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ABSTRACT

SPATIALLY VARIED OPEN-CHANNEL FLOW EQUATIONS

Recent development and improvement in numerical techniques and computer capability enables more accurate numerical solutions of spatially varied flow problems such as various phases of urban storm runoffs. Consequently, it is desirable to re-examine fundamentally the competency of the flow equations used in solving unsteady spatially varied flow problems. To achieve this goal, the continuity, momentum, and energy equations for unsteady nonuniform flow of an incompressible viscous nonhomogeneous fluid with lateral flow into or leaving a channel of arbitrary geometry in cross section and alignment are formulated in integral form for a cross section by using the Leibnitz rule. The resulted equations are then transformed into one-dimensional form by introducing the necessary correction factors and these equations can be regarded as the unified open-channel flow equations for incompressible fluids. The flow represented by these equations can be turbulent or laminar, rotational or irrotational, steady or unsteady, uniform or nonuniform, gradually or rapidly varied, subcritical or supercritical, with or without spatially and temporally variable lateral discharge. Flow equations for certain special cases are deduced from the derived general equations for the convenience of possible practical uses. Conventionally used various equations for open-channel flows are shown to be simplifications and approximations of special cases of the general equations. The inherent difference between the flow equations derived based on the energy and momentum concepts is discussed. Particular emphasis is given to the differences among the energy dissipation coefficient, the frictional resistance coefficient, and the total-head loss coefficient. Common hydraulic practice of using the Chezy, Manning, or Weisbach formulas to evaluate the dissipated energy gradient or the friction slope is only an approximation.

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SPATIALLY VARIED OPEN-CHANNEL FLOW EQUATIONS

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KEYWORDS—*energy equation/ flow resistance/ fluid mechanics/ homogeneous fluid/ hydraulics/ momentum equation/ nonhomogeneous fluid/ nonuniform flow/ *open-channel flow/ *spatially varied flow/ unsteady flow/ water flow

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NOTATION

- A = area of channel cross section;
- B = free surface width of channel cross section;
- D = Hydraulic depth;
- F = body force per unit mass;
- $\mathbb{F} = V_1/\sqrt{gD}$, Froude number of channel flow;
- f = Weisbach resistance coefficient given in Moody Diagram;
- f_e = energy dissipation coefficient (Eq. 112);
- f_H = total-head loss coefficient (Eq. 113);
- f_f = frictional resistance coefficient (Eq. 111);
- g = gravitational acceleration;
- H = approximate total head as defined in Eq. 101;
- H_B = true cross sectional total head of flow measured in terms of γ_a (Eq. 48);
- H_C = a reference head as defined in Eq. 50;
- $H_L = (U^2/2g) + (P_L/\gamma)$, piezometric head, with respect to horizontal reference datum, of lateral flow when joining channel flow;
- H_p = cross-sectional average piezometric head;
- h = depth of flow, measured along y-direction;
- i, j = orthogonal coordinate directions, i, j = 1, 2, 3;
- K = piezometric pressure correction factor or local potential energy flux correction factor (Eq. 25);
- K' = ambient piezometric pressure correction factor (Eqs. 26 and 33);
- k = direction of the normal of A;
- N = directional normal of a surface, positive outward;
- P = local piezometric pressure with respect to channel bottom (Eq. 23);
also, wetted perimeter;

$P_L = P_L + [\gamma(y \cos\theta + z_b)]_L$, piezometric pressure with respect to horizontal reference datum of lateral flow when joining channel flow;

$P_o = P + \gamma z_b$, local piezometric pressure with respect to horizontal reference datum;

\bar{p} = local temporal mean pressure intensity;

p_L = pressure intensity of lateral flow when joining channel flow;

q = rate of lateral flow per unit length of σ ;

R = hydraulic radius;

$Re = \rho V_L R / \mu$, Reynolds number of channel flow;

r = normal displacement of σ with respect to space or time projected on a plane parallel to A. positive outward;

S_e = dissipated energy gradient (Eq. 54);

S_f = friction slope (Eq. 27);

S_H = total head gradient (Eq. 102);

$S_o = -\partial z_b / \partial x$, channel bottom slope;

T_{ij} = force due to internal stresses τ_{ij} acting on A (Eqs. 29 and 34);

t = time;

U = velocity of lateral flow when joining channel flow;

\bar{u}_i = local temporal mean velocity component along x_i -direction;

u'_i = turbulent fluctuation of the i -th component of local velocity with respect to \bar{u}_i ;

\bar{V} = local temporal mean velocity vector;

V_i = i -th component of channel flow velocity averaged over cross section; (Eq. 8);

W = a symbol referred to conservative work done by internal and surface stresses as defined in Eq. 51;

x = longitudinal coordinate along channel bottom direction;

x_i = coordinate along i -th direction;

y = coordinate perpendicular to x on vertical plane;

z_b = elevation of channel bottom with respect to a horizontal reference datum;

α = convective kinetic energy flux correction factor (Eqs. 44 and 52);

β = momentum flux correction factor (Eqs. 24 and 32);

β' = local kinetic energy flux correction factor (Eq. 43);

γ = specific weight of fluid;

$\gamma_a = \rho_a g$;

ζ = unsteady pressure correction factor (Eq. 46);

η = convective potential energy flux correction factor (Eqs. 45 and 53);

θ = angle between channel bottom and horizontal plane;

λ = density correction factor (Eqs. 9 and 31);

μ = dynamic viscosity of fluid;

ρ = mass density of fluid;

ρ_a = cross-sectional average mass density (Eq. 6);

σ = perimeter bounding cross sectional area A; and

τ_{ij} = shear stresses for $i \neq j$, normal stresses for $i = j$, (Eq. 15).

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I. INTRODUCTION

Spatially varied open-channel flow can be defined as free surface flow in a channel with its discharge varying along the flow direction. In urban storm drainage systems, problems of sheet flow under rainfall or with infiltration, flow into road or roof gutters, and regulator overflow from sewers have been solved by using simplified approximate forms of one-dimensional equations for spatially varied flow. Other examples are routing of floods with lateral flows and side channel spillways or weirs. By making certain assumptions the simplified one-dimensional spatially varied flow equations can then be solved by stepwise integration for steady flow (6)^{*} and by the method of characteristics or finite-difference schemes for unsteady flow. Discussions and references on excellent previous studies on solving the quasi-linear hyperbolic partial differential equations for unsteady spatially varied flow can be found elsewhere (1, 4, 13, 19, 25, 27).

However, with the recent rapid progress in computer capability and numerical techniques, the possibility of solving more accurately open-channel flow problems is increasingly promising. Consequently, there is a renewed interest in critically examining the approximations involved due to the assumptions so that the accuracy of the numerical solutions would be compatible with the approximations involved in the equations. For example, conventionally used simplified spatially varied and other types of open-channel flow equations are actually for homogeneous fluids in two dimensional or prismatic channels. For polluted sewer flow, stratified flow in channels, or for flow in estuaries, such conditions often cannot be satisfied and the approximation involved may not be negligible for accurate numerical solutions.

* Numerals in parentheses refer to corresponding items in References.

Moreover, for highly nonuniform or unsteady flows, the assumptions on hydrostatic pressure and velocity distributions and resistance coefficient are often questionable. Hence, further fundamental considerations on spatially varied open-channel flow equations appear to be appropriate and timely, especially in view of today's increasing sophistication in engineering designs and stringency in available money to support such projects.

The present study is an attempt to formulate the continuity, momentum, and energy equations for an unsteady spatially varied free surface flow of an incompressible viscous nonhomogeneous fluid in a channel of arbitrary cross-sectional and alignment geometry by integrating the Reynolds equation of motion over a cross-sectional area of the channel. The resulted equations are then represented in one-dimensional form and can be considered as unified open-channel flow equations for incompressible fluids. Flow equations for special cases can be deduced from the derived equations. By careful examination of the resulted equations, the restrictions involved in the conventionally used flow equations, the physical significance of certain terms usually neglected, and the difference between momentum and energy approaches can be revealed. The existence of various "resistance" coefficients is also discussed. Hopefully, through the formulation and discussion of the flow equations the present study will provide some new information on the fundamentals of one-dimensional open-channel flow equations. As the present study is emphasized on the formulation of the equations and examination of the physical characteristics of the terms involved, no attempt is made to investigate the numerical methods to obtain solutions of the derived equations with any specified initial and boundary conditions.

II. RELATED PREVIOUS STUDIES

All but two of the early investigations related to the present subject on flow equations formulated were for incompressible homogeneous fluid in two-dimensional or prismatic channels and can be grouped into two categories: (a) derivation of one-dimensional equations for open-channel flow without lateral discharge by integration of equations of motion; and (b) one-dimensional formulation for spatially varied flow. To the knowledge of the writer no derivation of one-dimensional equations for nonhomogeneous fluids has been published.

In the first category of open-channel flow without lateral discharge, much has been done on formulation of the flow equations for incompressible homogeneous fluids. European hydraulicians, by using one-dimensional approach, derived various forms of open-channel flow equations before the turn of the century (6). Keulegan (16) was probably the first one to formally report a fundamental derivation of one-dimensional flow equations by integrating the equations of hydrodynamics over a control volume for a steady flow. He also showed that the one-dimensional dynamic equations can be derived by using either the momentum or the energy approaches. Rouse and McNown (22) vividly explained the difference between these two approaches. Farrell (10) further extended the formulation to an unsteady case.

In the second category of formulation of spatially varied flow equations by using the one-dimensional approach, Hinds (14) investigated the steady flow of water from side-channel spillways by neglecting frictional resistance. Favre (11) and Camp (3) each included frictional resistance in their studies. Beij (2) investigated the case of flow in roof gutters.

Keulegan (17) extended the problem to include certain unsteadiness effects in two-dimensional flow. Li (18) analyzed the case of steady flow in a trapezoidal channel with uniform lateral inflow, first neglecting the friction resistance then including it by using Chezy's formula with constant resistance coefficient. Traditionally, as in all the previously mentioned studies on lateral inflow the momentum approach is adopted; whereas for the case of lateral outflow the energy approach is used. Forchheimer (12) and De Marchi (8) are believed to be among the first to adopt the energy approach to derive one-dimensional flow equations with arbitrary assumptions on the total head or its gradient for the case of steady flow with decreasing discharge. Other early studies have been cited by Chow (6).

Later Iwagaki (15) adopted a pseudo-one-dimensional approach by expressing velocities in integrals over the depth to formulate the unsteady two-dimensional flow continuity and momentum equations. He then assumed the flow to be laminar and neglected the effect on momentum flux due to lateral inflow to solve for the water surface profile. Stoker (23) following strictly the one-dimensional approach, derived a more elaborate momentum equation for unsteady spatially varied flow, allowing variation of channel cross sections, and carefully avoided discussion on its application to lateral inflow or outflow cases. Chow (7) further included energy approach to derive one-dimensional unsteady spatially varied flow equations. Yen and Wenzel (28) accounted for the variations of pressure and velocity over a cross section to derive, by considering the momentum as well as energy of a control volume bounded by two cross sections, one-dimensional momentum and energy equations for steady spatially varied flow. They not only pointed out certain terms in the dynamic equations which were previously neglected, but also showed clearly the difference

between the momentum and energy approaches. They further pointed out that the traditional restriction of using momentum approach for lateral inflow and energy approach for lateral outflow is unnecessary.

Another school of researchers investigated the problem of spatially varied flow from the view point of rainfall-runoff relationships [see e.g., Chen and Chow (4)]. In this case the problem is further complicated by the discrete nature of the lateral input. However, most of these studies considered the unsteady flow case.

Only recently that the dynamic equations for unsteady spatially varied flow of incompressible homogeneous fluids have been formulated from a more fundamental viewpoint. Chen and Chow (4, 5) integrated Navier-Stokes equation over the depth of flow to obtain a one-dimensional momentum equation. However, because of certain assumptions involved in shear stresses and boundary conditions related to the lateral flow, their result is not sufficiently general. Strelkoff (24), on the other hand, gave an admirable derivation by integrating the equation of motion and the corresponding energy and continuity equations over a control volume bounded by two cross sections. However, instead of expressing each of the terms involved explicitly he lumped certain terms into a turbulent correction coefficient and a pressure correction coefficient in his resulted one-dimensional equations. He then proceeded to discuss the relative magnitude of these coefficients and demonstrated the differences between momentum and energy approaches for certain special cases. He, like Yen and Wenzel (28), pointed out the restriction of using momentum approach for lateral inflow and energy approach for lateral outflow is unnecessary. However, by using the turbulence and pressure coefficients instead of expressing explicitly

the terms in the flow equations, not only certain physical insight on the effects of unsteadiness and nonuniformity to the flow is lost, but also that the resulted equations cannot readily be reduced to special cases for application without complicated mathematical manipulations.

Thus it is clear that all the related previous studies on formulation of open-channel flow equations, with the exception of Strelkoff's, are either for special cases or with certain limitations or assumptions. The limitations or assumptions are usually on the geometry of the channel, the pressure and velocity distributions and the shear stresses and turbulence of the flow; in other words, the effects of nonuniformity and unsteadiness and the selection of resistance coefficient. Further limitations are also imposed on the characteristics of the lateral flow and the homogeneity of the fluid. Consequently, none of the previously formulated equations can be considered as a general unified open-channel flow equation for an incompressible fluid.

III. CONTINUITY EQUATIONS

The channel under consideration as shown schematically in Fig. 1 has no limitation on geometry of cross section or alignment. The liquid flowing in the channel is incompressible, viscous, and nonhomogeneous; i.e., both the density ρ and dynamic viscosity μ of the fluid may vary from point to point, and as a result of convection of the flow, from instant to instant at any point, but the density and viscosity of an infinitesimal incompressible element remain constant with time. The lateral flow through the free surface such as rainfall or evaporation and through the permeable boundary such as infiltration or seepage inflow can vary with both space and time and their fluid properties can also be nonhomogeneous but necessarily incompressible.

