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A SCHEME FOR STOCHASTIC STATE VARIABLE WATER RESOURCES SYSTEMS OPTIMIZATION

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ABSTRACT

A SCHEME FOR STOCHASTIC STATE VARIABLE WATER RESOURCES SYSTEMS OPTIMIZATION

This report describes the development of an analytical scheme for the formulation and optimization of water resources systems. The scheme being proposed and investigated is to model the stochastic input of annual as well as monthly streamflows to a hydrologic and water resources system, to formulate the system in a state variable format, and to optimize the stochastic state variable model so formulated by dynamic programming. For annual streamflows, a second-order autoregressive model with a data-based transformation is proposed, and both the maximum likelihood method and the Bayesian approach are used for estimating the model parameters. For monthly streamflows, two linear models are proposed, one is the regression model and the other is the functional relationship model, and their consideration of both uncorrelated and correlated errors and their techniques of generation by a stationary Markov process are discussed. The proposed state variable approach provides a generalized framework within which many different kinds of system models may be expressed and combined for the representation of a given hydrologic and water resources system. This simple yet general format is a major advantage of the proposed state variable modeling. While the annual or monthly streamflows are generated as stochastic inputs to the state variable system model by the proposed scheme, a new procedure of optimization of the system by stochastic dynamic programming is developed. Although the research effort should be further extended to the development of practical procedures for application, a few simple examples are given to illustrate the validity of such applications.

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CONTENTS

LIST OF FIGURES

LIST OF TABLES

I. INTRODUCTION

1-1. Objective of the Study.

The main objective of this study is to develop a scheme for the optimization of water resources systems as an advanced methodology for the optimum planning, design and operation of water resources systems. However, additional research will be needed to extend the proposed scheme to practical procedures that would be applied to the planning and design of actual water resources systems.

The scheme being proposed and investigated is to model the stochastic input of annual as well as monthly streamflows to a hydrologic and water resources system, to formulate the system in a state variable format, and to optimize the stochastic state variable model so formulated by the use of dynamic programming.

Since the physical mechanism of streamflow generation is not completely understood, this process may be treated as being stochastic, and approximate mathematical models can be derived using the historic data. Such models should be simple enough to employ a few parameters as possible and yet be flexible enough to capture all the essential features of the physical process.

On the basis of a stochastic model of the streamflow process, a Monte Carlo technique is used to generate a set of streamflow sequences that are equally as likely to occur as the observed streamflow sequence. The sequences so generated make possible a relatively comprehensive simulation study of the performance of a proposed hydrologic design as it responds to various possible sequences of streamflows, thus aiding the development of a well-balanced design. For that reason, streamflow generation used in conjunction with simulation provides a design evaluation technique which is superior to traditional methods that use only the observed streamflow sequence in evaluating hydrologic designs. Streamflow generation does not provide any new information about the streamflow process but it unveils and expands the available information in a manner suitable for a simulation study.

Both annual and monthly streamflows are frequently used as hydrologic inputs in planning and design studies of water resources systems. The monthly streamflows in particular, constitute a fairly general case because of their seasonal structure. Since conventional stochastic hydrologic modeling techniques do not adequately account for the model parameter uncertainties, one objective of this study is to explore the characteristics of such uncertainties and to incorporate them in the modeling process.

Once the stochastic inputs are modeled and can be generated, the hydrologic and water resources system should be modeled in a flexible format in order to accommodate the inputs and to optimize the output of the system. In this respect, a new state variable approach is investigated for serving in a generalized and flexible manner to accept the stochastic inputs and be optimized.

The use of dynamic programming for systems optimization is a powerful and well accepted tool. However, it is conventionally applicable only to deterministic systems. In this study the water resources system is made stochastic because of its stochastic inputs. Attempts have been made to develop stochastic dynamic programming techniques; however, they have not been too successful because many difficulties arise due to the probabilistic nature of the system. The few stochastic dynamic programming techniques that have been proposed previously are rather limited to simplified assumptions. In this study, investigation is made to develop a stochastic dynamic programming algorithm on the basis of more realistic assumptions so that the results of applying the proposed technique will be closer to the real world situation.

1-2. Scope of the Study.

The scope of this study is essentially reported in the subsequent five chapters. Chapter II will discuss the stochastic modeling and generation of annual streamflows. In this chapter, existing methods of stochastic modeling of streamflows are briefly described. In order to justify the use of available probabilistic theories which are mostly based on normal distributions, Box-Cox transformations of the variables are introduced. The transformed streamflows are then modeled by the second-order autoregressive model. Estimation of the model parameters is made by the method of maximum likelihood and by the Bayesian method. Then, as an illustration, a set of given annual streamflow data are analyzed by the proposed scheme.

Chapter III covers the stochastic modeling of monthly streamflows. In this chapter, the development in stochastic generation of monthly streamflows is briefly reviewed and the general linear model of full rank is recommended for use. Estimation of the model parameters and monthly streamflow generation are discussed for the case of uncorrelated errors as well as for the case of correlated errors. However, because of limitations in time, no further studies have yet been made on the development of the algorithms for applying the proposed scheme for generating monthly streamflows as stochastic inputs.

Chapter IV presents the state variable modeling of hydrologic and water resources systems. The basic concepts and equations for the state variable mathematical model are first described. Then, the formulation of deterministic state variable models and stochastic state variable models are presented with illustration by examples.

Chapter V discusses the optimization techniques of stochastic dynamic programming. The discussion includes stochastic transformation, formulation of recursive equations, chance constraints and steady state probabilities, and risk analysis. An example is given to illustrate the optimization procedure and to discuss the results of the optimization technique.

The report is concluded with summaries and conclusions, a list of references, and an appendix describing the physical and economic data for Watasheamu Dam and Reservoir, which are used in the example of Chapter V.

1-3. Acknowledgements.

The research effort reported here was conducted as a team directed by the project director Ven Te Chow. Various aspects of the study constituted also parts of the doctoral studies by the other co-authors. Chapter II is based on a part of the doctoral thesis by Dong Hee Kim, Chapter III is based on the doctoral research by Taylan A. Ula, and Chapters IV and V are based on the doctoral research by David R. Maidment.

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II. MODELING OF STOCHASTIC INPUT: ANNUAL STREAMFLOWS

2-1. Stochastic Generation and Modeling of Streamflows.

The objective of streamflow generation is to produce a set of synthetic streamflow sequences that would occur equally as likely as the observed streamflow sequence. Statistically, this amounts to the generation of a set of samples from the population defining the streamflow process. However, the characteristics of this population are not known but can be inferred from the information contained in the observed streamflow sequence. A dilemma here is that the observed streamflow sequence is itself a sample, and since it is typically short, the information it contains may not be representative and reliable or may be subject to large sampling errors. Only within these limitations, a model fitted to the observed streamflow sequence can be considered as representative of the stochastic characteristics of the streamflow and thus can be used to reproduce the streamflow process and generate synthetic streamflow sequences. Recognizing this fact, hydrologists have suggested several methods of streamflow generation for extending the information contained in a streamflow record for use in water resources systems planning and design.

The physical mechanism of the streamflow generating process is not completely known and a mathematical model is often derived on the basis of the available historic observations to approximate the underlying process. A model representing a stochastic process, such as streamflow, should be simple enough to employ as few parameters as possible and yet be flexible enough to capture all the essential features of the physical process. Basically two types of models are in use today in stochastic modeling of annual streamflows, namely the Markov models of Thomas and Fiering (1962) and the fractional Gaussian noise (FGN) models of Mandelbrot and Wallis

(1968). Several modifications and refinements have been proposed to these models and extensive literature reviews were made by Chow and Meredith (1968), Kisiel (1969), and Matalas and Wallis (1975) among others.

Both types of models belong to a general class of discrete time linear stochastic processes (Box and Jenkins, 1970). The Markov model is simple in structure but its applicability is limited because the theoretical autocorrelation function is not flexible enough to fit a wide range of sample autocorrelation functions. The FGN model is a continuous moving average process of infinite order. As the first step toward its practical application, the FGN model must be replaced by a model of discrete fractional Gaussian noise (DFGN) process. The DFGN model has a peculiar kernel structure which does not satisfy the Box and Jenkins (1970) definition of stationarity. As the second step toward its operational uses, the DFGN model of infinite order must be approximated by a process involving a finite number of terms. So far six different approximations to DFGN may be found; namely, Types I and II (Mandelbrot and Wallis, 1969), filtered Type II (Matalas and Wallis, 1970), the broken line process (Mejia, Rodriguez-Iturbe and Dawdy, 1972), fast fractional Gaussian noise (FFGN) (Mandelbrot, 1971; Chi, Neal and Young, 1973), and the ARMA (1,1) with the parameter ϕ_1 close to 1 (O'Connell, 1971).

Type I and II, filtered Type II, and the broken line process are moving average models of finite order. In general, moving average models are inferior to autoregressive models for the following reasons: (1) The autocorrelation function of an autoregressive model tails off while that of moving average model cuts off; (2) the autoregressive model is linear in the parameters while the moving average model is nonlinear in the parameters; and (3) the autoregressive model is expressed in terms of the observed past values while the moving average model is written in terms of the unobserved past random disturbances. The FFGN, as a generating process, requires the summation of a large number of terms. ARMA (1,1) with ϕ_1 close to 1.0 can be regarded as an approximation to the FGN but it may not be a good approximation to a natural streamflow process, since the model parameters are not estimated from the observed streamflow sequence.

In the present investigation, a new model, a second-order autoregressive process with a data-based transformation, is proposed to approximate the streamflow generating process. The properties of the new model are to be presented and the parameter estimation procedures be developed. Due to its simplicity in structure and its generality in descriptive fidelity, the new model seems to be very promising for streamflow simulation.

2-2. The Box-Cox Transformations.

Streamflow sequences seldom follow the normal probability distribution. In this situation, it is often found advantageous to transform the original sequence so that the transformed sequence can be adequately represented by a stochastic model based on the normal probability distribution theory. The problem associated with transformations is two-fold: First, a suitable transformation must be identified; and second, parameter estimators must be found with desirable properties, using the transformation. The procedure developed here considers these two aspects of the transformation problem simultaneously.

Consider a family of transformations proposed by Box and Cox (1964):

$$z_t = \begin{cases} \dfrac{y_t^{\lambda} - 1}{\lambda} & (\lambda \neq 0) \\ \log y_t & (\lambda = 0) \end{cases} \qquad (2\text{-}1)$$

where y_t denotes the streamflow value taken at time t. Let us suppose that, for some unknown value of λ, the transformed observations can be represented by the normal second-order autoregressive, AR(2), process:

$$\frac{y_t^\lambda - 1}{\lambda} = \phi_o + \phi_1 \frac{y_{t-1}^\lambda - 1}{\lambda} + \phi_2 \frac{y_{t-2}^\lambda - 1}{\lambda} + a_t \tag{2-2a}$$

or

$$z_t = \phi_o + \phi_1 z_{t-1} + \phi_2 z_{t-2} + a_t, \ (t=1,2,\ldots,n) \tag{2-2b}$$

where $\underline{\phi}' = (\phi_o, \phi_1, \phi_2)$, which denotes the transpose of $\underline{\phi}$, is a vector of unknown parameters associated with the transformed data and a_t is normally and independently distributed random disturbance, with mean zero and variance σ_a^2, or NID$(0,\sigma_a^2)$. Eq. (2-2b) can also be expressed in terms of the deviation of z_t from its mean, or

$$(z_t-\mu) = \phi_1(z_{t-1}-\mu) + \phi_2(z_{t-2}-\mu) + a_t \tag{2-2c}$$

where μ is the mean of the process z_t. By comparing Eq. (2-2c) with Eq. (2-2b), it can be seen that

$$\phi_o = \mu(1 - \phi_1 - \phi_2) \tag{2-3}$$

Although Eq. (2-2c) is a commonly used form in stochastic hydrology, Eq. (2-2b) will be used in the following analysis because it is linear in its parameters. Eq. (2-2c) contains the product of parameters, $\phi_1\mu$ and $\phi_2\mu$, and hence it is not linear in its parameters.

2-3. Properties of the Second-Order Autoregressive Model.

When μ is assumed to be zero, the AR(2) model of Eq. (2-2b) can be written as

$$z_t = \phi_1 z_{t-1} + \phi_2 z_{t-2} + a_t \tag{2-4}$$

For stationarity, the roots, $1/\alpha_1$ and $1/\alpha_2$, of the following equation

$$(1-\phi_1 B-\phi_2 B^2) = (1-\alpha_1 B)(1-\alpha_2 B) = 0 \tag{2-5}$$

must lie outside the unit circle. This is equivalent to requiring that the parameters ϕ_1 and ϕ_2 lie in the triangular region:

$$\begin{aligned} &\phi_2 + \phi_1 < 1 \\ &\phi_2 - \phi_1 < 1 \\ &-1 < \phi_2 < 1 \end{aligned} \tag{2-6}$$

as shown in Fig. 2-1.

The autocorrelation function for the AR(2) process is given by the second-order difference equation:

$$\rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2} \qquad k \geq 2 \tag{2-7}$$

with $\rho_o = 1$ and $\rho_1 = \phi_1/(1-\phi_2)$. The solution of the difference equation (2-7) (Stralkowski, Wu, and DeVor 1970) is

$$\begin{aligned} \rho_k &= C_1 \alpha_1^k + C_2 \alpha_2^k; \ \alpha_1, \alpha_2 \text{ being real, unequal} \\ &= (C_1 + C_2^k)\alpha^k; \ \alpha_1, \alpha_2 \text{ being real, equal} \\ &= C_1 \gamma^k \cos(k\theta + C_2); \ \alpha_1, \alpha_2 \text{ being complex} \end{aligned} \tag{2-8}$$

where C_1 and C_2 are constants and

$$\gamma = \sqrt{-\phi_2} \tag{2-9}$$

$$\theta = \cos^{-1}\left(\frac{\phi_1}{2\sqrt{-\phi_2}}\right) \tag{2-10}$$

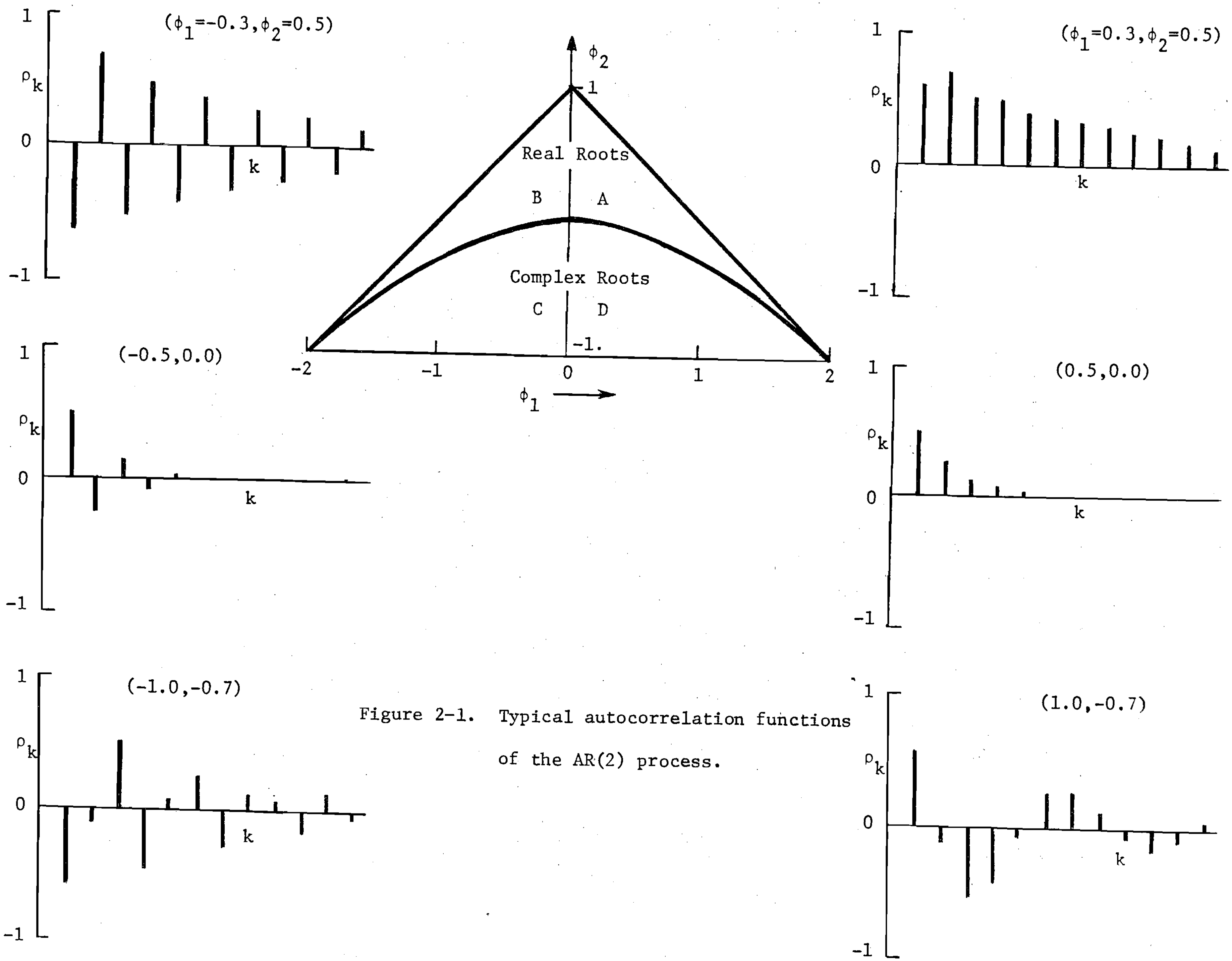

Figure 2-1. Typical autocorrelation functions of the AR(2) process.

The patterns in the autocorrelation function of the AR(2) process are also illustrated in Fig. 2-1. If the roots are real, which occurs when $\phi_1^2 + 4\phi_2 \geq 0$, either the autocorrelation function will be positive as it damps out to zero (region A), or it will alternate in sign as it damps to zero (region B). However, if the roots are complex ($\phi_1^2 + 4\phi_2 < 0$), the autocorrelation function is a damped sinusoidal wave (regions C and D).

The autocorrelation function for the AR(2) process is much more flexible than that for the Markov process. The AR(2) process may be used to fit either damped exponentials or damped sinusoidals in the sample autocorrelation functions. As early as 1927, Yule proposed a second order autoregressive model to describe the sequence of Wolfer's sunspot number over a 176 year period. Since the Yule's pioneering work, the AR(2) model has been extensively used to describe a variety of natural phenomena. For example, Quimpo (1968) used the AR(2) process in the modeling of daily river flows.

2-4. Estimation of Model Parameters.

Having chosen a model which will adequately represent the streamflow sequence, the next step is to estimate the model parameters. It is generally accepted that maximum likelihood estimates are asymptotically more efficient than moment estimates. The parameters of a streamflow generating model estimated from a short historic sequence, however, are not likely to be equal to their respective population values. We may treat this parameter uncertainty problem by employing the Bayesian approach of statistical inference. Bayesian inference provides a framework to pool all the available information to reduce the parameter uncertainty. More importantly, the Bayesian approach can make exact finite sample probability statements about the unknown parameters.

The model parameters are estimated by using both the maximum likelihood method and the Bayesian approach. A numerical example will be given in Section 2-5 to show how some of the techniques developed could be used in the analysis of actual hydrologic time series for water resources systems planning.

The parameters are to be estimated for the general AR(p) process which is given by an extended form of Eq. (2-2a) or Eq. (2-2b):

$$\frac{y_t^{\lambda}-1}{\lambda} = \phi_o + \phi_1 \frac{y_{t-1}^{\lambda}-1}{\lambda} + \phi_2 \frac{y_{t-2}^{\lambda}-1}{\lambda} + \dots + \phi_p \frac{y_{t-p}^{\lambda}-1}{\lambda} + a_t \qquad (2\text{-}11a)$$

or

$$z_t = \phi_o + \phi_1 z_{t-1} + \phi_2 z_{t-2} + \dots + \phi_p z_{t-p} + a_t \qquad (2\text{-}11b)$$

Eq. (2-11a) includes an AR(1) process as a special case with p=1. When p=0, the process reduces to a degenerate stochastic process where y_t is independently and identically distributed. Therefore the following procedure can also be applied to flood frequency analysis.

It is important to note that, for given λ, the autoregressive model shown by Eq. (2-11a) is linear in the parameters. Also at time t, $y_{t-1}, y_{t-2}, \dots, y_{t-p}$ on the right side of Eq. (2-11a) have already been observed and so they are not random variables but deterministic variables. Hence, this autoregressive model satisfies all the necessary conditions for the formulation of a normal linear regression model. Therefore, the general principles for the linear regression model are directly applicable to the autoregressive model.

Since λ is an unknown parameter, it will be estimated along with other unknown parameters, the ϕ's and σ_a. In what follows, the parameters are to be estimated by using both the maximum likelihood method and the Bayesian approach.

2-5. Maximum Likelihood Estimation of Parameters.

(1) The Likelihood Function. Before discussing the maximum likelihood method of estimating the parameters, the likelihood function will be defined. For this purpose, the joint probability density function (pdf) of $\underline{y}' = (y_1, y_2, \ldots, y_n)$ given the parameters λ, $\underline{\phi}$, and σ_a, can be factored as follows:

$$p(y_1, y_2, \ldots, y_n | \lambda, \underline{\phi}, \sigma_a) = p(y_1, y_2, \ldots, y_p | \lambda, \underline{\phi}, \sigma_a) \cdot p(y_{p+1}, y_{p+2}, \ldots, y_n | \lambda, \underline{\phi}, \sigma_a, y_1, y_2, \ldots, y_p) \tag{2-12}$$

It is seen from Eq. (2-11a) that $p(y_1, y_2, \ldots, y_p | \lambda, \underline{\phi}, \sigma_a)$ involves the unknown data $y_o, y_{-1}, \ldots, y_{1-p}$ occurring before beginning of the observations. A simple approximate method of resolving this difficulty is to consider $y_1, y_2, \ldots, y_p$ as fixed at their observed values; this is equivalent to assuming that

$$p(\underline{y} | \lambda, \underline{\phi}, \sigma_a) = p(\underline{y} | \lambda, \underline{\phi}, \sigma_a, y_1, y_2, \ldots, y_p) \tag{2-13}$$

where $\underline{y}' = (y_{p+1}, y_{p+2}, \ldots, y_n)$.

