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APPLICATION OF DDDP IN WATER RESOURCES PLANNING

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ABSTRACT

APPLICATION OF DDDP IN WATER RESOURCES PLANNING

This is the completion report for the second phase of a research program on advanced methodologies for water resources planning. It summarizes the various achievements accomplished during the period of the project. The main portion of the report, however, is devoted to the presentation of a working manual for use by practicing water resources engineers and analysts, showing the application of the discrete differential dynamic programming (DDDP) which has been developed in the project. For this portion of the report, the brief theoretical background of the DDDP methodology and a review of the principal aspects of its theory are included. Then, a detailed description of the DDDP methodology is given, giving emphasis to the key steps of its procedure. The DDDP methodology as a means to solve already formulated dynamic programming problems is proposed. For illustrative purposes, three examples are given to show the application of the DDDP methodology in the solution of optimization problems arising from the planning and operation of complex water resources projects.

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PREFACE

This is the final report of the OWRR Project B-060-ILL entitled "Advanced Methodologies for Water Resources Planning - Phase II," which is a continuation of "Advanced Methodologies for Water Resources Planning - Phase I" and covers a study period of July 1971 to January 1974. The final report for the Phase-I study was published as University of Illinois Water Resources Center Research Report No. 47 (UILU-WRC-71-0047) in November, 1971, and entitled "Methodologies for Water Resources Planning: DDDP and MLOM(TLOM)" by Ven Te Chow, the Principal Investigator (Publication Board No. 205750, Clearinghouse for Federal, Scientific and Technical Information, now named National Technical Information Service, U. S. Department of Commerce, Springfield, Virginia, 22151) [Chow, 1971].

The main objective of the Phase-II research is to develop a general analytical procedure of water resources planning by enhancing and extending the value of the results produced by Phase I of the research program on advanced methodologies for water resources planning. The overall objective of the entire research program is to investigate a number of advanced concepts in water resources planning which are of basic importance but have not been generally introduced into practice. Modern concept of water resources planning is to formulate water resources problems as hydroeconomic systems and then to optimize the systems by stochastic theory and operations research techniques.

Phase I of the research program has already produced new optimization tools for planning, including DDDP (Discrete Differential Dynamic Programming) and MLOM (Multi-Level Optimization Model). For purposes of the Phase-II research, these tools were further examined in order to enhance their effectiveness by extending them to the practical realm of water resources planning and by applying them to the analysis of several water resources problems to test hydrologic, economic, urban and other environmental aspects of the problems.

For Phase-II of the research program, the major achievements are as follows:

(1) Optimal operation of a flood-control reservoir system. The technique of discrete differential dynamic programming that was developed in the Phase-I research was applied successfully to an actual problem. The technique is applied to the operation of a system of three flood control reservoirs (Huntington, Salamonie, and Missisinewa) located in the Upper Wabash River basin in Indiana. This system of reservoirs is developed mainly for flood control, although seasonal pools are specified by the U. S. Army Corps of Engineers for recreation, fish and wildlife. The objective of this study is to determine a set of system operating rules which minimize annual flood damage under the assumption that the seasonal pool requirements are system constraints. First, a flood routing model was developed for the reservoir system. Then, the DDDP technique is used to determine the optimal operation. It is interesting to find that the estimated natural flood damage for the year 1969 would be \$520,288. By actual regulation of the reservoirs by the Corps of Engineers, the damage is reduced to \$239,708. With optimization by DDDP, the damage could be further reduced to \$184,746 but it is very close to the damage reduced by actual regulation. Finally, an operating model is formulated by multiple regression. Use of this operating model yields a damage of \$190,337, having a difference of only 3.03% from the direct optimization. This proves, for this example, that the DDDP technique can be a very useful tool to develop an operating rule

formulation for the problem of project expansion, the proposed methodology requires that each specific flood control planning problem be dealt with individually. Thus, the proposed methodology must be considered as a general guideline to arrive at optimum flood control plans in a specific planning situation. This is demonstrated through its application to an actual river basin in Illinois (Embarras River), using the technique of discrete differential dynamic programming developed in this project to deal with the computational complexity involved in the dynamic programming analysis for the sequential expansion of the structural components of the flood control project for the actual river basin.

This study is reported in detail in a doctoral thesis by G. Cortes-Rivera [see (h) in item (5), or Cortes-Rivera, 1973].

(4) Optimal planning of a water quality system. This study is to develop a methodology for the optimal planning of a water quality management system in a river basin where multiple sources of wastewater are interacting in the receiving water. The optimal system is defined as a least-cost combination of water quality management alternatives which if implemented will insure meeting the specified water quality standards. The optimization of the system can be attained by applying programming technique such as the DDDP mentioned above. To demonstrate the practical application of the system optimization technique, Salt Fork of Vermilion River located in eastern Illinois is selected for analysis. This river basin is chosen because of its adequate size, reasonably well-kept records and convenient location. Data from waste treatment plants in the basin collected from the Champaign County Sanitary District are analyzed. To properly model the probabilistic characteristic of the water quality system, it is necessary to base the analysis on short-interval, such as

daily, measurement of variables. Data from other basins are also being collected and analyzed to supplement missing data in the chosen basin. Daily records of flow rate and quality parameters of both wastewater and receiving water are studied by probabilistic analysis. The probabilistic analysis provides a mathematical model for water quality variables for which historical records are available. It also gives a basis for confidently estimating variables when historical records are incomplete or missing. It is found that in some cases wastes associated with natural watershed flow, agricultural runoff and storm water may contribute significantly to the deterioration of stream water quality. Some efforts are therefore made to collect information in this respect which is not generally available and difficult to find. This study is being completed and will be reported in a doctoral thesis by C. L. Yin during the spring semester of 1974.

(5) Publications, theses and consultation. One paper reporting optimal operation of water resources systems was presented at the International Symposium on Mathematical Models in Hydrology in Warsaw, Poland, and is published in the Symposium Proceedings. One report on DDDP and MLOM is published as a University of Illinois Water Resources Center Research Report. One paper on farm irrigation model is published in the American Society of Civil Engineers Proceedings, and another paper on multi-reservoir optimization model has been submitted for publication in the ASCE Proceedings.

Two doctoral theses were completed. A third doctoral study was finished and its thesis will be submitted during the spring semester of 1974.

of the reservoir system.

This study is reported in a doctoral thesis by J. E. Schaufelberger [see (g) in item (5), or Shaufelberger, 1971] on page 8.

(2) Development of a multi-reservoir optimization model. In this study, the analysis of river basin developments is viewed as a multi-level optimization problem. For the analysis, mixed integer programming is coupled with historical or stochastically generated, streamflow sequences to derive the optimal design for a complex river basin development. In formulating the model, emphasis is placed on the interrelationships which exist between the various components of the system and the coordination and integration of these components into a single economic unit. The proposed model is designed to determine simultaneously the optimal set and sizes of reservoirs in the system, the optimal target outputs for the tangible water uses, power and irrigation, and the optimal operating procedure for attaining these outputs subject to the technological constraints. Intangible water users, such as recreation and water quality control, are treated as optional constraints and their imputed values are obtained by a multiple solution technique. Part of the input to this model is provided by the irrigation sub-model developed in Phase I of this research program where effort has been made to optimize in two levels of irrigation subsystems. This continuing study is an extension of the two-level model (TLOM) and the model (MLOM) so developed for a multi-reservoir system may therefore be conceived as a higher level of system optimization. The objective here is to determine the optimal level of development of a river basin system which involves a number of potential dam sites and several competing water users. At this level of optimization it is assumed that the demand and benefit relationships for the various purposes have already been determined at a lower level of optimization.

This study is reported in a paper by Windsor and Chow [see (e) and (f) in item (5), or Windsor and Chow, 1972, 1973] on page 8.

(3) Application of systems modeling to flood control planning. For the study of the mathematical modeling and optimization of water resources project expansion, a methodology is developed for planning flood control projects which are composed of structural as well as non-structural elements. This methodology is based on mathematical programming techniques that allow the identification of optimal flood control plans over the planning period. In particular, a linear programming formulation is developed for the problem of finding the optimum extent to which non-structural measures should be applied, and an algorithm based on dynamic programming is utilized for the analysis of the optimum expansion of the structural components of the flood control project. The optimum flood control plan is obtained from the results of the linear and dynamic programming algorithms. These, in turn, incorporate flood-hydrology information resulting from regional flood-frequency analysis. In general, investigations on project-expansion problems in water resources planning have been mostly concentrated on planning water supply and wastewater treatment plants. Considering the flood control planning process as a problem of project expansion is a contribution to such an investigation in general, and to the study of non-structural alternatives, in particular. The proposed methodology for the first time introduces the use of mathematical programming techniques to the solution of flood control planning for finding the optimum combination of structural and non-structural alternatives. In fact, the planning methodology developed in this study can be followed in any situation where comprehensive flood control planning is desired or required. In this sense, it is a methodology of general applicability. However, insofar as it incorporates a dynamic programming

Several meetings discussing problems of the research were held with outside consultants and experts including: Dr. Walter O. Wunderlich and Dr. L. N. Fan, Systems Engineers, Tennessee Valley Authority; Ing. Milan Andjelić, Institute of Automation and Telecommunication, Belgrade, Yugoslavia; Ing. Vladimir Petrović, Technical Director, Morava River Corporation, Republic of Serbia, Belgrade, Yugoslavia; Dr. J. L. Serafim, President-Director, COBA, Lisbon, Portugal; and Dr. Warren A. Hall, then Professor of Civil Engineering, University of California, Riverside, California, and now Director, U.S. Water Resources Research Center.

The following is a list of publications and theses produced in the Phase-II research of the project:

(a) Chow, V. T., "General Report on Optimal Operation of Water Resources System," *Proceedings*, International Symposium on Mathematical Models in Hydrology, Warsaw, Poland, 26-31 July 1971, Separate Volume, pp. 1-9, July 1971.

(b) Chow, V. T., "Methodologies for Water Resources Planning: DDDP and MLOM(TLOM)," Water Resources Center, *Research Report No*. 47, UILU-WRC-71-0047, University of Illinois, Urbana, Illinois, 50 p., November 1971.

(c) Windsor, J. S., and V. T. Chow, "Model for Farm Irrigation in Humid Areas," *Journal of the Irrigation and Drainage Division, Proceedings,* American Society of Civil Engineers, Vol. 97, No. IR3, pp. 369-385, September 1971.

(d) Windsor, J. S., and V. T. Chow, "Model for Farm Irrigation in Humid Areas," *Transactions*, American Society of Civil Engineers, Vol. 137, pp. 687-688, 1972.

(e) Windsor, J. S., and V. T. Chow, "Multireservoir Optimization Model," *Journal of the Hydraulics Division, Proceedings*, American Society of Civil Engineers, Vol. 98, No. HY10, pp. 1827-1845, October, 1972.

(f) Windsor, J. S., and V. T. Chow, "Multireservoir Optimization Model," *Transactions*, American Society of Civil Engineers, Vol. 138, pp. 532-533, 1973.

(g) Schaufelberger, J. E., "A Systems Approach to the Operation of Flood Control Reservoirs," *Ph.D. Thesis*, University of Illinois at Urbana-Champaign, Urbana, Illinois, 1971.

(h) Cortes-Rivera, G., "Flood Control Project Planning by Mathematical Programming: A Project-Expansion Approach," *Ph.D. Thesis*, University of Illinois at Urbana-Champaign, Urbana, Illinois, 1973.

(6) Application of research results. This research project deals mainly with the practical application of the methodologies developed in the project. The DDDP technique was applied very successfully to a U. S. Army Corps of Engineers' project on the operation of three reservoirs (Huntington, Salamonie and Missisinewa) of the Upper Wabash River. Both DDDP and MLOM were applied, as class assignments at the University of Illinois, to the Lincoln Lake project on the Embarrass River near Charleston, Illinois which is also a Corps of Engineers' project that has caused a great deal of local concern.

The DDDP and MLOM techniques have been used by the Texas Water Development Board. Regarding this application, the following are excerpts from a letter dated February 2, 1972, from Mr. Arden O. Weiss, former Director of Systems Engineering Division of TWDB:

"The work described in the Office of Water Resources Research completion report, entitled "Methodologies for Water Resources Planning: DDDP and MLOM(TLOM)," and personal interactions during the evolution of the subject report were most beneficial to me and the other Systems Engineering staff of the Texas Water Development Board. This was especially true as we were formulating and implementing solution methodologies for analyzing the dynamic interaction response of the semiarid irrigated farmstead with the irrigation surface supply system.

"In this regard, the DDDP concepts are providing the Texas Water Development Board (and others as knowledge of the technique spreads) with the capability to pursue and solve problems normally thought to be outside the computational capability of dynamic programming techniques. Of particular merit is DDDP's applicability to (1) optimizing operating rules for large systems of reservoirs as in the proposed Texas Water System on an aggregated basin-by-basin basis, and (2) optimizing a single reservoir's operating rules at storage and purpose segmentation detail which previously has not been possible.

"Also, the MLOM concepts and procedures were used by the Systems Engineering staff as a point of departure in developing similar multilevel optimization procedures. The procedures being developed . . . are applicable to direct interface with the SIM series of models also being developed by the Texas Water Development Board with partial financial support of the Office of Water Resources Research."

Also of interest is the application of DDDP by the Tennessee Valley Authority to the operation of the TVA reservoir system. The optimal operation of a six-reservoir subsystem of the TVA was solved by the use of DDDP.

The interest in the application of DDDP and MLOM has been developed in other countries. For example, two engineers from Yugoslavia came to Urbana in April 1972 to study these techniques for their application to the Morava River basin development in Serbia.

(7) Preparation of a working manual. Phase II of the research program was supposed to be terminated originally in June 1973 but it was extended to January 1974 in order to develop a working manual on the application of the discrete differential dynamic programming for use by practicing water resources engineers, analysts and planners. Since most of the results of the Phase-II research have been presented elsewhere as listed in item (5) above, the working manual will constituent the major remaining portion of this report.

Many persons participated in this project and contributed to the research. In addition to the Principal Investigator, Ven Te Chow, the following research staff members were at various times involved in the project:

Gonzalo Cortes-Rivera, M.S., Ph.D., Research Assistant in Civil Engineering

Freddy Isambert, B.S., Research Assistant in Civil Engineering

Dong Hee Kim, M.S., Research Assistant in Civil Engineering

David R. Maidment, B.S., Research Assistant in Civil Engineering

John E. Schaufelberger, M.S., Graduate Student, Civil Engineering

Latino Torelli, M. S., Research Assistant in Civil Engineering

Taylan A. Ula, B.S., Research Assistant in Civil Engineering

James S. Windsor, Ph.D., Research Associate in Civil Engineering

Chang-lung Yin, M.S., Research Assistant in Civil Engineering

This report was mainly prepared by V. T. Chow and G. Cortes-Rivera. They wish to acknowledge, with great appreciation, the contributions of other staff members involved in the project as listed above, and also those of the authors of references quoted in this report.

I. INTRODUCTION

The remarkable progress in recent years in advanced methodologies for water resources planning and development has been made possible mainly through the use of systems analysis and operations research techniques in the solution of complex problems involved in the design and operation of modern water resources projects. Much of this progress is the outgrowth of research programs in universities and other research institutes. In fact, a number of successful applications of these techniques have been achieved by several governmental water planning agencies and private engineering consulting firms. Just as important as the development of these techniques is the transfer of their methodologies to the actual practice so as to benefit the planning and development of water resources projects.

One of the operations research techniques that are found to be applicable to the mathematical analysis of water resources systems is the dynamic programming, the reason being its ability to simulate the nonlinear, sequential-decision characteristic generally exhibited by most problems involved in the planning and operation of water resources projects. However, conventional dynamic programming algorithms are only capable of handling problems of very low dimensionality, i.e., a few state variables, thus severely limiting the usefulness of the dynamic programming technique in water resources systems analysis which often involves many variables. To overcome the difficulties arising from the high dimensionality of water resources systems, the methodology of discrete differential dynamic programming (DDDP) developed at the University of Illinois [Heidari, Chow and Meredith, 1971] may be used in many problems. The DDDP methodology is an iterative technique which permits the solution of high-dimensional dynamic programming problems within the range of computer time and memory capacities of high-speed digital computers now available.

This report is prepared to present a detailed discussion of the DDDP methodology in a form suitable for its application to actual water resources problems by practicing engineers and analysts. An attempt is made to bridge the gap between the theory and the practice in the use of the DDDP methodology. It is hoped that the report will serve as a working manual for practitioners.

The report will summarize in Chapter II the theoretical background of the DDDP methodology and a review of the principal aspects of its theory which has been developed and documented [Heidari, 1970; Heidari, Chow and Meredith, 1970; Chow, 1971]. This will be followed in Chapter III with a description in detail of the DDDP methodology, giving emphasis to the key steps of its procedure. No attempt is made to elaborate on the formulation of dynamic programming problems, but the DDDP methodology as a means to solve already formulated dynamic programming problems is proposed. Finally, in Chapter IV, three examples are provided to illustrate the application of the DDDP methodology in the solution of optimization problems arising from the planning and operation of complex water resources projects.

II. THEORETICAL ASPECTS OF DDDP

2-1. General Considerations.

Discrete differential dynamic programming (DDDP) is a computational method to obtain the solution of optimization problems which can be formulated in terms of dynamic programming (DP), requiring reasonably small computer time and memory. DDDP is based on an iterative technique in which the recursive equation of dynamic programming is used to find an improved solution by confining the search for the improved solution in the neighborhood of a trial solution. Thus, DDDP takes advantage of the knowledge of a previous trial solution to guide the search for a new improved solution.

The application of the DDDP methodology requires that the optimization problem be formulated according to the DP methodology. Since the problem formulation is independent of the computational method used in achieving its solution, DDDP is not an aid to the formulation of a given optimization problem but a powerful computational tool to obtain solution of the problem.

This section briefly describes the theoretical aspects of DDDP. Since DDDP is based on DP, the basic theory of DP will be first presented and then followed by a description of the DDDP.

2-2. The DP.

The DP, first introduced by Bellman [1957], is a methodology of optimization based on the principle of optimality to conceive the formulation of certain class of optimization problems. The problems suitable for DP analysis should allow themselves to be decomposed into a series of sequential problems of smaller magnitude, whose solutions are then combined to obtain the solution of the entire problem. Because of its theoretical characteristics, DP is a convenient tool to formulate sequential-decision problems; that is, problems in which a sequence of interrelated decisions must be determined.

A. Characteristics of DP Problems. To be suitable for DP analysis, an optimization problem should exhibit the following characteristics [Chow and Meredith, 1969]:

(a) It must be a problem which can be divided into stages with a decision required at each stage. In general terms, optimization problems consist of finding the value of N variables which optimize (i.e., maximize or minimize) the value of a given objective function while subject to a set of constraints. By the DP methodology an optimization problem is converted into a sequence of N simpler problems, each of them having to decide on the value of one of the N variables at a time.

Stages may represent points in space or time, or may represent abstract steps in the problem-solving process. In water resources systems analysis, optimization problems arise in the design (space-oriented optimization) or in the scheduling of construction and operation (time-oriented optimization) of the systems; thus, stages refer either to time or to space. In time-oriented problems, stages may be widely separated points in time, up to decades, for example, in construction-scheduling problems; or may be points in time closely following one another, for example, at one-day or shorter intervals, for optimization of the operation of flood control systems. In space-oriented optimization problems, stages may refer to different sites of water delivery, for example, in aqueduct-routing optimization; or to sites of water storage, for example, in problems of selection of dam sites.

(b) Each stage must have a state vector associated with it. The state vector is a set of state variables containing all the information about conditions of a system at a particular stage, and convey information about the variation of these conditions from one stage to the next. When two or more variables are necessary to describe the system conditions, the state vector is said to be multidimensional. Realistic mathematical formulation of optimization problems in water resources systems analysis usually involve the definition of a high-dimensional state vector.

(c) The effect of a decision vector at each stage is to transform the current state vector into a state vector associated with the next stage. The decision vector is a set of variables representing the alternative actions exerted upon a system at each stage. The effects of these actions are evaluated by a suitable measure of effectiveness at each stage and for any feasible state or condition of the system. The character of such a measure of effectiveness must reflect the objectives of the system's design or operation. A decision vector is multidimensional if there are two or more actions affecting the system simultaneously at a given stage.

(d) For given current stage and state of the problem, the optimal decision should be independent of decisions made in previous stages. That is, the information about previous stages relevant to the selection of optimal values for the decision variables is already contained in the values of the state variables at the current stage.

B. <u>The DP Recursive Equation</u>. The recursive equation is the basis for the formulation of any optimization problem by DP. It condenses the sequential nature of the DP methodology and reflects the principle of optimality [Bellman, 1957] which states:

> "an optimal set of decisions has the property that whatever the first decision is, the remaining decisions must be optimal with respect to the outcome which results from the first decision".