For a point (more precisely, an infinitesimal unit volume) in a turbulent unsteady flow the continuity equation is

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho \bar{u}_i}{\partial x_i} = 0 \quad (1)$$

or, since the fluid is incompressible (20) — not necessarily homogeneous,

$$\frac{\partial \bar{u}_i}{\partial x_i} = 0 \quad (2)$$

in which t is time; u_i is the local velocity component of the fluid along the x_i direction; and the bar represents temporal averaging over turbulent fluctuation. Repetition of the subscript i in a term implies summation over the three possible orthogonal coordinate directions, $i = 1, 2, 3$.

For a channel cross section with an area A (Fig. 2), the boundary condition is-

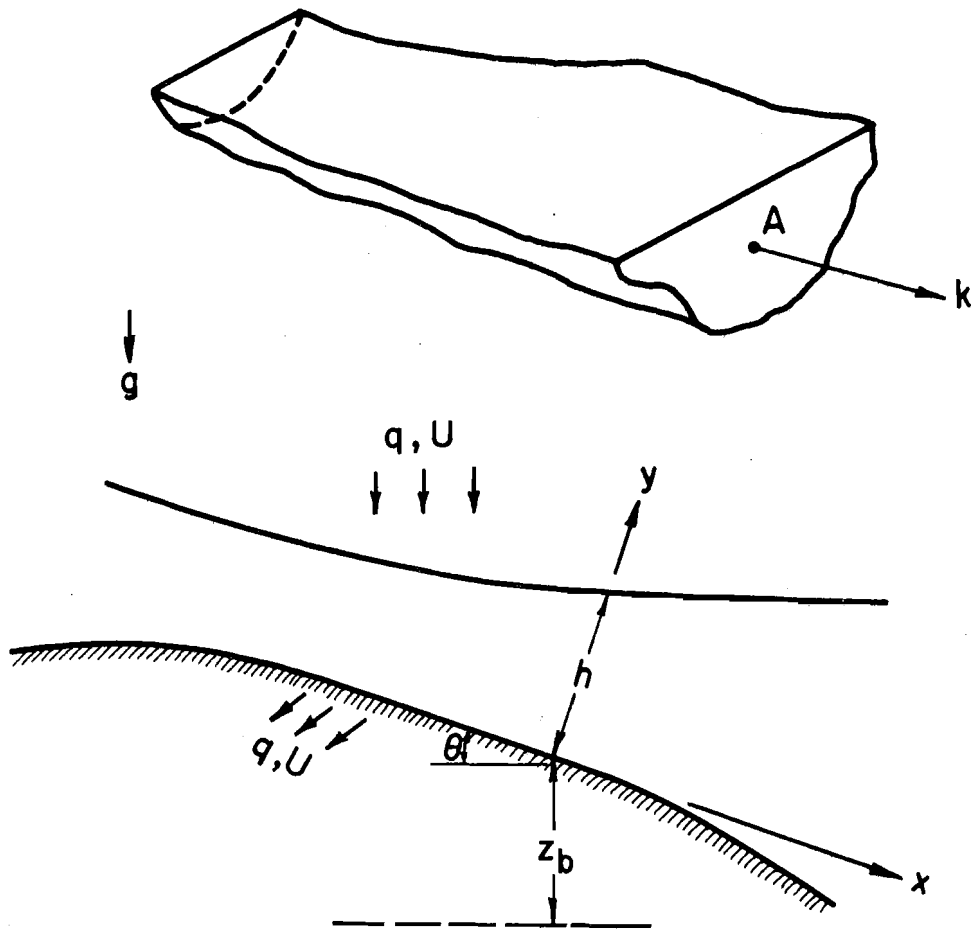


Fig. 1. Schematic Drawing of Spatially Varied Channel Flow

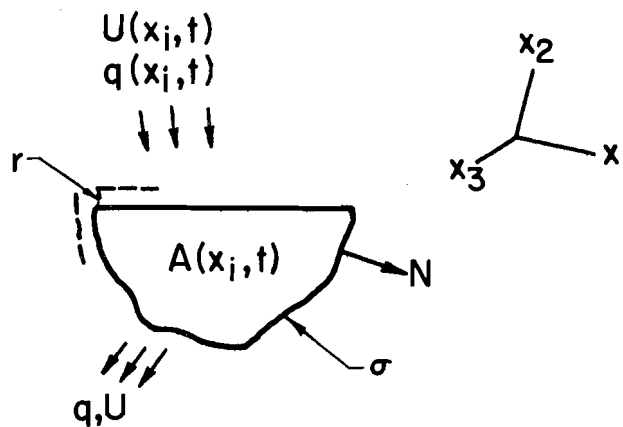


Fig. 2. Definition Sketch of a Channel Cross Section

$$\left[G \frac{\partial A}{\partial t} + G \bar{u}_i \frac{\partial A}{\partial x_i} \right]_{\sigma} = \int_{\sigma} G q \, d\sigma \quad (3)$$

in which σ is the perimeter bounding A ; G represents any continuous scalar quantity under consideration at σ , e.g., $G = \rho$ for mass conservation; and q is the rate of lateral flow into the channel per unit length of σ , having a dimension of length per unit time and being positive for inflow. Note that although q has a dimension of velocity it is not necessarily equal to any component of the velocity of the lateral flow which may be of discrete nature such as raindrops. It should also be noted here that the orientation of the coordinate system x_i can be arbitrarily chosen and in general A is a function of x_i as well as of t . However, in practice as a matter of convenience, it is usually so chosen that the direction normal of A coincides with one of the x_i 's.

The continuity equation for a channel cross section can be obtained by integrating the point continuity equation over A . By applying the Leibnitz rule, integration of Eq. 1 yields

$$\int_A \left(\frac{\partial \rho}{\partial t} + \frac{\partial \rho \bar{u}_i}{\partial x_i} \right) dA = \frac{\partial}{\partial t} \int_A \rho dA - \left[\rho \frac{\partial A}{\partial t} \right]_{\sigma} + \frac{\partial}{\partial x_i} \int_A \rho \bar{u}_i dA - \left[\rho u_i \frac{\partial A}{\partial x_i} \right]_{\sigma} = 0 \quad (4)$$

Or, with the boundary condition, Eq. 3 with $G = \rho$,

$$\frac{\partial}{\partial t} \int_A \rho \, dA + \frac{\partial}{\partial x_i} \int_A \rho \bar{u}_i \, dA - \int_{\sigma} \rho q \, d\sigma = 0 \quad (5)$$

which is the continuity equation in integral form for a channel cross section.

If the cross-sectional mean fluid density ρ_a is defined as

$$\rho_a = \frac{1}{A} \int_A \rho \, dA \quad (6)$$

then Eq. 5 can be written in one-dimensional form as

$$\frac{\partial}{\partial t} (\rho_a A) + \frac{\partial}{\partial x_i} (\lambda_{ki} \rho_a A V_i) = \int_{\sigma} \rho q \, d\sigma \quad (7)$$

in which the mean velocity component of the cross section, V_i , is defined as

$$V_i = \frac{1}{A} \int_A \bar{u}_i \, dA \quad (8)$$

and the density correction factor λ is defined as

$$\lambda_{ki} \rho_a A V_i = \int_A \rho \bar{u}_i \, dA \quad (9)$$

where k represents the direction of the normal of A which does not necessarily coincide with one of the coordinates ($i = 1, 2, 3$). Note that the subscripts of λ and other similar correction factors to be defined later are for the purpose of indicating their directional nature and repetition of these subscripts does not imply summation. For homogeneous fluids

$$\lambda_{ki} = 1.$$

Likewise, integration of Eq. 2 yields

$$\int_A \frac{\partial \bar{u}_i}{\partial x_i} \, dA = \frac{\partial}{\partial x_i} \int_A \bar{u}_i \, dA - [\bar{u}_i \frac{\partial A}{\partial x_i}]_{\sigma} - [\frac{\partial A}{\partial t}]_{\sigma} + [\frac{\partial A}{\partial t}]_{\sigma} = 0 \quad (10)$$

Hence, with the aid of Eq. 3, Eq. 10 yields

$$\frac{\partial A}{\partial t} + \frac{\partial (AV_i)}{\partial x_i} = \int_{\sigma} q \, d\sigma \quad (11)$$

If the channel cross section is taken such that its normal is along the direction $i = 1$, then, with $\lambda_{11} = \lambda$, the continuity equations in one-dimensional form, Eqs. 7 and 11, can be simplified as

$$\frac{\partial}{\partial t} (\rho_a A) + \frac{\partial}{\partial x} (\lambda \rho_a AV_1) = \int_{\sigma} \rho q \, d\sigma \quad (12)$$

and

$$\frac{\partial A}{\partial t} + \frac{\partial AV_1}{\partial x} = \int_{\sigma} q \, d\sigma \quad (13)$$

IV. MOMENTUM EQUATIONS

IV-1. Momentum Equation in Integral Form

For the channel and flow conditions described in the first paragraph of the preceding chapter, the momentum equation for a cross section can be obtained by integrating the Reynolds equation over the cross sectional area A. For a point in a turbulent flow the Reynolds equation of motion is

$$\rho \frac{\partial \bar{u}_i}{\partial t} + \rho \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} = \rho F_i - \frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} \quad (14)$$

in which F_i is the component of the body force per unit mass along the x_i direction; p is the local pressure intensity; and

$$\tau_{ij} = \mu \left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right) - \rho \overline{u'_i u'_j} \quad (15)$$

in which u' is the turbulent fluctuation of u with respect to \bar{u} .

With the aid of Eq. 3 and the Leibnitz rule, integration of the left side of Eq. 14 over A yields

$$\begin{aligned} \int_A \rho \left(\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} \right) dA &= \int_A \left(\frac{\partial \rho \bar{u}_i}{\partial t} + \frac{\partial \rho \bar{u}_i \bar{u}_j}{\partial x_j} \right) dA \\ &= \frac{\partial}{\partial t} \int_A \rho \bar{u}_i dA - \left[\rho \bar{u}_i \frac{\partial A}{\partial t} \right]_{\sigma} + \frac{\partial}{\partial x_j} \int_A \rho \bar{u}_i \bar{u}_j dA - \left[\rho \bar{u}_i \bar{u}_j \frac{\partial A}{\partial x_j} \right]_{\sigma} \\ &= \frac{\partial}{\partial t} \int_A \rho \bar{u}_i dA + \frac{\partial}{\partial x_j} \int_A \rho \bar{u}_i \bar{u}_j dA - \int_{\sigma} \rho u_i q d\sigma \end{aligned} \quad (16)$$

in which U_i = the component of velocity of the lateral flow along x_i direction at the instant of joining the channel flow.

Likewise,

$$\int_A \frac{\partial \bar{p}}{\partial x_i} dA = \frac{\partial}{\partial x_i} \int_A \bar{p} dA - [\bar{p} \frac{\partial A}{\partial x_i}]_{\sigma} = \frac{\partial}{\partial x_i} \int_A \bar{p} dA + \int_{\sigma} \bar{p} \frac{\partial r}{\partial x_i} d\sigma \quad (17)$$

in which r is the displacement of the boundary of A with respect to space or time along the direction of the normal N of σ projected on a plane parallel to A and N_i is the i -th directional cosine of the normal N of σ , N being positive outward (Fig. 2).

Furthermore, integration of the last term in Eq. 14 yields

$$\int_A \frac{\partial \tau_{ij}}{\partial x_j} dA = \frac{\partial}{\partial x_j} \int_A \tau_{ij} dA - [\tau_{ij} \frac{\partial A}{\partial x_j}]_{\sigma} = \frac{\partial}{\partial x_j} \int_A \tau_{ij} dA + \int_{\sigma} [\tau_{ij}]_{\sigma} N_j d\sigma \quad (18)$$

Therefore, by integrating Eq. 14 over A and with the aid of Eqs. 16, 17, and 18, the momentum equation for a channel cross section is

$$\begin{aligned} \frac{\partial}{\partial t} \int_A \rho \bar{u}_i dA + \frac{\partial}{\partial x_j} \int_A \rho \bar{u}_i \bar{u}_j dA - \int_{\sigma} \rho U_i q d\sigma &= \int_A \rho F_i dA - \frac{\partial}{\partial x_i} \int_A \bar{p} dA \\ - \int_{\sigma} \bar{p} \frac{\partial r}{\partial x_i} d\sigma + \frac{\partial}{\partial x_j} \int_A \tau_{ij} dA + \int_{\sigma} [\tau_{ij}]_{\sigma} N_j d\sigma & \end{aligned} \quad (19)$$

In a gravitational field the body force can be considered derivable from a potential such that

$$F_i = -g \frac{\partial}{\partial x_i} (z_b + y \cos \theta) = g S_{oi} - \frac{\partial}{\partial x_i} (gy \cos \theta) \quad (20)$$

in which g is the gravitational acceleration; z_b is the elevation of channel bed, i.e., the lowest point in the cross section, measured from a reference datum along the gravitational direction; y is the distance normal to the channel bed on a vertical plane measured from the lowest point in the cross section; θ is the angle between the channel bed and a horizontal plane (Fig. 1); and $S_{oi} = -\partial z_b / \partial x_i$ is the slope of channel bed along x_i direction. Hence,

$$\begin{aligned} \int_A \rho F_i dA &= g S_{oi} \int_A \rho dA - \frac{\partial}{\partial x_i} \int_A \rho g y \cos \theta dA - \int_{\sigma} \rho g y \cos \theta \frac{\partial r}{\partial x_i} d\sigma \\ &+ g \cos \theta \int_A y \frac{\partial \rho}{\partial x_i} dA \end{aligned} \quad (21)$$

Substitution of Eq. 21 into Eq. 19 yields the following momentum equation for a cross section in a channel of arbitrary geometry having an unsteady nonuniform free surface flow of an incompressible nonhomogeneous viscous fluid in it and with spatially and temporally variable lateral discharge.

$$\begin{aligned} \frac{\partial}{\partial t} \int_A \rho \bar{u}_i dA + \frac{\partial}{\partial x_j} \int_A \rho \bar{u}_i \bar{u}_j dA - \int_{\sigma} \rho U_i q d\sigma &= g S_{oi} \int_A \rho dA \\ - \frac{\partial}{\partial x_i} \int_A P dA - \int_{\sigma} P \frac{\partial r}{\partial x_i} d\sigma + \frac{\partial}{\partial x_j} \int_A \tau_{ij} dA \\ + \int_{\sigma} [\tau_{ij}]_{\sigma} N_j d\sigma + g \cos \theta \int_A y \frac{\partial \rho}{\partial x_i} dA \end{aligned} \quad (22)$$

in which the local piezometric pressure P in reference to the channel bottom is

$$P = \bar{p} + \rho g y \cos \theta \quad (23)$$

The physical meaning of the terms in Eq. 22 is as follows:

For the discharge through A and by referring to the i-th component, the first term represents the time rate of change of momentum flux; the second term is the rate of spatial change of convective momentum flux of the channel flow; the third term is due to the momentum influx of the lateral flow; the first term at the right of the equality sign simply represents the component of the gravitational force; the second term is the rate of spatial change of piezometric pressure acting on A, the third term represents the component of the force due to ambient piezometric pressure acting on the boundary surface; the fourth term is due to the change of deformation stresses of the flow with space; the fifth term is the component of the external surface stresses (such as those due to wind or lateral flow) acting on the boundary; and finally, the sixth term is due to the variation of density with space.