Since a_t in Eq. (2-11a) is NID$(0, \sigma_a^2)$, the joint pdf of $\underline{a}'$ $= (a_{p+1}, a_{p+2}, \ldots, a_n)$ may be written as

$$p(a_{p+1}, a_{p+2}, \ldots, a_n) = \frac{1}{(2\pi\sigma_a^2)^{(n-p)/2}} \exp\left\{-\frac{\sum_{t=p+1}^{n} a_t^2}{2\sigma_a^2}\right\} = \frac{1}{(2\pi\sigma_a^2)^{(n-p)/2}} \exp\left\{-\frac{\underline{a}'\underline{a}}{2\sigma_a^2}\right\} \tag{2-14}$$

From Eq. (2-11b) the following equations can be written:

$$\begin{bmatrix} a_{p+1} \\ a_{p+2} \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} z_{p+1} \\ z_{p+2} \\ \vdots \\ z_n \end{bmatrix} - \begin{bmatrix} 1 & z_p & z_{p-1} & \cdots & z_1 \\ 1 & z_{p+1} & z_p & \cdots & z_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & z_{n-1} & z_{n-2} & \cdots & z_{n-p} \end{bmatrix} \begin{bmatrix} \phi_o \\ \phi_1 \\ \vdots \\ \phi_p \end{bmatrix} \tag{2-15}$$

or in matrix notation:

$$\underline{a} = \underline{z} - \underline{x}\ \underline{\phi} \tag{2-16}$$

To write the joint pdf for $(y_{p+1}, y_{p+2}, \ldots, y_n)$, there is the need for the Jacobian of the transformation from $(a_{p+1}, a_{p+2}, \ldots, a_n)$ to $(y_{p+1}, y_{p+2}, \ldots, y_n)$, which can be obtained from Eq. (2-11a) as

$$J = \prod_{t=p+1}^{n} \left|\frac{\partial a_t}{\partial y_t}\right| = \prod_{t=p+1}^{n} y_t^{\lambda-1} \tag{2-17}$$

Then,

$$p(\underline{y} \mid \lambda, \underline{\phi}, \sigma_a) \approx p(\underline{y} \mid \lambda, \underline{\phi}, \sigma_a, y_1, y_2, \ldots, y_p)$$

$$= \frac{J}{(2\pi\sigma_a^2)^{(n-p)/2}} \exp\left\{-\frac{\sum_{t=p+1}^{n} [z_t - (\phi_o + \phi_1 z_{t-1} + \ldots + \phi_p z_{t-p})]^2}{2\sigma_a^2}\right\}$$

$$= \frac{J}{(2\pi\sigma_a^2)^{(n-p)/2}} \exp\left\{-\frac{[\underline{z} - \underline{x}\ \underline{\phi}]'[\underline{z} - \underline{x}\ \underline{\phi}]}{2\sigma_a^2}\right\} \tag{2-18}$$

When Eq. (2-18) is viewed as a function of the parameters, it is called the likelihood function, or

$$\ell(\lambda, \underline{\phi}, \sigma_a \mid \underline{y}) = \frac{J}{(2\pi\sigma_a^2)^{(n-p)/2}} \exp\left\{-\frac{[\underline{z} - \underline{x}\ \underline{\phi}]'[\underline{z} - \underline{x}\ \underline{\phi}]}{2\sigma_a^2}\right\} \tag{2-19}$$

It is often convenient to work with the log likelihood function which is given by

$$L(\lambda,\underline{\phi},\sigma_a \mid \underline{y}) = \log \ell(\lambda,\underline{\phi},\sigma_a \mid \underline{y})$$

$$= -\frac{(n-p)}{2}(\log 2\pi + \log \sigma_a^2) + \log J$$

$$- \frac{[\underline{z} - \underline{x}\,\underline{\phi}]'[\underline{z} - \underline{x}\,\underline{\phi}]}{2\sigma_a^2} \qquad (2\text{-}20)$$

(2) Maximum likelihood Estimates. The method of maximum likelihood is preferable to the method of moments since maximum likelihood estimates are, in general, asymptotically more efficient than moment estimates and are asymptotically unbiased (Kendall and Stuart, 1967). Maximum likelihood estimates may be adjusted to derive unbiased estimates.

The maximum likelihood estimates of $\lambda,\underline{\phi}$, and σ_a^2 can be obtained by maximizing the likelihood function of Eq. (2-19). For given λ, Eq. (2-19) is the likelihood for a standard least squares problem. Hence, the maximum likelihood estimates of the ϕ's are the least squares estimates of the ϕ's:

$$\hat{\underline{\phi}}(\lambda) = (\underline{x}'\underline{x})^{-1}\,\underline{x}'\underline{z} \qquad (2\text{-}21)$$

It is well known that $\hat{\underline{\phi}}(\lambda)$ are the unbiased minimum variance estimates of $\underline{\phi}(\lambda)$ (Scheffé, 1959). The maximum likelihood estimate of $\sigma_a^2(\lambda)$ is

$$\hat{\sigma}_a^2(\lambda) = \frac{[\underline{z} - \underline{x}\,\hat{\underline{\phi}}(\lambda)]'[\underline{z} - \underline{x}\,\hat{\underline{\phi}}(\lambda)]}{n-p} = \frac{S^2(\lambda)}{n-p} \qquad (2\text{-}22)$$

where

$$S^2(\lambda) = [\underline{z} - \underline{x}\,\hat{\underline{\phi}}(\lambda)]'[\underline{z} - \underline{x}\,\hat{\underline{\phi}}(\lambda)] \qquad (2\text{-}23)$$

An unbiased estimate of $\sigma_a^2(\lambda)$ is given by

$$s^2(\lambda) = \frac{S^2(\lambda)}{n-2p-1} \qquad (2\text{-}24)$$

The residual variance has n-2p-1 degrees of freedom since there are effectively n-p observations and p+1 degrees of freedom are lost in fitting the unknown constants $\underline{\phi}$. Eqs. (2-21) and (2-22) are substituted in Eq. (2-19) to obtain the maximized likelihood function as follows:

$$\ell_{max}(\lambda) \propto \left[\frac{S^2(\lambda)}{J^{2/(n-p)}}\right]^{-\frac{n-p}{2}} \qquad (2\text{-}25)$$

Since there are different dimensions for different values of λ, the presence of the Jacobian in Eq. (2-25) makes the term in the bracket dimensionless. Similarly, the maximized log likelihood function is given by

$$L_{max}(\lambda) = \log \ell_{max}(\lambda) = \text{const} - \frac{n-p}{2} \log \left[\frac{S^2(\lambda)}{J^{2/(n-p)}}\right] \qquad (2\text{-}26)$$

Since only the relative value of $\ell_{max}(\lambda)$ is of interest, the likelihood function is usually regarded as containing an arbitrary multiplicative constant. Likewise, the log likelihood function contains an arbitrary additive constant. Now $L_{max}(\lambda)$ is evaluated for various values of λ and the value $\hat{\lambda}$ which maximizes Eq. (2-26) is the maximum likelihood estimate for λ. Then, Eqs. (2-21) and (2-22), evaluated for $\lambda = \hat{\lambda}$, are the maximum likelihood estimates for $\underline{\phi}$ and σ_a^2, respectively.

There has been discussion about whether the parameters should be estimated from the original sample or from the transformed sample. Matalas (1967), Burges (1972), and Matalas and Wallis (1972) suggested calculating the sample statistics from the original data first, substituting them into the equations which relate the population parameters in the original and the transformed units, and then solving the equations for the parameters in the transformed units. In this procedure the parameters are estimated by the method of moments. However, in the method of maximum likelihood the parameters can be easily estimated from the transformed data. For the independent lognormal process, Finney (1941) and Aitchison and Brown (1963) showed that maximum likelihood estimates are much more efficient than moment estimates for large samples.

(3) Confidence Regions for ϕ_1 and ϕ_2 of the AR(2) Model. The AR(2) model is recommended for use in this study. The confidence of its estimated parameters ϕ_1 and ϕ_2 is now discussed. To a good degree of approximation, a $100(1-\alpha)\%$ joint confidence region for the parameters ϕ_1, ϕ_2 for given λ is given by Stralkowski, Wu, and DeVor (1970) as

$$[\phi_1(\lambda) - \hat{\phi}_1(\lambda)]^2 + 2\hat{\rho}_1(\lambda)\ [\phi_1(\lambda) - \hat{\phi}_1(\lambda)]$$

$$\cdot[\phi_2(\lambda) - \hat{\phi}_2(\lambda)] + [\phi_2(\lambda) - \hat{\phi}_2(\lambda)]^2$$

$$\leq \frac{2}{n-5}\ [1 - \hat{\phi}_1(\lambda)\hat{\rho}_1(\lambda) - \hat{\phi}_2(\lambda)\hat{\rho}_2(\lambda)].F_\alpha(2,n-5) \tag{2-27}$$

where $F_\alpha(2,n-5)$ is the upper 100α percentage point of an F distribution with (2,n-5) degrees of freedom. Eq. (2-27) is the equation of an ellipse and can be written in the form

$$\frac{u(\lambda)^2}{\beta_1(\lambda)^2} + \frac{v(\lambda)^2}{\beta_2(\lambda)^2} = 1 \tag{2-28}$$

where

$$\begin{bmatrix} u(\lambda) \\ v(\lambda) \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \phi_1(\lambda) - \hat{\phi}_1(\lambda) \\ \phi_2(\lambda) - \hat{\phi}_2(\lambda) \end{bmatrix} \tag{2-29}$$

and

$$\beta_1(\lambda) = \left[\frac{2[1 - \hat{\phi}_1(\lambda)\hat{\rho}_1(\lambda) - \hat{\phi}_2(\lambda)\hat{\rho}_2(\lambda)]\ F_\alpha(2,n-5)}{(n-5)[1 + \hat{\rho}_1(\lambda)]}\right]^{1/2}$$

$$\beta_2(\lambda) = \left[\frac{2[1 - \hat{\phi}_1(\lambda)\hat{\rho}_1(\lambda) - \hat{\phi}_2(\lambda)\hat{\rho}_2(\lambda)]\ F_\alpha(2,n-5)}{(n-5)[1 - \hat{\rho}_1(\lambda)]}\right]^{1/2} \qquad (2\text{-}30)$$

In Eq. (2-28), $u(\lambda)$ and $v(\lambda)$ are transformed coordinates centered at $\hat{\phi}_1(\lambda)$, $\hat{\phi}_2(\lambda)$ having directions 45° and 135° respectively and $\beta_1(\lambda)$ and $\beta_2(\lambda)$ are the one-half lengths of the principal axes of the ellipse.

2-6. Bayesian Estimation of Parameters.

(1) The Bayesian Theorem. The basic difference between the Bayesian and non-Bayesian (classical) approaches of statistical inference is that the Bayesian approach looks upon a population parameter as a random variable while the non-Bayesian approach looks upon a population parameter as an unknown constant. In the Bayesian approach, a pdf can be ascribed to a population parameter, which would contain any information known prior to taking the data plus the information obtained from the data.

Suppose that $p(\underline{z},\underline{\phi})$ is a joint pdf for a vector of observations $\underline{z}$ and a vector of parameters $\underline{\phi}$. Then,

$$p(\underline{z},\underline{\phi}) = p(\underline{z}|\underline{\phi})p(\underline{\phi}) = p(\underline{\phi}|\underline{z})p(\underline{z}) \qquad (2\text{-}31)$$

Given the observed data $\underline{z}$, the conditional distribution of $\underline{\phi}$ is

$$p(\underline{\phi}|\underline{z}) = \frac{p(\underline{\phi})p(\underline{z}|\underline{\phi})}{p(\underline{z})} \qquad (2\text{-}32)$$

which can be expressed alternatively as

$$p(\underline{\phi}|\underline{z}) \propto p(\underline{\phi}).p(\underline{z}|\underline{\phi}) \qquad (2\text{-}33)$$

or posterior pdf $\propto$ prior pdf x likelihood function. Eq. (2-33) is usually referred to as the Bayesian theorem. In this equation, $p(\underline{\phi})$, which tells us what is known about $\underline{\phi}$ without knowledge of the data, is called the prior distribution of $\underline{\phi}$. The term $p(\underline{z}|\underline{\phi})$, viewed as a function of $\underline{\phi}$, is the likelihood function. The term $p(\underline{\phi}|\underline{z})$ is called the posterior pdf of $\underline{\phi}$ given $\underline{z}$ and has all the prior and sample information incorporated in it. The posterior pdf is employed in the Bayesian approach to make inferences about parameters (Zellner, 1971; Box and Tiao, 1973).

(2) Posterior Pdf's for Parameters. As regards a prior pdf for the parameters, it is assumed that the information is diffuse and represented, following Box and Cox (1964), by

$$p(\lambda,\underline{\phi},\sigma_a) \propto \frac{1}{\sigma_a \; J^{(p+1)/(n-p)}} \qquad 0<\sigma_a<\infty;\ -\infty<\underline{\phi},\lambda<\infty \qquad (2\text{-}34)$$

where J is the Jacobian given by Eq. (2-17) and p is the order of an autoregressive process shown in Eq. (2-11b). Note that the ϕ's are not restricted to be within the stationarity region and thus the analysis applies without imposing the condition of stationarity on a given time series. When some prior information about the model parameters is available, the prior pdf in Eq. (2-34) can be altered to incorporate this information. As Benson and Matalas (1968) and Vicens, Rodriguez-Iturbe and Schaake (1974) reported that the regional regression models for the autocorrelation coefficient were not very successful in providing the prior information, the above diffuse prior is recommended.

On combining the prior pdf in Eq. (2-34) with the likelihood function in Eq. (2-19) using the Bayesian theorem, the posterior pdf for the parameters is obtained as:

$$p(\lambda,\underline{\phi},\sigma_a|\underline{\underline{y}}) \propto \frac{J^{(n-2p-1)/(n-p)}}{\sigma_a^{(n-p+1)}} \exp\{-\frac{[\underline{z} - \underline{x}\ \underline{\phi}]'[\underline{z} - \underline{x}\ \underline{\phi}]}{2\sigma_a^2}\} \tag{2-35}$$

It is to be noted that the likelihood function, Eq. (2-19), and the posterior pdf, Eq. (2-35), are proportional. Hence, it is expected that the results from the Bayesian approach will be comparable to those from the maximum likelihood method. Also in large samples, the maximum likelihood estimates are the approximate means of the posterior pdf of the parameters, a pdf that will usually be approximately normal (Zellner, 1971).

In the Bayesian approach of statistical analysis, inferences about unknown parameters are made through the use of the posterior pdf for the parameters. The joint posterior pdf shown in Eq. (2-35) can be analyzed conveniently by taking note of the following algebraic identity:

$$(\underline{z} - \underline{x}\ \underline{\phi})'(\underline{z} - \underline{x}\ \underline{\phi}) = (n-2p-1)s^2(\lambda) + [\underline{\phi} - \underline{\hat{\phi}}(\lambda)]'\ \underline{x}'\underline{x}\ [\underline{\phi} - \underline{\hat{\phi}}(\lambda)] \tag{2-36}$$

On substituting Eq. (2-36) in Eq. (2-35),

$$p(\lambda,\underline{\phi},\sigma_a|\underline{\underline{y}}) \propto \frac{J^{(n-2p-1)/(n-p)}}{\sigma_a^{(n-p+1)}} \exp\{-\frac{(n-2p-1)s^2(\lambda) + [\underline{\phi} - \underline{\hat{\phi}}(\lambda)]'\ \underline{x}'\underline{x}[\underline{\phi} - \underline{\hat{\phi}}(\lambda)]}{2\sigma_a^2}\} \tag{2-37}$$

Integrating Eq. (2-37) with respect to σ_a,

$$p(\lambda,\underline{\phi}|\underline{\underline{y}}) = \int_0^\infty p(\lambda,\underline{\phi},\sigma_a|\underline{\underline{y}})\ d\sigma_a$$

$$\propto J^{(n-2p-1)/(n-p)}\{(n-2p-1)s^2(\lambda) + [\underline{\phi} - \underline{\hat{\phi}}(\lambda)]'\ \underline{x}'\underline{x}\ [\underline{\phi} - \underline{\hat{\phi}}(\lambda)]\}^{-(n-p)/2} \tag{2-38}$$

It can be seen from Eq. (2-38) that the joint posterior pdf for $\underline{\phi}$, given λ, is in the multivariate Student t form. The marginal pdf for ϕ_i, given λ, is in the univariate Student t form with n-2p-1 degrees of freedom and with the following moments:

$$E[\phi_i|\lambda] = \hat{\phi}_i(\lambda) \qquad i=0,1,2,\dots p \tag{2-39}$$

$$\mathrm{Var}[\phi_i|\lambda] = \frac{(n-2p-1)}{(n-2p-3)} s^2(\lambda)k^{ii}$$

where k^{ii} is the i-th diagonal element of $(\underline{x}'\underline{x})^{-1}$.

Further, by integrating Eq. (2-37) with respect to $\underline{\phi}$,

$$p(\lambda,\sigma_a|\underline{y}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} p(\lambda,\underline{\phi},\sigma_a|\underline{y})d\underline{\phi}$$
$$\phi_o \quad \phi_1 \quad\quad \phi_p$$

$$\propto \frac{1}{\sigma_a^{(n-2p)}} \exp\left\{-\frac{(n-2p-1)s^2(\lambda)}{2\sigma_a^2}\right\} \tag{2-40}$$

For fixed λ, the marginal posterior pdf for σ_a is in the form of an inverted gamma function with the following moments:

$$E[\sigma_a|\lambda] = \frac{\sqrt{(n-2p-1)/_2}\ \Gamma\ [(n-2p-2)/_2]}{\Gamma[(n-2p-1)/2]} s(\lambda)$$

$$\mathrm{Var}[\sigma_a|\lambda] = \frac{(n-2p-1)s^2(\lambda)}{(n-2p-3)} - [E(\sigma_a|\lambda)]^2 \tag{2-41}$$

To obtain the marginal posterior pdf for λ, integrate Eq. (2-40) with respect to σ_a:

$$p(\lambda|\underline{y}) = \int_0^\infty p(\lambda,\sigma_a|\underline{y})d\sigma_a$$

$$\propto J^{(n-2p-1)/(n-p)}[s^2(\lambda)]^{-(n-2p-1)/2}$$

$$\propto \left[\frac{S^2(\lambda)}{J^{2/(n-p)}}\right]^{-(n-2p-1)/2} \tag{2-42}$$

This pdf can be analyzed numerically. In the logarithmic form,

$$\log p(\lambda|\underline{y}) = \text{const} - \frac{(n-2p-1)}{2}\log\left[\frac{S^2(\lambda)}{J^{2/(n-p)}}\right] \tag{2-43}$$

Thus, in comparison with Eq. (2-25), it can be seen that the maximum likelihood estimate of λ is identical to the modal value of the posterior pdf for λ.

(3) Contours of Posterior Pdf's for ϕ_1 and ϕ_2 of the AR(2) Model. The probability density function in Eq. (2-38) is a monotonically decreasing function of the quadratic form $[\underline{\phi} - \hat{\underline{\phi}}(\lambda)]'\underline{x}'\underline{x}[\underline{\phi} - \hat{\underline{\phi}}(\lambda)]$. Hence, the contours of $p[\underline{\phi}|\lambda,\underline{y}]$ are ellipsoidal in the parameter space of $\underline{\phi}$ (Box and Tiao, 1972).

For an AR(2) process, to a good degree of approximation, the ellipsoidal contour of $p(\phi_1,\phi_2|\lambda,\underline{y})$ is given by

$$[\phi_1(\lambda) - \hat{\phi}_1(\lambda)]^2 + 2\hat{\rho}_1(\lambda)[\phi_1(\lambda) - \hat{\phi}_1(\lambda)]$$

$$\cdot[\phi_2(\lambda) - \hat{\phi}_2(\lambda)] + [\phi_2(\lambda) - \hat{\phi}_2(\lambda)]^2$$

$$\leq \frac{2}{n-5}[1 - \hat{\phi}_1(\lambda)\hat{\rho}_1(\lambda) - \hat{\phi}_2(\lambda)\hat{\rho}_1(\lambda)]\cdot F_\alpha(2,n-5) \tag{2-44}$$

This equation will delineate a region containing an $100(1-\alpha)\%$ of the posterior pdf for ϕ_1 and ϕ_2 given λ. The ellipsoidal posterior region enclosed by the contour given by Eq. (2-44) is numerically equivalent to an $100(1-\alpha)\%$ confidence region for ϕ_1 and ϕ_2 given by Eq. (2-27).

2-7. Analysis of Annual Streamflow Data.

(1) The Annual Streamflow Data. In this section some of the procedures developed above are applied to an actual hydrologic time series. The annual (calendar year) streamflow discharges of the Sangamon River near Monticello, Illinois were taken from the U.S. Geological Survey Water-Supply Papers No. 1308, 1728, 1915, and 2115. The data are given in Table 2-1 and are plotted in Fig. 2-2. For ease of reference, the streamflow series from 1945 to 1969 is called Series A; the streamflow series from 1930 to 1969, Series B; and the streamflow series from 1915 to 1969, Series C.

(2) Model Identification. The sample autocorrelations for Series A, B and C are shown in Table 2-2 and the correlograms are plotted in Fig. 2-3. For Series A and B, the correlograms exhibit damped sinusoidal patterns with positive values of $\hat{\rho}_1$ and therefore an AR(2) model with parameter values in Area D in Fig. 2-1 should be appropriate. The correlogram for Series C does not reveal any apparent pattern and the series is fitted by both an AR(1) and AR(2) model. Computer programs are available for analyzing time series and multiple linear regression (IBM, 1970; University of Illinois, 1974). As is shown in Table 2-3, $\hat{\phi}_2$ in the AR(2) model is very close to zero and so Series C could very well be represented by the AR(1) model.

It is to be pointed out that although autocorrelations are not invariant under the transformation, the general pattern of the autocorrelations remain the same. Hence, the models identified above could also be employed with transformation.

(3) Estimation of λ. Since λ is a nonlinear parameter in the model of Eq. (2-2a), λ can not be estimated in a closed form. The residual sum of squares $S^2(\lambda)$ in Eq. (2-23) for a range of values of λ can be readily

Table 2-1. Annual Discharge of the Sangamon River Near Monticello, Illinois (1915-1969)

Year	Discharge in cfs	Year	Discharge in cfs	Year	Discharge in cfs
1915	281	1934	118	1953	172
1916	347	1935	421	1954	69.7
1917	221	1936	266	1955	197
1918	411	1937	437	1956	239
1919	323	1938	466	1957	456
1920	365	1939	476	1958	392
1921	309	1940	132	1959	329
1922	508	1941	388	1960	299
1923	561	1942	527	1961	393
1924	559	1943	574	1962	519
1925	262	1944	297	1963	153
1926	912	1945	298	1964	241
1927	1010	1946	452	1965	320
1928	375	1947	378	1966	272
1929	646	1948	318	1967	358
1930	360	1949	514	1968	555
1931	109	1950	658	1969	409
1932	161	1951	471		
1933	428	1952	422		

Table 2-2. Sample Autocorrelations for Series A, B and C

Series	Number of Observations	Lags	Autocorrelations						
A	25	1-7	0.42	-0.03	-0.18	-0.35	-0.28	-0.09	-0.01
		8-14	-0.04	0.05	0.19	0.20	0.08	-0.18	-0.34
B	40	1-7	0.25	-0.05	-0.13	-0.09	-0.16	-0.05	-0.01
		8-14	0.09	0.08	0.0	-0.04	-0.08	-0.06	-0.14
C	55	1-7	0.30	0.07	0.10	-0.03	-0.23	-0.05	-0.15
		8-14	-0.09	-0.04	-0.09	-0.04	-0.05	-0.06	-0.09

Table 2-3. The Fitted Models

Series	Model	Parameter Estimates	Model Equation
A	AR(2)	$\hat{\phi}_1 = 0.55$	$\bar{z}_t = 0.55\bar{z}_{t-1} - 0.27\bar{z}_{t-2} + a_t$
		$\hat{\phi}_2 = -0.27$	
B	AR(2)	$\hat{\phi}_1 = 0.28$	$\bar{z}_t = 0.28\bar{z}_{t-1} - 0.12\bar{z}_{t-2} + a_t$
		$\hat{\phi}_2 = -0.12$	
C	AR(1)	$\hat{\phi}_1 = 0.3$	$\bar{z}_t = 0.3\bar{z}_{t-1} + a_t$
C	AR(2)	$\hat{\phi}_1 = 0.31$	$\bar{z}_t = 0.31\bar{z}_{t-1} - 0.02\bar{z}_{t-2} + a_t$
		$\hat{\phi}_2 = -0.02$	

*$\bar{z}_t$ denotes the deviation from the mean of the process.

