STUDIES IN:
ENTANGLEMENT ENTROPY OF TWO DIMENSIONAL QUASI-TOPOLOGICAL QUANTUM FIELD THEORY
AND
GEOMETRY OF THE EXACT RENORMALIZATION GROUP AND HIGHER SPIN HOLOGRAPHY

BY

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DISSERTATION

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Abstract

Part I: Entanglement Entropy of 2d Quasi-Topological Quantum Field Theory

We compute the entanglement entropy of two-dimensional quasi-topological quantum field theories (QTFTs). These are theories in which the correlation functions depend on the topology and on the total area of the underlying space-time, but are blind to all local details of the geometry, and include topological quantum field theory (TFT) as their limiting case. We use two complimentary methods to compute the entanglement entropy; the first method is the replica trick and the other is to devise a novel tensor network representation, or more precisely, matrix product state (MPS) representation, of the quantum states of QTFT. We demonstrate that the two calculations are in agreement.

Part II: Geometry of the Exact Renormalization Group and Higher Spin Holography

We consider the Wilson-Polchinski exact renormalization group (RG) applied to the generating functional of single-trace operators at a free-fixed point in $d = 2 + 1$ dimensions. By exploiting the rich symmetry structure of free-field theory, we study the geometric nature of the RG equations and the associated Ward identities. The geometry, as expected, is holographic, with anti-de Sitter spacetime emerging correspondent with RG fixed points. In particular, we are able to cast the renormalization group equations as Hamilton equations of radial evolution in AdS_{d+1}. We solve these bulk equations of motion in terms of a boundary source and derive an on-shell bulk action. We demonstrate that it correctly encodes all of the correlation functions of the field theory, written as "Witten diagrams." Going further, we show that the field theory construction gives us a particular vector bundle over the $(d + 1)$-dimensional RG mapping space, called a jet bundle, whose structure group arises from the bilocal transformations of the bare fields in the path integral. The sources for quadratic operators constitute a connection on this bundle and a section of its endomorphism bundle. We make comparisons to Vasiliev-type higher spin theories. Detailed calculations are carried out for the case of Majorana fermions. Results and comments are presented for complex scalars. Additional details can be found in [1, 2].

Table of Contents

Chapter 1 Introduction to Part I: Entanglement Entropy in 2d QTFT 1

Chapter 2 Quasi-Topological Quantum Field Theories and YM_2 3
2.1 Quasi-topological quantum field theory by example: YM_2 3
2.2 Lattice formalism for general QTFT 4
2.3 Continuum limit of LQTFT as a deformation of LTFT 6
2.4 Example: Gauge theory 9

Chapter 3 Entanglement Entropy: Definitions and Examples 11
3.1 Entanglement entropy defined 11
3.2 Schmidt decomposition 12
3.3 Entanglement entropy via the replica trick 13
3.4 Example: The replica trick for 2d Yang-Mills 14
3.5 Generalization of entanglement entropy in QTFT to any genus 16

Chapter 4 MPS representations for QTFTs 18
4.1 Matrix Product States and Entanglement Entropy 18
4.2 MPS representations for QTFTs 20
4.2.1 MPS: the no-handles time-slice 20
4.2.2 MPS: general slicings 22
4.3 Example: Gauge theory (continued) 23
4.3.1 Gauge theory: MPS without handles 24
4.3.2 Gauge theory: MPS with handles 27
4.3.3 Fixing the normalization 27

Chapter 5 Entanglement Entropy Calculations for 2d Gauge Theory 30
5.1 Generalized YM_2 entanglement entropy via the replica trick 30
5.2 Entanglement entropy from the matrix product state 32

Chapter 6 Conclusion and Discussion For Part I 35

Chapter 7 Introduction to Part II: Geometry of ERG & Higher Spin Holography 36
7.1 Higher spin 37
7.1.1 A brief history of higher spin 37
7.1.2 Vasiliev's higher spin theories 37
7.2 The Klebanov-Polyakov conjecture 38

Chapter 8 The Bilocal Actions . . . 39
8.1 The free fixed points . . . 39
8.1.1 Free fixed point of the Majorana theory . . . 39
8.1.2 Free fixed point of the bosonic theory . . . 40
8.2 Deforming and regularizing the Majorana theory . . . 40
8.2.1 Making sense of the path integral normalization . . . 42
8.3 The Complex Scalar in 2+1 . . . 43

Chapter 9 Symmetries Of The Bilocal Actions . . . 44
9.1 Majorana Symmetries: $O(L_2)$, $CO(L_2)$ . . . 44
9.1.1 The bilocal gauge transformation . . . 44
9.1.2 The flat connection . . . 45
9.1.3 Extension to $CO(L_2)$ . . . 46
9.2 Bosonic Symmetries: $U(L_2)$, $CU(L_2)$ . . . 47

Chapter 10 From Renormalization Group to Holography . . . 49
10.1 RG flow of the Majorana theory . . . 51
10.1.1 Exact RG equations: Wilson-Polchinski renormalization in two steps . . . 51
10.1.2 Exact RG equations: extension to the bulk and emergence of AdS_{d+1} . . . 55
10.1.3 Callan-Symanzik equations . . . 57
10.1.4 The RG Ward identity . . . 59
10.2 Holography as Hamilton-Jacobi: the Majorana case . . . 59
10.3 Bosonic RG flow of the bosonic theory . . . 61
10.3.1 Exact RG equations: Wilson-Polchinski renormalization in two steps . . . 61
10.3.2 Exact RG equations: extension to the bulk and emergence of AdS_{d+1} . . . 62
10.3.3 Callan-Symanzik equations . . . 62
10.3.4 The RG Ward identity . . . 63
10.4 Holography as Hamilton-Jacobi: the bosonic case . . . 63

Chapter 11 Correlation Functions . . . 65
11.1 The two-point function of the Majorana theory . . . 65
11.2 Higher-point functions and Witten diagrams . . . 66
11.2.1 Higher-point functions of the Majorana theory . . . 66
11.2.2 Higher-point functions of the Bosonic theory . . . 71

Chapter 12 Formalizing with Jet Bundles . . . 74

Appendices . . . 76

Appendix A Singular Value Decomposition . . . 77
A.1 SVD in practice . . . 77
A.2 QR and LQ variations of SVD . . . 77
A.3 Canonicalization of MPS for generalized YM_2 and extraction of the entanglement entropy . . . 78

Appendix B Hamiltonian . . . 82
B.1 The Hamiltonian in QTFT . . . 82
B.2 The groundstate energies of generalized YM_2 . . . 83

Appendix C The Migdal Formalism for YM_2 and insertion of Wilson Loops . . 85
C.1 Migdal formalism . . . 85
C.2 Wilson Loop Insertions in Gauge Theory . . . 86
C.2.1 Contractible Wilson Loops . . . 87
C.2.2 Non-Contractible Wilson Loop . . . 87
C.2.3 Multiple Non-Contractible Wilson Loops . . . 88
C.3 Entanglement Entropy with Wilson Loops . . . 88
C.3.1 Entanglement Entropy with a Contractible Wilson Loop . . . 89
C.3.2 Entanglement Entropy with Non-Contractible Wilson Loop straddling regions A and B . . . 89
C.3.3 Entanglement Entropy with Non-Contractible Wilson Loop restricted to region A or region B . . . 90

References . . . 91

Chapter 1

Introduction to Part I: Entanglement Entropy in 2d QTFT

Entanglement entropy in quantum field theories has been a subject of ever increasing interest over the past twenty years. One of the original motivations was an attempt to better understand the area law scaling of black hole entropy [3, 4]. Entanglement entropy has since been established as a useful quantity to characterize quantum field theories and quantum many-body systems. One area of continual interest has been the scaling of entanglement entropy in the vicinity of quantum critical points [5]. Examples of more recent studies include the relationship of entanglement entropy to holography [6, 7], and entanglement entropy as a generalization of the c-theorem in higher dimensions [8]. In gauge theories, it was demonstrated that entanglement entropy can be used to detect confinement-deconfinement transitions [9]. Another popular direction of study has been entanglement entropy in topological field theories [10, 11, 12]; in particular, the topological entanglement entropy has become a vital (numerical) tool to diagnose topological order that can potentially be realized in (2+1) dimensional lattice quantum systems [13, 14].

In this paper, we will focus on two-dimensional quasi-topological quantum field theories (QTFTs) [15, 16, 17]. These are theories in which the correlation functions depend on the topology and on the total area of the underlying space-time, but are blind to all local details of the geometry. This is a slight generalization of the more familiar concept of a topological quantum field theory (TFT). In fact, every QTFT can be understood as a deformation of some TFT.

We will use two complimentary methods to compute the entanglement entropy; the first method is the replica trick which involves the evaluation of the path integral on replicated Riemann surfaces. The other method is to develop a tensor network representation, or more precisely, matrix product state (MPS) representation, of the quantum ground states of QTFT. A subsequent Schmidt decomposition (singular value decomposition) gives us the entanglement entropy. We will show that the two calculations are in agreement.

Tensor network algorithms [18] have been developed as an efficient representation of quantum ground states of lattice many-body systems. The simplest example of a tensor network is the matrix product state (MPS). It is the tool of choice in implementing density matrix renormalization group (DMRG) and is particularly well-suited for the description of ground states of gapped systems [19]. In fact, it is known that ground states of any gapped 1+1 dimensional quantum lattice system can be faithfully represented by an MPS [20]. In recent years, MPS and tensor network representations have been applied to gauge field theories as well as topologically-ordered lattice quantum systems; see, for example, [21, 22, 23]. In this paper, we will seek a MPS representation of QTFTs in two dimensions, thereby extending the notion of a tensor network beyond the context of spin lattice systems (Fig. 1.1). After showing how one can define a sensible MPS construction for a quasi-topological field theory, we will perform entanglement entropy calculations using a standard MPS technique (namely singular value decomposition (SVD)).

The structure of this paper is as follows. In Section 2 we introduce the concept of a two-dimensional quasi-topological quantum field theory and the canonical example, 2d Yang-Mills

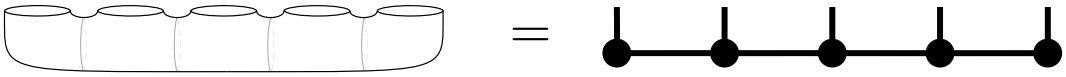


Figure 1.1: A schematic matrix product representation of a QTFT.

(YM_2). We then develop a general formalism in which we can study QTFT [16, 17]. This development requires first introducing a lattice construction, although in practice we will only use the formalism in the continuum limit. In Section 3 we discuss entanglement entropy. We show how to extract this quantity directly from a Schmidt decomposition as well as from the replica trick. In Section 4, we discuss matrix product states and how to represent the states of a QTFT as MPS. In Section 5, we return to entanglement entropy calculations in QTFT, now using the MPS structure to organize direct calculations via Schmidt decomposition. In particular, we calculate entanglement entropy for generalized YM_2 using two different methods – the replica trick and singular value decomposition – and find them to be in agreement, as one would hope. In Section 6, we summarize our results and discuss future directions of work.

Chapter 2

Quasi-Topological Quantum Field Theories and YM_2

2.1 Quasi-topological quantum field theory by example: YM_2

We begin our study of QTFT with the most well known example, two-dimensional Yang-Mills theory (YM_2). This is governed by the action

$$S[A] = -\frac{1}{2e^2}\int_\Sigma \mathrm{Tr}\,(F\wedge\star F) = -\frac{1}{4e^2}\int_\Sigma \sqrt{g}\; d^2x\; g^{\mu\sigma}g^{\nu\varrho}\;\mathrm{Tr}\,(F_{\mu\nu}F_{\sigma\varrho})\,. \tag{2.1}$$

Here $g(x)$ is a background metric on some Riemann surface Σ and $F = dA + A\wedge A$ is the curvature 2-form constructed from the gauge field in the usual way. In writing (2.1), it is implicit that we have chosen a gauge group G and constructed a principle bundle $P_{\Sigma,G}$ fibered in this group. To make the quasi-topological nature of the YM_2 action explicit, we must first rearrange things a bit.[1] To do this, recall that the curvature, being a gauge-covariant 2-form, must be proportional to the volume form ϵ. Therefore, we may write $F = f\epsilon$, where f transforms in the adjoint representation of G. Equation (2.1) now reduces to

$$S = -\frac{1}{2e^2}\int_\Sigma \epsilon\;\mathrm{Tr}(f^2). \tag{2.2}$$

The explicit metric dependence has disappeared, replaced by the volume form. Apparently, the YM_2 action sees areas but not distances. This action is invariant under area-preserving diffeomorphisms and is therefore a quasi-topological field theory. In the weak coupling limit $e \to 0$, this theory becomes strictly topological. This is not obvious from (2.1) but can be demonstrated explicitly by the introduction of an auxiliary field. [2] The existence of such a limit is a general property of QTFTs and will be manifest in our construction below.

It has long been known that two-dimensional Yang-Mills (in the continuum) can be reduced to an exactly solvable problem in lattice gauge theory via a certain procedure, which we will refer to as the Migdal formalism, involving triangulation of the space-time [24]. The key principle behind this procedure is the invariance of the theory under area-preserving diffeomorphisms (*i.e.*, its quasi-topological nature). A structured account of this formalism, which is not a primary method of computation in this paper, has been relegated to Appendix (C.1). That said, the Migdal formalism is the quickest way to derive the exact partition function of YM_2 on a closed Riemann surface of

[1] Here we closely follow the presentation found in [15].

[2] Following [15], we introduce an adjoint-valued scalar field ϕ and integrate over it:

$$Z = \int \mathcal{D}A\; e^{-S[A]} = \int \mathcal{D}A\; e^{-\frac{1}{2e^2}\int_\Sigma \epsilon \mathrm{Tr} f^2} = \int \mathcal{D}\phi\mathcal{D}A\; e^{-\frac{e^2}{2}\int_\Sigma \epsilon\;\mathrm{Tr}\;\phi^2 - i\int_\Sigma \mathrm{Tr}(\phi F)} \xrightarrow{e\to 0} \int \mathcal{D}\phi\mathcal{D}A\; e^{-i\int_\Sigma \mathrm{Tr}(\phi F)},$$

which is fully metric independent.

genus g and area ϱ. The correct expression, as can be found in appendix C.1, is

$$Z(\varrho, g) = \sum_{R \in \text{irrep(G)}} d_R^{2-2g} e^{-\frac{e^2 \varrho \, c_2(R)}{2}}. \tag{2.3}$$

The sum is over all irreducible representations of the gauge group G. The symbols d_R and $c_2(R)$ are, respectively, the dimension and the quadratic Casimir of representation R. The Migdal equation is a well known result and we will use expression (2.3) to perform YM_2 entanglement entropy calculations via the replica trick in §3. We will, however, rederive the YM_2 partition function in §4 using a different formalism, which we now introduce.

2.2 Lattice formalism for general QTFT

We will now develop a formalism in which many subsequent calculations will be done. The steps are as follows. First we introduce the notion of a lattice quasi-topological quantum field theory (LQTFT). We then take the continuum limit and learn that every quasi-topological field theory is a deformation of some underlying lattice topological field theory (LTFT). Furthermore, the set of all LTFTs is in one-to-one correspondence with the set of associative algebras. We can therefore build customized QTFTs by choosing any associative algebra we please and determining the set of allowed deformations (to be discussed below). This recipe will be demonstrated in the context of gauge theory, where, as it turns out, the underlying algebra is the group ring $\mathbb{C}[G]$. The formalism developed here is adopted from [16, 17]. The reader may refer to these references for a more thorough construction.

Let $\Sigma_{g,\varrho}$ be a compact, oriented surface of genus g and area ϱ. We triangulate this surface and assign to each trianglular plaquette an identical area $\epsilon = \frac{\varrho}{n}$, where n is the total number of triangles. (The identical area constraint is a convenience, not a necessity.) To each of the edges of a triangle we assign an index i and associate to that triangle a complex-valued weight C_{ijk}. The ordering (ijk) defines an orientation; all cyclic permutations are equivalent: $C_{ijk} = C_{jki} = C_{kij}$. Neighboring triangles are glued together along shared edges using a gluing operator g^{ij}. (See Fig. 2.1.) For now, the C_{ijk} and g^{ij} are arbitrary and unrelated, but soon we will relate the two objects through a shared algebraic structure. Thus a given triangulation is assigned a particular algebraic expression involving the C_{ijk} and g^{ij}.

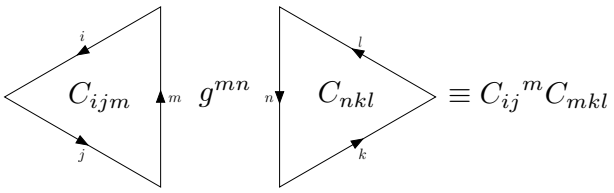


Figure 2.1: Gluing two triangles along one edge. After gluing, a raised index can be understood as labeling an oriented edge with a flipped arrow (*i.e.,* clockwise directed).

An alternative *dual triangulation* formalism replaces each triangle with a double-lined 3-point vertex and each gluing operator with a double-lined 2-point vertex (Fig. 2.2). The double-lined notation encodes the local orientation of the surface. (Allowing the two parallel lines to cross would indicate a disorienting twist in the topology. We will not consider such surfaces.)

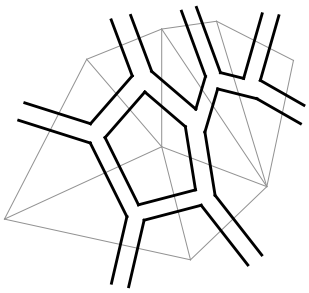


Figure 2.2: An example dual triangulation.

The partition function is the sum over all configurations with appropriate gluings enforced:

$$Z = \prod_{\text{triangles}} C_{ijk} \prod_{\text{glued links}} g^{ab}.$$

So far this construction is very general. Now, to enforce invariance of the partition function under area-preserving diffeomorphisms, we will require correlation functions to remain invariant under local area-preserving rearrangements of the geometry. For instance, the two ways of bipartitioning a square plaquette shown in Fig. 2.3 are equivalent. This equivalence is called *flip symmetry*. The process of flipping between these two equivalent representations is called the flip move.[3]

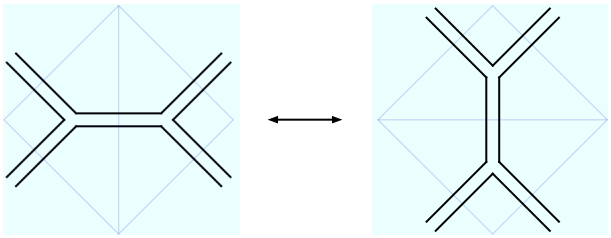


Figure 2.3: Flip symmetry.

Correlation functions are understood as follows. Given an operator $\mathcal{O}_i$, the insertion of such an operator into a correlation function is a puncture of the surface $\Sigma_{g,\varrho}$, with the boundary labeled by an index i. From here we can construct what will be the building blocks of our matrix product states. First, we construct the *2-point function*, $\eta_{ij} = \langle \mathcal{O}_i \mathcal{O}_j \rangle_{g=0}$, which is just a cylinder (*i.e.*, a two holed sphere) with its boundaries labeled i and j. The simplest triangulation is depicted in Fig. 2.4.

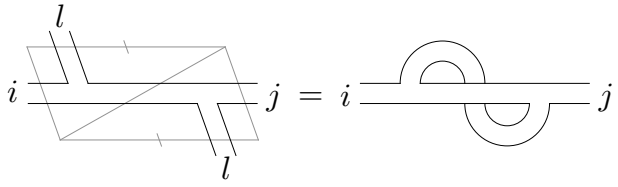


Figure 2.4: Minimal triangulation for the 2-point function.

[3] In the case of a purely topological field theory, there is an additional symmetry that must be enforced on the triangulated spacetime. It is called *bubble symmetry* and is depicted in Fig. 2.6. The meaning of that symmetry is that the TFT is uneffected by adding or removing area from the underlying spacetime.

The associated weight is[4]

$$\eta_{ij} = C_{ik}{}^{l} C_{lj}{}^{k}.$$

The *3-point function*, $\mathcal{N}_{ijk} = \langle \mathcal{O}_i \mathcal{O}_j \mathcal{O}_k \rangle_{g=0}$, is topologically the pair-of-pants. To assemble this object, we take a triangle and wrap two of its edges about two disjoint cylinders (Fig. 2.5).

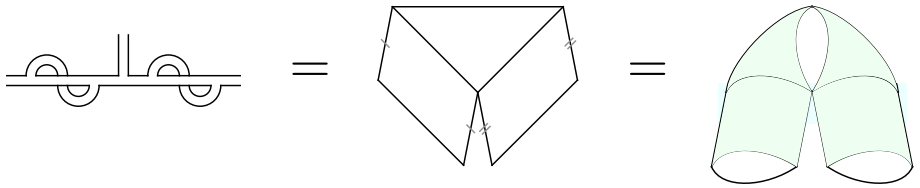


Figure 2.5: Minimal triangulation for the three point function.

The associated weight is

$$\mathcal{N}_{ijk} = \eta_i{}^{i'} \eta_j{}^{j'} C_{i'j'k} = \eta_k{}^{k'} \eta_i{}^{i'} C_{i'jk'} = \eta_j{}^{j'} \eta_k{}^{k'} C_{ij'k'},$$

All three of these expressions are equivalent, as can be demonstrated by flip symmetry.[5] Also by flip symmetry, it can be shown that $\mathcal{N}$ is totally symmetric, not just cyclically symmetric, on its three indices.

2.3 Continuum limit of LQTFT as a deformation of LTFT

We now discuss the continuum limit and show that every QTFT can be understood as a deformation of some lattice topological quantum field theory (LTFT).

In the continuum limit, the total area $\varrho = n\epsilon$ is held fixed while $\epsilon \to 0$ and $n \to \infty$. To properly understand this limit, it is of fundamental importance to understand the effect of $\epsilon \to 0$ on "bubbles", which are objects of weight $K_{ij} \equiv C_{ik}{}^{l} C_{jl}{}^{k}$.[6] As illustrated in Fig. 2.6, as ϵ goes to zero, these bubbles effectively collapse – *i.e.,* to zeroeth order in ϵ, a bubble takes on the reduced role of a gluing operator. In other words, bubbles are disks; they encode regions of area but are topologically trivial.

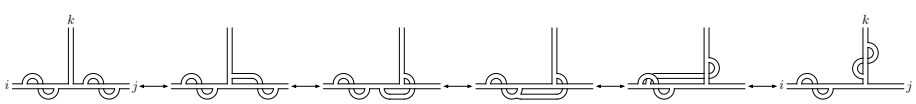
We therefore identify the weight of the metric with that of a bubble in the continuum limit:

$$g_{ij} = K_{ij}(0) \equiv C_{ik}{}^{l}(0) C_{jl}{}^{k}(0). \tag{2.4}$$

Here we have introduced the notation $C_{ijk}(\epsilon)$ to mean the weight of a triangular plaquette of area ϵ (with ϵ approaching zero in the continuum limit). Relationship (2.4) is referred to as *bubble symmetry*. A lattice theory with both bubble symmetry and flip symmetry is a called a *lattice topological field theory* [16]. It is in the continuum limit of such theories that we recover the

[4]The raised indeces imply contractions with g^{ij}, as explained in Fig. 2.1.

[5] For instance, to demonstrate $\eta_i{}^{i'} \eta_j{}^{j'} C_{i'j'k} = \eta_k{}^{k'} \eta_i{}^{i'} C_{i'jk'}$, we have the following sequence of flip moves:



[6]Bubbles can also be thought of as cones.

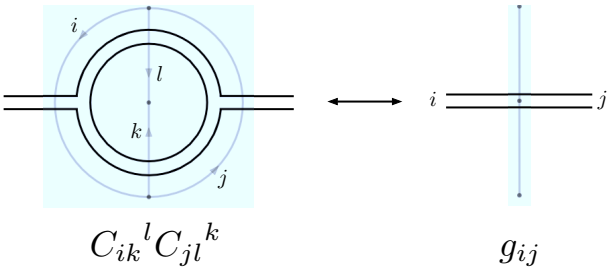


Figure 2.6: In the limit $\epsilon \to 0$, the bubble, K_{ij}, approaches zero area, and hence should contribute the same weight as the metric, g_{ij}. (Note: collapsing a bubble does not change the global topology.)

familiar set of topological field theories. The $C_{ijk}(0)$ can therefore be understood as weights of triangular plaquettes in some LTFT.

It is clear from (2.4) that for any given LTFT, the entire content of the theory, aside from the topology of the underlying space-time, is encoded in the set $\{C_{ij}{}^{k}(0)\}$. We now introduce the aforementioned associative algebra, which we will denote A_0. The set of basis elements of this algebra will be denoted $\{\phi_i\}$, while the structure constants of the algebra are to be identified with the weights of triangular plaquettes in the LTFT (but with one index raised), $\{C_{ij}{}^{k}(0) \,|\, \phi_i\phi_j = C_{ij}{}^{k}(0)\phi_k\}$. The condition of associativity is guaranteed by flip symmetry, as one can easily check. Furthermore, it can be shown that the set of associative algebras and LTFT's are in one-to-one correspondence [16].

Now, if a quasi-topological lattice theory is to have a well-defined (smooth) continuum limit, it must be true that $K_{ij}(\epsilon)$ can be expressed as a perturbation away from the corresponding bubble $K_{ij}(0)$ of the associated LTFT:

$$K_{ij}(\epsilon) = K_{ij}(0) + \epsilon\partial_\epsilon K_{ij}(\epsilon)|_{\epsilon=0} + \mathcal{O}(\epsilon^2) \equiv g_{ij} + 2\epsilon B_{ij} + \mathcal{O}(\epsilon^2). \tag{2.5}$$

Here we have made the convenient definition

$$B_{ij} \equiv \frac{1}{2}\partial_\epsilon K_{ij}(\epsilon)|_{\epsilon=0} = \frac{1}{2}\partial_\epsilon \left(C_{ik}{}^{l}(\epsilon)C_{jl}{}^{k}(\epsilon)\right)\Big|_{\epsilon=0}. \tag{2.6}$$

We can also express the weight of a triangular face Δ of finite area ϱ_Δ in this perturbative language. The idea is to start with the simplest triangulation of Δ (*i.e.*, a single plaquette), and then take the continuum (refinement) limit. In the language of dual triangulations, this corresponds to introducing more and more bubbles, while holding the total area of the triangular face fixed (Fig. 2.7).

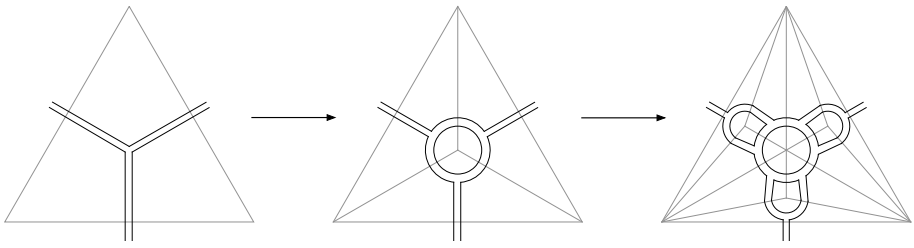


Figure 2.7: The first three steps in the continuum bubble-refinement of a triangular face.

By repeated use of the flip move, we can then push these bubbles out into one long bubbly

string (Fig. 2.8).

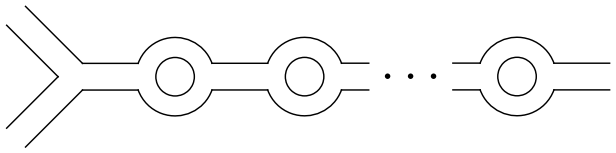


Figure 2.8: The large bubble limit of Fig. 2.7 rearranged so that all bubbles appear single file. If the total number of vertices in this diagram is n_Δ and the total area is ϱ_Δ, then there are $\frac{n_\Delta - 1}{2}$ bubbles, each of area $2\frac{\varrho_\Delta}{n_\Delta}$.

In this configuration, the total weight of the triangular face Δ is easy to read off:[7]

$$\lim_{n_\Delta \to \infty} C_{ijl}(\epsilon) \left[K(\epsilon)^{\frac{n}{2}}\right]_k^{\ l} = C_{ijl}(0) \lim_{n_\Delta \to \infty} \left[\left(\mathbf{1} + \frac{2\varrho_\Delta}{n_\Delta} B\right)^{\frac{n_\Delta - 1}{2}}\right]_k^{\ l} = C_{ijl}(0) \left(e^{\varrho_\Delta B}\right)_k^{\ l},$$

In conclusion,

$$\mathscr{C}_{ijk}(\varrho_\Delta) = C_{i'jk}(0) \left[e^{\varrho_\Delta B}\right]_i^{\ i'}.$$

We use a script $\mathscr{C}_{ijk}$ here to denote the fact that this is the weight of a triangular face of finite area in the continuum limit, while $C_{ijk}(0)$ is the weight of an individual triangular plaquette of vanishingly small area (*i.e.,* it is an LTFT triangle weight). In what follows, we will be less careful and reuse notation from the lattice theory to denote weights in the continuum theory. For example, η_{ϱ_α} will mean the weight of a cylinder in the continuum theory, with total area ϱ_α. It is easy to show that

$$\boxed{\left(\eta_{\varrho_\alpha}^{i_\alpha}\right)_j = C_{i'k}{}^m(0) C_{mj}{}^k(0) \left(e^{\varrho_\alpha B}\right)_{i_\alpha}^{\ i'} \equiv \left(\eta_0^{i'}\right)_j \left(e^{\varrho_\alpha B}\right)_{i_\alpha}^{\ i'}} \tag{2.7}$$

$$\boxed{\left(\mathcal{N}_{\varrho_\alpha}^{i_\alpha}\right)_{j_2}^{j_1} = C_{j_1 k_1}^{\ \ l_1}(0) C_{l_1 j_1'}^{\ \ k_1}(0) C_{i' j_1'}^{\ \ j_2'}(0) C_{j_2 k_2}^{\ \ l_2}(0) C_{l_2 j_2'}^{\ \ k_2}(0) \left(e^{\varrho_\alpha B}\right)_{i_\alpha}^{\ i'} \equiv \left(\mathcal{N}_0^{i'}\right)_{j_2}^{j_1} \left(e^{\varrho_\alpha B}\right)_{i_\alpha}^{\ i'}} \tag{2.8}$$

Here we have also introduced the notation $\left(\eta_0^i\right)_j$ and $\left(\mathcal{N}_0^i\right)_{j_2}^{j_1}$ to mean, respectively, the weights of the cylinder and three-holed sphere in the corresponding topological theory.

This structure is useful because we can quickly write down the weights for different perturbations away from the same underlying LTFT. The matrix B parametrizes this family of LQTFT's. Note that the form of the B matrix is not completely arbitrary. For a given LTFT, it is constrained by the following conditions:

$$B = B^{\mathbf{T}}, \qquad \left[B, C_i^0\right] = 0, \qquad \left[B, C_i^{0\mathbf{T}}\right] = 0, \tag{2.9}$$

where $\left(C_i^0\right)_j^{\ k} \equiv C_{ij}^{\ \ k}(0)$. One can check the first two constraints directly from equations (2.6) and (2.5), respectively. These are, in fact, the only two independent constraints on B [17]. The third constraint is clearly the transpose of the second constraint (subject to the first constraint), but it proves useful enough to write as a separate equation.

[7] Here we use $K_i{}^j(0) = g_i{}^j = \delta_i{}^j$ and the fact that there is one bubble for every two (fundamental) triangles.

2.4 Example: Gauge theory

We now follow the above prescription for a specific example. We choose the LTFT algebra A_0 to be the group ring

$$A_0 = \mathbb{C}[G] = \bigoplus_{g\in G} \mathbb{C}\phi_g,$$

where G is our desired gauge group. In this scenario, the algebra inherits multiplication directly from the group G:

$$\phi_i\phi_j = \gamma\,\phi_{ij} \quad \text{for } i,j \in G. \tag{2.10}$$

We have included an additonal factor γ, which we will need for normalization purposes later on. (Note that this is just the usual group ring with a rescaled basis: $\phi_i \to \frac{\phi_i}{\gamma}$, with $\gamma \in \mathbb{C}$. Eventually we will take $\gamma = |G|^{-1}$, but for now we leave it arbitrary.) From (2.10) we determine the structure constants of A_0:

$$\phi_i\phi_j = C_{ij}{}^k(0)\phi_k = \gamma\,\phi_{ij} \quad \Longrightarrow \quad C_{ij}{}^k(0) = \gamma\,\delta(ij,k),$$

where we have introduced the notation $\delta(i,j) \equiv \delta_i{}^j$. Using (2.4), we also construct the metric, inverse metric, and LTFT triangle weights:

$$g_{ij} = \gamma^2\,|G|\delta(i,j^{-1}), \quad g^{ij} = \frac{1}{\gamma^2\,|G|}\delta(i,j^{-1}), \quad C_{ijk}(0) = \gamma^3\,|G|\delta(ijk,\mathbf{1}), \tag{2.11}$$

where $|G|$ is the order of the group.[8] We determine the possible forms of the deformation matrix B from the constraints (2.9). Introducing the notation $B(i,j) \equiv B_i{}^j$, the constraints become

$$B(i,j) = B(j,i), \quad B(i,j) = B(j^{-1}i,\mathbf{1}), \quad B(i,j) = B(ij^{-1},\mathbf{1}). \tag{2.12}$$

From this, we see that $B(i,j)$ depends on its arguments only in terms of the specific product ij^{-1} and, furthermore, it is a class function – *i.e.*, $B(iji^{-1},\mathbf{1}) = B(j,\mathbf{1})$ – so it may be character expanded as

$$B(i,j) = \sum_R B_R\,d_R\,\chi_R(ij^{-1}). \tag{2.13}$$

The sum runs over all irreducible representations of G. The group character, $\chi_R(g)$, is the trace of group element g in representation R. The complex coefficients B_R are unconstrained and parametrize a family of quasi-topological quantum field theories. For computational convenience, we have extracted from B_R a number d_R, which is the dimension of representation R. As will become evident later (see §4.3), what we are building here is a family of QTFTs commonly referred to as *generalized YM*$_2$. In particular, the choice $B_R = -\frac{e^2 c_2(R)}{2}$ will reproduce the usual Yang-Mills theory in two dimensions. (Note, we will often use the generic nomenclature "gauge theory" in place of "generalized YM$_2$".)