IV-2. Momentum Equation in One-Dimensional Form

In order to transform the momentum equation, Eq. 22, into one-dimensional form, it is necessary to define the following correction coefficients. The momentum flux correction factor, or the Boussinesq coefficient, β_{kij} , for the cross section A can be defined as

$$\beta_{kij} \rho_a V_i V_j A = \int_A \rho \bar{u}_i \bar{u}_j dA \quad (24)$$

The piezometric pressure correction factor K to account for nonhydrostatic pressure distribution is defined as

$$K \rho_a g A h \cos \theta = \int_A P dA \quad (25)$$

in which h is the depth of flow measured above the lowest point in the cross section along y direction normal to the channel bed. If the density ρ does not vary within A and the pressure distribution is hydrostatic, then $K = 1$. By referring to Eqs. 17 and 21, the ambient piezometric pressure correction factor K'_{ki} can be defined as

$$K'_{ki} \rho_a g h \cos\theta \frac{\partial A}{\partial x_i} = - \int_{\sigma} p \frac{\partial r}{\partial x_i} d\sigma \quad (26)$$

Again, if the ambient piezometric pressure acting on the boundary σ is hydrostatically distributed, $K'_{ki} = 1$.

The friction slope is defined as

$$S_{fi} = - \frac{1}{\rho_a g A} \int_{\sigma} [\tau_{ij}]_{\sigma} N_j d\sigma \quad (27)$$

Hence, with the aid of Eqs. 6 to 9 and 24 to 27, Eq. 22 yields the following one-dimensional momentum equation

$$\begin{aligned} & \lambda_{ki} \frac{\partial v_i}{\partial t} + v_i \left(\frac{\partial \lambda_{ki}}{\partial t} - v_j \lambda_{ki} \frac{\partial \lambda_{kj}}{\partial x_i} \right) + v_j \frac{\partial}{\partial x_j} (\beta_{kij} v_i) \\ & + (\beta_{kij} - \lambda_{ki} \lambda_{kj}) \frac{v_i}{\rho_a A} \frac{\partial}{\partial x_j} (\rho_a A v_j) + \frac{1}{\rho_a A} \int_{\sigma} (\lambda_{ki} v_i - U_i) \rho q d\sigma \\ & = g S_{oi} - g S_{fi} - g \frac{\partial}{\partial x_i} (Kh \cos\theta) + (K'_{ki} - K) g h \cos\theta \frac{1}{A} \frac{\partial A}{\partial x_i} \\ & + \frac{1}{\rho_a A} \frac{\partial \Gamma_{ij}}{\partial x_j} - K g h \cos\theta \frac{1}{\rho_a} \frac{\partial \rho_a}{\partial x_i} + \frac{g \cos\theta}{\rho_a A} \int_A y \frac{\partial \rho}{\partial x_i} dA \end{aligned} \quad (28)$$

in which the force due to internal stresses T_{ij} is

$$T_{ij} = \int_A \tau_{ij} dA \quad (29)$$

If the cross section is taken such that $k = i = 1$ and $x_1 = x$ the general channel flow direction, then the derivative of any cross-sectional mean quantity with respect to x_2 or x_3 is equal to zero. Hence, Eq. 28 can be simplified as

$$\begin{aligned} & \lambda \frac{\partial V_1}{\partial t} + V_1 \frac{\partial}{\partial x} (\beta V_1) + (\beta - \lambda^2) \frac{V_1}{\rho_a A} \frac{\partial}{\partial x} (\rho_a A V_1) + (K - K') gh \cos \theta \frac{1}{A} \frac{\partial A}{\partial x} \\ & + g \frac{\partial}{\partial x} (Kh \cos \theta) = g S_o - g S_{fx} + \frac{1}{\rho_a A} \frac{\partial T_{11}}{\partial x} + \frac{1}{\rho_a A} \int_{\sigma} (U_1 - \lambda V_1) \rho q d\sigma \\ & - V_1 \left(\frac{\partial \lambda}{\partial t} - V_1 \lambda \frac{\partial \lambda}{\partial x} \right) - Kgh \cos \theta \frac{1}{\rho_a} \frac{\partial \rho_a}{\partial x} + \frac{g \cos \theta}{\rho_a A} \int_A y \frac{\partial \rho}{\partial x} dA \end{aligned} \quad (30)$$

in which

$$\lambda = \lambda_{11} = \frac{1}{\rho_a A V_1} \int_A \rho \bar{u}_1 dA \quad (31)$$

$$\beta = \beta_{111} = \frac{1}{\rho_a V_1^2 A} \int_A \rho \bar{u}_1^2 dA \quad (32)$$

$$K' = K'_{11} = \frac{-1}{\rho_a gh \cos \theta} \int_{\sigma} P \frac{\partial r}{\partial x} d\sigma \quad (33)$$

$$T_{11} = \int_A \left(2 \frac{\partial \bar{u}_1}{\partial x} - \overline{\rho u_1'^2} \right) dA \quad (34)$$

and $S_o = S_{ox} = \sin \theta$.

Equation 30 can be regarded as the general one-dimensional momentum equation for open-channel flow of an incompressible fluid. The first term in Eq. 30 is due to local acceleration of the flow. The second and third terms represent the effect of convective acceleration, due partly to the variation of channel geometry and partly to the spatial variation of the density of the fluid. The fourth and fifth terms are due to action of the piezometric pressure. The first term at the right of the equality sign is the effect of gravity. The second term represents the frictional force acting on the boundary, including not only that part on the solid boundary but also the stresses acting on the free surface such as those due to wind or rainfall. The third term is the spatial change of the longitudinal component of the total internal stresses acting on the cross section, excluding the pressure, but including the effect of change of viscosity of the nonhomogeneous fluid, and this term is conventionally neglected. The fourth term at the right represents the effect of the lateral flow and the last three terms are due to nonhomogeneous density of the fluid. Conventionally used various forms of open-channel flow equations derived based on the momentum concept are simplifications and approximations of Eq. 30. Further discussion on special cases of Eq. 30 will be given later.

V. ENERGY EQUATIONS

V-1. Energy Equation in Integral Form

The energy equation of mean motion for a point in a turbulent flow field of an incompressible nonhomogeneous viscous fluid is

$$\frac{\partial}{\partial t} \left(\rho \frac{\bar{V}^2}{2} \right) + \frac{\partial}{\partial x_j} \left(\rho \frac{\bar{V}^2}{2} \bar{u}_j \right) = \rho \bar{u}_i F_i - \bar{u}_i \frac{\partial \bar{p}}{\partial x_i} + \bar{u}_i \frac{\partial \tau_{ij}}{\partial x_j} \quad (35)$$

in which

$$\bar{V}^2 = \bar{u}_i \bar{u}_i \quad (36)$$

and other symbols have been defined previously.

With the aid of the boundary condition, Eq. 3, with $G = \rho \bar{V}^2 / 2$ and the Leibnitz rule, integration of the two terms at the left of Eq. 35 yields

$$\begin{aligned} & \int_A \left[\frac{\partial}{\partial t} \left(\rho \frac{\bar{V}^2}{2} \right) + \frac{\partial}{\partial x_j} \left(\rho \frac{\bar{V}^2}{2} \bar{u}_j \right) \right] dA \\ &= \frac{\partial}{\partial t} \int_A \rho \frac{\bar{V}^2}{2} dA - \left[\rho \frac{\bar{V}^2}{2} \frac{\partial A}{\partial t} \right]_{\sigma} + \frac{\partial}{\partial x_j} \int_A \rho \frac{\bar{V}^2}{2} \bar{u}_j dA - \left[\rho \frac{\bar{V}^2}{2} \bar{u}_j \frac{\partial A}{\partial x_j} \right]_{\sigma} \\ &= \frac{\partial}{\partial t} \int_A \rho \frac{\bar{V}^2}{2} dA + \frac{\partial}{\partial x_j} \int_A \rho \frac{\bar{V}^2}{2} \bar{u}_j dA - \int_{\sigma} \rho \frac{U^2}{2} q d\sigma \end{aligned} \quad (37)$$

in which U is the velocity of the lateral flow when joining the channel flow. Likewise, for the pressure term in Eq. 35, with the aid of the

continuity equation and the Leibnitz rule,

$$\begin{aligned}
\int_A \bar{u}_i \frac{\partial \bar{p}}{\partial x_i} dA &= \int_A \frac{\partial \bar{u}_i \bar{p}}{\partial x_i} dA = \frac{\partial}{\partial x_i} \int_A \bar{p} \bar{u}_i dA - [\bar{p} \bar{u}_i \frac{\partial A}{\partial x_i}]_\sigma \\
&= \frac{\partial}{\partial x_i} \int_A \bar{p} \bar{u}_i dA - [\bar{p} (\frac{\partial A}{\partial t} + \bar{u}_i \frac{\partial A}{\partial x_i})]_\sigma + [\bar{p} \frac{\partial A}{\partial t}]_\sigma \\
&= \frac{\partial}{\partial x_i} \int_A \bar{p} \bar{u}_i dA - \int_\sigma p_L q d\sigma + \frac{\partial}{\partial t} \int_A \bar{p} dA - \int_A \frac{\partial \bar{p}}{\partial t} dA
\end{aligned} \tag{38}$$

in which p_L is the pressure intensity of the lateral flow when joining the channel flow and the last term is due to unsteadiness of the flow.

For the last term in Eq. 35,

$$\begin{aligned}
\int_A \bar{u}_i \frac{\partial \tau_{ij}}{\partial x_j} dA &= \int_A \frac{\partial}{\partial x_j} (\bar{u}_i \tau_{ij}) dA - \int_A \tau_{ij} \frac{\partial \bar{u}_i}{\partial x_j} dA \\
&= \frac{\partial}{\partial x_j} \int_A \bar{u}_i \tau_{ij} dA + \int_\sigma [\bar{u}_i \tau_{ij}]_\sigma N_j d\sigma - \int_A \tau_{ij} \frac{\partial \bar{u}_i}{\partial x_j} dA
\end{aligned} \tag{39}$$

From Eqs. 35, 37, 38, and 39 the energy equation for a channel cross section with area A is

$$\begin{aligned}
\frac{\partial}{\partial t} \int_A \rho \frac{\bar{V}^2}{2} dA + \frac{\partial}{\partial x_j} \int_A \rho \frac{\bar{V}^2}{2} \bar{u}_j dA - \int_\sigma (\rho \frac{U^2}{2} + p_L) q d\sigma \\
= \int_A \rho \bar{u}_i F_i dA - \frac{\partial}{\partial x_i} \int_A \bar{p} \bar{u}_i dA - \frac{\partial}{\partial t} \int_A \bar{p} dA + \int_A \frac{\partial \bar{p}}{\partial t} dA \\
+ \frac{\partial}{\partial x_j} \int_A \bar{u}_i \tau_{ij} dA + \int_\sigma [\bar{u}_i \tau_{ij}]_\sigma N_j d\sigma - \int_A \tau_{ij} \frac{\partial \bar{u}_i}{\partial x_j} dA
\end{aligned} \tag{40}$$

In a gravitational field, with the specific weight of the fluid $\gamma = \rho g$, from Eq. 20 and by using the Leibnitz rule and Eqs. 1 and 3, one obtains

$$\begin{aligned} \int_A \rho \bar{u}_i F_i dA &= - \int_A \frac{\partial}{\partial x_i} [\gamma \bar{u}_i (z_b + y \cos \theta)] dA + \int_A g (z_b + y \cos \theta) \frac{\partial \rho \bar{u}_i}{\partial x_i} dA \\ &= - \frac{\partial}{\partial x_i} \int_A \gamma (z_b + y \cos \theta) \bar{u}_i dA + \int_\sigma [\gamma (z_b + y \cos \theta)]_L q d\sigma - \frac{\partial}{\partial t} \int_A \gamma (z_b + y \cos \theta) dA \end{aligned} \quad (41)$$

since $\partial(z_b + y \cos \theta)/\partial t = 0$. Hence, from Eqs. 40 and 41, the energy equation in integral form over a cross section for a nonuniform unsteady free surface flow of an incompressible nonhomogeneous viscous fluid in a channel of arbitrary geometry and with spatially and temporally variable lateral discharge is

$$\begin{aligned} \frac{\partial}{\partial t} \int_A \left(\rho \frac{\bar{V}^2}{2} + P_o \right) dA + \frac{\partial}{\partial x_j} \int_A \rho \frac{\bar{V}^2}{2} \bar{u}_j dA + \frac{\partial}{\partial x_i} \int_A P_o \bar{u}_i dA - \int_\sigma \left(\rho \frac{U^2}{2} + P_L \right) q d\sigma \\ = \int_A \frac{\partial \bar{p}}{\partial t} dA + \frac{\partial}{\partial x_j} \int_A \bar{u}_i \tau_{ij} dA + \int_\sigma [\bar{u}_i \tau_{ij}]_\sigma N_j d\sigma - \int_A \tau_{ij} \frac{\partial \bar{u}_i}{\partial x_j} dA \end{aligned} \quad (42)$$

in which $P_o = \bar{p} + \gamma(y \cos \theta + z_b) = P + \gamma z_b$ is the local piezometric pressure, and $P_L = p_L + [\gamma(y \cos \theta + z_b)]_L$ is the piezometric pressure of the lateral flow when joining to the channel flow, both referred to the horizontal reference datum $z = 0$.