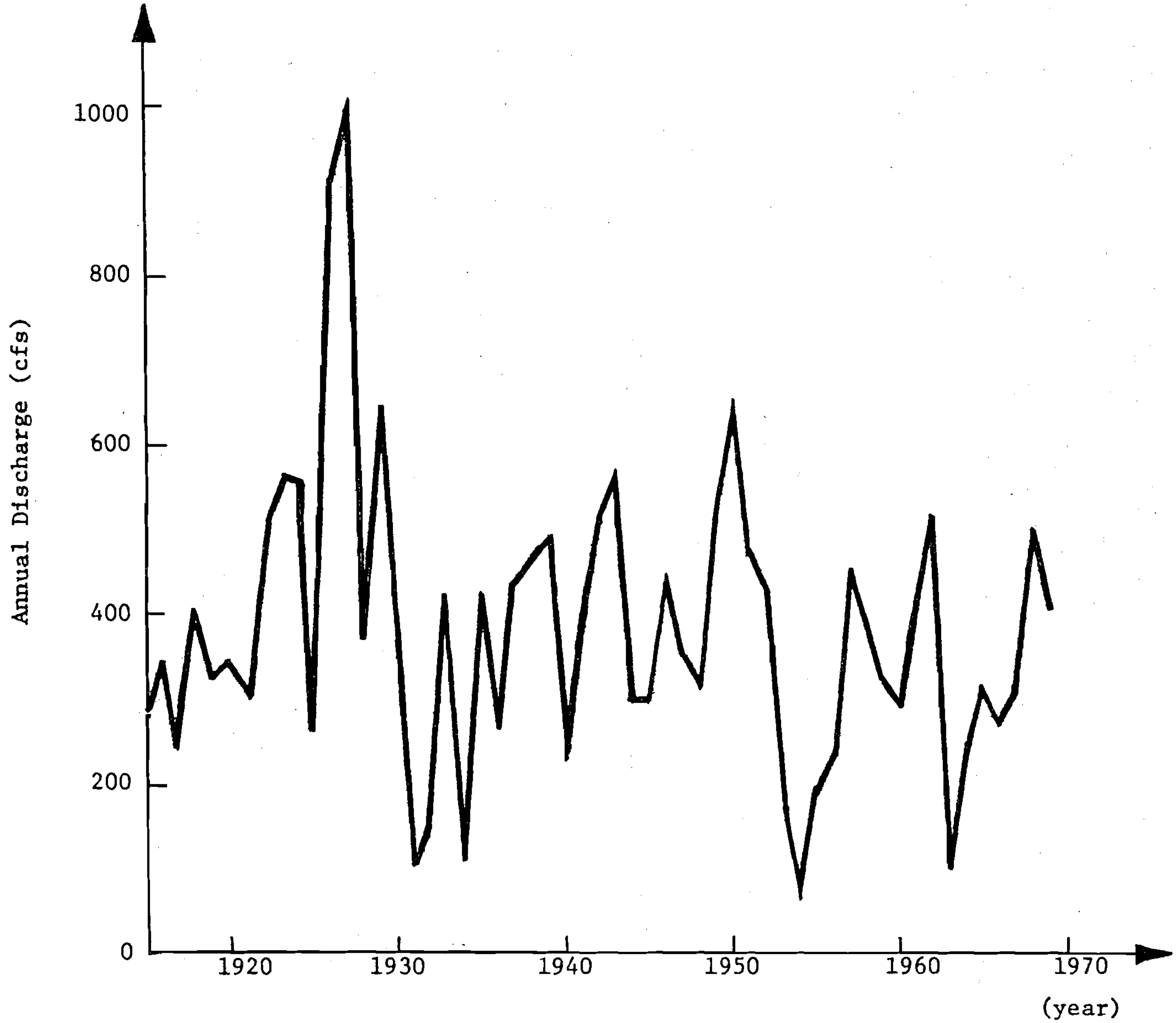

Figure 2-2. Annual discharge of the Sangamon River near Monticello, Illinois (1915-1969).

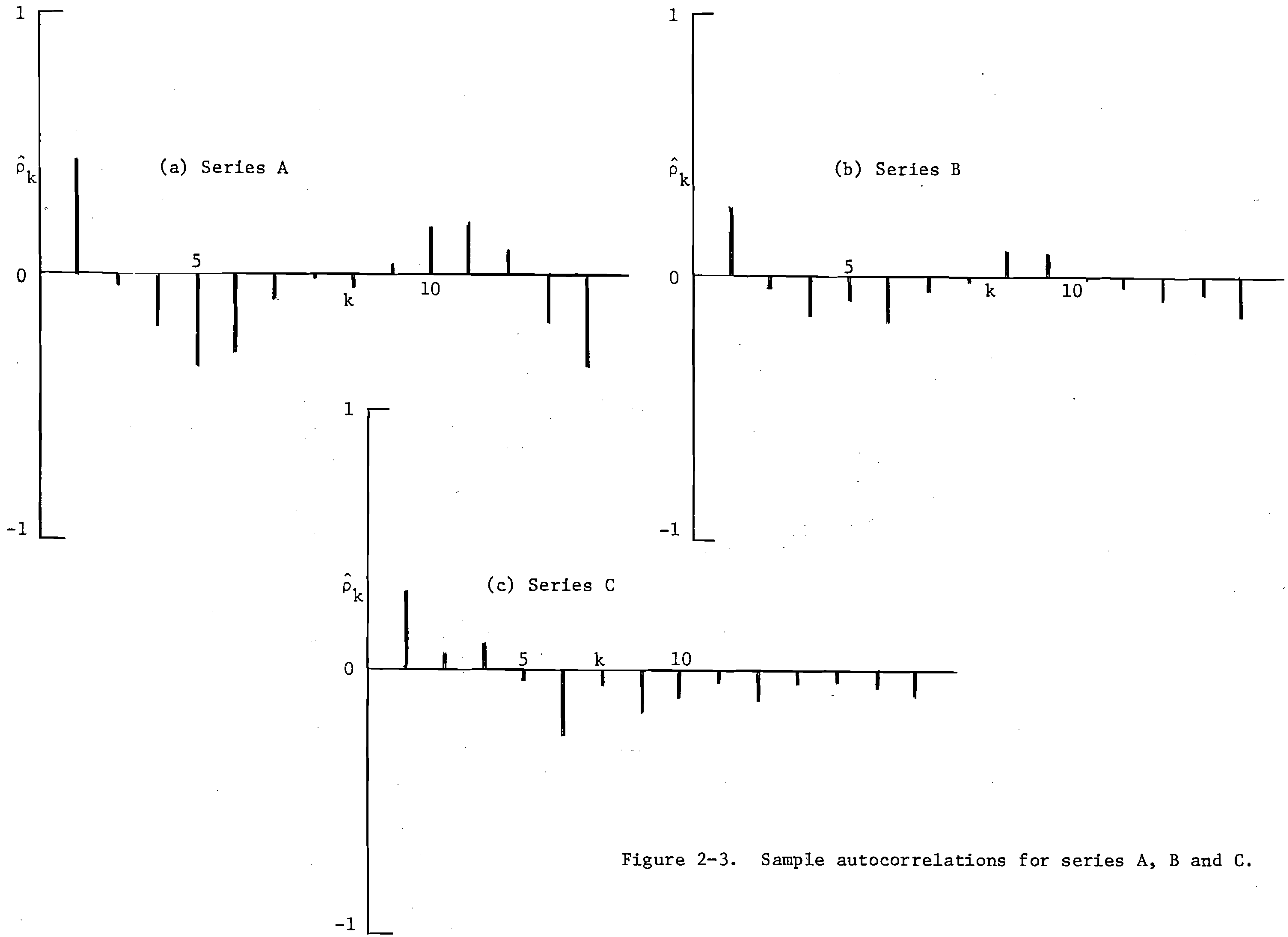

Figure 2-3. Sample autocorrelations for series A, B and C.

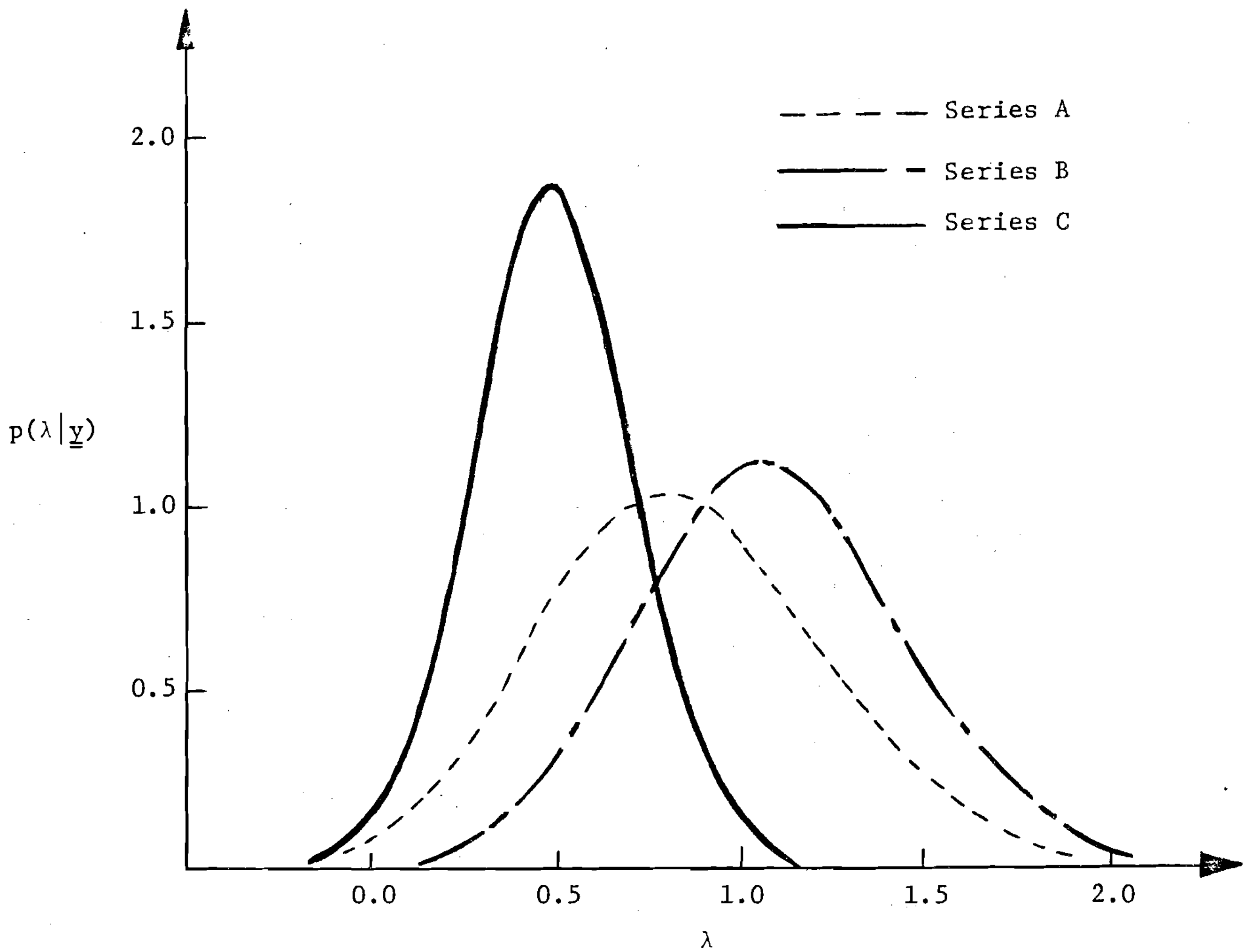

Figure 2-4. Posterior distributions of λ for series A, B and C.

calculated by using a computer program for the multiple linear regression (IBM, 1970; University of Illinois, 1974). Table 2-4 shows values of $\ell_{max}(\lambda)$ in Eq. (2-25) and of $p(\lambda|\underline{y})$ in Eq. (2-42) over a range of λ where the density is appreciable. As expected, maximum likelihood estimate of λ is identical to the modal value of the posterior pdf for λ for each series. The posterior pdf's for λ shown in Fig. 2-4 are approximately normally distributed. It may be noted that log transformation (λ=0), which is quite often used in hydrology, is not appropriate for any of the series. The suitable transformations are: $\hat{\lambda}$ = 0.75 for Series A, $\hat{\lambda}$ = 1.0 for Series B, and $\hat{\lambda}$ = 0.5 for Series C

(4) Estimation of $\underline{\phi}$ and σ_a given λ. For given λ, the model is linear in the parameters and the model parameters can be readily estimated by the least squares method. For each series, λ=0 (log transformation), $\lambda=\hat{\lambda}$, and λ=1 (no transformation) are applied and are fitted by AR(1) and AR(2) model. An AR(1) model is included because of its popular use in hydrology. For an AR(1) process, the likelihood function of Eq. (2-19) involves n-1 observations. Hence, to have the same number of effective observations for both AR(1) and AR(2) models, one less observation is employed for the AR(1) model than those for the AR(2) model. Table 2-5 gives the parameter estimates for various combinations of a number of observations, models, and λ's.

(5) Confidence Regions for ϕ_1 and ϕ_2 of the AR(2) Model. Approximately 95% confidence regions for ϕ_1 and ϕ_2 for Series A, B and C in the original units and in the transformed units are constructed, using Eq. (2-27), as shown in Figs. 2-5 and 2-6, respectively. They are all within the stationarity boundary. Sampling variabilities of the parameter estimates are well illustrated since the estimates of ϕ_1 and ϕ_2 for

Table 2-4. Values of $\ell_{max}(\lambda)$ and of $p(\lambda|\underline{y})$ Over a Range of λ for Series A, B and C

	Series A		Series B		Series C	
λ	$\ell_{max}(\lambda)$ $\times 10^{+57}$	$p(\lambda\|\underline{y})$	$\ell_{max}(\lambda)$ $\times 10^{+104}$	$p(\lambda\|\underline{y})$	$\ell_{max}(\lambda)$ $\times 10^{+156}$	$p(\lambda\|y)$
-0.125	3.8	0.03			4.8	0.03
0.0	11.6	0.08			23.9	0.13
0.125	29.3	0.18	4.5	0.02	83.4	0.44
0.25	60.8	0.34	13.1	0.07	198.3	0.98
0.375	105.3	0.55	32.3	0.16	329.2	1.59
0.5	154.4	0.77	66.7	0.31	381.6	1.83
0.625	194.4	0.94	117.0	0.53	312.9	1.51
0.75	213.7	1.01	175.9	0.76	183.3	0.91
0.875	207.9	0.99	228.6	0.97	77.5	0.41
1.0	181.3	0.88	258.4	1.09	23.9	0.13
1.125	143.1	0.72	256.5	1.08	5.5	0.03
1.25	103.3	0.54	225.0	0.96		
1.375	68.7	0.38	175.2	0.76		
1.5	42.5	0.25	121.9	0.55		
1.625	24.4	0.15	76.2	0.35		
1.75	13.5	0.10	43.0	0.21		
1.875	6.1	0.04	21.9	0.11		
2.0			10.6	0.05		

Table 2-5. Model Parameter Estimates

Data from	to	NOB	ENOB	Model	λ	ϕ_0	ϕ_1	ϕ_2	s
1946	1969	24	23	AR(1)	0.0	3.147	0.454		0.4555
					0.75	58.10	0.449		30.50
					1.0	197.8	0.437		129.79
1945	1969	25	23	AR(2)	0.0	4.010	0.566	-0.262	0.4508
					0.75	74.72	0.563	-0.278	30.08
					1.0	253.0	0.547	-0.274	128.18
1931	1969	39	38	AR(1)	0.0	4.383	0.243		0.4859
					1.0	267.9	0.254		137.19
1930	1969	40	38	AR(2)	0.0	4.632	0.255	-0.058	0.4918
					1.0	298.2	0.281	-0.116	138.17
1916	1969	54	53	AR(1)	0.0	4.274	0.269		0.5053
					0.5	25.96	0.285		8.897
					1.0	268.9	0.304		173.9
1915	1969	55	53	AR(2)	0.0	4.396	0.276	-0.028	0.5102
					0.5	26.56	0.291	-0.023	8.983
					1.0	274.6	0.311	-0.022	175.6

*NOB is the abbreviation for "number of observations".
ENOB is the abbreviation for "effective number of observations".

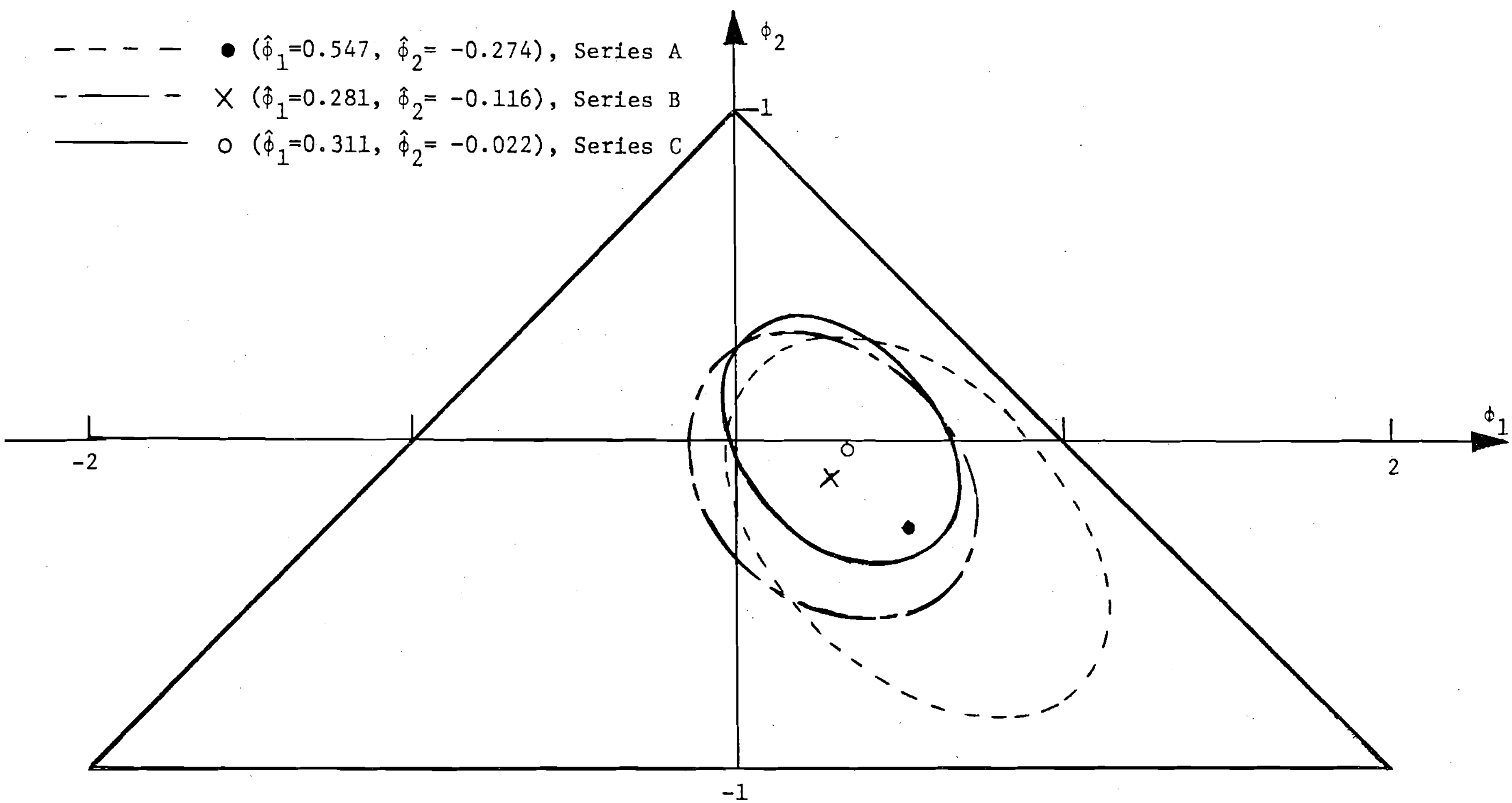

Figure 2-5. 95% confidence regions, and also 95% contours of the posterior pdf, for ϕ_1 and ϕ_2 of the AR(2) model for series A, B and C in the original units (λ = 1.0).