As a final step, we determine the exact form of the matrices η^{i_α} and $\mathcal{N}^{i_\alpha}$, as given by (2.7) and (2.8):

$$\left(\eta^{i_\alpha}_{\varrho_\alpha}\right)_j = C_{i'k}{}^m(0)C_{mj}{}^k(0)\left(e^{\varrho_\alpha B}\right)_{i_\alpha}^{\ i'} = \gamma^2\sum_{k,R} e^{\varrho_\alpha B_R} d_R\chi_R(i_\alpha k j k^{-1}), \tag{2.14}$$

[8] For notational simplicity, we choose to work here, and throughout most of the paper, in the langauge of finite groups. However, everything done here should carry over in the obvious way to compact Lie groups (*e.g.*, $|G| \longrightarrow \mathbf{vol}(G)$).

$$\begin{aligned}\left(\mathcal{N}_{\varrho_\alpha}^{i_\alpha}\right)_{j_2}^{j_1} &= g^{l_1 l_1'} g^{k_1 k_1'} C_{k_1 l_1'}{}^{j_1}(0) C_{k_1' l_1}{}^{j_1'}(0) C_{i' j_1'}{}^{j_2'}(0) C_{j_2 k_2}{}^{l_2}(0) C_{l_2 j_2'}{}^{k_2}(0) \left(e^{\varrho_\alpha B}\right)_{i_\alpha}^{i'} \\ &= \frac{\gamma}{|G|^2} \sum_{k_1,k_2,R} e^{\varrho_\alpha B_R} d_R \chi_R(i_\alpha k_1^{-1} j_1^{-1} k_1 k_2^{-1} j_2 k_2), \end{aligned} \tag{2.15}$$

where we have re-summed the exponentials as

$$\left(e^{\varrho_\alpha B}\right)_{i_\alpha}^{i'} = \delta_{i_\alpha}{}^{i'} + \varrho_\alpha B_{i_\alpha}{}^{j} + \frac{1}{2}\varrho_\alpha^2 \sum_j B_{i_\alpha}{}^{j} B_j^{i'} + \ldots = \sum_R e^{\varrho_\alpha B_R} d_R \chi_R(i_\alpha i'^{-1}).$$

We will continue this example when we build up matrix product state representations for quasi-topological field theories in §4.

Chapter 3

Entanglement Entropy: Definitions and Examples

In this section we provide a brief overview of entanglement entropy. This will keep the paper self-contained as well as provide the conceptual framework in which entanglement entropy can be understood in the context of quasi-topological quantum field theories.

3.1 Entanglement entropy defined

Entropy is a familiar concept from statistical mechanics. Von Neumann defined the entropy of a quantum state ρ to be

$$S(\rho) = -\mathrm{Tr}(\rho \log \rho), \tag{3.1}$$

where ρ is the density matrix.

In this paper our focus will be on quantum pure states: $\rho = |\psi^-\rangle\langle\psi^+|$.[9,10] As a whole, a system in a pure state has zero von Neumann entropy. However, if there exists an observer for whom only a subset of the observables of the entire system are accessible, the entropy, as measured by that observer, will in general be nonzero. This is the concept of the *entanglement entropy*, a refinement of Von Neumann's entropy. The usual construction is as follows. Start by bipartitioning the system at some fixed time, calling one piece A and the other B. More specifically, consider the situation in which the Hilbert space can be written as $\mathscr{H}_{\text{total}} = \mathscr{H}_A \otimes \mathscr{H}_B$. In a local quantum field theory, this splitting is usually accomplished by distinguishing a spatial region A from its complement B. From the perspective of an observer who only has access to the observables of subsystem A, the state of the total system appears to be $\rho_A = \mathrm{Tr}_B\, \rho$, where the partial trace is only over the degrees of freedom of B. In general, this *reduced density* matrix, ρ_A, will be in some quantum mixed state.[11] We define the entanglement entropy of subsystem A to be the Von Neumann entropy of the reduced state:

$$S_A(\rho) = -\mathrm{Tr}_A(\rho_A \log \rho_A). \tag{3.2}$$

[9] Note that we are using plus and minus labels to distinguish the bra from the ket. We will elaborate on the meaning of this distinction in §3.5.

[10] In conventional field theories, such states are usually accessed by taking the system to zero temperature, hence projecting to the ground state. We will not be taking that route. The meaning of temperature in our construction will be discussed later.

[11] A mixed state is a state of the form $\rho = \sum_i p_i \left|\psi^{(i)-}\right\rangle\!\left\langle\psi^{(i)+}\right|$ where at least two of the p_i are nonzero. The p_i are classical ensemble weights ($0 \le p_i \le 1,\ \sum_i p_i = 1$) as well as the eigenvalues of ρ. A pure state is one in which p_i is one for exactly one state and zero for all others; thus, explaining why expression (3.1) is zero for a pure state.

3.2 Schmidt decomposition

For a given pair of pure quantum states $|\psi^-\rangle$ and $\langle\psi^+|$, defined over the same time-slice, a useful expression for the entanglement entropy can be obtained from their Schmidt decompositions:

$$\left|\psi^-\right\rangle = \sum_{i,j} \Psi^-_{ij} \left|i\right\rangle_A \otimes \left|j\right\rangle_B , \qquad \left\langle\psi^+\right| = \sum_{i,j} \left\langle i\right|_A \otimes \left\langle j\right|_B \Psi^+_{ij},$$

where $\{|i\rangle_A\}$ and $\{|j\rangle_B\}$ are orthonormal bases for $\mathscr{H}_A$ and $\mathscr{H}_B$ respectively. The $\{\Psi^\pm_{ij}\}$ are sets of complex-valued coefficients. If we can somehow write the kets (bras) of $\mathscr{H}_{\text{total}}$ ($\bar{\mathscr{H}}_{\text{total}}$) in this form, the calculation of entanglement entropy is trivial. To see this, first focus on the ket and perform a *singular value decomposition* (SVD) on Ψ_{ij}:

$$\left|\psi^-\right\rangle = \sum_{i,j,k} U^-_{ik} S^-_{kk} V^-_{kj} \left|i\right\rangle_A \otimes \left|j\right\rangle_B = \sum_k s^-_k \left(\sum_i U^-_{ik} \left|i\right\rangle_A\right) \otimes \left(\sum_j V^-_{kj} \left|j\right\rangle_B\right) \equiv \sum_k s^-_k \left|k^-\right\rangle_A \otimes \left|k^-\right\rangle_B .$$

Here, U^- is *left-normalized* ($U^{-\dagger}U^- = \mathbf{1}$), V^- is *right-normalized* ($V^-V^{-\dagger} = \mathbf{1}$), and S^- is square diagonal with diagonal elements $\{s^-_k\}$ known as *singular values*. Such a decomposition is guaranteed to exist. (Refer to §A.1 to see how such a decomposition is achieved in practice.) The exact same procedure can be used to decompose the bra: $\langle\psi^+| = \sum_l \langle l^+|_B \otimes \langle l^+|_A \, s^+_l$. The density matrix is then

$$\rho = \left|\psi^-\right\rangle\!\left\langle\psi^+\right| = \left(\sum_k s^-_k \left|k^-\right\rangle_A \otimes \left|k^-\right\rangle_B\right)\left(\sum_l \left\langle l^+\right|_B \otimes \left\langle l^+\right|_A s^+_l\right).$$

The reduced density matrix is found by tracing out the B degrees of freedom,

$$\rho_A = \mathrm{Tr}_B\, \rho \equiv \sum_j {}_B\!\left\langle k^+(j)\right|\rho\left|k^-(j)\right\rangle_B = \sum_k s^-_k s^+_k \left|k^-\right\rangle_{A\ A}\!\left\langle k^+\right| \equiv \sum_k s^2_k \left|k^-\right\rangle_{A\ A}\!\left\langle k^+\right| . \tag{3.3}$$

The notation $|k^-(j)\rangle_B$ reminds us that $|k^-(j)\rangle_B = V^-_{kj}\,|j\rangle_B$, where $|j\rangle_B$ is the original orthonormal basis for $\mathscr{H}_B$ (and similarly for the bra). In the final equality, we have made the convenient definition

$$s^2_k \equiv s^-_k s^+_k . \tag{3.4}$$

Since ρ_A is diagonal in the A basis, we can now read off the entanglement entropy:

$$S_A = -\mathrm{Tr}_A(\rho_A \log \rho_A) = -\sum_k (s^2_k \log s^2_k). \tag{3.5}$$

The set $\{s^2_k\}$ encodes the *entanglement spectrum*, which is a refinement of the entanglement entropy itself.[12]

[12] The entanglement spectrum can be described in the following way: If we were to reinterpret ρ_A in (3.3) as a sum over configurations of a statistical ensemble at temperature β_E^{-1}, then the s^2_k would be eigenvalues of the Boltzmann operator $e^{-\beta_E \hat{H}_E}$. The entanglement spectrum, as defined in [25], is the set of eigenvalues of the "entanglement Hamiltonian" $\hat{H}_E$; namely the set $\{E_k \,|\, E_k = -\frac{1}{\beta_E}\log s^2_k\}$. It is conventional to set the arbitrary parameter β_E to one.

3.3 Entanglement entropy via the replica trick

In this work, we will focus on entanglement entropy calculations in quantum field theory. We will employ two different methods for calculating entanglement entropies. One method relies on the Schmidt decomposition of states. The other method is the *replica trick*. This trick allows one to deal with the logarithm in (3.2), without having to explicitly diagonalize the density matrix.

The replica trick is the identification

$$S_A \equiv -\mathrm{Tr}_A(\rho_A \log \rho_A) = -\lim_{n\to 1}\left[\frac{d}{dn}\mathrm{Tr}_A \rho_A^n\right]. \tag{3.6}$$

We can naively justify this trick by the seemingly innocuous manipulations

$$\lim_{n\to 1}\left[\frac{d}{dn}\mathrm{Tr}_A\, \rho_A^n\right] = \lim_{n\to 1}\left[\frac{d}{dn}\sum_i \lambda_i^n\right] = \lim_{n\to 1}\sum_i \lambda_i^n \log\lambda_i = \mathrm{Tr}_A(\rho_A \log \rho_A),$$

where λ_i are the eigenvalues of ρ_A. This identification is useful because we can interpret $\mathrm{Tr}_A\, \rho_A^n$ as a path integral on a glued, n-sheeted geometry to be constructed below. However, this interpretation is also naive, because we implicitly assume that n has a unique analytic continuation from the natural numbers (at least in the vicinity of $n = 1$). Although the replica trick has earned a reputation as a reliable method of computation, the assumption of uniqueness is not true in general and care should be taken.[13] One of the nice features of the current work is that we will be able to check our replica trick results against a more direct calculation in which we explicitly diagonalize ρ. That calculation will be carried out in §4 and §5.

We now explain the replica trick in more detail. As a simple example, consider the case of a scalar field in two dimensions.[14] The construction is as follows. We put the total system in a thermal state $\rho = e^{-\beta\hat{H}}$, so that the field theory lives on the strip $\mathbb{R}\times[0,\beta]$. Later, we will take the zero temperature limit $\beta\to\infty$. We partition the system spatially, into parts A and B, at (imaginary) time $\tau = 0$. By then specifying field configurations ϕ_0 and ϕ_β at times $\tau = 0$ and $\tau = \beta$, we can write components of the density matrix as the following Euclidean path integral:

$$\rho_{\phi_0(x),\phi_\beta(x)} = Z(\beta)^{-1}\int_{\tau=0}^{\tau=\beta}[d\phi]\, e^{-S_{\mathrm{E}}}\,\delta(\phi(x,0)-\phi_0(x))\;\delta(\phi(x,\beta)-\phi_\beta(x))\,.$$

The partial trace over B corresponds to identifying $\phi(x,0)$ and $\phi(x,\beta)$ for $x\in B$ (Fig. 3.1):

$$(\rho_A)_{\phi_0(x|x\in A),\phi_\beta(x|x\in A)} = \int_{\tau=0}^{\tau=\beta}[d\phi_0]_{x\in B}[d\phi_\beta]_{x\in B}\;\delta(\phi_0(x)-\phi_\beta(x))\;\rho_{\phi_0(x),\phi_\beta(x)}.$$

Finally, we construct $\mathrm{Tr}\,\rho_A^n$ by cyclically pasting n copies of ρ_A in the following manner (Fig. 3.2):

$$\mathrm{Tr}\,\rho_A^n = \int\prod_{k=1}^{n}\left\{[d\phi_0^{(k)}][d\phi_\beta^{(k)}]\;\delta\Big(\phi_0^{(k)}(x)-\phi_\beta^{(k+1)(x)}\Big)_{x\in A}\;\delta\Big(\phi_0^{(k)}(x)-\phi_\beta^{(k)(x)}\Big)_{x\in B}\;\rho_{\phi_0^{(k)}(x),\phi_\beta^{(k+1)}(x)}\right\},$$

where $\phi^{(n+1)}$ and $\phi^{(1)}$ are identified.

At this point, we can take the zero temperature limit $\beta\to\infty$. Denoting $\mathrm{Tr}\,\rho_A^n$ as $\frac{Z_n}{Z_1^n}$ – *i.e.*, the

[13] See [26], for example, for a more in depth discussion of such issues.

[14] The following example is lifted from [12].

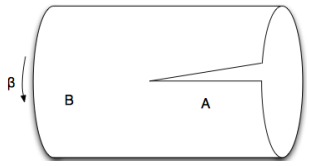


Figure 3.1: The geometry of the reduced density matrix ρ_A.

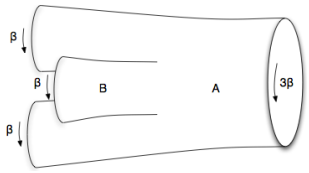


Figure 3.2: An example of a glued geometry. This is $\mathrm{Tr}\,\rho_A^3$.

normalized partition function on the n-sheeted surface – we arrive at

$$S_A = -\lim_{n\to 1} \frac{d}{dn} \frac{Z_n}{Z_1^n}. \tag{3.7}$$

3.4 Example: The replica trick for 2d Yang-Mills

The replicated geometry in the previous section was easy to draw, which made it a useful example. However, the focus of this paper is QTFT and so a more relevant example is 2d Yang-Mills.[15] In order to make use of Migdal's formula, equation (2.3), we need the traced geometries to be closed Riemann surfaces. We therefore start with YM_2 living on a finite length cylinder $S^1 \times [0, \beta]$, with axis along the (imaginary) time direction. At time $\tau = 0$ we split the region spatially into parts A and B. We will denote by I the number of disjoint intervals of region A. We then fix the states at times $\tau = 0$ and $\tau = \beta$ and trace out the B degrees of freedom. The result is a torus with I cuts in it (one for each disjoint subregion of A).

We next construct the glued geometry by making n replicas of the cut torus and pasting them together cyclically. Geometrically this is difficult to draw, although the result is topologically simple; it is again a Riemann surface. The genus of this glued surface is determined by the Riemann-Hurwitz formula:[16]

$$g_{n,I} = n(g + I - 1) - I + 1, \qquad \varrho_{n,I} = n\varrho. \tag{3.8}$$

[15] While this work was in preparation, the following calculation, as well as that for higher genus, appeared in [27]. Our calculations were done independently; however see [27] for discussions of the strong and weak coupling limits.

[16] Given a map $\Sigma_n \to \Sigma$, the Riemann-Hurwitz formula reads

$$\chi(\Sigma_n) = n\,\chi(\Sigma) - \sum_P (e_P - 1),$$

where $\chi(\Sigma) = 2 - 2g$ is the Euler characteristic of Riemann surface Σ, index P runs over the set of ramification points in Σ_n (*i.e.,* each P maps to a branch point on Σ), and e_P is the ramification index at P. In the case at hand, we have $2I$ ramification points (one for each boundary of A), each of which has ramification index n.

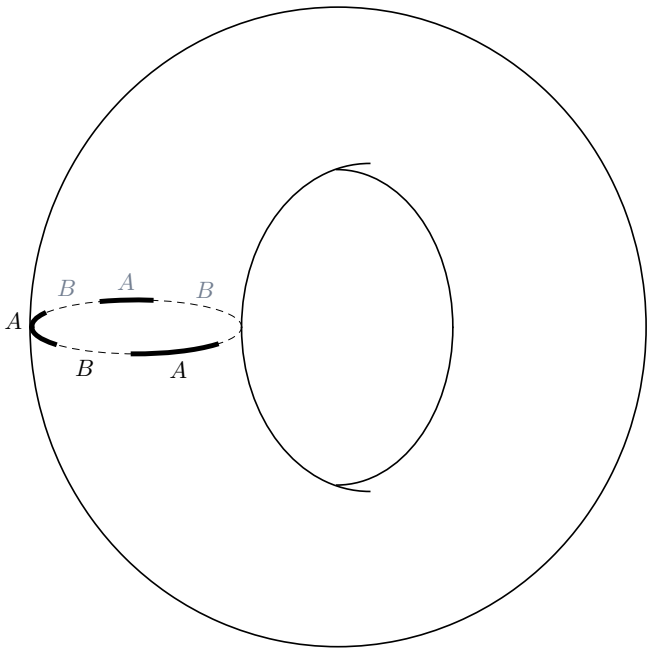


Figure 3.3: The geometry of ρ_A for the case $I = 3$. In this example, the partial trace over the B degrees of freedom yields a torus with three cuts. The locations of these cuts coincide with the disjoint set labeled by A.

The calculation of the entanglement entropy is now trivial. Combining (2.3) with (3.7), we have

$$S_A^{(T^2)} = -\lim_{n\to 1}\frac{d}{dn}\left.\frac{\sum_{R'} d_{R'}^{2-2g_{n,I}} e^{\varrho_{n,I} B_{R'}}}{\left(\sum_{R'} d_{R'}^{2-2g} e^{\varrho B_{R'}}\right)^n}\right|_{\substack{g=1\\(\text{torus})}} = -\sum_R \frac{e^{-\frac{e^2\varrho\, c_2(R)}{2}}}{\sum_{R'} e^{-\frac{e^2\varrho\, c_2(R)}{2}}} \log\left[\frac{d_R^{-2I} e^{-\frac{e^2\varrho\, c_2(R)}{2}}}{\sum_{R'} e^{-\frac{e^2\varrho\, c_2(R)}{2}}}\right]. \tag{3.9}$$

As one might expect, the entanglement entropy depends on the number of disjoint intervals comprising spatial region A, but not on the length of these intervals.[17] At this point, one would usually take the temperature to zero, forcing the total system into its groundstate. We might do that in this case by recognizing that the area of the torus is $\varrho \sim \beta\, L$, where L characterizes the circumference of space and β is the inverse temperature. In this way, the low temperature limit corresponds to the large area ($\varrho \to \infty$) limit of the theory, or equivalently, to the $e \to \infty$, strong coupling limit. However, one might question the validity of discussing the *length* of the timelike direction. After all, in QTFTs, physical observables of the theory cannot depend on details of the timelike components of the metric, it is only the determinant of the metric which has real physical meaning. (More will be said about this shortly.) In this paper we will avoid this confusion by keeping our areas finite.

[17] While the number of intervals, I, of subregion A appears naturally in the replica trick calculation, one might wonder where this dependence will enter in a direct calculation from the definition $S_A = -\mathrm{Tr}_A(\rho_A \log \rho_A)$. This question was addressed in [28], where it was shown that in gauge theories a naive splitting of the Hilbert space into $\mathscr{H}_A \otimes \mathscr{H}_B$ fails. The I dependence of S_A comes from contributions from non-gauge-invariant degrees of freedom residing on the entangling surface (*i.e.,* the points separating subregions A and B).

3.5 Generalization of entanglement entropy in QTFT to any genus

In the previous section, we calculated the partition function of YM_2 on $S^1 \times [0, \beta]$. This corresponded to taking a trace in the imaginary time direction, effectively compactifying the cylinder into a torus. However, the calculation of the entanglement entropy depended on the traced geometries, not on the details of how we arrived at those geometries. In fact, if we naively take the calculation in (3.9) at face value, we can calculate the entanglement entropy for (unreplicated) topologies of arbitrary genus. The result is simple to compute:[18]

$$S_A = -\sum_R \frac{d_R^{2-2g} e^{-\frac{e^2 \varrho\, c_2(R)}{2}}}{\sum_{R'} d_R'^{2-2g} e^{-\frac{e^2 \varrho\, c_2(R)}{2}}} \log \left[\frac{d_R^{2-2g-2I} e^{-\frac{e^2 \varrho\, c_2(R)}{2}}}{\sum_{R'} d_{R'}^{2-2g} e^{-\frac{e^2 \varrho\, c_2(R)}{2}}} \right]. \tag{3.10}$$

A more structured way to think about this is the following. We don't have to think of S^1 as space and $\mathbb{R}$ as (imaginary) time. We might instead prefer to think in the parametrization of Euclidean space-time depicted in Fig. 3.4.

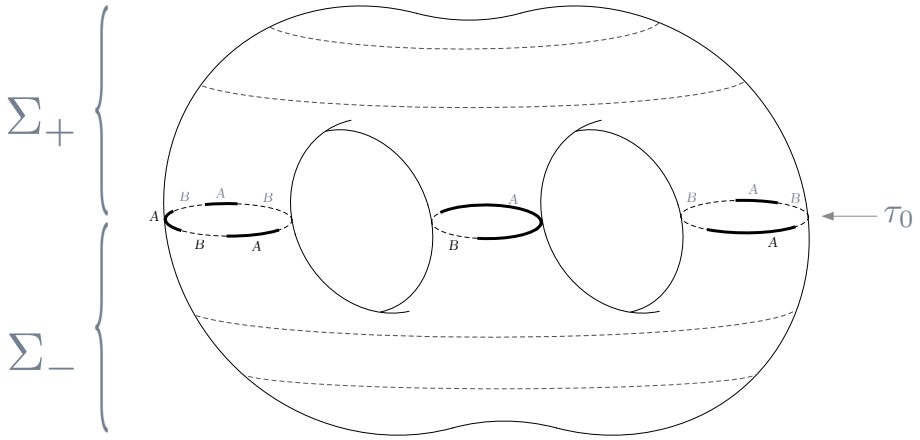


Figure 3.4: The dashed lines represent equal time slices. As time evolves, spatial regions may separate or merge. In this example, we are showing the geometry of the reduced density matrix ρ_A, where space at time τ_0 is the union of three disjoint circles.

The density matrix $\rho = |\psi^-\rangle\langle\psi^+|$ should now be understood in the following way. To construct the ket $|\psi^-(\tau_0)\rangle$ at some time τ_0, fix $\psi^-(\tau_0)$ and path integrate over all degrees of freedom localized over Σ_-.[19] To construct the bra $\langle\psi^+(\tau_0)|$, fix $\psi^+(\tau_0)$ and path integrate over all degrees of freedom localized over Σ_+. The meaning of the minus and plus labels is now clear; they distinguish the degrees of freedom attached to the lower vs. upper surfaces. This interpretation is natural and has been a cornerstone of topological quantum field theory since the early days of TFT (e.g. [29, 15]). From ρ, construct the reduced density matrix ρ_A by tracing out the B degrees of freedom at time τ_0. This corresponds, geometrically, to gluing Σ_- to Σ_+ over the B intervals. The result is demonstrated by example in Fig. 3.4. The replica trick now proceeds in the obvious way. We

[18] In actuality, there is an important caveat. We are here only considering the case in which none of the disjoint subregions of A (or B) are closed loops. The reverse scenario, in which subregions A and B are each, independently, a collection of S^1's, will be the focus of our calculations in §5.

[19] In the context of the QTFT lattice contruction introducd in §2, fixing $\psi^-(\tau_0)$ means specifying the set of boundary labels $\{i_\alpha\}$ for the lower surface at time-slice τ_0.

make n copies of the half-pasted geometry and then glue cyclically along the A intervals. The path integration associated with the fully glued geometry is the n-sheeted partition function Z_n and the entanglement entropy S_A is easily computed from equation (3.7). The result is, of course, equation (3.10).

Finally, although the usual notion of temperature has become obscured in this construction, the procedure is still well-defined and the entanglement entropies we calculate are still physically meaningful.

Chapter 4

MPS representations for QTFTs

As discussed earlier, the replica trick is one of the standard methods for calculating entanglement entropies. In this section we consider an alternative to the replica trick: the matrix product state construction of QTFTs and subsequent Schmidt decomposition. It will act as a check on our replica trick calculations.

4.1 Matrix Product States and Entanglement Entropy

We start by considering MPS in its native environment. Consider a 1D spin chain (*i.e.*, one spatial dimension). We will take $|\sigma_1 \cdots \sigma_N\rangle \equiv |\sigma_1\rangle \otimes \cdots \otimes |\sigma_N\rangle$ as a basis for the Hilbert space. Each spin σ_α can take on some set of particular values, for example $\sigma_\alpha \in \{\uparrow, \downarrow\}$. In this basis, a general quantum state can be written as the superposition

$$|\psi\rangle = \sum_{\sigma_1,\ldots,\sigma_N} \psi_{\sigma_1,\ldots,\sigma_N} |\sigma_1 \cdots \sigma_N\rangle \,, \tag{4.1}$$

where $\psi_{\sigma_1,\ldots,\sigma_N}$ is a numerical coefficient. This coefficient is a multilocal object. We can give it a more local flavor by decomposing it into a product of local factors; one factor to be associated with each spin site α. The idea of MPS, in particular, is to rewrite the coefficient as a product of matrices:

$$\psi_{\sigma_1,\ldots,\sigma_N} = M^{\sigma_1} M^{\sigma_2} \cdots M^{\sigma_{N-1}} M^{\sigma_N}, \tag{4.2}$$

where each M^{σ_α} is a matrix. In this conventional notation, the σ_i attached to each matrix should be understood as both a label, indicating the spin site the matrix is associated with, and an index, enumerating the physical states of that spin. This means that the matrix denoted M^{σ_α} is in general a different matrix than the one denoted M^{σ_β} (for $\alpha \neq \beta$). Also, note that in order for this product to produce a number, M^{σ_1} and M^{σ_N} must be row and column vectors, respectively. (To represent a system with periodic boundary conditions, use instead a trace: $\psi^{\text{PBC}}_{\sigma_1,\ldots,\sigma_N} = \text{Tr}[M^{\sigma_1} \cdots M^{\sigma_N}]$.)

There is a conventional diagrammatic representation for an MPS. Each $(M^{\sigma_\alpha})^a{}_b$ of the matrix product is represented by a solid dot to which is attached one vertical edge and two horizontal edges. Physically, we can think of the dot as representing the spin site itself. The vertical edge represents the state, labeled by the physical index σ_α, while the horizontal edges are labeled by the matrix indices. For the case of row and column vectors, the vertex has just one attached horizontal edge (Fig. 4.1).

Kets are represented by pasting these pieces together into a chain (*i.e.*, the matrix product) (Fig. 4.2).

Bras are represented similarly, but with the diagram flipped upside down. Tracing is represented by pasting vertically (Fig. 4.3).

MPS is popular as a computational tool because it has well developed approximation algorithms

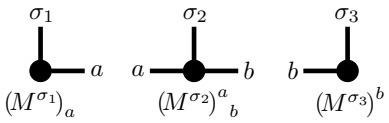


Figure 4.1: Diagrammatic representation of a row vector, a matrix and a column vector, respectively.

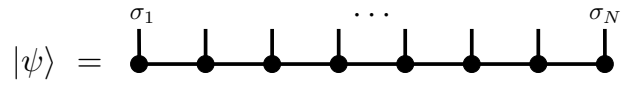


Figure 4.2: Diagrammatic representation of a matrix product state: $|\psi\rangle = \sum_{\sigma_1,\ldots,\sigma_N} M^{\sigma_1} \cdots M^{\sigma_N} |\sigma_1 \cdots \sigma_N\rangle$

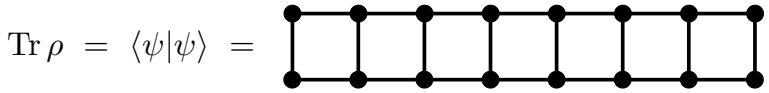


Figure 4.3: Diagrammatic representation of $\mathrm{Tr}\,\rho$. The pasting of vertical edges represents integration over all possible degrees of freedom localized on each site.

which can be used to efficiently find ground states of gapped systems via a variational approach [19]. However, that will not be the focus of this paper. Rather we will develop a recipe to find exact MPS representations of general quasi-topological systems. We will then use the MPS framework as an organizational tool for direct entanglement entropy calculations via Schmidt decomposition.

In our MPS representation of a state $|\psi\rangle$, it looks like we have something very similar to a Schmidt decomposition. This is in fact true, but there is a technical problem here involving normalization. To see this, start by partitioning the Hilbert space into two parts: part A, encompassing the first a spin degrees of freedom; and part B, encompassing the remaining $N - a$. Then rewrite (4.1) in the form

$$|\psi\rangle = \sum_{\{i_\alpha\},\{j_\beta\}} \Psi_{\{i_\alpha\}\{j_\beta\}} \, |\{\phi_{i_\alpha}\}\rangle_A \otimes \left|\{\phi_{j_\beta}\}\right\rangle_B .$$

If each basis vector $|\phi_{i_\alpha}\rangle$ can be in one of n configurations, then Ψ is a matrix with n^a rows and n^{N-a} columns, running over all possible configurations of the A system and the B system, respectively. Our goal is to follow the procedure laid out in §3.1 and decompose Ψ into the form

$$\Psi_{\{i_\alpha\}\{j_\beta\}} = \sum_k A_{\{i_\alpha\}k} S_{kk} B_{k\{j_\beta\}} \tag{4.3}$$

such that S is square diagonal, $A^*_{k\{i_\alpha\}} A_{\{i_\alpha\}l} = \delta_{kl}$, and $B_{k\{j_\beta\}} B^*_{\{j_\beta\}l} = \delta_{kl}$. Attempting to perform this decomposition all at once, while enforcing the left and right normalization conditions, is in practice a difficult combinatorics problem, one which must be independently examined for each system considered. We would like a more structured recipe.

The solution is to break the work into pieces by scanning through the matrix product in (4.2), enforcing left/right normalization on one M^{σ_α} at a time. In this way, we gradually work our way into the form given in (4.3). This form is called the mixed-canonical matrix product state. The details of this procedure will be laid out in section §5.2.

4.2 MPS representations for QTFTs

In the context of QTFT, we do not of course have a lattice of spins on which to define an MPS. States in the Hilbert space can be thought of as associated with the algebra elements (or group elements in the case of YM_2) assigned to a boundary of a triangulation. The path integral corresponding to a genus g surface can be thought of as the normalization of the ground state. In any QFT, cutting open the path integral on a space-like surface (at constant time, say) gives rise to a wave-functional in a basis given by the space-dependent fields. In a QTFT, if we define 'space' by cutting open the genus g surface along ℓ cycles, this data reduces to the set of algebra elements on each of the ℓ loops.

4.2.1 MPS: the no-handles time-slice

We begin with an illustrative example in which we consider a slicing that separates a genus g surface into two topologically identical surfaces, each with $g+1$ boundaries and no handles, as shown in Fig. 4.4. A more general analysis will follow.

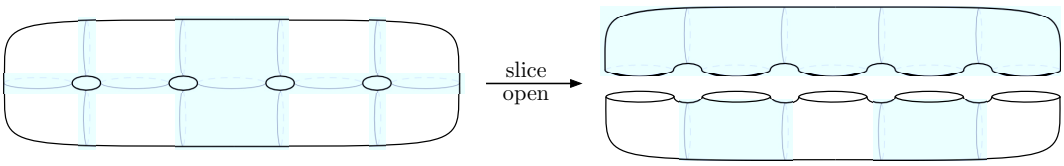


Figure 4.4: Slicing the Riemann surface $\Sigma_{g,\varrho}$ into two surfaces, each of which have $g+1$ boundaries and no handles. More generally there can be some number of uncut handles h_-, h_+ remaining in the lower or upper surfaces.

We start by focusing on the lower surface, conceptually breaking it up into cylinders and pants, as depicted in Fig. 4.5. To each boundary we assign an index. Each index enumerates basis elements of some associative algebra which we are free to choose.

Figure 4.5: Conceptually breaking up the lower half of Fig. 4.4 into cylinders $\eta^{i_\alpha}_{\varrho_\alpha}$ and pants $\mathcal{N}^{i_\alpha}_{\varrho_\alpha}$.

We might also visualize this setup in the dual triangulation formalism, drawn in Fig. 4.6.

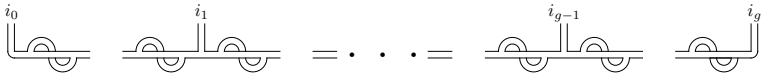


Figure 4.6: Dual triangularization of Fig. 4.5.

Physical states live on the boundary of the surface. It is natural to take as a basis for the Hilbert space a tensor product of vector spaces localized over each disjoint boundary loop,

$$\mathscr{H} = \otimes_{\alpha=0}^{g} \mathscr{H}_\alpha \simeq \otimes^{g+1} \mathscr{H}_{S^1}.$$

In particular, we will denote states as

$$\left|\psi^-\right\rangle = \sum_{i_0,\ldots,i_g} \psi_-^{i_0 i_1 \ldots i_g} \left|\phi_{i_0}\right\rangle \otimes \left|\phi_{i_1}\right\rangle \otimes \cdots \otimes \left|\phi_{i_g}\right\rangle \equiv \sum_{i_0,\ldots,i_g} \psi_-^{i_0 i_1 \ldots i_g} \left|\phi_{i_0}\phi_{i_1}\cdots\phi_{i_g}\right\rangle.$$

The coefficient is a matrix product, which we will write in various ways:

$$\begin{aligned}\psi_-^{i_0 i_1 \dots i_g} &\sim \eta^{i_0 \mathbf{T}} \mathcal{N}^{i_1} \mathcal{N}^{i_2} \cdots \mathcal{N}^{i_{g-1}} \eta^{i_g} \\ &= (\eta_{\varrho_0}^{i_0 \mathbf{T}})_{j_1} (\mathcal{N}_{\varrho_1}^{i_1})^{j_1}{}_{j_2} (\mathcal{N}_{\varrho_2}^{i_2})^{j_2}{}_{j_3} \cdots (\mathcal{N}_{\varrho_{g-1}}^{i_{g-1}})^{j_{g-1}}{}_{j_g} (\eta_{\varrho_g}^{i_g})^{j_g} \\ &\equiv (\eta_{\varrho_0})_{i_0 j_1} (\mathcal{N}_{\varrho_1})_{i_1}{}^{j_1}{}_{j_2} (\mathcal{N}_{\varrho_2})_{i_2}{}^{j_2}{}_{j_3} \cdots (\mathcal{N}_{\varrho_{g-1}})_{i_{g-1}}{}^{j_{g-1}}{}_{j_g} (\eta_{\varrho_g})_{i_g}{}^{j_g}. \end{aligned} \tag{4.4}$$

The ket is then[20]

$$\left|\psi^-\right\rangle = \frac{1}{\sqrt{Z}} \sum_{i_0,\dots,i_g} \eta_{\varrho_0}^{i_0 \mathbf{T}} \mathcal{N}_{\varrho_1}^{i_1} \cdots \mathcal{N}_{\varrho_{g-1}}^{i_{g-1}} \eta_{\varrho_g}^{i_g} \left|\phi_{i_0} \phi_{i_1} \cdots \phi_{i_g}\right\rangle. \tag{4.5}$$

The $\frac{1}{\sqrt{Z}}$ is an overall normalization factor, where Z is the partition function.