The physical meaning of the terms in Eq. 42 is as follows: For the discharge through A, the first term is the time rate of change of kinetic and potential energy; the second and the third terms are the rate of spatial change of convective kinetic and potential energy fluxes,

respectively, by the channel flow; and the fourth term is the energy influx due to the lateral flow. The terms on the right of Eq. 42 are as follows: The first is the rate at which work is done due to change of internal pressure intensity over A; the second is the rate of spatial change of work which is done by deformation stresses; the third is due to the work done by external stresses on the boundary; and the fourth and last term is the rate at which work is done by deformation stresses to overcome the velocity gradients, i.e., the rate of energy dissipation. It is interesting to note that the nonhomogeneous nature of the fluid is not expressed explicitly in the energy equation.

V-2. Energy Equation in One-Dimensional Form

To transfer Eq. 40 into one-dimensional form, in addition to the correction factors defined previously, the local kinetic energy flux correction factor β' is defined as

$$\beta' \frac{1}{2} \rho_a \bar{v}_i \bar{v}_i A = \int_A \frac{\rho}{2} \bar{v}^2 dA \quad (43)$$

The convective kinetic energy flux correction factor, or the Coriolis coefficient, α_{kj} , can be defined as

$$\alpha_{kj} \frac{\rho_a}{2} \bar{v}_i \bar{v}_i \bar{v}_j A = \int_A \frac{\rho}{2} \bar{v}^2 \bar{u}_j dA \quad (44)$$

The local potential energy flux correction factor η' is equal to the piezometric pressure correction factor K defined in Eq. 25. The convective potential energy flux correction factor η_{ki} is defined as

$$\eta_{ki} \gamma_a AV_i h \cos \theta = \int_A P \bar{u}_i dA \quad (45)$$

in which $\gamma_a = \rho_a g$ is the cross sectional average specific weight of the fluid. For homogeneous fluid with hydrostatic pressure distribution, $\eta_{ki} = 1$. Finally, the unsteady pressure correction factor ζ is defined as

$$\zeta \gamma_a A \frac{\partial h}{\partial t} = \int_A \frac{\partial p}{\partial t} dA \quad (46)$$

Substitution of Eqs. 43 through 46 and 25 into Eq. 42 and dividing the result by $\rho_a g$ yields the one-dimensional energy equation per unit mass of the incompressible fluid

$$\begin{aligned} \frac{\partial H_B}{\partial t} + \frac{H_B}{\rho_a A} \frac{\partial \rho_a A}{\partial t} + V_i \frac{\partial}{\partial x_i} \left(\alpha_{ki} \frac{1}{2g} V_j V_j + \eta_{ki} h \cos \theta + \lambda_{ki} z_b \right) \\ + \left(\alpha_{ki} \frac{V_j V_j}{2g} + \eta_{ki} h \cos \theta + \lambda_{ki} z_b \right) \frac{1}{\rho_a A} \frac{\partial}{\partial x_i} (\rho_a AV_i) \\ - \frac{1}{\gamma_a A} \int_{\sigma} \left(\rho \frac{U^2}{2} + P_L \right) q d\sigma = \zeta \frac{\partial h}{\partial t} + \frac{1}{\gamma_a A} \left(\frac{\partial}{\partial x_j} \int_A \bar{u}_i \tau_{ij} dA \right. \\ \left. + \int_{\sigma} [\bar{u}_i \tau_{ij}]_{\sigma} N_j d\sigma - \int_A \tau_{ij} \frac{\partial \bar{u}_i}{\partial x_j} dA \right) \end{aligned} \quad (47)$$

in which the total head of the flow H_B measured in terms of the cross sectional averaged specific weight γ_a is

$$H_B = \beta' \frac{V_i V_i}{2g} + Kh \cos \theta + z_b \quad (48)$$

For a flow cross section with its normal along one of the coordinate direction $x_1 = x$, Eq. 47 can be simplified as

$$\begin{aligned} \frac{\partial H_B}{\partial t} + \frac{H_B}{\rho_a A} \frac{\partial \rho_a A}{\partial t} + V_1 \frac{\partial H_C}{\partial x} + \frac{H_C}{\rho_a A} \frac{\partial}{\partial x} (\rho_a A V_1) - \frac{1}{\gamma_a A} \int_{\sigma} (\rho \frac{U^2}{2} + P_L) q \, d\sigma \\ = \zeta \frac{\partial h}{\partial t} - V_1 S_e + \frac{W}{\gamma_a A} \end{aligned} \quad (49)$$

in which

$$H_C = \alpha \frac{V_1 V_1}{2g} + \eta h \cos \theta + \lambda z_b \quad (50)$$

$$W = \frac{\partial}{\partial x} \int_A \bar{u}_i \tau_{i1} \, dA + \int_{\sigma} [\bar{u}_i \tau_{ij}]_{\sigma} N_j \, d\sigma \quad (51)$$

$$\alpha = \alpha_{11} = \frac{1}{\frac{V_1 V_1}{\rho_a \frac{1}{2}} V_1 A} \int_A \frac{\rho}{2} \overline{V^2 u_1} \, dA \quad (52)$$

$$\eta = \eta_{11} = \frac{1}{\gamma_a A V_1 h \cos \theta} \int_A P \bar{u}_1 \, dA \quad (53)$$

$\lambda = \lambda_{11}$ (Eq. 31), and the dissipated energy gradient is

$$S_e = \frac{1}{\gamma_a A V_1} \int_A \tau_{ij} \frac{\partial \bar{u}_i}{\partial x_j} \, dA \quad (54)$$

Equation 49 may be considered as the general one-dimensional energy equation per unit mass for open-channel flow of an incompressible fluid. Like the original equation (Eq. 42) from which it is derived, the first two terms in Eq. 49 represent the time rate of change of total energy; the next two terms account for the convective energy exchange; and

the fifth term is the energy influx from the lateral flow. To the right of the equality sign in Eq. 49, the first term represents the work which is done by the change of internal pressure with respect to time reflected as change of depth with time; the second term accounts for energy dissipation; and the last term includes the rate of spatial change of work which is done by internal deformation stresses and the work done by external stresses on the boundary, respectively.

V-3. One-Dimensional Dynamic Equation from Energy Approach

In hydraulics, a one-dimensional dynamic equation somewhat similar to the momentum equation is often derived based on the energy approach. Unfortunately, because of the similarities in their simplified forms, the equations derived respectively based on the energy and momentum approaches are often misunderstood and misused in solving problems. To facilitate later discussion on this subject, the one-dimensional dynamic equation for incompressible fluids is derived in this Section.

For a channel cross section with its normal along one of the coordinate direction $x_1 = x$ and with a non-erodible bed so that both θ and z_b are independent of time and $\partial z_b / \partial x = -S_o$, Eq. 49, after dividing by V_1 and substitution of Eq. 12 and rearranging, can be simplified as

$$\begin{aligned} & \frac{\beta'}{g} \frac{V_i}{V_1} \frac{\partial V_i}{\partial t} + \frac{V_i V_i}{2gV_1} \frac{\partial \beta'}{\partial t} + \frac{h \cos \theta}{V_1} \frac{\partial K}{\partial t} + \frac{1}{V_1} (K \cos \theta - \zeta) \frac{\partial h}{\partial t} + \frac{\partial}{\partial x} (\eta h \cos \theta) \\ & + \frac{V_i V_i}{2g} \frac{\partial \alpha}{\partial x} + \frac{\alpha V_i}{g} \frac{\partial V_i}{\partial x} + [(\alpha - \lambda \beta') \frac{V_i V_i}{2g} + (\eta - \lambda K) h \cos \theta] \frac{1}{\rho_a AV_1} \frac{\partial \rho_a AV_1}{\partial x} \\ & - (\beta' \frac{V_i V_i}{2g} + K h \cos \theta) \frac{\partial \lambda}{\partial x} = \lambda S_o - S_e + \frac{W}{\gamma_a AV_1} + \frac{1}{\rho_a AV_1} \int_{\sigma} (H_L - H_B) \rho q \, d\sigma \end{aligned}$$

in which $H_L = (U^2/2g) + (P_L/\gamma_L)$ is the local piezometric head, with respect to the horizontal datum $z = 0$, for the lateral flow when joining or leaving the main channel flow. Equation 55 is, not surprisingly, considerably more complicated than any of the conventionally used corresponding equations (6, 7, 28) as at this stage no restrictions are made on Eq. 55 provided the fluid is incompressible.

VI. SPECIAL CASES FOR NONHOMOGENEOUS FLUIDS

The continuity, momentum, and energy equations derived in the preceding three chapters, both in integral and in one-dimensional forms, are general equations for open-channel flows for incompressible fluids. The flow can be turbulent or laminar, rotational or irrotational, steady or unsteady, uniform or nonuniform, gradually or rapidly varied, sub-critical or supercritical, with or without spatially and temporally variable lateral flows. The fluid can be nonhomogeneous, such as problems involving stratified flows, water and thermo pollutions, estuary salinities, and lateral flow of different fluid properties. The channel can be of arbitrary shape and longitudinal and lateral alignment. In fact, the channel bed can be erodible (in this case the terms $(Kh/V_1)(\partial \cos \theta / \partial t)$ and $(1/V_1)(\partial z_b / \partial t)$ should be added to the left side of Eq. 55).

However, complete as they appear, these differential equations cannot readily be used in solving engineering problems because of their highly nonlinear nature and the present knowledge in numerical solution techniques and computer capability. Nevertheless, it is extremely rare that such complete equations are necessary in engineering practice, although it is worthwhile to know what terms can be neglected for various cases. In this chapter one-dimensional flow equations for certain special cases of incompressible nonhomogeneous fluids are deduced for the flow cross section with its directional normal along x_1 direction. The corresponding equations for incompressible homogeneous fluids are given in the following chapter. These equations are deduced for the purposes of possible easy adaptation to solve problems but also to reveal the assumptions and limitations involved in conventionally used equations.

VI-1. Steady Spatially Varied Flow of Nonhomogeneous Fluid

For steady flow of nonhomogeneous liquid with lateral flow, the continuity equation can be obtained from Eq. 12 as

$$\frac{d}{dx} (\lambda \rho_a AV_1) = \int_{\sigma} \rho q \, d\sigma \quad (56)$$

Or, from Eq. 13

$$\frac{d(AV_1)}{dx} = \int_{\sigma} q \, d\sigma \quad (57)$$

The momentum equation is obtained from Eq. 30 with the aid of Eq. 56 to yield

$$\begin{aligned} & \frac{d}{dx} (Kh \cos\theta) + \frac{V_1^2}{g} \frac{d\beta}{dx} + [(K - K')h \cos\theta - \beta \frac{V_1^2}{g}] \frac{1}{A} \frac{dA}{dx} \\ & = S_o - S_{fx} + \frac{1}{\gamma_a A} \frac{\partial T_{11}}{\partial x} + \frac{1}{\gamma_a A} \int_{\sigma} (U_1 - 2 \frac{\beta}{\lambda} V_1) \rho q \, d\sigma \\ & + 2 \frac{\beta}{\lambda} \frac{V_1^2}{g} \frac{d\lambda}{dx} + (\beta \frac{V_1^2}{g} - Kh \cos\theta) \frac{1}{\rho_a} \frac{d\rho_a}{dx} + \frac{\cos\theta}{\rho_a A} \int_A y \frac{\partial \rho}{\partial x} \, dA \end{aligned} \quad (58)$$

From Eqs. 49 and 56, the energy equation per unit mass is

$$V_1 \frac{dH_c}{dx} - V_1 H_c \frac{1}{\lambda} \frac{d\lambda}{dx} - \frac{1}{\rho_a A} \int_{\sigma} (H_L - \frac{H_c}{\lambda}) \rho q \, d\sigma = -V_1 S_e + \frac{W}{\gamma_a A} \quad (59)$$

VI-2. Unsteady Nonuniform Flow of Nonhomogeneous Fluid

For unsteady flow, without lateral discharge, of nonhomogeneous incompressible fluid, the flow equations can be obtained from Eqs. 12, 30 and 49 by dropping the terms involving q and U . The continuity equation is

$$\frac{\partial}{\partial t} (\rho_a A) + \frac{\partial}{\partial x} (\lambda \rho_a AV_1) = 0 \quad (60)$$

Or, from Eq. 13,

$$\frac{\partial A}{\partial t} + \frac{\partial (AV_1)}{\partial x} = 0 \quad (61)$$