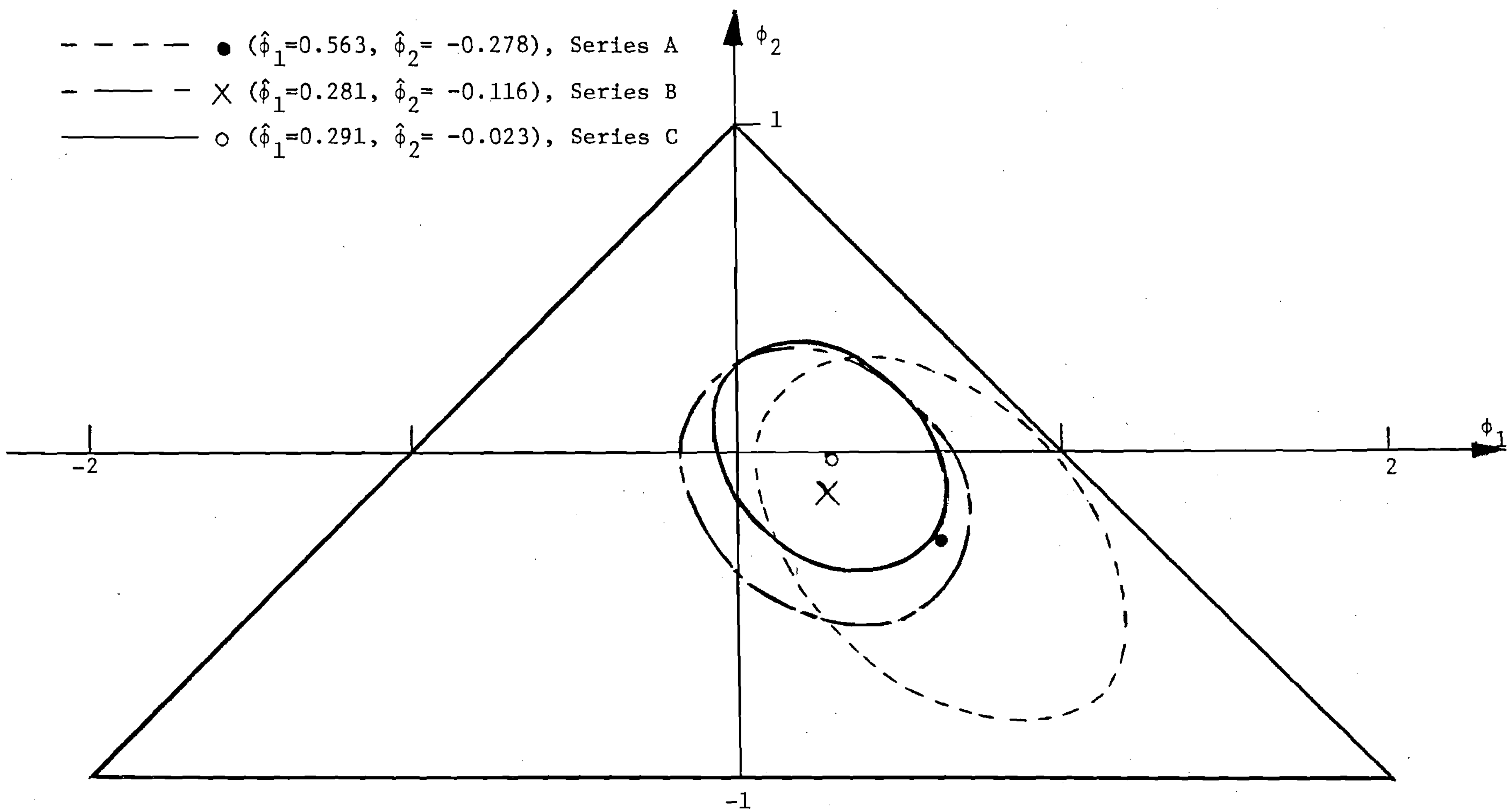

Figure 2-6. 95% confidence regions, and also 95% contours of the posterior pdf, for ϕ_1 ϕ_2 of the AR(2) model for series A, B and C in the transformed units ($\lambda = \hat{\lambda}$).

Series A lie just inside (Fig. 2-5) and outside (Fig. 2-6) of the 95% confidence region for Series C.

(6) <u>Contours of Posterior pdf's for ϕ_1 and ϕ_2 of the AR(2) Model.</u> As was stated in Section 2-6(3), the posterior distribution for ϕ_1 and ϕ_2 is identical to the confidence distribution of ϕ_1 and ϕ_2. Hence, Figs. 2-5 and 2-6 also delineate a region containing 95% of the posterior probability for ϕ_1 and ϕ_2 given the value of λ for Series A, B and C. The mean values for ϕ_1 and ϕ_2, namely, $\hat{\phi}_1$ and $\hat{\phi}_2$, are the least squares quantities given by Eq. (2-21).

III. MODELING OF STOCHASTIC INPUT: MONTHLY STREAMFLOWS

3-1. Stochastic Generation of Monthly Streamflows.

The objective of stochastic generation of streamflows has been discussed previously in Section 2-1. The techniques discussed in Chapter II are not suitable for the generation of monthly streamflows because the proposed model cannot take into account of the seasonal structure of streamflows. Several techniques have been proposed for the generation of monthly streamflows. These techniques involve first making a reliable estimate of the streamflow parameters and then using these parameters in a model to generate monthly streamflows. The following is a brief review of some of these techniques of importance.

Benson and Matalas (1967) proposed a regionalization technique to estimate the streamflow parameters at sites where very short or no streamflow records are available. By using all available long streamflow records in a region, they employed regression equations to relate the means, standard deviations, and skewness and correlation coefficients of monthly and annual streamflows to the physical and climatic characteristics of the corresponding basins. They suggested that these relationships can be used to obtain estimates of streamflow parameters at any site, within the same region, having a very short or no streamflow record. However, they could not obtain satisfactory relationships for the skewness and correlation coefficients of monthly streamflows.

Another approach to improve the estimate of streamflow parameters is to augment a streamflow record by correlating it with longer streamflow records at nearby sites. Fiering (1962, 1963) considered the case where a short record is augmented by regression using two longer records of equal length. Assuming that these records are samples from a trivariate normal distribution, he compared the variances of sample means and sample variances obtained from the original and the augmented records. He showed that, depending on certain conditions, augmentation can reduce or increase the variances of sample mean and sample variance. Therefore, under some conditions, augmentation may produce poorer estimates of mean and variance than could be obtained from the original record alone. Hence, its utility is limited. In a similar study, Matalas and Jacobs (1964) included an error term in the regression equation and considered the case where a short record is augmented using a longer record. They noted that the inclusion of the error term improves the efficiency of augmentation. Gilroy (1970) generalized Matalas and Jacob's formulation to the case where a short record is augmented using any number of longer records of equal length. Formulations of Fiering, Matalas and Jacobs, and Gilroy are based on the assumption that observations in individual records are identically and independently distributed. For that reason, their results are at best applicable to annual streamflows for which the assumption of independence can sometimes be justified. Frost and Clarke (1973) relaxed the assumption of independence and considered the estimation of parameters of a first-order autoregressive series when a longer first-order autoregressive series which is cross correlated with the shorter series is available.

Another approach to improve estimates of streamflow parameters is the Bayesian approach. Bayesian estimation aims at improving estimates of parameters by pooling all available information about the parameters. In the Bayesian approach, population parameters are treated as random variables, and their distributions are specified by "prior" distributions. Any supplementary information about population parameters is incorporated into the analysis through prior distributions. Estimates of parameters are then based on "posterior" distributions which take into account both the supplementary information, which enters through prior distributions, and the sample information, which enters through the likelihood function. Along this line, Lenton et al. (1974) considered a Bayesian estimation of serial correlation coefficient of annual streamflows described by a first-order autoregressive model. In the Bayesian approach, because of analytical difficulties, the choice of prior distributions is usually dictated by mathematical rather than physical considerations. The degree of complexity in the Bayesian approach increases significantly when one considers more than one parameter, since in that case dependence between parameters should be taken into account. It should also be mentioned that the Bayesian approach is still a controversial topic, especially among mathematical statisticians.

Although estimates of streamflow parameters can be improved by supplementing the sample information, the sampling errors associated with these estimates can never be eliminated. Statistical estimates are always subjected to sampling errors. Therefore, a realistic approach to streamflow generation should recognize this fact and explicitly account for the sampling errors inherent in the estimates of model parameters. The present study of modeling the stochastic input to a water resources system is motivated mainly by this consideration.

Most hydrologists have avoided the sampling error problem by assuming that the population parameters are equal to their estimates. This approach implies full confidence in the information used to obtain the estimates and, hence, cannot be entirely justified either physically or statistically. Beard (1965) attempted to account for sampling errors by using a different set of parameter estimates for each generated streamflow sequence. He claimed that different sets of estimates can be generated by using the sampling distributions of the estimators. However, a problem here is that some of the sampling distributions he used depend on the unknown population parameters, but he did not explain how he resolved this problem. Another problem is that some of the sampling distributions he used are not independent of each other; therefore, estimates cannot be generated independently. Beard did not properly account for this dependence either. Even if different sets of estimates can be generated properly, this approach does not account for the effects of sampling errors on individual streamflow sequences, because values of estimates are held fixed for individual sequences.

In the Bayesian estimation, the sampling error problem does not exist because Bayesian estimates are conditioned on the observed sample. However, parameter uncertainties are still taken into account by treating population parameters as random variables. Parameter uncertainties can be incorporated into the analysis through Bayesian distributions which are obtained by integrating over the parameter space the product of the underlying probability density function, which is conditioned on population parameters, and the joint posterior probability density function of the population parameters. Vicens et al. (1974) applied this approach to the generation of annual streamflows using independent normal and first-order autoregressive processes.

The approach to be discussed in this chapter is the use of a linear model for generating monthly streamflows for which sampling errors of model parameters are taken into account.

3-2. Linear Models for Monthly Streamflow Generation.

The linear models employed in this study have the form

$$y = \beta_1 + \beta_2 x_2 + \beta_3 x_3 + \ldots + \beta_p x_p + e \qquad (3\text{-}1)$$

where y is a random variable, $x_2, x_3, \ldots, x_p$ are nonrandom variables, $\beta_1, \beta_2, \ldots, \beta_p$ are unknown parameters, and e (error) is a random variable with zero mean.

In this study, linear models are essentially used for relating monthly streamflows to one another. In Eq. (3-1), y represents the streamflow (or its, e.g., logarithmic, transformation) in a particular calendar month, and $x_2, x_3, \ldots, x_p$ represent the streamflows (or their transformations) in p-1 preceding calendar months. A different linear model is used for each calendar month relating the streamflow in that calendar month to the streamflows in a number of preceding calendar months. This will result in twelve different linear models; they may not be of the same order, that is, p may take a different value for each model. The linear model for a particular month can then be used for generating a streamflow value for that calendar month each year on the basis of streamflow values already generated for the p-1 preceding calendar months in that year. Starting with the most recent observations in the record, and applying the above procedure sequentially for each calendar month and cyclically for each year, a monthly streamflow sequence of any desired length can be generated. This sequence is then divided into smaller sequences of equal length, resulting in a set of monthly streamflow sequences that may be used in systems analysis of water resources projects.

To apply linear models of the form (3-1) to monthly streamflows, two different interpretations can be given:

(1) Functional-relationship Model. Suppose that streamflow, y, in a particular calendar month can be expressed approximately as a function of streamflows, $x_2, x_3, \ldots, x_p$, for p-1 preceding calendar months as

$$y \approx \beta_1 + \sum_{i=2}^{p} \beta_i x_i \tag{3-2}$$

where $\beta_1, \beta_2, \ldots, \beta_p$ are unknown parameters to be estimated. It is further assumed that, for fixed values of $x_2,x_3,\ldots,x_p$, the true value of y will fluctuate about its approximate value $\beta_1 + \sum_{i=2}^{p} \beta_i x_i$ in a random fashion. With these assumptions, one can write

$$y = \beta_1 + \sum_{i=2}^{p} \beta_i x_i + e \tag{3-3}$$

where e is a random variable with zero mean representing the error of approximation. In Eq. (3-3), $x_2,x_3,\ldots,x_p$ are variables in the ordinary sense, that is, they are nonrandom; and y is a random variable because it is a function of random variable e. The distribution of y depends on the nonrandom variables $x_2,x_3,\ldots,x_p$, on the unknown parameters $\beta_1,\beta_2,\ldots,\beta_p$, and on the distribution of random variable e. With these considerations, the model defined by Eq. (3-3), which is referred to as the functional-relationship model, fits into the definition of the linear model (3-1).

(2) Regression Model. In the functional-relationship model, streamflows are related in a mathematical sense through the function (3-2). However, streamflows can be also related in a statistical sense by taking into account of their joint probability distribution.

Suppose that the streamflow, y, in a particular calendar month and the streamflows, $x_2,x_3,\ldots,x_p$, in the p-1 preceding calendar months are jointly distributed random variables such that the conditional distribution of y given $x_2,x_3,\ldots,x_p$ can be defined by the equation

$$y = \beta_1 + \sum_{i=2}^{p} \beta_i x_i + e \tag{3-4}$$

where the random variable y is conditional on the random variables $x_2,x_3,\ldots,x_p$, e is a random variable with zero mean, $x_2,x_3,\ldots,x_p$ are variables representing the given values of random variables $x_2,x_3,\ldots,x_p$, and $\beta_1,\beta_2,\ldots,\beta_p$ are the unknown parameters. The model defined by Eq. (3-4) also fits into the definition of the linear model (3-1). In a multivariate normal distribution, for example, conditional distributions can be expressed in the form (3-4). The generation of monthly streamflows can be regarded as a problem of conditional inference since given the streamflows in a number of preceding months, the streamflow in a particular month is inferred; and hence, conditional distributions of the streamflows are appropriate. In Eq. (3-4), $\beta_1 + \sum_{i=2}^{p} \beta_i x_i$ is the conditional mean of random variable y given the random variables $x_2,x_3,\ldots,x_p$. Therefore, there is a linear relationship between the conditional mean of random variable y and the given values of random variables $x_2,x_3,\ldots,x_p$. The plane defined by this relationship is called the regression plane of y on $x_2,x_3,\ldots,x_p$. For that reason, model (3-4) is usually referred to as a regression model.

The regression model is different from the functional-relationship model in some aspects. In the regression model, streamflows are related in a sabbatical sense through their joint probability distribution. Although there is a functional relationship between the conditional mean of random variable y and the given values of random variables $x_2,x_3,\ldots,x_p$, there is, in general, no functional relationship between the random variables themselves. In Eq. (3-4), $x_2,x_3,\ldots,x_p$ are variables representing the given values of random variables $x_2,x_3,\ldots,x_p$ in the conditional distribution of random variable y. They are not variables in the ordinary sense because they have a random origin. However, they can be treated as nonrandom variables with the interpretation that statistical inferences are conditional on the observed values of random variables $x_2,x_3,\ldots,x_p$. The differences between the functional-relationship model and the regression model, and their implications are discussed by Kendall (1951) and Graybill (1961). In spite of the differences, the functional-relationship and regression models are similar from a statistical point of view, and both can be considered as the same type of linear models defined by Eq. (3-1).

3-3. The General Linear Model of Full Rank

Regarding the linear model (3-1), suppose that n sets of observations are available on $y, x_2, x_3, \ldots, x_p$; the relationships between these observations can be written as

$$y_j = \beta_1 + \beta_2 x_{j2} + \beta_3 x_{j3} + \ldots + \beta_p x_{jp} + e_j, \quad j=1,2,\ldots,n \tag{3-5}$$

where $y_j, x_{j2}, x_{j3}, \ldots, x_{jp}$ are the j-th set of observations on $y, x_2, x_3, \ldots, x_p$, and e_j is the unobservable value of random variable e corresponding to this set of observations. In this study, $y_j, x_{j2}, x_{j3}, \ldots, x_{jp}$ represent the streamflow values (or their transformations) in the j-th year of the record for a given set of p successive calendar months. Eqs. (3-5) can be written in a matrix form as

$$Y = X\beta + e \tag{3-6}$$

where

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{bmatrix}, \quad X = \begin{bmatrix} 1 & x_{12} & x_{13} & \cdots & x_{1p} \\ 1 & x_{22} & x_{23} & \cdots & x_{2p} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 1 & x_{n2} & x_{n3} & \cdots & x_{np} \end{bmatrix}, \quad \beta = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \cdot \\ \cdot \\ \cdot \\ \beta_p \end{bmatrix}, \quad e = \begin{bmatrix} e_1 \\ e_2 \\ \cdot \\ \cdot \\ \cdot \\ e_n \end{bmatrix} \tag{3-7}$$

With the assumption that the rank of matrix X is equal to $p \leq n$, i.e., full rank, the model defined by (3-6) is referred to as the general linear model of full rank. This is the model proposed in this study. The assumption of full rank is usually satisfied when one is dealing with continuous variables such as is the case in this study here. If the rank of matrix X is equal to p_1 where $p_1<p$, then $p-p_1$ of the variables $x_2,x_3,\ldots,x_p$ are linear functions of the others. In that case, one can exclude these variables from the analysis without any loss of information and thus reduce the rank of matrix X to the full rank p_1.

To make any progress with the general linear model of full rank, some assumptions concerning the distribution of random vector e should be made. Two types of the model will be considered; one assuming uncorrelated errors and the other assuming correlated errors.

3-4. Linear Model with Uncorrelated Errors.

In this case, e is assumed to have a multivariate normal distribution with mean vector $E(e) = 0$ and covariance matrix $cov(e) = \sigma^2 I$, where 0 is the nx1 null vector, I is the nxn identity matrix, and σ^2 is an unknown constant. This assumed condition will be denoted by "e is $N(0,\sigma^2 I)$," where N indicates a multivariate normal distribution. This assumption is equivalent to saying that $e_1,e_2,\ldots,e_n$ are independently, identically, and normally distributed with mean zero and unknown variance σ^2.

For the functional-relationship model (3-3), $e_1,e_2,\ldots,e_n$ represent random errors. It is reasonable to assume that these errors are identically distributed. Assumption of normality of errors can be justified by appealing to the central limit theorem and by reasoning that a random error represents the sum of a large number of independent random errors. Assumption of independence of errors can be justified in this study on the basis that they correspond to the streamflow observations at one year apart; hence their time dependence can be neglected. However, this assumption will be relaxed later in considering the case in which $e_1,e_2,\ldots,e_n$ can be correlated.

For the regression model (3-4), assume that the conditional distribution of random variable y, given the random variables $x_2, x_3, \ldots, x_p$, is normal with mean $\beta_1 + \sum_{i=2}^{p} \beta_i x_i$ and variance σ^2, where σ^2 does not depend on $x_2, x_3, \ldots, x_p$. This is equivalent to saying that in Eq. (3-4) the random variable e is normally distributed with mean zero and constant variance σ^2. In a multivariate normal distribution, for example, the conditional distributions satisfy this assumption. For this case, the assumption that $e_1, e_2, \ldots, e_n$ are independently, identically, and normally distributed with mean zero and variance σ^2 implies that $y_j, x_{j2}, x_{j3}, \ldots, x_{jp}$ $(j=1,2,\ldots,n)$ are random samples taken from the joint distribution of random variables $y, x_2, x_3, \ldots, x_p$.

When e is $N(0, \sigma^2 I)$, it follows from (3-6) that Y is $N(X\beta, \sigma^2 I)$. This means that $y_1, y_2, \ldots, y_n$ are independently and normally distributed with constant variance σ^2 and that $E(y_j) = \beta_1 + \sum_{i=2}^{p} \beta_i x_{ji}$, where E is the expectation operator. With the above assumptions concerning the distribution of random vector e, the general linear model of full rank can be written as

$$\begin{aligned} &Y \text{ is } N(X\beta, \sigma^2 I) \\ &\text{rank } (X) = p \end{aligned} \qquad (3\text{-}8)$$

This model has been discussed extensively by Graybill (1961), including the results that may be obtained by using this model.

3-5. Streamflow Generation by Linear Models with Uncorrelated Errors.

As mentioned earlier, the linear model for a particular calendar month will be used for generating a streamflow value for that calendar month each year on the basis of streamflow values already generated for the p-1 preceding calendar months in that year. Suppose that y_k $(k=n+1, n+2, \ldots)$ represents the (k-n)th streamflow to be generated for a particular calendar month and that $x_{k2}, x_{k3}, \ldots, x_{kp}$ are the corresponding streamflow values already generated for the p-1 preceding calendar months. Based on model (3-8), y_k is normally distributed with mean $\beta_1 + \sum_{i=2}^{p} \beta_i x_{ki}$ and variance σ^2. If $\beta_1, \beta_2, \ldots, \beta_p$ and σ^2 were known, then a value for streamflow y_k could be generated by drawing a random number from the above normal distribution. Since $\beta_1, \beta_2, \ldots, \beta_p$ and σ^2 are unknown, they must first be estimated using the corresponding monthly streamflow observations in the record which constitute the elements of matrices Y and X in model (3-8). Then, these estimates can be used together with $x_{k2}, x_{k3}, \ldots, x_{kp}$ to generate a value for streamflow y_k.

The unknown parameters in model (3-8), $\beta_1, \beta_2, \ldots, \beta_p$ and σ^2 can be estimated by their maximum-likelihood estimates. The maximum-likelihood estimator of β_i, $\hat{\beta}_i$, is given by the i-th element of the px1 vector $\hat{\beta}$ which is

$$\hat{\beta} = (X'X)^{-1}X'Y \tag{3-9}$$

where (´) is the transpose operator. A desirable property of estimator $\hat{\beta}_i$ is that it is the minimum-variance unbiased estimator of β_i. In other words, it is unbiased, i.e., $E(\hat{\beta}_i) = \beta_i$, and, for a given sample size, it has the minimum variance among all unbiased estimators of β_i. The maximum-likelihood estimator of σ^2 (corrected for bias) is

$$\hat{\sigma}^2 = \frac{Y'[I-X(X'X)^{-1}X']Y}{n-p} \tag{3-10}$$

where $\hat{\sigma}^2$ is the minimum-variance unbiased estimator of σ^2.

Going back to the streamflow generation problem, one may assume that the population parameters $\beta_1, \beta_2, \ldots, \beta_p, \sigma^2$ are equal to their maximum-likelihood estimates $\hat{\beta}_1, \hat{\beta}_2, \ldots, \hat{\beta}_p, \hat{\sigma}^2$. Under this assumption, y_k is normally distributed with mean $\hat{\beta}_1 + \sum_{i=2}^{p} \hat{\beta}_i x_{ki}$ and variance $\hat{\sigma}^2$. Or, equivalently,

$$\frac{y_k - x_k'\hat{\beta}}{\hat{\sigma}} \text{ is } N(0,1) \tag{3-11}$$

where N(0,1) indicates a normal distribution with zero mean and unit variance, and x_k is a pxl vector such that $x_k' = (1, x_{k2}, x_{k3}, \ldots, x_{kp})$. A value for streamflow y_k can then be generated by drawing a random number from the standard normal distribution, multiplying it by $\hat{\sigma}$, and adding to it $x_k'\hat{\beta}$. Hydrologists have taken this approach in using regression models for the streamflow generation.

The above approach avoids the sampling error problem by assuming that the population parameters are known. The maximum-likelihood estimates enter into the analysis as the known values of population parameters. They are treated as known constants, and sampling errors inherent in these estimates are unaccounted for. Since maximum-likelihood estimators $\hat{\beta}_1, \hat{\beta}_2, \ldots, \hat{\beta}_p, \hat{\sigma}^2$ are unbiased estimators of the corresponding population parameters $\beta_1, \beta_2, \ldots, \beta_p, \sigma^2$, the assumption that population parameters are equal to their maximum-likelihood estimates amounts to assuming that the maximum-likelihood estimators have zero variances. This might be a reasonable assumption if maximum-likelihood estimators have relatively small variances. But, there seems to be no easy means of evaluating this assumption because distributions of the maximum-likelihood estimators depend on the unknown population parameters.