Let us assume that the optimization problem can be divided analytically into N discrete stages and that each stage n (n=1, 2, ..., N with n=1 for the first stage, etc.) has associated with it a state vector $\bar{S}_n$. According to the problem characteristic (c) mentioned above, $T_n(\ )$ can be defined as a transformation function which acts on the state vector $\bar{S}_n$ to convert it into a state vector $\bar{S}_{n-1}$ associated with stage n-1, because of the action of the decision vector $\bar{D}_{n-1}$ in the (n-1)th stage. Expressing in mathematical terms:

$$\bar{S}_{n-1} = T_n(\bar{S}_n, \bar{D}_{n-1});\ n = 2,\ldots,N \qquad (2\text{-}1)$$

where $\bar{S}_n \in \{\bar{S}\}_n$, $\bar{S}_{n-1} \in \{\bar{S}\}_{n-1}$, and $\bar{D}_{n-1} \in \{\bar{D}\}_{n-1}$; $\{\bar{S}\}_n$, $\{\bar{S}\}_{n-1}$ and $\{\bar{D}\}_{n-1}$ being, respectively, the admissible values of the state vector for stage n, the state vector for stage n-1, and the decision vector for stage n-1.

Throughout this discussion, a forward algorithm for DP is assumed. For this algorithm, the computation begins with the analysis of the first stage (n=1) and continues sequentially until the analysis of the last stage (n=N) is completed. On the contrary, backward algorithm begins with the analysis of the last stage (n=1) and continues sequentially until the analysis of the first stage (n=N) is performed. In this case, the order of the stages is reversed, or n is counted backward starting with n=1 for the last stage.

Let us further define $R_n(\bar{S}_n, \bar{D}_{n-1})$ as the return, or the measure of effectiveness, of the decision-induced state transformation for stage n. If the problem's objective is to optimize (maximize or minimize) a given function for the effectiveness of the sequence of system transformations, then, the objective function, z, can be expressed as:

$$z = \underset{\bar{D}_n \in \{\bar{D}\}_n}{\text{Optimize}}\ f[R_1(\bar{S}_1,\bar{D}_o), \ldots, R_n(\bar{S}_n,\bar{D}_{n-1}), \ldots, R_N(\bar{S}_N,\bar{D}_{N-1})] \qquad (2\text{-}2)$$

where f[ ] is a function representing the combined effect of system transformations proceeding from an initial or original state vector $\bar{S}_o$ to a final state vector $\bar{S}_N$ resulting from a sequence of decision vectors $\bar{D}_o, \ldots, \bar{D}_{N-1}$.

Usually, the solution of Eq.(2-2)is subject to a set of constraints or conditions of diverse nature (budgetary, physical, political or institutional) imposed upon the system. For example, physical constraints may specify limits to the magnitude of the state or decision variables, budgetary constraints may indicate ceilings to expenditures associated with the decisions, and political or institutional constraints may restrict the system to the real-world situation.

To solve Eq.(2-2) by DP, an N-stage problem is solved sequentially stage by stage of the decision process. For the decomposition of $\bar{D}_n$, two sufficient conditions are [Nemhauser, 1966]: separability, i.e.,

$$f[R_1(\bar{S}_1,\bar{D}_o), R_2(\bar{S}_2,\bar{D}_1), \ldots, R_N(\bar{S}_N,\bar{D}_{N-1})] =$$

$$f_1\{R_1(\bar{S}_1,\bar{D}_o), f_2[R_2(\bar{S}_2,\bar{D}_1), \ldots, R_T(\bar{S}_N,\bar{D}_{N-1})]\} \qquad (2\text{-}3)$$

where f_1 and f_2 are real-valued functions; and monotonicity, i.e.,

$$f_1\{R_1(\bar{S}_1,\bar{D}_o), f_2'[R_2(\bar{S}_2,\bar{D}_1), \ldots, R_T(\bar{S}_N,\bar{D}_{N-1})]\} \geq$$

$$f_1\{R_1(\bar{S}_1,\bar{D}_o), f_2''[R_2(\bar{S}_2,\bar{D}_1), \ldots, R_N(\bar{S}_N,\bar{D}_{N-1})]\} \qquad (2\text{-}4)$$

where $f_2'[\ \ ] \geq f_2''[\ \ ]$, and for all values of $R_1(\bar{S}_1,\bar{D}_o)$. Thus, for the decomposition of $\bar{D}_n$,

$$\underset{\bar{D}_n \in \{\bar{D}\}_n}{\text{Optimize}}\ f[R_1(\bar{S}_1,\bar{D}_o),\ R_2(\bar{S}_2,\bar{D}_1),\ \ldots,\ R_N(\bar{S}_N,\bar{D}_{N-1})]$$

can be converted into

$$\underset{D_o \in \{\bar{D}\}_o}{\text{Optimize}}\ f_1\ \{R_1(\bar{S}_1,\bar{D}_o),\ \underset{\bar{D}_1,\ldots,\ \bar{D}_{N-1}}{\text{Optimize}}\ f_2[R_2(\bar{S}_2,\bar{D}_1),\ \ldots,\ R_N(\bar{S}_N,\bar{D}_{N-1})]\} \quad (2\text{-}5)$$

Additive functions are typical in the analysis of systems for which the objective of their design or operation is to maximize the sum of the returns or the effectiveness of the system transformation, at all the stages. In such a case, the objective function is given by

$$f[R_1(\bar{S}_1,\bar{D}_o),\ \ldots,\ R_N(\bar{S}_N,\bar{D}_{N-1})] = \sum_{n=1}^{N} R_n(\bar{S}_n,\bar{D}_{n-1}) \quad (2\text{-}6)$$

and satisfying the conditions of separability and monotonicity, f[ ] is therefore decomposable.

Now let $F_N(\bar{S}_N)$ be defined as the maximum return from the system transformations from some initial state $\bar{S}_o$ to some final state $\bar{S}_N$. Then,

$$F_N(\bar{S}_N) = \underset{\substack{\bar{S}_n \in \{\bar{S}\}_n \\ \bar{D}_n \in \{\bar{D}\}_n}}{\text{Max}} \sum_{n=1}^{N} R_n(\bar{S}_n,\bar{D}_{n-1}) \quad (2\text{-}7)$$

subject to the transformation functions, Eq. (2-1), and all the constraints imposed on the system. Because the objective function is decomposable,

$$F_N(\overline{S}_N) = \max_{\substack{\overline{S}_N \in \{\overline{S}\}_N \\ \overline{D}_{N-1} \in \{\overline{D}\}_{N-1}}} [R_N(\overline{S}_N, \overline{D}_{N-1}) + \max_{\substack{\overline{S}_n \in \{\overline{S}\}_n \\ \overline{D}_n \in \{\overline{D}\}_n}} \sum_{n=1}^{N-1} R_n(\overline{S}_n, \overline{D}_{n-1})] \qquad (2\text{-}8)$$

By definition

$$F_{N-1}(\overline{S}_{N-1}) = \max_{\substack{\overline{S}_n \in \{\overline{S}\}_n \\ \overline{D}_n \in \{\overline{D}\}_n}} \sum_{n=1}^{N-1} R_n(\overline{S}_n, \overline{D}_{n-1}) \qquad (2\text{-}9)$$

then

$$F_N(\overline{S}_N) = \max_{\substack{\overline{S}_N \in \{\overline{S}\}_n \\ \overline{D}_N \in \{D\}_n}} [R_N(\overline{S}_N, \overline{D}_{N-1}) + F_{N-1}(\overline{S}_{N-1})] \qquad (2\text{-}10)$$

subject to Eq. (2-1) and the constraints imposed on the system. For any stage n (n=1, 2, ..., N), Eq. (2-10) becomes the DP recursive equation where $F_o(\overline{S}_o)$ is a known quantity associated with the initial state vector $\overline{S}_o$ representing the original condition of the system.

C. Computer Memory Required for Solving the DP Recursive Equation. For a digital computer solution of the DP recursive equation, Eq. (2-10), enough storage capacity is required to keep $F_{n-1}(\overline{S}_{n-1})$ and $F_n(\overline{S}_n)$, for all feasible values of the state vectors $\overline{S}_{n-1}$ and $\overline{S}_n$ at two consecutive stages, n-1 and n, and the values $\overline{D}^*_{n-1}(\overline{S}_n)$ of the decision vector $\overline{D}_{n-1}$ which satisfies Eq. (2-10) for every feasible value of the state vector $\overline{S}_n$, n=1, 2, ..., N. This memory requirement may be enormous, as illustrated in the following example:

Suppose that the appropriate formulation of an optimization problem involves N=12 stages, a four-dimensional (M=4) state vector, and a four-dimensional (T=4) decision vector. This is a typical case in the analysis of the yearly operation of a four-reservoir water resources system on a monthly basis. Assume further that each variable of the state vector can be quantized into Q=10 levels. Then, the amount of numbers which must be stored in the computer's memory is, at least, equal to $2Q^M$ for $F_n(\overline{S}_n)$ and $F_{n-1}(\overline{S}_{n-1})$ and equal to TNQ^M for $\overline{D}^*_{n-1}(\overline{S}_n)$, n=1, 2, ..., N. For the given example, the total numbers for $F_n(\overline{S}_n)$, $F_{n-1}(\overline{S}_{n-1})$, and $\overline{D}^*_{n-1}(\overline{S}_n)$ amount to 500,000. Storing these numbers will require at least two million bytes (2,000K), a quantity far larger than the total memory capacity of available digital computers.

It can be seen from the preceding example that the computer memory required for the solution of the recursive equation increases linearly with the dimensionalities of the decision vector and the number of stages, geometrically with the number of quantized values of the state variables, and exponentially with the dimensionality of the state vector. The most critical factor in the determination of the required computer memory is obviously the dimensionality of the state vector which affects the memory exponentially. However, for a given problem, the dimensionality of the state and decision vectors and the number of stages are fixed by the appropriate formulation of the problem, and the accuracy of the solution depends on the fineness of the division of the state and decision variables into quantized values. There exists then, a compromise between the accuracy of a solution and the laboriousness of obtaining it. For a minimum desirable number of quantized values of the state variables, the DP solution of a particular problem may not be possible if the entire set of quantized values of the state variables is considered at a time. In such cases, DDDP allows a compatible computer solution of DP problems with high-dimensional state and decision vectors, without any reduction in the number of quantized values of state and decision variables.

2-2. The DDDP

DDDP is an iterative procedure by which the DP recursive equation may be solved within a restricted set of quantized values of the state variables, thus continuously approaching the optimal solution corresponding to the entire set of quantized values of the state variables. The collection of restricted quantized values of the state variables at all the stages composes what is called a corridor. The composition of corridors varies from one iteration to the next in such a way as to obtain convergence of the algorithm toward the optimal solution for the entire set of quantized values of the state variables. The path of the iterations through a corridor is called a trajectory.

Let the state vector $\overline{S}_n$ at stage n (n = 1, 2, ..., N) be composed of M state variables ($S_{1,n}$, ..., $S_{m,n}$, ..., $S_{M,n}$) each of which is quantized into Q_m values (m = 1, 2, ..., M). Thus, the entire set of quantized values of the variables is

$$
\begin{array}{llllllll}
S_{1,n}: & s_{1,n,1}, & s_{1,n,2}, & \ldots, & s_{1,n,j}, & \ldots, & s_{1,n,Q_1}, & n = 1, \ldots, N \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \\
S_{m,n}: & s_{m,n,1}, & s_{m,n,2}, & \ldots, & s_{m,n,j}, & \ldots, & s_{m,n,Q_m}, & n = 1, \ldots, N \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \\
S_{M,n}: & s_{M,n,1}, & s_{M,n,2}, & \ldots, & s_{M,n,j}, & \ldots, & s_{M,t,Q_M}, & n = 1, \ldots, N
\end{array}
$$

where $s_{m,n,j}$ is the j-th quantized value of the m-th variable at the n-th stage.

Let $q'_{m,n}$ be the number of the Q_m quantized values of the m-th state variable which is considered at a time at stage n, and let $\Delta q_{m,n}$, $(1 \leq \Delta q_{m,n} \leq Q_m$; m = 1, 2, ..., M; n = 1, 2, ..., N) be the spacing, in terms of numbers of quantized values, between adjacent members of the restricted set of the $q'_{m,n}$ numbers. This spacing specifies the width of the corridor for each state variable at every stage.

Let it also be assumed that the optimal trajectory resulting from the (k-1)-th iteration is given by $s^{(k-1)}_{1,t,Q^*_{1,n}}$, $s^{(k-1)}_{2,n,Q^*_{2,n}}$, ..., $s^{(k-1)}_{m,n,Q^*_{m,t}}$, ..., $s^{(k-1)}_{M,n,Q^*_{M,n}}$, (n = 1, 2, ..., N), where $Q^*_{m,n}$, (m = 1, 2, ..., M), represents the location of the value of the m-th state variable within its own entire set of quantized values at stage n, and that $q'_{m,n}$, (m = 1, 2, ..., M; n = 1, 2, ..., N), is taken as 3 for illustrative purposes. The corridor for the k-th iteration is then delineated by

$$
\begin{array}{llll}
S_{1,n}: & s^{(k-1)}_{1,n,L_{1,n}}, & s^{(k-1)}_{1,n,C_{1,n}}, & s^{(k-1)}_{1,n,U_{1,n}} \\
\cdots & \cdots & \cdots & \cdots \\
S_{m,n}: & s^{(k-1)}_{m,n,L_{m,n}}, & s^{(k-1)}_{m,n,C_{m,n}}, & s^{(k-1)}_{m,n,U_{m,n}} \\
\cdots & \cdots & \cdots & \cdots \\
S_{M,n}: & s^{(k-1)}_{M,n,L_{M,n}}, & s^{(k-1)}_{M,n,C_{M,n}}, & s^{(k-1)}_{M,n,U_{M,n}}
\end{array}
$$

where, if $1 \leq Q^*_{m,n} \leq Q_m$,

n = 1, 2, ..., N

$$L_{m,n} = \text{Max}\ (1;\ Q^*_{m,n} - \Delta q_{m,n}),\ m = 1, 2, \ldots, M \qquad (2\text{-}11)$$

$$C_{m,n} = Q^*_{m,n},\ m = 1, 2, \ldots, M \qquad (2\text{-}12)$$

and $$U_{m,n} = \text{Min}\ (Q_m;\ Q^*_{m,n} + \Delta q_{m,n}),\ m = 1, 2, \ldots, M \qquad (2\text{-}13)$$

if $Q^*_{m,n} = 1$,

$$L_{m,n} = Q^*_{m,n}, \quad m = 1, 2, \ldots, M \tag{2-14}$$

$$C_{m,n} = \text{Min}\ (Q_m;\ Q^*_{m,n} + \Delta q_{m,n}), \quad m = 1, 2, \ldots, M \tag{2-15}$$

and $$U_{m,n} = \text{Min}\ (Q_m;\ Q^*_{m,n} + 2\ \Delta q_{m,n}), \quad m = 1, 2, \ldots, M \tag{2-16}$$

and, if $Q^*_{m,n} = Q_m$,

$$L_{m,n} = \text{Max}\ (1;\ Q^*_{m,n} - 2\ \Delta q_{m,n}), \quad m = 1, 2, \ldots, M \tag{2-17}$$

$$C_{m,n} = \text{Max}\ (1;\ Q^*_{m,n} - \Delta q_{m,n}), \quad m = 1, 2, \ldots, M \tag{2-18}$$

and $$U_{m,n} = Q^*_{m,n}, \quad m = 1, 2, \ldots, M \tag{2-19}$$

The corridor for values of $q'_{m,n}$ other than 3 can be developed following a similar reasoning.

As stated above, a trajectory, either initial or optimal, is required for the formation of the corridor for the k-th iteration. Associated with such a trajectory there is a value $F^*_{(k-1)}$ of the objective function which is either calculated from the transformations of the system as specified by the initial trial trajectory (if k = 1) or obtained from the solution of the recursive equation within the corridor for the (k-1)-th iteration (if $2 \le k \le k_{max}$, where k_{max} is a given maximum number of iterations). The solution of the recursive equation within the corridor of the k-th iteration yields a value F^*_k for the objective function, which when compared with the value of the objective function for previous iterations will determine the convergence of the algorithm toward the optimal solution. If the convergence is obtained, the trajectory yielding F^*_k represents the solution to the optimization problem.

The procedure just described can be applied using any value of the spacing parameter $\Delta q_{m,t}$. To ensure that the solution by DDDP of the optimization problem is the one corresponding to the entire set of quantized values of the state variables, the iterative procedure must be performed several times, in cycles, using at each cycle values for the spacing parameters smaller than those used in the preceding cycle. The smallest value of the spacing parameters should be 1, so that the corridors of the final cycle are composed of contiguous values in the original set of quantized values of the state variables.

The above discussions are the bare essentials of the theoretical aspects of DDDP. For details, the reader should be referred to the work of Heidari, Chow and Meredith [1971].

III. THE DDDP PROCEDURE

Based on the theoretical aspects of DDDP given in the previous chapter, a practical procedure of applying the DDDP methodology is developed in this chapter. A brief general presentation of the procedure will be followed by a detailed analysis of its major steps.

3-1. Description of the Procedure.

The general scheme of the DDDP procedure may be concisely represented by a block diagram in Fig. 1. This diagram shows essentially a computer program to carry out the compilations for DDDP. As can be seen from this figure, the procedure is composed of cycles, which in turn are made up of iterations. Each computation cycle corresponds to the process, starting from a trial trajectory, to search for the optimal trajectory within all corridors of a given width. The computation cycle is complete when the search process has converged to the optimal trajectory according to a convergence criterion to be discribed later.

For each iteration of a cycle, the optimal trajectory within a given corridor and its return are determined by the DP methodology. A new iteration is needed if the convergence criterion for the cycle is not satisfied. The number of cycles for the entire procedure and the allowable maximum number of iterations per cycle are specified in advance.

The DDDP procedure terminates either when the convergence criterion for the last cycle is satisfied, or when the maximum number of iterations for such a cycle is exceeded. In the former case, the optimal solution of the entire optimization problem is the optimal solution obtained in the last cycle. In the latter case, only a near-optimal solution is achieved and the entire procedure may then be repeated either allowing a larger number of iterations per cycle, or modifying the convergence criterion.

If the maximum number of iterations for the intermediate cycles is exceeded, the DDDP procedure assigns the current optimal trajectory (a near-optimal trajectory for the cycle) as the trial trajectory for the next cycle.

The scheme of the DDDP procedure shown in Fig. 1 is of general applicability. The development of each of its major steps, however, should depend on the optimization problem under consideration. The remaining sections of this chapter are devoted to the discussions in some detail the major steps of the procedure.

3-2. The Initial Trial Trajectory.

The DDDP procedure begins with the establishment of an initial trial trajectory and its return (see block 1 in Fig. 1). A trajectory is the sequence of transformations of the state vector throughout the entire period N of system analysis. A trajectory is feasible if it satisfies all constraints imposed on the system, and it is optimal if, besides being feasible, it optimizes the objective criterion of system performance.

The basic idea behind the selection of an initial trial trajectory is to provide, for the process of searching the optimal trajectory, both a starting point and a region (i.e., the corridor around the trial trajectory) where the optimal trajectory is expected to lie.

Thus, the initial trial trajectory being the first approximation of the optimal trajectory, should be feasible and also as close as possible to the optimal trajectory. These requirements are to insure that at least one of the alternative trajectories to be examined for optimality has satisfied all the constraints and that the time of computation to reach the optimal trajectory is as short as possible. While the satisfaction of the feasibility requirements can always be assessed, it is not possible to determine in advance how actually close to the optimality is the initial trial trajectory. In general, a satisfactory initial trial trajectory may be established either by engineering judgment or by system decomposition to determine a near-optimal initial trial trajectory.

Engineering judgment relies on information on hydrologic and other data concerning the behavior of the system. For example, in the analysis of the operation of a reservoir network, a trial sequence of monthly releases or storages may be established from the knowledge on the occurrence of wet and dry months or periods of a year. The return corresponding to the trial trajectory established by this approach is then readily calculated by using the values of the state and decision variables of the trial trajectory in the expression of the problem's objective function, Eq. (2-2).

The second approach to determine a satisfactory initial trial trajectory consists of dividing the original system into a number of subsystems, obtaining the trajectories which optimize each subsystem independently, and considering the set of these optimal trajectories as an approximation of the optimal trajectory for the entire system. The optimization of each subsystem should be subject to the same constraints of the original system.

This system-decomposition approach [Larson, 1968], is also applicable to problems where there is small interaction between various state and decision variables. In this case, the transformation function for a given state variable, say the m-th, must depend on a few of the other state and decision variables; namely, those with an index higher than m. Thus,

$$S_{m,n+1} = T_m(S_{m,n}, S_{m+1,n}, \ldots, S_{M,n}; D_{m,n}, D_{m+1,n}, \ldots, D_{M,n}; n) \qquad (3\text{-}1)$$

where $S_{m,n+1}$ and $S_{m,n}$ represent the values of the m-th state variable at stages n+1 and n, respectively; $D_{m,n}$ is the value of the m-th decision variable at stage n; $T_m(\ )$ is the transformation function for the m-th variable; and there are M decision variables and M state variables.