The associated bra (representing states living on the boundary of the upper surface) is

$$\left\langle\psi^+\right| = \frac{1}{\sqrt{Z}} \sum_{i_0,\dots,i_g} \left\langle\phi_{i_0} \phi_{i_1} \cdots \phi_{i_g}\right| \bar{\eta}_{\bar{\varrho}_g}^{i_g \mathbf{T}} \bar{\mathcal{N}}_{\bar{\varrho}_{g-1}}^{i_{g-1}} \cdots \bar{\mathcal{N}}_{\bar{\varrho}_1}^{i_1} \bar{\eta}_{\bar{\varrho}_0}^{i_0}, \tag{4.6}$$

where ϱ_α and $\bar{\varrho}_\alpha$ are independent areas and the bars over the weights indicate orientation reversal of the boundary loops:

$$\bar{\eta}_i{}^j = \eta^i{}_j, \qquad \bar{\mathcal{N}}_i{}^j{}_k = \mathcal{N}^i{}_j{}^k.$$

An important point here is that orientation reversal of boundary loops is not the same thing as complex conjugation. Furthermore, the areas of the upper and lower surfaces are not equal for arbitrary time-slicings. For these reasons, the matrix products appearing in (4.5) and (4.6) are not generally conjugate to one another. We will see this explicitely in our MPS construction for gauge theory in §4.3.

Enforcing $\operatorname{Tr}\rho = \langle\psi^+|\psi^-\rangle = 1$, we calculate the partition function,

$$Z = \eta^{i_g k_g} \mathcal{N}^{i_{g-1}}{}_{k_g}{}^{k_{g-1}} \cdots \mathcal{N}^{i_1}{}_{k_2}{}^{k_1} \eta^{i_0}{}_{k_1} \eta_{i_0 j_1} \mathcal{N}_{i_1}{}^{j_1}{}_{j_2} \cdots \mathcal{N}_{i_{g-1}}{}^{j_{g-1}}{}_{j_g} \eta_{i_g}{}^{j_g},$$

where we have used $\left\langle\phi_{j_\beta}\middle|\phi_{i_\alpha}\right\rangle = \delta_{i_\alpha j_\beta}$. This does indeed correspond to integration over the whole surface $\Sigma_{g,\varrho}$.[21]

Before continuing, it is worth pointing out that the structure of the MPS just introduced is somewhat trivial in the case of quasi-topological quantum field theory. In particular, for the time-slicing just studied, the set of $\mathcal{N}^{i_\alpha}$'s are in fact all identical (i.e., $\mathcal{N}^{i_\alpha} = \mathcal{N}^{i_\beta}$ for all α, β), and more generally, in QTFT, the $\mathcal{N}^{i_\alpha}$'s all commute. This implies that the matrix product ordering of the weights (or, equivalently, the tensor product ordering of the basis vectors) is irrelevant. Intuitively, this is the statement that in QTFT the geometry is "flexible", meaning we can smoothly deform the space-time so as to effectively interchange any two boundary loops. There is no concept of nearest-neighbors. This will be demonstrated explicitely later on in the context of YM_2. It is possible, however, to add additional structure to the matrix product state so as to reduce the degree of this flexibility, while still remaining in the framework of QTFT. For example, in gauge

[20] The projection of the ket into some basis $\{\phi_{i_0}\phi_{i_1}\cdots\phi_{i_g}\}$ returns the wavefunction (4.4). That is,

$$\left\langle\phi_{i_0} \phi_{i_1} \cdots \phi_{i_g}\middle|\psi^-\right\rangle = \frac{1}{\sqrt{Z}} \eta_{\varrho_0}^{i_0 \mathbf{T}} \mathcal{N}_{\varrho_1}^{i_1} \cdots \mathcal{N}_{\varrho_{g-1}}^{i_{g-1}} \eta_{\varrho_g}^{i_g}$$

.

[21] In the case of (quasi-)topological field theory, there is an additional constraint on the partition function which we have ignored here. This will be addressed in the context of gauge theory at the end of §4.3.

theories one can add Wilson loops to the picture (in various different ways). This will be discussed further in the conclusion with some related calculations exhibited in Appendices C.2 and C.3.

4.2.2 MPS: general slicings

In the previous section, we considered a rather contrived scenario. We started with a genus g surface and then only considered the case in which all g handles are simultaneously sliced open. This scenario was depicted in Fig. 4.4. If we considered stretching this space-time out along the time direction (so that the cut cylinders were connected only in the distant past and distant future), the "no-handles" MPS just developed would look very much like that of a non-interacting $1D$ spin chain. More generally, however, we should embrace the additional structure of the full space-time picture and allow for physical states to merge and separate over subsequent time-slices. In this way, the total Hilbert space takes on the structure of a Fock space: $\mathscr{H}_{\text{total}} \simeq \oplus_{\ell=0}^{g+1} \mathscr{H}_\ell$, where ℓ denotes the number of boundary loops in a given sector.

We must now extend our analysis to include sliced geometries with some number of handles left intact in the lower and/or upper surfaces. To deal with this added complexity, we introduce the notation $\Sigma^{(-)}_{h_-,\ell,\varrho_-}$ and $\Sigma^{(+)}_{h_+,\ell,\varrho_+}$ to denote the two surfaces, where the $\mp$ refers to a choice of orientation and the set $\{h_\mp, \ell, \varrho_\mp\}$ refers respectively to the number of handles, number of boundaries, and area of each surface. We do not distinguish between ℓ_- and ℓ_+ since they must be equal.

As a warm up, consider a g-handled surface $\Sigma_{g,\varrho}$ cut open along g cycles with one handle remaining in the lower surface, $\Sigma_{g,\varrho} = \Sigma^{(+)}_{0,g,\varrho_+} \cup \Sigma^{(-)}_{1,g,\varrho_-}$. We depict the upper and lower surfaces in Fig. 4.7 and Fig. 4.8, respectively.

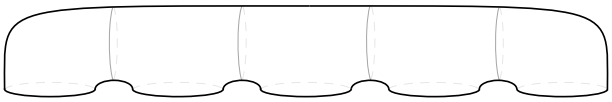


Figure 4.7: An example surface $\Sigma^{(+)}_{h_+,\ell,\varrho_+} = \Sigma^{(+)}_{0,g,\varrho_+}$ for the case $g = 5$.

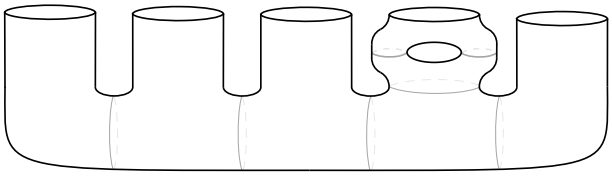


Figure 4.8: An example surface $\Sigma^{(-)}_{h_-,\ell,\varrho_-} = \Sigma^{(-)}_{1,g,\varrho_-}$ for the case $g = 5$.

In Fig. 4.8 we have, for definiteness, associated the uncut handle with a particular boundary site; and hence associating it to a particular element of the matrix product in the MPS representation of the state. Of course, in quasi-topological field theory the actual location of the handle is irrelevant, but this detail should (and will) work itself out.

To explore how this new handle affects our MPS representation, we first extend our notation. Previously we had $\eta^{i_\alpha}_{\varrho_\alpha}$ and $\mathcal{N}^{i_\alpha}_{\varrho_\alpha}$ representing the two-point and three-point functions on the sphere. We now add to these an additional index h_α, so that we can talk about two-point and three-point functions on surfaces of arbitrary genus: $\eta^{i_\alpha}_{h_\alpha,\varrho_\alpha}$ and $\mathcal{N}^{i_\alpha}_{h_\alpha,\varrho_\alpha}$.

In this bulkier notation, the MPS representation of a ket with no handles added is

$$\left|\psi^-_{(0,0,\ldots,0,0)}\right\rangle = \frac{1}{\sqrt{Z}} \sum_{i_0,\ldots,i_{\ell-1}} \eta^{i_0\mathbf{T}}_{0,\varrho_0} \mathcal{N}^{i_1}_{0,\varrho_1} \cdots \mathcal{N}^{i_{\ell-2}}_{0,\varrho_{\ell-2}} \eta^{i_{\ell-1}}_{0,\varrho_{\ell-1}} \left|\phi_{i_0}\phi_{i_1}\cdots\phi_{i_{\ell-1}}\right\rangle. \tag{4.7}$$

The string of 0's labeling the ket tell us that there are zero handles attached to each of the boundary sites.

Now we introduce the *handle operator* acting on boundary site α,[22]

$$H_\alpha = \sum_{i_\alpha, i'_\alpha} \mathrm{Tr}(\mathcal{N}^{i_\alpha}_{0,0} \bar{\mathcal{N}}^{i'_\alpha}_{0,0}) \left|\phi_{i_\alpha}\right\rangle\!\left\langle\phi_{i'_\alpha}\right|, \tag{4.8}$$

with the property that it increases the number of handles attached to site α by one,

$$|\psi^-_{(0,\ldots,0,\underset{\alpha}{\overset{\uparrow}{1}},0,\ldots,0)}\rangle = H_\alpha \left|\psi^-_{(0,\ldots,0)}\right\rangle.$$

Recall that the $\bar{\mathcal{N}}^{i'_\alpha}$ is a three-holed sphere with boundary loops reversed. Also, note that we have chosen not to assign any area to the handle itself. For a QTFT, this construction is still completely general. Applying the handle operator to a ket encoding a surface with zero handles and g boundaries yields

$$\left|\psi^-_{(0,\ldots,0,1,0,\ldots,0)}\right\rangle = \frac{1}{\sqrt{Z}} \sum_{i_0,\ldots,i_g} \eta^{i_0\mathbf{T}}_{0,\varrho_0} \mathcal{N}^{i_1}_{0,\varrho_1} \cdots \mathcal{N}^{i_{\alpha-1}}_{0,\varrho_{\alpha-1}} \mathcal{N}^{i_\alpha}_{1,\varrho_\alpha} \mathcal{N}^{i_{\alpha+1}}_{0,\varrho_{\alpha+1}} \cdots \mathcal{N}^{i_{g-2}}_{0,\varrho_{g-2}} \eta^{i_{g-1}}_{0,\varrho_{g-1}} \left|\phi_{i_0}\cdots\phi_{i_{g-1}}\right\rangle, \tag{4.9}$$

where

$$\mathcal{N}^{i_\alpha}_{1,\varrho_\alpha} = \sum_{i'_\alpha} \mathrm{Tr}(\mathcal{N}^{i_\alpha}_{0,0} \bar{\mathcal{N}}^{i'_\alpha}_{0,0}) \mathcal{N}^{i'_\alpha}_{0,\varrho_\alpha}.$$

Of course it is easy to extend this analysis to full generality. A state with any number of additional handles can be generated as follows:

$$\left|\psi^-_{(h_0,\ldots,h_{\ell-1})}\right\rangle = \left(\prod_{\alpha=0}^{\ell-1} H_\alpha^{h_\alpha}\right)\left|\psi_{(0,\ldots,0)}\right\rangle = \frac{1}{\sqrt{Z}} \sum_{i_0,\ldots,i_g} \eta^{i_0\mathbf{T}}_{h_0,\varrho_0} \mathcal{N}^{i_1}_{h_1,\varrho_1} \cdots \mathcal{N}^{i_{\ell-2}}_{h_{\ell-2},\varrho_{\ell-2}} \eta^{i_{\ell-1}}_{h_\ell,\varrho_{\ell-1}} \left|\phi_{i_0}\phi_{i_1}\cdots\phi_{i_{\ell-1}}\right\rangle, \tag{4.10}$$

where ℓ is the number of boundary loops for the time-slice over which this state lives.

4.3 Example: Gauge theory (continued)

Continuing with the example we started in §2.4, we now explicitly construct the matrix product states for a quasi-topological gauge theory.

[22] In equation (4.8) we are using shorthand notation. The operator H_α acts trivially on all sites $\beta \neq \alpha$:

$$\sum_{i_0,\ldots,i_{\ell-1}} \sum_{i'_0,\ldots,i'_{\ell-1}} \delta^{i'_0 i_0} \cdots \delta^{i'_{\alpha-1} i_{\alpha-1}} \mathrm{Tr}(\mathcal{N}^{i_\alpha}_{0,0} \bar{\mathcal{N}}^{i'_\alpha}_{0,0}) \delta^{i'_{\alpha+1} i_{\alpha+1}} \cdots \delta^{i'_{\ell-1} i_{\ell-1}} \left|\phi_{i_0}\cdots\phi_{i_{\ell-1}}\right\rangle\!\left\langle\phi_{i'_0}\cdots\phi_{i'_{\ell-1}}\right|$$

4.3.1 Gauge theory: MPS without handles

It is convenient to first consider the simple setup depicted in Fig. 4.5, in which there are no uncut handles in the lower surface. A matrix product state representation is then given by equation (4.7). Once we have this worked out, we can easily build up more general states by acting with the handle operator.

It is intuitively clear that for a QTFT the ordering of the boundary loops, $(\phi_{i_0}, \phi_{i_1}, \ldots, \phi_{i_{\ell-1}})$, is arbitrary, since they can always be rearranged via area preserving diffeomorphisms. This can be demonstrated explicitly in the dual-triangulation formalism from Fig. 4.6 by repeated use of the flip move. In the language of MPS, the commutativity of the boundary loops translates to $\left[\mathcal{N}^{i_\alpha}_{0,\varrho_\alpha}, \mathcal{N}^{i_\beta}_{0,\varrho_\beta}\right] = 0$. We therefore should be able to simultaneously diagonalize the entire product $\mathcal{N}^{i_1}_{0,\varrho_1} \cdots \mathcal{N}^{i_{\ell-2}}_{0,\varrho_{\ell-2}}$.

Hence, our initial challenge is to diagonalize expression (2.15), repeated here for convenience:

$$\left[\mathcal{N}^i_{0,\varrho}\right]^{j_1}_{\ j_2} = \frac{\gamma}{|G|^2} \sum_R e^{\varrho B_R}\, d_R\, \chi_R(i \sum_{k_1} k_1^{-1} j_1^{-1} k_1 \sum_{k_2} k_2^{-1} j_2 k_2). \tag{4.11}$$

The first thing to notice is that this matrix does not really depend on the particular group elements j_1, j_2 indexed, but rather on the conjugacy classes $[j_1^{-1}]$, $[j_2]$. In particular, it depends on a sum over the orbits of these conjugacy classes – *e.g.*, $\sum_{k_1} k_1^{-1} j_1^{-1} k_1$. These sums are maps from the set of conjugacy classes to the center of the group algebra $Z(\mathbb{C}[G])$. According to Schur's lemma, these abelian elements are proportional to the identity. This fact can be used to easily decompose the trace in (4.11).[23] The result is

$$\begin{aligned}\left[\mathcal{N}^i_{\varrho}\right]^{j_1}_{\ j_2} &= \frac{\gamma}{|G|^2} \sum_R e^{\varrho B_R} d_R^{-1}\ \chi_R(i)\ \chi_R\left(\sum_{k_1} k_1^{-1} j_1^{-1} k_1\right)\ \chi_R\left(\sum_{k_2} k_2^{-1} j_2 k_2\right) \\ &= \gamma \sum_R e^{\varrho B_R} d_R^{-1}\ \chi_R(i)\ \chi_R\left(j_1^{-1}\right)\ \chi_R\big(j_2\big)\,. \end{aligned} \tag{4.12}$$

We can then write down the components of $\mathcal{N}^i_{0,\varrho}$ explicitly for a group $G = \{g_1, g_2, \ldots, g_{|G|}\}$:

$$\mathcal{N}^i_{0,\varrho} = \gamma \sum_R e^{\varrho B_R} d_R^{-1} \chi_R(i) \begin{pmatrix} \chi_R(g_1^{-1})\chi_R(g_1) & \chi_R(g_1^{-1})\chi_R(g_2) & \cdots & \chi_R(g_1^{-1})\chi_R(g_{|G|}) \\ \chi_R(g_2^{-1})\chi_R(g_1) & \chi_R(g_2^{-1})\chi_R(g_2) & \cdots & \chi_R(g_2^{-1})\chi_R(g_{|G|}) \\ \vdots & \vdots & \ddots & \vdots \\ \chi_R(g_{|G|}^{-1})\chi_R(g_1) & \chi_R(g_{|G|}^{-1})\chi_R(g_2) & \cdots & \chi_R(g_{|G|}^{-1})\chi_R(g_{|G|}) \end{pmatrix}.$$

This is a $|G| \times |G|$ matrix and there will be a total of $|G|$ eigenvectors. However, since many of the columns are degenerate, being labeled by elements of the same conjugacy class, many eigenvalues will be zero. Hence, there are at most $|Z(\mathbb{C}[G])|$ nonzero eigenvalues, where $|Z(\mathbb{C}[G])|$ is the cardinality of the center of the group algebra $\mathbb{C}[G]$.[24] Labeling the non-zero eigenvectors as $\vec{\chi}_R$,

[23] Given elements $A \in R[G]$ and $N \in R[Z(G)]$, we can make the following manipulations:

$$\mathrm{tr}(A \cdot N) = \mathrm{tr}(A \cdot n\mathbf{1}_G) = n\,\mathrm{tr}(A) = \frac{1}{d_R}\mathrm{tr}(A)\mathrm{tr}(n\mathbf{1}_G) = \frac{1}{d_R}\mathrm{tr}(A)\mathrm{tr}(N),$$

[24] The notation $|Z|$ will be used as shorthand for any element of the following equality

$$|Z| \equiv |Z(\mathbb{C}[G])| = \text{number of conjugacy classes in } G = \text{number of irreducible representations of } G$$

one can check that

$$\mathcal{N}^i_{0,\varrho}\,\vec{\chi}_R = \gamma\left(e^{\varrho B_R}\frac{|G|}{d_R}\chi_R(i)\right)\vec{\chi}_R, \qquad \vec{\chi}_R = \begin{pmatrix} \chi_R(g_1^{-1}) \\ \chi_R(g_2^{-1}) \\ \vdots \\ \chi_R(g_{|G|}^{-1}) \end{pmatrix}. \tag{4.13}$$

Choosing a convenient indexing order for the group elements, we can write the diagonolized $\mathcal{N}^i_{0,\varrho}$ as

$$\mathcal{N}^i_{0,\varrho} = \gamma \begin{pmatrix} e^{\varrho B_{R_1}}\frac{|G|}{d_{R_1}}\chi_{R_1}(i) & & & & & & \\ & \ddots & & & & & \\ & & e^{\varrho B_{R_{|Z|}}}\frac{|G|}{d_{R_{|Z|}}}\chi_{R_{|Z|}}(i) & & & & \\ & & & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & \end{pmatrix}.$$

Now we need to put the $\eta^i_{0,\varrho}$ into the same basis. The diagonalization of the $\mathcal{N}^i_{0,\varrho}$ corresponded to unitary tranformation matrices inserted into the MPS in the following way:

$$\begin{aligned} \left|\psi^-_{(0,\ldots,0)}\right\rangle &= \sum_{i_0,\ldots,i_{\ell-1}} \eta^{i_0\mathbf{T}}_{0,\varrho_0}\mathcal{N}^{i_1}_{0,\varrho_1}\cdots\mathcal{N}^{i_{g-1}}_{0,\varrho_{\ell-2}}\eta^{i_{\ell-1}}_{0,\varrho_{\ell-1}}\left|\phi_{i_0}\cdots\phi_{i_{\ell-1}}\right\rangle \\ &= \sum_{i_0,\ldots,i_{\ell-1}} (\eta^{i_0\mathbf{T}}_{0,\varrho_0}U)(U^{-1}\mathcal{N}^{i_1}_{0,\varrho_1}U)(U^{-1}\cdots U)(U^{-1}\mathcal{N}^{i_{\ell-2}}_{0,\varrho_{\ell-2}}U)(U^{-1}\eta^{i_{\ell-1}}_{0,\varrho_{\ell-1}})\left|\phi_{i_0}\cdots\phi_{i_{\ell-1}}\right\rangle, \end{aligned} \tag{4.14}$$

where U is constructed from the normalized eigenvectors of $\mathcal{N}^i_{0,\varrho}$:

$$U = \frac{1}{\sqrt{|G|}}\begin{pmatrix} \chi_{R_1}(g_1^{-1}) & \cdots & \cdots & \chi_{R_{|Z|}}(g_1^{-1}) & 0 & \cdots & \cdots & 0 \\ \chi_{R_1}(g_2^{-1}) & \cdots & \cdots & \chi_{R_{|Z|}}(g_2^{-1}) & \sqrt{\frac{G}{2}} & \cdots & \cdots & 0 \\ \chi_{R_1}(g_3^{-1}) & \cdots & \cdots & \chi_{R_{|Z|}}(g_3^{-1}) & -\sqrt{\frac{G}{2}} & \cdots & \cdots & 0 \\ \chi_{R_1}(g_4^{-1}) & \cdots & \cdots & \chi_{R_{|Z|}}(g_4^{-1}) & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \chi_{R_1}(g_{|G|-1}^{-1}) & \cdots & \cdots & \chi_{R_{|Z|}}(g_{|G|-1}^{-1}) & 0 & \cdots & \cdots & \sqrt{\frac{G}{2}} \\ \chi_{R_1}(g_{|G|}^{-1}) & \cdots & \cdots & \chi_{R_{|Z|}}(g_{|G|}^{-1}) & 0 & \cdots & \cdots & -\sqrt{\frac{G}{2}} \end{pmatrix}. \tag{4.15}$$

U is a $|G|\times|G|$ unitary matrix.[25] We use this matrix to transform the η^i's into the same basis as the $\mathcal{N}^i$'s. In the old basis, we have

$$\left[\eta^i_{0,\varrho}\right]_j = \gamma^2\sum_R e^{\varrho B_R}d_R\chi_R\left(i\sum_k kjk^{-1}\right) = \gamma^2|G|\sum_R e^{\varrho B_R}\chi_R(i)\chi_R(j) \doteq |G|\sum_R e^{\varrho B_R}\chi_R(i)\begin{pmatrix}\chi_R(g_1)\\ \vdots \\ \chi_R(g_{|G|})\end{pmatrix}.$$

[25] The columns on the right are the eigenvectors corresponding to zero eigenvalues. The form of the matrix given in (4.15) is just an example. Each of the columns on the right contain a $\begin{pmatrix}\sqrt{\frac{G}{2}}\\ -\sqrt{\frac{G}{2}}\end{pmatrix}$ pair, connecting two rows indexing group elements in the same conjugacy class. For the example given, g_2 and g_3 must be in the same conjugacy class.

In the diagonal basis we have

$$\eta^i_{0,\varrho} \to U^\dagger \eta^i_\varrho = \gamma^2 \begin{pmatrix} \sqrt{|G|}\sum_R e^{\varrho B_{R_1}} \chi_R(i) \sum_g \chi_R(g)\chi_{R_1}(g^{-1}) \\ \vdots \\ \sqrt{|G|}\sum_R e^{\varrho B_{R_{|Z|}}} \chi_R(i) \sum_g \chi_R(g)\chi_{R_{|Z|}}(g^{-1}) \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \gamma^2 |G|^{\frac{3}{2}} \begin{pmatrix} e^{\varrho B_{R_1}} \chi_{R_1}(i) \\ \vdots \\ e^{\varrho B_{R_{|Z|}}} \chi_{R_{|Z|}}(i) \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

At this point it is clear that all of these trailing zeros really are superfluous and we truncate our matrices,

$$\eta^i_{0,\varrho} = \gamma^2 |G|^{\frac{3}{2}} \begin{pmatrix} e^{\varrho B_{R_1}} \chi_{R_1}(i) \\ \vdots \\ e^{\varrho B_{R_{|Z|}}} \chi_{R_{|Z|}}(i) \end{pmatrix}, \qquad \mathcal{N}^i_{0,\varrho} = \gamma |G| \begin{pmatrix} e^{\varrho B_{R_1}} d^{-1}_{R_1} \chi_{R_1}(i) & & \\ & \ddots & \\ & & e^{\varrho B_{R_{|Z|}}} d^{-1}_{R_{|Z|}} \chi_{R_{|Z|}}(i) \end{pmatrix}, \tag{4.16}$$

such that the matrix indices j_1,j_2 now run only over conjugacy classes, while the label i still runs over the entire group G.

Before we put the handles in, some comments are in order. First, from the MPS perspective, the rank of the matrices sets an upper bound on the entanglement entropy of the state[26] and so the diagonalization process and subsequent truncation has effectively found a tighter bound on the scaling of the entanglement entropy. Of course we haven't made any approximations here, so we don't really need to talk about setting bounds. We will calculate the entanglement entropy exactly in §5.2. A more precise statement is that we have found a suitable basis for the *entanglement Hamiltonian* $\hat{\mathcal{H}}_E$ (see footnote 12). As will become obvious in our entanglement entropy calculation in §5.2, the diagonal components of these diagonalized matrices contribute independently to the entanglement spectrum.

A second observation is from the QTFT perspective. What we have just discovered is that, of the set of states we started with (*i.e.*, the basis elements $\{\phi_i \,|\, \phi_i \in \mathbb{C}[G]\}$), only a small subset of them contribute to calculations of physical observables. More specifically, the basis elements of $\{\phi_i \,|\, \phi_i \in Z(\mathbb{C}[G])\}$ define the set of physical states, while all other basis elements correspond to spurious states. This fact was noted in [16] and was, in essence, stated as a theorem of QTFT. One may then object that the "physical index" i (of $\eta^i and \mathcal{N}^i$) should also be restricted to run only over a basis of the commutative algebra $Z(\mathbb{C}[G])$. In fact, this is an option. Such a restriction can be made but it is not a requirement and is in some respect a matter of interpretation. Retaining the full set of states (spurious included) on the i index, matches onto the usual convention of lattice gauge theory (as well as the Migdal formalism), in which group elements assigned to plaquette (or lattice) edges are chosen directly from the set of group elements, not from the center of the group algebra.

[26] The bound is simply $S_A \leq \log(\text{rank})$ where rank is the size of the auxiliary space in a MPS. [19].

4.3.2 Gauge theory: MPS with handles

We now add handles. This is a simple extension. The trace in H_α (equation (4.8)) is basis independent, so we may as well evaluate it in the diagonal basis given in (4.16):

$$\mathrm{Tr}(\mathcal{N}^{i_\alpha}_{0,0}\bar{\mathcal{N}}^{i'_\alpha}_{0,0}) = \gamma^2\,|G|^2\,\mathrm{Tr}\left[\begin{pmatrix} d_{R_1}^{-1}\chi_{R_1}(i_\alpha) & & \\ & \ddots & \\ & & d^{-1}_{R_{|Z|}}\chi_{R_{|Z|}}(i_\alpha)\end{pmatrix}\begin{pmatrix} d_{R_1}^{-1}\chi_{R_1}(i'^{-1}_\alpha) & & \\ & \ddots & \\ & & d^{-1}_{R_{|Z|}}\chi_{R_{|Z|}}(i'^{-1}_\alpha)\end{pmatrix}\right].$$

The handle operator is then

$$H_\alpha = \gamma^2\,|G|^2\sum_{i_\alpha,i'_\alpha}\sum_R d_R^{-2}\,\chi_R(i_\alpha)\chi_R(i'^{-1}_\alpha)\,\big|\phi_{i_\alpha}\big\rangle\big\langle\phi_{i'_\alpha}\big|\,.$$

Applying this operator to the state with no handles in the lower surface yields a matrix product state of the form (4.9), with

$$\begin{aligned}[\mathcal{N}^{i_\alpha}_{1,\varrho_\alpha}]^{j_\alpha}{}_{j_{\alpha+1}} &\doteq \gamma^3\,|G|^3\sum_{i'_\alpha}\sum_R d_R^{-2}\,\chi_R(i_\alpha)\chi_R(i'^{-1}_\alpha)\begin{pmatrix} e^{\varrho_\alpha B_{R_1}}d_{R_1}^{-1}\chi_{R_1}(i'_\alpha) & & \\ & \ddots & \\ & & e^{\varrho_\alpha B_{R_{|Z|}}}d^{-1}_{R_{|Z|}}\chi_{R_{|Z|}}(i_\alpha)\end{pmatrix}\\
&= \gamma^2\,|G|^3\begin{pmatrix} d_{R_1}^{-2} & & \\ & \ddots & \\ & & d^{-2}_{R_{|Z|}}\end{pmatrix}\cdot\gamma\,|G|\begin{pmatrix} e^{\varrho_\alpha B_{R_1}}d_{R_1}^{-1}\chi_{R_1}(i_\alpha) & & \\ & \ddots & \\ & & e^{\varrho_\alpha B_{R_{|Z|}}}d^{-1}_{R_{|Z|}}\chi_{R_{|Z|}}(i'_\alpha)\end{pmatrix}\\
&\doteq \frac{\gamma^2|G|^3}{d^2_{R_{j_\alpha}}}\delta^{j_\alpha}{}_{j'_\alpha}\left[\mathcal{N}^{i_\alpha}_{0,\varrho_\alpha}\right]^{j'_\alpha}{}_{j_{\alpha+1}} \qquad \text{(no sum on } j_\alpha).\end{aligned}$$

A state with any number of additional handles can be generated by repeated application of the handle operator, as written in equation (4.10). From the current example, it is easy to see that each additional handle simply contributes a factor of $\frac{\gamma^2|G|^3}{d^2_{R_j}}\delta^j{}_k$, so

$$\eta^{i_\alpha}_{h_\alpha,\varrho_\alpha} = \gamma^{2(h_\alpha+1)}\,|G|^{3(h_\alpha+\frac{1}{2})}\begin{pmatrix} e^{\varrho_\alpha B_{R_1}}d_{R_1}^{-2h_\alpha}\chi_{R_1}(i_\alpha)\\ \vdots \\ e^{\varrho_\alpha B_{R_{|Z|}}}d^{-2h_\alpha}_{R_{|Z|}}\chi_{R_{|Z|}}(i_\alpha)\end{pmatrix}, \tag{4.17}$$

$$\mathcal{N}^{i_\alpha}_{h_\alpha,\varrho_\alpha} = \gamma^{2h_\alpha+1}\,|G|^{3h_\alpha+1}\begin{pmatrix} e^{\varrho_\alpha B_{R_1}}d_{R_1}^{-(1+2h_\alpha)}\chi_{R_1}(i_\alpha) & & \\ & \ddots & \\ & & e^{\varrho_\alpha B_{R_{|Z|}}}d^{-(1+2h_\alpha)}_{R_{|Z|}}\chi_{R_{|Z|}}(i_\alpha)\end{pmatrix}. \tag{4.18}$$

4.3.3 Fixing the normalization

We will now determine the normalization factor $\frac{1}{\sqrt{Z}}$ and fix the value of γ.

Multiplying out the matrix product gives

$$\Big|\psi^-_{h_-,\varrho_-}\Big\rangle = \frac{1}{\sqrt{Z}}\,\gamma^{2h_-+\ell+2}\,|G|^{3h_-+\ell+1}\sum_{i_0,\ldots,i_{\ell-1}}\sum_R e^{\varrho_- B_R}d_R^{2-\ell-2h_-}\prod_{\alpha=0}^{\ell-1}\chi_R(i_\alpha)\,\big|\phi_{i_0}\cdots\phi_{i_{\ell-1}}\big\rangle\,. \tag{4.19}$$

At this point, we have simplified our notation, $(h_0^-, \ldots, h_{\ell-1}^-) \to h_-$. As anticipated, only the total number of handles is relevant, not the locations at which they are inserted. (Recall that h_- and h_+ denote the total number of uncut handles in the lower and upper surfaces, respectively.)

The bra is constructed identically,

$$\left\langle \psi^+_{h_+,\varrho_+} \right| = \frac{1}{\sqrt{Z}}\, \gamma^{2h_+ + \ell + 2}\, |G|^{3h_+ + \ell + 1} \sum_{i_0,\ldots,i_{\ell-1}} \left\langle \phi_{i_0} \cdots \phi_{i_{\ell-1}} \right| \sum_R e^{\varrho_+ B_R} d_R^{2-\ell-2h_+} \prod_{\alpha=0}^{\ell-1} \chi_R(i_\alpha^{-1}).$$

Note, the ket and bra are not complex conjugate to each other, as mentioned previously. This is because the area dependent exponentials, $e^{\varrho_- B_R}$ and $e^{\varrho_+ B_R}$, are in general not complex conjugate. In the topological limit ($B_R \to 0$), these exponentials vanish and the usual relationship is restored.

The genus of the total surface $\Sigma_{g,\varrho}$ is $g = \ell - 1 + h_- + h_+$. The normalization factor is computed from the inner product by enforcing $\operatorname{Tr} \rho = \left\langle \psi^+_{h_+,\varrho_+} \middle| \psi^-_{h_-,\varrho_-} \right\rangle = 1$:

$$Z = \gamma^{2g+6}\, |G|^{3g+5} \sum_R e^{\varrho B_R} d_R^{2-2g}. \tag{4.20}$$

We will now fix the value of γ. The key point is that in topological field theory we are not free to choose any overall normalization for the partition function. In particular, the value of the partition function on the torus is highly constrained by general principles of quantum field theory [29]. To see this, first recall that the Hamiltonian of a purely topological quantum field theory is zero, $\hat{H}^{\text{topo}} = 0$. Momentarily going back to the thermal parametrization of Euclidean space-time (with periodic imaginary time), and restricting our attention to generalized YM_2 on the torus in the topological limit, we can write the partition function in the familiar form

$$Z^{\text{topo}}_{T^2} = \operatorname{Tr}_{\mathscr{H}_{S^1}} e^{-\beta \hat{H}^{\text{topo}}} = \operatorname{Tr}_{\mathscr{H}_{S^1}} \mathbf{1} = \dim \mathscr{H}_{S^1}. \tag{4.21}$$

On the other hand, if we compute the partition function (for $g = 1$, $B_R = 0$) using equation (4.20), we find:

$$Z^{\text{topo}}_{T^2,\gamma} = |\gamma|^8\, |G|^8 \sum_R 1 = |\gamma|^8 |G|^8\, |Z(\mathbb{C}[G])| = \gamma^8 |G|^8\, \dim \mathscr{H}_{S^1}.$$

In order to make these two equations match, we must take[27]

$$\gamma = \frac{1}{|G|}. \tag{4.22}$$

The partition function is thus

$$Z = |G|^{g-1} \sum_R e^{\varrho\, B_R} d_R^{2-2g}. \tag{4.23}$$

This differs from the Migdal formula, equation (2.3), by the prefactor $|G|^{g-1}$. We will have more to say about this in the context of our entanglement entropy calculations in the coming section. For now, we simply point out that while the usual Migdal equation cannot be be obtained exactly from the QTFT lattice formalism developed in §2, that equation still satisfies the torus normalization condition given in (4.21). More generally, this normalization condition will remain satisfied

[27] Note that γ is a constant. As defined in equation (2.10), it is an overall multiplicative factor on the set of structure constants of the group algebra $\mathbb{C}[G]$. It cannot be a function of g or ϱ.

with any additional factor of the form $|G|^{\text{const}\cdot\chi(\Sigma_g)+B_0\varrho}$ multiplying the partition function. The representation-independent object B_0 is any constant with dimensions of inverse area that vanishes in the topological limit (e.g. in YM_2, $B_0 = \text{const}\cdot e^2$). These arbitrary factors can be interpreted as counterterms which one may add to the action.

Chapter 5

Entanglement Entropy Calculations for 2d Gauge Theory

In §3.5 we wrote down the entanglement entropy for 2d Yang-Mills on a space-time of arbitrary genus. However, in doing so we had assumed a particular type of A/B partitioning in which none of the individual disjoint regions of A (or B) was a closed loop. Now, in light of our MPS construction (see Fig. 4.4), it is natural to consider the reverse scenario, in which A and B are each, independently, the union of a disjoint set of S^1's. We first perform this calculation via the replica trick, followed by the same calculation carried out in the language of MPS.