A more realistic approach would be one which recognizes and accounts for the sampling errors inherent in the estimates of model parameters. With this objective in mind, consider the problem of predicting y_k on the basis of sample observations $y_1, y_2, \ldots, y_n$. Let $g(y_1, y_2, \ldots, y_n)$ be a predictor of y_k, where g is any function of sample observations. Among all such predictors, one can choose a desirable predictor by using the minimum-variance unbiasedness criterion. The minimum-variance unbiased predictor of y_k is the predictor $\hat{g}(y_1,y_2,\ldots,y_n)$ which minimizes the variance of $[y_k-g(y_1,y_2,\ldots,y_n)]$ over all functions g with the condition that the expected value of $[y_k-g(y_1,y_2,\ldots,y_n)]$ is zero. It can be shown that $x_k'\hat{\beta}$ is the minimum-variance unbiased predictor of y_k. One can show that $y_k-x_k'\hat{\beta}$ is normally distributed with mean zero and variance $\sigma^2[1+x_k'(X'X)^{-1}x_k]$. Therefore, $z = \frac{y_k-x_k'\hat{\beta}}{\sigma\sqrt{1+x_k'(X'X)^{-1}x_k}}$ is N(0,1). This relation involves the unknown parameter σ; however, one can delete it easily. It can be shown that $v = \frac{(n-p)\hat{\sigma}^2}{\sigma^2}$ is $\chi^2(n-p)$, i.e., chi-square distribution with n-p degrees of freedom, and that it is independent of z. From the definition of t-distribution, it then follows that $\frac{z}{\sqrt{\frac{v}{n-p}}}$ is t(n-p), i.e., t-distribution with n-p degrees of freedom. Therefore,

$$\frac{y_k-x_k'\hat{\beta}}{\hat{\sigma}\sqrt{1+x_k'(X'X)^{-1}x_k}} \text{ is } t(n-p) \tag{3-12}$$

In the above approach, the maximum-likelihood estimators enter into the analysis as the estimators of unknown population parameters. They are treated as random variables, and their sampling distributions are taken into account. In this way, the approach recognizes and accounts for the sampling errors inherent in the maximum-likelihood estimates.

By using sample values of $\hat{\beta}$ and $\hat{\sigma}$, one can use relation (3-12) to generate a value for streamflow y_k. This can be done by drawing a random number from t-distribution with n-p degrees of freedom, multiplying it by $\hat{\sigma}\sqrt{1+x'_k(X'X)^{-1}x_k}$, and adding to it $x_k'\hat{\beta}$. Relation (3-12) is for a particular calendar month, and it will be used for generating a streamflow value for that calendar month each year. In relation (3-12), $\hat{\sigma}$ is a random variable and $\hat{\beta}$ is a random vector. For that reason, repeated use of relation (3-12) is allowed if values of $\hat{\beta}$ and $\hat{\sigma}$ are varied randomly each time. But, the streamflow record provides only one set of values for $\hat{\beta}$ and $\hat{\sigma}$. However, if relation (3-12) is used to generate values for streamflow y_k repeatedly with the same sample values of $\hat{\sigma}$ and $\hat{\beta}$, the sampling errors associated with these sample values are in effect accounted for by treating them as random variables and by taking into account their sampling distributions. Relation (3-12) is recommended for use with this interpretation. This interpretation is also reasonable by considering the problem from a Bayesian point of view. Zellner (1971) derived relation (3-12) by using a Bayesian approach. In this approach, $\hat{\beta}$ and $\hat{\sigma}$ enter into relation (3-12) not as random variables but as fixed sample values. Therefore, relation (3-12) can be used repeatedly with the same sample values of $\hat{\beta}$ and $\hat{\sigma}$ without any difficulty.

From any continuous record of streamflows, relation (3-12) can be used to generate a set of monthly streamflow sequences. The characteristics of these sequences can then be compared with those of the sequences generated by using relation (3-11). The comparison will indicate whether accounting for sampling errors would introduce any significant changes on the generated streamflows.

Relations (3-11) and (3-12) can be also compared analytically to see how they differ as far as the generation of streamflows are concerned. The mean of t(n-p) is zero for n-p>1, and its variance is $\frac{n-p}{n-p-2}$ for n-p>2. It can be safely assumed that n-p>2. Therefore, generating a value for streamflow y_k through relation (3-12) amounts to drawing a random number from a normal distribution with mean $x_k'\hat{\beta}$ and variance $\frac{n-p}{n-p-2}\,[1+x_k'(X'X)^{-1}x_k]\hat{\sigma}^2$. On the other hand, generating a value for streamflow y_k through relation (3-11) amounts to drawing a random number from normal distribution with mean $x_k'\hat{\beta}$ and variance $\hat{\sigma}^2$. Thus, the means are the same but the variances differ by the ratio

$$\frac{n-p}{n-p-2}\quad[1+x_k'(X'X)^{-1}x_k] \tag{3-13}$$

It can be shown that $(X'X)^{-1}$ is a positive definite matrix. It then follows that $x_k'(X'X)^{-1}x_k > 0$ if x_k is not a null vector and that $x_k'(X'X)^{-1}x_k = 0$ otherwise. Therefore, the ratio (3-13) is always greater than one, which means that the variance of the distribution from which y_k is generated is greater for case (3-12) than it is for case (3-11). This is as expected because the variance for case (3-12) includes the variances of estimates as well as the variance of y_k. As n-p increases, t(n-p) approaches N(0,1). Therefore, for large values of n-p and for values of ratio (3-13) close to unity, relation (3-12) reduces to relation (3-11).

Ratio (3-13) provides a quantitative measure for assessing the significance of sampling errors in generation of streamflows. In a practical analysis using streamflow records, its behavior can be examined, and its variation with respect to calendar months and sample sizes can be evaluated.

3-6. Linear Model with Correlated Errors.

In Section (3-4), the general linear model of full rank is considered under the assumption that the random vector e is $N(0,\sigma^2 I)$. A less restrictive assumption is that e is $N(0,\sigma^2 V)$, where V is a known nxn nonsingular matrix. This assumption implies that $e_1, e_2, \ldots, e_n$ are normally distributed with zero means, known ratios of variances, and known correlations. This is a more general case as compared to the previous one because it does not require that $e_1, e_2, \ldots, e_n$ be uncorrelated or that they have equal variances.

When e is $N(0,\sigma^2 V)$, it follows from (3-6) that Y is $N(X\beta,\sigma^2 V)$. Therefore, with the new assumptions concerning the distribution of random vector e, the general linear model of full rank can be written as

$$Y \text{ is } N(X\beta, \sigma^2 V)$$
$$\text{rank } (X) = p \tag{3-14}$$

The previous model (3-8) is, obviously, a special case of model (3-14) with $V = I$.

Model (3-14) can be reduced to model (3-8) with a simple transformation. It can be shown that there exists an nxn nonsingular matrix P such that $PVP' = I$. One can then use matrix P to show that

$$\tilde{Y} \text{ is } N(\tilde{X}\beta, \sigma^2 I)$$
$$\text{rank } (\tilde{X}) = p \tag{3-15}$$

where $\tilde{Y} = PY$ and $\tilde{X} = PX$ (Scheffé, 1959). This is the same as model (3-8) except that Y and X are replaced by $\tilde{Y}$ and $\tilde{X}$. Therefore, the results pertaining to model (3-8) also apply to model (3-14) provided that Y and X are replaced by $\tilde{Y}$ and $\tilde{X}$. Matrix P does not need to be known because it usually appears in expressions as $P'P$ which is equal to V^{-1}.

Consider, for example, the maximum-likelihood estimators $\tilde{\beta}$ and $\tilde{\sigma}^2$ of parameters β and σ^2 in model (3-14). They can be obtained from Eqs. (3-9) and (3-10) by substituting $\tilde{Y}$ and $\tilde{X}$ for Y and X. The result is that

$$\tilde{\beta} = (X'V^{-1}X)^{-1}X'V^{-1}Y \tag{3-16}$$

and

$$\tilde{\sigma}^2 = \frac{Y'[V^{-1}-V^{-1}X(X'V^{-1}X)^{-1}X'V^{-1}]Y}{n-p} \tag{3-17}$$

where $\tilde{\beta}$ and $\tilde{\sigma}^2$ are minimum-variance unbiased estimators of β and σ^2 for model (3-14).

The correlated errors in the linear model may be considered as autoregressive errors, thus resulting in a linear model with autoregressive errors. In this study, the errors, $e_1, e_2, \ldots, e_n$ correspond to the streamflow observations at one year apart, which might be correlated due to some time dependence. Since they are one year apart, they are free of seasonal effects. Therefore, it is reasonable to assume that their correlation structure is stationary. As a special case of model (3-14), it is assumed that $e_1, e_2, \ldots, e_n$ are generated by a stationary first-order normal autoregressive process (stationary Markov process). If this assumption turns out to be unsatisfactory, the analysis can be extended to higher order stationary normal autoregressive processes. This model is discussed by Johnston (1972). Regression models have been used for streamflow generation always with the assumption that errors are uncorrelated; this model will indicate whether there are significant correlations between errors and whether accounting for these correlations will introduce significant changes on the generated streamflows.

The errors $e_1, e_2, \ldots, e_n$ are said to be generated by a stationary Markov process if, for all t,

$$e_t = \alpha\, e_{t-1} + u_t \tag{3-18}$$

where α is a constant such that $|\alpha|<1$, and $\{u_t\}$ is a sequence of random variables independently, identically, and normally distributed with mean zero and variance σ_u^2. It can then be shown that sequence $\{e_t\}$ constitutes a normal process and that, for all t,

$$E(e_t) = 0$$

$$\mathrm{Var}(e_t) = \sigma^2 = \frac{1}{1-\alpha^2}\,\sigma_u^2$$

$$\mathrm{Cov}(e_t, e_{t+s}) = \alpha^s \sigma^2 \;, \qquad s = 0,1,\ldots \tag{3-19}$$

$$\mathrm{Cor}(e_t, e_{t+s}) = \alpha^s \;, \qquad s = 0,1,\ldots$$

where Var, Cov and Cor are the variance, covariance, and correlation operators, respectively. It follows from the last relation that α is the first-order autocorrelation coefficient of the sequence $\{e_t\}$; and for better recognition, it will be denoted by ρ.

It follows that if $e_1, e_2, \ldots, e_n$ are generated by the stationary Markov process (3-18), then the random vector e is $N(0, \sigma^2 V)$, where

$$V = \begin{bmatrix} 1 & \rho & \rho^2 & \cdots & \rho^{n-1} \\ \rho & 1 & \rho & \cdots & \rho^{n-2} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \rho^{n-1} & \rho^{n-2} & \rho^{n-3} & \cdots & 1 \end{bmatrix} \tag{3-20}$$

Therefore, with the assumption that the random vector e is generated by the stationary Markov process (3-18), the general linear model of full rank can be written as

$$Y \text{ is } N(X\beta, \sigma^2 V)$$

$$\text{rank } (X) = p$$

$$V = \begin{bmatrix} 1 & \rho & \rho^2 & \cdots & \rho^{n-1} \\ \rho & 1 & \rho & \cdots & \rho^{n-2} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \rho^{n-1} & \rho^{n-2} & \rho^{n-3} & \cdots & 1 \end{bmatrix} \tag{3-21}$$

Model (3-21) is a special case of model (3-14).

To be able to use model (3-21), the matrix V should be known. This amounts to knowing ρ. However, ρ is not known but can only be estimated. As an approximation, one can replace ρ by its estimate and proceed as if it is known. Various methods for estimating ρ have been discussed by Johnston (1972). If ρ is zero, then V is the identity matrix; therefore, model (3-21) reduces to model (3-8). A method for testing the hypothesis $\rho = 0$ has been developed by Durbin and Watson (1950,1951). This test can be used to check whether correlations between errors are significant. The inverse of matrix V in model (3-21) is given by

$$V^{-1} = \frac{1}{1-\rho^2}\begin{bmatrix} 1 & -\rho & & & & \\ -\rho & 1+\rho^2 & -\rho & & & \\ & -\rho & 1+\rho^2 & -\rho & & \\ & & \ddots & \ddots & \ddots & \\ & & & -\rho & 1+\rho^2 & -\rho \\ & & & & -\rho & 1 \end{bmatrix} \qquad (3\text{-}22)$$

The maximum-likelihood estimates $\tilde{\beta}$ and $\tilde{\sigma}^2$ of parameters β and σ^2 in model (3-21) can be obtained by using Eqs. (3-16) and (3-17) together with matrix (3-22).

3-7. Streamflow Generation by Linear Models with Correlated Errors.

Relation (3-11) was derived for model (3-8) with the assumption that the population parameters $\beta_1, \beta_2, \ldots, \beta_p, \sigma^2$ are equal to their maximum-likelihood estimates. Employing the same assumption, the analogous relation for model (3-21) is

$$\frac{y_k - x_k'\tilde{\beta}}{\tilde{\sigma}} \text{ is } N(0,1) \qquad (3\text{-}23)$$

A value for streamflow y_k can then be generated by drawing a random number from the standard normal distribution, multiplying it by $\tilde{\sigma}$, and adding to it $x_k'\tilde{\beta}$

In relation (3-11), streamflow y_k is generated from the normal distribution with mean $x_k'\hat{\beta}$ and variance $\hat{\sigma}^2$. In relation (3-23), streamflow y_k is generated from normal distribution with mean $x_k'\tilde{\beta}$ and variance $\tilde{\sigma}^2$. Therefore, as far as generation of streamflows are concerned, relations (3-11) and (3-23) can be compared in terms of quantities $x_k'\hat{\beta},\hat{\sigma}^2,x_k'\tilde{\beta}$ and $\tilde{\sigma}^2$. In the practical analysis using streamflow records, relation (3-23) can be used to generate a set of monthly streamflow sequences. The characteristics of these sequences can then be compared with those of the sequences generated by using relation (3-11). The comparison will then indicate whether accounting for correlations between errors would introduce any significant changes on the generated streamflows.

A relation analogous to relation (3-12) of model (3-8) cannot be derived for model (3-21) because complications arise due to the fact that errors are correlated. Goldberger (1962) obtained the minimum-variance linear unbiased predictor of y_k for the general case, model (3-14), and, also, for the special case, model (3-21). But, one cannot proceed with that to derive a relation analogous to relation (3-12) because the unknown parameter σ^2 cannot be eliminated. However, as an approximation, one can replace σ^2 by its maximum-likelihood estimate $\tilde{\sigma}^2$ and proceed the analysis as if it is known. Since both ρ and σ^2 are to be replaced by their estimates, β might as well be replaced by its maximum-likelihood estimate $\tilde{\beta}$ and the relation (3-23) be used.

IV. STATE VARIABLE MODELING OF WATER RESOURCES SYSTEM

4-1. Basic Concepts of State Variable Modeling.

The stochastic models of annual and monthly streamflows as described in Chapters II and III can be used as input to a water resources system model. In this chapter, a state variable approach is proposed to model the water resources system. This approach to model a dynamic system in general has been originally developed in the field of automatic control (Athans and Falb, 1966; Ogata, 1967; Gupta and Hasdorff, 1970).

In applying the state variable approach, the systems may be linear or nonlinear, time-variant or time-invariant, deterministic or stochastic, and of multiple inputs and multiple outputs. A diverse range of system models may therefore be derived as special cases of the general state variable forms. For a system to be amenable to state variable analyses, however, it must be lumped. This means that the system must evolve in only one dimension such as time or space and be describable by ordinary differential or difference equations. Water resource systems are usually distributed and are properly described by partial differential equations. For optimization purposes however, it is normally quite satisfactory to approximate the distributed system behavior by using linked lumped systems.

The system concept used in the state variable approach may be shown in Fig. 4-1. It shows that some input flow medium enters the plant or structure of the system where it is modified by physical processes until it leaves the system as output. Since the system is dynamic, the input vector $\underline{u}$ and the output vector $\underline{y}$ are both functions of time.

The system structure is given explicit representation as a vector $\underline{x}$, $\underline{x} = \{x_1, x_2, x_3, \ldots, x_n\}$, of state variables which is a function of time. The "state" of the system at any given time t_1 is given by the values

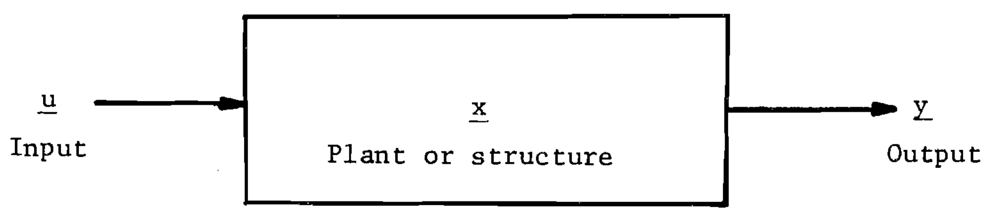

Figure 4-1. Basic systems concept.

of the state variables: $x_1(t_1)$, $x_2(t_1)$, . . ., $x_n(t_1)$, which constitute the state vector $\underline{x}(t_1)$. This is the fundamental concept of state variable modeling. Commonly used "state" quantities include the level of stock in an inventory and the volume of water in a reservoir. In water resources systems, the state variables are usually expressed in volumetric or mass units while the input and output variables are volume or mass flow rates. The state of the system is a measure of the level of activity in each of its components and can be thought of as the interface between the past and the future of the system's time history. Conceptually, the idea of "state" used in state variable methodology is the same as that used in matrix-type Markov chain models.

In the mathematical formulation of the model, the change of the state of the system over time in response to the inputs is described by a set of ordinary, first-order differential or difference equations, in matrix form called the "state equation". The state of the system and in some cases the inputs, are related to the outputs through the "output equation", which is algebraic.

Where vector-matrix operations are used, the notation employed is that a lower case letter, a, is a scalar; a lower case letter underlined, $\underline{a}$, is a vector; and an upper case letter, A, is a matrix.

4-2. Deterministic State Variable Model

(1) Formulation. For practical purposes, the basic form of the state variable model is given by Eqs. (4-1) and (4-2).

$$\dot{\underline{x}}(t) = A\,\underline{x}(t) + B\,\underline{u}(t) \tag{4-1}$$

$$\underline{y}(t) = C\,\underline{x}(t) + D\,\underline{u}(t) \tag{4-2}$$

where

$$\underline{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}, \quad \dot{\underline{x}}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix}$$

$$y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_r(t) \end{bmatrix}, \quad \underline{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_p(t) \end{bmatrix}$$

Since there are n state variables, p input variables, and r output variables, A is an n x n matrix, B an n x p matrix, C an r x n matrix, and D an r x p matrix. The elements of A, B, C, and D, may be functions of the current state vector and time. Eqs. (4-1) and (4-2) thus form a nonlinear and time-variant, dynamic system model. The matrix elements may be functions of time alone in which case the model is linear and time-variant. If the matrix elements are all constant, then the model is linear and time-invariant.

In the formulation of actual water resources system models, an appropriate special case of Eqs. (4-1) and (4-2) must be chosen. Then, two approaches may be taken: The first approach is to break the system down into more manageable subsystems. For each subsystem, a state variable model is formulated according to a physical reasoning, then these models are linked together using the matrix format to produce a model of the whole system (Muzik, 1974). In the second approach, the physical laws are assumed to be too complex for exact representation by an aggregate of simple models so instead an abstract structure is hypothesized whose characteristics are inferred from the input and output data available (Chow, 1964; Duong, Wynn, and Johnson, 1975).

In order to apply the model of Eqs. (4-1) and (4-2) to actual water resource systems using a digital computer, the time horizon must be discretized into stages of length Δt. The differential $\dot{\underline{x}}(t)$ may be approximated by the difference $\Delta\underline{x}(k)/\Delta t$ where k is the stage index. The state at the next stage, $\underline{x}(k+1)$, may be found from Eq. (4-3), or

$$\underline{x}(k+1) = \underline{x}(k) + \Delta\underline{x}(k) \tag{4-3}$$

The analogous discrete time model to Eqs. (4-1) and (4-2) is therefore formulated as

$$\underline{x}(k+1) = A\underline{x}(k) + B\underline{u}(k) \tag{4-4}$$

and

$$\underline{y}(k) = C\underline{x}(k) + D\underline{u}(k) \tag{4-5}$$

To use Eqs. (4-4) and (4-5) as a water resources system model, the input sequence $\underline{u}(k)$ is known for the period of analysis made up of K stages, k = 1, 2, ..., K. The initial state of the system $\underline{x}(1)$ is also known. The computations are performed recursively from stage to stage. Beginning with k = 1, Eq. (4-5) may be used to calculate $\underline{y}(1)$ and Eq. (4-4) to calculate $\underline{x}(2)$. Proceeding to the next stage, y(2) and x(3) may be calculated using $\underline{x}(2)$ and $\underline{u}(2)$ and so on.

For the linear, time-invariant, discrete time model, Eqs. (4-4) and (4-5), to be stable, the eigenvalues of the A matrix in the model must all lie within the unit circle (Koenig, Tokad and Kesavan, 1967). The stability requirements for nonlinear and time-variant models are discussed by Willems (1970).

It is normally necessary to use a parameter optimization method to obtain the best fit of the model to a set of data. If the model formulated is based on physical hypotheses, this parameter fitting may often be accomplished using linear regression. Nonlinear programming methods (Himmelblau, 1972) may also be used.

(2) An Example. The application of the state variable modeling approach to a hydrologic system is illustrated with the following example. The objective is to formulate a deterministic model of the direct storm runoff. The formulation used is adapted from Chow and Kulandaiswamy (1971) and illustrated with data from the U.S. Army Corps of Engineers (1954). These data comprise the rainfall hyetograph and streamflow hydrograph for the storm of April 4-5, 1941 on the 247-square mile Wills Creek watershed near Cumberland, Maryland. The storm duration of 65 hours is divided into hourly stages, k, k=1,2,...,65. The model has one input variable, u(k), the volume of effective rainfall in stage k; one output variable, y(k), the volume of direct streamflow in stage k; and three state variables, $\underline{x}(k) = \{x_1(k), x_2(k), x_3(k)\}$. The formulation is given in Eqs. (4-6) and (4-7):

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -\alpha_1 & -\alpha_2 & -\alpha_3 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(k) \qquad (4\text{-}6)$$

$$y(k) = [\alpha_1 \quad \alpha_4 \quad \alpha_5] \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} \qquad (4\text{-}7)$$

Several methods were used to choose the optimum values of the five parameters, α_1, α_2, ..., α_5. Chow and Kulandaiswamy (1971) used a linear regression approach employing a watershed storage function which is a linear combination of the parameters and derivatives of x(k) and y(k). A linear programming approach is now formulated for the same problem. The objective function and constraints in this linear programming are given by Eqs. (4-8) and (4-9) respectively.

$$\min F = \sum_{k=1}^{K} [p(k) + q(k)] \tag{4-8}$$

$$S(k) + p(k) - q(k) = S_r(k), \quad k=1,2,\ldots,K \tag{4-9}$$

where p(k) and q(k) are the positive and negative errors respectively, between the actual storage $S_r(k)$ and the computed storage S(k). The storage equation itself at each stage forms a further set of K equality constraints.

To allow negative parameter values, each parameter must be included as the sum of a negative and a positive dummy variable. With a small variable cost attached, these are added into the objective function so that only one of the two dummy variables will appear in the solution.

This linear program was solved using the ALPS linear programming code on the IBM 360/75J digital computer of the University of Illinois at Urbana. The results are very close to those obtained by the linear regression. It is concluded, however, that the linear regression is an easier approach to implement because of the problems with negative parameter values in the linear programming formulation.