The dynamic equation based on momentum approach is

$$\begin{aligned} \frac{\lambda}{g} \frac{\partial V_1}{\partial t} + \frac{V_1}{g} \frac{\partial}{\partial x} (\beta V_1) + (\beta - \lambda^2) \frac{V_1}{\gamma_a A} \frac{\partial}{\partial x} (\rho_a AV_1) + (K - K') h \cos \theta \frac{1}{A} \frac{\partial A}{\partial x} \\ + \frac{\partial}{\partial x} (Kh \cos \theta) = S_o - S_{fx} + \frac{1}{\gamma_a A} \frac{\partial T_{11}}{\partial x} - \frac{V_1}{g} \left(\frac{\partial \lambda}{\partial t} - V_1 \lambda \frac{\partial \lambda}{\partial x} \right) \\ - Kh \cos \theta \frac{1}{\rho_a} \frac{\partial \rho_a}{\partial x} + \frac{\cos \theta}{\rho_a A} \int_A y \frac{\partial \rho_a}{\partial x} dA \end{aligned} \quad (62)$$

From Eqs. 49 and 60, the energy equation per unit mass is

$$\begin{aligned} \frac{\partial H_B}{\partial t} + V_1 \frac{\partial H_c}{\partial x} + (H_c - \lambda H_B) \frac{1}{\rho_a A} \frac{\partial}{\partial x} (\rho_a AV_1) - H_B V_1 \frac{\partial \lambda}{\partial x} \\ = \zeta \frac{\partial h}{\partial t} - V_1 S_e + \frac{W}{\gamma_a A} \end{aligned} \quad (63)$$

VI-3. Steady Nonuniform Flow of Nonhomogeneous Fluid

One-dimensional equations for steady nonuniform flow of non-homogeneous liquid without lateral discharge can be obtained from Eqs. 56 through 59 by dropping terms involving lateral flow or from Eqs. 60 through 63 by eliminating the unsteady terms. The continuity equation is

$$\frac{d}{dx} (\lambda \rho_a AV_1) = 0 \quad (64)$$

Or,

$$\frac{d(AV_1)}{dx} = 0 \quad (65)$$

Consequently,

$$\frac{d}{dx} (\lambda \rho_a) = 0 \quad (66)$$

The dynamic equation based on momentum approach is

$$\begin{aligned} \frac{d}{dx} (Kh \cos\theta) + \frac{V_1^2}{g} \frac{d\beta}{dx} + [(K - K')h \cos\theta - \beta \frac{V_1^2}{g}] \frac{1}{A} \frac{dA}{dx} = S_o - S_{fx} \\ + \frac{1}{\gamma_a A} \frac{dT_{11}}{dx} + 2 \frac{\beta}{\lambda} \frac{V_1^2}{g} \frac{d\lambda}{dx} + (\beta \frac{V_1^2}{g} - Kh \cos\theta) \frac{1}{\rho_a} \frac{d\rho_a}{dx} + \frac{\cos\theta}{\rho_a A} \int_A y \frac{\partial \rho}{\partial x} dA \end{aligned} \quad (67)$$

The corresponding energy equation per unit mass is

$$V_1 \frac{dH_c}{dx} - V_1 H_c \frac{1}{\lambda} \frac{d\lambda}{dx} = - V_1 S_e + \frac{W}{\gamma_a A} \quad (68)$$

VI-4. Steady Uniform Flow of Nonhomogeneous Fluid

For steady uniform flow of nonhomogeneous fluid with parallel stream lines and without lateral flow, if θ is a function of x , the flow equations are essentially the same as those for the preceding case, Eqs. 64 to 68. However, if θ , i.e., the channel slope, is a constant independent of x , the continuity equation can be expressed as

$$\frac{dA}{dx} = 0 \quad (69a)$$

$$\frac{dV_1}{dx} = 0 \quad (69b)$$

$$\frac{d}{dx} (\lambda \rho_a) = 0 \quad (69c)$$

Essentially this implies that the flow is in a prismatic channel and the depth is constant. From Eqs. 67, 69, 23, and 25, the corresponding dynamic equation based on momentum approach is

$$S_o - S_{fx} = \frac{V_1^2}{\gamma_a} \frac{d}{dx} (\beta \rho_a) + \frac{1}{\gamma_a A} \frac{d}{dx} \left(\int_A \bar{p} dA - T_{11} \right) \quad (70)$$

Or, perhaps more appropriately, since $\partial \bar{u}_1 / \partial x = 0$, Eq. 70 can be written as

$$S_o - S_{fx} = \frac{1}{\gamma_a A} \frac{d}{dx} \int_A (\rho_a \bar{u}_1^{-2} + \rho_a \bar{u}_1'^2 + \bar{p}) dA \quad (71)$$

Likewise, from Eqs. 68, 69, 50, 51, 52, and 53, and by noting that $V_i V_i = V_1^2$, one obtains the dynamic equation based on energy approach as

$$\lambda S_o - S_e = \frac{V_1^2}{2\gamma_a} \frac{d}{dx} (\alpha \rho_a) + \frac{h \cos \theta}{\rho_a} \frac{d}{dx} (\eta \rho_a) - \frac{d}{dx} \int_A \bar{u}_i \tau_{i1} dA \quad (72)$$

Since $\bar{u}_i = \bar{u}_1$ and $\partial \bar{u}_i / \partial x = 0$, Eq. 72 may also be written as

$$\lambda S_o - S_e = \frac{1}{\gamma_a AV_1} \frac{d}{dx} \int_A \left(\frac{\rho}{2} \bar{u}_1^2 + P + \overline{\rho u_1'^2} \right) \bar{u}_1 dA \quad (73)$$

Equations 70 to 73 are obviously different from the corresponding equations for homogeneous fluids. Even for a flow with only lateral density variations, i.e., $\lambda \neq 1$ but $\partial \rho / \partial x = 0$, only the friction slope is equal to the channel slope ($S_o = S_{fx}$) and $S_o = S_e / \lambda$. For steady uniform flow of a homogeneous fluid in a prismatic channel of constant slope, $S_o = S_{fx} = S_e$ and $\lambda = 1$. The difference is due to the spatial variation of the fluid density for the nonhomogeneous fluid case.

VII. SPECIAL CASES FOR HOMOGENEOUS FLUIDS

For an incompressible homogeneous fluid, neither the density nor the viscosity changes with either time or space. Hence, $\rho = \rho_a$ and the density correction factor λ , as defined in Eq. 9, is equal to unity. Consequently, the continuity equation in one-dimensional form, Eq. 7 becomes

$$\frac{\partial A}{\partial t} + \frac{\partial (AV_i)}{\partial x_i} = \int_{\sigma} q \, d\sigma \quad (74)$$

It should be noted here that Eq. 74 is identical with Eq. 11 which has been derived for nonhomogeneous fluids.

From Eq. 28, the corresponding momentum equation is

$$\begin{aligned} \frac{\partial V_i}{\partial t} + v_j \frac{\partial}{\partial x_j} (\beta_{kij} V_i) + (\beta_{kij} - 1) \frac{V_i}{A} \frac{\partial}{\partial x_j} (AV_j) + \frac{1}{A} \int_{\sigma} (V_i - U_i) q \, d\sigma \\ = gS_{oi} - gS_{fi} - g \frac{\partial}{\partial x_i} (Kh \cos\theta) + (K'_{ki} - K) gh \cos\theta \frac{1}{A} \frac{\partial A}{\partial x_i} + \frac{1}{\rho A} \frac{\partial T_{ij}}{\partial x_j} \end{aligned} \quad (75)$$

Likewise, from Eq. 47, the energy equation per unit mass is

$$\begin{aligned} \frac{\partial H_B}{\partial t} + \frac{H_B}{A} \frac{\partial A}{\partial t} + v_i \frac{\partial}{\partial x_i} \left(\alpha_{ki} \frac{v_j v_j}{2g} + \eta_{ki} h \cos\theta + z_b \right) \\ + \left(\alpha_{ki} \frac{v_j v_j}{2g} + \eta_{ki} h \cos\theta + z_b \right) \frac{1}{A} \frac{\partial (AV_i)}{\partial x_i} - \frac{1}{A} \int_{\sigma} H_L q \, d\sigma = \zeta \frac{\partial h}{\partial t} \\ + \frac{1}{\gamma A} \left(\frac{\partial}{\partial x_j} \int_A \bar{u}_i \tau_{ij} \, dA + \int_{\sigma} [\bar{u}_i \tau_{ij}]_{\sigma} N_j \, d\sigma - \int_A \tau_{ij} \frac{\partial \bar{u}_i}{\partial x_j} \, dA \right) \end{aligned} \quad (76)$$

Equations 74 to 76 are presented without specific restriction on the orientation of the cross section as they may be useful for curvilinear

flows such as flow in meandering channels. For a flow cross section with its directional normal along the direction $x_1 = x$, the one-dimensional flow equations can be simplified from Eqs. 74 to 76. Such equations are given for the following special cases.

VII-1 Unsteady Spatially Varied Flow of Homogeneous Fluid

For the case of unsteady spatially varied flow of incompressible homogeneous fluid in open channels, such as channel flow with rainfall or infiltration, the flow equations can be obtained from Eqs. 74 to 76, or from Eqs. 12, 30, and 49 by setting $\lambda = 1$ and $\rho = \rho_a = \text{constant}$. The continuity equation is

$$\frac{\partial A}{\partial t} + \frac{\partial (AV_1)}{\partial x} = \int_{\sigma} q \, d\sigma \quad (77)$$

which is the same as Eq. 13 for nonhomogeneous fluids.

The momentum equation is

$$\begin{aligned} \frac{\partial V_1}{\partial t} + V_1^2 \frac{\partial \beta}{\partial x} + (2\beta - 1)V_1 \frac{\partial V_1}{\partial x} + [(\beta - 1)V_1^2 + (K - K')gh \cos\theta] \frac{1}{A} \frac{\partial A}{\partial x} \\ + g \frac{\partial}{\partial x} (Kh \cos\theta) = gS_o - gS_{fx} + \frac{1}{\rho A} \frac{\partial T_{11}}{\partial x} + \frac{1}{A} \int_{\sigma} (U_1 - V_1)q \, d\sigma \end{aligned} \quad (78)$$

The energy equation per unit mass is

$$\begin{aligned} \frac{\partial H_B}{\partial t} + (H_C - H_B) \frac{1}{A} \frac{\partial}{\partial x} (AV_1) + V_1 \frac{\partial H_C}{\partial x} - \frac{1}{A} \int_{\sigma} (H_L - H_B)q \, d\sigma \\ = \zeta \frac{\partial h}{\partial t} - V_1 S_e + \frac{W}{\gamma A} \end{aligned} \quad (79)$$

The correction factors and H_B and H_C have been defined in Eqs. 32, 48, 50, 52, and 53 with $\rho_a = \rho$ and $\lambda = 1$ for homogeneous fluid. Various simplified forms of Eq. 78 have been used together with the corresponding continuity equation to solve problems numerically. For example, the conventionally used unsteady spatially varied flow equation in St. Venant equation form (1, 4, 13, 19, 25, 27) is

$$\frac{\partial V_1}{\partial t} + V_1 \frac{\partial V_1}{\partial x} + g \cos \theta \frac{\partial h}{\partial x} = g(S_o - S_{fx}) + \frac{1}{A} \int_{\sigma} (U_1 - V_1) q \, d\sigma \quad (80)$$

In order to reduce Eq. 78 to Eq. 80, the following assumptions are necessary:

- (a) The velocity is uniformly distributed over the cross sectional area A so that $\beta = 1$.
- (b) The pressure distribution is hydrostatic so that $K = K' = 1$.
- (c) The channel slope S_o , and hence θ , is constant independent of x .
- (d) The variation with respect to the flow direction x of the internal normal stresses acting on the cross section, $\partial T_{11} / \partial x$, is relatively negligible.

These assumptions essentially require that there is no rapid changes in flow cross sections. Particularly, there should be no flow separation and the flow should not be highly curvilinear. In other words, Eq. 80 should only be used for spatially and temporally gradually varied flows, and it is inaccurate around the region where the Froude number of the flow is close to unity. For the cases which the four required assumptions for Eq. 80 cannot be approximately satisfied, it is often desirable to evaluate the problem from a three-dimensional view point instead of attempting to estimate the variations of T_{11} and the correction factors.

The simplified continuity equation often used together with Eq. 80 in solving problems is

$$\frac{\partial h}{\partial t} + D \frac{\partial v_1}{\partial x} + v_1 \frac{\partial h}{\partial x} = \frac{1}{B} \int_{\sigma} q \, d\sigma \quad (81)$$

The necessary assumptions involved to reduce Eq. 77 to Eq. 81 are as follows:

- (a) The cross sectional area $A = BD$, in which B is the free surface width as defined previously and D is the hydraulic depth.
- (b) The channel is sufficiently wide so that $D \approx h$; or, the side walls of the channel are sufficiently steep, so that $\partial D/\partial t \approx \partial h/\partial t$ and $\partial D/\partial x \approx \partial h/\partial x$.
- (c) There is no rapid changes or discontinuity of the free surface width with respect to space and time, i.e., both $\partial B/\partial t$ and $\partial B/\partial x$ terms are relatively negligible.

Thus, it is interesting to know that the necessary assumptions for Eqs. 80 and 81 are completely different despite the fact that they are often used together in solving problems. Needless to say, a solution obtained by using these two equations should have all the assumptions involved in both equations satisfied. Unfortunately, these two equations have often been used ignorantly without fully realizing the assumptions and limitations imposed.

The friction slope S_{fx} appearing in the momentum equation (Eq. 78), and consequently in Eq. 80 is often mistaken as the dissipated energy gradient S_e . There is no simple way without making serious assumptions to reduce the energy equation (Eq. 79) into the St. Venant equation form. Further discussion on this subject will be given later in Chapter VIII.