5.1 Generalized YM_2 entanglement entropy via the replica trick

In (4.23), we have written a formula for the gauge theory partition function which depends on the genus of the Riemann surface:

$$Z = |G|^{g-1} \sum_R e^{\varrho\, B_R} d_R^{2-2g}.$$

The replica trick entanglement entropy calculation is now straightforward. We divide the system spatially into two parts, A and B, where A consists of some number of boundary loops and B consists of the rest (Fig. 5.1). (As we shall see, the specific number of loops included in each region will not actually matter.)

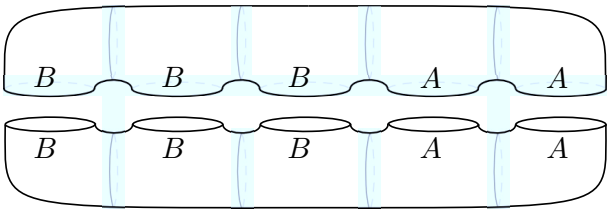


Figure 5.1: A geometric interpretation of the density matrix $\rho = |\psi^-\rangle\langle\psi^+|$. In this example we have a $g = 4$ Riemann surface with zero uncut handles. Uncut handles do not change the analysis in any way; the result would be the same.

We then construct the n-sheeted Riemann surface by the usual procedure: trace over the B degrees of freedom, make n replications, and trace cyclically along the A cuts (Fig. 5.2). Of course this is essentially the same calculation we did in §3.4, except that we no longer have any ramification points. Thus, in determining the genus of the pasted geometry, Riemann-Hurwitz[28] yields the simple result

$$g_n = n(g-1) + 1,$$

which, unlike the previous calculation, makes no reference to the number of disjoint regions of A.

[28] See footnote 16

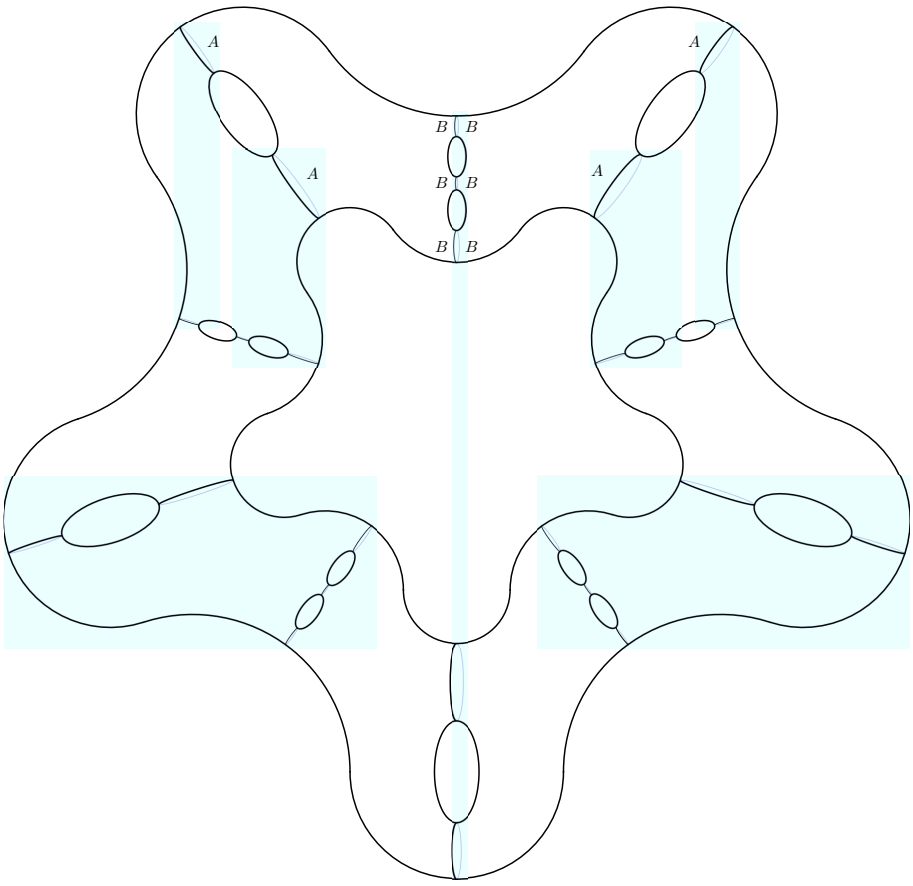


Figure 5.2: Example: A $g = 4$ surface cut open on two cycles (labeled A). The glued surface drawn is the $n = 5$ replication. Notice that only the total number of boundaries in $A \cup B$ matters in determining the topology of the glued geometry.

(Fig. 5.2 illustrates why the number of boundary loops assigned to A vs. B is irrelevant. The replication and gluing process yields the same glued topology regardless of the relative number of boundary loops assigned to each region.) Of course, the area of the n-sheeted geometry is still given by

$$\varrho_n = n\varrho.$$

Using (3.7), the entanglement entropy is then

$$\begin{aligned} S_{\mathrm{YM}_2} &= -\lim_{n\to 1} \frac{d}{dn} \frac{|G|^{[(g-1)n+1]-1} \sum_R d_R^{2-2[(g-1)n+1]} e^{n\varrho B_R}}{\left(|G|^{g-1} \sum_{R'} d_{R'}^{2-2g} e^{\varrho B_{R'}}\right)^n} \\ &= -\sum_R \frac{d_R^{2-2g} e^{\varrho B_R}}{\sum_{R'} d_R'^{2-2g} e^{\varrho B_{R'}}} \log \left[\frac{d_R^{2-2g} e^{\varrho B_R}}{\sum_{R'} d_{R'}^{2-2g} e^{\varrho B_{R'}}} \right]. \end{aligned} \tag{5.1}$$

In the next section, we will calculate the same entanglement entropy via a Schmidt decomposition. An interesting point is that while the replica trick calculation is sensitive to the overall

normalization of the partition function, and thus requires the torus normalization condition (4.22)[29] , the direct Schmidt decomposition calculation is not (*i.e.,* the direct approach is insensitive to the chosen normalization of the underlying group algebra, $\phi_i\phi_j = \gamma\phi_{ij}$). This insensitivity is what one would naively expect given the QTFT construction, where different choices of γ clearly correspond to isomorphic algebras.

5.2 Entanglement entropy from the matrix product state

Our goal is to extract the entanglement entropy from the MPS by putting it in the form of a Schmidt decomposition. We will use a standard procedure for putting matrix product states into mixed canonical form. Those familiar with this procedure can skim through to the results.

We do this by scanning through the matrix product, first from left to right, then from right to left, employing QR and LQ decomposition respectively[30]. We thereby squeeze the singular values to the center of the product. These values are then separated and identified as the entanglement spectrum of the system. We will now go through this process in a very schematic way. In Appendix A.3 we will go through the same procedure again for the specific example of generalized YM_2, but with computational details filled in.

We begin with equation (4.5),

$$\left|\psi^-_{h_-,\varrho_-}\right\rangle = \frac{1}{\sqrt{Z}} \sum_{i_0,\ldots,i_{\ell-1}} \eta^{i_0\mathbf{T}}_{h_0,\varrho_0} \mathcal{N}^{i_1}_{h_1,\varrho_1} \cdots \mathcal{N}^{i_{a-1}}_{h_{a-1},\varrho_{a-1}} \cdot \mathcal{N}^{i_a}_{h_a,\varrho_a} \cdots \mathcal{N}^{i_{g-1}}_{h_{\ell-2}\varrho_{\ell-2}} \eta^{i_{\ell-1}}_{\varrho_{\ell-1}} \left|\phi_{i_0}\cdots\phi_{i_{\ell-1}}\right\rangle .$$

We have added a dot to indicate the point in the matrix product at which the A degrees of freedom end and the B degrees of freedom begin. As indicated by the index labels, we are choosing region A to consist of a boundary loops and region B to consist of $\ell - a$ boundary loops. We will start the process of canonicalization from the left side of the matrix product.

Recall that the leftmost matrix element of the matrix product ket, $\eta^{i_0\mathbf{T}}_{h_0,\varrho_0}$, is really a set of $|G|$ row vectors indexed by i_0. The first step of canonicalization is to reshape this set of vectors into a matrix and then perform a QR decomposition:

$$\left[\eta^{i_0\mathbf{T}}_{h_0,\varrho_0}\right]_{j_1} \longrightarrow [\eta^{\mathbf{T}}_{h_0,\varrho_0}]^{i_0}_{\ j_1} = [Q_0]^{i_0}_{\ k_1}[R_{h_0,\varrho_0}]^{k_1}_{\ j_1}.$$

The QR decomposition guarentees that Q_0 is left-normalized: $Q_0^\dagger Q_0 = \mathbf{1}$.[31] We next reshape the matrix $[Q_0]^{i_0}_{\ k_1}$ back into a set of $|G|$ row vectors $\{A^{i_0}\}$:

$$[Q_0]^{i_0}_{\ k_1} \longrightarrow \left[A^{i_0}\right]_{k_1} .$$

[29] In the context of generalized YM_2, the torus normalization, equality (4.22), enforces the condition $Z = |G|^{-\frac{1}{2}\chi(\Sigma_g)}\sum(\cdots)_R$. Under replication, the partition function becomes $Z_n = |G|^{-\frac{1}{2}n\chi(\Sigma_g)}\sum(\cdots)^n_R$. The prefactor, $|G|^{-\frac{1}{2}\chi(\Sigma_g)}$, thereby cancels in the ratio $\frac{Z_n}{Z^n}$. More generally, the prefactor of any partition function of the form $Z = |G|^{\mathrm{const}\cdot\chi(\Sigma_g)}\sum(\cdots)_R$ will cancel in the replica trick calculation. So, in this context, the role of the torus normalization condition is precisely to ensure that the chosen overall normalization of the partition function is irrelevant. (As a side note, prefactors of the form $|G|^{\varrho B_0}$ also share this feature. However, any such prefactor can be trivially absorbed into the deformation parameter: $B_R \to B_R + B_0 \log|G|$.)

[30] QR and LQ decomposition are variations of singular value decomposition (SVD). Definitions are given in Appendix A.2.

[31] The labels $\{h_0, \varrho_0\}$ are fixed to the R matrix, indicating that the data encoding the number of handles and area of the cylinder is being pushed right – it is not left behind in the left-normalized matrix Q_0. Eventually all of this data will be squeezed out into the singular values.

The set of vectors $\{A^{i_0}\}$ is left-normalized in the sense that $\sum_{i_0} A^{i_0\dagger}A^{i_0} = \mathbf{1}$. The matrix R_{h_0,ϱ_0} is then paired with the next element of the matrix product,

$$\left|\psi^-_{h_-,\varrho_-}\right\rangle = \frac{1}{\sqrt{Z}} \sum_{i_0,\ldots,i_{\ell-1}} A^{i_0}\left(R_{h_0,\varrho_0}\mathcal{N}^{i_1}_{h_1,\varrho_1}\right)\mathcal{N}^{i_2}_{h_2,\varrho_2}\cdots\mathcal{N}^{i_{\ell-2}}_{h_{\ell-2},\varrho_{\ell-2}}\eta^{i_{\ell-1}}_{h_{\ell-1},\varrho_{\ell-1}}\left|\phi_{i_0}\cdots\phi_{i_{\ell-1}}\right\rangle,$$

to form a new set of matrices $\{R_{h_0,\varrho_0}\mathcal{N}^{i_1}_{h_1,\varrho_1}\}$, indexed by i_1. We then reshape this set of matrices into a single matrix, followed by another QR decomposition:

$$[R_{h_0,\varrho_0}\mathcal{N}^{i_1}_{h_1,\varrho_1}]^{j_1}_{j_2} \longrightarrow [R_{h_0,\varrho_0}\mathcal{N}_{h_1,\varrho_1}]^{(i_1j_1)}_{j_2} = [Q_1]^{(i_1j_1)}_{k_2}[R_{h_0+h_1,\varrho_0+\varrho_1}]^{k_2}_{j_2}$$

The index "(i_1j_1)" runs over all $|G|\times|Z(G)|$ combinations of the values of i_1 and j_1. Of course, Q_1 is left-normalized and we can decompose it into a set of left-normalized matrices $\{A^{i_1}\}$:

$$[Q_1]^{(i_1j_1)}_{k_2} \longrightarrow \left[A^{i_1}\right]^{j_1}_{k_2},$$

after which our MPS takes the form

$$\left|\psi^-_{h_-,\varrho_-}\right\rangle = \sum_{i_0,\ldots,i_{\ell-1}} A^{i_0}A^{i_1}\left(R_{h_0+h_1,\varrho_0+\varrho_1}\mathcal{N}^{i_2}_{h_2,\varrho_2}\right)\cdots\mathcal{N}^{i_{\ell-2}}_{h_{\ell-2},\varrho_{\ell-2}}\eta^{i_{\ell-1}}_{h_{\ell-1},\varrho_{\ell-1}}\left|\phi_{i_0}\cdots\phi_{i_{\ell-1}}\right\rangle,$$

Note, A^{i_0} and A^{i_1} are a different set of matrices. We can perform this process iteratively, as many times as we need. Eventually we arrive at

$$\left|\psi^-_{h_-,\varrho_-}\right\rangle = \frac{1}{\sqrt{Z}} \sum_{i_0,\ldots,i_{\ell-1}} A^{i_0}A^{i_1}\cdots A^{i_{a-1}}R_{h_A,\varrho_A}\cdot\mathcal{N}^{i_a}_{h_a,\varrho_a}\cdots\mathcal{N}^{i_{\ell-2}}_{h_{\ell-2},\varrho_{\ell-2}}\eta^{i_{\ell-1}}_{h_{\ell-1},\varrho_{\ell-1}}\left|\phi_{i_0}\cdots\phi_{i_{\ell-1}}\right\rangle,$$

where $R_{h_A,\varrho_A} \equiv R_{h_0+\ldots+h_{a-1}\,,\,\varrho_0+\ldots+\varrho_{a-1}}$.

We then perform an identical procedure starting from the right side of the matrix product ket, except we employ LQ decomposition where previously we performed QR. For instance, after the first iteration from the right, the ket MPS will transform as

$$\begin{aligned}\left|\psi^-_{h_-,\varrho_-}\right\rangle &= \sum_{i_0,\ldots,i_{\ell-1}} A^{i_0}A^{i_1}\cdots A^{i_{a-1}}R_{h_A,\varrho_A}\cdot\mathcal{N}^{i_a}_{h_a,\varrho_a}\cdots\mathcal{N}^{i_{\ell-2}}_{h_{\ell-2},\varrho_{\ell-2}}\eta^{i_{\ell-1}}_{\varrho_{\ell-1}}\left|\phi_{i_0}\cdots\phi_{i_{\ell-1}}\right\rangle \\ &\longrightarrow \sum_{i_0,\ldots,i_{\ell-1}} A^{i_0}A^{i_1}\cdots A^{i_{a-1}}R_{h_A,\varrho_A}\cdot\mathcal{N}^{i_a}_{h_a,\varrho_a}\cdots\mathcal{N}^{i_{\ell-2}}_{h_{\ell-2},\varrho_{\ell-2}}L_{h_{\ell-1},\varrho_{\ell-1}}B^{i_{\ell-1}}\left|\phi_{i_0}\cdots\phi_{i_{\ell-1}}\right\rangle,\end{aligned}$$

where B^{i_g} is right-normalized: $\sum_{i_g} B^{i_g}B^{i_g\dagger} = \mathbf{1}$, as required for the Schmidt decomposition.

After all interations are complete, the ket MPS takes the form

$$\left|\psi^-_{h_-,\varrho_-}\right\rangle = \frac{1}{\sqrt{Z}} \sum_{i_0,\ldots,i_{\ell-1}} A^{i_0}A^{i_1}\cdots A^{i_{a-1}}R_{h_A,\varrho_A}L_{h_B,\varrho_B}B^{i_a}\cdots B^{i_{\ell-2}}B^{i_{\ell-1}}\left|\phi_{i_0}\cdots\phi_{i_{\ell-1}}\right\rangle,$$

where all the A^{i_α} are left-normalized and all the B^{i_β} are right-normalized. We then explicitly take the product $R_{h_A,\varrho_A}L_{h_B,\varrho_B}$ and decompose the (diagonal) result into a direct sum. We find

$$\left|\psi^-_{h_-,\varrho_-}\right\rangle = \sum_R s^-_R(h_-,\varrho_-) \sum_{i_0,\ldots,i_{\ell-1}} (A^{i_0}A^{i_1}\cdots A^{i_{a-1}})_R(B^{i_a}\cdots B^{i_{g-1}}B^{i_g})^R\left|\phi_{i_0}\cdots\phi_{i_{\ell-1}}\right\rangle, \qquad (5.2)$$

where the $\{s_R^-\}$ are the singular values normalized with respect to Z.

We then follow the exact same procedure to canonicalize the matrix product bra:

$$\left\langle \psi^+_{h_+,\varrho_+} \right| = \sum_R s_R^+(h_+,\varrho_+) \sum_{i_0,\ldots,i_{\ell-1}} (\bar{A}^{i_0}\bar{A}^{i_1}\cdots\bar{A}^{i_{a-1}})_R (\bar{B}^{i_a}\cdots\bar{B}^{i_{g-1}}\bar{B}^{i_g})^R \left|\phi_{i_0}\cdots\phi_{i_{\ell-1}}\right\rangle . \tag{5.3}$$

The procedure laid out here is a standard recipe for bringing a matrix product state to mixed-canonical form. The interesting feature here is that, at least for the case of 2d gauge theory, this procedure can be carried out exactly. The details of the canonicalization of the gauge theory MPS is laid out in Appendix A.3. The results are summarized as follows:

$$\begin{aligned}
A^{i_0} &\doteq \left\{ |G|^{-\frac{1}{2}} \Big(\chi_{R_1}(g_1),\cdots,\chi_{R_{|Z|}}(g_1)\Big), \ldots, |G|^{-\frac{1}{2}} \Big(\chi_{R_1}(g_{|G|}),\cdots,\chi_{R_{|Z|}}(g_{|G|})\Big) \right\}, \\
A^{i_\alpha} &\doteq \left\{ |G|^{-\frac{1}{2}} \begin{pmatrix} \chi_{R_1}(g_1) & & \\ & \ddots & \\ & & \chi_{R_{|Z|}}(g_1) \end{pmatrix}, \ldots, |G|^{-\frac{1}{2}} \begin{pmatrix} \chi_{R_1}(g_{|G|}) & & \\ & \ddots & \\ & & \chi_{R_{|Z|}}(g_{|G|}) \end{pmatrix} \right\} \quad \forall\ \alpha \in [1, a-1], \\
B^{i_\beta} &\doteq \left\{ |G|^{-\frac{1}{2}} \begin{pmatrix} \chi_{R_1}(g_1) & & \\ & \ddots & \\ & & \chi_{R_{|Z|}}(g_1) \end{pmatrix}, \ldots, |G|^{-\frac{1}{2}} \begin{pmatrix} \chi_{R_1}(g_{|G|}) & & \\ & \ddots & \\ & & \chi_{R_{|Z|}}(g_{|G|}) \end{pmatrix} \right\} \quad \forall\ \beta \in [a, \ell-2], \\
B^{i_{\ell-1}} &\doteq \left\{ |G|^{-\frac{1}{2}} \begin{pmatrix} \chi_{R_1}(g_1) \\ \vdots \\ \chi_{R_{|Z|}}(g_1) \end{pmatrix}, \ldots, |G|^{-\frac{1}{2}} \begin{pmatrix} \chi_{R_1}(g_{|G|}) \\ \vdots \\ \chi_{R_{|Z|}}(g_{|G|}) \end{pmatrix} \right\}.
\end{aligned}$$

The barred matrices of expression (5.3) are identical to the unbarred matrices, but with all group elements inverted (*e.g.*, $g_1 \to g_1^{-1}$).[32] The singular values are

$$s_R^- \in \frac{|G|^{\frac{\ell-2}{2}+h_-}}{\sqrt{|G|^{g-1}\sum_{R'} d_{R'}^{2-2g} e^{\varrho_- B_{R'}}}} \cdot \left\{ d_{R_1}^{2-\ell-2h_-} e^{\varrho_- B_{R_1}}, \ldots, d_{R_{|Z|}}^{1-g-2h_-} e^{\varrho_- B_{R_{|Z|}}} \right\},$$

$$s_R^+ \in \frac{|G|^{\frac{\ell-2}{2}+h_+}}{\sqrt{|G|^{g-1}\sum_{R'} d_{R'}^{2-2g} e^{\varrho_+ B_{R'}}}} \cdot \left\{ d_{R_1}^{2-\ell-2h_+} e^{\varrho_+ B_{R_1}}, \ldots, d_{R_{|Z|}}^{1-g-2h_+} e^{\varrho_+ B_{R_{|Z|}}} \right\}.$$

From this we calculate $s_R^2 \equiv s_R^- s_R^+$:

$$s_R^2 \in \frac{1}{\sum_{R'} d_{R'}^{2-2g} e^{\varrho\, B_{R'}}} \cdot \left\{ d_{R_1}^{2-2g} e^{\varrho\, B_{R_1}}, \ldots, d_{R_{|Z|}}^{2-2g} e^{\varrho\, B_{R_{|Z|}}} \right\},$$

where, again, $g = \ell - 1 + h_- + h_+$ and $\varrho = \varrho_- + \varrho_+$.

The entanglement spectrum is $\left\{E_R | E_R = -\log s_R^2\right\}$. Finally, from equation (3.5), the entanglement entropy is

$$S_A = -\sum_R s_R^2 \log s_R^2 = -\sum_R \frac{d_R^{2-2g} e^{\varrho\, B_R}}{\sum_{R'} d_{R'}^{2-2g} e^{\varrho\, B_{R'}}} \log \left[\frac{d_R^{2-2g} e^{\varrho\, B_R}}{\sum_{R'} d_{R'}^{2-2g} e^{\varrho\, B_{R'}}} \right], \tag{5.4}$$

which exactly matches the replica trick result in (5.1).

[32] $\{g_1, \ldots, g_{|G|}\}$ are the full set of elements of G, the chosen gauge group. The ordering is irrelevant.

Chapter 6

Conclusion and Discussion For Part I

We have demonstrated that it is possible to extract the exact entanglement entropy for generalized YM_2 with and without use of the replica trick (at least for a certain type of A/B partitioning). Although YM_2 is a nearly trivial theory, it should be possible to extend this type of analysis to any 2d TFT and its family of quasideformations (assuming the underlying associative algebra is not too pathological). In this spirit, a study of 2d topological sigma models [30, 31] is currently under investigation.

Within the framework of YM_2, an obvious extension of this work is to add Wilson loops. As suggested in [15], these can be added in several ways, and the resulting path integrals are analogous to statistical weights of certain lattice statistical mechanical systems. (For an illustration, see calculations in Appendices C.2 and C.3.) It will thus be interesting to see if the resulting theories can be identified with known lattice systems of interest in condensed matter physics. We hope that under the MPS framework developed in this paper, more significant progress along these lines could be made. It is possible that the entanglement entropy may act as a useful guide in this context. For example, a set of Wilson loops can be put along non-trivial cycles of the Riemann surface $\Sigma_{g,\varrho}$ in Fig. 4.4. Such Wilson loops can change the way disjoint spatial regions (labeled by $i_0, \cdots, i_g$ in Fig. 4.5) are entangled. While in the absence of such Wilson loops, a spatial region interacts (is entangled) with all the other spatial regions equally, by inserting Wilson loops in a particular way, we can give more structure as to how different spatial regions are entangled. These matters are also currently under investigation. If successful, it would be interesting to extend such calculations to a more general set of QTFTs, where gauge invariance is generalized to a symmetry of the associated Hopf algebra [32].

Chapter 7

Introduction to Part II: Geometry of ERG & Higher Spin Holography

Guage/gravity duality has become a modern-day cornerstone of theoretical physics. From its origins in string theory, it has found application in a diverse set of physics sub-disciplines, probably most visibly in condensed matter theory. Despite its amazing utility and a profusion of circumstantial evidence as to its veracity, the underlying mechanisms that make the tool work have in many respects remained shrouded in mystery. There does exist, however, quite a bit of speculation on the subject. One of the most enduring (and endearing) pieces of holographic lore is that holography is somehow a geometrization of the renormalization group. The gist of this idea is that scale transformations of the d-dimensional quantum field theory (QFT) correspond to radial motion in the $(d+1)$-dimensional bulk space-time, and, in particular, that different RG flows correspond to different geometries, which are asymptotically anti-de Sitter (AdS) if the RG flow begins (or ends) near a fixed point. Much has been written about this. Of particular importance to the current work are two early papers [33, 34], which propose a relationship between RG flow and Hamilton-Jacobi theory of the bulk radial evolution. Other early literature on the subject includes [35, 36, 37, 38, 39, 40] and more recently [41, 42, 43, 44, 45].

The goal in this paper is to make precise the intuition behind such speculations. We use brute force calculations and rigorous arguments to build up enough mathematical scaffolding to hold together both sides of the duality. Our most powerful computational tool for bridging the gap is the *exact renormalization group* (ERG). This technique was pioneered by Polchinski in [46]. It is a particularly clean method for carrying out renormalization group analysis, starting from small deformations away from a free-fixed point. The fact that we are choosing such a tool may sound peculiar to those familiar with holographic research. Traditionally, AdS/CFT relates simple geometric constructions in the bulk to strongly coupled dynamics in the dual boundary theory. However, this tradition stems not from rigid principles of holographic conjecture, but rather from pragmatic considerations. Holography has seen its greatest successes as a tool for extracting information from traditionally intractable quantum field theories. While historically significant, that's not our concern here. Rather we are looking to put gauge/gravity conjectures on a firmer mathematical footing. To this end, we turn things around and start from very simple quantum field theories, where computation is possible.

We will study two dualities, one for Majorana fermions and the other for complex scalars. Both of these will have global symmetries; in the fermionic case, we consider an $O(N)$ global symmetry; for the bosons, we consider a $U(N)$ global symmetry. We will couple to these theories bilocal $O(N)$-singlet (or $U(N)$-singlet), single-trace operators. The motivation here comes from an old conjecture of Klebanov and Polyakov [47], stating that such theories are dual to bulk higher spin gauge theories of the Vasiliev-type [48, 49, 50, 51]. We will have more words to say about such theories below.

If we want to definitively demonstrate that there is an exact holographic duality, we will need to find a one-to-one mapping of the sources in the d-dimensional quantum field theory onto fields

in a $(d+1)$-dimensional space-time. Furthermore, this higher dimensional space should be asymptotically AdS_{d+1} and this geometry should correspond to the RG fixed-point of the quantum field theory. More specifically, there must be a specific connection on a bundle over a $(d+1)$-dimensional space, that can be interpreted as being equivalent to having an AdS metric.

The layout of this thesis is as follows. In the remainder of the introduction, we will discuss a bit of the history of higher spin and in particular, Vasiliev's higher spin gauge theories. We'll also have a few more words to say about the Klebanov-Polyakov conjecture. In section 8 we introduce the actions for the $O(N)$ and $U(N)$ vector models at their free-field fixed points. We then deform the these theories by including bilocal single-trace operators. We choose our operators such that the new sources resemble in some way the field content of the relevant Vasiliev theory. We also introduce a non-standard form of regularization as preparation for ERG calculations of the quantum theories. In section 9 we discover that the bilocal actions of section 8 are invariant under a set of unfamiliar symmetry groups, which we call $O(L_2)$ and $U(L_2)$. Under the actions of these groups, one of the bilocal sources transforms in a way reminiscent of a connection. The other sources transform by conjugation. We develop Ward identities to describe the invariance of the partition functions under these transformations. In section 10 we get to the real meat of the paper. We develop a set of extremely powerful renormalization group equations, including an RG Ward identity. It's at this point that we see the emergence of a higher dimensional space-time. We then introduce the Hamilton-Jacobi formalism and use it to extract a bulk action. In section 12, we discuss some of the more mathematically formal details of the bulk geometry.

7.1 Higher spin

7.1.1 A brief history of higher spin

Theories containing fields of spin greater than spin-2 are usually referred to as *higher spin* theories. Such theories have garnered attention from physicists since the early years of quantum theory. For instance, in 1936, Dirac wrote a paper exploring generalizations of his spin-$\frac{1}{2}$ wave equation to arbitrarily high spin [52]. A few years later, the subject was taken up by Fierz and Pauli [53], who demonstrated that in the presence of an interacting electromagnetic field, mathematical inconsistencies developed in Dirac's higher-spin generalizations. They were able to work around these inconsistencies by moving to a Lagrangian formalism and introducing a tower of lower-spin auxiliary fields. This work was further formalized, generalized and improved upon over the years (see for example [54, 55, 56, 57, 58, 59]). In the case of massless fields, much of this effort culminated in the work of Fronsdal and Fang [60, 61], who developed a consistent theory for non-interacting higher-spin gauge fields. However, promoting such theories to include interactions ran into a snag in the form a series of no-go theorems (see [62, 63, 64, 65, 66]). A loophole exists, however. These theorems only apply in flat space-time. Moving to AdS space, Vasiliev developed a consistent theory of interacting higher-spin gauge fields [48, 49, 50, 51], which we now discuss.

7.1.2 Vasiliev's higher spin theories

We will not need to go into the gritty details of Vasiliev's higher spin constructions. We just pause to make note of a few relevant points.

A Vasiliev-type higher spin theory is a set of equations describing the dynamics of two one forms, often denoted $\mathcal{W}(x|Y,Z) = \mathcal{W}_I(x|Y,Z)dx^I$ and $\mathcal{S} = \mathcal{S}^i_A(x|Y,Z)dZ^A_i$, and a zero-form, sometimes denoted $B(x|Y,Z)$ (or, in other cases, $A(x|Y,Z)$). As mentioned previously, such theories live over AdS space-time. The variables $\{Y^A_i\}$ and $\{Z^A_j\}$ are $Sp(2) \times O(2,d)$ variables, with $A, B, \ldots \in$

$O(2,d)$ and $i,j \in Sp(2)$. One point worth noting is that the $\{Z_i^A\}$ are a set of auxiliary variables, introduced by Vasiliev to keep the equations of motion compact. These variables must be set to zero at the end of calculations, before extracting physical data. In light of this, it is important to note that while $\mathcal{W}$ is a connection over space-time (coordinatized by $\{x^I\}$), $\mathcal{S}$ is a one form over the space of these auxiliary coordinates $\{Z_i^A\}$. Therefore, while $\mathcal{W}$ and B encapsulate physical degrees of freedom, the $\mathcal{S}$ represent some kind of purely gauge degrees of freedom. We will have more to say about this later.

One last point to be made is that while consistency of the Vasiliev system requires an infinite tower of higher spin fields, this infinite tower can exist in one of two ways. The first is a tower with one spin for every non-negative integer ($s = 0, 1, 2, \ldots$). The second possibility is formulated as a consistent truncation of the first case; the truncation yields a theory with one spin for each non-negative *even* integer ($s = 0, 2, 4, \ldots$). This second flavor of higher spin theory is referred to as a *minimal* Vasiliev-type higher spin theory. It is this latter type of higher spin theory that will be of interest to us in this paper.

7.2 The Klebanov-Polyakov conjecture

The quintessential example of AdS/CFT duality relates type IIB string theory on $\mathrm{AdS}_5 \times S^5$ space-time to $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory in four dimensions [67]. While much circumstantial evidence exists to support this duality, a rigorous proof of this conjecture as an exact correspondence is well on its way to becoming the unrealized dream of a generation of physicists. Perhaps the true hurdle in producing such a proof is that both sides of the conjectured duality are independently very complicated theories. For this reason, it would be nice to find a simpler example of the AdS/CFT correspondence for which we have full control over the theory from at least one side of the duality. Such a duality was first proposed by Klebanov and Polyakov in 2002 [47]. The conjecture relates a classical Vasiliev-type higher-spin gauge theory in AdS_4 to the well known bosonic $O(N)$ vector model in three dimensions. More specifically, as is pointed out in [47], by restricting the action of the $O(N)$ vector model to contain only $O(N)$ singlet conserved currents,

$$J_{(\mu_1 \cdots \mu_s)} = \phi^a \partial_{(\mu_1} \cdots \partial_{\mu_s)} \phi^a,$$

one arrives at a theory with one current for each value of even spin ($s = 0, 2, 4, \ldots$). This spectrum of currents is then in one-to-one correspondence with the spectrum of massless higher-spin fields in the minimal bosonic higher spin theory on AdS_4.

While the $O(N)$ vector model is a drastically simpler theory than $\mathcal{N} = 4$ Super Yang-Mills, fleshing out the exact mapping to Vasiliev theory has proven non-trivial. For some of the early studies, see [47, 68, 69, 70]. Somewhat more recently, interest was reignited, possibly due to detailed calculations appropriately matching tree level three-point functions of Vasiliev's higher spin gauge theory in AdS_4 to correlators of the expected dual bosonic $O(N)$ vector model in three dimensions (see [71], and the review paper [72]). In 2011, a paper was released by Douglas *et al.* claiming to show a derivation of a Vasiliev-like higher spin gauge theory in AdS_{d+1} via the exact renormalization group (ERG) of a bosonic $U(N)$ vector model in d dimensions [73]. While amassing huge amounts of attention (see, for example, [74, 75] among many many others), the details remained sketchy.

In everything that follows, our goal is to put the ERG to higher-spin holography program on solid footing.

Chapter 8

The Bilocal Actions

We start by introducing the Majorana $O(N)$ and complex scalar $U(N)$ vector models, writing down the respective actions at the free-fixed point for each theory. Following this, we introduce regularization factors (à la Polchinski [46]), then deform away from the fixed points by adding bilocal $O(N)$ singlet (or $U(N)$ singlet), single-trace operators, in preparation of ERG calculations.

8.1 The free fixed points

For the sake of concreteness we will often work in $2+1$ dimensional Minkowski spacetime $(\mathbb{R}^3, \eta)$. However, most of our discussions generalize to other dimensions and nontrivial geometries in the obvious way. For this reason, we will often use the generic notation $(M_d, g^{(0)})$ to denote the background geometry.

8.1.1 Free fixed point of the Majorana theory

The Majorana theory will consist of N Majorana fermions with an $O(N)$ symmetry. At the free-fixed point, the action takes the form

$$S^0_{\text{Maj}} = \frac{1}{2}\int d^3x\; \tilde{\psi}^m_\alpha(x)\, i(\gamma^\mu)^\alpha{}_\beta \partial_\mu\, \psi^\beta_m(x) = \frac{1}{2}\int d^3x\; \tilde{\psi}^m(x)\, i\partial\!\!\!/\, \psi_m(x), \tag{8.1}$$

where $m = 1, \ldots, N$ and $\alpha, \beta = 0, 1$ (for $d = 3$). Before defining $\tilde{\psi}^m$, we choose the following (real) basis for the the Dirac algebra, $C\ell(2,1)$:

$$\gamma^0 = i\sigma_0 \equiv \epsilon, \qquad \gamma^1 = \sigma_1, \qquad \gamma^2 = \sigma_2,$$

from which we then define

$$\tilde{\psi}^m_\alpha \equiv \psi^\beta_m \epsilon_{\beta\alpha}.$$

This relationship is important. When we perform functional differentiation in the Majorana theory, the fields $\tilde{\psi}$ and ψ are *not* to be treated as independent variables.