Two direct search methods, the relaxation method and the steapest descent method, may be also used to find optimum parameter set. To apply these methods the following objective function is formulated:

$$\min F = \sum_{k=1}^{K} [y(k) - y_r(k)]^2 \tag{4-10}$$

where $y_r(k)$ is the actual streamflow. An iterative scheme is employed whereby a parameter set is chosen, the model response and the value of the objective function are computed, and then a new parameter set is chosen, using the search method until a satisfactory solution is obtained. It is found that the relaxation method is superior to all other approaches. This method consists of freezing all parameters except one, allowing this one to vary and then freezing it and releasing the next one, and so on. A comparison between the modeled streamflow and the actual streamflow is shown in Fig. 4-2 for the two methods of fitting the parameters. The model itself is computationally very efficient since the simulation of one storm hydrograph took only 0.04 seconds of computer time for this example.

4-3. Stochastic State Variable Model.

(1) Formulation. To extend the discrete time model of Eqs. (4-4) and (4-5) to incorporate stochastic inputs, a vector of random variables $w(k) = \{w_1(k), w_2(k), \ldots, w_m(k)\}$ is defined. This vector is usually considered to be independently, normally distributed with zero mean and unit variance. These variables are joined to the matrix equations to produce a stochastic discrete time state variable model as shown in Eqs. (4-11) and (4-12) (Aoki, 1967; Meditch, 1969; Aström, 1970; Schweppe, 1973):

$$\underline{x}(k+1) = A\underline{x}(k) + B\underline{u}(k) + G\underline{w}(k) \tag{4-11}$$

$$\underline{y}(k) = C\underline{x}(k) + D\underline{u}(k) \tag{4-12}$$

This model is very suitable for representing the behavior of water resources systems subject to stochastic inputs. Stochastic models of hydrologic time series data may be developed by using methods such as those described in Chapters 2 and 3. To employ such a stochastic input model in an optimization study for planning purposes, it may be combined with a model

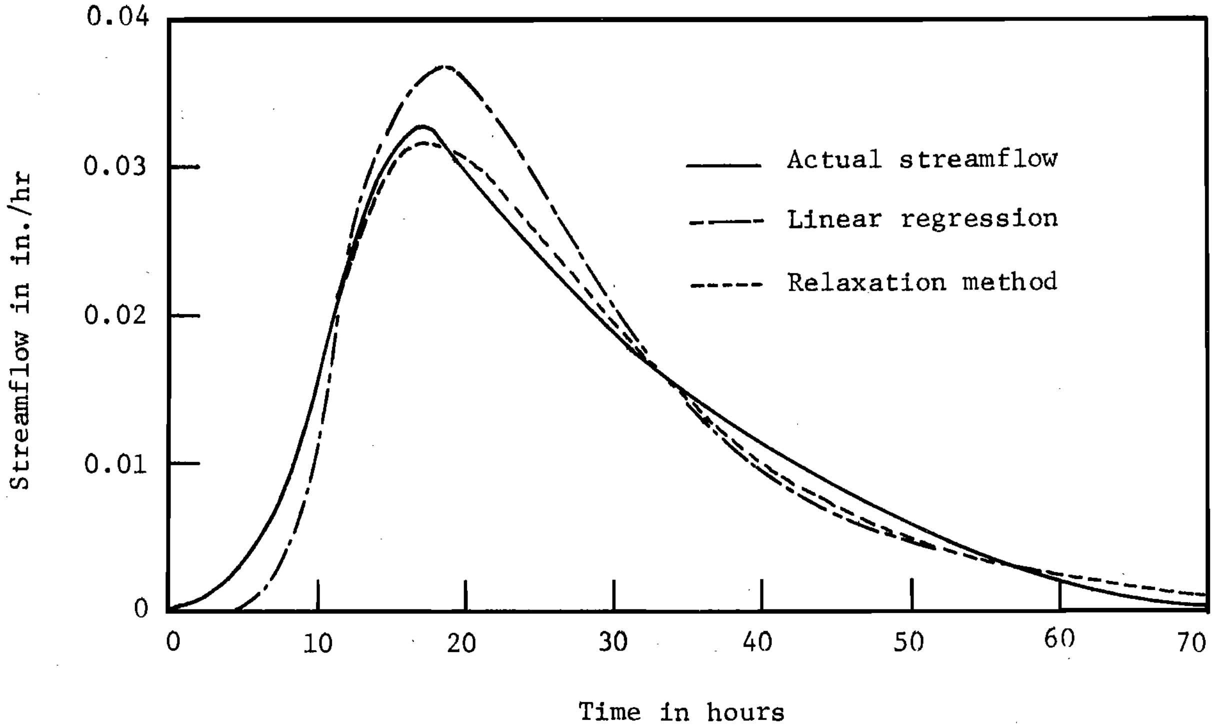

Figure 4-2. Comparison of results of parameter fitting methods.

of the system itself and then subjected to mathematical programming procedures. This is the main objective of Chapter 5. The first step in such an approach is to represent the stochastic hydrologic system model in state variable form. Autoregressive and moving average models are well suited to this purpose.

The linear autoregressive model of p-th order is given by Eq. 2-11b. The output of the model in the last p stages is defined as the values of the p state variables. In state variable format this model appears as follows:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ \vdots \\ x_p(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & \ddots & & 0 \\ 0 & \cdots & & & 0 & 1 \\ \phi_p & \cdots & & & \phi_2 & \phi_1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ \vdots \\ x_p(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} (w(k) \qquad (4\text{-}13a)$$

$$y(k) = [0 \;\; 0 \;\cdots\; 0 \;\; 1] \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_p(k) \end{bmatrix} \qquad (4\text{-}13b)$$

A linear moving average model of q-th order may be written as

$$y(k) = [\theta_1 w(k-1) + \theta_2 w(k-2) + \ldots + \theta_q w(k-q) \qquad (4\text{-}14)$$

The random disturbances which have occurred in the last q stages are taken as the state variables. The state variable form of this model is shown as follows:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ \\ x_q(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ & & & & & 0 \\ 0 & & & & & 1 \\ 0 & \cdots & & & & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ \\ x_q(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} w(k) \qquad (4\text{-}15a)$$

$$y(k) = [-\theta_q, -\theta_{q-1}, \ldots, -\theta_1] \begin{bmatrix} x_1(k) \\ x_2(k) \\ \\ x_q(k) \end{bmatrix} \qquad (4\text{-}15b)$$

Since both parts of an autoregressive moving average (ARMA) model are representable in state variable form it is reasonable to expect that they may be joined in this framework. A direct synthesis of the two models may lead to unnecessary duplication of the state variables however. Consider an ARMA (1,1) normalized process, which is identical to the autoregressive integrated moving average (ARIMA) (1,0,1) process:

$$y(k) = \phi_1 y(k-1) + w(k) - \theta_1 w(k-1) \qquad (4\text{-}16)$$

In the state variable form, only one state variable is needed; thus

$$x(k+1) = \phi_1 x(k) + (\phi_1 - \theta_1) w(k) \qquad (4\text{-}17a)$$

$$y(k) = x(k) + w(k) \qquad (4\text{-}17b)$$

Once an appropriate state variable model of the hydrologic inputs has been formulated it may be combined with another state variable model representing the behavior of the system in response to the inputs. The basic principle for this second model is usually the equation of continuity of flow, though more complex system models can be formulated by the state variable approach if necessary.

(2) An Example. A mathematical model of a storage reservoir, Fig. 4-3, subject to stochastic inputs is now formulated according to the state variable approach. This model is to be incorporated into a stochastic dynamic programming algorithm in Chapter 5 to find the optimal release policy for the reservoir. The data used for this model in the example application are given in the Appendix.

The volume of storage at the beginning of stage k is denoted by x(k). During stage k the release is denoted by u(k) and the inflow by q(k). Using the principle of conservation of mass:

$$x(k+1) = ax(k) - u(k) + q(k) \qquad (4\text{-}18)$$

where a is a coefficient accounting for seepage, evaporation, and spillway losses. The output equation expresses the volume of outflow from the reservoir as

$$y(k) = cx(k) + u(k) \qquad (4\text{-}19)$$

where c is a coefficient relating spillway flow to storage volume, x(k). Assuming that q(k) is an independently distributed random variable with known mean $\mu(k)$ and variance $\sigma^2(k)$, Eqs. (4-18) and (4-19) may be expressed according to the general format of Eqs. (4-11) and (4-12) as

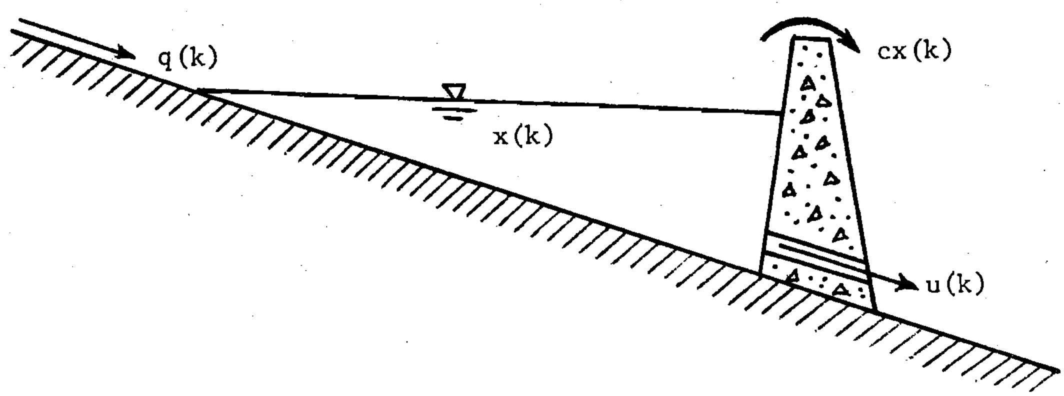

Figure 4-3. A storage reservoir system.

$$x(k+1) = ax(k) + [-1 \ 1]\begin{bmatrix} u(k) \\ \mu(k) \end{bmatrix} + \sigma(k)w(k) \tag{4-20a}$$

$$y(k) = cx(k) + [1 \ \ 0] \begin{bmatrix} u(k) \\ \mu(k) \end{bmatrix} \tag{4-20b}$$

If the inflows q(k) to the storage reservoir of Fig. 4-3 are considered to be serially correlated with first-order correlation coefficient r(k); they may be described by the following equation (Fiering and Jackson, 1971):

$$\frac{q(k)-\mu(k)}{\sigma(k)} = r(k) \left[\frac{q(k-1)-\mu(k-1)}{\sigma(k-1)}\right] + (1-r^2(k))^{1/2} \ w(k) \tag{4-21}$$

Let

$$x_1(k) = \frac{q(k-1)-\mu(k-1)}{\sigma(k-1)}$$

Hence, from Eq. (4-21)

$$x_1(k+1) = r(k)x_1(k) + (1-r^2(k))^{1/2} \ w(k) \tag{4-22}$$

and

$$q(k) = \sigma(k)r(k)x_1(k) + \mu(k) + \sigma(k)(1-r^2(k))^{1/2} \ w(k) \tag{4-23}$$

Combining Eq. (4-18), (4-22) and (4-23), the complete state equation is formulated as

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} r(k) & 0 \\ \sigma(k)r(k) & a \end{bmatrix}\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u(k) \\ \mu(k) \end{bmatrix}$$
$$\begin{bmatrix} (1-r^2(k))^{1/2} \\ \sigma(k)(1-r^2(k))^{1/2} \end{bmatrix} w(k) \tag{4-24}$$

V. WATER RESOURCES SYSTEMS OPTIMIZATION BY STOCHASTIC DYNAMIC PROGRAMMING

5-1. Stochastic Transformation of Water Resources Systems.

The objective of the optimization is to choose the decision policy for the system which is best according to its performance function. This function measures the degree of attainment of the goals set for the system. The policy may be in the form of a set of charts which specify the required decisions in stage k, $\underline{u}(k)$, as a function of the state of the system at the beginning of the stage, $\underline{x}(k)$. In this study, a stochastic dynamic programming technique is proposed to optimize the water resources system.

To apply dynamic programming for system optimization, the behavior of the system must be described by a state transformation equation. This equation expresses the output values of the state variables at each stage as a function of the input values of these variables and the inputs and decisions occurring during the stage. The state variable approach may be used to formulate such equations. When the inputs at each stage are stochastic, the output values of the state variables become random variables describable by probability distributions.

For discussion purposes, consider the system described by Eq. (4-20). The feasible range of the state variable x(k), $x_{min}(k)$ to $x_{max}(k)$, may be divided into I intervals, i = 1, 2, ..., I; each of size $\Delta x(k)$. Likewise, the feasible range of x(k+1) may be divided into J intervals of length $\Delta x(k+1)$. In the dynamic programming procedure, assume that the optimal decision at interval i of x(k) is to be chosen from a set of decisions U(k). The expected value E[x(k+1)] and variance Var[x(k+1)] of the resulting state, x(k+1), may be found from Eq. (4-20) for any specified decision u(k). The resulting probability distribution for the output state x(k+1) may appear as in Fig. (5-1).

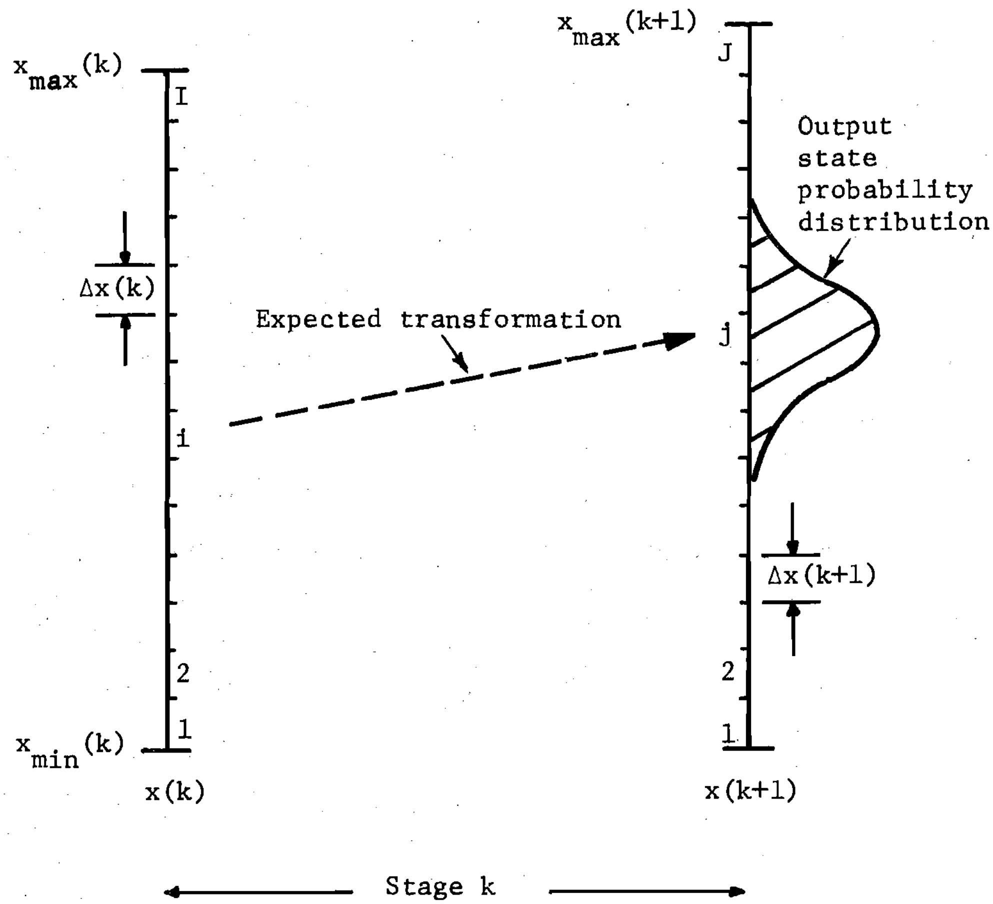

Figure 5-1. Stochastic transformation.

The "expected transformation" is that which would occur if the mean inflow occurred. Since the input state is assumed known, the standard deviation of the output state probability distribution is the same as that for the flow distribution at this stage, $\sigma(k)$.

5-2. The Recursive Equation.

If $v_i(k)$ is the value of being at interval i of x(k), the recursive equation of dynamic programming may be written as

$$v_i(k) = \underset{u(k)\epsilon U(k)}{\mathrm{opt}}[R(x(k),\ u(k)) + \sum_{j=1}^{J} p_{ij}(k)v_j(k+1)] \qquad (5\text{-}1)$$

where $p_{ij}(k)$ is the probability that the state of the system x(k+1) will fall within interval j, and $v_j(k+1)$ is the corresponding value attached to this interval from the optimization of stage k+1. If x(k+1) is assumed to be normally distributed, $p_{ij}(k)$, may be readily calculated from a normal distribution table since E[x(k+1)] and Var[x(k+1)] are known. If x(k+1) has some other distribution, then the calculations are more complicated but still feasible.

If the flow means, $\mu(k)$, and standard deviations, $\sigma(k)$, are constant and independent of the stage, the sequence of values of the state variable, x(k), k = 1, 2, ..., K, constitutes a single-step Markov process. The optimal policy can be found by a successive approximations method of White (1963). The principle upon which this method is based is as follows: If Eq. (5-1) is applied recursively backward over a number of stages, it is found that the relationship between $v_i(k)$ and $v_i(k+1)$ becomes approximately linear as follows:

$$v_i(k) = v_i(k+1) + g_i;\ i = 1,\ 2,\ \ldots,\ I \qquad (5\text{-}2)$$

where g_i is a so called "gain" for state i. To find the optimal policy, the computations are carried out for sufficient stages until all the individual state gains converge to a single value, independent of the state.

Su and Deininger (1972) have extended White's method to incorporate periodic Markov processes. These have statistics which vary from stage to stage but which are periodic after N stages, e.g., $\mu(k+N) = \mu(k)$. This formulation is appropriate for the optimization of monthly decisions for reservoir operation over a year. If the gain is now interpreted as the increase in value over a period of N stages, the same optimization approach as for the single-step Markov process can be applied.

5-3. Chance Constraints and Steady State Probabilities.

Since the output state at each stage is described as a random variable, the state boundary constraints which confine the behavior of the system within the feasible range of the state variables, must be set on a probabilistic or "chance constraint" basis. Thus at any stage, if the probability of the reservoir emptying out completely is greater than an allowable limit, then the decision under consideration is rejected. Similarly, the decision is rejected if the probability of overtopping the reservoir is greater than an allowable limit (Askew, 1974a, 1974b).

Once a policy has been chosen for the system, the stochastic nature of the behavior of the storage is a fixed Markov process. Consequently the steady state probabilities of occupancy of any storage state may be found by using the transition probability matrices from each stage. At each stage and state during the optimization of the recursive Eq. (5-1), the transition probabilities, p_{ij}, resulting from the optimal decision are stored. Once a complete period of N stages of policy optimization has been performed backward over the stages, the probability distribution of the storage in each stage may be found by calculating forward over the stages.

The probability distribution of the input states at the first stage, $p_i(1)$, may initially be assumed to be uniform. The probability distribution of the output states at stage 1,$p_j(1)$, may be found from the following equation with k=1:

$$p_j(k) = \sum_{i=1}^{I} p_i(k)p_{ij}(k); \; j=1,2,...,J. \qquad (5\text{-}3)$$

For stage 2, the output state probability distribution for stage 1 becomes the input state probability distribution, $p_i(2)$. Hence, $p_j(2)$ may be found and the calculations for all succeeding stages proceed analogously. From the results of the last stage, the probability distribution of the input states at the first stage is reinitialized and the cycle repeated until the probability distributions at each stage converge to steady values. This usually requires 3 or 4 cycles of computation over all N stages. As the policy converges, so do the steady state probability distributions.

5-4. Risk Analysis.

The procedure of deriving the steady state probability distributions provides a link between the classical theories of reservoir storage based on Markov chain analysis (Moran, 1959; Chow, 1964; Lloyd, 1967) and operational policy determinations made using dynamic programming. With this information it is possible to assess the risk of failure of the system at any point in the planning horizon if "failure" is defined to have occurred when the values of the state variables fall into an unacceptable range, e.g., an empty reservoir. The policy derivation may then be made sensitive to the risk of system failure as well as to the optimization of the usual net benefits function. This may be accomplished by using penalty functions to preclude policies involving a high risk of failure or by manipulating the chance constraints on the state variables so as to allow optimization of the system policy within constraints imposed by failure considerations (Klemes, 1975).

5-5. An Example.

(1) Description of the System. The system selected to demonstrate the proposed methodology is the proposed Watasheamu Dam and Reservoir (U.S. Bureau of Reclamation, 1962). The project is located on the East Fork of the Carson River in Nevada. The reservoir has a capacity of 160,000 acre-feet and a mean inflow of 259,000 acre-feet per year. The example considers operation for hydroelectric power, irrigation and flood control. Details of the physical and economic data used in the example are given in the Appendix.

The volume of storage in the reservoir is taken as a state variable, the decision variable is the release to be made through the dam in each stage of one month duration. The feasible range of the state variable at each stage is divided into intervals of 10,000 acre-feet. A flood storage reservation at the top of the dam is made in those months where flood control is necessary. The number of storage intervals varies from 10 to 17 per stage depending on the magnitude of the flood storage reservation. The feasible range of the decision variable at each stage is taken as 0 to 100,000 acre-feet.

The state variable model for the system is Eq. (4-20) where the loss coefficient "a" depends on $x(k)$ and k and accounts for evaporation. The means and standard deviations for each stage are estimated from the historic inflow data at the site. The probability distribution of the output state at each stage, $x(k+1)$, is normal with mean, $ax(k) - u(k) + \mu(k)$, and standard deviation, $\sigma(k)$.

The objective of the optimization is to maximize the benefits obtained from the operation of the system. The hydropower benefits are the value of the energy generated at each stage which is a nonlinear function of storage and release. Power generation ceases when the elevation of water in the reservoir is below a specified minimum. The benefits from irrigation are estimated by piecewise linear functions of the release. Details are given in the Appendix. The combined objective function is thus nonlinear and discontinuous.

(2) Optimization Procedure and Results. The optimization procedure used is a backward, stochastic dynamic programming algorithm. In outline, the procedure is as follows: The value of all output states at the last stage is initialized to zero. Computations proceed backwards by stages until a stable policy and expected annual returns from the operation of the system have been found. Although the statistical parameters of the inflow distribution vary from month to month within a year, they are periodic from year to year. One year of twelve stages of computation therefore constitutes an iteration of the optimization procedure. If the value of all output states at the last stage for each iteration is set equal to the value of the corresponding input states at the beginning of the first stage of the previous iteration, the cumulative expected value grows as shown in Fig. 5-2. From the figure, it may be seen that after two or three iterations, the increase in value for any state over one year is constant and independent of the state. This is the expected annual return from the operation of the system or "gain". The cumulative return decreases with storage however as can be seen from the difference between the two lines in Fig. 5-2. Since the lines are parallel after the policy has stabilized, the difference between them can be interpreted as the relative value of beginning operation in the highest storage state instead of the lowest.