Thus, obtaining a trial trajectory by this approach is to reduce the solution of an optimization problem to that of a series of optimization problems, each possibly having one single-dimensional state variable. As indicated in Fig. 2, each subsystem corresponds to one of the M state variables of the original problem. The optimization of the m-th subsystem incorporates the already obtained optimal trajectories of the preceding subsystems m+1 through M. As an example, for the (M-1)-th subsystem, the optimization problem is to find the trajectories $\hat{S}_{M-1,n}$, n = 1, 2, ..., N, or equivalently, the decision sequence $\hat{D}_{M-1,n}$, n = 1, 2, ..., N, for which the objective function of the subsystem, i.e.,

$$f_{M-1}[r_{M-1}(S_{M-1,1}, D_{M-1,1}, 1), \ldots, r_{M-1}(S_{M-1,N}, D_{M-1,N}, N)]$$

has an optimal value (denoted by $\hat{J}_{M-1}$). Such an objective function is composed of the returns

$$r_{M-1}(S_{M-1,n}, D_{M-1,n}, n);\ n = 1, 2, \ldots, N$$

for the transformations of the subsystem at stages 1 through N; and the transformation equation of the subsystem, that is,

$$S_{M-1,n+1} = T_m(\hat{S}_{M,n}, \hat{D}_{M,n}, S_{M-1,n}, D_{M-1,n}, n);\ n = 1, 2, \ldots, N-1$$

incorporates both the optimal trajectory $[\hat{S}_{M,n}]$, n = 1, 2, ..., N, and the optimal decision sequence $[\hat{D}_{M,n}]$, n = 1, 2, ..., N of the M-th subsystem.

Once the optimizations of all M subsystems are performed, the initial trial trajectory for the total problem is given by the set of optimal trajectories for all subsystems as follows: $[\hat{S}_{m,n}]$, m = 1, 2, ..., M; n = 1, 2, ..., N with the associated trial decision sequence $[\hat{D}_{m,n}]$, m = 1, 2, ..., M; n = 1, 2, ..., N.

Finally, the value of the objective function corresponding to the initial trial trajectory is obtained by providing the values of the state and decision variables of the initial trial trajectory to the objective function, Eq. 2-2.

In the system-decomposition approach to find an initial trial trajectory, the expressions of the objective functions for the optimization of subsystems are of a general nature. For optimization problems of water resources systems, however, most objective functions of the subsystems are additive; i.e., of the form:

$$f_m[r_m(S_{m,1},D_{m,1},1),\ \ldots,\ r_m(S_{m,N},D_{m,N},N)] = \sum_{n=1}^{N} r_m(S_{m,n},D_{m,n},n) \qquad (3\text{-}2)$$

If the return for the transformation of the entire system is a linear (additive) function of the returns for the transformations of the individual subsystems at all stages, that is

$$R_n(\overline{S}_n,\overline{D}_n) = \sum_{m=1}^{M} r_m(S_{m,n},D_{m,n},n);\ n = 1,\ 2,\ \ldots,\ N \qquad (3\text{-}3)$$

and, in addition, there exists no interaction between subsystems, or

$$S_{m,n+1} = T_m(S_{m,n},\ D_{m,n},n);\ n = 1,\ 2,\ \ldots,\ N;\ m = 1,\ 3,\ \ldots,\ M \qquad (3\text{-}4)$$

then, the solution of the optimization problem for the entire system corresponds precisely to the collection of optimal solutions for the subsystems; or, optimal trajectory: $[\hat{S}_{m,n}]$; m = 1, 2, ..., M; n = 1, 2, ..., N with the optimal value, J^*, of the objective function being

$$J^* = \sum_{m=1}^{M} \hat{J}_m \tag{3-5}$$

where $\hat{J}_m$ is the optimal value of the objective function in the optimization of the m-th subsystem. Thus, for this extreme case, the initial trial trajectory coincides with the optimal trajectory.

3-3. Construction of Corridors.

After the determination of an initial trial trajectory, the next step of the DDDP procedure consists of constructing a corridor around it, which specifies the limiting values of the state variables used in the optimization of the system. In the procedure, two actions regarding the corridor should be taken: (a) the selection of an initial width of the corridor (see block 2 in Fig. 1) and its subsequent modifications (see block 9 in Fig. 1), and (b) the actual construction of the corridor around a trial trajectory (see block 4 in Fig. 1).

A. Selection of Corridor Widths. As stated previously, a corridor specifies the values of each state variable which are considered at a time or stage in the optimization process. Any of such values should belong to the feasible range of variation of the state variable under consideration. In order to reduce the computer memory requirement to a minimum in the solution of a multi-dimensional DP problem, the number of values ($q_{m,n}$, using the notation of Section 2-3) of each state variable considered at a time should be as small as possible. The most satisfactory number has been proved to be three [Heidari, Chow, and Meredith, 1971; Shaufelberger, 1971; Cortes, 1973; Tauxe, Hall and Yeh, 1973].

For a given corridor, the difference between adjacent values of a state variable is the width of the corridor, as far as that variable is concerned. Each variable may have a different corridor width, depending on its nature and its range of variation.

Fig. 3, where again the notation corresponds to that used in Section 2-3, illustrates the concept of corridor width. The figure shows the corridor for the m-th state variable during the k-th iteration of a computation cycle. The trial trajectory is given by the values of the optimal trajectory resulting from the (k-1)-th iteration; that is, $[s^{(k-1)}_{m,n,Q^*_{m,2}}]$, n = 1, 2, ..., N. It shows that the width of the corridor, being constant in this case, is equal to $\Delta q_{m,n}$ (= 3 in Fig. 3) times the smallest increment of the m-th state variable. This increment depends on the number of values into which a state variable is divided for analysis. This number is usually large, generally at least 50. If they are so considered at a time, the solution of a multi-state optimization problem becomes computationally infeasible by the conventional DP.

In the specification of the initial corridor width and its subsequent modifications, a coarse-grid technique is followed, which selects a large width for the first cycle of the DDDP procedure, and smaller widths for the subsequent cycles. In general, the larger the corridor width and the closer the trial trajectory to the optimal trajectory, the smaller the number of iterations are required to reach convergence to the optimal trajectory for a given cycle. Consequently, the use of larger widths for earlier cycles ensures that the optimal trajectories for such cycles are obtained within a small number of iterations. Furthermore, since the trial trajectory for any later cycle is the optimal trajectory for its preceding cycle and thus closer to optimality than any arbitrary one, smaller corridor widths can be used in later cycles to search for the optimal trajectory, also within a small number of iterations. In this manner, the total number of iterations in the process of finding the optimal solution for the entire problem can be kept within reasonable limits.

The most appropriate number of computation cycles, or the most appropriate number of different corridor widths, has been reported to be at least 6, the corridor width used in a given cycle being from 70 to 50 percent of the width used in the preceding cycle. Accordingly, the coarse-grid technique can be summarized as follows: For the first cycle (j=1),

$$(\Delta q_{m,n})_j = \frac{Q_m}{x}; \; m = 1, 2, \ldots, M; \; n = 1, 2, \ldots, N \tag{3-6}$$

where $\Delta q_{m,n}$ and Q_m have been defined in Section 2-3, and x is a factor which specifies the initial corridor width. For example, if x = 10, the boundaries of the initial corridor for a state variable cover one-fifth of the range of variation of the variable. For subsequent cycles (j = 2, ..., NC; NC = number of cycles),

$$(\Delta q_{m,n})_j \cong \frac{1}{2} (\Delta q_{m,n})_{j-1}; \; m = 1, 2, \ldots, M; \; n = 1, 2, \ldots, N \tag{3-7}$$

B. Design of the Corridor. In general, a corridor composed of 3 values of the state variable should be constructed symmetrically around the trial trajectory whenever possible. However, asymmetrical corridors may result if any of the boundaries of the corridor exceeds the limits (upper or lower) of the variable. A 2-valued corridor is produced if the trial trajectory passes through the upper or lower limit of the state variable. Fig. 4 illustrates the construction of the corridor for a variable quantized in 9 (= Q_m) values, and with the corridor width $\Delta q_{m,n} = 2$ for n = 1, 2, ..., 7. It can be seen that the corridor is symmetrical at stages n = 1, 2, 4, 6, 7; asymmetrical at stage n = 5, and 2-valued at stage n = 3.

Although, conceptually speaking, the corridor width for a given variable may change from stage to stage during the same computation cycle, it is customarily to treat the corridor width as a constant for all stages throughout the cycle. Also, it is generally recommended to construct 3-valued corridors for all state variables. The corridor width, nevertheless, may be different for each variable.

3-4. Optimization Within a Corridor

As indicated in block 5 of Fig. 1, after the construction of a corridor around the trial trajectory, the optimal trajectory, and its return, within the corridor should be found. This is done by means of a conventional DP algorithm for the search of the optimal trajectory, however restricting the computations only to those values of the state variables defined by the corridor.

Thus, the solution of the DP recursive equation, Eq. 2-10, is carried out inside the corridor, subject to the transformation equations, Eq. 2-1, and all the constraints imposed on the system. The solution of the DP recursive equation gives the optimal value of the objective function (i.e., the optimal return) and can be obtained by a DP algorithm of a suitable direction (backward or forward). The optimal trajectory is retrieved by means of computations which follow a direction contrary to the one used in the solution of the DP recursive equation.

The step indicated in block 5 of Fig. 1 is performed for every iteration of a computation cycle. Hence, it is essential that the optimization algorithm be as efficient as possible. A discussion on the various means of increasing the efficiency of DP algorithms is beyond the scope of this study, but a detailed treatment of such search procedures can be found in a number of texts on optimization [for example, Wilde, 1964].

3-5. Tests for Convergence

The purpose of these tests [see block 6 of Fig. 1] is twofold: (a) to determine, for the last computation cycle of the DDDP procedure, whether or not the optimal trajectory has been approached; and (b) to decide, for any intermediate cycle, if the optimal trajectory resulting from a given iteration of the cycle represents a significant improvement over the trial trajectory of the cycle.

A. Tests for Intermediate Cycles. Theoretically, the trial trajectory for any cycle should be the optimal one corresponding to the previous cycle, so that the trial trajectory represents the best approximation of the overall optimal trajectory available at the beginning of the cycle. As previously indicated, the optimal trajectory for each cycle is approached iteratively; each iteration provides a trajectory whose return is at least equal to the return of the trajectory for the preceding iteration. However, the improvement in return from trajectories of consecutive iterations decreases as the number of iterations increases, the largest improvement corresponding to the first iteration; that is,

$$(F_1^*-F_o^*) > (F_i^*-F_{i-1}^*); \quad i = 2, 3, \ldots, I \qquad (3\text{-}8)$$

where F_n^*, n = 1, 2, ..., I, represents the return from the optimal trajectory for the n-th iteration of a given cycle, I is the maximum number of iterations per cycle, and F_o^* is the return associated with the initial trial trajectory.

Then, a measure of the improvement of the returns from consecutive iterations, relative to the improvement obtained in the first iteration, is given by

$$\delta_i = \frac{F_i^* - F_{i-1}^*}{F_1^* - F_o^*} ; \quad i = 1, 2, \ldots, I \tag{3-9}$$

If, during any of the intermediate cycles, the iterative process yields a value of δ_i which does not represent a significant improvement in the return; that is, whenever

$$\delta_i \leq \varepsilon; \quad i = 1, 2, \ldots, I \tag{3-10}$$

then the computation cycle should be terminated. A value of 0.10 for ε has been found satisfactory in this study.

It should be noted that, while this test may prevent the optimal trajectory for an intermediate cycle from being approached, it does provide a satisfactory trial trajectory for the next cycle within a small number of iterations. Since an iteration includes the solution of an optimization problem (Section 3-4), this test will give a satisfactory trial trajectory at substantial savings in computer time.

B. Test for the Final Cycle. This test is designed to determine the convergence of the DDDP algorithm toward the solution for the optimization problem. It consists of assessing, for every iteration of the final cycle, if

$$\frac{F_i^* - F_{i-1}^*}{F_{i-1}^*} \leq \lambda; \quad i = 1, 2, \ldots, I \tag{3-11}$$

where λ is an arbitrary convergence parameter and all other terms have been defined before. For practical purposes, a value of $\lambda = 0.001$ can be considered adequate.

As soon as the convergence criterion of Eq. (3-11) is obtained, the DDDP procedure stops, then, the optimal solution of the optimization problem is represented by the trajectory which yields the optium return F_i^*.

IV. ILLUSTRATIVE EXAMPLES

4-1. Example 1: Operation of a Multi-Purpose, Multi-Unit System.

This example, adapted from Heidari, Chow, and Meredith [1971], is employed to illustrate in detail the DDDP procedure discussed in the previous chapter.

A. Description of the System. The water resources system as shown in Fig. 5 consists of four reservoirs which control the flow in two streams primarily for the purposes of hydropower production and irrigation water supply. The operation of the reservoirs is subject to seasonal storage requirements for flood control, which dictate the maximum storage capacity of the reservoirs; and to requirements for recreation and fish conservation in the reservoirs, which impose a minimum storage capacity. Table 1 shows the feasible range of variation of the storage in the four reservoirs on a monthly basis.

The natural inflows into reservoirs 1 and 2 are given in Fig. 6 and represent the hydrologic input to the system.

Prior water rights downstream of each reservoir require that reservoir releases be not less than a specified minimum amount. Furthermore, the capacity of the power generators sets a maximum limit to reservoir releases. Maximum and minimum limits for the reservoir releases in this system are given in Table 2.

The system is to be operated on a monthly basis and the reservoirs begin and end their yearly operational cycle with given amounts of water stored as shown in Table 3.

The net revenues from hydropower and irrigation obtained from the operation of the system are assumed proportional to the reservoir releases. The unit net revenue function for each reservoir is presented in Table 4.

Table 1. Range of Feasible Storage Capacities for Reservoirs of Example 1 (1000 ac-ft)

Month	Capacity of Reservoir 1, S_1		Capacity of Reservoir 2, S_2		Capacity of Reservoir 3, S_3		Capacity of Reservoir 4, S_4	
	max	min	max	min	max	min	max	min
1	1200	100	1800	100	800	100	1500	100
2	1200	100	1700	100	800	100	1500	100
3	1200	100	1500	100	800	100	1500	100
4	1000	100	1500	100	800	100	1500	100
5	900	100	1500	100	800	100	1500	100
6	800	100	1200	100	800	100	1500	100
7	800	100	1200	100	800	100	1500	100
8	900	100	1500	100	800	100	1500	100
9	1000	100	1700	100	800	100	1500	100
10	1000	100	1800	100	800	100	1500	100
11	1200	100	1800	100	800	100	1500	100
12	1200	100	1800	100	800	100	1500	100

Table 2. Range of Feasible Reservoir Releases for Example 1

1000 ac-ft	D_1 Release from Reservoir 1	D_2 Release from Reservoir 2	D_3 Release from Reservoir 3	D_4 Release from Reservoir 4
Max/mo	400	450	450	800
Min/mo	0.5	0.5	0.5	0.5

Table 3. Storage in the Reservoirs at the Beginning and Ending of a Year's Cycle for Example 1

Reservoir N	Storage, 1000 ac-ft
1	600
2	600
3	600
4	800

Table 4. Unit Net Revenue Functions for Example 1
($/1000ac-ft/mo)

Month	Reservoir 1 (Hydropower)	Reservoir 2 (Hydropower)	Reservoir 3 (Hydropower)	Reservoir 4 (Hydropower and Irrigator)
1	110	140	100	260
2	100	110	100	290
3	100	100	120	360
4	120	100	180	440
5	180	120	250	420
6	250	180	220	400
7	220	250	200	380
8	200	220	180	410
9	180	200	220	360
10	220	180	180	310
11	180	220	140	270
12	140	180	110	250

B. Formulation of the Optimization Problem. The objective of the optimization of the water resources system is to obtain the maximum net revenue from the yearly (12-month) operation of the system for hydropower production and irrigation water supply; that is, to

$$\max \sum_{t=1}^{12} \sum_{m=1}^{4} C_m(t)\ D_m(t) \qquad (4\text{-}1)$$

where $C_m(t)$ is the net revenue from a unit release from the m-th reservoir during the t-th month (Table 4) and $D_m(t)$ is the release from the m-th reservoir during the t-th month. This maximization is subject to the following requirements:

(a) Storage Constraints. The storage at the beginning of the first month of any year should be a known quantity (Table 3):

$$\begin{aligned} S_m(1) &= 600{,}000 \text{ ac-ft};\ m = 1,\ 2,\ 3 \\ S_4(1) &= 800{,}000 \text{ ac-ft} \end{aligned} \qquad (4\text{-}2)$$

Because the system is operated on a monthly basis but with yearly cycles, $S_m(13) = S_m(1)$, m = 1, 2, 3, 4.

For all other months, the storage in the reservoirs should belong to the set of admissible storages as indicated in Table 1; that is,

$$S_m^{min}(t) \leq S_m(t) \leq S_m^{max}(t);\ t = 2,\ \ldots,\ 12 \qquad (4\text{-}3)$$

where $S_m(t)$ is the storage in the m-th reservoir at the beginning of the t-th month; and $S_m^{min}(t)$ and $S_m^{max}(t)$ are, respectively, the minimum and maximum storages in the m-th reservoir at the beginning of the t-th month.

(b) Release Constraints. The reservoir releases during any month should belong to the range of feasible releases (Table 2):

$$D_m^{min}(t) \leqq D_m(t) \leqq D_m^{max}(t); \; t = 1, 2, \ldots, 12 \tag{4-4}$$

where $D_m(t)$ represents the volume released from the m-th reservoir during the t-th month; and $D_m^{min}(t)$ and $D_m^{max}(t)$ are, respectively, the lower and upper limits for the release from the m-th reservoir during the t-th month.

(c) System Transformation Functions. They are expressed by the principle of continuity for each reservoir:

$$\begin{aligned} S_1(t) &= S_1(t+1) + D_1(t) - I_1(t); \; t = 1, 2, \ldots, 12 \\ S_2(t) &= S_2(t+1) + D_2(t) - I_2(t); \; t = 1, 2, \ldots, 12 \\ S_3(t) &= S_3(t+1) + D_3(t) - D_2(t); \; t = 1, 2, \ldots, 12 \\ S_4(t) &= S_4(t+1) + D_4(t) - D_1(t) - D_3(t); \; t = 1, 2, \ldots, 12 \end{aligned} \tag{4-5}$$

where $I_1(t)$ and $I_2(t)$ are the inflows into reservoirs 1 and 2, respectively, during the t-th month (Fig. 6), and all other terms have been defined previously.

In the DP formulation of this problem, there exist 12 stages (t = 1, 2, ..., 12); a four-dimensional state vector

$$\bar{S}_t = \begin{bmatrix} S_1(t) \\ S_2(t) \\ S_3(t) \\ S_4(t) \end{bmatrix} \tag{4-6}$$

and a four-dimensional decision vector

$$\overline{D}_t = \begin{bmatrix} D_1(t) \\ D_2(t) \\ D_3(t) \\ D_4(t) \end{bmatrix} \qquad (4\text{-}7)$$

The DP recursive equation is expressed, for any stage t, as

$$F_{t+1}(\overline{S}_{t+1}) = \max \{ \sum_{m=1}^{4} C_m(t)\, D_m(t) + F_t(\overline{S}_t) \} \qquad (4\text{-}8)$$

subject to the constraints of Eqs. (4-2) to (4-5).

C. Solution by DDDP. (a) Quantization of State Variables. The smallest division for each state variable is selected as 1000 ac-ft. Hence, the total number of quantized values for each variable changes with the month, within the limits indicated in Table 5. By the DDDP procedure, only a few of these variables are considered at a time.

(b) Corridors. In applying the DDDP procedure, 8 cycles are considered, and for each cycle a maximum number of iterations, 15, is permitted. The corridor widths for each cycle are chosen as shown in Table 6. Using these widths, 3-valued corridors are constructed around the trial trajectories.

(c) Initial Trial Trajectory. In Table 4, it can be observed that the net revenue per unit of reservoir release is greater for the months 5 through 10 than for other months of the year. Therefore, an initial trial trajectory is constructed which allows large releases during those months. The trajectory so generated is presented in Table 7 and its return is $2,702,700.

