + \dot{b}_i(t)) = \max_{u_i}\left\{x - C_i(u_i) + \frac{\partial V_i(x,t)}{\partial x}\left(\sum u_i - \alpha x\right)\right\} \tag{3.19}$$

$$= \max_{u_i}\left\{x - \frac{1}{2}u_i^2 + a_i(t)(u_1 + u_2(t) - \alpha x)\right\} \tag{3.20}$$

$$= \max_{u_i}\left\{(1 - \alpha a_i(t))\,x - \frac{1}{2}u_i^2 + a(i)(t)(u_1 + u_2)\right\} \tag{3.21}$$

Collecting terms and equation the coefficients of the powers to 0, we obtain the following system of differential equations:

$$\dot{a}_i(t) = -1 + (r+\alpha)a_i(t) \tag{3.22}$$

$$\dot{b}(t) = rb_i(t) + a_i(t) - a_i(t)(a_1(t) + a_2(t)) \tag{3.23}$$

Note, the boundary condition depends on the terminal payoff so $a_i(T) = Q_i$ and $b_i(T) = 0$. We solve this first-order ODE analytically to obtain the Nash equilibrium solutions:

$$u_i^* = a_i = \frac{1}{r+\alpha} + \left(Q_i - \frac{1}{r+\alpha}\right)e^{(r+\alpha)(t-T)} \tag{3.24}$$

It is interesting to note the limiting behavior of this solution (i.e., going from a finite-horizon game T to an infinite horizon game). As $t \to \infty$, we have $u_i^* = \frac{1}{r+\alpha}$ which implies that the strategies of the players in the finite horizon eventually converges to the strategies of the infinite horizon game which is totally dependent on the discount rate and the depreciation rate of information.

3.3 Search Games

One type of differential games that is used to model search and capture operations is called a *search game*. A search game is a two-person non-zero sum game in which a *searcher* chooses continuous trajectories to find a *hider* who seeks cover within a certain area which the searcher is aware of. The goal of the hider is to maximize capture time whereas the searcher wants to minimize it.

Search games were first introduced by Isaacs [1] with the classical *Princess-Monster*. In the Princess-Monster game, a monster tries to capture a princess in a dark room $\mathcal{D}$ of arbitrary shape. The original problem has the monster moving along continuous trajectories at a set speed and the princess moving along continuous trajectories at arbitrary speed. Capture happens if the distance between the monster and princess is smaller than some threshold.

Although they draw many similarities, search games different from *pursuit-evasion games* in that there is no direct visual contact between the players (i.e., a pursuer sees where the evader is going while a searcher does not). Search games have extensive applications in operations research, graph theory and computer science (see [22] for a survey). Fokkink and Lindelauf [23] provided several applications of search games to the study of counter-terrorism. In particular, they discussed the solution approach for single-agent (i.e., one searcher) discrete and differential games.

Let us describe a single-agent *moving fugitive game* which will give us insight on how to best formulate a capture strategy for a hidden terrorist.

Example 3.3.1 (Moving Fugitive) *A terrorist, denoted H, attempts to hide in a finite number of locations. The authorities, S, are aware of when H moves but does not know to which location. The value of the game is the probability that capture occurs. Let $Q(t)$ be the probability distribution function of the capture time τ. Whenever H moves, a new stage of the game begins and the probability of capture changes according to*

$$\dot{Q}(t) = g(t)(1 - Q(t)) \tag{3.25}$$

with initial condition $Q(0) = 0$ and $g(\cdot)$ is any unbounded strictly increasing function. Here, $g(\cdot)$ can be viewed as a search function. What can we conclude about the terrorist's behavior?

We can solve this differential equation directly to obtain

$$Q(t) = 1 - exp\left\{-\int_0^t g(s)ds\right\} \tag{3.26}$$

The terrorist H will need to determine the time h to start a new stage of the game (i.e., when to move). Let $Q(h)$ be the probability that H is captured before time h so $1 - Q(h)$ is the probability that a new stage will begin. So, the problem of H is

$$\max_h \, (1 - Q(h))(h + \tau) + \int_0^h t\dot{Q}(t)dt \tag{3.27}$$

where τ is the expected capture time. Differentiating, we find that $\tau = \frac{(1-Q(h)}{\dot{Q}(h)} = \frac{1}{g(h)}$ so the optimal time for h must satisfy

$$h = g^{-1}(1/\tau) \tag{3.28}$$

We see that the expression in (3.27) is equal to τ upon plugging (3.25) and (3.26). Since $g(\cdot)$ is strictly increasing and unbounded, an interesting conclusion can be drawn about the terrorist's behavior. If the capture time τ is small, then the time the terrorist will remain dormant is long. Otherwise, if τ is large, then the terrorist's resting time is short.

Since the hider will remain dormant as long as necessary , a natural extension of this game is to look at the incetives which will cause a hider to move. Such extension can be found in [24]. One approach is to view $Q(\cdot)$ as the probability of capture by betrayal. Then, if we define $\dot{Q}(t)$ to be increasing with time, the hider will have an incentive to move to avoid capture.

Another approach is to introduce a better search for when the searcher is looking in the correct location. That is, suppose there exists some $f(\cdot)$ such that $f(t) > g(t)$ for all t. This will make the expected capture time depend on the choice of the hider as well as the searcher at the start of each stage. Owen and McCormick [24] proved that the expected capture time of this stochastic game will converge as the number of moves by the hider approaches infinity.

Chapter 4

Conclusions

We have discussed some applications of differential games to the study of counter-terrorism. From our examples, we've found that we can gather much insight on government-terrorist behavior despite sometime not being able to derive an explicit solution.

For resource-allocation and search games, a natural extension of these games is to include multiple players, say multiple TOs and incorporate co-operative behavior. Terror groups such as Islamic State of Iraq and Syria (ISIS) and Al-Qaeda have formed elaborate cooperative networks and pooled resources. They share intelligence, operatives, logistics and training facilities as well as a common enemy. The resource-allocation dynamics can incorporate multiple inputs which either drain or increase the resource stock. On the other hand, government agencies such as Interpol and the CIA also share resources and so search games with multiple searchers with differing dynamics could be considered.

Another possible extension is to examine the time horizon of the game. Terrorist leaders often do not face a time constraint whereas government leaders are faced with term limits. Time horizon can also be randomized by making T a random variable with some probability distribution Ω. Each player's optimization problem would then be to maximize the expected value of the payoff function. It turns out that games with random time horizon can be recast as a discounted game over the inifinite time horizon [25]. An interesting application would be to use real terrorist data to estimate the density of Ω for the problem.

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