It will also prove useful to introduce the orthonormal frame,

$$\underline{e}^{(0)}_a = \delta^\mu_a \partial_\mu,$$

with $a = 0, 1, 2$ (or more generally $a = 0, \ldots, d-1$). The dual coframe is denoted $e^a_{(0)}$ and is related to the metric in the usual way,

$$g^{(0)} = \eta_{ab}\, e^a_{(0)} \otimes e^b_{(0)}.$$

8.1.2 Free fixed point of the bosonic theory

For bosons, we are going to study a theory of N complex scalar fields. The action of the free fixed point is

$$S^0_{\text{Scal}} = \int d^3x \; \phi^*_m(x) \, \Box_{(x)} \, \phi(x)^m. \tag{8.2}$$

This action has a global $U(N)$ symmetry. We are using $\Box \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu$ as the d'Alembertian.

When we perform functional differentiation, it is important to remember that the complex fields ϕ^* and ϕ *are* treated as independent variables.

8.2 Deforming and regularizing the Majorana theory

Our end goal is to find a higher spin theory in the holographic bulk. Vasiliev's higher spin theory will be our guiding light, although not necessarily our landing point. With this in mind, we deform the Majorana fixed point action (8.1) by adding quadratic bilocal, single-trace operators,

$$\hat{\Pi} = \frac{1}{2}\tilde{\psi}^m(x)\psi(y), \qquad \hat{\Pi}^\mu = \frac{1}{2}\tilde{\psi}^m(x)\gamma^\mu\psi(y).$$

We source these operators as

$$S_{\text{Maj}} = \frac{1}{2}\int_x \tilde{\psi}_m(x) \, i\not{\partial}_{(x)} \, \psi^m(x) + \frac{1}{2}\int_{x,y} \tilde{\psi}_m(x) \left(A(x,y) + \not{W}(x,y)\right) \psi^m(y) \tag{8.3}$$

The idea here is that the bilocal zero-form $A(x,y)$ and one-form $W(x,y)$ encode an infinite set of *local* sources via their "quasi-local" expansions:

$$A(x,y) \cong \sum_{s=0}^{\infty} A^{a_1\cdots a_s}(x) \; \partial^{(x)}_{a_1} \cdots \partial^{(x)}_{a_s}\delta(x-y)$$

$$W_\mu(x,y) \cong \sum_{s=0}^{\infty} W_\mu{}^{a_1\cdots a_{s-1}}(x) \; \partial^{(x)}_{a_1} \cdots \partial^{(x)}_{a_{s-1}}\delta(x-y).$$

We will make these statements more precise later. For now, the intuition is that as we make our way towards a holographic interpretation, we expect the set of local sources $\{A^{a_1\cdots a_s}\}$ and $\{W_\mu{}^{a_1\cdots a_{s-1}}\}$ will come to define boundary values for an infinite set of fields of ever higher spin. It is also worth noting that in the Majorana theory, the action is only sensitive to the symmetric part of $A(x,y)$ and the antisymmetric part of $W_\mu(x,y)$.[33] We may therefore restrict the quasi-local expansions to sums over even values of s only. As of now, s is simply a summation parameter in the quasi-local expansions, but from the bulk perspective, this will correspond to a higher spin theory of even spins. So we are already seeing some signs of a minimal Vasiliev higher spin theory.

Moving on, we construct the generating functional for the corresponding quantum theory by exponentiating the above action and inserting it into a path integral. Before doing so, we must introduce some form of regularization. The traditional approach used in Wilsonian renormalization

[33] These statements follow from the following relationships:

$$\int_{x,y}\tilde{\psi}(x)A(x,y)\psi(y) = \int_{x,y}\tilde{\psi}(y)A(x,y)\psi(x), \qquad \int_{x,y}\tilde{\psi}(x)\not{W}(x,y)\psi(y) = -\int_{x,y}\tilde{\psi}(y)\not{W}(x,y)\psi(x),$$

where we have used $\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}$, $\epsilon_{\alpha\beta}\gamma^\beta{}_\gamma = \epsilon_{\gamma\beta}\gamma^\beta{}_\alpha$ and $\psi_\alpha(x)\psi_\beta(y) = -\psi_\alpha(y)\psi_\beta(x)$.

is to first Fourier transform, then introduce an arbitrary upperbound Λ on the set of Fourier modes over which the path integral is performed:

$$\int^{\infty} \prod_k [d\psi_k] \cdot e^{i \sum_k S[\psi_k]} \xrightarrow{\text{reg}} \int^{\Lambda} \prod_k [d\psi_k] \cdot e^{i \sum_k S[\psi_k]} \qquad \text{(standard approach)}$$

We will take a nuanced approach. Following [46], we insert an operator K_F^{-1} into the kinetic term of (8.3),

$$S_{\text{Maj}}^{\text{reg}} = \frac{1}{2} \int_x \tilde{\psi}_m(x)\, K_F^{-1}(-\Box_{(x)}/M^2) i\not{\partial}_{(x)}\, \psi^m(x) + \frac{1}{2} \int_{x,y} \tilde{\psi}_m(x) \left(A(x,y) + \not{W}(x,y)\right) \psi^m(y), \quad (8.4)$$

where this new operator, K_F, is defined via its Fourier transform, $\tilde{K}_F$:

$$\tilde{K}_F(s) \longrightarrow \begin{cases} 1 & s < 1 \\ 0 & s > 1 \end{cases} \quad (8.5)$$

Before continuing, we introduce some convenient notation. We define the regulated kinetic operator

$$P_{F;\mu}(x,y) \equiv K_F^{-1}(-\Box_{(x)}/M^2) i\partial_\mu^{(x)} \delta(x-y), \quad (8.6)$$

as well as the composite operator

$$D_\mu(x,y) \equiv P_{F;\mu}(x,y) + W_\mu(x,y).$$

$D_\mu(x,y)$ is suggestively structured like a covariant derivative. In fact, as we will later discover, there is formally a sense in which it is. For the moment, it is just nice compact notation. We also introduce the functional product, or bullet product,

$$(f \cdot g)(x,y) \equiv \int_u f(x,u) g(u,y).$$

It is conceptually appealing to think of the bullet product as functional matrix multiplication, where space-time coordinates take the role of matrix indices.

With our new notation, the regulated action takes a very compact form,

$$S_{\text{Maj}}^{\text{reg}} = \frac{1}{2} \tilde{\psi}_m \cdot \left(\not{P}_F + \not{W} + A\right) \cdot \psi^m + U = \frac{1}{2} \tilde{\psi}_m \cdot \left(\not{D} + A\right) \cdot \psi^m + U. \quad (8.7)$$

Note that we have also introduced a source for the identity operator $\mathbf{1}$, *i.e.,* a cosmological constant,

$$U \equiv \int_{x,y} \mathcal{U}(x,y) \equiv \int_{x,y} \mathcal{U}_0\, \delta(x-y).$$

We can think of this extra (constant) term as being extracted from the overall normalization of the path integral, which we will introduce next. The reason we extract this piece now is that later we will need it to track shifts in the overall normalization of the path integral during the process of renormalization (due to anomalous transformations of the path integral measure). What this means will become more clear when we discuss symmetries of the action in section 9 and then perform detailed RG calculations in section 10.

Finally, we define the generating functional for the quantum theory,

$$Z[M, g^{(0)}, U, A, W_\mu] = \mathcal{N} \int [d\psi] e^{iS^{\rm reg}_{\rm Maj}[\psi]} = (\det P_F)^{-N/2} \int [d\psi] e^{iS^{\rm reg}_{\rm Maj}[\psi]}$$

The overall normalization has been chosen in such as way as to make the path integral well defined. More specifically, without this normalization, the path integral evaluates to zero. We carry out a careful analysis of the normalization in the next section.

8.2.1 Making sense of the path integral normalization

We will now carefully check that the proposed normalization of the path integral makes sense. As previously stated, the normalization is split into two pieces, $\mathcal{N}$ and e^{iU}, and we will here be focused on the former,

$$\mathcal{N} = Z_0^{-1} \equiv \left(\int [d\psi] e^{-S^{0;\rm reg}_{\rm Maj}[\psi]} \right)^{-1} \equiv (\det P_F)^{-N/2} .$$

$S^{0;\rm reg}_{\rm Maj}$ is the free fixed point action (equation (8.1), with a regulated derivative, $P_{F;\mu}$, in place of the partial derivative).

Although we have been, and will be, primarily concerned with the Lorentzian theory, in order to make rigorous sense of the path integral normalization, we work momentarily in the Euclidean theory. We therefore write the partition function as

$$Z = \frac{1}{Z_0} \int [d\psi] e^{-S^{\rm reg}_{\rm Maj}[\psi]} \qquad \left(\text{where } Z_0 = \int [d\psi] S^{0;\rm reg}_{\rm Maj}[\psi] \right). \tag{8.8}$$

To see that we have chosen a sensible normalization, we expand the fields ϕ^m in eigenmodes of the operator[34]

$$K(-\Box_x/M^2) D^{(0)}_\mu(x,y) \equiv \partial_\mu \delta(x-y) + K(-\Box_x/M^2) W^{(0)}_\mu(x,y)$$

so that

$$\begin{aligned} K(-\Box_x/M^2) D^{(0)}_\mu \cdot \Psi_a &= \lambda_{a;\mu} \Psi_a \\ \psi^{m;\alpha}(x) = \sum_a \varphi^{m;\alpha}_a \Psi_a(x), \qquad \tilde{\psi}^m_\alpha(x) &= \sum_a \tilde{\varphi}^m_{a;\alpha} \Psi_a(x) \end{aligned} \tag{8.9}$$

with orthogonality conditions

$$\Psi_a \cdot \Psi_b = \delta_{ab}, \qquad \sum_a \Psi_a(x) \Psi_a(y) = \delta(x-y)$$

Plugging (8.9) back into action (8.7) and exploiting orthogonality, we find

$$S^{\rm reg}_{\rm Maj} = \frac{1}{2} \sum_{a,b} \varphi^m_a \left[K^{-1}_{\frac{\lambda_a}{M}} \lambda_a \delta_{ab} + A_{ab} + W_{ab} \right] \varphi^m_b,$$

where $K_{\frac{\lambda_a}{M}} \equiv K\left(-\frac{\lambda_a^2}{M^2}\right)$, $A_{ab} \equiv \Phi^a \cdot A \cdot \Phi^b$, and $W_{ab;\mu} \equiv \Phi^a \cdot W_\mu \cdot \Phi^b$. This sum contains many

[34] Note: $D^{(0)}_{F;\mu}(x,y) = K^{-1}(-\Box_x/M^2) D^{(0)}_\mu(x,y)$. The operator $D^{(0)}_\mu$ does not contain the inverse cutoff function K^{-1} and is therefore everywhere "finite".

divergent contributions; in particular, every mode with $\lambda_a > M$ contributes a divergence. Therefore, without proper normalization, any euclidean path integral with weights e^{-S} will vanish. Our choice of normalization for the path integral in (8.8) is a natural way to remove these divergences. Since

$$S^{0;\mathrm{reg}}_{\mathrm{Maj}} = \frac{1}{2}\sum_a \varphi^m_a K^{-1}_{\frac{\lambda_a}{M}} \lambda_a \varphi^m_a,$$

we have

$$\begin{aligned} Z &= \frac{\int \Pi_m [d\psi^m] e^{-S^{0;\mathrm{reg}}_{\mathrm{Maj}}[\psi^m]}}{\int \Pi_m [d\psi^m] e^{-S^{\mathrm{reg}}_{\mathrm{Maj}}[\psi^m]}} = \frac{\int \Pi_{m,a} [d\varphi^m_a] e^{-S^{0;\mathrm{reg}}_{\mathrm{Maj}}[\varphi^m_a]}}{\int \Pi_{m,a} [d\varphi^m_a] e^{-S^{\mathrm{reg}}_{\mathrm{Maj}}[\varphi^m_a]}} \\ &= \frac{\left[\det\left(\frac{1}{2}K^{-1}_{\frac{\lambda_a}{M}}\lambda_a\delta_{ab} + \frac{1}{2}A_{ab} + \frac{1}{2}\not{W}_{ab}\right)\right]^{\frac{N}{2}}}{\left[\det\left(\frac{1}{2}K^{-1}_{\frac{\lambda_a}{M}}\lambda_a\delta_{ab}\right)\right]^{\frac{N}{2}}}, \end{aligned}$$

where the determinants are over the matrices indexed by a, b. The divergences due to high energy modes are now canceled. In fact, for the set of modes with $\lambda_a > M$, contributions to the above quotient of determinants simply amount to repeated multiplication by unity (*i.e.,* $Z = \det[\mathbf{1} + O(K_{\frac{\lambda_a}{M}})]^{\frac{N}{2}}$)

8.3 The Complex Scalar in 2+1

We enhance the boson theory in a way similar to what we did for the Majorana case. We will regularize the kinetic using K_F^{-1} and introduce two bilocal sources, a one-form $W_\mu(x,y)$ and a zero form $B(x,y)$. Once again, these sources have their quasi-local expansions. We now use them to source bilocal single-trace operators,

$$\hat{\Pi} = \phi^*_m \cdot B \cdot \phi^m, \qquad \hat{\Pi}^\mu = \phi^*_m \cdot (D^\mu \cdot \phi^m) - (D^\mu \cdot \phi^*_m) \cdot \phi^m.$$

where we are again defining $D_\mu(x,y) \equiv P_{F;\mu}(x,y) + W_\mu(x,y)$. However, in order to maintain consistency with [2], the definition of this operator will differ from the Majorana case by a factor of i:

$$P_{F;\mu} \equiv K_F^{-1}(-\Box_{(x)}/M^2)\partial^{(x)}_\mu \delta(x-y) \qquad \text{(definition for bosonic theory)}$$

Putting it all together, we arrive at the regulated, deformed bilocal action

$$S^{\mathrm{reg}}_{\mathrm{Bos}} = \frac{1}{2}\phi^*_m \cdot (-D_\mu \cdot D^\mu + B) \cdot \phi^m + U.$$

We now see that our choice of bilocal operators has "covariantized" the kinetic operator, which is what we were secretly aiming for.

The generating functional for the corresponding quantum theory is

$$Z = Z_0^{-1} \int [d\phi\, d\phi^*] e^{iS^{\mathrm{reg}}_{\mathrm{Bos}}[\phi,\phi^*]} = (\det(-P_F^2))^N \int [d\phi\, d\phi^*] e^{iS^{\mathrm{reg}}_{\mathrm{Bos}}[\phi,\phi^*]}.$$

Chapter 9

Symmetries Of The Bilocal Actions

In this section look for symmetries of the bilocal actions. We start by considering redefinitions of the integration variables of the path integrals (namely the elementary fields) and asking how the sources transform under such redefinitions. By restricting our attention to certain non-local, linear transformations of the integration variables, we find that W_μ transforms, in some sense, like a connection, and that the scalar fields A and B transform via conjugation. Recognizing that redefinition of integration variables is a trivial procedure, we are able to develop powerful Ward identities, relating the generating functionals at different values of the sources.

9.1 Majorana Symmetries: $O(L_2)$, $CO(L_2)$

9.1.1 The bilocal gauge transformation

To make better contact with Vasiliev's higher spin theory, we now look for an interpretation of the one-form W_μ as a connection. We can develop such an interpretation by staying true to our bilocal formalism and looking for symmetry in the bilocal action. We start our search for symmetry by considering the natural field redefinition:

$$\psi^m_\alpha \mapsto \mathcal{L} \cdot \psi^m_\alpha,$$

where $\mathcal{L}$ is, for now, any functional map $L_2(\mathbb{R}^3, g^{(0)}) \mapsto L_2(\mathbb{R}^3, g^{(0)})$. We assign to the group of such linear maps the name $GL(L_2(\mathbb{R}^d, \eta))$. Plugging this into the Majorana action (8.7), we find

$$\begin{aligned} S^{\text{reg}}_{\text{Maj}}[\mathcal{L}\cdot\psi, A, W] &= \frac{1}{2}\,(\mathcal{L}\cdot\tilde{\psi})\cdot\big(\not{P}_F + \not{W} + A\big)\cdot(\mathcal{L}\cdot\psi) \\ &= \frac{1}{2}\,\tilde{\psi}\cdot\mathcal{L}^{\mathbf{T}}\cdot\big(\not{P}_F + \not{W} + A\big)\cdot\mathcal{L}\cdot\psi, \end{aligned} \tag{9.1}$$

where $\mathcal{L}^{\mathbf{T}}(x,y) \equiv \mathcal{L}(y,x)$. We are interested in how the sources, W and A, transform under this linear mapping. However, the source-independent kinetic term should remain invariant (*i.e.,* $\frac{1}{2}\,\tilde{\psi}\cdot\not{P}_F\cdot\psi \to \frac{1}{2}\,\tilde{\psi}\cdot\not{P}_F\cdot\psi$). We can *almost* reach the correct form by rearranging (9.1) in the following way:

$$S^{\text{reg}}_{\text{Maj}}[\mathcal{L}\cdot\psi, A, W] = \frac{1}{2}\,\tilde{\psi}\cdot\big(\mathcal{L}^{\mathbf{T}}\cdot\mathcal{L}\cdot\not{P}_F + \mathcal{L}^{T}\cdot\not{W}\cdot\mathcal{L} + \mathcal{L}^{\mathbf{T}}\cdot[P_{F;\mu},\mathcal{L}]_{\cdot} + \mathcal{L}^{\mathbf{T}}\cdot A\cdot\mathcal{L}\big)\cdot\psi. \tag{9.2}$$

It is now clear that we need to place some additional restriction on $\mathcal{L}$. The least restricting thing we can do is require

$$\mathcal{L}^{\mathbf{T}}\cdot\mathcal{L} = \mathbf{1}, \tag{9.3}$$

which is functional shorthand for $\int_z \mathcal{L}(z,x)\mathcal{L}(z,y) = \delta(x-y)$. This group of orthogonal maps will be called $O(L_2(\mathbb{R}^d, \eta))$ or simply $O(L_2)$ for short. From (9.3) we have that $\mathcal{L}^{\mathbf{T}} = \mathcal{L}^{-1}$, from which we can write down

$$Z[M,\, g^{(0)},\, U,\, A,\, W_\mu] = Z[M,\, g^{(0)},\, U,\, \mathcal{L}^{-1}\cdot A\cdot\mathcal{L},\, \mathcal{L}^{-1}\cdot W_\mu\cdot\mathcal{L} + \mathcal{L}^{-1}\cdot[P_{F;\mu}, \mathcal{L}]_.] \tag{9.4}$$

We now see that W_μ transforms like an $O(L_2)$ "connection". Given the strange bilocal nature of the $O(L_2)$ group action, one might question if this connection is really a connection in the usual geometrical sense. We will have more to say about this in section (12).

We can recover another familiar looking expression by moving to the infinitesimal version of the Ward identity. Taking

$$\mathcal{L}(x,y) = \delta(x-y) - \epsilon(x,y)$$

and plugging it into (9.4), we have, to first order in ϵ,

$$Z\Big[M,\, g^{(0)},\, U,\, A,\, W_\mu\Big] = Z\Big[M,\, g^{(0)},\, U,\, A + [\epsilon, A]_.\,,\, W_\mu + [\epsilon, D_\mu]_.\Big]\,.$$

From this we can write down the infinitesimal Ward identity,

$$\mathrm{Tr.}\left\{[D_\mu,\epsilon]_.\cdot\frac{\delta}{\delta W_\mu} + [\epsilon, A]_.\cdot\frac{\delta}{\delta A}\right\} Z[M,\, g^{(0)},\, U,\, A,\, W_\mu] = 0.$$

Hence, the partition function is invariant under the simultaneous transformations

$$\delta W_\mu = [D_\mu, \epsilon]_.\,, \qquad \delta A = [\epsilon, A]_.\,. \tag{9.5}$$

These expressions look exactly like what one would expect from an infinitesimal gauge transformation.

There is a subtle but important point to be made here. Because we are writing expressions in position space (rather than momentum space) transformations like (9.5) will make a mess of things, mixing modes from above and below the UV cutoff. Of course, above the cutoff, the contribution from modes is supposed to be suppressed. To express that fact, we need to impose a constraint on the transformation parameter ϵ to disallow this mixing from happening. A compact way to do this is

$$[\epsilon, M d_M P_{F;\mu}]_. = 0. \tag{9.6}$$

This works because $P_{F;\mu}$ is a step function in energy at the cutoff; hence, $d_M P_{F;\mu}$ only has support at the cutoff.

9.1.2 The flat connection

The $O(L_2)$ Ward identity given in (9.4), is a relationship of great significance. For example, in section (8), we wrote down the action for the free-fixed point of the Majorana theory, $S^{0;\mathrm{reg}}_{\mathrm{Maj}} = \frac{1}{2}\tilde{\psi}_m \cdot \not{P}_F \cdot \psi^m$. This action represents the origin of coupling space, where $A = W_\mu = 0$. In the quantum theory, the corresponding partition function is $Z[M, g^{(0)}, 0, 0]$. Now, if we refer to equation (9.4), we come to appreciate that this is not a unique way to write the partition function for the free-fixed point. In fact, for any choice of $\mathcal{L}$ obeying the orthogonality condition, we have the equivalency

$$Z\Big[M, g^{(0)}, 0, 0\Big] = Z\Big[M, g^{(0)}, 0, \mathcal{L}^{-1}\cdot[P_{F;\mu}, \mathcal{L}]_.\Big]\,.$$

This equality is precise, meaning that both descriptions of the fixed-point are equally valid. They differ only by a (non-physical) gauge transformation. In light of this, it becomes convenient to pull apart W_μ as follows:

$$W_\mu = W_\mu^{(0)} + \widehat{W}_\mu \tag{9.7}$$

$$dW^{(0)} + W^{(0)} \wedge W^{(0)} = 0. \tag{9.8}$$

What we have done is split the connection W_μ into $W_\mu^{(0)}$ and $\widehat{W}_\mu$, which are, respectively, a flat $O(L_2)$ connection and an $O(L_2)$ tensor. By calling $\widehat{W}_\mu$ a tensor, we mean that it transforms via conjugation under an $O(L_2)$ gauge transformation[35]. Note that the exterior derivative used in equation (9.8) is regulated via the kinetic operator,

$$d = dx^\mu \, [P_{F;\mu}, \,]_. \, .$$

When performing RG calculations, the splitting of W_μ will become important, for it is the flow of the tensorial sources A and $\widehat{W}_\mu$ away from the free fixed point $Z[M, g^{(0)}, 0, W^{(0)}]$ which is meaningful.

Before continuing, we introduce one more useful piece of notation. We will often find it convenient to write the covariant derivative with respect to the flat connection,

$$D_\mu^{(0)} \equiv P_{F;\mu} + W_\mu^{(0)}.$$

9.1.3 Extension to $CO(L_2)$

We now explore an enlarged symmetry, which we call $CO(L_2)$. Again we consider linear tranformations of the field, $\psi_\alpha^m \mapsto \mathcal{L} \cdot \psi_\alpha^m$, but we relax the orthogonality condition, allowing for a conformal factor of the form

$$\mathcal{L}^{\mathbf{T}} \cdot \mathcal{L} = \Omega^2. \tag{9.9}$$

With space-time coordinate labels, this constraint reads $\int_z \mathcal{L}(z,x)\mathcal{L}(z,y) = \Omega^2(x)\delta(x-y)$. In general Ω can be any local function. For now, we focus on the case where Ω is a constant, $\Omega = \lambda_\psi^\delta$, where $\Delta_\psi = \frac{d-1}{2}$ is the scaling dimension of ψ^m. To understand the action of a $CO(L_2)$ transformation, it turns out to be useful to introduce a conformal factor z into the metric, so that

$$g^{(0)} = z^{-2}\eta_{\mu\nu}.$$

This allows us to think in terms of a flat background metric $\eta_{\mu\nu}$. We also choose to rescale the sources as $A \mapsto z^{d+1}A$, $W \mapsto z^d W$. This guarantees that the kinetic term and source terms of the action scale identically[36]. With the conformal factors inserted, the action now becomes

$$S_{\text{Maj}}^{\text{reg}} = \frac{1}{2z^{d-1}}\tilde{\psi} \cdot (\not{P}_F + \not{W} + A) \cdot \psi,$$

[35]

$$W_\mu = W_\mu^{(0)} + \widehat{W}_\mu \;\longrightarrow\; \mathcal{L}^{-1} \cdot W_\mu \cdot \mathcal{L} + \mathcal{L}^{-1} \cdot [P_{F;\mu}, \mathcal{L}]_. = \Big(\mathcal{L}^{-1} \cdot W_\mu^{(0)} \cdot \mathcal{L} + \mathcal{L}^{-1} \cdot [P_{F;\mu}, \mathcal{L}]_.\Big) + \Big(\mathcal{L}^{-1} \cdot \widehat{W} \cdot \mathcal{L}\Big)$$

[36] Note that in order to preserve the defining relationship for the Dirac delta function, $\int d^d x \sqrt{g}\, \delta^{(d)}(x-y) \equiv 1$, we must also make the replacement $\delta^{(d)}(x-y) \mapsto z^d \delta^{(d)}(x-y)$

and the regularized derivative $P_{F;\mu}$ now becomes

$$P_{F;\mu} = K_F^{-1}\left(-\frac{z^2\eta^{\mu\nu}\partial_\mu\partial_\nu}{M^2}\right).$$

Under a $CO(L_2)$ transformation, the action then transforms as

$$\begin{aligned} S^{\rm reg}_{\rm Maj}[\psi] &\mapsto S^{reg}_{Maj}[\mathcal{L}\cdot\psi] \qquad (9.10)\\ &= \frac{1}{2z^{d-1}}\tilde{\psi}^m\cdot\mathcal{L}^T\cdot(\not{P}_{F;\mu}+\not{W}_\mu+A)\cdot\mathcal{L}\cdot\psi^m \\ &= \frac{1}{2z^{d-1}}\tilde{\psi}^m\cdot(\mathcal{L}^T\cdot\mathcal{L}\cdot\not{P}_{F;\mu}+\mathcal{L}^T\cdot W_\mu\cdot\mathcal{L}+\mathcal{L}^T\cdot[P_{F;\mu},\mathcal{L}].+\mathcal{L}^T\cdot A\cdot\mathcal{L})\cdot\psi^m \\ &= \frac{1}{2(\lambda^{-1}z)^{d-1}}\tilde{\psi}^m\cdot(\not{P}_{F;\mu}+\mathcal{L}^T\cdot W_\mu\cdot\mathcal{L}+\mathcal{L}^T\cdot[P_{F;\mu},\mathcal{L}].+\mathcal{L}^T\cdot A\cdot\mathcal{L})\cdot\psi^m. \qquad (9.11)\end{aligned}$$

We have learned that a $CO(L_2)$ transformation can be thought of as a "gauge" transformation on the sources, plus a Weyl transformation of the background metric, $z\mapsto\lambda^{-1}z$, along with a rescaling of the cutoff $M\mapsto\lambda^{-1}M$. In addition, unlike in the simple $O(L_2)$ case, we must now consider the possibility that the measure of the path integral exhibits a non-trivial Jacobian under a $CO(L_2)$ transformation. We track this possible anomaly by absorbing the possible Jacobian factor $\mathcal{J}$ into the constant term in the action. In other words, we allow for a shift in the cosmological constant:

$$U\mapsto\hat{U}=U-i\,\ln\mathcal{J}.$$

Finally, we can summarize these results in terms of a Ward identity:

$$Z\left[M,z,U,A,W_\mu\right]=Z\left[\lambda^{-1}M,\lambda^{-1}z,\widehat{U},\mathcal{L}^{-1}\cdot A\cdot\mathcal{L},\mathcal{L}^{-1}\cdot W_\mu\cdot\mathcal{L}+\mathcal{L}^{-1}\cdot[P_{F;\mu},\mathcal{L}]\right].$$

9.2 Bosonic Symmetries: $U(L_2)$, $CU(L_2)$

Much of the analysis for the boson carries through identically. We again consider linear functional transformations of the type

$$\phi^m\mapsto\mathcal{L}\cdot\phi^m.$$

However, since we are dealing with complex scalars, the gauge groups of interest are now the unitary group $U(L_2)$,

$$\mathcal{L}^\dagger\cdot\mathcal{L}=\mathbf{1},$$

and conformal unitary group $CU(L_2)$,

$$\mathcal{L}^\dagger\cdot\mathcal{L}=\Omega.$$

Under the expanded symmetry of $CU(L_2)$, the Ward identity for the complex Boson is

$$Z\left[M,z,U,B,W_\mu\right]=Z\left[\lambda^{-1}M,\lambda^{-1}z,\widehat{U},\mathcal{L}^{-1}\cdot B\cdot\mathcal{L},\mathcal{L}^{-1}\cdot W_\mu\cdot\mathcal{L}+\mathcal{L}^{-1}\cdot[P_{F;\mu},\mathcal{L}]\right],$$

just like in the Majorana theory. However, there is an additional background symmetry which exists for the bosonic case. To see this, note that the action (with appropriate conformal factors

inserted) is

$$S_{\text{bos}}^{\text{reg}} = \frac{1}{z^{d-2}}\phi_m^* \cdot \left(-D_\mu^{(0)} \cdot D^{(0)\mu} + B - \left\{\widehat{W}^\mu, D_\mu^{(0)}\right\}_{\cdot} - \widehat{W}_\mu \cdot \widehat{W}^\mu\right) \cdot \phi^m.$$

The first term in parentheses is the kinetic term, with the flat connection $W^{(0)}$ absorbed. The rest of the action includes the source terms B and $\widehat{W}$, all of which transform tensorial. In this sense, the structure is similar to the Majorana case. However, unlike in the Majorana theory, we no longer have the Dirac algebra to define a basis $(\mathbf{1}, \gamma^0, \gamma^1, \gamma^2)$ on which to project the sources. This lack of structure gives us freedom to bundle the source terms, defining a new source $\mathcal{B}$ as follows:

$$\mathcal{B} = B - \left\{\widehat{W}^\mu, D_\mu^{(0)}\right\}_{\cdot} - \widehat{W}_\mu \cdot \widehat{W}^\mu. \tag{9.12}$$

Of course, this is just one possible reassignment of the sources. More precisely, what we have found is an additional symmetry of the bosonic theory. This can be encoded in the independent identity

$$Z\left[M, z, U, B, \widehat{W}_\mu + \Lambda_\mu\right] = Z\left[M, z, B - \{\Lambda_\mu, D_\mu\}_{\cdot} - \Lambda_\mu \cdot \Lambda^\mu \cdot W_\mu^{(0)}, \widehat{W}_\mu]\right].$$

Using relationship (9.12) is in fact equivalent to using this additional symmetry to shift $\widehat{W}_\mu$ to zero; it leaves us with a theory in terms of sources $\mathcal{B}$ and $W^{(0)}$ only. We will henceforth use (9.12) to simplify our action:

$$S_{\text{bos}}^{\text{reg}} = \frac{1}{z^{d-2}}\phi_m^* \cdot \left(-D_\mu^{(0)} \cdot D^{(0)\mu} + \mathcal{B}\right) \cdot \phi^m + U.$$

With this choice, the $CU(L_2)$ Ward identity also simplifies:

$$Z\left[M, z, U, \mathcal{B}\right] = Z\left[\lambda^{-1}M, \lambda^{-1}z, \widehat{U}, \mathcal{L}^{-1} \cdot \mathcal{B} \cdot \mathcal{L}\right]$$

Chapter 10

From Renormalization Group to Holography

Since the inception of AdS/CFT, there has been much speculation about its connection to renormalization group. A concrete demonstration of this connection has, however, been lacking. In this section we put these speculations on firmer ground by demonstrating that, at least for the case at hand, the correspondence can be made rigorous.

We first recall the usual notion of renormalization of a quantum field theory as laid out by Wilson. In this approach, we think of the couplings of an action as being, not constants, but functions of the energy scale at which the physical system may be probed. The usual recipe for carrying out Wilsonian renormalization is a multi-step process. In the initial step, which we will call step zero, the theory is regularized by imposing a UV cutoff M on the path integral. This removes the very high energy Fourier modes of the quantum fields. In step one of renormalization, the cutoff is then lowered by integrating out all modes between λM and M, where λ is a number slightly smaller than unity ($\lambda = 1 - \varepsilon$). Finally, in step two, the space-time coordinates are scaled so as to bring the cutoff of the new path integral back to value M. This allows one to make a fair comparison between the original path integral and the renormalized path integral.

In this work, we follow an approach conceptually identical to traditional Wilsonian renormalization, but with some technical modifications. For starters, in **step zero**, we cutoff the path integral, not by brazenly ignoring the highest energy modes, but by use of the cutoff function K_F^{-1} inserted into the kinetic term of the action. This gives us the regularized action presented in the previous section[37]:

$$S_0 = \frac{1}{2}\,\tilde{\psi}\cdot i\not{\partial}\cdot\psi \;\longrightarrow\; S_0^{\text{reg}} = \frac{1}{2}\,\tilde{\psi}\cdot \not{P}_{F;\mu}\cdot\psi.$$

Next, in **step one**, we "integrate out" the high energy modes by scaling the argument of the cutoff function, $M \to \lambda M$, and allowing the couplings to transform as needed such that the value of the partition function remains unchanged,

$$Z[M, z, U(z), A(z;x,y), W_\mu(z;x,y)] = Z\left[\lambda M, z, \widetilde{U}(z), \widetilde{A}(z;x,y), \widetilde{W}_\mu(z;x,y)\right]. \tag{10.1}$$

We have used tildes to denote the modified sources. In **step two**, the final step, we scale the metric by shifting the conformal factor as $z \to \lambda^{-1} z$ via a $CO(L_2)$ transformation, rather than by direct manipulation of the space-time coordinates. This has the effect of bringing the cutoff back to its original value. In this step, we also allow for a generalization of the usual Wilsonian approach; in addition to scaling, we allow arbitrary translations. We will lay out the details of these steps explicitly in the coming section.