VII-2. Steady Spatially Varied Flow of Homogeneous Fluid

For steady spatially varied flow of homogeneous fluid having constant density and viscosity, the corresponding flow equations can be obtained by dropping the unsteadiness terms in Eqs. 77, 78, and 79 for the unsteady case or by setting $\lambda = 1$ and $\rho = \rho_a = \text{constant}$ in Eqs. 56, 58, and 59 for the steady nonhomogeneous case. The continuity equation is

$$\frac{d(AV_1)}{dx} = \int_{\sigma} q \, d\sigma \quad (82)$$

The momentum equation is

$$\begin{aligned} \frac{V_1^2}{g} \frac{d\beta}{dx} + [(K - K')h \cos\theta - \beta \frac{V_1^2}{g}] \frac{1}{A} \frac{dA}{dx} + \frac{d}{dx} (Kh \cos\theta) \\ = S_o - S_{fx} + \frac{1}{\gamma A} \frac{dT_{11}}{dx} + \frac{1}{gA} \int_{\sigma} (U_1 - 2\beta V_1) q \, d\sigma \end{aligned} \quad (83)$$

The energy equation per unit mass is

$$V_1 \frac{dH_C}{dx} - \frac{1}{A} \int_{\sigma} (H_L - H_C) q \, d\sigma = -V_1 S_e + \frac{W}{\gamma A} \quad (84)$$

As discussed in Chapter II, many previous derivations have been made by different investigators to formulate the flow equations for steady spatially varied flow of an incompressible homogeneous fluid. So far the most complete equations for such flow are those derived by Yen and Wenzel (28). To reduce Eq. 83 to the dynamic equation derived from the momentum consideration by Yen and Wenzel (Eq. 12 in Ref. 28, with appropriate adjustments for the differences in definition for q and the correction factors), the following

assumptions are implicitly made:

- (a) The channel is straight and very wide, or alternatively, being prismatic with sufficiently steep side walls and no rapid change of free surface so that $dh/dx \approx dD/dx \gg (D/B)dB/dx$, and hence

$$\frac{dA}{dx} \approx \frac{A}{D} \frac{dh}{dx} \quad (85)$$

- (b) The variation with respect to x of the internal normal stress acting on A , dT_{11}/dx , is relatively negligible.

The most commonly used form of dynamic equations for gradually varied flow with lateral inflow is (6)

$$\frac{dh}{dx} = \frac{S_o - S_{fx} - \frac{2V_1}{gA} \int_{\sigma} q \, d\sigma}{1 - \frac{V_1^2}{gD}} \quad (86)$$

To reduce Eq. 83 to Eq. 86, in addition to the two assumptions just mentioned, the following assumptions are necessary:

- (c) The pressure distribution is hydrostatic so that $K = K' = 1$.
- (d) The velocity is uniformly distributed over A so that $\beta = 1$.
- (e) The channel slope S_o , and hence θ , is constant and small, i.e., $\cos\theta \approx 1$.
- (f) There is no x component of the velocity of the lateral flow, i.e., $U_1 = 0$.

Likewise, to reduce Eq. 84 to the dynamic equation derived from the energy approach by Yen and Wenzel (Eq. 33 in Ref. 28) the following

assumptions are involved:

- (a) The channel flow has only longitudinal velocity component, i.e., $V_i = V_1$.
- (b) The assumption on channel geometry involved in Eq. 85 is valid.
- (c) The rate of change with respect to x of the work which is done by the internal stresses over A is negligible, i.e.,

$$\frac{d}{dx} \int_A \bar{u}_i \tau_{i1} dA \approx 0 \quad (87)$$

- (d) The rate of work which is done by the external stresses on the boundary σ is negligible, i.e.,

$$\int_{\sigma} [\bar{u}_i \tau_{ij}]_{\sigma} N_j d\sigma \approx 0 \quad (88)$$

This assumption usually does not impose much restriction. On the free surface the shear stress is usually negligible unless there is strong wind or large tangential component of the lateral flow. For the solid boundary part of σ , the infiltration or seepage velocity \bar{u}_i is usually small.

To further reduce the energy equation to the commonly used dynamic equation for steady flow with lateral outflow for homogeneous fluid based on the energy approach (6)

$$\frac{dh}{dx} = \frac{S_o - S_e - \frac{\alpha V_1}{Ag} \int_{\sigma} q d\sigma}{1 - \frac{\alpha V_1^2}{gD}} \quad (89)$$

in addition to the four assumptions just mentioned, the following assumptions are necessary:

- (e) The pressure distribution is hydrostatic so that $\eta = 1$.
- (f) The change with respect to the longitudinal direction of the velocity distribution on A is small so that the $d\alpha/dx$ term is relatively negligible.
- (g) The channel slope is constant and small, and hence $\cos\theta \approx 1$.
- (h) The piezometric head of the lateral flow, P_L/γ , is equal to $h \cos\theta + z_b$; and the velocity of the lateral flow, U, is numerically equal to $\sqrt{\alpha} V_1$.

Strelkoff (24) has given an elaborated explanation on why Eq. 86 can be used as an acceptable approximation for the lateral inflow case and Eq. 89 for the lateral outflow case.

VII-3. Unsteady Nonuniform Flow of Homogeneous Fluid

For an unsteady nonuniform flow of a homogeneous fluid with constant ρ and μ and without lateral flow, the flow equations can be obtained from Eqs. 77, 78, and 79 by dropping the terms involving q and U or from Eqs. 60, 62, and 63 by setting $\lambda = 1$ and $\rho = \rho_a = \text{constant}$. The continuity equation is the same as Eq. 61 for the corresponding non-homogeneous fluid case:

$$\frac{\partial A}{\partial t} + \frac{\partial (AV_1)}{\partial x} = 0 \quad (90)$$

The momentum equation is

$$\begin{aligned} \frac{\partial V_1}{\partial t} + V_1^2 \frac{\partial \beta}{\partial x} + (2\beta - 1)V_1 \frac{\partial V_1}{\partial x} + [(\beta - 1)V_1^2 + (K - K')gh \cos\theta] \frac{1}{A} \frac{\partial A}{\partial x} \\ + g \frac{\partial}{\partial x}(Kh \cos\theta) = gS_o - gS_{fx} + \frac{1}{\rho A} \frac{\partial T_{11}}{\partial x} \end{aligned} \quad (91)$$

The energy equation per unit mass is

$$\frac{\partial H_B}{\partial t} + (H_C - H_B) \frac{1}{A} \frac{\partial}{\partial x}(AV_1) + V_1 \frac{\partial H_C}{\partial x} = \zeta \frac{\partial h}{\partial t} - V_1 S_e + \frac{W}{\gamma A} \quad (92)$$

The St. Venant equations (6, 10, 16, 23, 24)

$$\frac{\partial h}{\partial t} + h \frac{\partial V_1}{\partial x} + V_1 \frac{\partial h}{\partial x} = 0 \quad (93)$$

$$\frac{\partial V_1}{\partial t} + V_1 \frac{\partial V_1}{\partial x} + g \frac{\partial h}{\partial x} = g(S_o - S_{fx}) \quad (94)$$

are apparently approximations of Eqs. 90 and 91, respectively. Since Eq. 93 can be obtained from Eq. 81 and Eq. 94 from Eq. 80 by dropping the lateral flow terms, the assumptions involved to obtain Eqs. 93 and 94 are exactly the same as those for Eqs. 81 and 80, respectively.

It may be appropriate to mention here that Eq. 91 (or Eq. 78 if precipitation is considered), after neglecting the $\partial T_{11}/\partial x$ term and adding the Coriolis force term, becomes the one-dimensional equation of motion for hurricane and storm surges in coastal regions and estuaries (26). However, in practice further assumptions such as neglecting the bed friction or the convective acceleration are made so that numerical computation is possible.

VII-4. Steady Nonuniform Flow of Homogeneous Fluid

For steady flow of homogeneous liquid without lateral flow, the continuity equation is

$$\frac{d(AV_1)}{dx} = 0 \quad (95)$$

The momentum equation is

$$\begin{aligned} \frac{d}{dx}(Kh \cos\theta) + [(K - K')h \cos\theta - \beta \frac{V_1^2}{g}] \frac{1}{A} \frac{dA}{dx} + \frac{V_1^2}{g} \frac{d\beta}{dx} \\ = S_o - S_{fx} + \frac{1}{\gamma A} \frac{dT_{11}}{dx} \end{aligned} \quad (96)$$

The energy equation per unit mass is

$$V_1 \frac{dH_C}{dx} = -V_1 S_e + \frac{W}{\gamma A} \quad (97)$$

By assuming $V_i = V_1$, from Eqs. 97 and 95, the dynamic equation based on the energy approach is

$$\frac{d}{dx}(\eta h \cos\theta) - \frac{\alpha V_1^2}{Ag} \frac{dA}{dx} + \frac{V_1^2}{2g} \frac{d\alpha}{dx} = S_o - S_e + \frac{W}{\gamma AV_1} \quad (98)$$

The conventional nonuniform flow equation for backwater computation (6, 23)

$$\frac{dh}{dx} = \frac{S_o - S}{1 - \frac{V_1^2}{gD}} \quad (99)$$

is obviously an approximation of either Eq. 96 or Eq. 98. In Eq. 99 S is "the slope of the flow." In fact, it is probably this equation without specifically defining S that causes many confusions in nonuniform flow computation. Both α and β have been multiplied to the term V_1^2/gD by different investigators and S has been regarded as the friction slope, the energy slope, as well as the water surface slope. Clearly, if Eq. 99 is deduced from Eq. 96, $S = S_{fx}$ is the friction slope and the following assumptions are involved:

- (a) The pressure distribution is hydrostatic so that $K = K' = 1$.
- (b) The channel has constant slope and approximately $\cos\theta = 1$.
- (c) The assumption on channel geometry involved in Eq. 85 is valid.
- (d) The cross sectional velocity distribution is uniform so that $\beta = 1$.

Or, alternatively, if the denominator is $1 - (\beta V_1^2/gD)$ instead of $1 - (V_1^2/gD)$, the variation of the velocity distribution over A with respect to x is small so that the $d\beta/dx$ term is relatively negligible.

- (e) The variation with respect to x of the internal normal stress acting on the cross section, dT_{11}/dx , is relatively negligible.

Likewise, if Eq. 99 is deduced from Eq. 98, $S = S_e$ is the dissipated energy gradient and the following assumptions are involved:

- (a), (b), and (c): same as Assumptions (a), (b), and (c) from Eq. 96 to Eq. 99, and $\eta = 1$.
- (d) The cross sectional velocity distribution is uniform so that $\alpha = 1$. Or, alternatively, if the denominator is $1 - (\alpha V_1^2/gD)$ instead of $1 - (V_1^2/gD)$, the variation with respect to x of the

cross sectional velocity distribution is small so that the $d\alpha/dx$ term is negligible.

(e) The assumptions involved in Eqs. 87 and 88 hold.

From Eqs. 95, 96, and 98, it is obvious that for steady uniform flow of homogeneous fluid, $S_o = S_{fx} = S_e$.

VIII. COMPARISON BETWEEN MOMENTUM AND ENERGY
APPROACHES AND VARIOUS FLOW GRADIENTS

Despite the similarities in appearance between their simplified one-dimensional forms such as Eqs. 96 and 98, the flow equations derived based on the momentum approach is basically different from those based on the energy approach (10, 16, 22, 24, 28). The differences are clearly shown in the original momentum and energy equations in integral form, Eqs. 22 and 42. Although basically both are derived from Newton's second law, the momentum equation is a vector relationship in which all the forces acting on the cross section should be considered, including the external forces acting on the boundary surface which may perform no work. The energy equation, contrarily, is a scalar relationship taking into account not only the convective energy transfer and work done by external forces but also the work which is done by the internal forces. In their respective one-dimensional forms, Eqs. 30 and 49 or 55, this inherent difference between the momentum and energy approaches are reflected in the correction factors as well as in certain terms.

First, in the momentum approach, Eq. 30, only the component of the velocity along the direction considered, V_1 , affects the momentum balance, whereas in the energy approach, Eq. 55, all the three components of the flow velocity V_i , $i = 1, 2, 3$, are involved. Secondly, in Eq. 30, there is only one momentum flux correction factor, β , which is a second order tensor, whereas in Eq. 55 two energy flux correction factors, α and β' , the former being a vector and the latter being a scalar, are involved. Only for the special case $\bar{u}_1 = \bar{u}_2 = \bar{u}_3 = 0$ that β' , which involves summation over $i = 1, 2, 3$, is numerically equal to β but their physical meanings are different. In general, $\alpha \neq \beta' \neq \beta$, which indicates the

different effects of velocity distribution on momentum and energy. Thirdly, in Eq. 30, there is only one pressure correction factor K which is a scalar quantity, whereas in Eq. 55, both K and the convective potential energy flux correction factor η , which is a vector, appear. This indicates the difference in effect of piezometric pressure distribution on momentum and energy. Fourthly, for a given mean velocity V_1 the effect of unsteadiness as reflected by the local acceleration, is a function of λ for the momentum case while for the energy case it is a function of β' and K but not of λ . Fifthly, for nonhomogeneous liquid the cross sectional change of density with respect to time, as indicated by $\partial\lambda/\partial t$ in Eq. 30, affects the momentum balance but it has no direct effect on the energy balance. Sixthly, in the momentum equation, only the appropriate component of the velocity of the lateral flow is considered, while in the energy approach, the total magnitude of the lateral flow velocity vector is of interest. Seventhly, as indicated by the $\zeta \partial h/\partial t$ term in Eq. 55, for a stationary boundary the ambient pressure does no work and hence it vanishes in the energy equation, while as shown by K' in Eq. 30, the effect of the ambient pressure appears in the momentum equation as long as there is a spatial change in flow cross section, i.e., nonuniform flow. Eighthly, Eq. 30 is readily applicable to loose boundary with erosion or deposition, whereas terms to account for the energy required for the change of the boundary, $(Kh/V_1) (\partial \cos\theta/\partial t)$ and $(1/V_1) (\partial z_b/\partial t)$, should be added to the left side of Eq. 55. Ninthly, except for the slopes of the channel bottom S_o and the free surface $\partial h/\partial x$, no other gradients are common to both Eqs. 30 and 55.