Table 5. Number of Quantized Values for State Variables in Example 1

Variable	Number of Values	
	max	min
S_1	1100	700
S_2	1700	1100
S_3	700	700
S_4	1400	1400

Table 6. Corridor Widths for State Variables in Example 1

Cycle	Corridor Width, 1000 ac-ft			
	S_1	S_2	S_3	S_4
1	440	440	440	440
2	220	220	220	220
3	110	110	110	110
4	55	55	55	55
5	30	30	30	30
6	15	15	15	15
7	6	6	6	6
8	1	1	1	1

Table 7. Initial Trial Trajectory for Example 1

Month	Storage at beginning of month (1000 ac-ft)				Release during month (1000 ac-ft)			
	S_1	S_2	S_3	S_4	D_1	D_2	D_3	D_4
1	600	600	600	800	50	40	40	60
2	600	600	600	830	50	20	20	50
3	650	650	600	850	150	50	50	50
4	700	800	600	1000	200	50	50	150
5	800	950	600	1100	350	250	100	750
6	800	1100	750	800	250	250	200	450
7	800	1200	800	800	200	300	400	600
8	800	1200	700	800	125	450	350	475
9	800	1000	800	800	125	130	230	555
10	800	1000	700	600	75	220	120	395
11	800	900	800	400	75	150	250	25
12	900	850	700	700	400	320	420	720
13	600	600	600	800				

(d) Tests for Convergence. The following convergence parameters are used: $\varepsilon = 0.10$ and $\lambda = 0.001$.

(e) Results. Fig. 7 shows the overall convergence of the procedure toward the optimal solution. It can be seen that no intermediate cycle more than 4 iterations is needed to converge to a satisfactory trial trajectory for the next cycle. The optimal solution is presented in Table 8. It can be noted that releases at or near the maximum reservoir releases should be made during the months of highest unit net revenue (Table 4). The net return obtained from the optimal operation of the system is \$3,079,800.

Because, in this particular example, the objective function, Eq. (4-1), is linear, it is possible to check the accuracy of the DDDP solution against the results of a linear programming (LP) formulation of the problem. The optimal solution by the LP methodology is \$3,082,665. Therefore, the DDDP solution differs less than 0.1% from the LP solution. For all practical purposes, this difference is negligible.

4-2. Example 2: Operation of a Multireservoir Flood Control System

In this section, the optimization of the daily operation of a flood control system is presented. The example, adapted from Shaufelberger [1971], illustrates the system-decomposition approach to derive the initial trial trajectory by the DDDP procedure.

A. The Flood Control System. The three-reservoir flood control system is sketched in Fig. 8. There exist 4 flood-damaging reaches (DR_m; m = 1, ..., 4), and 5 streams whose flows (I_5, I_9, I_{11}, I_{12}, I_{13}) are not controlled by the reservoirs. The purpose of this reservoir system is to modify the natural flows in the controlled streams (I_1, I_2, I_3)

Table 8. Optimal Trajectory and Releases for Example 1
(1000 ac-ft)

Month	Storage at beginning of month				Release during month			
	S_1	S_2	S_3	S_4	D_1	D_2	D_3	D_4
1	600	600	600	800	27	205	5	2
2	623	435	800	830	5	2	2	2
3	718	503	800	835	5	5	5	73
4	913	698	800	772	366	5	250	800
5	847	893	555	588	397	93	450	800
6	800	1200	198	635	398	350	448	800
7	652	1200	100	681	396	448	447	799
8	456	1052	101	725	44	450	144	800
9	537	852	407	113	11	221	450	200
10	651	761	178	374	395	4	25	2
11	331	877	157	792	5	446	1	1
12	501	531	602	797	1	1	3	1
13	600	600	600	800				

so as to minimize the annual flood damages in the flood-damaging reaches. Although the three reservoirs are basically single-purpose flood control reservoirs, seasonal pools have been specified for recreation and wild life. Fig. 9 presents the variation of the storage available for flood control as a result of the seasonal pool requirements. A minimum release for each reservoir has been established at 20 cfs. The maximum release for a given storage in each reservoir depends on the capacity of its outlet works. In Fig. 10, the maximum releases are shown as functions of the storage in each reservoir.

Routing equations describing the flow of water through the system from the distributed inputs to the system output are obtained by a multiple linear regression analysis of historic streamflow records as follows:

(a) Discharge at Reach 2.

$$I_7(t) = 0.25\ I_7(t-1) + 0.75\ D_1(t) - 0.114\ I_4(t) + 1.164\ I_5(t) + 0.75\ D_2(t) + 0.287\ I_6(t) \qquad (4\text{-}9)$$

where $I_7(t)$ and $I_7(t-1)$ are the flows at reach 2 during days t and t-1; $D_1(t)$ and $D_2(t)$ are the releases from reservoirs 1 and 2 during day t; $I_4(t)$, $I_5(t)$ and $I_6(t)$ are the flows at gaging stations 4, 5, and 6 during day t.

(b) For Discharge at Reach 3.

$$I_{10}(t) = 0.302\ I_{10}(t-1) + 0.700\ I_7(t) + 0.700\ D_3(t) + 1.130\ I_9(t) \qquad (4\text{-}10)$$

where $I_{10}(t)$ and $I_{10}(t-1)$ are the discharges at reach 3 during days t and t-1; $D_3(t)$ is the release from reservoir 3 during day t; $I_9(t)$ is the streamflow at gaging station 9 during day t; and $I_7(t)$ has been defined above.

(c) For Discharge at Reach 4.

$$I_{14}(t) = 0.297\ I_{14}(t-1) + 0.358\ I_7(t-1) + 0.350\ I_7(t) + 0.403\ [I_{11}(t-1) + I_{12}(t-1)] + 0.360\ [I_{11}(t) + I_{12}(t)] + 0.895\ I_{13}(t) \qquad (4\text{-}11)$$

where $I_{14}(t)$ and $I_{14}(t-1)$ are the discharges at reach 4 during days t and t-1; and $I_{11}(t)$, $I_{11}(t-1)$, $I_{12}(t)$, $I_{12}(t-1)$, $I_{13}(t)$ and $I_{13}(t-1)$ are the streamflows at gaging stations 11, 12, and 13, respectively, during days t and t-1.

Flood damages accrue in the form of non-crop and crop damages. Non-crop damages are given in Fig. 11 as functions of the flow at each reach; crop damages are functions of both the discharge at the damage reaches and the time of the flood occurrence, as illustrated in Figs. 12 and 13.

B. Formulation of the Optimization Problem. The operational objective of the multi-unit flood control system is minimization of flood damages; that is,

$$\text{Min} \sum_{j=1}^{4} \sum_{t=1}^{T=365} DAM_j[\bar{D}(t)] \qquad (4\text{-}12)$$

where $DAM_j[\bar{D}(t)]$ represents the crop and non-crop damages in the j-th reach as a result of the set $\bar{D}_m(t)$, m = 1, 2, 3, of releases from the reservoir system during the t-th day.

As can be seen from the damage functions in Figs. 11 to 13 with the routing equations (4-9) to (4-11), the objective function or Eq. (4-12) is nonlinear. Furthermore, because flood damage in any reach is a function of both river discharge and time, the problem is one of sequential decisions both in time and space. As shown by the lagged variables in the routing equations (4-9) to (4-11), water released from the reservoirs require time to flow through the system. Thus, a forward DP algorithm for the T-day operational process (T=365) may be developed, with daily reservoir releases as the decision variables, storage volumes as the state variables, and time t as the stage variable. The DP recursive equation may be written as:

$$F_{t+1}(\bar{S}_{t+1}) = \text{Min} \{ \sum_{j=1}^{4} DAM_j[\bar{D}(t)] + F_t(\bar{S}_t) \} \tag{4-13}$$

where $F_{t+1}(\bar{S}_{t+1})$ and $F_t(\bar{S}_t)$ are the minimum damages from the beginning of the year to the beginning of days t+1 and t, respectively, as functions of the sets of storage volumes $\bar{S}_{t+1}$ and $\bar{S}_t$, respectively, at the beginning of these days; $DAM_j[\bar{D}(t)]$ has been defined before; and the sets $\bar{S}_{t+1}$ and $\bar{S}_t$ are given by

$$\bar{S}_{t+1} = S_m(t+1);\ m = 1, 2, 3$$

$$\bar{S}_t = S_m(t);\ m = 1, 2, 3 \tag{4-14}$$

The solution of the DP recursive equation (4-13) is subject to following conditions:

(a) Storage Constraints.

$$SP_m(t) \le S_m(t) \le S_m^{max}(t);\ m = 1, 2, 3;\ t = 1, 2, \ldots, T \tag{4-15}$$

where $S_m(t)$ is the storage in the m-th reservoir at the beginning of the m-th day; $SP_m(t)$ is the seasonal pool requirement in the m-th reservoir at the beginning of the t-th day; $S_m^{max}(t)$ is the maximum storage for flood control in reservoir m at the beginning of the t-th day. The range of variation of the usable storage volume for flood control is given in Fig. 9. Furthermore, the system operation is subject to the requirement that the reservoirs be at their seasonal pools at the beginning of the first day of each year; that is,

$$S_m(1) = S_m(366) = SP_m(t);\ m = 1, 2, 3 \qquad (4\text{-}16)$$

(b) Release Constraints.

$$20 \text{ cfs} \le D_m(t) \le D_m^{max}\ [S_m(t)];\ m = 1, 2, 3;\ t = 1, 2, \ldots, T \qquad (4\text{-}17)$$

where $D_m(t)$ has been defined before; and $D_m^{max}\ [S_m(t)]$ is the maximum release from the m-th reservoir during the t-th day as a function of the storage volume $S_m(t)$ at the beginning of that day (Fig. 10).

(c) System Transformation Functions. These are the expressions for the principle of continuity for each reservoir:

$$S_m(t+1) = S_m(t) + I_m(t) - D_m(t);\ m = 1, 2, 3;\ t = 1, 2, \ldots, T \qquad (4\text{-}18)$$

where $I_m(t)$ is the natural flow into the m-th reservoir during day t, and all other terms have been defined before.

C. Solution by DDDP. (a) Initial Trial Trajectory. In order to find a close-to-optimal system operating policy, a spatial incremental dynamic programming (SIDP) technique is developed, based on the concepts of the system-decomposition approach (Section 3-2). The technique begins with the division of the flood control system into three spatial subsystems, each of which contains only one reservoir, as illustrated in Fig. 14. Starting with the uppermost subsystem, the algorithm is carried through down the river, in an incremental manner, optimizing the operation of each subsystem over all the T days, subject to the suboptimal operating policies of the upstream subsystems.

The SIDP methodology begins with subsystem A optimizing the operation of reservoir 1, and considering only flood damages at reach 1. The operation of reservoir 2 is optimized considering only subsystem B, taking into account the damages in reach 2 exclusively, with daily releases from reservoir 1 as obtained according to the previously determined operating policy. In optimizing the operation of reservoir 2, daily streamflows in reach 2 are established according to the routing equation (4-9). Next, the operation of reservoir 3 is optimized considering subsystem C. Daily releases from the upstream reservoirs are fixed in accordance with previously determined release policies. The streamflow at reach 3 is determined with routing equation (4-10) and that at reach 4 with routing equation (4-11). Only damages in those two reaches are considered in the optimization of subsystem C.

The operation of each reservoir is optimized individually by DDDP optimization technique. Because each subsystem has only one state variable, it could also be optimized by conventional DP. By using DDDP, however, a finer grid of state-variable values can be defined. The initial trial trajectory is that defined by the seasonal pool storage for each reservoir. A single-cycle DDDP algorithm using a corridor width of 1.6 kilo ac-ft for each state variable and an 11-valued corridor is developed.

The algorithm terminates when the incremental reduction in damage iteration is less than one thousandth the value of damage for the last iteration.

(b) Corridors. In applying the DDDP procedure for the optimization of the entire system, 2 computation cycles are considered; and for each cycle, an arbitrary maximum number of iterations is allowed. The corridor widths used for each cycle are as follows:

Computation Cycle	Corridor Width, 1000 ac-ft			
	S_1	S_2	S_3	S_4
1	3.2	3.2	3.2	3.2
2	1.6	1.6	1.6	1.6

Using these widths, 3-valued corridors are constructed around the trial trajectories.

(c) Test for Convergence. For both computation cycles, the test for convergence corresponding to a final cycle (Section 3-5,B) is adopted with $\lambda = 0.001$.

(d) Results. The result of optimization of the system for the water year 1957 is presented in Fig. 15, where curve A gives the result corresponding to a DDDP procedure without the SIDP algorithm (Section 4-2,C,a), and curve B indicates the result of including such an algorithm in the DDDP procedure. The initial trial trajectory for the DDDP procedure is represented by the sequence of seasonal pool storages in the three reservoirs. The number of computation cycles and the corridor widths are also those given above. By using the system-decomposition approved (Section 3-2) to find the initial trial trajectory, computer time requirements are reduced from 83 minutes to 38 minutes. It may be noted that the number of iterations required to reach the optimal solution with the SIDP algorithm is less than half the number required without it and that there is less than one percent difference between the two solutions.

4-3. Example 3: Planning for Expansion of a Flood Control Project

This example, adapted from Cortes [1973], illustrates the use of DDDP procedure to solve high-dimensional DP problems which arise in the analysis of water resources systems planning.

A. Description of the System. The system for a flood control project consists of structural components designed to control floodwaters physically, and non-structural components to reduce damages caused by overflowing waters. Fig. 16 shows the location of the project components on a river basin. A detention reservoir and two downstream levees form the structural components of the system. The agricultural land-use in the tracts behind and protected by the levees forms the non-structural component.

A previous parametric linear programming analysis of the non-structural components [Cortes, 1973] has provided the optimal land-use policy as well as the net economic benefit from that policy, for any sequence of expression of the structural components. The optimization problem to be solved by the DDDP procedure then consists of finding the sequence of expansion of the structural components so that the net economic benefits are maximized over the planning period.

Table 9 presents the physical characteristics for the alternative magnitudes of the structural components of the system. The total capacity of the detention reservoir corresponds to the capacity required to route safely the spillway design flood through the reservoir, with a 5-ft freeboard, for a given capacity at the spillway crest level. Since, for all capacities, the outlet works are assumed the same, the only variable

Table 9. Physical Data for Structural Components

Order	Detention Reservoir Capacity 1000 ac-ft At Spillway Crest	Detention Reservoir Capacity 1000 ac-ft Total	Land Required for Detention Reservoir (acres)
1	0	0	0
2	10	208.3	3785
3	20	245.3	4435
4	30	272.8	4939
5	40	296.5	5373
6	50	319.1	5710
7	60	339.3	5998

Order	Elevation at Levee 3 (feet above msl)	Volume (10^3 cu yd)	Land Required for Levee 3 (acres)
1	460.0	0	0
2	462.5	55.2	22.9
3	465.0	114.9	28.2
4	467.5	196.7	33.6
5	470.0	301.6	39.2
6	472.5	430.6	45.0
7	475.0	584.7	51.0
8	477.5	765.0	57.1
9	480.0	972.5	63.4

Order	Elevation at Levee 4 (feet above msl)	Volume (10^3 cu yd)	Land Required for Levee 4 (acres)
1	424.0	0	0
2	426.5	382.2	158.4
3	429.0	782.0	191.6
4	431.5	1,717.4	293.5
5	434.0	2,676.2	348.2
6	436.5	3,770.3	394.3
7	439.0	6,502.7	440.3
8	441.5	7,331.2	485.6
9	444.0	8,147.8	531.4
10	446.5	9,975.2	577.1
11	449.0	11,982.7	622.6
12	451.5	14,181.1	668.4

in the design of the detention reservoir is its storage capacity.

The design variable for the levees is their elevation at their respective control gaging stations along the river. The areas required for the construction of the levees, as well as their embankment volumes are calculated using topographical information and typical cross sections of the levees. The area required for the construction of the reservoir is calculated according to the land inundated in the passage of the 10-year flood through the detention reservoir.

Economic benefits of the project will result from (a) reduction of direct and indirect flood damages; and (b) efficient utilization of land and properties within the floodplain. For a given degree of flood protection, direct damages increase with time according to the economic growth of the floodplain. Indirect damages are usually evaluated as a percentage of direct damages. In this example, an arithmetic growth rate of 2 percent per year and an indirect-damage factor of 25 percent are assumed, considering the agricultural character of the floodplain.

The expected annual direct damage is a function of the magnitude of the structural components. Figs. 17 to 19 are a sample of three-dimensional direct damage functions, from which other similar functions can be developed, corresponding to floodplain conditions existing in the beginning of the planning period.

Expected annual land-enhancement benefits from the project are the difference between the expected net annual income with the project and the expected net annual income if without the project. The expected annual income varies with the magnitude of the project. Figs. 20 and 21 are samples of a three-dimension income function, showing the expected income variation.

Reservoir construction costs are evaluated according to the following relation adapted from Dawes and Wathne [1968]:

$$C_R = 15.26HWS^{0.5444} \tag{4-19}$$

where C_R is the construction cost in dollars, S is the total storage capacity in acre feet, and HW is the Handy-Whitman cost index for the year for which costs are estimated. The cost of construction of levees are evaluated as

$$C_1 = pV \tag{4-20}$$

where C_1 is the construction cost in dollars, p is the unit cost in dollars per cubic yard, and V is the embankment volume in cubic yards.

Land acquisition costs are computed as

$$CL_t = CL_o(1+r_d)^t \tag{4-21}$$

where CL_t is the unit land-value at year t from the beginning of the planning period, r_d is the annual rate of land-value differential inflation, and CL_o is the unit land-value at the beginning of the planning period. Operation, maintenance, and replacement costs are assumed to be fixed percentages (1.0 for detention reservoir and 2.0 for levees) of construction cost.

B. Formulation of the Optimization Problem. The purpose of the analysis of the expansion process is to determine the sequence of expansions of the structural components of a flood control project which would maximize the net economic benefits from the project throughout the period of analysis. It is assumed that the project is to be developed for a planning period of N years, and that any expansion of the project, if made at all, will be made every Δt years. The planning period is then divided into T planning sub-periods, each of duration Δt years; that is,

$$N = T\Delta t \tag{4-22}$$

Expansions of the project will be considered at the beginning of each sub-period, so that it will be effective throughout that and the following sub-periods.

Let the state variables represent the magnitude of the structural components of the project and also the land available to construct them. Considering the land available as a state variable allows the analysis of the effects of land-value differential inflation on the timing of construction and expenditures for the project. A 6-dimensional state vector $\bar{S}_t$ at stage t can be defined as

$$\bar{S}_t = \begin{bmatrix} S_{1,t} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ S_{6,t} \end{bmatrix}$$

$$\bar{S}_t = [S_{m,t}] \tag{4-23}$$

where $S_{m,t}$; m = 1, 2,...,6, are magnitudes of the structural components and the amount of land available for the project during stage t.

In addition, let $\bar{D}_t$ be a 6-dimensional decision vector representing the expansion of the project at the beginning of the stage t; that is,

$$\bar{D}_t = \begin{bmatrix} D_{1,t} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ D_{6,t} \end{bmatrix}$$

or

$$\bar{D}_t = [D_{m,t}] \tag{4-24}$$

where $D_{m,t}$; m = 1, 2,...,6, are the expressions of the project components at the beginning of the t-th stage.

The, $R(\bar{S}_t,\bar{D}_t,t)$ can be defined as the value, at the beginning of stage t, of the net economic benefit from the project during that stage, and can be evaluated as

$$\begin{aligned} R(\bar{S}_t,\bar{D}_t,t) = {} & DAM[(\bar{S}_t)_o,t] - DAM[\bar{S}_t,\bar{D}_t,t] \\ & + G_{\bar{S}_t,t} - G_{(\bar{S}_t)_o,t} \\ & - [C(\bar{D}_t,t) + OMR(\bar{S}_t,\bar{D}_t,t)] \end{aligned} \tag{4-25}$$

where $DAM[(\bar{S}_t)_o,t]$ is the expected flood damage during stage t which would occur in the floodplain if no structural components were in place, i.e., if the state during stage t were $(\bar{S}_t)_o$; $DAM(\bar{S}_t,\bar{D}_t,t)$ is the expected flood damage during stage t which would occur if the state were $\bar{S}_t$ during that stage as a result of $\bar{D}_t$; $G_{\bar{S}_t,t}$ is the net income from the agricultural use of the land in the floodplain during stage t for the optimal land-use policy (see earlier mentioning in this Section about parametric linear programming analysis of the non-structural components) associated with the state $\bar{S}_t$; $G_{(\bar{S}_t)_o,t}$ is the net income from the agricultural use of the land for the optimal land-use policy associated with the conditions of no structural components in place; $C(\bar{D}_t,t)$ is the construction and land acquisition costs of the expansion $\bar{D}_t$ at stage t; and $OMR\ (\bar{S}_t,\bar{D}_t,t)$ is the operation, maintenance and replacement costs during stage t of the project components of magnitude $\bar{S}_t$ as a result of the expansion $\bar{D}_t$.

Thus, the objective function of the planning optimization problem can be written as follows:

$$\text{Max} \sum_{t=1}^{T} [R(\bar{S}_t,D_t,t)/(1+r)^{(t-1)\Delta t} \qquad (4\text{-}26)$$

where r is the annual rate of discount, and all other terms have been defined before.