This relative value function may be used at the beginning of the first stage of an iteration to reinitialize, for the next iteration, the

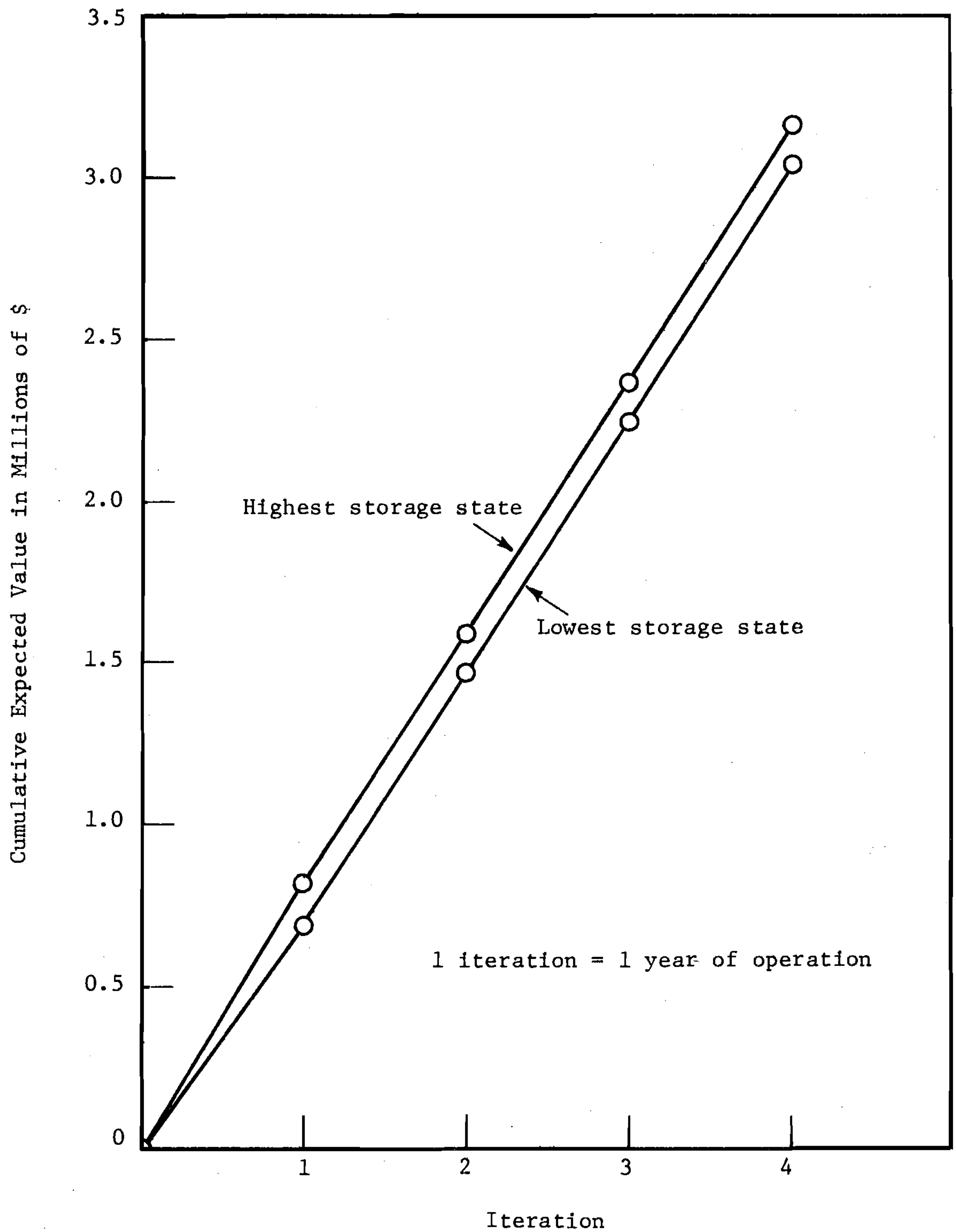

Figure 5-2. Expected value growth with iteration for two reservoir states.

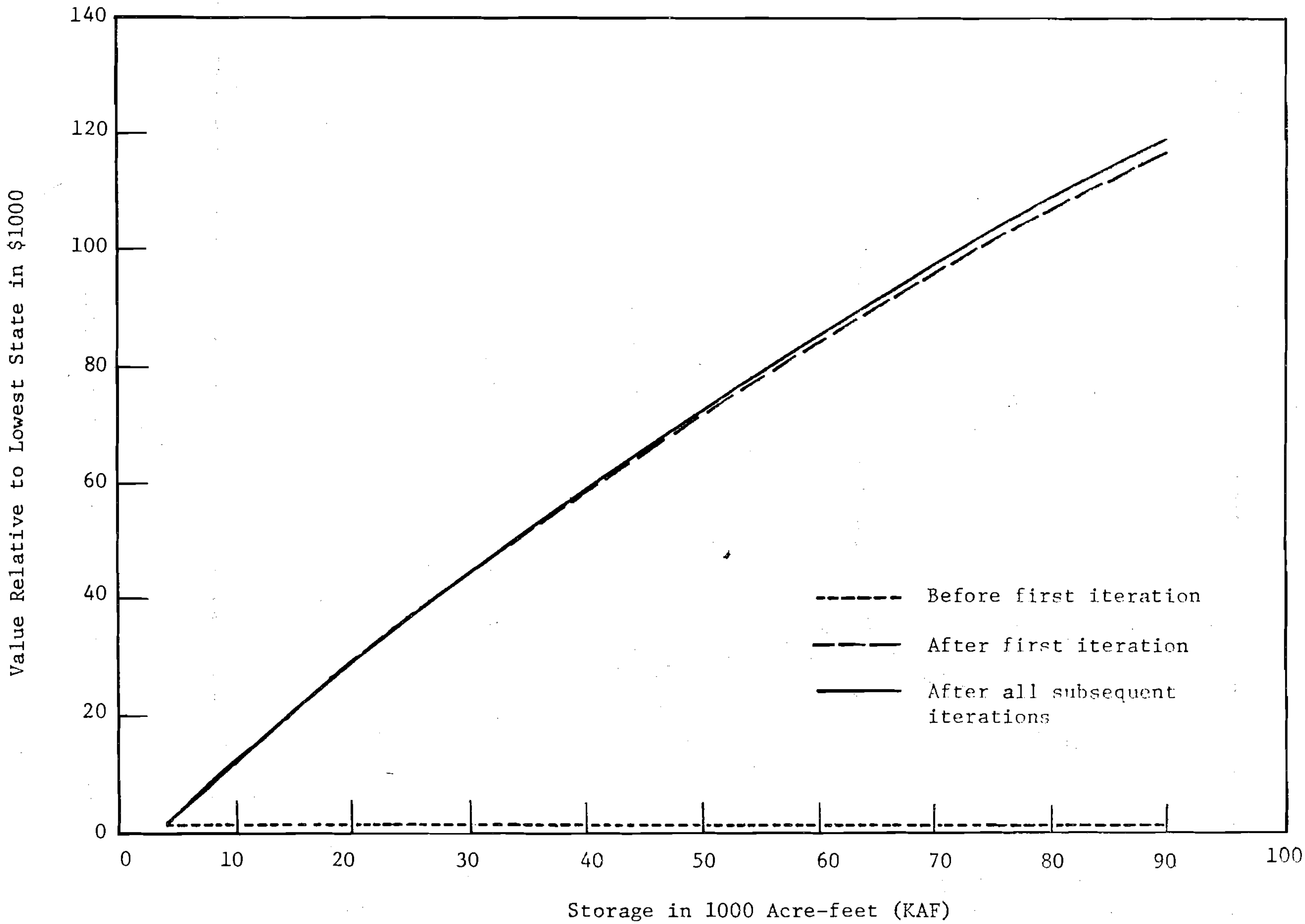

Figure 5-3. Relative value of beginning operation with initial storage.

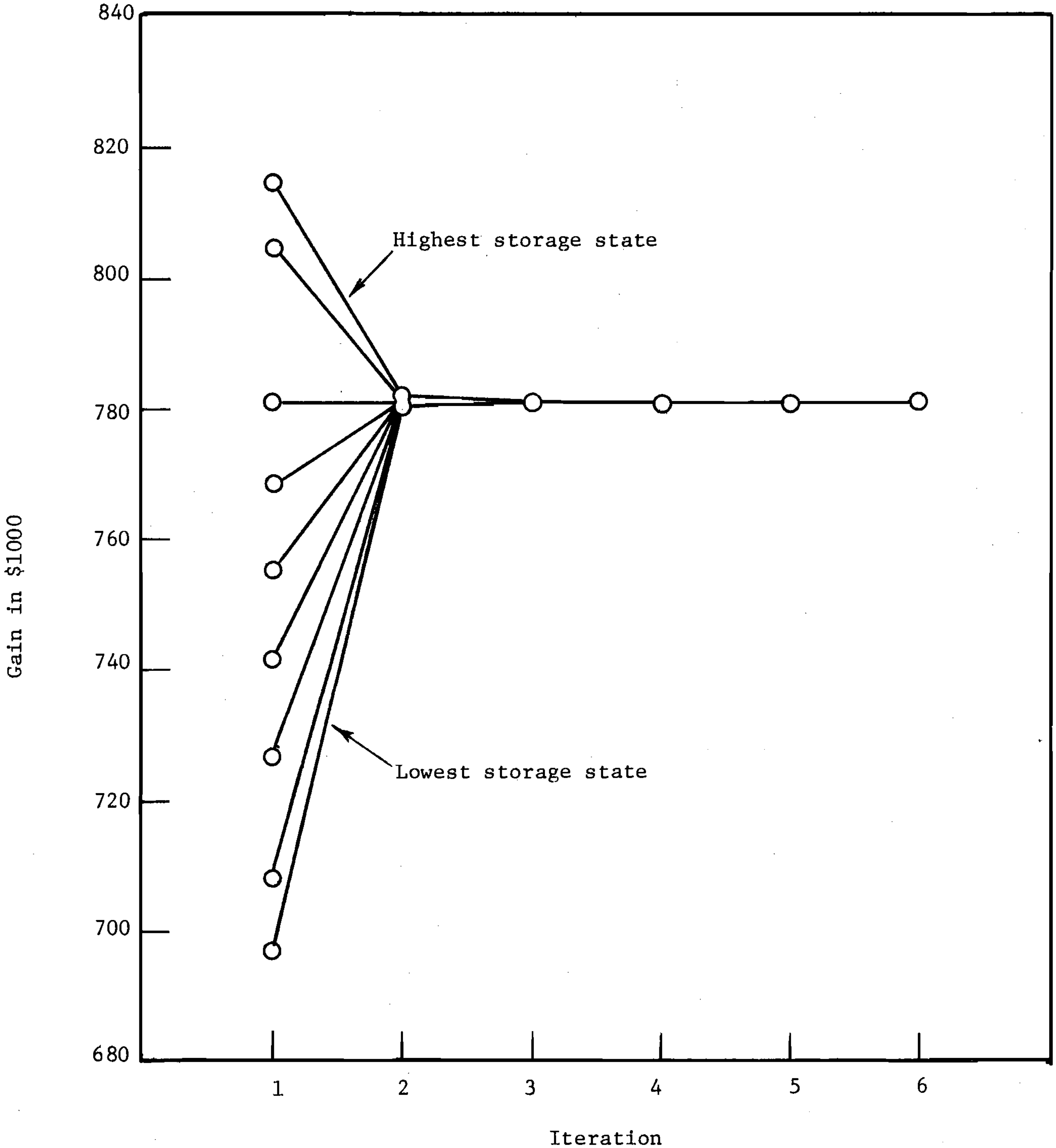

Figure 5-4. Gain vs. Iteration for various states.

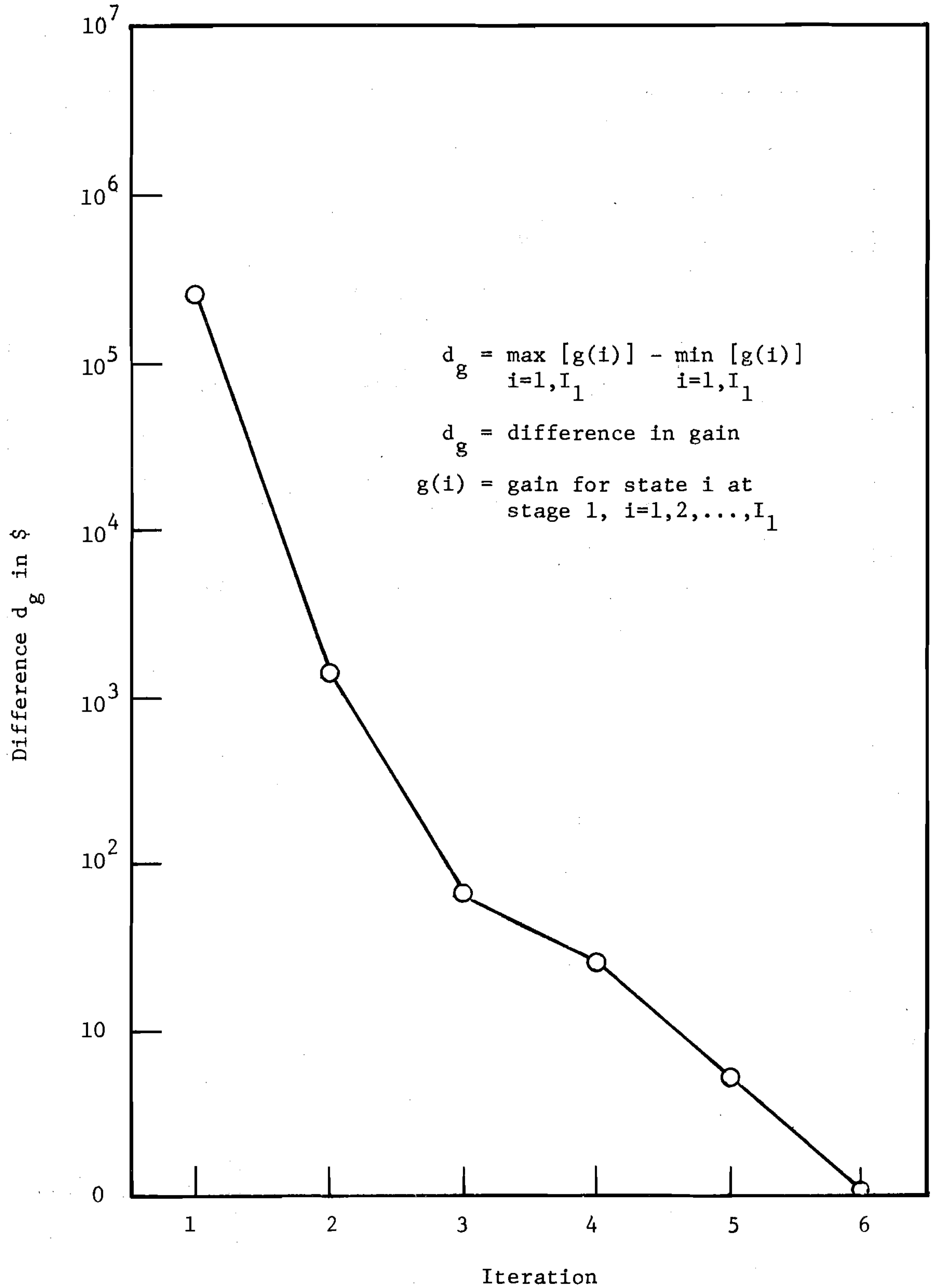

Figure 5-5. Difference between greatest and smallest gains vs. Iteration.

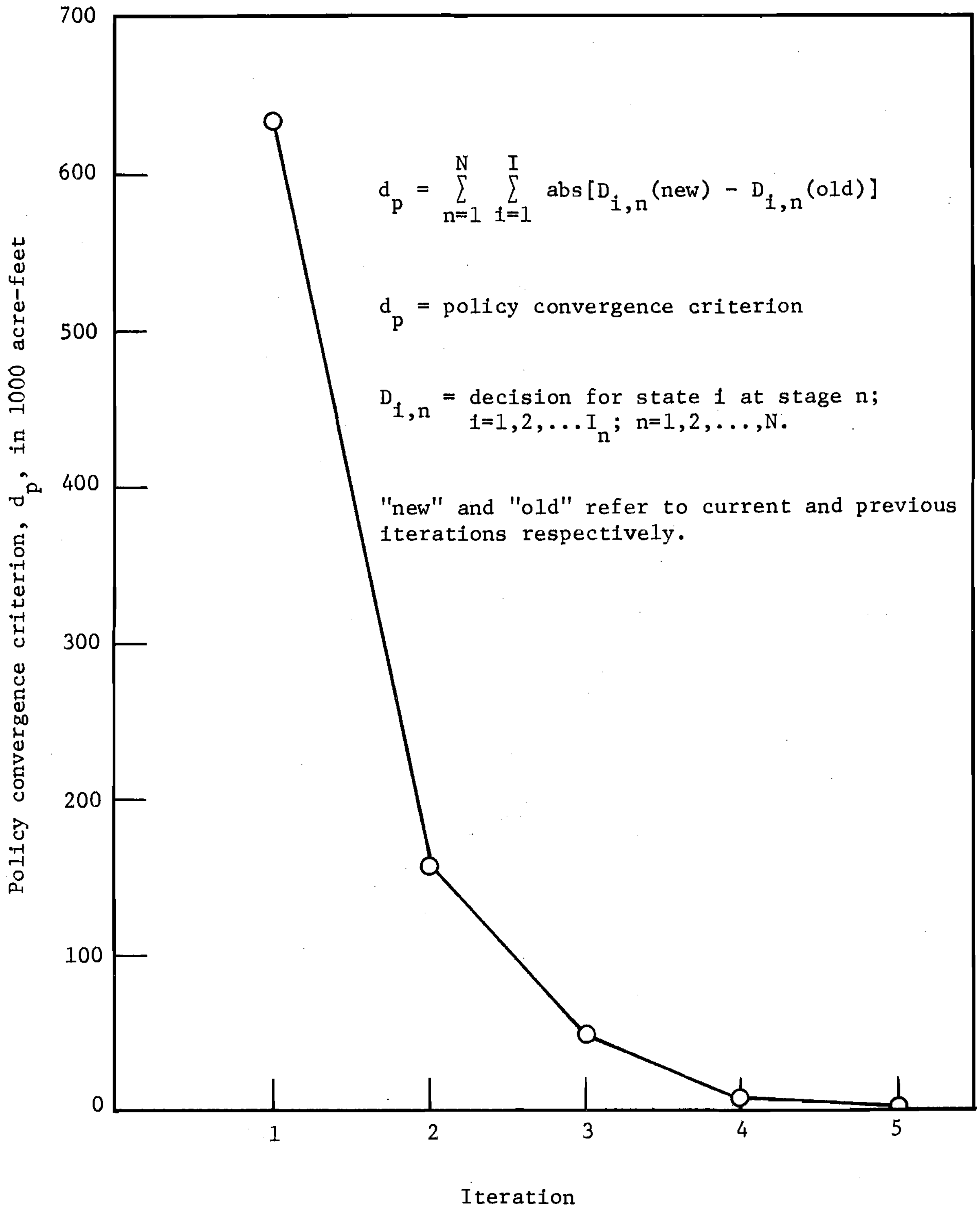

Figure 5-6. The policy convergence.

values of the output states at the last stage. The rapid stabilization of the relative value function is seen in Fig. 5-3. The shape of the function in this figure is probably due to the fact that a given release generates more energy when made from a higher elevation or storage but the marginal worth of increasing the storage decreases with increasing storage since the additional increments of elevation decrease.

The individual state gains, calculated according to Eq. (5-2) applied over each period of 12 stages, also converge quickly as can be seen in Fig. 5-4.

A convergence criterion may be defined as the difference between the highest and lowest gains on an iteration. This criterion is shown as a function of iteration in Fig. 5-5. Another measure of convergence is the change in the derived policy from one iteration to the next. A policy convergence criterion is defined as the sum of the absolute values of the differences between the policy obtained on the current iteration and that of the previous iteration. The rapid convergence of the policy using this criterion is shown in Fig. 5-6.

To find the optimal decision from a state space point a simple search procedure is used which consists of testing each decision in a set defined around the previously optimal decision for this state space point. The number of decisions used is 10 and the difference between adjacent decisions, the step size, is set at 10,000 acre-feet for the first iteration. The step size is halved after each iteration to a minimum of 1000 acre-feet. Thus, the range of this set is initially the entire feasible decision range and successively smaller portions of it in later iterations. The policy derived is shown in Table 5-1. The form of the optimal policy reflects the nature of the system. As available storage rises so does the

Storage in KAF \ Month	Jan	Feb	Mar	Apr	May	June	July	Aug	Sept	Oct	Nov	Dec
3.05	0	0	0	0	0	0	0	4	3	5	0	0
10	0	0	0	0	0	0	0	8.75	8.75	11.25	0	0
20	0	0	0	0	0	0	0	18.75	18.75	20	0	0
30	0	0	0	0	11.25	0	0	28.75	28.75	20	0	0
40	0	0	0	0	22.5	2	0	38.5	38.5	20	0	0
50	0	0	0	0	25	15	4	10	45	20	0	0
60	0	0	0	0	25	25	13.75	18.5	45	20	0	0
70	0	0	0	0	25	30	24.75	35	36.25	20	1	4
80	0	0	0	0	25	30	33.75	45	45	20	11.5	14.25
90	5	0	0	0	25	30	43.75	45	45	20	22.5	24.75
100	-	6.25	0	0	25	30	45	45	45	20	31.5	-
110	-	-	0	5	33.75	30	45	40.25	45	20	41.25	-
120	-	-	-	15	43.75	30	45	45	45	20	51.25	-
130	-	-	-	25	53.75	30	45	45	45	20	-	-
140	-	-	-	35	63.75	36.25	45	45	45	24.75	-	-
150	-	-	-	45	73.75	47.25	45	45	45	33.75	-	-
157.5	-	-	-	52.5	81.25	54	45	45	45	41.25	-	-

Table 5-1. Optimal Release Policy in Thousands of Acre-Feet (KAF) per Month for Each Storage State and Month

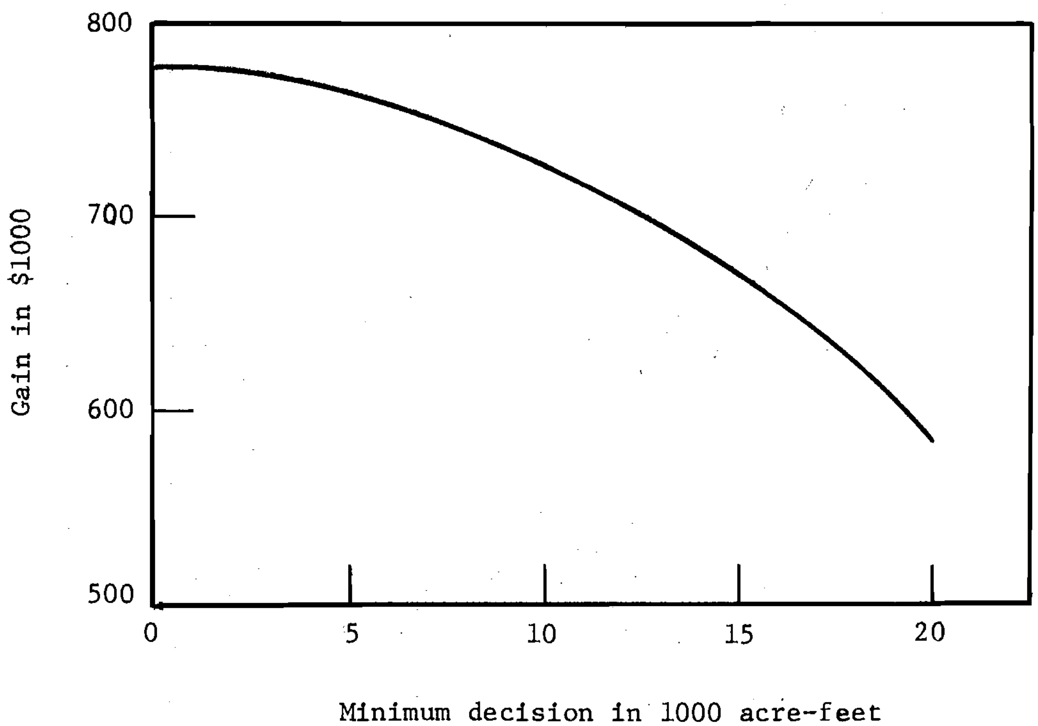

Figure 5-7. Effect on gain of changing minimum feasible decision.