In fact, the difference between the friction slope S_{fx} and the dissipated energy gradient S_e illustrates characteristically the difference

between the momentum and the energy concepts. The friction slope accounts for the resistance due to external boundary stresses as defined by Eq. 27 whereas the dissipated energy gradient accounts for the rate of energy dissipation due to internal stresses working over a velocity gradient field as defined in Eq. 54. The relationship between S_{fx} and S_e can be obtained from Eqs. 30 and 55,

$$\begin{aligned}
S_e = & \lambda S_{fx} + \frac{\lambda^2}{g} \frac{\partial V_1}{\partial t} - \frac{1}{V_1} \frac{\partial H_B}{\partial t} + \frac{\zeta}{V_1} \frac{\partial h}{\partial t} + \frac{\partial}{\partial x} [(\lambda K - \eta)h \cos\theta] - \frac{\partial}{\partial x} \left(\alpha \frac{V_i V_i}{2g} \right) \\
& + \frac{\lambda V_1}{g} \frac{\partial}{\partial x} (\beta V_1) + [(\lambda \beta' - \alpha) \frac{V_i V_i}{2g} + (\beta - \lambda^2) \lambda \frac{V_1^2}{g} \\
& + (\lambda K - \eta)h \cos\theta] \frac{1}{\rho_a A V_1} \frac{\partial}{\partial x} (\rho_a A V_1) + \lambda (K - K')h \cos\theta \frac{1}{A} \frac{\partial A}{\partial x} \\
& - \frac{\lambda}{\gamma_a A} \frac{\partial T_{11}}{\partial x} + \frac{W}{\gamma_a A V_1} + \frac{\lambda}{g} V_1 \frac{\partial \lambda}{\partial t} + \left(\beta' \frac{V_i V_i}{2g} - \lambda^2 \frac{V_1^2}{g} + Kh \cos\theta \right) \frac{\partial \lambda}{\partial x} \\
& + Kh \cos\theta \frac{\lambda}{\rho_a} \frac{\partial \rho_a}{\partial x} - \frac{\lambda \cos\theta}{\rho_a A} \int_A y \frac{\partial \rho}{\partial x} dA \\
& + \frac{1}{\rho_a A V_1} \int_{\sigma} (H_L - H_B + \lambda^2 \frac{V_1^2}{g} - \frac{\lambda}{g} U_1 V_1) \rho q d\sigma
\end{aligned} \tag{100}$$

In hydraulics, a total head H is often used in one-dimensional analysis. The total head of the flow at a cross section is defined as

$$H = H_p + z_b + \frac{V_1^2}{2g} \tag{101}$$

in which H_p is the cross sectional average piezometric head of the channel flow with respect to the channel bottom, i.e., to the lowest point of the

channel cross section. Hence, the total head gradient S_H is

$$S_H = -\frac{\partial H}{\partial x} = S_o - \frac{\partial H_p}{\partial x} - \frac{\partial}{\partial x} \left(\frac{V_1^2}{2g} \right) \quad (102)$$

Consequently, from Eqs. 30 and 102,

$$\begin{aligned} S_H = & S_{fx} + \frac{\partial}{\partial x} (Kh \cos\theta - H_p) + \frac{\lambda}{g} \frac{\partial V_1}{\partial t} + \frac{V_1^2}{g} \frac{\partial \beta}{\partial x} + (\beta - 1) \frac{\partial}{\partial x} \left(\frac{V_1^2}{2g} \right) \\ & + (\beta - \lambda^2) \frac{V_1}{\gamma_a A} \frac{\partial}{\partial x} (\rho_a AV_1) + (K - K') h \cos\theta \frac{1}{A} \frac{\partial A}{\partial x} \\ & - \frac{1}{\gamma_a A} \frac{\partial T_{11}}{\partial x} + \frac{1}{\gamma_a A} \int_{\sigma} (\lambda V_1 - U_1) \rho q \, d\sigma + \frac{V_1}{g} \left(\frac{\partial \lambda}{\partial t} - V_1 \lambda \frac{\partial \lambda}{\partial x} \right) \\ & + Kh \cos\theta \frac{1}{\rho_a} \frac{\partial \rho_a}{\partial x} - \frac{\cos\theta}{\rho_a A} \int_A y \frac{\partial \rho}{\partial x} \, dA \end{aligned} \quad (103)$$

From Eqs. 55 and 102,

$$\begin{aligned} S_H = & \frac{S_e}{\lambda} + \frac{1}{\lambda V_1} \left(\frac{\partial H_B}{\partial t} - \zeta \frac{\partial h}{\partial t} \right) + \frac{\partial}{\partial x} \left(\frac{\eta}{\lambda} h \cos\theta - H_p \right) - \frac{\partial}{\partial x} \left(\frac{V_1^2}{2g} \right) + \frac{1}{\lambda} \frac{\partial}{\partial x} \left(\alpha \frac{V_1 V_i}{2g} \right) \\ & + \left(\frac{H_c}{\lambda} - H_B \right) \frac{1}{\rho_a AV_1} \frac{\partial}{\partial x} (\rho_a AV_1) - (H_B - z_b + \frac{\eta}{\lambda} h \cos\theta) \frac{1}{\lambda} \frac{\partial \lambda}{\partial x} \\ & - \frac{W}{\lambda \gamma_a AV_1} - \frac{1}{\lambda \rho_a AV_1} \int_{\sigma} (H_L - H_B) \rho q \, d\sigma \end{aligned} \quad (104)$$

Similar expressions, of course, can be obtained for the gradients of H_B and H_C . However, they are omitted here because of their infrequent use in practice. Nevertheless, it should be mentioned here that the true cross sectional averaged total head measured in terms of the cross sectional averaged specific weight, γ_a , is H_B as defined in Eq. 48. However, because of its

frequent appearance in steady flow equations, H_C as defined in Eq. 50 is often mistaken as the total head of the flow. Worse still, in practice, perhaps as a matter of convenience, the approximate total head H as defined in Eq. 101 is usually used. For flow of homogeneous liquid with approximately hydrostatic pressure distribution, H_C differs from H only by the factor α in the velocity head term. But in general, H_B , H_C , and H , and consequently their corresponding gradients with respect to x , are different.

From Eqs. 100 and 103 or 104 it can be shown that in general for open-channel flow S_{fx} , S_e , and S_H are not equal and none is equal to the channel slope S_o or the free surface slope with respect to a horizontal datum, $(\partial h/\partial x) = S_o$. Apparently, the differences are due to the unsteadiness and nonuniformity of the flow and nonhomogeneous properties of the fluid. Even for steady flow of homogeneous liquid with $V_i = V_1$ and $V_2 = V_3 = 0$, by noting that in this case $\lambda = 1$, $\beta' = \beta$, $H_p = Kh \cos\theta$ (Eq. 25), and with the aid of Eq. 82, Eqs. 100, 103, and 104 can only be simplified respectively to the following:

$$\begin{aligned}
 S_e = S_{fx} + \frac{d}{dx} [(K-\eta)h \cos\theta] + \frac{V_1^2}{2g} \frac{d}{dx} (2\beta - \alpha) \\
 + [(\alpha - \beta) \frac{V_1^2}{g} + (K - K') h \cos\theta] \frac{1}{A} \frac{dA}{dx} - \frac{1}{\gamma A} \frac{dT_{11}}{dx} + \frac{W}{\gamma AV_1} \\
 + \frac{1}{AV_1} \int_{\sigma} [H_L - H_C + (2\beta - \alpha) \frac{V_1^2}{g} - \frac{1}{g} U_1 V_1] q \, d\sigma \quad (105)
 \end{aligned}$$

$$\begin{aligned}
 S_H = S_{fx} + \frac{V_1^2}{g} \frac{d\beta}{dx} + [(1 - \beta) \frac{V_1^2}{g} + (K - K') h \cos\theta] \frac{1}{A} \frac{dA}{dx} \\
 - \frac{1}{\gamma A} \frac{dT_{11}}{dx} + \frac{1}{gA} \int_{\sigma} [(2\beta - 1) V_1 - U_1] q \, d\sigma \quad (106)
 \end{aligned}$$

and

$$S_H = S_e + \frac{d}{dx} [(\eta - K) h \cos\theta] + \frac{V_1^2}{2g} \frac{d\alpha}{dx} + (1 - \alpha) \frac{V_1^2}{gA} \frac{dA}{dx} - \frac{W}{\gamma AV_1} - \frac{1}{AV_1} \int_{\sigma} [H_L - H_C + (1 - \alpha) \frac{V_1^2}{g}] q \, d\sigma \quad (107)$$

In addition to the just mentioned five slopes (S_{fx} , S_e , S_H , S_o , and $\partial h/\partial x$), there are three other gradients which are of interest from hydraulics viewpoint: One is the gradient of the so-called acceleration head, $(\partial V_1/\partial t)/g$; the second is the gradient of the cross sectional average piezometric head with respect to the channel bottom, $\partial H_p/\partial x$; and the third is the hydraulic gradient, $(\partial H_p/\partial x) - S_o$, which is simply the slope of the hydraulic grade line or the gradient of H_p referred to a horizontal reference instead of the channel bottom. Only for steady uniform flow of homogeneous fluid with $q = 0$ are S_o , S_{fx} , S_e , S_H , and the hydraulic gradient equal to one other and $\partial h/\partial x = \partial H_p/\partial x = \partial V_1/\partial t = 0$. In general, none of these eight gradients is equal to any one of the other gradients and, for example, for certain spatially varied flows such as sheet flow under rainfall the values of S_e can be of order of magnitude larger than S_H , S_{fx} , or S_o (29). Unfortunately, in hydraulic computations care is usually not being taken in distinguishing and selecting the appropriate gradients to be used in the corresponding approximate flow equations. Often the "slope" in these equations is approximated by using the Chezy or Manning formulas, or simply by S_o or the free surface slope.

In their complete one-dimensional form, the momentum and energy equations, Eqs. 30 and 49, can be applied to open-channel flows of incompressible fluid without restriction. In practice, however, it is necessary to neglect the relatively small terms in these equations so that solutions are

possible. For example, for a hydraulic jump of a homogeneous liquid, the energy dissipation, and hence S_e , is large and usually not readily known, consequently the energy relationship, Eq. 98, is not suitable to be used. On the other hand, the magnitude of $S_o - S_{fx}$ as well as those for $(K - K')$, $d\beta/dx$, and dT_{11}/dx terms in the momentum relationship is relatively small, and hence they can be neglected. Consequently, Eq. 96 is reduced to

$$\frac{d}{dx} (Kh \cos\theta) = \beta \frac{V_1^2}{gA} \frac{dA}{dx} = -\beta \frac{V_1}{g} \frac{dV_1}{dx} \quad (108)$$

If hydrostatic pressure distribution is assumed, $K = 1$. For a two-dimensional flow, the area per unit width is $A = h$ and the discharge, $V_1 h$, is constant independent of x . Furthermore, if $\theta = \text{constant}$ and $d\beta/dx$ negligible, Eq. 108 yields

$$\frac{d}{dx} \left(\frac{h^2}{2} \cos\theta \right) = -\frac{d}{dx} \left[\frac{\beta}{g} (V_1 h) V_1 \right] \quad (109)$$

Integration of Eq. 109 from a section immediately before the jump with depth h_1 to the section after the jump with a depth h_2 , the sequent depth relationship for the hydraulic jump is obtained

$$\frac{h_2}{h_1} = \frac{1}{2} \left(\sqrt{1 + 8F_1^2} - 1 \right) \quad (110)$$

in which $F_1 = V_1 / \sqrt{gh_1 / \beta \cos\theta}$ is the approaching flow Froude number. Similar techniques can be adopted to apply Eq. 96 to abrupt expansion of the channel. However, it is beyond the scope of this report to cite the numerous possible applications of the flow equations derived.

As mentioned in Chapter II, Strelkoff, in his excellent paper (24), also pointed out the fundamental difference between the momentum and energy approaches. He integrated the point form continuity, momentum, and energy equations for homogeneous liquid over a control volume and by using the Gauss transformation to obtain the equations in differential-integral form. When taking the limit of the length of the control volume approaching zero, Strelkoff's flow equations would be transformed to Eqs. 5, 22, and 42 with $\rho = \text{constant}$. He then proceeded to define a turbulence correction factor and a nonuniformity and unsteadiness correction factor to one dimensionalize the momentum and energy equations and discussed the nature of these two factors for various special cases. Consequently, his resulted equations appear considerably different from their counterparts in this study, Eqs. 78 and 79. The two correction-factors as defined by Strelkoff may be convenient in utilizing detail laboratory turbulence and velocity measurements to detect the nature of their variations. The present study, on the other hand, derived the equations by integration over the cross section with the aid of the Leibnitz rule and the resulted one-dimensionalized equations are derived having correction factors defined with the intention to preserve the physical nature and easy adoption for practical uses.

IX. RESISTANCE TO OPEN-CHANNEL FLOW

One of the difficulties involved in engineering computation and design of open-channel flow problems is the determination of the flow resistance. Rouse (21) gave a vivid discussion on the subject. He subdivided open-channel flow resistance into surface resistance, form resistance, wave resistance, and resistance due to unsteadiness (local acceleration). He further cleverly demonstrated the effects of nonuniformity and unsteadiness on flow resistance and the difficulties involved in determining it. The flow resistance he referred to is "...that part of the slope corresponding to the local dissipation rate in unsteady flow" (21), which is obviously the dissipated energy gradient, S_e , defined by Eq. 54 in this study. The friction slope, S_{fx} , named as traditionally used, as defined in Eq. 27 referring to the resistance force due to boundary shear stresses, is apparently corresponding to the surface resistance discussed by Rouse.