The DP recursive equation for a forward DP formulation is then expressed as

$$F_t(\bar{S}_t) = \max \left[\frac{R(\bar{S}_t,\bar{D}_t,t)}{(1+r)^{(t-1)\Delta t}} + F_{t-1}\ (\bar{S}_{t-1})\right] \qquad (4\text{-}27)$$

where $F_t(\bar{S}_t)$ and $F_{t-1}(\bar{S}_{t-1})$ represent the maximum present values of the net economic benefits of the project when it is in a state $\bar{S}_t$ during stage t, and in a state $\overline{S}_{t-1}$ during stage t-1, respectively. This maximization is subject to the following conditions:

(a) Constraints on the State Variables.

$$0 \leq S_{m,t} \leq S_m^{max}; \; t = 1, 2, \ldots, T; \quad m = 1, 2, \ldots, 6 \qquad (4\text{-}28)$$

where S_m^{max} is the largest feasible magnitude of the m-th project component (Table 9).

$$S_{m,t} = S_{m,t-1} = S_{m,t-2} = \ldots = S_{m,t-\Delta p_m} \qquad (4\text{-}29)$$

which is the expression of an institutional constraint and where $S_{m,t-j}$; $j = 1, 2, \ldots, \Delta p_m$ is the magnitude of the m-th structural component during the (t-j)-th planning subperiod, and Δp_m is the number of consecutive planning subperiods during which the m-th structural component can not be modified.

(b) Restrictions on the Decision Variables.

$$0 \leq D_{m,t} \leq S_m^{max}, \; S_{m,t-1}; \; m = 1, 2, \ldots, 6; \quad t = 1, 2, \ldots, T \qquad (4\text{-}30)$$

(c) Budgetary Constraints.

$$C(\overline{D}_t, t) \leq BGT_t; \; (t-1, 2, \ldots, T) \qquad (4\text{-}31)$$

where BGT_t represents the budget available at the beginning of planning subperiod t to meet the construction and land acquisition costs $C(\overline{D}_t, t)$.

(d) State-Transformation Function.

$$\bar{S}_t = T_t(\bar{S}_{t-1}, \bar{D}_t) = \bar{S}_{t-1} + \bar{D}_t \qquad (4\text{-}32)$$

or equivalently in expanded form,

$$S_{m,t} = S_{m,t-1} + D_{m,t}; \quad m = 1, 2, \ldots, 6; \quad t = 1, 2, \ldots, T \qquad (4\text{-}33)$$

where all terms have been defined before.

C. Solution by DDDP. (a) Initial Trial Trajectory. For this example on planning water resources systems, the initial trial trajectory corresponds to one of the many alternatives on planning; namely, the do-nothing alternatives. This particular trajectory is as valid and as good as any other arbitrarily chosen, as was confirmed by using different initial trial trajectories.

(b) Corridors. With 3 computation cycles, each allowing a maximum of 10 iterations, the DDDP procedure incorporates 3-valued corridors for each state variable. The corridor widths used in the procedure, which are independent of the stage variable t in this case, are as follows:

Computation Cycle	Corridor Widths ($\Delta q_{m,t}$; m = 1, 2, ..., 6)					
	Reservoir Capacity	Levee 3 Elevation	Levee 4 Elevation	Land for Reservoirs	Land for Levee 3	Land for Levee 4
1	3	4	6	3	4	6
2	2	2	3	2	2	3
3	1	1	1	1	1	1

The quantized values for each state variable in the DP formulation are given in Table 10.

Table 10. Quantized Values of the State Variables in the DP Formulation

Order	Reservoir Capacity (ac-ft)	Elevation of Levee 3 (ft above msl)	Elevation of Levee 4 (ft above msl)	Land for Construction of Detention Reservoir (ac)	Land for Construction of Levee 3 (ac)	Land for Construction of Levee 4 (ac)
1	0	460.0	424.0	0	0.00	0.00
2	10,000	462.5	426.5	3,785	22.88	158.41
3	20,000	465.0	429.0	4,435	28.18	191.61
4	30,000	467.5	431.5	4,737	33.61	293.47
5	40,000	470.0	434.0	5,373	39.24	348.18
6	50,000	472.5	436.5	5,716	45.00	394.29
7	60,000	475.0	439.0	5,998	50.99	440.28
8		477.5	441.5		57.12	485.56
9		480.0	444.0		63.43	531.39
10			446.5			577.07
11			449.0			622.55
12			451.5			668.38

(c) Convergence Tests. In this example, the following convergence parameters are used: $\varepsilon = 0.10$ and $\lambda = 0.001$.

(d) Results. Optimal expansions of the structural components of the flood control project, as identified by the DDDP procedure, can be obtained for a number of budgetary and institutional constraints, for various values of the rate of discount, and for various values of the rate of land-value differential inflation. Thus, the DDDP analysis makes it possible the investigation of the sensitivity of the expansion of the project in response to changes in economic factors for which it might not be feasible to assign specific values at the time the analysis is made. Here only results for various budgetary constraints are presented. Further results are presented elsewhere [Cortes, 1973].

Throughout the sensitivity analysis for the budgetary constraints, other factors are assumed and kept invariable as follows: $r = 5.5$ percent per year; $r_d = 3.3$ percent per year; $\Delta p_m = 1$; $m = 1, 2, \ldots, 6$.

Fig. 22 shows the optimal expansion of the structural elements of the project under the following budgetary constraints:

A. BGT_t unconstrained; $t = 1, 2, \ldots, T$

B. BGT_t = \$12,000,000; $t = 1, 2, \ldots, T$

C. BGT_t = \$5,000,000; $t = 1, 2, \ldots, T$

For case A, the optimal solution consists of installing, at the beginning of the planning period, the detention reservoir with a capacity of 60,000 acre feet at the spillway crest level (339,271 acre-feet of total storage capacity), levee 3 with an elevation of 475.0 ft above msl, and levee 4 with an elevation of 4440 ft above msl. No modifications should be made to the structural components throughout the planning period. With a reduction in the budget available (Case B), the construction of the detention reservoir is no longer a part of the optimal expansion, and the construction of the levees should be carried out by stages. The final elevation of levee 4 is larger than its final elevation for the case of unconstrained budget. Further reduction in the budget available (Case C) results in the construction of levee 3 in one stage and the delay of the construction and staging of levee 4. The final elevations of the levees remain unaltered by the reduction in the available budget from Case B to Case C.

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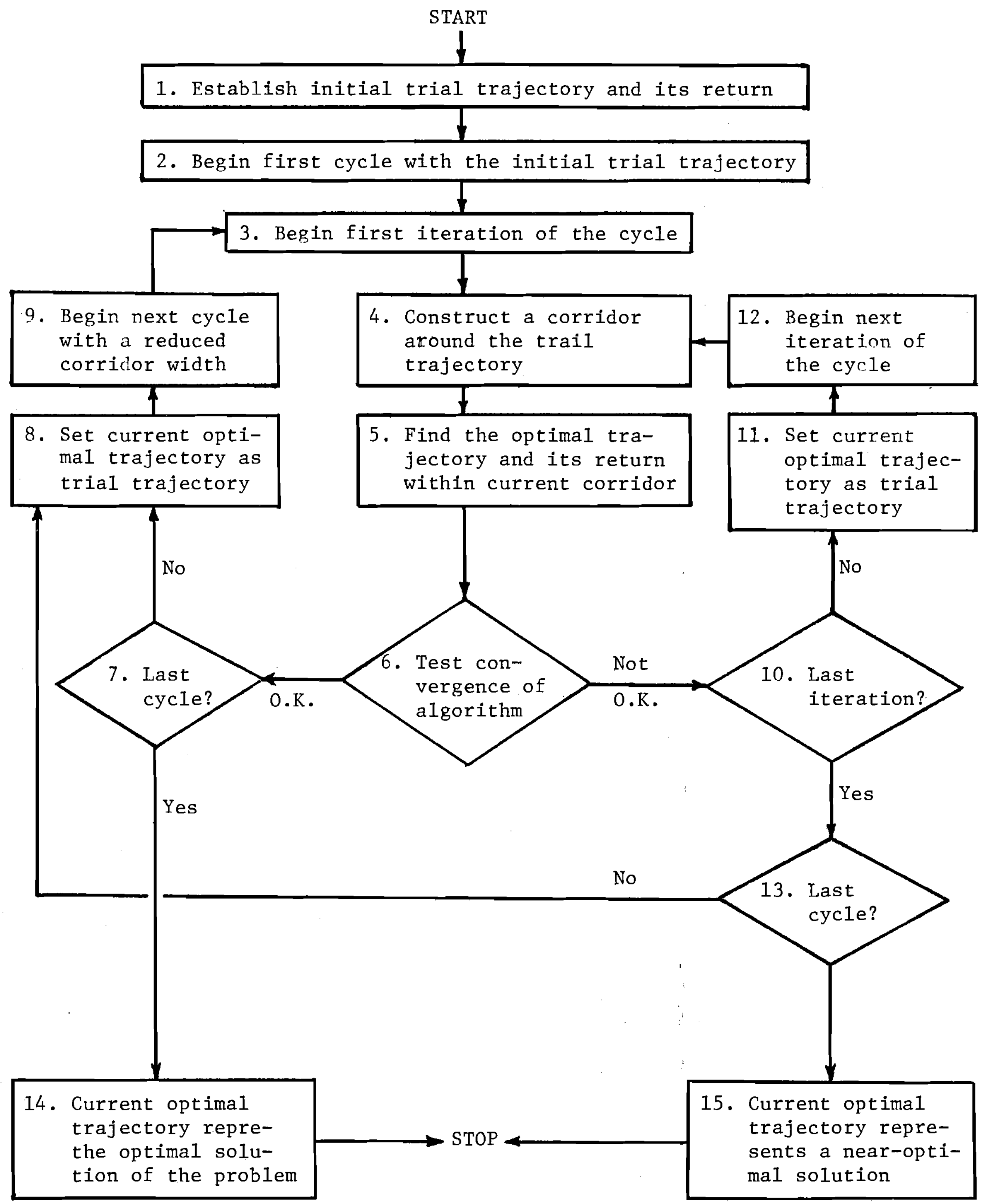
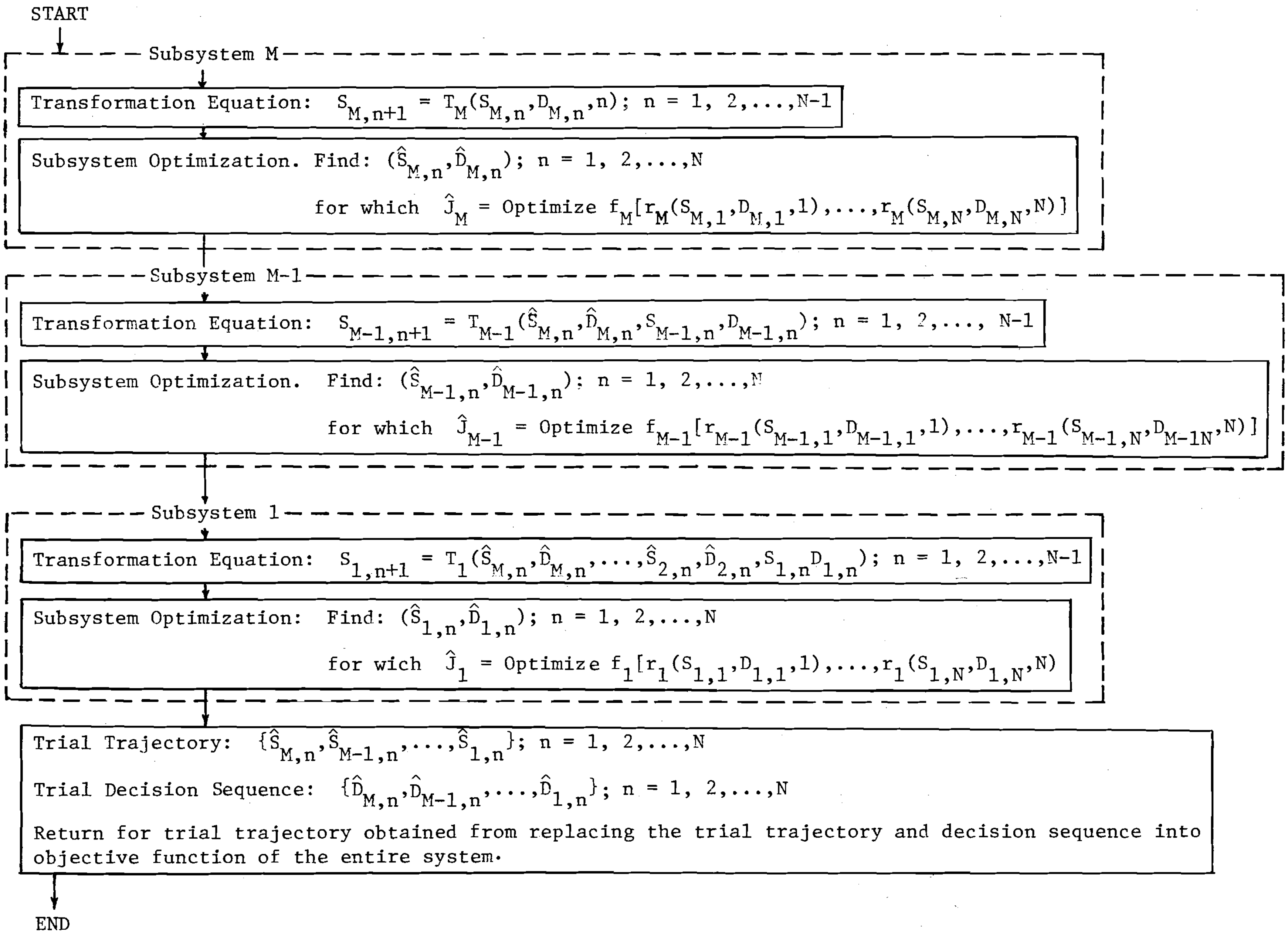


Fig. 1. General scheme of the DDDP procedure



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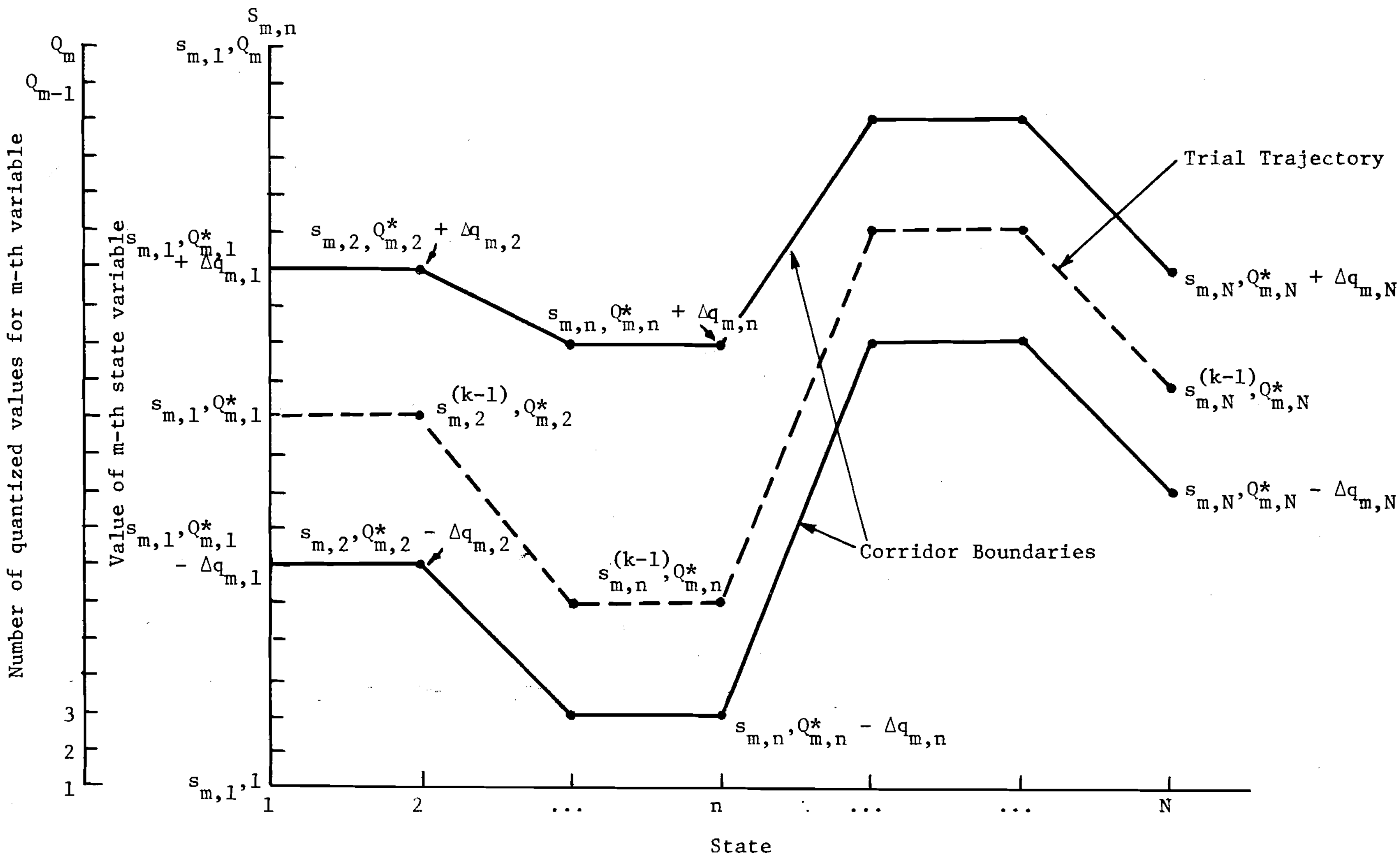

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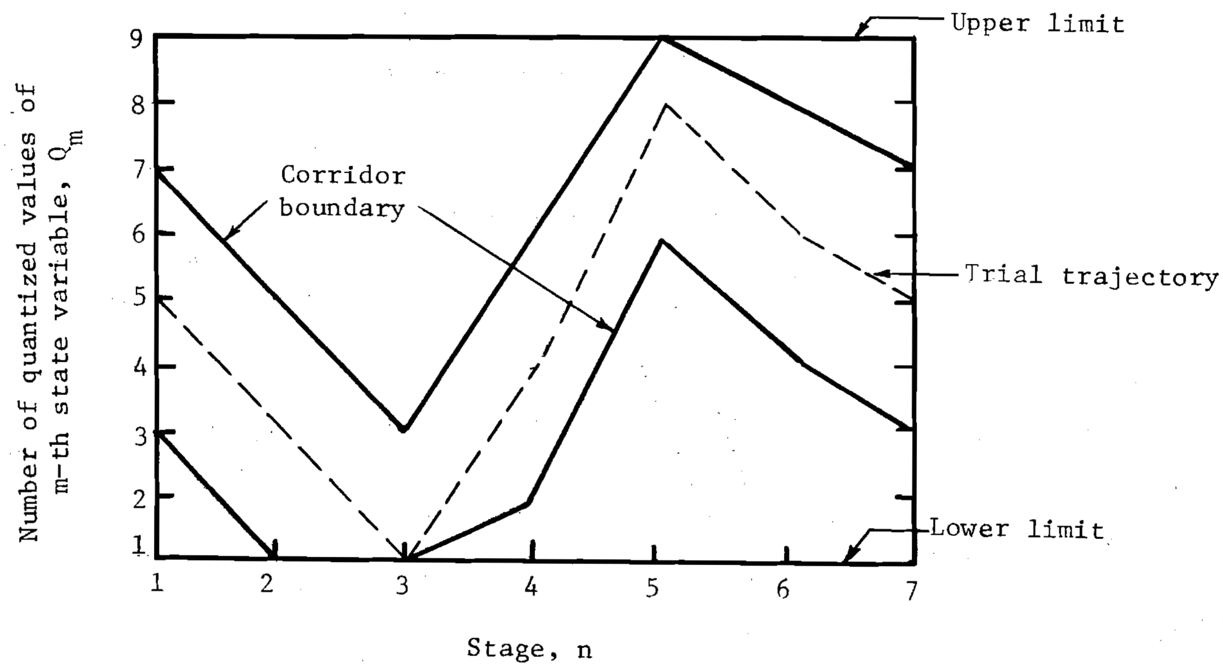