For purpose of comparison, after completing step two of the renormalization procedure, we re-label the final sources as $W_\mu(\lambda^{-1}z; x+\varepsilon\xi, y+\varepsilon\xi)$, $A(\lambda^{-1}z; x+\varepsilon\xi, y+\varepsilon\xi)$ and $U(\lambda^{-1}z)$. In

[37] We are presenting the renormalization recipe in the Majorana language. The recipe for the complex scalar field theory is conceptually the same.

summary, the RG procedure is

$$
\begin{aligned}
&Z[M, z, U(z), A(z;x,y), W_\mu(z;x,y)] \\
&\overset{(1)}{=} Z\left[\lambda M, z, \widetilde{U}(z), \widetilde{A}(z;x,y), \widetilde{W}_\mu(z;x,y)\right] && (10.2)\\
&\overset{(2)}{=} Z\Big[M, \lambda^{-1}z, \widehat{\widetilde{U}}(z), \mathcal{L}^{-1}\cdot\widetilde{A}(z;x,y)\cdot\mathcal{L}, \mathcal{L}^{-1}\cdot\widetilde{W}_\mu(z;x,y)\cdot\mathcal{L} + \mathcal{L}^{-1}\cdot[P_{F;\mu}, \mathcal{L}]\Big] && (10.3)\\
&= Z\Big[M, \lambda^{-1}z, U(\lambda^{-1}z), A(\lambda^{-1}z; x+\varepsilon\xi, y+\varepsilon\xi), W_\mu(\lambda^{-1}z; x+\varepsilon\xi, y+\varepsilon\xi)\Big] && (10.4)
\end{aligned}
$$

So, we are equating the initial and final partition functions at the same cutoff M, but at different values of z. The first equality is just step one of RG, equation (10.1), written again for clarity. The second equality is step two of RG, the $CO(L_2)$ transformation. This equality includes the notation $\widehat{\widetilde{U}}$, encoding the possibility of a $CO(L_2)$ Weyl anomaly. The significance of the anomaly will be clarified shortly when we carry out a detailed RG computation, and even furthermore when we calculate correlation functions in section (11). The third equality is simply a re-labeling of the sources, as described above.

A pictorial represention of the two step RG process is shown in Fig. 10.1. In the first step, we rescale the cutoff, $M \to \lambda M$, where $\lambda = 1-\varepsilon$ and $\varepsilon \ll 1$. In such a process, the size of local regions of space-time do not change; rather the microscopic physical description of the theory becomes more coarsely grained (*i.e.,* the degrees of freedom are reduced). In step two, we conformally rescale the metric by taking $z \to \lambda^{-1}z$. In contrast to step one, this process *does* shrink local regions of space-time. The important point here is that the grainularity of space-time is now effectively returned to its original level. This allows for a legitimate comparison of the generating functional over the original and final descriptions of space-time.

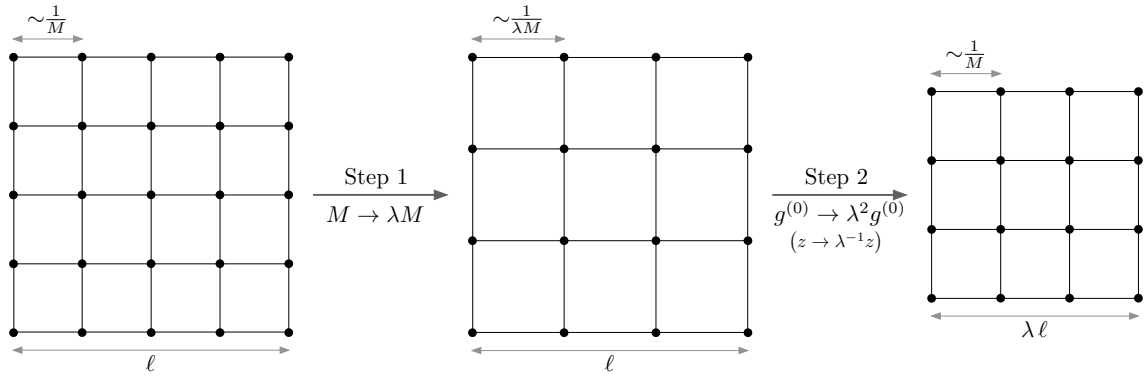


Figure 10.1: A schematic description of the two step RG process. We have indicated the cutoff in terms of the lattice spacing.

Fig. 10.2 depicts how one can now develop a holographic interpretation of RG by taking the conformal factor z as a new coordinate in a space-time of one-higher dimension. The bulk can be built up layer-by-layer via repeated application of the two step renormalization. Note that Fig. 10.2 is just a stacked version of Fig. 10.1 (with the depiction of the intermediate step removed). Renormalization of the d-dimensional boundary theory thereby corresponds to motion into the $(d+1)$-dimensional bulk.

As one moves into the bulk via RG transformations, the values of A and W_μ change according to relationship (10.4). For the moment we will be careful and continue to naively think in terms of

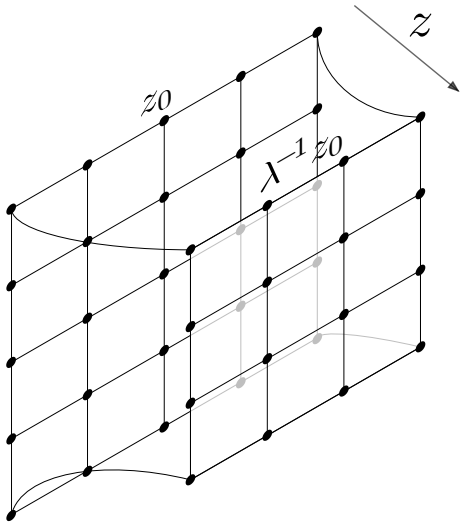


Figure 10.2: It is useful to think of different values of z as corresponding to different copies of space-time. From this point of view, a holographic interpretation naturally emerges out of the renormalization group.

the full RG space as just a series of d-dimensional copies of the original space-time, each element of which is labeled by its value in z. What we will find is that under RG, $W_\mu(z)$ naturally evolves into a legitimate connection over a $(d+1)$-dimensional space-time, in which the stack of d-dimensional spaces are naturally understood as foliations in the z-coordinate of the higher-dimensional space.

10.1 RG flow of the Majorana theory

In this section we carry out exact renormalization group calculations in fine detail. We start with the Majorana theory.

To cut back on notation a bit, it is convenient to split up the action as

$$\begin{aligned}
S_{\text{reg}}^{Maj} &= S_0 + S_{int} + U, \\
S_0 &= \frac{1}{2}\int_{x,y} \tilde{\psi}^m(x) \not{D}^{(0)}(M;x,y)\psi^m(y), \quad \left\{ \begin{array}{l} D_\mu^{(0)}(M;x,y) = P_{F;\mu}(M;x,y) + W_\mu^{(0)}(x,y) \\ P_{F;\mu}(M;x,y) = K_F^{-1}(-\Box_x/M^2) i\partial_\mu^x \delta(x-y) \end{array} \right. \\
S_{int} &= \frac{1}{2}\int_{x,y} \tilde{\psi}^m(x) \left[A(x,y) + \widehat{W}(x,y) \right] \psi^m(y), \\
U &= \int_{x,y} \mathcal{U}(x,y) \equiv \int_{x,y} \mathcal{U}_0\, \delta(x-y).
\end{aligned} \tag{10.5}$$

The partition function will be defined as $Z = Z_0^{-1}\int [d\psi] e^{iS}$, normalized with respect to $Z_0 \equiv \int [d\psi] e^{iS_0}$. It is important to remember that for Majorana fields, unlike (complex) Dirac fields, ψ and $\tilde{\psi}$ are *not* independent objects ($\tilde{\psi}^m_\beta \equiv \psi^{m;\alpha}\epsilon_{\alpha\beta}$).

10.1.1 Exact RG equations: Wilson-Polchinski renormalization in two steps

STEP 1: We begin the renormalization procedure by lowering the cut-off from M to λM, where $\lambda < 1$. In the language of traditional Wilsonian renormalization, this amounts to integrating out a shell of fast modes; however, that which is integrated out is effectively reabsorbed back into the

action via a shift in the values of the sources. More specifically, the Polchinski formalism demands the following equality is respected:

$$Z[M, z, A, W_\mu, \mathcal{U}] = Z[\lambda M, z, \widetilde{A}, \widetilde{W}_\mu, \widetilde{\mathcal{U}}]. \tag{10.6}$$

What is being said here is that we're adjusting the values of the sources in such a way as to keep the value of the path integral independent of the value of the cutoff. We will now determine exactly how these sources transform. Working infinitesimally, we take $\lambda = 1 - \varepsilon$ in (10.6), yielding

$$0 = \delta_\varepsilon Z = \delta_\varepsilon \left(Z_0^{-1} \int [d\psi] e^{iS} \right) = -Z_0^{-1} \left(\delta_\varepsilon \int [d\psi] e^{iS_0} \right) Z_0^{-1} \int [d\psi] e^{iS} + Z_0^{-1} \left(\delta_\varepsilon \int [d\psi] e^{iS} \right) \tag{10.7}$$

where the variations of the path integrals can be expanded by chain rule:

$$\begin{aligned} \delta_\varepsilon \int [d\psi] e^{iS_0} &= \int [d\psi] M d_M e^{iS_0} \\ \delta_\varepsilon \int [d\psi] e^{iS} &= \int [d\psi] \left(e^{i(S_{int}+U)} M d_M e^{iS_0} + e^{iS_0} \mathrm{Tr} \left\{ \delta_\varepsilon A \cdot \frac{\delta}{\delta A} + \delta_\varepsilon \widehat{W}_\mu \cdot \frac{\delta}{\delta \widehat{W}_\mu} + \delta_\varepsilon \mathcal{U} \cdot \frac{\delta}{\delta \mathcal{U}} \right\} e^{i(S_{int}+U)} \right). \end{aligned} \tag{10.8}$$

These expressions use the dot notation defined previously, $f \cdot g \equiv \int_x f(x) g(x)$, as well as a functional trace, $\mathrm{Tr} f(x,y) \equiv \int_{x,y} \delta(x-y) f(x,y)$.[38] It is convenient to define

$$\not{\Delta}(x,y) \equiv \gamma^\mu \Delta_\mu(x,y) = M d_M \left(\not{D}^{(0)} \right)^{-1} (x,y). \tag{10.9}$$

Given our choice of S_0, we get

$$M d_M e^{iS_0} = -\frac{i}{2} \int_{x,y} \left(\not{D}^{(0)} \cdot \psi \right)^\alpha (x) \epsilon_{\alpha\beta} \left(\not{\Delta} \right)^\beta{}_\gamma (x,y) \left(\not{D}^{(0)} \cdot \psi \right)^\gamma (y)\ e^{iS_0}, \tag{10.10}$$

where we have supressed the $O(N)$ vector indices, and explicitly shown some of the spinor indices. Using

$$\epsilon^{\alpha\beta} \frac{\delta S_0}{\delta \psi^\beta(x)} = \left(\not{D}^{(0)} \cdot \psi \right)^\alpha (x),$$

we may re-write (10.10) as

$$M d_M e^{iS_0} = -\frac{i}{2} \int_{x,y} \left(\not{\Delta} \right)^\alpha{}_\gamma (x,y) \epsilon^{\gamma\beta} \left(\frac{\delta^2}{\delta\psi^\alpha(x)\delta\psi^\beta(y)} e^{iS_0} - i \frac{\delta^2 S_0}{\delta\psi^\alpha(x)\delta\psi^\beta(y)} e^{iS_0} \right). \tag{10.11}$$

Substituting this expression back into (10.7), the term proportional to $\frac{\delta^2 S_0}{\delta\psi^\alpha(x)\delta\psi^\beta(y)}$ cancels out. This is the effect of normalization. The other term, $\frac{\delta^2}{\delta\psi^\alpha(x)\delta\psi^\beta(y)} e^{iS_0}$, can be integrated by parts so

[38] Note, when a functional derivative appears as part of a functional product, the contraction is

$$\left(f \cdot \frac{\delta}{\delta g} \right)(x,y) = \int_u f(x,u) \frac{\delta}{\delta g(y,u)}$$

that the functional derivatives hit $e^{iS_{int}}$. The result is

$$0 = \frac{1}{Z_0}\int [d\psi]\; e^{iS_0}\;\mathrm{Tr}\left\{\frac{i}{2}(\not{\Delta})^{\alpha}{}_{\gamma}\epsilon^{\gamma\beta}\cdot\frac{\delta^2}{\delta\psi^{\alpha}\delta\psi^{\beta}} + \delta_{\varepsilon}A\cdot\frac{\delta}{\delta A} + \delta_{\varepsilon}\widehat{W}_{\mu}\cdot\frac{\delta}{\delta\widehat{W}_{\mu}} + \delta_{\varepsilon}\mathcal{U}\cdot\frac{\delta}{\delta\mathcal{U}}\right\} e^{i(S_{int}+U)}. \tag{10.12}$$

Using the explicit forms of S_{int} and U from equations (10.5), this expression reduces to

$$\begin{aligned} 0 = \Big\langle & -\frac{i}{2}\,N\int_{x,y}\not{\Delta}^{\alpha}{}_{\beta}(x,y)\;(A+\widehat{\not{W}})^{\beta}{}_{\alpha}(x,y) \\ & -\frac{1}{2}\,\tilde{\psi}\cdot(A+\widehat{\not{W}})\cdot\not{\Delta}\cdot(A+\widehat{\not{W}})\cdot\psi \\ & +\frac{1}{2}\,\tilde{\psi}\cdot\delta_{\varepsilon}A\cdot\psi + \frac{1}{2}\,\tilde{\psi}\cdot\delta_{\varepsilon}\widehat{\not{W}}\cdot\psi + \delta_{\varepsilon}U\Big\rangle. \end{aligned} \tag{10.13}$$

Now restricting our attention to 2+1 dimensions, we evaluate the various gamma matrix products (making repeated use of $\gamma^{\mu}\gamma^{\nu} = \eta^{\mu\nu} + \epsilon^{\mu\nu\lambda}\gamma_{\lambda}$) to obtain

$$0 = \int_{x,y}\Big(\delta_{\varepsilon}A - \beta^{(A)}\Big)(x,y)\;\Pi(x,y) + \int_{x,y}\Big(\delta_{\varepsilon}\widehat{W} - \beta^{(W)}_{\mu}\Big)(x,y)\;\Pi^{\mu}(x,y) + \Big(\delta_{\varepsilon}U - \beta^{(U)}\Big)\,\langle 1\rangle,$$

where we have defined

$$\beta^{(A)} = A\cdot\Delta^{\mu}\cdot\widehat{W}_{\mu} + \widehat{W}_{\mu}\cdot\Delta^{\mu}\cdot A + \epsilon^{\mu\nu\lambda}\widehat{W}_{\mu}\cdot\Delta_{\nu}\cdot\widehat{W}_{\lambda} \tag{10.14}$$

$$\beta^{(W)}_{\mu} = A\cdot\Delta_{\mu}\cdot A + \epsilon_{\mu\nu\lambda}\Big(A\cdot\Delta^{\nu}\cdot\widehat{W}^{\lambda} + \widehat{W}^{\nu}\cdot\Delta^{\lambda}\cdot A\Big) + \widehat{W}_{\nu}\cdot\Delta^{\nu}\cdot\widehat{W}_{\mu} - \widehat{W}_{\nu}\cdot\Delta_{\mu}\cdot\widehat{W}^{\nu} + \widehat{W}_{\mu}\cdot\Delta_{\nu}\cdot\widehat{W}^{\nu} \tag{10.15}$$

$$\beta^{(U)} = -iN\,\mathrm{Tr}\Big\{\Delta_{\mu}\cdot\widehat{W}^{\mu}\Big\}. \tag{10.16}$$

Taking Π, Π^{μ} and $\langle 1\rangle$ to be independent objects, we deduce that

$$\delta_{\varepsilon}A = \beta^{(A)}, \qquad \delta_{\varepsilon}\widehat{W} = \beta^{(W)}_{\mu}, \qquad \delta_{\varepsilon}U = \beta^{(U)}.$$

STEP 2: Next, we perform a $CO(L_2)$ scale transformation, such that $z \to (1+\varepsilon)z$. We will accompany this with an arbitrary spatial translation. One motivation for including this extra transformation is that we are aiming here for a holographic interpretation, and while the scale transformation will naturally fill the role of a *lapse* in the bulk (from the Hamiltonian point of view with time along the radial direction), the translation plays the role of a *shift*. More concretely, we will write the bilocal translation as $x^{\mu} \to x^{\mu} + \varepsilon\xi^{\mu}$, $y^{\mu} \to y^{\mu} + \varepsilon\xi^{\mu}$ and, as we will soon find, $\vec{\xi}$ becomes a shift vector in the bulk frame. From the boundary point of view, there is no reason not to include the translation, since the theory is inherently translation invariant.

Under this pair of transformations (scale plus translation), the sources A and W_{μ} transform. We have already explored what these transformations look like; for instance, under a $CO(L_2)$ transformation, A transforms like a tensor and W_{μ} transforms like a connection. Now we need to get our hands a bit dirtier. To get started, we parametrize the infinitesimal $CO(L_2)$ plus spatial translation as

$$\mathcal{L} = \mathbf{1} + \varepsilon z W_z + \varepsilon\xi^{\mu}W_{\mu}. \tag{10.17}$$

We have here introduced a new function $W_z(z;x,y)$. Subjecting (10.17) to the orthogonality constraint (9.9) tells us that $z(W_z + W_z^{\mathbf{T}}) = -2\Delta_{\psi}\mathbf{1}$. This notation is meant to be suggestive;

we will soon see that W_z naturally fills the role of the dz component of a bulk connection with components (W_z, W_μ). In fact, we can already see signs of this; the parametrization chosen in (10.17) seems to resemble an infinitesimal Wilson line in a $(d+1)$-dimensional space,

$$\mathcal{L} = \mathbf{1} + \int_0^1 dt \left(\frac{dz}{dt} W_z + \frac{dx^\mu}{dt} W_\mu \right) + \mathcal{O}(\varepsilon^2),$$

covariantly transporting sources from $(z; x^\mu, y^\mu)$ to $(z + \varepsilon z; x^\mu + \varepsilon\xi^\mu, y^\mu + \varepsilon\xi^\mu)$ along the path $(z(t); x^\mu(t), y^\mu(t)) = (z + t\varepsilon z, x^\mu + t\varepsilon\xi^\mu, y^\mu + t\varepsilon\xi^\mu)$. See figure (10.3) for a pictorial depiction of this process.

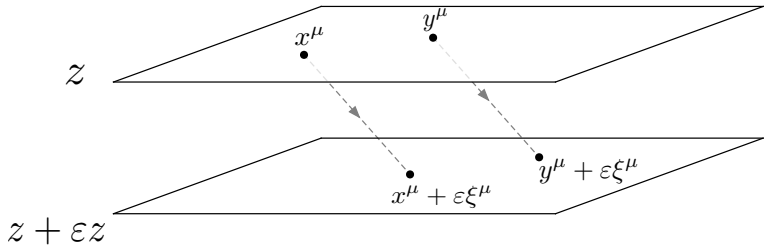


Figure 10.3: A pictorial representation of the Wilson line interpretation – the $CO(L_2)$ transformation in step 2 of RG may be thought of as an infinitesimal Wilson line, covariantly transporting sources from z to $z + z\varepsilon$.

We now apply the infinitesimal $CO(L_2)$ transformation plus spatial translation of (10.17). This brings the partition function back to the original cut-off M, while the conformal factor of the background metric changes as $z \mapsto \lambda^{-1} z$. We then label the final sources as $A(\lambda^{-1}z; x+\varepsilon\xi, y+\varepsilon\xi)$, $W_\mu(\lambda^{-1}z; x+\varepsilon\xi, y+\varepsilon\xi)$ and $U(\lambda^{-1}z)$, so that

$$A(\lambda^{-1}z; x+\varepsilon\xi, y+\varepsilon\xi) = A(z; x, y) - \varepsilon z \, [W_z, A]_. - \varepsilon\xi^\mu \, [W_\mu, A]_. + \varepsilon\beta^{(A)} + O(\varepsilon^2)$$

$$W_\mu(\lambda^{-1}z; x+\varepsilon\xi, y+\varepsilon\xi) = W_\mu(z) + \varepsilon z \, [P_{F;\mu} + W_\mu, W_z]_. + \varepsilon\xi^\nu \, [P_{F;\mu} + W_\mu, W_\nu]_. + \varepsilon\beta_\mu^{(W)} + O(\varepsilon^2)$$

$$U(\lambda^{-1}z) = U(z) + \varepsilon\beta_U - i\varepsilon N \operatorname{Tr}\{\Delta_z \cdot W^z\}.$$

We have here introduced the notation Δ_z to denote a (possible) $CO(L_2)$ anomaly. In particular, Δ_z should be thought of as the anomaly for a single Majorana fermion, hence the scaling of the full anomaly with N. Note that given the structure of β_U, it seems as if Δ_z naturally combines with Δ_μ into $\Delta_I = (\Delta_z, \Delta_\mu)$. Furthermore, we redefine $\not{\Delta}$ as $\not{\Delta} = \frac{M}{z} d_z \left(\not{D}^{(0)} \right)^{-1}$. The result is

$$A(z + \varepsilon z; x + \varepsilon\xi, y + \varepsilon\xi) = A(z; x, y) + [A, \varepsilon z W_z + \varepsilon\xi^\mu W_\mu]_. + \varepsilon z \beta^{(A)} + O(\varepsilon^2) \tag{10.18}$$

$$W_\mu(z + \varepsilon z; x + \varepsilon\xi, y + \varepsilon\xi) = W_\mu(z; x, y) + [P_{F;\mu} + W_\mu, \varepsilon z W_z + \varepsilon\xi^\nu W_\nu]_. + \varepsilon z \beta_\mu^{(W)} + O(\varepsilon^2). \tag{10.19}$$

$$U(z + \varepsilon z) = U(z) + \varepsilon z \beta_U - i\varepsilon N \operatorname{Tr}\{\Delta_z \cdot W^z\}.$$

In equation (10.19), we have a relationship for the full connection $W_\mu(z; x, y)$ under an RG transformation. To make sense of how the pieces $W_\mu^{(0)}(z; x, y)$ and $\widehat{W}(z; x, y)$ independently transform, we can demand that the transformation of $W_\mu^{(0)}$ leaves the connection flat. In other words,

the gauge transformation induced by RG on $W_\mu^{(0)}$ must be pure gauge:

$$W_\mu^{(0)}(z+\varepsilon z; x+\varepsilon\xi, y+\varepsilon\xi) = W_\mu^{(0)}(z;x,y) + \varepsilon\left[D_\mu^{(0)}, zW_z^{(0)} + \xi^\nu W_\nu^{(0)}\right] + O(\varepsilon^2) \tag{10.20}$$

This expresses the fact that the theory is RG invariant at the fixed point.

Finally, the desired RG equations can be extracted by expanding (10.20), (10.18) and (10.19) and taking $\varepsilon \to 0$. For instance, starting with the RG equation for A, the left-hand side expands to

$$A(z+\varepsilon z; x+\varepsilon\xi, y+\varepsilon\xi) = A(z;x,y) + \varepsilon\left(z\partial_z + \xi^\mu P_{F;\mu}\right) A(z;x,y) + O(\varepsilon^2).$$

Comparing to the right-hand side of (10.18), we have, in the limit $\varepsilon \to 0$,

$$\left(z\partial_z + \xi^\mu P_{F;\mu}\right) A = [A, zW_z + \xi^\mu W_\mu]_. + z\beta^{(A)}.$$

At this point, having extracted what we could from the path integral, we should also take the cutoff M to infinity. The effect of this is to replace the regulated derivative $P_{F;\mu}$ with an ordinary partial derivative ∂_μ.

Following the same procedure for $W^{(0)}$ and $\widehat{W}_\mu$, we have the ERG equations:

$$\left(\partial_z + z^{-1}\xi^\nu\partial_\nu\right) W_\mu^{(0)} + \left[W_z^{(0)} + z^{-1}\xi^\nu W_\nu^{(0)}, D_\mu^{(0)}\right]_. = 0 \tag{10.21}$$

$$(\partial_z + z^{-1}\xi^\mu\partial_\mu)A + \left[W_z + z^{-1}\xi^\mu W_\mu, A\right]_. = \beta^{(A)} \tag{10.22}$$

$$(\partial_z + z^{-1}\xi^\nu\partial_\nu)W_\mu + \left[W_z + z^{-1}\xi^\nu W_\nu, D_\mu\right]_. = \beta^{(\widehat{W})} \tag{10.23}$$

10.1.2 Exact RG equations: extension to the bulk and emergence of $\mathbf{AdS}_{d+1}$

We will now argue that (10.22) and (10.23) can be promoted from boundary RG equations for bilocal couplings A and W_μ, to bulk equations of motion for a pair of bilocal bulk fields $\mathcal{A}$ and $\mathcal{W}_I$.

By repeated application of the *infinitesimal* version of the ERG equations, we can in principle extend A and W_μ from any given value of z to any other value of z. Through this process, the connection $W_\mu\, dx^\mu$ is naturally enhanced to include a new piece of information pointing along the dz direction, namely W_z. This extension, W_z, plays an essential role in the RG transformations. It would therefore be nice to interpret the collection (W_z, W_μ) as the components of a larger connection in a $(d+1)$-dimensional space. However, in order for this extended object to be a true $CO(L_2)$ connection in the extended space, we must check that it still transforms properly under z-dependent $CO(L_2)$ transformations, *i.e.,* transformations of the form $\mathcal{L}(z;x,y)$. This can in fact be shown true. If we write an infinitesimal z-dependent transformation as $\mathcal{L}(z;x,y) = \mathbf{1} - \alpha(z;x,y)$ and continue to enforce constraint (9.6),

$$[\alpha, z d_z P_{F;\mu}]_. = 0,$$

then the transformation properties of W_μ and $W_\mu^{(0)}$ remain the same,

$$\delta W_\mu = [D_\mu, \alpha]_., \qquad \delta W_\mu^{(0)} = \left[D_\mu^{(0)}, \alpha\right]_.,$$

as long as we demand that W_z and $W_z^{(0)}$ also transform in the expected ways,

$$\delta W_z = [D_z, \alpha]_. \,, \qquad \delta W_z^{(0)} = \left[D_z^{(0)}, \alpha\right]_. \,.$$

In writing these constraints, we have defined $D_z \equiv d_z + W_z$ and $D_z^{(0)} \equiv d_z + W_z^{(0)}$. Finally, we can definitively say that the connection naturally extends to a $(d+1)$-dimensional connection, which we denote $\mathcal{W} = \mathcal{W}_I \, dx^I$. Similarly, the pseudoscaler A extends to a bulk field, which we will denote by $\mathcal{A}$.

We now extend the RG equations into the bulk. Our goal is to write covariant expressions in the higher dimensional space. We can achieve this goal by introducing a bit of sensible notation. On the boundary, we have the frame $\underline{e}_a^{(0)} = \delta_a^\mu \partial_\mu$. When extended into the bulk, we must add a z component. We can guess a sensible form for the frame by looking at the RG equations, (10.21), (10.22) and (10.23), which both contain the differential operator $(\partial_z + z^{-1}\xi^\mu \partial_\mu)$. Choosing $\underline{e}_z^{(0)} = \partial_z + z^{-1}\xi^\mu \partial_\mu$, the vector ξ^μ now taks on the role of a shift vector in the bulk frame, as foreshadowed previously. Most importantly, (10.21), (10.22) and (10.23), can now be cleanly promoted to bulk equations of motion,

$$i_{\underline{e}_z^{(0)}} (\boldsymbol{d}\mathcal{W}^{(0)} + \mathcal{W}^{(0)} \wedge_. \mathcal{W}^{(0)}) = 0 \tag{10.24}$$

$$i_{\underline{e}_z^{(0)}} (\boldsymbol{d}\mathcal{A} + [\mathcal{W}, \mathcal{A}]_.) = \beta_z^{(\mathcal{A})} \tag{10.25}$$

$$i_{\underline{e}_z^{(0)}} (\boldsymbol{d}\mathcal{W} + \mathcal{W} \wedge_. \mathcal{W}) = \beta_a^{(\mathcal{A})} e_a^{(0)}, \tag{10.26}$$

where we have defined the bulk one-form $\mathcal{W} \equiv \mathcal{W}_I dx^I \equiv \mathcal{W}_\mu dx^\mu + \mathcal{W}_z dz$, the bulk exterior derivative $\boldsymbol{d} \equiv dx^\mu \partial_\mu + dz \partial_z$, and the bulk interior product $i_{\underline{v}}$. One can straightforwardly check that (10.25) and (10.26) reduce to (10.22) and (10.23) by explicit application of the interior product, followed by restriction to the boundary.

Finally, we interpret (10.25) and (10.26) as $z-$components of the fully covariant expressions. We can also write down the enforced flatness condition

$$\boxed{\begin{aligned} \mathcal{F}^{(0)} &\equiv \boldsymbol{d}\mathcal{W}^{(0)} + \mathcal{W}^{(0)} \wedge \mathcal{W}^{(0)} = 0 \\ \mathcal{D}\mathcal{A} &\equiv \boldsymbol{d}\mathcal{A} + [\mathcal{W}, \mathcal{A}] = \boldsymbol{\beta}^{(\mathcal{A})} \\ \mathcal{F} &\equiv \boldsymbol{d}\mathcal{W} + \mathcal{W} \wedge_. \mathcal{W} = \boldsymbol{\beta}^{(\widehat{\mathcal{W}})} \end{aligned}} \tag{10.27}$$

$$\tag{10.28}$$

$$\tag{10.29}$$

where $\beta^{(\mathcal{A})}$ has been promoted to the 1-form $\boldsymbol{\beta}^{(\mathcal{A})} = \beta^{(\mathcal{A})} e_{(0)}^z + \beta_a^{(\mathcal{A})} e_{(0)}^a$ and similarly $\beta_\mu^{(\mathcal{W})}$ to the 2-form $\boldsymbol{\beta}^{(\mathcal{W})} = \beta_a^{(\mathcal{W})} e_{(0)}^z \wedge e_{(0)}^a + \beta_{ab}^{(\mathcal{W})} e_{(0)}^a \wedge e_{(0)}^b$. The transverse components of $\boldsymbol{\beta}^{(\mathcal{A})}$ and $\boldsymbol{\beta}^{(\mathcal{W})}$ not appearing in the original RG equations (10.25) and (10.26) are constrained by consistency to satisfy their own flow equations, namely the Bianchi identities,

$$\mathcal{D}\boldsymbol{\beta}^{(\mathcal{A})} = \left[\boldsymbol{\beta}^{(\mathcal{W})}, \mathcal{A}\right], \qquad \mathcal{D}\boldsymbol{\beta}^{(\mathcal{W})} = 0.$$

Equation (10.27) tells us that $\mathcal{W}^{(0)}$ is a flat connection on M_{d+1}. This is the point at which AdS_{d+1} finally emerges. Concretely, in a suitable choice of local coordinates, a natural choice for $\mathcal{W}^{(0)}$ is

$$\mathcal{W}^{(0)} = -\frac{dz}{z} D(x,y) + \frac{dx^\mu}{z} P_\mu(x,y),$$

where we define

$$P_\mu(x,y) = \partial_\mu^{(x)}\delta(x-y), \qquad D(x,y) = (x^\mu\partial_\mu^{(x)} + \Delta_\psi + \frac{d}{2})\delta(x-y).$$

This choice of $\mathcal{W}^{(0)}$ may be regarded as a Cartan connection on M_{d+1}, or equivalently as the Maurer-Cartan form of $O(2,d)$. It precisely corresponds to the AdS_{d+1} in the Poincaré patch. For further details on Cartan geometry, see [76].