As discussed in the preceding chapter, in addition to S_e and S_{fx} , the total head gradient is also often used as an indication of the flow resistance. In practice the gradient corresponding to the flow resistance is usually expressed alternatively in some form of resistance factor for more general use. Among the various proposed resistance factors, Weisbach's f , Manning's n , and Chezy's C are the most popular ones used in open-channel flows. Their values as well as the relationship among them have been well established for steady uniform flow of homogeneous fluids. Particularly, the Weisbach f can be found from the Moody diagram as a function of the Reynolds number of the flow and the relative boundary roughness. However, the validity of using the Moody diagram values for cases other than steady uniform flow has always been subject to question.

According to Weisbach's pipe flow resistance concept, a frictional resistance coefficient f_f corresponding to S_{fx} can be defined based on the momentum concept

$$f_f = - \frac{8}{\rho V_1^2} \frac{\int_{\sigma} [\tau_{li}]_{\sigma} N_i d\sigma}{P} = \frac{8gRS_{fx}}{V_1^2} \quad (111)$$

in which P is the wetted perimeter of the cross section and $R = A/P$ is the hydraulic radius of the section. The energy dissipation coefficient f_e can be defined as

$$f_e = \frac{8gRS_e}{V_1^2} \quad (112)$$

and the total head loss coefficient f_H as

$$f_H = \frac{8gRS_H}{V_1^2} \quad (113)$$

Only for steady uniform flow of homogeneous fluid without lateral discharge is $f_f = f_e = f_H$ and the value is equal to the corresponding Weisbach f given in the Moody diagram. In general the values of f_f , f_e , and f_H , just as S_{fx} , S_e , and S_H , are not equal and are different from that given by the Moody diagram for the same Reynolds number and relative roughness. The amount of deviations of f_f , f_e , and f_H from the Moody diagram value due to unsteadiness, nonuniformity of flow, nonhomogeneous fluid, and lateral discharge have yet to be determined individually as well as collectively. Limited information on the effect of unsteadiness on flow resistance can be found in Ref. (21). Certain cases of negligible turbulence or nonuniformity effects were discussed by Keulegan (16) and Strelkoff (24).

Recently, Yen and Wenzel (29) found that for a two-dimensional steady Stokes flow of an incompressible viscous homogeneous liquid under creeping motion on a uniformly sloped plane with lateral flow,

$$f_f = \frac{24}{\mathbb{R}} + 4 \left[\frac{q}{V_1} \left(C_1 - \frac{U_1}{V_1} \right) \right]_{z_b, h} \quad (114)$$

$$f_e = \frac{24}{\mathbb{R}} + 4 \left[\frac{q}{V_1} \left(3C_1 - 3 \frac{U_1}{V_1} - \frac{C_2}{2} + \frac{U_2^2}{V_1^2} \right) \right]_{z_b, h} \quad (115)$$

$$f_H = \frac{24}{\mathbb{R}} + 12 \left[\frac{q}{V_1} \left(C_1 - \frac{U_1}{V_1} \right) \right]_{z_b, h} \quad (116)$$

in which $\mathbb{R} = \rho V_1 R / \mu$ is the Reynolds number of the flow, the subscripts z_b and h indicate that the terms inside the bracket are to be evaluated at both the bottom and the free surface, and

$$C_1 = \frac{9}{5} - \frac{3}{80} \mathbb{R} f_f + \frac{1}{1920} \mathbb{R}^2 f_f^2 \quad (117)$$

$$C_2 = \frac{27}{7} - \frac{9}{56} \mathbb{R} f_f + \frac{27}{8960} \mathbb{R}^2 f_f^2 - \frac{1}{71680} \mathbb{R}^3 f_f^3 \quad (118)$$

They also showed some experimental results of steady sheet flow of water under rainfall which is reproduced here as Fig. 3.

Although at the present only limited quantitative information is available on the flow resistance other than the steady uniform case, nonetheless useful qualitative information can be obtained from careful examination of Eqs. 30, 55, 100, and 102 to 107. For example, from Eq. 100 it is obvious that the lateral flow can play an important role in the difference between S_e and S_{fx} , and hence f_e and f_f . From Eqs. 100 and 104, since the

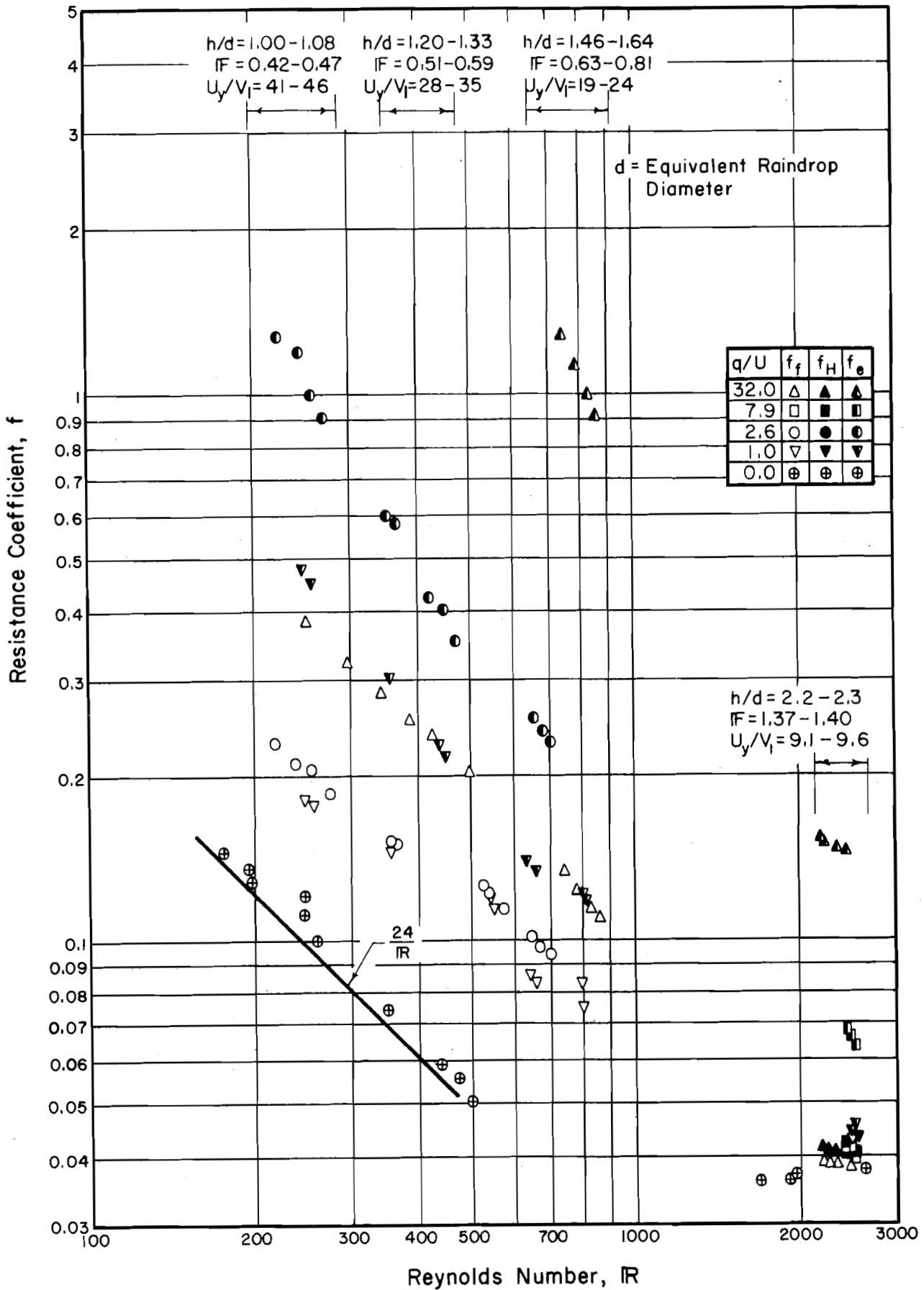


Fig. 3. Resistance Coefficients for Steady Sheet Flow under Rainfall (29)

magnitude of $H_L - H_B$ is usually of the order of $U^2/2g - V_1^2/2g$, for the case of large lateral inflow velocity, i.e., for $|U| \gg V_1$, f_e can be much greater than either f_f or f_H . Such is the case of sheet flow under rainfall as shown in Fig. 3, for which the major part of the raindrop energy brought into the channel flow is simply dissipated through the disturbance the drops generated. Thus, with the exception of rare cases, usually for channel flow with lateral inflow, $f_e > f_H > f_f$. Likewise for channel flow with lateral outflow, usually $f_e < f_H < f_f$, which can also be observed from Eqs. 114 to 116.

On the other hand, the effect of nonhomogeneous fluid on the differences between f_e and f_H and f_f depends mainly on the magnitude of λ and the sign of $\partial\rho_a/\partial x$. The effect of nonhomogeneous density on the flow resistance is due primarily to the modification of pressure distribution and only indirectly to the velocity distribution. For steady uniform stratified flow with $q = 0$, f_e is greater than f_f since from Eqs. 71 and 73, $S_e = \lambda S_f$. In open channels, the change of density, if any, is often small in comparison to the liquid density. Consequently, the value of λ is usually smaller than that for β and mostly is between 1.0 and 1.1, and $\partial\rho_a/\partial x$ and $\partial\lambda/\partial x$ are also small. Contrarily, the effect of nonhomogeneous viscosity on the flow resistance is through the modifications of stresses (Eq. 15) and energy dissipation process through viscosity, and this effect is not reflected explicitly in the flow equations.

It may be appropriate to note here that, in view of the usually relatively small variation of density in the flow and low velocity near the channel bottom, $\alpha \geq \beta' \geq \beta \geq \lambda$. In fact, one advantage of using the one-dimensionalized flow equations is that the values of α , β' , and β usually do not vary rapidly with respect to space and their values can often be roughly estimated. Consequently, sufficient computational accuracy can

often be met by including the estimated values of α , β' , and β without considering their spatial variations. Moreover, for a wide channel or a rectangular cross section with constant density over A (but ρ can vary with x), if the piezometric pressure P increases with increasing depth, then $\eta > K > 1$; whereas if P decreases with increasing depth, $\eta < K < 1$.

Physically, the effects of unsteadiness and nonuniformity of the flow on resistance is through modifications of the pressure and velocity distributions of the flow as reflected by the pressure and velocity correction factors appearing in the appropriate flow equations. It is plausible that any change in pressure would correspond to acceleration and any change in velocity gradient would result in different energy dissipation rates and shear distribution. Again, like the effect of density, the effects of unsteadiness and nonuniformity of the flow on the differences between f_e and f_f or f_H depend on whether the flow is locally or convectively accelerating or decelerating. Conceivably, for an accelerating flow, f_e is smaller than f_f or f_H while for a decelerating flow f_e may be considerably greater than f_H or f_f .

Nevertheless, considerable efforts are needed in the future to establish quantitatively more general and useful information on resistance to open-channel flows. As can be seen from Eqs. 100 and 103 or 104, accurate flow resistance coefficients determination requires detail measurements of the flow and it is by no means an easy task.

X. CONCLUSIONS

Continuity, momentum, and energy equations in integral form for a cross section of a spatially varied flow of an incompressible nonhomogeneous fluid in open channels can be derived by integrating the corresponding differential form continuity equation, Reynolds equation, and energy equation with the aid of the Leibnitz rule and appropriate boundary conditions. The resulting equations, Eqs. 5, 19 and 40, give a clear insight on the mechanics of the flow and clearly show the differences between momentum and energy concepts. However, engineering applications of these integral equations require detail information on velocity, pressure, stress, and density distributions of the flow which is often unavailable. By using cross sectional mean values of the flow, one-dimensional continuity, momentum, and energy equations can be obtained from the corresponding equations in integral form. The derived one-dimensional flow equations are more suitable for simplification with appropriate assumptions for numerical computation than the original integral equations.

The one-dimensional equations derived in the present study, Eqs. 12, 13, 30, and 49 can be regarded as unified general open-channel flow equations for incompressible fluid with the terms due to various effects such as density variation, lateral flow, and local acceleration expressed explicitly. Corresponding flow equations for special cases such as the following can easily be obtained from the general equations: (a) unsteady spatially varied flow of homogeneous fluid; (b) unsteady nonuniform flow of homogeneous fluid without lateral discharge; (c) steady spatially varied flow of homogeneous fluid; (d) steady nonuniform flow of homogeneous fluid without lateral discharge; (e) steady uniform flow of homogeneous fluid; (f) unsteady nonuniform flow of nonhomogeneous fluid without lateral discharge; (g) steady

spatially varied flow of nonhomogeneous fluid; (h) steady nonuniform flow of nonhomogeneous fluid without lateral discharge; and (i) steady uniform flow of nonhomogeneous fluid without lateral discharge. Conventionally used open-channel flow equations can be obtained by simplifications of the herein derived flow equations based on certain assumptions.

The one-dimensional dynamic equation based on the momentum concept, Eq. 30, is inherently different from that based on the energy concept, Eq. 55. In particular, the dissipated energy gradient S_e is different from the friction slope S_{fx} ; and both are different from the total-head gradient $-\partial H/\partial x$, the hydraulic gradient $(\partial H_p/\partial x) - S_o$, and the channel bottom slope S_o . Relationship between any two of these gradients can be derived from the general flow equations. Only for steady uniform flow of homogeneous fluid without lateral flow that these five gradients are equal to one another. Some qualitative nature of the dissipated energy coefficient f_e , friction resistance coefficient f_f , and total head loss coefficient f_H can be observed from the derived equations. However, further study is needed on the quantitative variations of f_f , f_e and f_H due to unsteadiness and nonuniformity of flow, nonhomogeneous fluid, and lateral flow.

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