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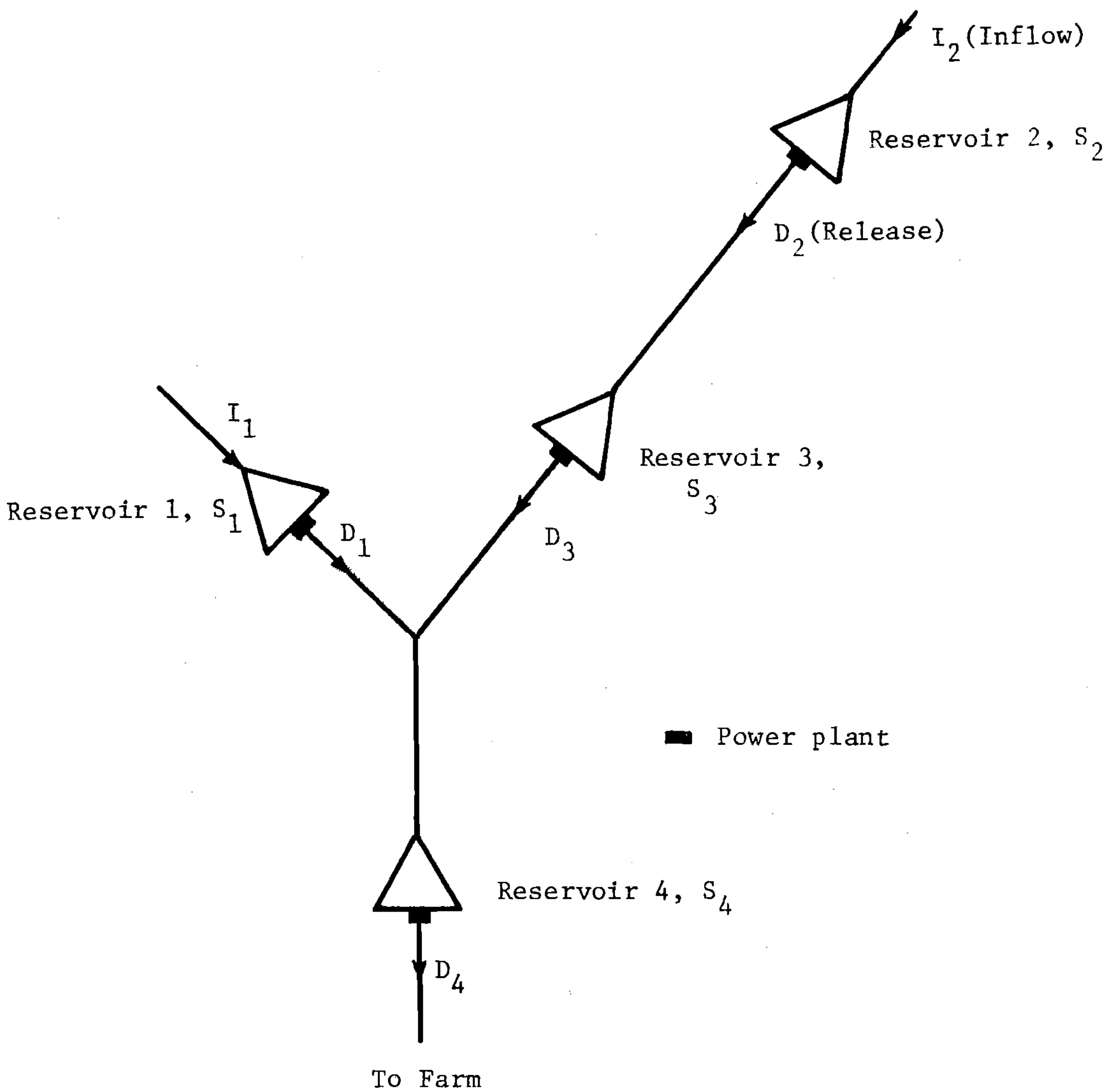

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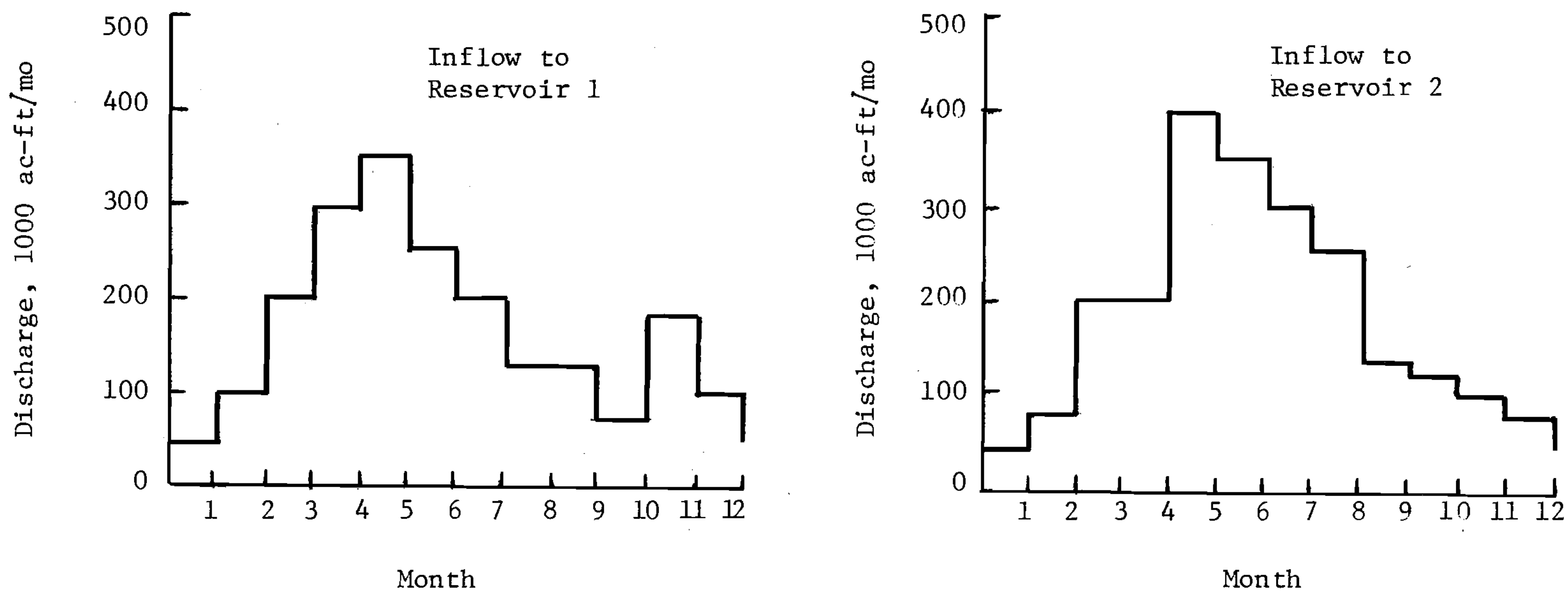

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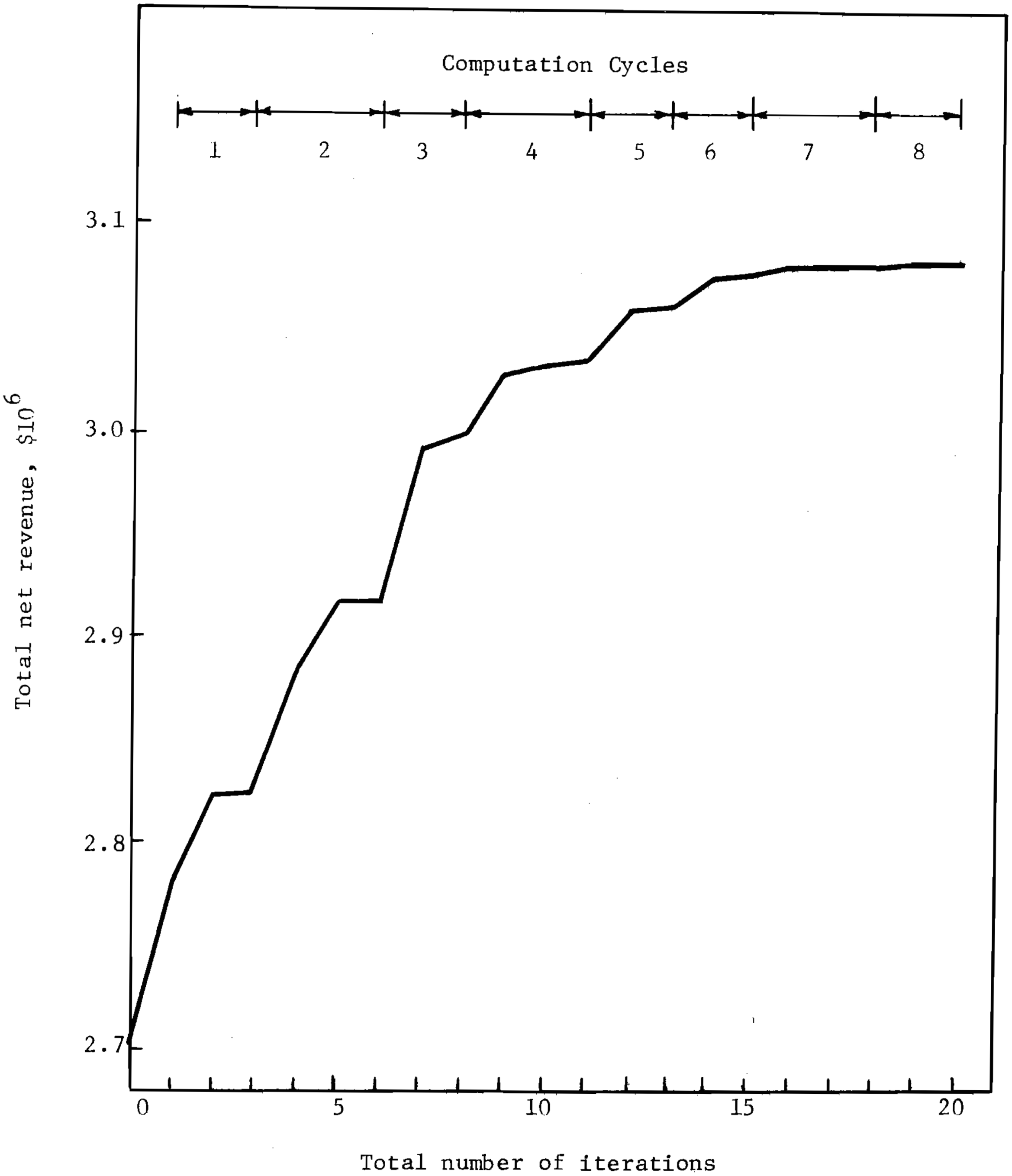

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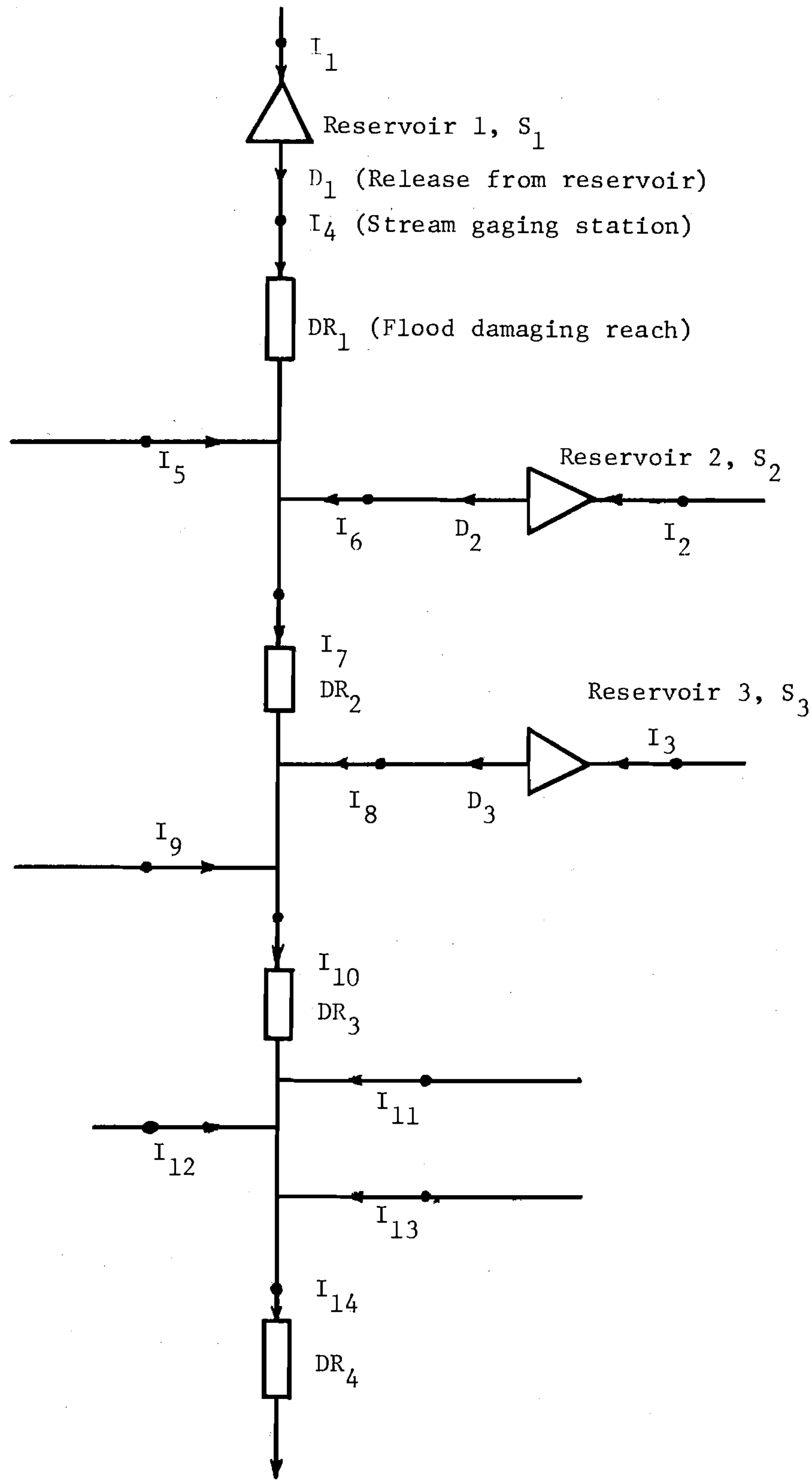

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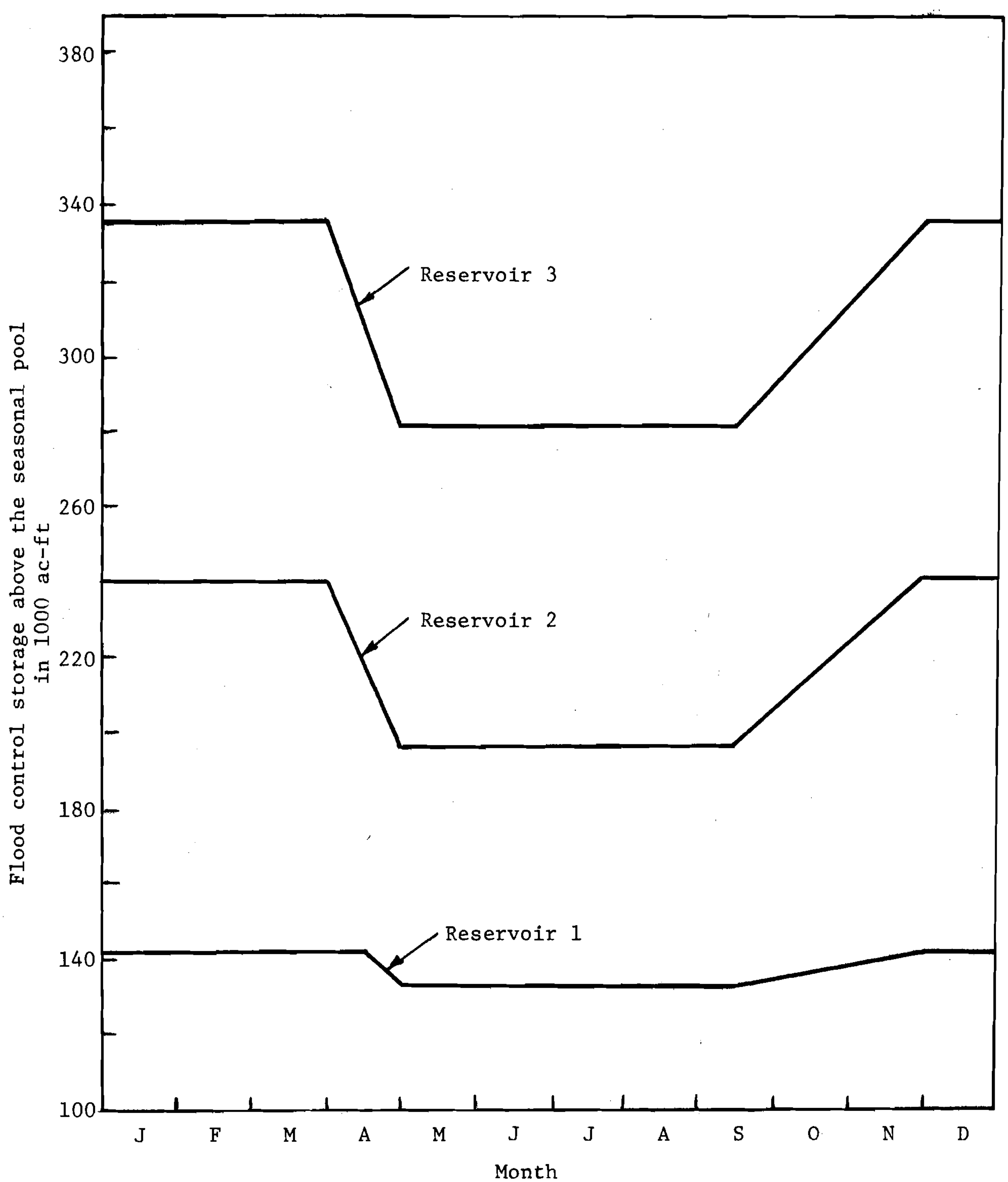

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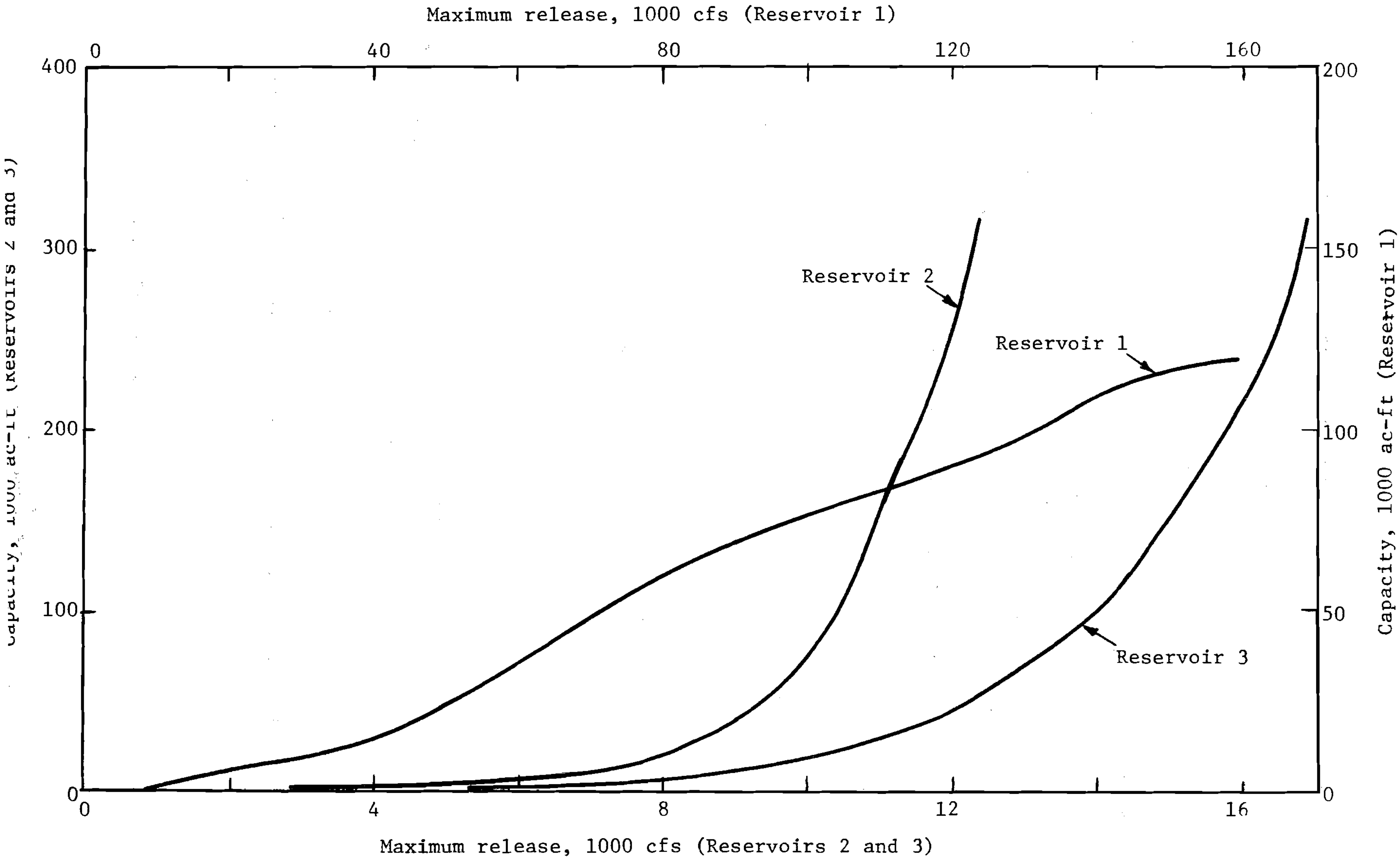