10.1.3 Callan-Symanzik equations

We can similarly derive the Callan-Symanzik equations for the RG flow. The Majorana action has two bilocal operators:

$$\hat{\Pi}(x,y) = \frac{1}{2}\psi^\alpha(x)\epsilon_{\alpha\beta}\psi^\beta(y), \qquad \hat{\Pi}^\mu(x,y) = \frac{1}{2}\psi^\alpha(x)\epsilon_{\alpha\beta}(\gamma^\mu)^\beta{}_\delta\psi^\delta(y).$$

For a generic operator $\mathcal{O}$, with normalized expectation

$$\langle\mathcal{O}\rangle \equiv \frac{\int[\psi]\mathcal{O}e^{iS}}{\int[\psi]e^{iS}},$$

one can straightforwardly check from an argument similar to the one described above, that

$$Md_M\langle\mathcal{O}\rangle = \frac{1}{2}\int_{u,v}\not{\Delta}^\gamma{}_\delta(u,v)\epsilon^{\delta\eta}\left\langle -\frac{\delta S_{int}}{\delta\psi^\gamma(u)}\frac{\delta\mathcal{O}}{\delta\psi^\eta(v)} - \frac{\delta\mathcal{O}}{\delta\psi^\gamma(u)}\frac{\delta S_{int}}{\delta\psi^\eta(v)} + i\frac{\delta^2\mathcal{O}}{\delta\psi^\gamma(u)\delta\psi^\eta(v)}\right\rangle$$

For the case of quadratic interactions, as before we have

$$\frac{\delta S_{int}}{\delta\psi^\gamma(u)} = \int_z \epsilon_{\gamma\beta}\left[A(u,z)\delta^\beta{}_\delta + \widehat{W}_\mu(u,z)(\gamma^\mu)^\beta{}_\delta\right]\psi^\delta(z)$$

Let us also consider the general quadratic operator $\mathcal{O}_M = \frac{1}{2}\psi^\alpha(x)\epsilon_{\alpha\beta}M^\beta{}_\delta\psi^\delta(y)$. We have

$$\frac{\delta\mathcal{O}_M}{\delta\psi^{m;\gamma}(u)} = \frac{1}{2}\left(\delta^{(d)}(x-u)\epsilon_{\gamma\beta}M^\beta{}_\delta\psi^{m;\delta}(y) - \psi^{m;\alpha}(x)\epsilon_{\alpha\beta}M^\beta{}_\gamma\delta^{(d)}(y-u)\right),$$

$$\frac{\delta^2\mathcal{O}_M}{\delta\psi^{m;\gamma}(u)\delta\psi^{m;\eta}(v)} = \frac{N}{2}\left(\delta^{(d)}(x-u)\delta^{(d)}(y-v)\,\epsilon_{\eta\beta}M^\beta{}_\gamma - \delta^{(d)}(x-v)\delta^{(d)}(y-u)\,\epsilon_{\gamma\beta}M^\beta{}_\eta\right),$$

where the N appears from tracing over $O(N)$ indices. Thus, after step one of RG we have

$$\begin{aligned}\delta_\varepsilon\langle\mathcal{O}_M\rangle = &-i\frac{N}{2}\not{\Delta}^\beta{}_\gamma(x,y)M^\gamma{}_\beta \\ &-\frac{1}{2}\int_{u,v,z}\left\langle\psi^\kappa(z)\epsilon_{\kappa\rho}\left[A(z,v)\delta^\rho_\eta + \widehat{W}_\mu(z,v)(\gamma^\mu)^\rho{}_\eta\right]\not{\Delta}^\eta{}_\delta(v,x)\epsilon^{\delta\gamma}\epsilon_{\gamma\beta}M^\beta{}_\tau\psi^\tau(y)\right\rangle \\ &-\frac{1}{2}\int_{u,v,z}\left\langle\psi^\alpha(x)\epsilon_{\alpha\beta}M^\beta{}_\gamma\not{\Delta}^\gamma{}_\delta(y,v)\epsilon^{\delta\eta}\epsilon_{\eta\rho}\left[A(v,z)\delta^\rho_\kappa + \widehat{W}_\mu(v,z)(\gamma^\mu)^\rho{}_\kappa\right]\psi^\kappa(z)\right\rangle\end{aligned}$$

We may now write down separate equations for M either $\mathbf{1}$ or γ^μ. Since both the operators transform tensorially under $CO(L_2)$, after step 2 we get:

$$\begin{aligned}\Pi(z+\varepsilon z; x+\varepsilon\xi, y+\varepsilon\xi) = \Pi(z;x,y) + [\Pi, \varepsilon z W_z + \varepsilon\xi^\mu W_\mu] \\ + \varepsilon\left(\Delta_\nu \cdot A \cdot \Pi^\nu - \Pi^\nu \cdot A \cdot \Delta_\nu\right) - \varepsilon\left(\Delta^\mu \cdot \widehat{W}_\mu \cdot \Pi + \Pi \cdot \widehat{W}_\mu \cdot \Delta^\mu\right) \\ + \varepsilon\epsilon^{\mu\nu\lambda}\left(\Delta_\mu \cdot \widehat{W}_\nu \cdot \Pi_\lambda + \Pi_\mu \cdot \widehat{W}_\nu \cdot \Delta_\lambda\right) + O(\varepsilon^2)\end{aligned} \tag{10.30}$$

$$\begin{aligned}\Pi^\mu(z+\varepsilon z; x+\varepsilon\xi, y+\varepsilon\xi) = \Pi^\mu(z;x,y) - i\varepsilon N\Delta^\mu + [\Pi^\mu, \varepsilon z W_z + \varepsilon\xi^\mu W_\mu] \\ + \varepsilon\left(\Delta^\mu \cdot A \cdot \Pi + \Pi \cdot A \cdot \Delta^\mu\right) + \varepsilon\epsilon^{\mu\nu\sigma}\left(\Delta_\nu \cdot A \cdot \Pi_\sigma + \Pi_\nu \cdot A \cdot \Delta_\sigma\right) \\ - \varepsilon\left(\Delta^\nu \cdot \widehat{W}_\nu \cdot \Pi^\mu + \Pi^\mu \cdot \widehat{W}_\nu \cdot \Delta^\nu\right) - \varepsilon\left(\Delta_\nu \cdot \widehat{W}^\mu \cdot \Pi^\nu + \Pi^\nu \widehat{W}^\mu \Delta_\nu\right) \\ - \varepsilon\left(\Delta^\mu \cdot \widehat{W}_\nu \cdot \Pi^\nu + \Pi^\nu \cdot \widehat{W}_\nu \cdot \Delta^\mu\right) + \varepsilon\epsilon^{\mu\nu\lambda}\left(\Delta_\nu \cdot \widehat{W}_\lambda \cdot \Pi + \Pi \cdot \widehat{W}_\nu \cdot \Delta_\lambda\right)\end{aligned} \tag{10.31}$$

The Callan-Symanzik equations can be written in a more compact form by making the definitions

$$\begin{aligned}\gamma(x,y;u,v) &\equiv \frac{\delta\beta^{(A)}(u,v)}{\delta A(x,y)} = \delta(x-u)\Delta^\mu \cdot \widehat{W}_\mu(y,v) + \widehat{W}_\mu \cdot \Delta^\mu(u,x)\delta(y-v) \\ \gamma_\mu(x,y;u,v) &\equiv \frac{\delta\beta_\mu^{(\widehat{W})}(u,v)}{\delta A(x,y)} = \frac{\delta\beta^{(A)}(u,v)}{\delta\widehat{W}^\mu(x,y)} \\ &= \delta(x-u)\Delta_\mu \cdot A(y,v) + A \cdot \Delta_\mu(u,x)\delta(y-v) \\ &\quad + \epsilon_\mu{}^{\nu\lambda}\left(\delta(x-u)\Delta_\nu \cdot \widehat{W}_\lambda(y,v) + \widehat{W}_\nu \cdot \Delta_\lambda(u,x)\delta(y-v)\right) \\ \gamma_{\mu\nu}(x,y;u,v) &\equiv \frac{\delta\beta_\nu^{(\widehat{W})}(u,v)}{\delta\widehat{W}^\mu(x,y)} \\ &= \epsilon_{\mu\lambda\nu}\delta(x-u)\Delta^\lambda \cdot A(y,v) + \epsilon_{\nu\lambda\mu}A \cdot \Delta^\lambda(u,x)\delta(y-v) \\ &+ \delta(x-u)\Delta_\mu \cdot \widehat{W}_\nu(y,v) + \widehat{W}_\nu \cdot \Delta_\mu(u,x)\delta(y-v) \\ &- \delta(x-u)\Delta_\nu \cdot \widehat{W}_\mu(y,v) - \widehat{W}_\mu \cdot \Delta_\nu(u,x)\delta(y-v) \\ &+ \delta(x-u)\Delta_\lambda \cdot \widehat{W}^\lambda(y,v)\eta_{\mu\nu} + \widehat{W}_\lambda \cdot \Delta^\lambda(u,x)\delta(y-v)\eta_{\mu\nu}\end{aligned}$$

We will denote the bulk extensions of the momenta Π and Π^μ as $\mathcal{P}$ and $\mathcal{P}^\mu$ respectively. Comparing the ε terms on both sides of equations (10.30) and (10.31), and then taking $\varepsilon \mapsto 0$, we obtain the bulk Callan-Symanzik equations:

$$[\mathcal{D}_{\underline{e}_z^{(0)}}, \mathcal{P}](x,y) = \left\{\text{Tr. } \boldsymbol{\gamma}(x,y;u,v)\cdot\mathcal{P}(v,u) + \text{Tr. } \boldsymbol{\gamma}_\mu(x,y;u,v)\cdot\mathcal{P}^\mu(v,u)\right\} \tag{10.32}$$

$$\left[\mathcal{D}_{\underline{e}_z^{(0)}}, \mathcal{P}^\mu\right](x,y) = \{-N\Delta^\mu(x,y) + \text{Tr. } \boldsymbol{\gamma}^\mu(x,y;u,v)\cdot\mathcal{P}(v,u) + \text{Tr. } \boldsymbol{\gamma}^\mu{}_\nu(x,y;u,v)\cdot\mathcal{P}^\nu(v,u)\} \tag{10.33}$$

where as before $\mathcal{D} = \boldsymbol{d} + \mathcal{W}$. We are also using $\boldsymbol{\gamma}(z;x,y;u,v) \equiv -\frac{\delta\boldsymbol{\beta}^{(\mathcal{A})}(z;u,v)}{\delta\mathcal{A}(z;y,x)}$ to mean the bulk extension of $\gamma(x,y;u,v)$, while $\boldsymbol{\gamma}_\mu$ and $\boldsymbol{\gamma}_{\mu\nu}$ are extensions for γ_μ and $\gamma_{\mu\nu}$.

10.1.4 The RG Ward identity

We started our RG investigations by writing down a Ward identity (see equation (10.4)):

$$\begin{aligned} &Z[M, z, U(z), A(z; x, y), W_\mu(z; x, y)] \\ &\quad = Z\Big[M, \lambda^{-1}z, U(\lambda^{-1}z), A(\lambda^{-1}z; x + \varepsilon\xi, y + \varepsilon\xi), W_\mu(\lambda^{-1}z; x + \varepsilon\xi, y + \varepsilon\xi)\Big]. \end{aligned}$$

We can now expand the right hand side of this equation to extract an order ε Ward identity,

$$\begin{aligned} \frac{\partial}{\partial z} Z = -\mathrm{Tr}\left\{ \left(\left[A, W_{\underline{e}_z^{(0)}} \right]_. + \beta^{(A)} \right) \cdot \frac{\delta}{\delta A} + \left(\left[D_\mu, W_{\underline{e}_z^{(0)}} \right]_. + \beta_\mu^{(W)} \right) \cdot \frac{\delta}{\delta W_\mu} \right\} Z \\ + N\,\mathrm{Tr}\left\{ \Delta^\mu \cdot \widehat{W}_\mu + \Delta^z \cdot \widehat{W}_{\underline{e}_z^{(0)}} \right\} Z, \end{aligned} \tag{10.34}$$

where the beta functions were derived in a previous section (see equations (10.16)). This identity will be of extreme significance in our discussions of Hamilton-Jacobi.

10.2 Holography as Hamilton-Jacobi: the Majorana case

Now that we have motivated the existence of a higher-dimensional space, we are free to take a bulk perspective. Our ambitious first task will be to construct a bulk action using the information we know about the boundary. We will tackle this using the Hamilton-Jacobi approach to holography [33, 34]. The defining property of holography is contained in the equation

$$Z[z_*, A(z_*), W_\mu(z_*)] = e^{iS_{\mathrm{HJ}}[z_*, A(z_*), W_\mu(z_*)]}, \tag{10.35}$$

where we have chosen a constant hypersurface $z = z_*$ on which to evaluate this expression. S_{HJ} is the classical Hamilton-Jacobi functional for the bulk theory where the radial direction is playing the role of time. From Hamilton-Jacobi theory, S_{HJ} is to be understood as the bulk action taken on-shell with appropriate boundary conditions applied. In our case, a natural set of boundary conditions that we have available to us equate the bulk momenta to connected vacuum expectation values of the boundary operators,

$$\mathcal{P}|_{z=\epsilon} = \Pi \equiv \frac{\delta S_{HJ}}{\delta A}, \qquad \mathcal{P}^\mu|_{z=\epsilon} = \Pi^\mu \equiv \frac{\delta S_{HJ}}{\delta W_\mu}.$$

Recall that $\mathcal{P}$ and $\mathcal{P}^\mu$ are the conjugate momenta to the bulk fields $\mathcal{A}$ and $\mathcal{W}_\mu$.

We now pursue the bulk (off-shell) action, by way of the Hamilton-Jacobi equation

$$\frac{\partial}{\partial z} S_{\mathrm{HJ}} = -\mathcal{H},$$

where $\mathcal{H}$ is the bulk Hamiltonian. The key observation is that this equation is of very similar form to the RG Ward identity (10.34). In fact, we can make these two equations coincide with minimal effort. Using (10.35) with $z_* = \epsilon$, the left-hand side of the Ward identity can be expanded to

$$\left.\frac{\partial}{\partial z} Z\right|_{z=\epsilon} = \left.\frac{\partial}{\partial z} e^{iS_{\mathrm{HJ}}}\right|_{z=\epsilon} = \left. i \frac{\partial}{\partial z} S_{\mathrm{HJ}}\, Z\right|_{z=\epsilon}. \tag{10.36}$$

The right-hand side of the Ward identity is of the form

$$
\begin{aligned}
&-\mathrm{Tr.}\left\{(\ldots)\cdot\frac{\delta}{\delta A}+(\ldots)\cdot\frac{\delta}{\delta W_\mu}+N\ (\ldots)\right\}Z\\
&=-i\mathrm{Tr.}\left\{(\ldots)\cdot\frac{\delta S_{\mathrm{HJ}}}{\delta A}+(\ldots)\cdot\frac{\delta S_{\mathrm{HJ}}}{\delta W_\mu}-iN\ (\ldots)\right\}Z\Big|_{z=\epsilon}\\
&=-i\mathrm{Tr.}\left\{\left(\left[A,W_{\underline{e}_z^{(0)}}\right]_{\cdot}+\beta^{(A)}\right)\cdot\frac{\delta}{\delta A}+\left(\left[D_\mu,W_{\underline{e}_z^{(0)}}\right]_{\cdot}+\beta_\mu^{(W)}\right)\cdot\frac{\delta}{\delta W_\mu}\right.\\
&\qquad\qquad\left.+N\left(\Delta^\mu\cdot\widehat{W}_\mu+\Delta^z\cdot\widehat{W}_{\underline{e}_z^{(0)}}\right)\right\}Z\Big|_{z=\epsilon}
\end{aligned}
\tag{10.37}
$$

Equating (10.36) to (10.37), we can pick off the bulk Hamiltonian

$$
\mathcal{H}=\mathrm{Tr.}\left\{\left(\left[\mathcal{A},\mathcal{W}_{\underline{e}_z^{(0)}}\right]_{\cdot}+\beta^{(\mathcal{A})}\right)\cdot\mathcal{P}+\left(\left[\mathcal{D}_\mu,\mathcal{W}_{\underline{e}_z^{(0)}}\right]_{\cdot}+\beta_\mu^{(\widehat{\mathcal{W}})}\right)\cdot\mathcal{P}^\mu+N\left(\Delta^\mu\cdot\widehat{\mathcal{W}}_\mu+\Delta^z\cdot\widehat{\mathcal{W}}_{\underline{e}_z^{(0)}}\right)\right\}
$$

One can check directly that the dz components of the RG equations (10.25,10.26) and the Callan-Symanzik equations (10.32,10.33) are in fact now reproduced by the Hamilton equations of motion:

$$
\begin{aligned}
d_z\mathcal{A}&=\frac{\delta\mathcal{H}}{\delta\mathcal{P}},\qquad & d_z\mathcal{W}_\mu&=\frac{\delta\mathcal{H}}{\delta\mathcal{P}^\mu},\\
d_z\mathcal{P}&=-\frac{\delta\mathcal{H}}{\delta\mathcal{A}},\qquad & d_z\mathcal{P}^\mu&=-\frac{\delta\mathcal{H}}{\delta\mathcal{W}_\mu}.
\end{aligned}
$$

Of course, the bulk Hamiltonian should also be able to reproduce the transverse components of the RG equations. We can insert these by hand, introducing additional non-dynamical Lagrange multipliers $\mathcal{Q}^\mu$ and $\mathcal{Q}^{\mu\nu}$:

$$
\mathcal{H}_{\mathrm{constraint}}=-\mathrm{Tr.}\left\{\left([\mathcal{D}_\mu,\mathcal{A}]_{\cdot}-\beta_\mu^{(\mathcal{A})}\right)\cdot\mathcal{Q}^\mu+\left(\mathcal{F}_{\mu\nu}-\beta_{\mu\nu}^{(\widehat{\mathcal{W}})}\right)\cdot\mathcal{Q}^{\mu\nu}\right\}.
$$

The full Hamiltonian is then $(\mathcal{H}+\mathcal{H}_{\mathrm{constraint}})$. An unusual property of this Hamiltonian is that it is linear, rather than quadratic, in momentum. However, we are still free to construct a bulk "phase space action" in the usual "$p\dot{q}-H$" way:

$$
I=\int dz\ \mathrm{Tr.}\left\{\mathcal{P}^I\cdot\left([\mathcal{D}_\mu,\mathcal{A}]_{\cdot}-\boldsymbol{\beta}_I^{(\mathcal{A})}\right)+\mathcal{P}^{IJ}\cdot\left(\mathcal{F}_{IJ}-\boldsymbol{\beta}_{IJ}^{(\mathcal{W})}\right)-N\Delta_I\cdot\widehat{\mathcal{W}}^I\right\}.
\tag{10.38}
$$

In writing this action, we have used the notation $\mathcal{P}^I\equiv\{\mathcal{P},\mathcal{Q}^\mu\}$, $\mathcal{P}^{IJ}\equiv\{\mathcal{P}^\mu,\mathcal{Q}^{\mu\nu}\}$[39]. It should give us some confidence that variation of this action does yield the RG and Callan-Symanzik equations, provided we gauge fix all the Lagrange multipliers to zero.

[39] It should be noted that an action of this sort has been proposed before in various contexts. See [77, 78, 79, 80].

10.3 Bosonic RG flow of the bosonic theory

We now address renormalization group flow of the Bosonic theory:

$$\begin{aligned} S_{\rm reg}^{Bos} &= S_0 + S_{\rm int} + U \\ S_0 &= -\phi^*_m \cdot D^{(0)}_\mu \cdot D^{(0)}_\mu \cdot \phi^m \\ S_{\rm int} &= \phi^*_m \cdot \mathcal{B} \cdot \phi^m \\ U &= \int_{x,y} \mathcal{U}(x,y) \equiv \int_{x,y} \mathcal{U}_0\, \delta(x-y). \end{aligned}$$

The partition function will be defined as $Z = Z_0^{-1} \int [d\phi\, d\phi^*] e^{iS^{Bos}_{\rm reg}[\phi,\phi^*]}$, normalized with respect to $Z_0 \equiv \int [d\phi\, d\phi^*] e^{iS_0[\phi,\phi^*]}$.

10.3.1 Exact RG equations: Wilson-Polchinski renormalization in two steps

The exact RG equations are derived in two steps.

STEP 1: We lower the cutoff but allow the couplings to shift in order that the partition function remains unchanged:

$$Z[M, z, \mathcal{B}, \mathcal{U}] = Z[\lambda M, z, \tilde{\mathcal{B}}, \tilde{\mathcal{U}}].$$

Working infinitesimally with $\lambda = 1 - \varepsilon$, we are solving the problem $\delta_\varepsilon Z = 0$. Using the Polchinski formalism, we arrive at the expression

$$0 = (\delta_\varepsilon \mathcal{B} - \beta^{(\mathcal{B})}) \cdot \Pi + (\delta_\varepsilon U - \beta^{(U)}) \cdot \langle \mathbf{1} \rangle \,,$$

where

$$\beta^{(\mathcal{B})} = \mathcal{B} \cdot \Delta_B \cdot \mathcal{B}, \qquad \beta^{(U)} = -iN\, {\rm Tr}\, (\Delta_B \cdot \mathcal{B})$$

Taking the coefficients of Π and $\langle \mathbf{1} \rangle$ to be independently zero, we have

$$\delta_\varepsilon \mathcal{B} = \beta^{(\mathcal{B})}, \qquad \delta_\varepsilon U = \beta^{(U)}.$$

STEP 2: We perform an infinitesimal $CU(L_2)$ transformation to bring the the cutoff from λM back to value M. Again working infinitesimally, we this time choose to parametrize the transformation as $\mathcal{L} = 1 + \varepsilon z W^{(0)}_z$. After performing the $CU(L_2)$ transformation, we relabel the source $\tilde{\mathcal{B}}(z)$ as $\mathcal{B}(z + \varepsilon z)$ and the source $\tilde{U}$ as $U(z + \varepsilon z)$. We also redefine Δ_B as $\Delta_B = \frac{M}{z} d_M \left({D^{(0)}}^2 \right)^{-1}$. The resulting equalities are

$$\mathcal{B}(z + \varepsilon z) = \mathcal{B}(z) - \varepsilon \left[W^{(0)}_z, \mathcal{B} \right]_. + \varepsilon z \beta^{(\mathcal{B})}$$

$$U(z + \varepsilon z) = U(z) + \varepsilon \beta^{(U)}.$$

The desired ERG equations are extracted by comparing terms at order ε:

$$\partial_z W^{(0)}_\mu - P_{F,\mu} W^{(0)}_z + \left[W^{(0)}_z, W^{(0)}_\mu \right]_. = 0$$

$$\partial_z \mathcal{B} + \left[W^{(0)}_z, \mathcal{B} \right]_. = \beta^{(\mathcal{B})}$$

10.3.2 Exact RG equations: extension to the bulk and emergence of AdS_{d+1}

We can following the same arguments made in the Majorana case to build up a bulk space-time populated with bulk fields. In particular, we can form a bulk connection $\mathcal{W}^{(0)} = \mathcal{W}^{(0)}_I dx^I$, by grouping components $(W^{(0)}_z, W^{(0)}_\mu)$ into a single one form over a $(d+1)$-dimensional space-time. Similarly, $\mathcal{B}$ gets promoted to a scalar in the bulk, which we will denote as $\mathfrak{B}$. At this point, we can also take the cutoff M to infinity, so that the regulated derivative $P_{F;\mu}$ becomes a regular partial derivative ∂_μ. With this, the ERG equations become

$$\partial_z \mathcal{W}^{(0)}_\mu - \partial_\mu \mathcal{W}^{(0)}_z + \left[\mathcal{W}^{(0)}_z, \mathcal{W}^{(0)}_\mu\right]_. = 0$$

$$\partial_z \mathfrak{B} + \left[\mathcal{W}^{(0)}_z, \mathfrak{B}\right]_. = \beta^{(\mathfrak{B})}.$$

We can then promote these equations to fully covariant bulk equations,

$$\mathcal{F}^{(0)} \equiv \mathbf{d}\mathcal{W}^{(0)}\mathcal{W}^{(0)} \wedge_. \mathcal{W}^{(0)} = 0$$

$$\mathcal{D}^{(0)}\mathfrak{B} = \mathbf{d}\mathfrak{B} + \left[\mathcal{W}^{(0)}, \mathfrak{B}\right]_. = \boldsymbol{\beta}^{(\mathfrak{B})},$$

where the transverse components of the one-form $\boldsymbol{\beta}^{(\mathfrak{B})} = \beta^{(\mathfrak{B})}_\mu dx^\mu + \beta^{(\mathfrak{B})} dz$ are determined by the Bianchi identity

$$\mathcal{F}^{(0)} \equiv \mathbf{d}\mathcal{W}^{(0)}\mathcal{W}^{(0)} \wedge_. \mathcal{W}^{(0)} = 0 \tag{10.39}$$

$$\mathcal{D}^{(0)}\boldsymbol{\beta}^{(\mathfrak{B})} \equiv \mathbf{d}\boldsymbol{\beta}^{(\mathfrak{B})} + \left[\mathcal{W}^{(0)}, \boldsymbol{\beta}^{(\mathfrak{B})}\right]_. = 0 \tag{10.40}$$

where we are using the definition $\mathbf{d} \equiv dx^\mu \left[\partial_\mu, \ \right]_. + dz\partial_z$, guaranteeing that $\mathbf{d}^2 = 0$.

Once again, the flatness condition (10.39) is our gateway to AdS_{d+1} space. We can, for instance, choose a solution of the form

$$\mathcal{W}^{(0)}(z; x, y) = -\frac{dz}{z} D(x, y) + \frac{dx^\mu}{z} P_\mu(x, y) \tag{10.41}$$

$$D(x, y) = \frac{1}{2}\left(x^\mu \partial^{(x)}_\mu - y^\mu \partial^{(y)}_\mu + 2\Delta_\phi\right)\delta^d(x - y), \qquad P_\mu(x, y) = \partial^{(x)}_\mu \delta^d(x - y).$$

$\mathcal{W}^{(0)}$ is therefore the AdS_{d+1} connection.

10.3.3 Callan-Symanzik equations

In the bosonic theory, there is only one operator to consider,

$$\hat{\Pi}(x, y) = \phi^*_m(y)\phi^m(x).$$

It is a straightforward calculation to demonstrate

$$M d_M \Pi \equiv M d_M \left\langle \hat{\Pi} \right\rangle = \mathrm{Tr}_. \left\{ \Delta_B \cdot \left\langle \frac{\delta S_1}{\delta \phi^*_m} \frac{\delta \hat{\Pi}}{\delta \phi^m} + \frac{\delta \hat{\Pi}}{\delta \phi^*_m} \frac{\delta S_1}{\delta \phi^m} - i \frac{\delta^2 \hat{\Pi}}{\delta \phi^*_m \delta \phi^m} \right\rangle \right\}.$$

The right hand side can be calculated explicitly. The result is

$$\delta_\varepsilon \Pi = i\varepsilon z \, N \, \Delta_B + \varepsilon z \, \mathrm{Tr}\left\{\gamma \cdot \Pi\right\},$$

where we have defined

$$\gamma(x,y;u,v) \equiv -\frac{\delta\beta^{(\mathcal{B})}(u,v)}{\delta\mathcal{B}(y,x)} = -\delta(x-u)\left(\Delta_B\cdot\mathcal{B}\right)(y,v) - \left(\mathcal{B}\cdot\Delta_B\right)(u,x)\,\delta(v-y).$$

We then perform an infinitesimal $CU(L_2)$ transformation, again parametrized as $\mathcal{L} = 1+\varepsilon z W_z^{(0)}$. The result is

$$\Pi(z+\varepsilon z;x,y) = \Pi(z;x,y) - \varepsilon\, z\left[W_z^{(0)},\Pi\right]_. + i\varepsilon z\; N\,\Delta_B + \varepsilon\, z\,\mathrm{Tr}\left\{\gamma(x,y;u,v)\cdot\Pi(v,u)\right\}$$

As with the beta function derived above, this relationship can be extended into the bulk. Denoting the bulk momentum as $\mathcal{P}$, we have

$$\mathcal{D}_z^{(0)}\mathcal{P} \equiv \partial_z\mathcal{P} + \left[\mathcal{W}_z^{(0)},\mathcal{P}\right]_. = iN\,\Delta_B + \mathrm{Tr}\left\{\boldsymbol{\gamma}(x,y;u,v)\cdot\mathcal{P}(v,u)\right\}$$

where $\boldsymbol{\gamma}(z;x,y;u,v) \equiv -\frac{\delta\boldsymbol{\beta}^{(\mathfrak{B})}(z;u,v)}{\delta\mathfrak{B}(z;y,x)}$ is the bulk extension of γ.

10.3.4 The RG Ward identity

Just like in the Majorana case, we can write down and order ε Ward identity:

$$-\frac{\partial}{\partial z}Z = \mathrm{Tr.}\left\{\left(\left[\mathcal{B},W_z^{(0)}\right]_. + \beta^{(\mathcal{B})}\right)\cdot\frac{\delta}{\delta\mathcal{B}} + \left[D_\mu^{(0)},W_z^{(0)}\right]_.\cdot\frac{\delta}{\delta W_\mu^{(0)}}\right\}Z + N\mathrm{Tr.}\;\left(\Delta_B\cdot\mathcal{B}\right)Z. \quad (10.42)$$

10.4 Holography as Hamilton-Jacobi: the bosonic case

Once again, the definiting property of holography is encapsulated in the equation

$$Z[z_*,W^{(0)}(z_*),\mathcal{B}(z_*)] = e^{iS_{HJ}[W^{(0)}(z_*),\mathcal{B}(z_*)]}$$

where S_{HJ} is the Hamilton-Jacobi equation. Using the boundary condition

$$\mathcal{P}|_{z=\epsilon} = \Pi \equiv \frac{\delta S_{HJ}}{\delta\mathcal{B}}$$

By comparing to the RG Ward identity (10.42), we can extract the Hamiltonian:

$$\mathcal{H} = \mathrm{Tr.}\left\{\left(\left[\mathcal{A},\mathcal{W}_z^{(0)}\right]_. + \boldsymbol{\beta}_z^{(\mathcal{A})}\right)\cdot\mathcal{P} + \left[\mathcal{D}_\mu^{(0)},\mathcal{W}_z^{(0)}\right]_.\cdot\mathcal{P}^\mu\right\} - iN\;\mathrm{Tr.}\left(\Delta_B\cdot\mathcal{A}\right). \quad (10.43)$$

We put in by hand the data for the transverse components of the RG equations:

$$\mathcal{H}_{\mathrm{constraint}} = \mathrm{Tr.}\left\{\left(\mathcal{D}_\mu^{(0)}\mathcal{A} - \boldsymbol{\beta}_\mu^{(\mathcal{A})}\right)\cdot\mathcal{Q}^\mu + \mathcal{F}_{\mu\nu}^{(0)}\cdot\mathcal{Q}^{\mu\nu}\right\}$$

where $\mathcal{Q}^\mu$ and $\mathcal{Q}^{\mu\nu}$ are Lagrange multipliers. The total Hamiltonian is $\mathcal{H}+\mathcal{H}_{\mathrm{constraint}}$. Using "$p\dot{q}-H$", we write down a phase space action:

$$I = \int_\infty^\epsilon dz\;\mathrm{Tr.}\left\{\mathcal{P}^I\cdot\left(\mathcal{D}_I^{(0)}\mathcal{A} - \boldsymbol{\beta}_I^{(\mathcal{A})}\right) + \mathcal{P}^{IJ}\cdot\mathcal{F}_{IJ}^{(0)} + iN\;\Delta_B\cdot\mathcal{A}\right\}, \quad (10.44)$$

where $\mathcal{P}^I \equiv \{\mathcal{P}, \mathcal{Q}^\mu\}$, $\mathcal{P}^{IJ} \equiv \{\mathcal{P}^\mu, \mathcal{Q}^{\mu\nu}\}$.

Chapter 11

Correlation Functions

In moving from the boundary to a bulk theory, we have been careful to be consistent every step along the way. However, as a final check on our bulk action, we should ask if we really have arrived at a theory holographically dual to the $O(N)$ boundary theory we started with. To demonstrate that this goal has in fact been fulfilled we check that the bulk action reproduces all of the boundary correlation functions. As a quick warm up, we start with a quick check on the two-point function.

11.1 The two-point function of the Majorana theory

In this section we will calculate the $O(N)$ trace of the boundary CFT's two-point function the Callan-Symanzik equations. Recall that these equations can be extracted directly from the bulk action (10.38) by variation, and so this will in some very small sense be a test of the bulk action. We start by expanding the Majorana two point-function in the usual Dirac algebra basis $\{\delta, \vec{\gamma}\}$:

$$S_\alpha{}^\beta(y,x) \equiv \left\langle \psi_\alpha^m(y)\tilde{\psi}^{m,\beta}(x)\right\rangle_{\text{CFT}} = C_A\,\Pi(x,y)\delta_\alpha{}^\beta + C_W\,\Pi_\mu(x,y)(\gamma^\mu)_\alpha{}^\beta, \tag{11.1}$$

In writing this expansion, we have extracted field-dependent pieces Π and Π_μ from the expansion coefficients. We are left only to determine the quantities C_A and C_W. To do this, we can multiply left- and right-hand sides of (11.1) by $\delta_\beta{}^\alpha$ and $(\gamma^\nu)_\beta{}^\alpha$ and evaluate the resulting $O(N)$ traces. We find that $C_A = C_W = -\frac{2}{d_{C\ell}}$, where the minus sign arises from the Grassmanian nature of the Fermionic fields[40]. In three space-time dimensions, $d_{C\ell} = 2$, so

$$S_\alpha{}^\beta(y,x) = -\Pi(x,y)\delta_\alpha{}^\beta - \Pi_\mu(x,y)(\gamma^\mu)_\alpha{}^\beta.$$

We can determine expression for $\Pi(x,y)$ and $\Pi_\mu(x,y)$ (the boundary values of $\mathcal{P}$ and $\mathcal{P}^\mu$) by integrating the Callen-Symanzik equations at the free-field fixed point $A = \widehat{W} = 0$. In this limit, the equations are extremely simple:

$$\begin{aligned}\left[\mathcal{D}^{(0)}, \Pi\right]_. &= 0\\ \left[\mathcal{D}^{(0)}, \Pi^\mu\right]_. &= -N\Delta^\mu dz.\end{aligned} \tag{11.2}$$

If we multiply equation (11.2) by γ^μ, we can use the definition of Δ_μ as given in equation (10.9),

$$\not{\Delta} = -z\partial_z(iD_F^{(0)})^{-1},$$

[40] $d_{C\ell}$ is the dimension of the γ-matrices. In space-time dimension 3, $d_{C\ell}$ is 2.

and intergrate with respect to z to obtain

$$S = iN(D_F^{(0)})^{-1}.$$

This is exactly what we would expect at the fixed point (*i.e.,* at the boundary). The two point function is the inverse Dirac operator, and we get N copies because we have computed the $O(N)$ trace.

11.2 Higher-point functions and Witten diagrams

We will now tackle the problem of calculating correlation functions more systematically. In doing so, we will be able to reproduce all correlation functions for the boundary theory.

11.2.1 Higher-point functions of the Majorana theory

We have derived a bulk action in equation (10.38). Note, unlike what one expects from a regular gravity theory, this action is not pure constraint. The first two terms of the action do vanish on-shell. However, the final term survives, yielding

$$I_{\text{o.s.}} = N\int_\epsilon^\infty dz\ \text{Tr.}\left\{\Delta_I \cdot \widehat{\mathcal{W}}^I_{\text{o.s.}}\right\} \xrightarrow{\widehat{\mathcal{W}}_z=0} N\int_\epsilon^\infty dz\ \text{Tr.}\left\{\Delta_\mu \cdot \widehat{\mathcal{W}}^\mu_{\text{o.s.}}\right\}. \tag{11.3}$$

We can extract $\widehat{\mathcal{W}}^I_{\text{o.s.}}$ by solving the RG equations (10.28) and (10.29). We can make progress by focusing on just the z components of these equations:

$$\left[\mathcal{D}_z^{(0)}, \mathcal{A}\right]_\cdot = \boldsymbol{\beta}_z^{(\mathcal{A})}, \qquad \left[\mathcal{D}_z^{(0)}, \mathcal{W}\right]_\cdot = \boldsymbol{\beta}_{z\mu}^{(\widehat{\mathcal{W}})}. \tag{11.4}$$

We now proceed to solve these equations perturbatively. This will prove not just convenient, but also conceptually enlightening. We start by writing the fields $\mathcal{A}$ and $\widehat{\mathcal{W}}$ as expansions in a small arbitrary parameter α,

$$\mathcal{A} = \sum_{n=1}^\infty \alpha^n \mathcal{A}^{(n)}, \qquad \widehat{\mathcal{W}} = \sum_{n=1}^\infty \alpha^n \widehat{\mathcal{W}}^{(n)}. \tag{11.5}$$

We are thinking of this as an expansion around the free-fixed point, $\mathcal{A} = \widehat{\mathcal{W}}_\mu = 0$. For boundary condition on equations (11.4), we choose to fixe the fields at $z = \epsilon$ to some small values

$$\mathcal{A}(\epsilon; x, y) = a^{(0)}(x, y), \qquad \widehat{\mathcal{W}}_\mu(\epsilon; x, y) = \hat{w}_\mu^{(0)}(x, y). \tag{11.6}$$

In reality, we also have another set of equations beyond (11.4), namely the Callan-Symanzik equations for $\mathcal{P}$ and $\mathcal{P}^\mu$. For this reason, we also consider another set of expansions,

$$\mathcal{P} = \mathcal{P}^{(0)} + \sum_{n=1}^\infty \alpha^n \mathcal{P}^{(n)}, \qquad \mathcal{P}^\mu = \mathcal{P}^{\mu(0)} + \sum_{n=1}^\infty \alpha^n \mathcal{P}^{\mu(n)}.$$

Furthermore, we now have another set of boundary conditions to choose. A natural choice is to fix the conjugate momenta in the infrared to

$$\mathcal{P}(\infty; x, y) = \mathcal{P}^\mu(\infty; x, y) = 0.$$

Noting that the beta functions (10.14) and (10.15) are summations over terms quadratic in the fields $\mathcal{A}$ and $\widehat{\mathcal{W}}$, we have, at first order in α,

$$\left[\mathcal{D}_z^{(0)}, \mathcal{A}^{(1)}\right]_\cdot = 0, \qquad \left[\mathcal{D}_z^{(0)}, \mathcal{W}^{(1)}\right]_\cdot = 0. \tag{11.7}$$

To solve these, we introduce a *boundary-to-bulk* Wilson line $K(z)$,

$$K(z) = \mathbf{P}_\cdot \left\{ e_\cdot^{-\int_\epsilon^z dz' \, \mathcal{W}_{z'}^{(0)}(z')} \right\}.$$

The notation $\mathbf{P}_\cdot$ means path ordering with respect to the bullet operation; similarly, $e_\cdot$ means exponentiation with respect to the bullet operation. $K(z)$ has the nice properties that it satisfies the simple differential equation

$$\partial_z K(z) + \mathcal{W}(z) \cdot K(z) = 0, \tag{11.8}$$

and has boundary solution

$$K(\epsilon) = \mathbf{1}. \tag{11.9}$$

Now, returning to (11.7), we simplify the equation by inserting various factors of $K(z)$. We start by conjugating the entire expression:

$$\begin{aligned}
0 &= K^{-1} \cdot \left[\mathcal{D}_z^{(0)}, \mathcal{A}^{(1)}\right]_\cdot \cdot K \\
&= K^{-1} \cdot \partial_z \mathcal{A}^{(1)} \cdot K + K^{-1} \cdot \left[W_z^{(0)}, \mathcal{A}^{(1)}\right]_\cdot \cdot K \\
&= K^{-1} \cdot \partial_z \mathcal{A}^{(1)} \cdot K + K^{-1} \cdot \left[W_z^{(0)} \cdot K \cdot K^{-1}, \mathcal{A}^{(1)}\right]_\cdot \cdot K \\
&= K^{-1} \cdot \partial_z \mathcal{A}^{(1)} \cdot K - K^{-1} \cdot \left[\partial_z K \cdot K^{-1}, \mathcal{A}^{(1)}\right]_\cdot \cdot K \\
&= K^{-1} \cdot \partial_z \mathcal{A}^{(1)} \cdot K - K^{-1} \cdot \partial_z K \cdot K^{-1} \cdot \mathcal{A}^{(1)} \cdot K + K^{-1} \cdot \mathcal{A}^{(1)} \cdot \partial_z K \\
&= K^{-1} \cdot \partial_z \mathcal{A}^{(1)} \cdot K + \partial_z K^{-1} \cdot \mathcal{A}^{(1)} \cdot K + K^{-1} \cdot \mathcal{A}^{(1)} \cdot \partial_z K \\
&= K^{-1} \cdot \partial_z \left(K \cdot \mathcal{A}^{(1)} \cdot K^{-1} \right) \cdot K,
\end{aligned} \tag{11.10}$$

where we have used (11.8) and $\partial_z K^{-1} = -K^{-1} \cdot \partial_z K \cdot K^{-1}$. We now know that $K \cdot \mathcal{A}^{(1)} \cdot K^{-1}$ is z-independent. From boundary conditions (11.6), we find

$$\mathcal{A}^{(1)}(z) = K(z) \cdot a^{(0)} \cdot K^{-1}(z), \qquad \widehat{\mathcal{W}}^{(1)}(z) = K(z) \cdot \hat{w}^{(0)} \cdot K^{-1}(z).$$

where we have also written down the solution for the connection $\widehat{\mathcal{W}}^{(1)}$. We further note that from (11.10), we have learned that conjugation by $K(z)$ effectively converts the covariant derivative $\left[\mathcal{D}_z^{(0)}, \ \right]_\cdot$ to a regular partial derivative ∂_z.