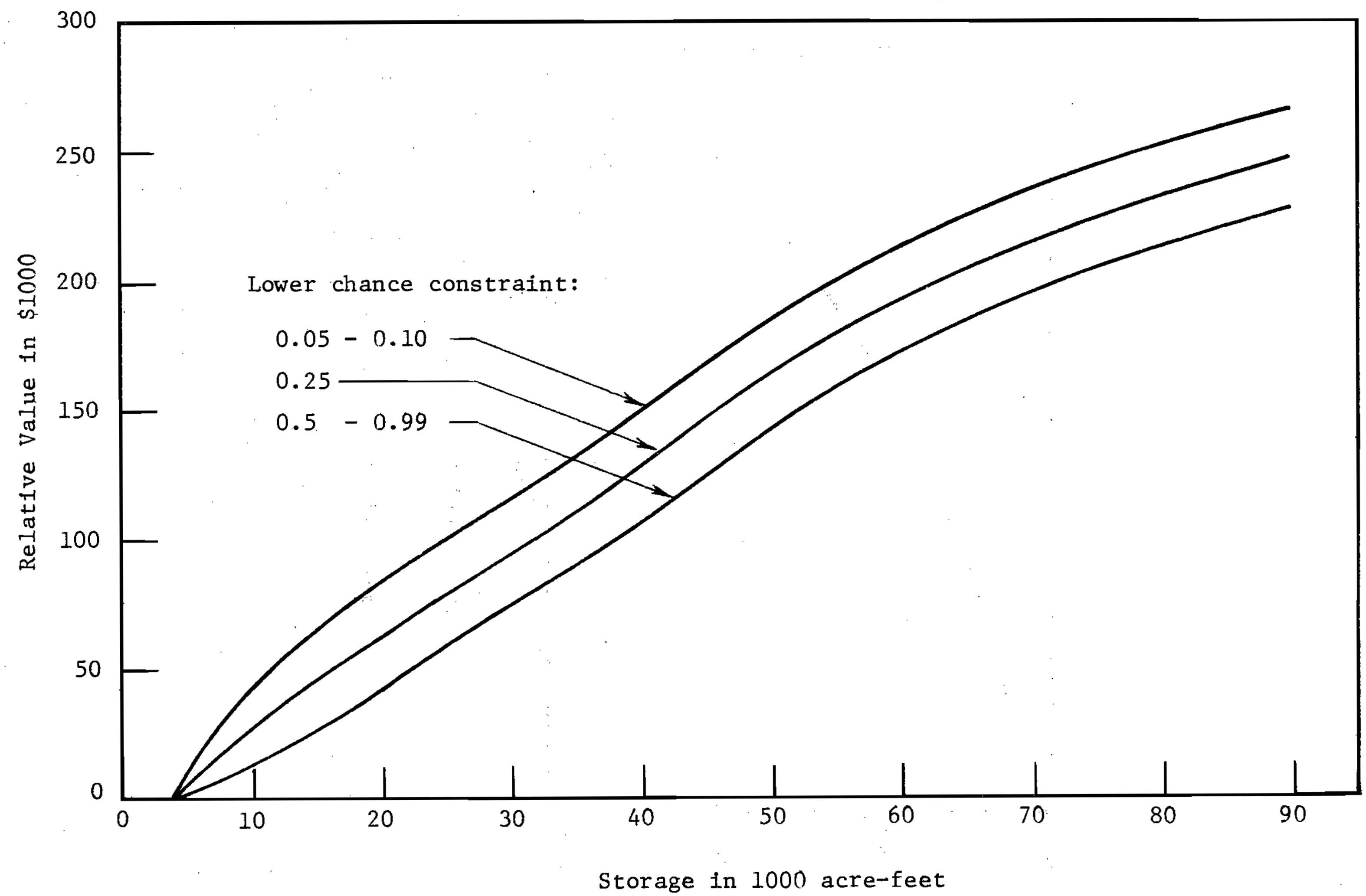

Figure 5-8. Relative value vs. storage for various values of lower chance **constraint**.

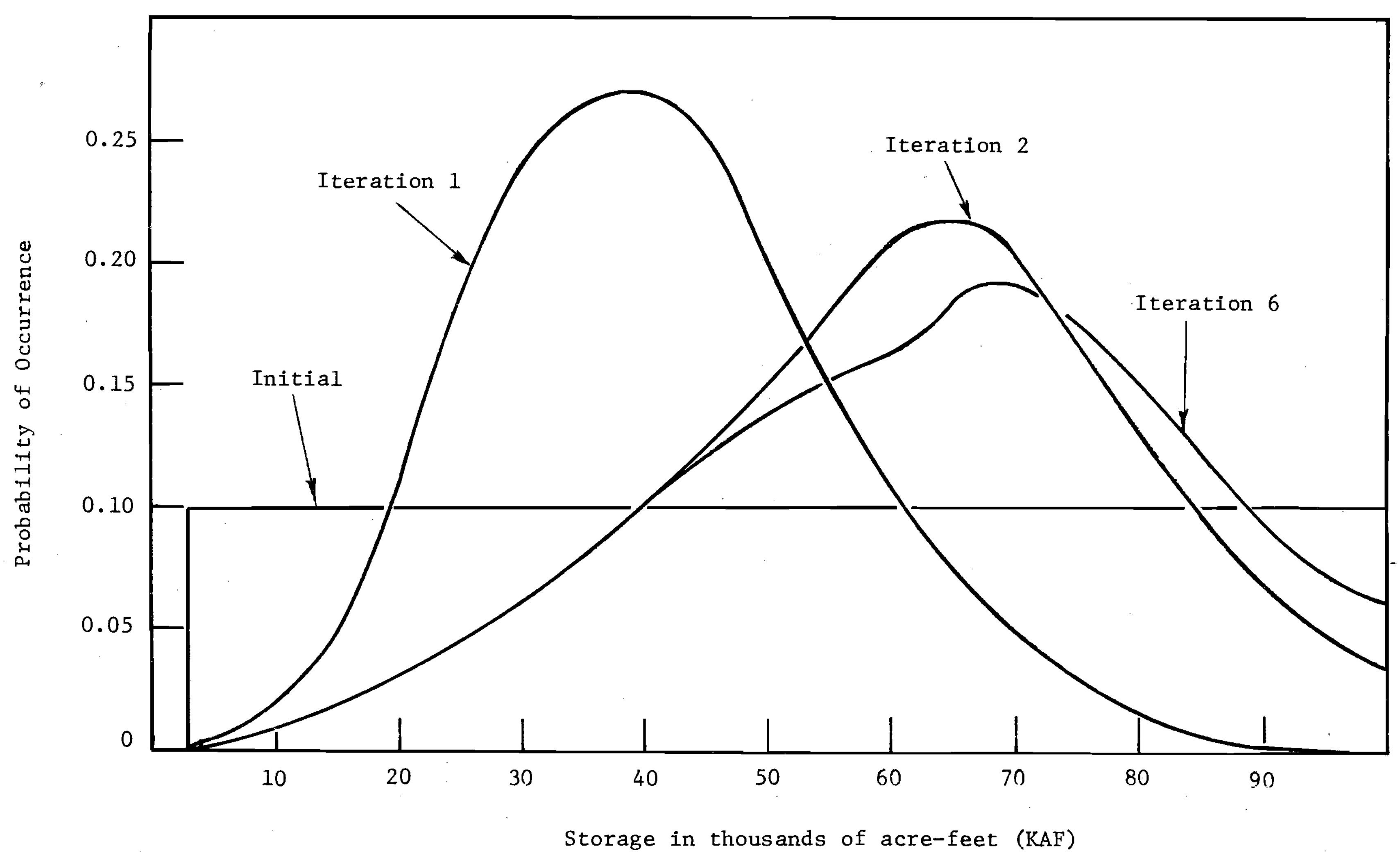

Figure 5-9. Stage 1 storage probability of occurrence for various policy iterations.

release to be made. The major releases are made in the irrigation season, May through October.

The effect of raising the minimum feasible decision at each stage is to constrain the choice of policy. Consequently the gain of the system falls as shown in Fig. 5-7.

The effect of varying the chance constraint for the lower boundary of the storage space on the relative value function is shown in Fig. 5-8. This shows that as the constraint becomes more stringent, it is more valuable to begin operation in a higher storage state. The gain of the system is not significantly affected by the variation of this chance constraint.

After each iteration of policy determination, the steady state probabilities of storage at all stages are calculated using Eq. (5-3). The resulting probability distribution for storage at the beginning of stage 1, January, is shown in Fig. 5-9 for several policy iterations. The probability distribution is rather more sensitive to changes in policy than is the relative value function.

For this Markov chain analysis, two extra storage states must be defined at each stage to account for the probability of emptying the reservoir or overtopping the maximum desirable storage. To avoid making these states into "trapping states" from which no exit is possible, it is assumed that identical transitions may be made from these states as for the adjacent states in the interior of the dynamic programming state space. This does not necessarily lead to a zero steady state probability of occurrence of the two exterior states however. Although the probability of emptying the reservoir in January is negligible in Fig. 5-9, there is more than 5% probability that the reservoir will be above 95 KAF in storage which means that it will infringe on the flood storage reservation with this probability throughout the project life when the derived policy is used. If it were desired to have more detailed information about the degree to which the flood storage reservation would be infringed upon, this could be found by defining extra states in the flood storage region for the Markov chain analysis.

VI. SUMMARIES AND CONCLUSIONS

6-1. On Stochastic Modeling of Annual Streamflows

In this part of the study, a brief review of the various models proposed for streamflow simulation has been presented. Basically two types of models are in use today in annual streamflow modeling, namely, the Markov models and the fractional Gaussian noise (FGN) models. Both models, however, belong to a general class of discrete time linear stochastic processes. The Markov model is simple in its structure but its applicability is limited because the theoretical autocorrelation function is not flexible enough to fit a wide range of sample autocorrelation functions. The FGN model is a continuous moving average process of infinite order. As the first step toward practical application, the FGN model must be replaced by a discrete fractional Gaussian noise (DFGN) model. The latter has a peculiar kernal structure which does not satisfy the Box and Jenkins's (1970) definition of stationarity. As the second step toward operational uses, the DFGN model of infinite order must be approximated by a process involving a finite number of terms; consequently, it becomes stationary. So far six different approximations to the DFGN model have been proposed.

In general, moving average models are inferior to autoregressive models for the following reasons: (1) The autocorrelation function of autoregressive model tails off while that of moving average model cuts off, (2) the autoregressive model is linear in the parameters while the moving average model is nonlinear in the parameters, and (3) the autoregressive model is expressed in terms of the observed past values while the moving average model is written in terms of the unobserved past random disturbances. A so-called fast FGN model requires the summation of a large number of terms. The ARMA(1,1) model with ϕ_1 close to 1 can be regarded as an approximation to the FGN model but, if ϕ_1=1, it may not be an approximation to natural streamflow processes since ϕ_1 as a parameter of the model is not estimated from the observed historical streamflow sequences.

In this study, a new model, a second order autoregressive process with a data-based transformation is proposed to approximate the underlying streamflow generating process. Due to its simplicity and generality, the new model seems to be very promising for annual streamflow modeling.

Both the maximum likelihood method and the Bayesian approach are used to estimate the model parameters and statistical inferences about the parameters are made. For large samples, the maximum likelihood estimates have smaller variance than the method of moments estimates and are therefore statistically more efficient.

Since streamflow sequences seldom follow the normal probability distribution, it is often found advantageous to transform the original sequence so that the transformed sequence can be adequately represented by a stochastic model based on the normal probability distribution theory. The problem associated with transformations is two-fold: First, to identify a suitable transformation; and second, to determine whether the parameters would be better estimated from the transformed data or from the original data. The proposed procedure considers both aspects of the transformation problem simultaneously.

The parameters of a streamflow generating model estimated from a short historic sequence are not likely to be equal to their respective population values. This parameter uncertainty problem may be treated by employing the Bayesian approach of statistical inference. The Bayesian inference provides a framework to pool all the available information to reduce the parameter uncertainty. More importantly, the Bayesian approach gives a probability distribution rather than unique values for the unknown parameters. It is noted that the likelihood function is proportional to the posterior probability density function (pdf) with diffuse prior pdf. Hence, the results from the Bayesian approach are comparable to those from the maximum likelihood method. Also, in large samples, the maximum likelihood estimates are the approximate means of the posterior pdf of the parameters, a pdf that will usually be approximately normal. Confidence regions for the parameters of the AR(2) models are constructed. The contours of the posterior distributions for the parameters are identical to the confidence regions for the parameters.

6-2. On Stochastic Modeling of Monthly Streamflows.

This part of the study considers the generation of monthly streamflow sequences. A monthly time interval is chosen for two reasons; first, monthly streamflows are frequently used in design studies of hydrologic systems; and second, monthly streamflows constitute a fairly general case because of their seasonal structure. Seasonal structure is a characteristic of all streamflow events except the annual ones.

In hydrology, many models have been considered for the generation of monthly streamflows, including regression models, autoregressive models, FGN models, broken line models and multiplicative seasonal ARIMA models. In this study, a new linear model approach is proposed.

The proposed approach accounts for the sampling errors associated with the estimates of model parameters. Conventional regression models, which do not account for sampling errors, are shown to be special cases of the proposed model. Regression models used for streamflow generation always assume that the errors are uncorrelated. In addition to this conventional case, this study considers a more general case in which errors are assumed to be generated by a stationary Markov process.

6-3. On State Variable Modeling.

The state variable approach to the mathematical modeling of hydrologic or water resources systems provides a generalized framework within which many different kinds of system models may be expressed and combined for the representation of a given system. Linear or nonlinear, time invariant or time-variant, deterministic or stochastic systems with multiple inputs and multiple outputs may be modeled by this approach. This simple yet general format is a major advantage of the state variable modeling.

The state variable models are particularly appropriate for their incorporation into optimization procedures to determine the optimum policy for the system. This may be accomplished by embedding the state variable model within an analytic optimization procedure. Dynamic programming and the Pontryagin maximum principle are two procedures which are excellent for this purpose. This may also be accomplished when the optimization involves repeated simulations of the system's behavior in which case the state variable approach may be used to formulate the simulation model. The state variable models are well suited to digital computation. This computational efficiency is often a critical factor in an optimization study where the computer limitations prohibit the use of very sophisticated mathematical models (Chow, Maidment,and Tauxe, 1975).

A disadvantage of the state variable approach is that since its mathematical format is general, there may be easier ways to formulate models of specific phenomena. For example, the storm runoff model presented in Section 4-2(2) could be much simpler to solve by the well known unit hydrograph concept for some simple cases instead of going through the more elaborate procedure of matrix representation.

6-4. On Systems Optimization by Stochastic Dynamic Programming.

The principal advantage of the combined stochastic state variable dynamic programming is that the stochastic nature of the water resource system inputs is embedded directly within the optimization procedure without requiring computationally expensive system simulations or discretized probability distributions. Using the powerful state variable modeling approach, the stochastic model of the system inputs is made a part of the mathematical model of system instead of being external to it as is required by other procedures. The optimization procedure thus accounts for the uncertainties in the system inputs which are truly stochastic in character. The methodology allows for the synthesis of the theory of Markov chains with the dynamic programming in a practical way so that the tradeoffs between system optimization and risk of failure may be evaluated.

As is to be expected, some problems are encountered in applying the methodology to a practical system. The first is the so-called "negative flow" problem. Whenever a normal distribution is used for the flows, there is a finite chance of having a negative inflow. Since this is clearly not a natural behavior, some adjustment must be made to preclude negative flows. The procedure adopted is to distribute the probability of negative inflows over the positive inflow range of the probability distribution by proportion to the positive probability ordinate values.

A second problem occurs when the standard deviation of the output state variable is much smaller than the intervals into which this variable is discretized for dynamic programming purposes. The situation can arise where the probability distributions resulting from two different decisions can both lie completely within one output state interval.

It is not easy to differentiate between the values of two decisions involved. This problem is overcome by defining a set of 5 subintervals within the state interval in question and using linear interpolation to assign a value to each subinterval for the expected value computations. In general, however, it is considered desirable to discretise the state space into intervals no larger than twice the standard deviation of the inflow distribution at that stage.

6-5. Recommendations for Future Studies.

It has been stated in the beginning of this report that the present investigation is only to propose and develop an analytical scheme for the optimization of water resources systems with stochastic inputs. Future research is therefore needed on the application of the proposed scheme to practical water resources projects and on the modification and extension of the scheme as a result of the future applied research.

The immediate extension of the study on stochastic modeling of annual streamflows would be a development of seasonal multisite streamflow simulation models. Another area of further research is the investigation of the combined use of regional and at-site hydrologic data to reduce the parameter uncertainty.

For the monthly streamflow generation, adequate algorithms should be developed and the results be interpreted also from a Bayesian point of view. The algorithms for the conventional regression model and the new algorithms should be compared numerically to see whether accounting for sampling errors would introduce any significant changes in the generated streamflows.

More applications of the streamflow generation techniques along with the simulation of water resources systems to various real-life situations would be highly desirable so as to better assess their potential.

It should be noted that the simultaneous optimization and Markov chain procedure opens up a new field in the treatment of the relationship between the system optimization and the risk of failure. Important questions which remain to be investigated include: How sensitive is the risk of failure to changes in the policy near the optimum? How can other kinds of risk such as the failure to meet a required decision be incorporated into the analysis? How sensitive are the steady state probability distributions to the formulation of the model for the stochastic inputs?

The direct application of the stochastic state variable dynamic programming to multiple unit systems is limited by computational requirements. It is considered that these limitations could be overcome to a large extent by using the steady state probability distributions for each single unit as a means of isolating individual units for analysis. The optimum for the combined system would be approached by successive approximations, each unit being optimized using the previously derived policies for the others.

The discrete differential dynamic programming, DDDP, (Heidari, Chow, Kokotovic and Meredith, 1971; Heidari, Chow and Meredith, 1971; Chow and Cortes-Rivera, 1974) is a powerful optimization technique which overcomes many of the computational limitations of conventional dynamic programming. The incorporation of the stochastic state variable model into this procedure would allow DDDP to consider stochastic inputs.

The above recommendations and suggestions are made in the realization that this study has not investigated all of the possible improvements and extensions that can be made to the proposed approach and its applications to actual systems.

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APPENDIX. PHYSICAL AND ECONOMIC DATA FOR WATASHEAMU DAM AND RESERVOIR

The data in this appendix were derived from Butcher and Fordham (1970).

A-1. Evaporation.

Evaporation is accounted for using the loss coefficient "a" in Eq. (4-20). For each storage the volume of evaporation loss was calculated as the product of the reservoir area and an estimated depth of evaporation as shown in the following:

Month	Jan	Feb	Mar	Apr	May	June	July	Aug	Sept	Oct	Nov	Dec
Evap.	0.036	0.072	0.157	0.215	0.322	0.502	0.716	0.716	0.502	0.301	0.143	0.072

where the evaporation values are expressed in feet/month.

The reservoir area as a function of storage was calculated using Eq. (A-1). (A-1).

$$\text{area} = 89.67 + 0.01559x(k) - 0.3619 \times 10^{-7}x^2(k) \qquad \text{(A-1)}$$

where area is in acres and x(k) is storage in acre-feet.

A-2. Energy.

To find the energy generated by a given release, the average elevation of reservoir during the stage, $\bar{h}$, is needed. The storage-elevation functions used for this purpose are given in Eqs. (A-1) and (A-2).

$$h = 58.2 + 1.83 \times 10^{-3}x(k) - 4.64 \times 10^{-9}x^2(k)$$
$$30{,}000 \leq x(k) \leq 160{,}000 \qquad \text{(A-2)}$$

$$h = 4.06 + 4.76 \times 10^{-3}x(k) - 5.19 \times 10^{-8}x^2(k) + 2.83 \times 10^{-13}x^3(k)$$
$$- 5.57 \times 10^{-19}x^4(k) \qquad \text{(A-3)}$$
$$x(k)<30{,}000;\ x(k) > 160{,}000$$

where h = elevation in feet. The energy in Kw-hr is computed using Eq. (A-4)

$$\text{energy} = 0.76808\ u(k)\bar{h} \tag{A-4}$$

where u(k) is the release in acre-feet/month. The price of energy is taken as 0.71 cents/kw-hr. Energy production ceases when $\bar{h}$ is less than 138 feet.

A-3. Irrigation.

Irrigation water is assumed to be sold at $2.50 per acre-foot up to a specified maximum volume in each irrigation month as follows:

Month.	May	June	July	Aug	Sept	Oct
Max Vol.	25	30	45	45	45	20

where maximum volume of irrigation sales in each month is expressed in KAF.

A-4. Flow Statistics.

The mean, standard deviation and serial correlation with the previous month of the inflows in each month are shown below.

Month	Jan	Feb	Mar	Apr	May	June
Mean	9948	10908	14630	35420	67794	58406
St. Dev.	6865	7553	5909	13120	23760	27720
Ser. Corr.	0.754	0.452	0.237	0.355	0.500	0.729
Month	**July**	**Aug**	**Sept**	**Oct**	**Nov**	**Dec**
Mean	23086	8251	5205	4858	8078	12365
St. Dev.	16670	4511	2336	1534	11460	16090
Ser. Corr.	0.934	0.920	0.955	0.802	0.211	0.669

where flow statistics are in acre-feet per month.

A-5. Storage Range and Discretisation.

In some months, flood storage is reserved at the top of the reservoir as follows:

Month	Nov	Dec	Jan	Feb	Mar	Apr
Flood Reservation	35	65	65	55	45	35

where flood reservation is expressed in thousands of acre-feet.

The total storage below the spillway is 160 KAF. 1.1 KAF of dead storage exist below the lowest outlet point in the dam. The state space for storage was made up of 10 KAF intervals except at the bottom where the first interval is from 1.1 KAF to 5 KAF.