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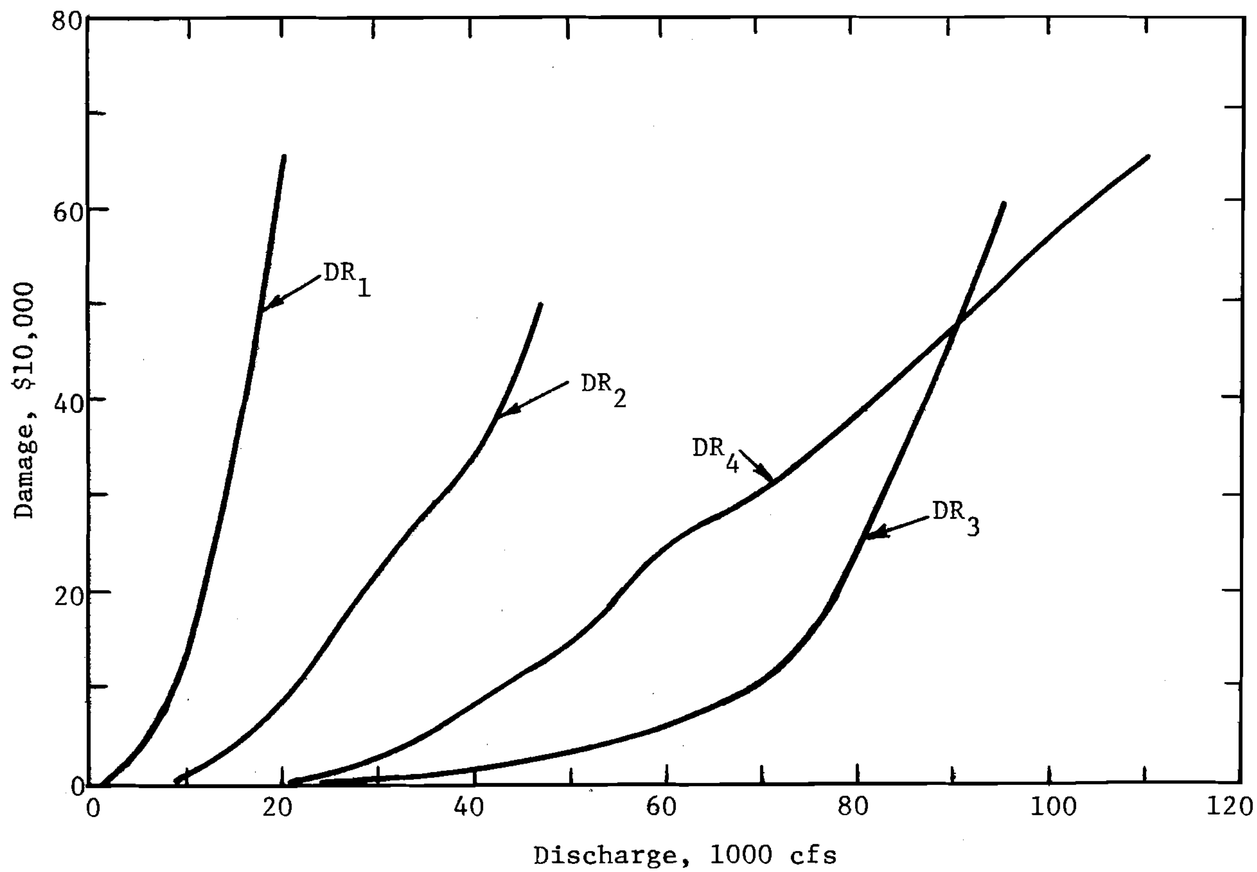

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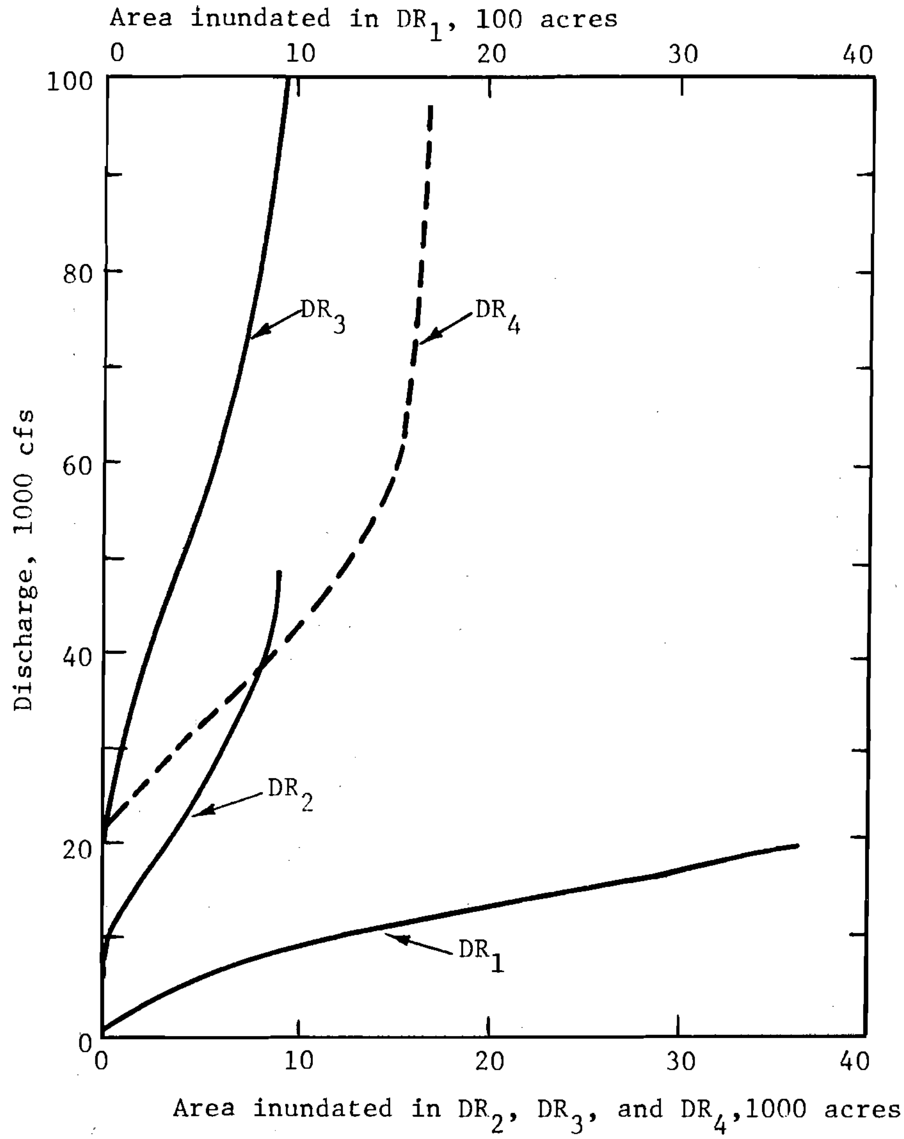

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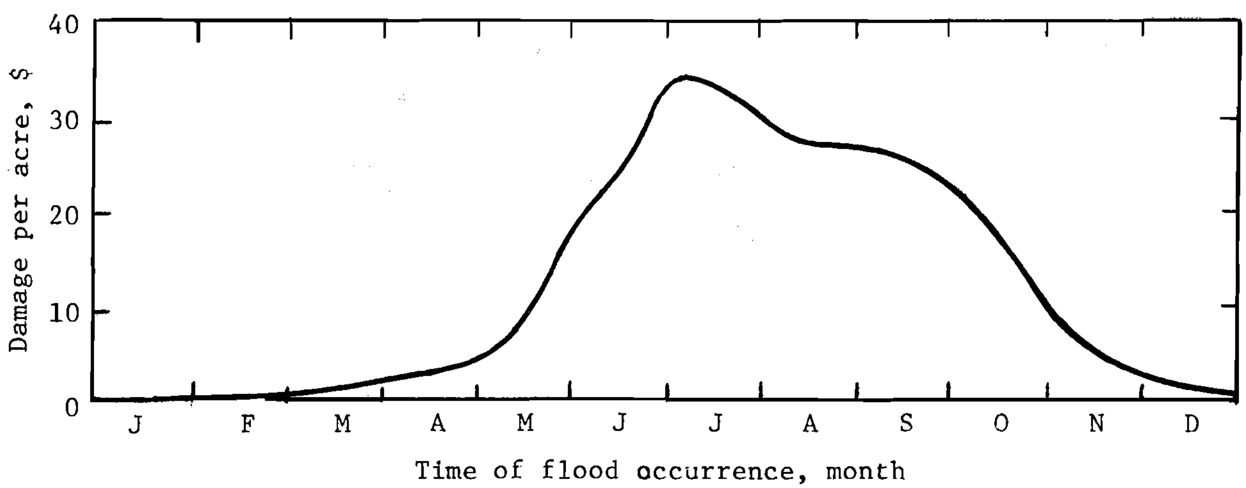

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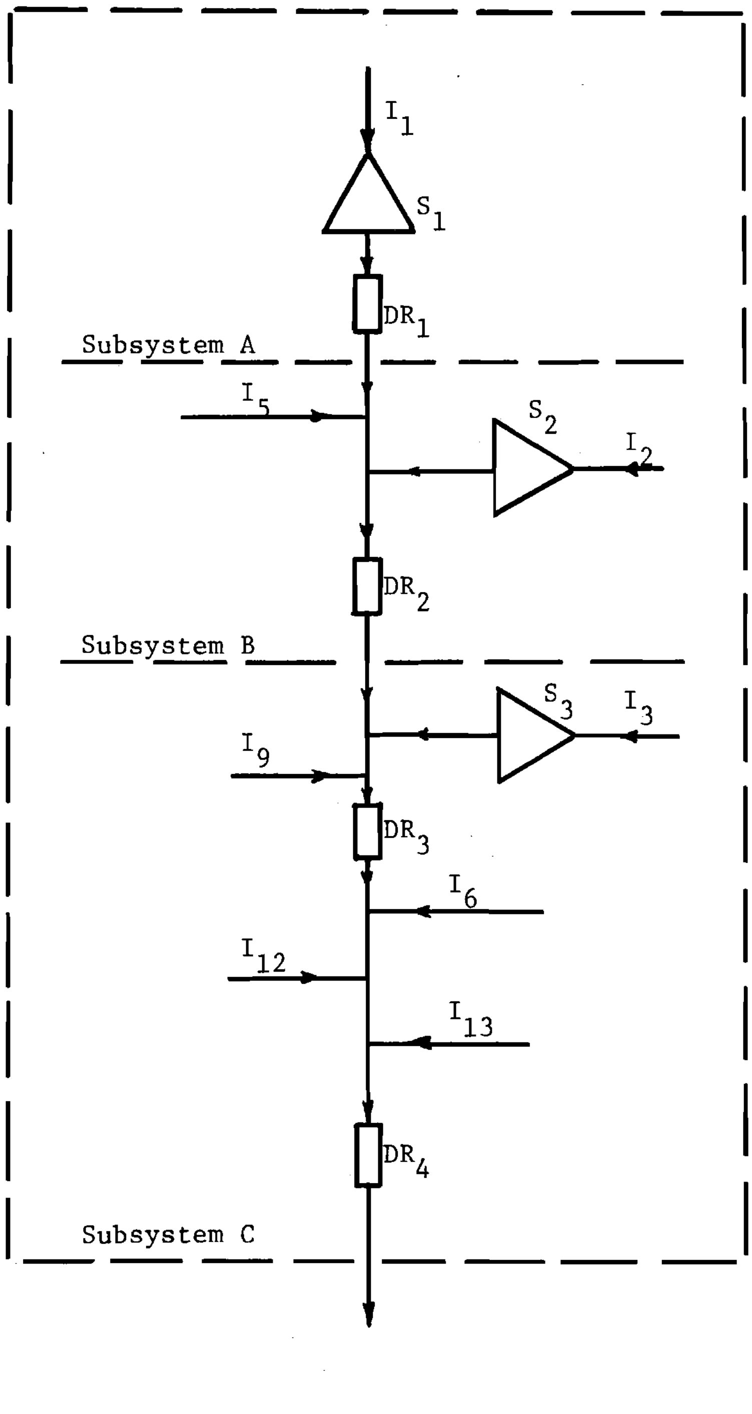

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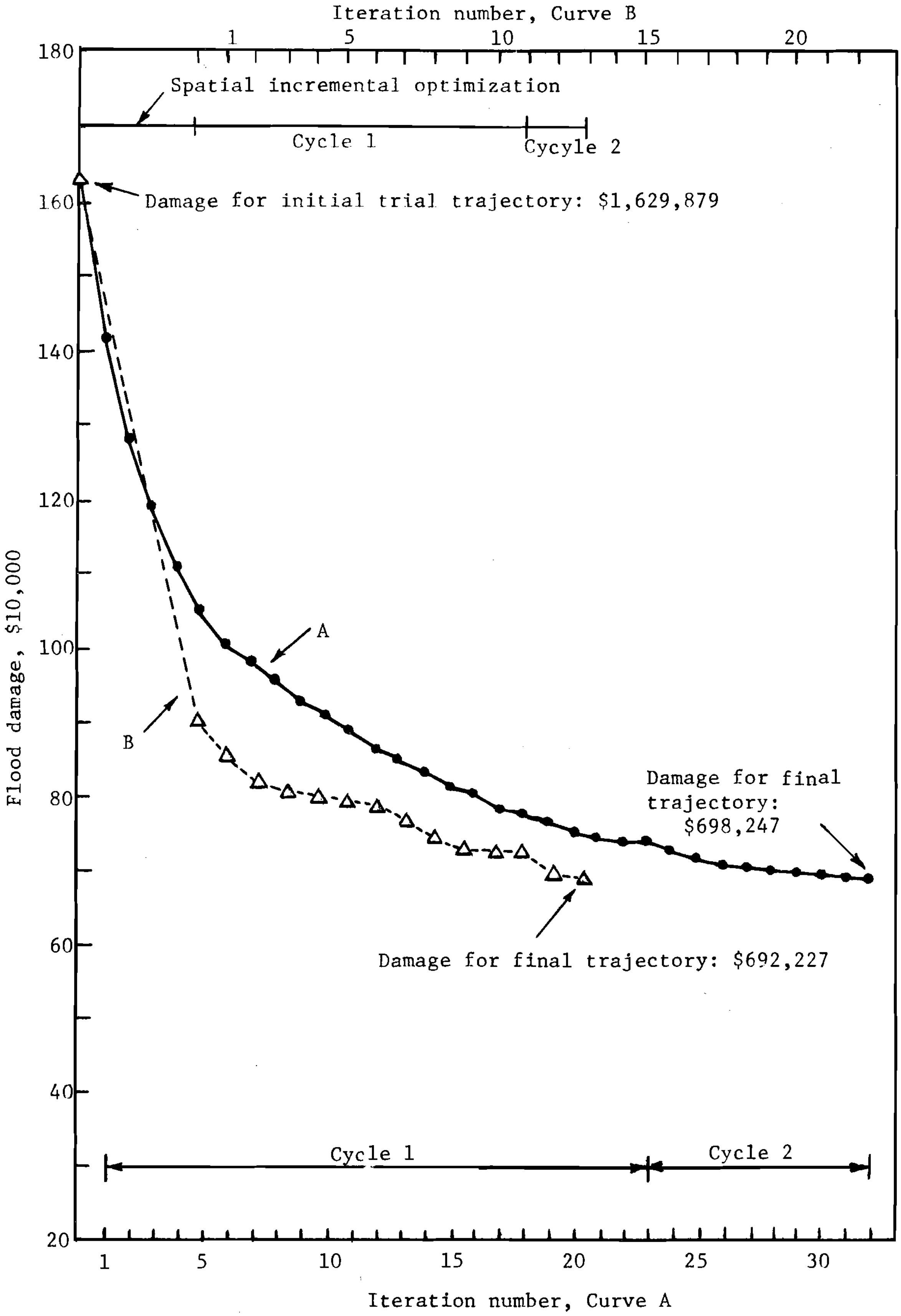

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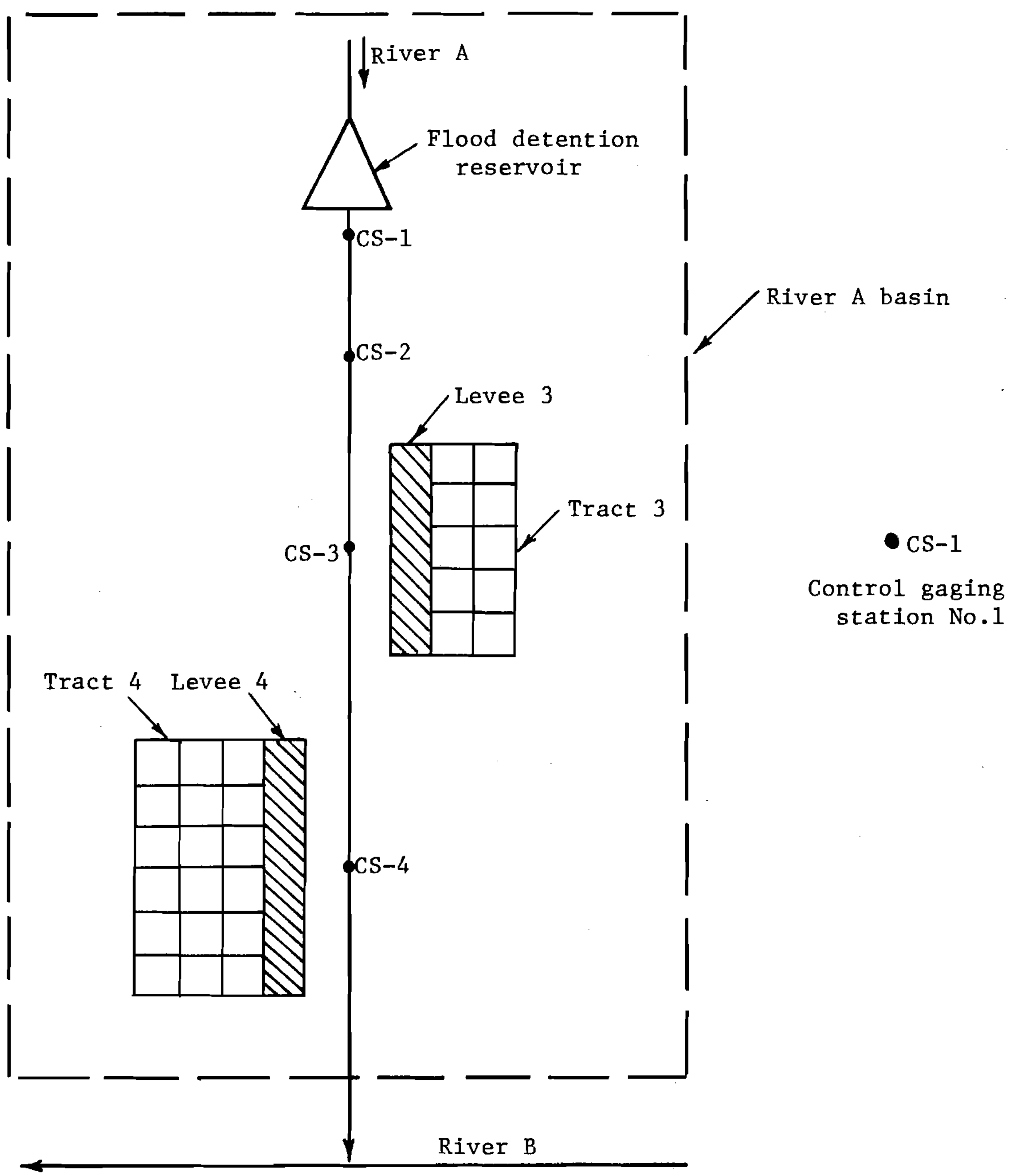

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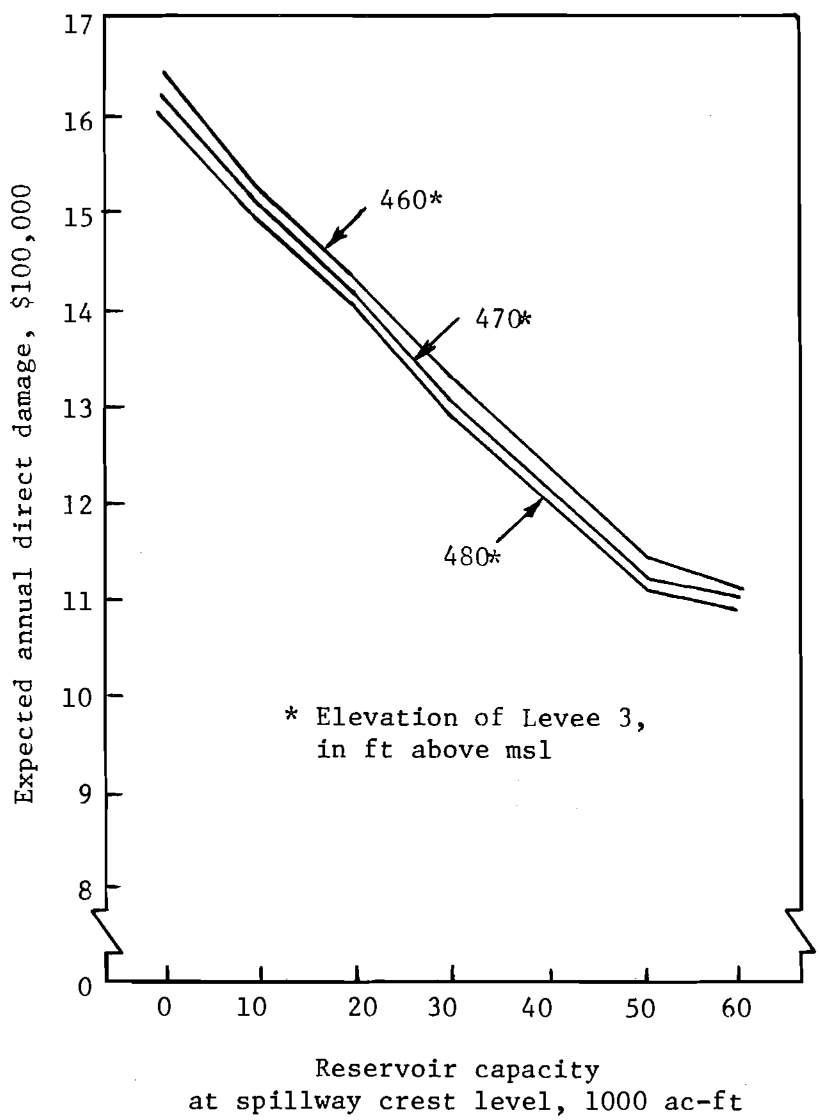

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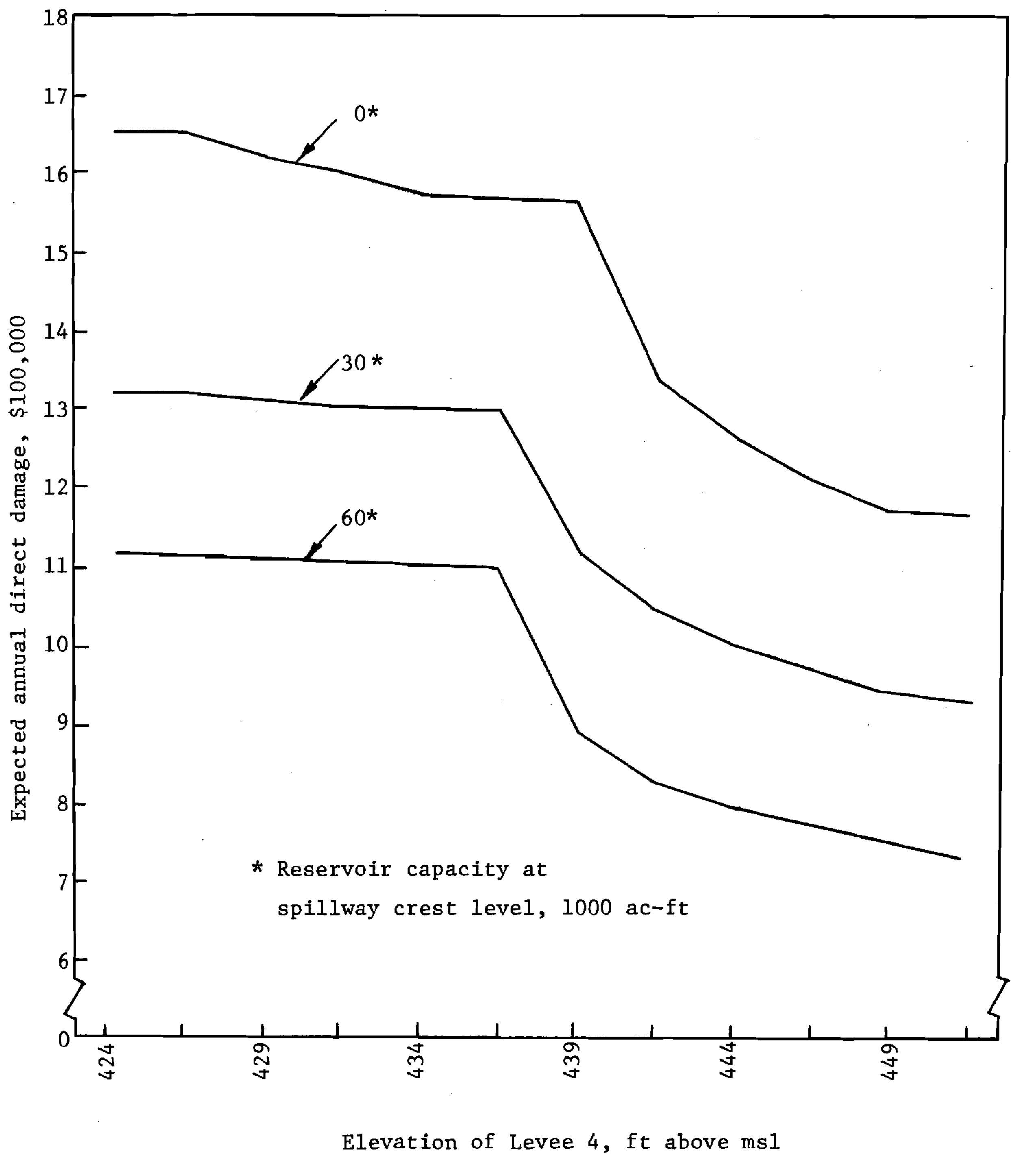

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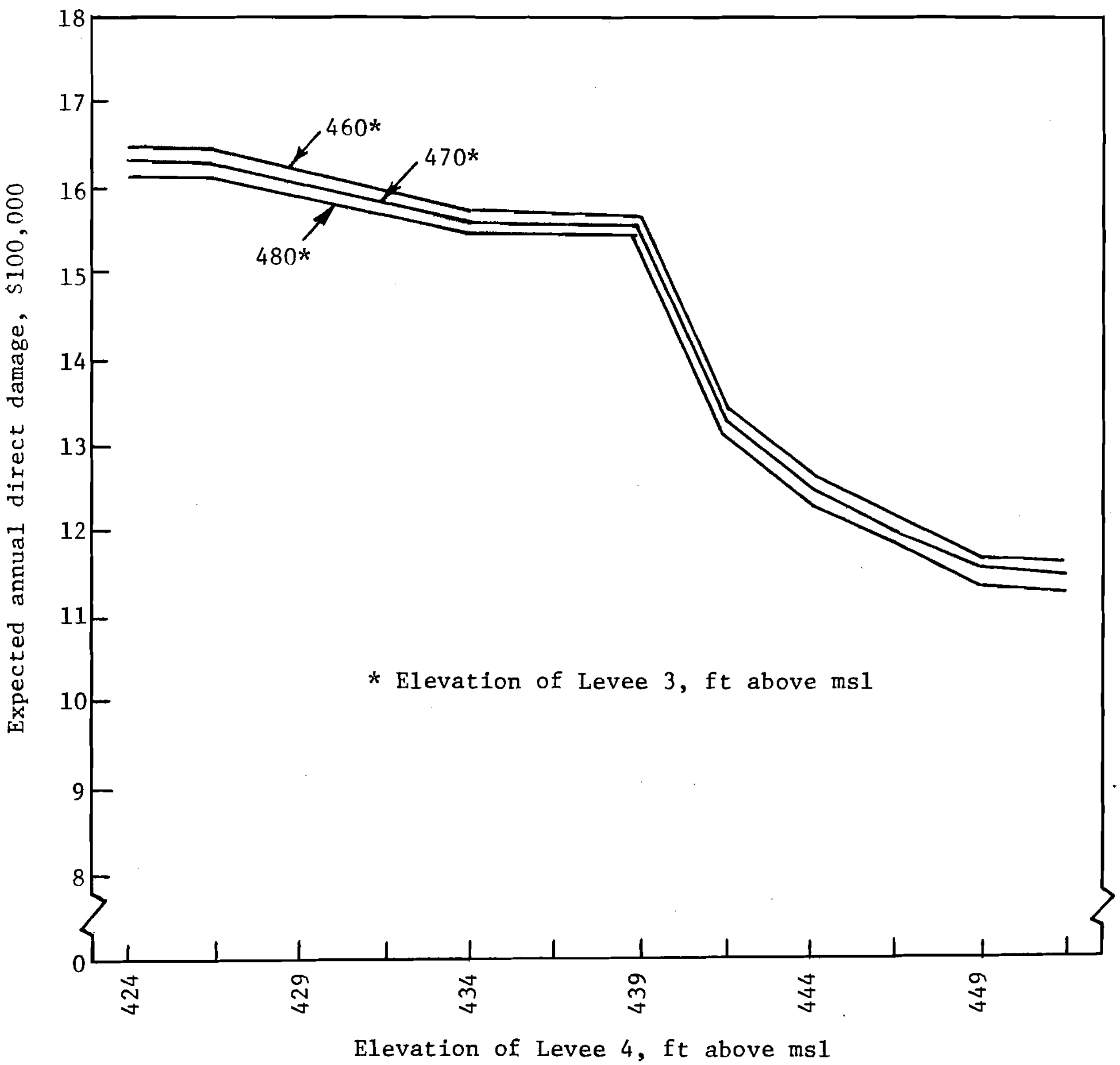

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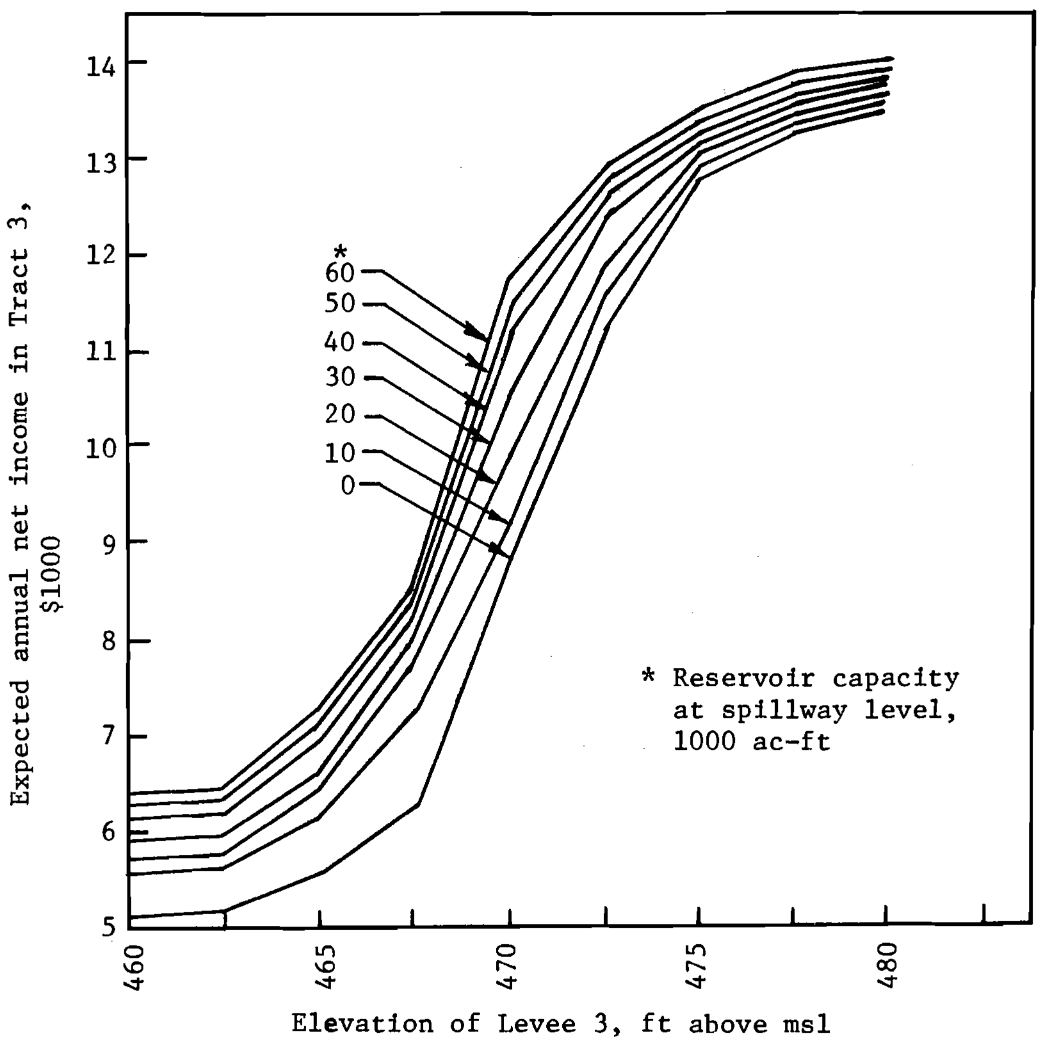

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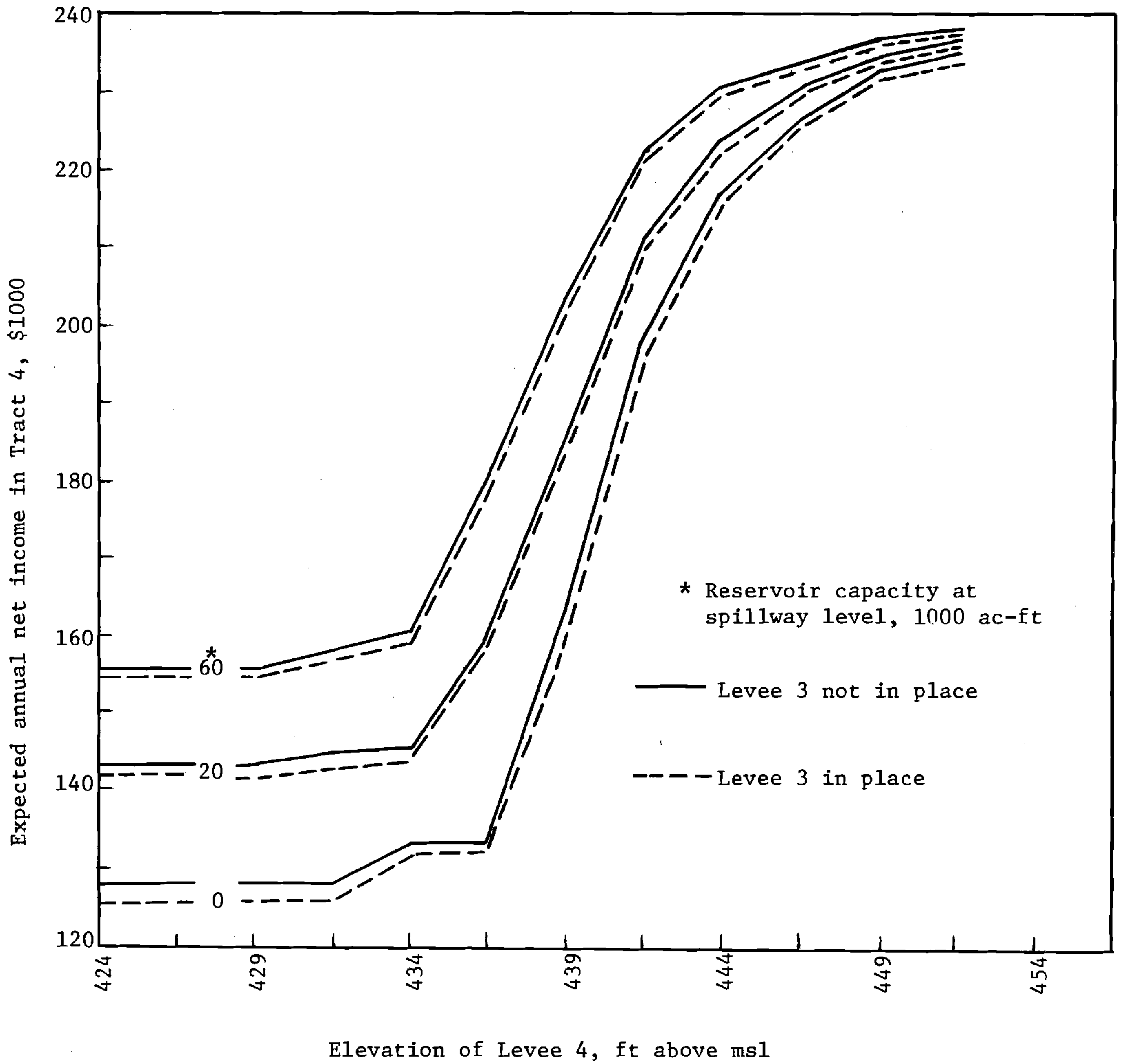

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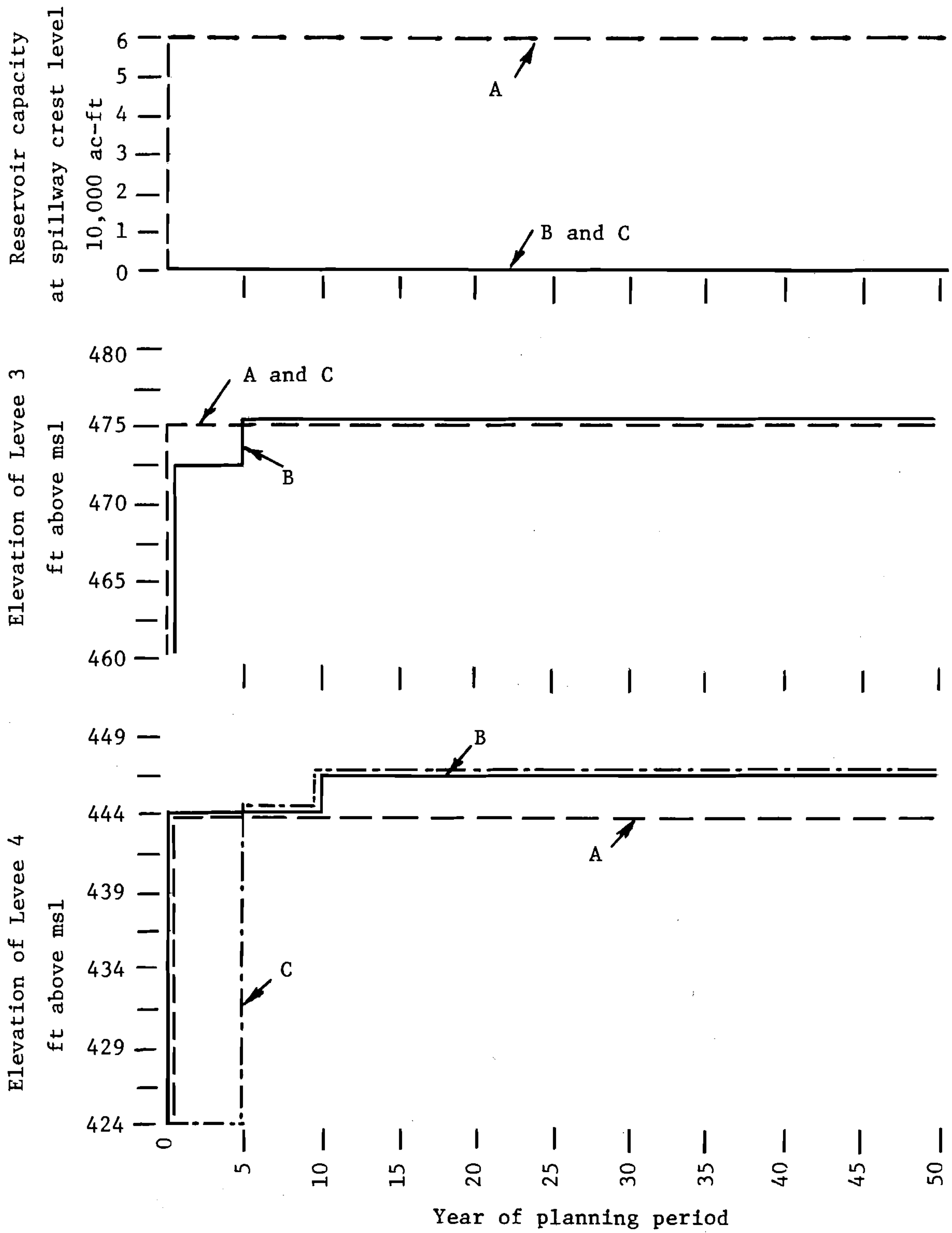

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