To first order, our on-shell action is now

$$I_{\text{o.s.}}^{(1)} = N \int_\epsilon^\infty dz \ \text{Tr.} \left\{ \Delta^\mu(z) \cdot K(z) \cdot \hat{w}_\mu^{(0)} \cdot K^{-1}(z) \right\} \equiv N \int_\epsilon^\infty dz \ \text{Tr.} \left\{ H^\mu(z) \cdot w_\mu^{(0)} \right\}, \tag{11.11}$$

where we have made the convenient definition

$$H^\mu(z) \equiv K^{-1}(z) \cdot \Delta^\mu(z) \cdot K(z).$$

We clean up the notation further by defining the *Wilsonian Green function* for the boundary field theory:

$$g^\mu(z; x, y) = \int_\epsilon^z dz' \; H^\mu(z; x, y),$$

so that the on-shell action is now of the form

$$I^{(1)}_{\text{o.s.}} = N dz \text{ Tr.} \left\{ g^\mu(\infty) \cdot w^{(0)}_\mu \right\} \equiv N dz \text{ Tr.} \left\{ g^\mu_{(0)} \cdot w^{(0)}_\mu \right\}. \tag{11.12}$$

We have introduced the notation $g^\mu_{(0)}(x, y)$ to indicate that this is a boundary quantity.[41] The on-shell action is now written in terms of boundary quantities only. For purposes of generalization, it turns out to be useful to rewrite equation (11.12) as

$$I^{(1)}_{\text{o.s.}} = N dz \text{ Tr.}_{,\gamma} \left\{ \not{g}^\mu_{(0)} \cdot (a^{(0)} + \not{w}^{(0)}_\mu) \right\}, \tag{11.14}$$

where the trace is now both over the space-time coordinates and the γ matrices. The computation we have carried out up to this point can be summarized in pictorial form as a Witten diagram (see fig. 11.1).

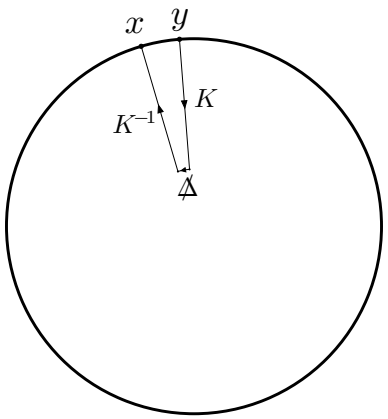


Figure 11.1: The Witten diagram representation for the boundary one-point function. The arrows indicate radial orientation, while the turnaround in the bulk represents an insertion of $\not{\Delta}$.

We can continue solving the RG equations order by order. In general, at kth order, the problem we are trying to solve has the form

$$\left[\mathcal{D}^{(0)}_z, \mathcal{A}^{(k)} \right]_. = \Phi^{(k)}_{\mathcal{A}}, \qquad \left[\mathcal{D}^{(0)}_z, \mathcal{W}^{(k)}_\mu \right]_. = \Phi^{(k)}_{\widehat{\mathcal{W}};\mu}. \tag{11.15}$$

[41] In fact, it is simple to show that $g^\mu_{(0)}(x, y)$ is essentially, up to a factor of N, just the free field propagator of the CFT:

$$\begin{aligned} \left\langle \psi_m(y) \tilde{\psi}^m(x) \right\rangle_{\text{CFT}} &= -\mathcal{P}^{(0)}(\epsilon; x, y) - \gamma^\mu \mathcal{P}^{(0)}_\mu(\epsilon; x, y) \\ &= \left. \frac{\delta I_{\text{o.s.}}}{\delta a_{(0)}(y, x)} \right|_{a_{(0)}=0} + \gamma^\mu \left. \frac{\delta I_{\text{o.s.}}}{\delta w^\mu_{(0)}(y, x)} \right|_{w^\mu_{(0)}=0} = N \not{g}_{(0)}(x, y) \end{aligned} \tag{11.13}$$

We can solve these covariantized differential equations using the same method as we used at first order. First we turn the covariant derivative into a regular partial derivative by conjugating tensorially by K,

$$K^{-1}(z)\cdot\left[\mathcal{D}_z^{(0)},\mathcal{A}^{(k)}\right]_\cdot\cdot K(z)=\partial_z(K^{-1}(z)\cdot\mathcal{A}^{(k)}(z)\cdot K(z)),$$

to obtain

$$\partial_z(K^{-1}(z)\cdot\mathcal{A}^{(k)}(z)\cdot K(z))=K^{-1}(z)\cdot\Phi_{\mathcal{A}}^{(k)}(z)\cdot K(z).$$

For $k\geq 2$, we have the boundary condition $\mathcal{A}^{(k)}(\epsilon)=0$, since the boundary condition (11.6) has already been satisfied at first order. The solution to this partial differential equation is

$$\mathcal{A}^{(k)}(z)=\int_\epsilon^\infty dz'\;G(z;z')\cdot\Phi_{\mathcal{A}}^{(k)}(z')\cdot G^{-1}(z';z),$$

where we have defined the *ingoing bulk-to-bulk* Wilson line

$$G(z;z')=\Theta(z-z')\,K(z)\cdot K^{-1}(z')$$

and the *outgoing bulk-to-bulk* Wilson line

$$G^{-1}(z';z)=\Theta(z-z')\,K(z')\cdot K^{-1}(z).$$

Of course, we have a solution for $\widehat{\mathcal{W}}_\mu^{(k)}$ of identical form:

$$\widehat{\mathcal{W}}_\mu^{(k)}(z)=\int_\epsilon^\infty dz'\;G(z;z')\cdot\Phi_{\widehat{\mathcal{W}};\mu}^{(k)}(z')\cdot G^{-1}(z';z).$$

Let's now see how this works at second order. The right-hand-side of (11.15) expands to

$$\begin{aligned}\Phi_{\widehat{\mathcal{W}}}^{(2)\mu}(z')&=\mathcal{A}^{(1)}\cdot\Delta^\mu\cdot\mathcal{A}^{(1)}+\varepsilon^{\mu\nu\lambda}(\mathcal{A}^{(1)}\cdot\Delta_\nu\cdot\widehat{\mathcal{W}}_\lambda^{(1)}+\widehat{\mathcal{W}}_\nu^{(1)}\cdot\Delta_\lambda\cdot\mathcal{A}^{(1)})\\&\quad+\widehat{\mathcal{W}}_\mu^{(1)}\cdot\Delta^\nu\cdot\widehat{\mathcal{W}}_\nu^{(1)}-\widehat{\mathcal{W}}_\nu^{(1)}\cdot\Delta_\mu\cdot\widehat{\mathcal{W}}^{(1)\nu}+\widehat{\mathcal{W}}_\nu^{(1)}\cdot\Delta^\nu\cdot\widehat{\mathcal{W}}_\mu^{(1)}\\&=K\cdot\Big(a^{(0)}\cdot H^\mu\cdot a^{(0)}+\varepsilon^{\mu\nu\lambda}(a^{(0)}\cdot H_\nu\cdot\hat{w}_\lambda^{(0)}+\hat{w}_\nu^{(0)}\cdot H_\lambda\cdot a^{(0)})\\&\quad+\hat{w}_\mu^{(0)}\cdot H^\nu\cdot w_\nu^{(0)}-\hat{w}_\nu^{(0)}\cdot H_\mu\cdot\hat{w}^{(0)\nu}+\hat{w}_\nu^{(0)}\cdot H^\nu\cdot\hat{w}_\mu^{(0)}\Big)\cdot K^{-1}.\end{aligned}$$

The on-shell action at second order is then

$$\begin{aligned}I_{\text{o.s.}}^{(2)}&=N\int_\epsilon^\infty dz\;\text{Tr.}\left\{\Delta_\mu\cdot\widehat{\mathcal{W}}_\mu^{(2)}\right\}\\&=N\int_\epsilon^\infty dz\int_\epsilon^\infty dz'\Theta(z-z')\;\text{Tr.}\left\{\Delta_\mu(z)\cdot K(z)\cdot K^{-1}(z')\Phi_{\widehat{\mathcal{W}}}^{(2)\mu}(z')\cdot K(z')\cdot K^{-1}(z)\right\}\\&=N\int_\epsilon^\infty dz\int_\epsilon^z dz'\;\text{Tr.}\Big\{H_\mu(z)\cdot\Big(a^{(0)}\cdot H^\mu\cdot a^{(0)}+\varepsilon^{\mu\nu\lambda}(a^{(0)}\cdot H_\nu(z')\cdot\hat{w}_\lambda^{(0)}+\hat{w}_\nu^{(0)}\cdot H_\lambda(z')\cdot a^{(0)})\\&\qquad+\hat{w}_\mu^{(0)}\cdot H^\nu(z')\cdot w_\nu^{(0)}-\hat{w}_\nu^{(0)}\cdot H_\mu(z')\cdot\hat{w}^{(0)\nu}+\hat{w}_\nu^{(0)}\cdot H^\nu(z')\cdot\hat{w}_\mu^{(0)}\Big)\Big\}\end{aligned}$$

In anticipation of the desired result, we can do a bit of reverse engineering; we make the notation

more compact by using the Dirac algebra in $2+1$ dimensions to organize the above terms[42]:

$$\begin{aligned} I^{(2)}_{\text{o.s.}} &= N\int_\epsilon^\infty dz \int_\epsilon^z dz'\ \text{Tr}_{\cdot,\gamma}\left\{ \not{H}(z)\cdot(a^{(0)}+\not{w}^{(0)})\cdot \not{H}(z')\cdot(a^{(0)}+\not{w}^{(0)})\right\} \\ &= \frac{N}{2}\text{Tr}_{\cdot,\gamma}\left\{ \not{g}^{(0)}\cdot(a^{(0)}+\not{w}^{(0)})\cdot \not{g}^{(0)}\cdot(a^{(0)}+\not{w}^{(0)})\right\} \end{aligned}$$

Again, we can represent this in the form of Witten diagram (see Fig. 11.2).

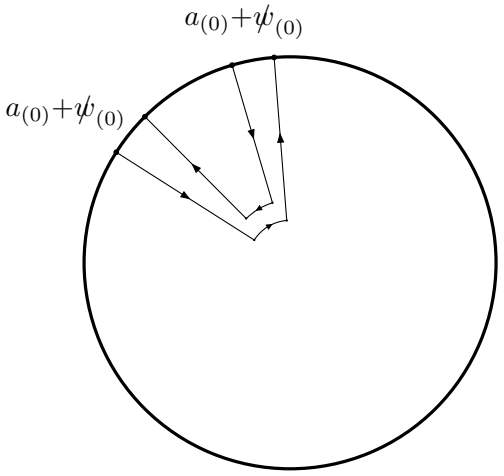


Figure 11.2: The Witten diagram representing the second-order term $I^{(2)}_{\text{o.s.}}$. The $a^{(0)}+\not{w}^{(0)}$ are boundary insertions of the ultraviolet bilocal source $a^{(0)}+\not{w}^{(0)}$.

More generally, at arbitrary k, the on-shell action is of the form

$$I^{(k)}_{\text{o.s.}} = N\int_\epsilon^\infty dz_1 \int_\epsilon^{z_1} dz_2 \cdots \int_\epsilon^{z_{k-1}} d_k\ \text{Tr}_{\cdot,\gamma}\Big\{ \not{H}(z_1)\cdot(a^{(0)}+\not{w}^{(0)})\cdot\cdots\cdot \not{H}(z_k)\cdot(a^{(0)}+\not{w}^{(0)}) + P(z_1,\ldots,z_k)\Big\},$$

where $P(z_1,\ldots,z_k)$ are permutations of the set $\{z_i\}$. Using $H_\mu(z)=\partial_z g_\mu(z)$, we can trivially perform the integrations to yield

$$I^{(k)}_{\text{o.s.}} = \frac{N}{k}\text{Tr}_{\cdot,\gamma}\left(\not{g}^{(0)}\cdot(a^{(0)}+\not{w}^{(0)})\right)^k.$$

As a final example, the three point function is depicted in Fig. 11.3.

We can sum up these terms exactly to arrive at

$$I_{\text{o.s.}} = N\text{Tr}_\gamma\left(\not{g}^{(0)} - a^{(0)} - \not{w}^{(0)}\right),$$

which precisely reproduces the boundary generating functional.

[42] To check this claim, one can work backward, applying $\gamma^\mu\gamma^\nu = \eta^{\mu\nu}\mathbf{1} + \varepsilon^{\mu\nu\lambda}\gamma_\lambda$, $\text{Tr}_\gamma\mathbf{1} = dim_{Cl} = 2$, $\text{Tr}_\gamma\gamma_\mu = 0$ to develop the following identities:

$$\begin{aligned} \text{Tr}_\gamma\left(\gamma^\mu\gamma^\nu\right) &= 2\eta^{\mu\nu} \\ \text{Tr}_\gamma\left(\gamma^\mu\gamma^\nu\gamma^\sigma\right) &= 2\varepsilon^{\mu\nu\sigma} \\ \text{Tr}_\gamma\left(\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\rho\right) &= 2\eta^{\mu\nu}\eta^{\sigma\rho} - 2\eta^{\mu\rho}\eta^{\mu\sigma} + 2\eta^{\mu\sigma}\eta^{\nu\rho} \end{aligned}$$

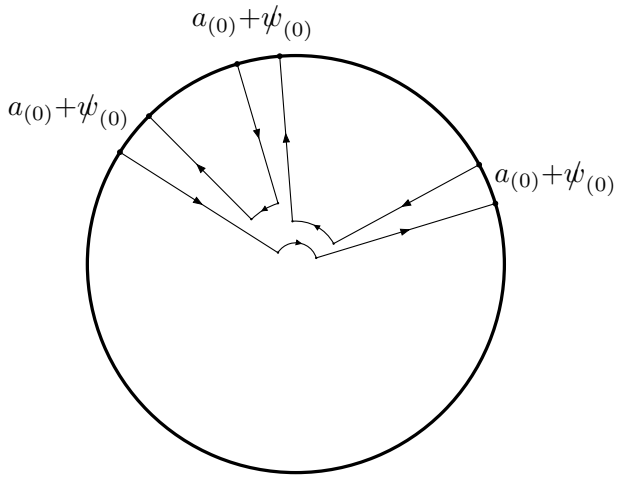


Figure 11.3: The Witten diagram for the bulk on-shell action at third order.

11.2.2 Higher-point functions of the Bosonic theory

For completeness, we also include the same analysis for the Bosonic theory. In this case, the process is actually simpler because we have only one bi-local source $\mathcal{B}$ and no Dirac algebra, so the higher-order terms organize in a more obvious way.

Starting from the bosonic action (10.44), we go on-shell to obtain:

$$I_{o.s.} = -iN \int_{\epsilon}^{\infty} dz \ \mathrm{Tr}\left(\Delta_B \cdot \mathcal{B}_{o.s.}\right)$$

$\mathcal{B}_{o.s.}$ is a solution to the bulk equation of motion $\mathcal{D}^{(0)}\mathcal{B} = \boldsymbol{\beta}^{(\mathcal{B})}$. Following the same procedure as in the Majorana case, we fix the value of $\mathcal{B}$ at the at the ultraviolet cutoff $z = \epsilon$:

$$\mathcal{B}(\epsilon; x, y) = b^{(0)}(x, y), \tag{11.16}$$

and we fix $\mathcal{P}$ in the infra-red:

$$\lim_{z\to\infty} \mathcal{P}(z; x, y) = 0 \tag{11.17}$$

We introduce a formal organizing parameter α, writing

$$\mathcal{B} = \alpha\mathcal{B}_{(1)} + \alpha^2\mathcal{B}_{(2)} + \cdots$$

$$\mathcal{P} = \mathcal{P}_{(0)} + \alpha\mathcal{P}_{(1)} + \alpha^2\mathcal{P}_{(2)} + \cdots,$$

and solve the equations of motion order by order in α, later setting α to one. Focusing on the

z-component of the equation of motion, we have

$$\Big[\mathcal{D}_z^{(0)}, \mathcal{B}_{(1)}\Big]_. = 0 \tag{11.18}$$

$$\Big[\mathcal{D}_z^{(0)}, \mathcal{B}_{(2)}\Big]_. = \mathcal{B}_{(1)} \cdot \Delta_B \cdot \mathcal{B}_{(1)} \tag{11.19}$$

$$\Big[\mathcal{D}_z^{(0)}, \mathcal{B}_{(3)}\Big]_. = \mathcal{B}_{(2)} \cdot \Delta_B \cdot \mathcal{B}_{(1)} + \mathcal{B}_{(1)} \cdot \Delta_B \cdot \mathcal{B}_{(2)} \tag{11.20}$$

$$\vdots$$

The first equation is homogeneous and has the solution

$$\mathcal{B}_{(1)}(z) = K(z) \cdot b_{(0)} \cdot K^{-1}(z), \tag{11.21}$$

where the boundary-to-bulk Wilson line is again defined as $K(z) = \mathscr{P}_. \exp\Big(- \int_\epsilon^z dz' \, \mathcal{W}_z^{(0)}(z')\Big)$.

At this order, the on-shell action is simply

$$I_{o.s.}^{(1)} = -iN \int_\epsilon^\infty dz \ \mathrm{Tr} \ \Delta_B \cdot \mathcal{B}_{(1)} = -iN \int_\epsilon^\infty dz \ \mathrm{Tr} \ \big(K^{-1} \cdot \Delta_B \cdot K \cdot b_{(0)}\big) = -iN \ \mathrm{Tr} \ g_{(0)} \cdot b_{(0)}. \tag{11.22}$$

where $g_{(0)}$ is defined in the same way as for the Majorana case.

In general, at any given order, we can write

$$\Big[\mathcal{D}_z^{(0)}, \mathcal{B}_{(k)}\Big]_. = \Phi_{(k)}(z)$$

where $\Phi_{(k)}$ is the inhomogeneous term at the corresponding order. We solve this by conjugating by K, reducing the covariant derivative to an ordinary derivative

$$\partial_z\Big(K^{-1}(z) \cdot \mathcal{B}_{(k)}(z) \cdot K(z)\Big) = K^{-1}(z) \cdot \Phi_{(k)}(z) \cdot K(z)$$

Using the boundary condition to be $\mathcal{B}_{(k)}(\epsilon) = 0, \ \forall k \geq 2$, the above equation yields

$$\mathcal{B}_{(k)}(z) = \int_\epsilon^\infty dz' \ G(z;z') \cdot \Phi_{(k)}(z') \cdot G^{-1}(z';z),$$

with the ingoing bulk-to-bulk Wilson line $G(z;z')$ and out going bulk-to-bulk Wilson $G^{-1}(z';z)$ defined, again, like in the Majorana theory. Collecting everything together, we get the integral equation

$$\mathcal{B}(z) = K(z) \cdot b_{(0)} \cdot K^{-1}(z) + \int_\epsilon^\infty dz' \ G(z;z') \cdot \beta^{(\mathcal{B})}[\mathcal{B}](z') \cdot G^{-1}(z';z)$$

So, the second order calculation is now

$$\mathcal{B}_{(2)} = \int_\epsilon^z dz' \ K(z) \cdot b_{(0)} \cdot K^{-1}(z') \cdot \Delta_B(z') \cdot K(z') \cdot b_{(0)} \cdot K^{-1}(z)$$

and thus, the on-shell action at second order is given by

$$I^{(2)}_{o.s} = -iN \int_\epsilon^\infty dz \int_\epsilon^z dz' \ \mathrm{Tr}\left(K^{-1}(z)\cdot\Delta_B(z)\cdot K(z)\cdot b_{(0)}\cdot K^{-1}(z')\cdot\Delta_B(z')\cdot K(z')\cdot b_{(0)}\right) \tag{11.23}$$

$$= -iN \int_\epsilon^\infty dz \int_\epsilon^z dz' \ \mathrm{Tr}\left(H(z)\cdot b_{(0)}\cdot H(z')\cdot b_{(0)}\right) \tag{11.24}$$

The z integrations yield

$$I^{(2)}_{o.s} = -iN \int_\epsilon^\infty dz \int_\epsilon^z dz' \ \mathrm{Tr}\left(H(z)\cdot b_{(0)}\cdot \partial_{z'} g(z')\cdot b_{(0)}\right) \tag{11.25}$$

$$= -iN \int_\epsilon^\infty dz \mathrm{Tr}\left(\partial_z g(z)\cdot b_{(0)}\cdot g(z)\cdot b_{(0)}\right) \tag{11.26}$$

$$= -i\frac{N}{2} \int_\epsilon^\infty dz \ \partial_z \mathrm{Tr}\left(g(z)\cdot b_{(0)}\cdot g(z)\cdot b_{(0)}\right) \tag{11.27}$$

which further integrates to

$$I^{(2)}_{o.s} = -i\frac{N}{2}\ \mathrm{Tr}\left(g_{(0)}\cdot b_{(0)}\cdot g_{(0)}\cdot b_{(0)}\right).$$

This result reproduces the correct two-point functions of the free field theory.

This procedure can be followed to arbitrary order. One finds the k^{th}-order term has the form

$$I^{(k)}_{o.s.} = -iN \int_\epsilon^\infty dz_1 \int_\epsilon^{z_1} dz_2 ... \int_\epsilon^{z_{k-1}} dz_k \ \mathrm{Tr}\left(H(z_1)\cdot b_{(0)}\cdot H(z_2)\cdot b_{(0)}\cdot ...\cdot H(z_k)\cdot b_{(0)} + \text{permutations}\right)$$

The permutations include all of the distinct orderings of $\{H(z_2), ..., H(z_k)\}$. Proceeding with the z-integrals as before, we find the on-shell action at this order is given by

$$I^{(k)}_{o.s.} = -i\frac{N}{k}\ \mathrm{Tr}\left(g_{(0)}\cdot b_{(0)}\right)^k$$

Collecting all the terms, we note that the on-shell action

$$I_{o.s.} = -iN\Big(\mathrm{Tr}\left(g_{(0)}\cdot b_{(0)}\right) + \frac{1}{2}\ \mathrm{Tr}\left(g_{(0)}\cdot b_{(0)}\cdot g_{(0)}\cdot b_{(0)}\right) + \frac{1}{3}\ \mathrm{Tr}\left(g_{(0)}\cdot b_{(0)}\cdot g_{(0)}\cdot b_{(0)}\cdot g_{(0)}\cdot b_{(0)}\right) + \cdots\Big)$$

precisely reproduces the boundary generating functional

$$Z[b_{(0)}]/Z[0] = e^{iI_{o.s.}} = \det{}^{-N}\left(1 - g_{(0)}\cdot b_{(0)}\right). \tag{11.28}$$

Thus we conclude that the holographic formulation correctly reproduces all of the correlation functions of the boundary field theory.

Chapter 12

Formalizing with Jet Bundles

We have shown that the one-form coupling W_μ can be extended to a bulk field $\mathcal{W}_\mu$ which transforms under the group $CO(L_2)$, or $CU(L_2)$, like a connection. As such, it is reasonable to believe that $\mathcal{W}_\mu$ really is a connection. However, in physics, a connection is usually understood to be a Lie-algebra valued one-form, and while we have identified "gauge groups" of the Majorana and bosonic actions, the details are a bit fuzzy. The biggest problem is the non-local nature of the group actions. To understand the confusion, first consider the simplest possible gauge group, $U(1)$. Under the action of an infinitesimal $U(1)$ transformation, a scalar field ϕ transforms as

$$\phi(x) = i\epsilon(x)\phi(x).$$

Here, ϕ is thought of as a section of a vector bundle associated to a principle $U(1)$ bundle, and the transformation may be thought of as a vertical group action. In contrast, under an infinitesimal $U(L_2)$ transformation, a scalar field ϕ^m transforms as

$$\delta\phi^m(x) = \int_y \epsilon(x,y)\phi^m(y). \tag{12.1}$$

The geometrical meaning of this transformation is not immediately clear. At each point x, the new value of $\phi^m(x)$ depends, not just on the old value of $\phi^m(x)$, but on the old values of $\{\phi^m(y)\}$ for all y in the common support of ϕ^m and ϵ. However, there is a way back to a local picture. If ϕ^m is a C^∞ function, then in principle, we have everything we need to know stored locally as long as we have access to all of the ϕ^m derivatives. Therefore, what we really need is a vector bundle with fibers that at each point x keeps track of $\phi^m(x)$ and all of its derivatives. Such a construction does exist in mathematics. It is called an *infinite jet bundle*. Loosely speaking, the infinite jet bundle is a vector bundle of which the fiber at a point p consists of all equivalence classes of functions which have the same derivatives at p. Schematically, an element Φ of the fiber at p correspondent to the function ϕ looks like

$$\Phi^m[\phi](p) = \left(\phi^m(x), \frac{\partial\phi^m}{\partial x^\mu}(p), \frac{\partial^2\phi^m}{\partial x^\mu \partial x^\nu}(p), \dots\right),$$

and is called the *jet* of ϕ at p. The space of all jets at a point constitutes the fiber of the infinite jet bundle at that point. Going back to equation (12.1), we see the action of ϵ on ϕ^m can be represented in terms of a linear and local action on its jet $\Phi^m[\phi]$. This is why we can think of $U(L_2)$ transformations as gauge transformations acting on the infinite jet bundle, satisfying the $U(L_2)$ condition. Given this interpretation, the one form $\mathcal{W}_\mu$ is naturally identified as a connection one form over the infinite jet bundle, while the zero form $\mathfrak{B}$ can be thought of as a section of its endomorphism bundle. Indeed, this interpretation fits nicely with our intuition for quasilocal

expansions of the bilocal sources,

$$W_\mu(x,y) \simeq \sum_{s=1}^{\infty} W_\mu^{a_1\cdots a_{s-1}}(x)\partial_{a_1}^{(x)}\cdots\partial_{a_{s-1}}^{(x)}\delta^d(x-y)+\cdots$$

$$B(x,y) \simeq \sum_{s=1}^{\infty} B^{a_1\cdots a_{s-1}}(x)\partial_{a_1}^{(x)}\cdots\partial_{a_{s-1}}^{(x)}\delta^d(x-y)+\cdots$$

The above quasilocal expansions basically express the fact that both $\mathcal{W}_\mu$ and $\mathfrak{B}$ are valued in the endomorphism bundle of the jet bundle. In other words, a given choice of endomorphism over each jet defines the vertical group action. More concretely, if we want to think in familiar language, we can return to the idea of a connection as a one-form valued in the Lie-algebra of the gauge group. Traditionally, that would look like $W = W_\mu^\alpha T^\alpha dx^\mu$, where $\{T^\alpha\}$ is a set of basis elements for the Lie-algebra. The quasi-local expansions can be thought of in the same spirit, with the set of differential operators $\{T^{(s)} \simeq \partial_{(x)}^s\delta^d(x-y)\}$ playing the role of the Lie-algebra elements.

We will not go into the gritty details here required to carefully build up all the necessary bundle structure. That work was done in large part in [1]. In particular, in [1], it was shown that by shifting ones attention away from the associated vector bundle (*i.e.,* the infinite jet bundle) and back to the principle bundle itself, something very similar to the full field content of Vasiliev's higher spin theory emerges. In particular, the horizontal components of the connection correspond to the one-form sources in the field theory, while the vertical components (the Faddeev-Popov ghosts) of the connection correspond to auxiliary pure-gauge degrees of freedom (see [81] for explanations). This seems to be the most likely origin of Vasiliev's mysterious $\mathcal{S}$ one-form.

Appendices

Appendix A

Singular Value Decomposition

A.1 SVD in practice

Singular value decomposition (SVD) is a procedure for turning any matrix M into a product

$$M = USV$$

where U is *left-normalized*, V is *right-normalized*, and S is square diagonal:

$$U^\dagger U = \mathbf{1}, \qquad VV^\dagger = \mathbf{1}, \qquad S = \text{diag}\,(s_1, s_2, \cdots, s_N)\,.$$

The s_i are called *singular values*. Finding U, S, and V is straightforward. Working in the S basis,

$$\hat{e}_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i^{\text{th}} \text{ slot},$$

we have

$$MM^\dagger(U\hat{e}_i) = (USV)(V^\dagger S^\dagger U^\dagger)U\hat{e}_i = US^2\hat{e}_i = s_i^2(U\hat{e}_i).$$

From this we learn:
1) The $\{s_i^2\}$ are eigenvalues of $MM^\dagger$. (Note: $MM^\dagger$ is Hermitian and hence $s_i^2 \in \mathbb{R}$.)
2) The columns of matrix U are the eigenvectors of $MM^\dagger$.
3) $V = S^{-1}U^\dagger M$. (V can also be constructed from the eigenvectors of $M^\dagger M$.)

A.2 QR and LQ variations of SVD

Sometimes we do not need the full singular value decomposition, $M = USV$, of a matrix M. Such a situation arises repeatedly in the normalization process described in §5.2. Here we define two variations of SVD, referred to as QR and LQ decomposition.

QR decomposition is SVD with U renamed Q and the product SV grouped into one matrix called R:

$$M = QR, \qquad Q^\dagger Q = \mathbf{1}.$$

LQ decomposition is SVD with the product US grouped into one matrix called L and V renamed Q:

$$M = LQ, \qquad QQ^\dagger = \mathbf{1}.$$

A.3 Canonicalization of MPS for generalized YM_2 and extraction of the entanglement entropy

Our goal is to determine the generalized YM_2 entanglement entropy from the MPS representation. We start with the ket,

$$\left|\psi^-_{h_-,\varrho_-}\right\rangle = \frac{1}{\sqrt{Z}} \sum_{i_0,\ldots,i_{\ell-1}} \eta^{i_0\mathbf{T}}_{h_0,\varrho_0} \mathcal{N}^{i_1}_{h_1,\varrho_1} \cdots \mathcal{N}^{i_{a-1}}_{h_{a-1},\varrho_{a-1}} \cdot \mathcal{N}^{i_a}_{h_a,\varrho_a} \cdots \mathcal{N}^{i_{g-1}}_{h_{\ell-2}\varrho_{\ell-2}} \eta^{i_{\ell-1}}_{\varrho_{\ell-1}} \left|\phi_{i_0}\phi_{i_1}\cdots\phi_{i_{\ell-1}}\right\rangle. \tag{A.1}$$

We must put (A.1) into mixed-canonical form. We follow the procedure laid out in §5.2, first left-normalizing the matrices encoding the A degrees of freedom, then right-normalizing the matrices encoding the B degrees of freedom, then extracting the remnant singular values. The extra dot "$\cdot$" appearing in the middle of expression (A.1) indicates the point in the matrix product at which the A degrees of freedom end and the B degrees of freedom begin.

Starting from (4.17), with $\gamma = \frac{1}{|G|}$, we know

$$[\eta^{i_0\mathbf{T}}_{h_0,\varrho_0}]_{j_1} \doteq |G|^{h_0-\frac{1}{2}} \left(e^{\varrho_0 B_{R_1}} d_{R_1}^{-2h_0} \chi_{R_1}(i_0), \cdots, e^{\varrho_0 B_{R_{|Z|}}} d_{R_{|Z|}}^{-2h_0} \chi_{R_{|Z|}}(i_0)\right).$$

We reshape $\eta^{i_0\mathbf{T}}_{h_0,\varrho_0}$ into a matrix and perform a QR decomposition:

$$\begin{aligned}
[\eta^{\mathbf{T}}_{\varrho_0}]^{i_0}_{j_1} &\doteq |G|^{h_0-\frac{1}{2}} \begin{pmatrix} e^{\varrho_0 B_{R_1}} d_{R_1}^{-2h_0} \chi_{R_1}(g_1) & \cdots & e^{\varrho_0 B_{R_{|Z|}}} d_{R_{|Z|}}^{-2h_0} \chi_{R_{|Z|}}(g_1) \\ \vdots & & \vdots \\ e^{\varrho_0 B_{R_1}} d_{R_1}^{-2h_0} \chi_{R_1}(g_{|G|}) & \cdots & e^{\varrho_0 B_{R_{|Z|}}} d_{R_{|Z|}}^{-2h_0} \chi_{R_{|Z|}}(g_{|G|}) \end{pmatrix} \\
&= |G|^{-\frac{1}{2}} \begin{pmatrix} \chi_{R_1}(g_1) & \cdots & \chi_{R_{|Z|}}(g_1) \\ \vdots & & \vdots \\ \chi_{R_1}(g_{|G|}) & \cdots & \chi_{R_{|Z|}}(g_{|G|}) \end{pmatrix} \cdot |G|^{h_0} \begin{pmatrix} e^{\varrho_0 B_{R_1}} d_{R_1}^{-2h_0} & & \\ & \ddots & \\ & & e^{\varrho_0 B_{R_{|Z|}}} d_{R_{|Z|}}^{-2h_0} \end{pmatrix} \\
&\equiv [Q_0]^{i_0}_{\ k_1} [R_{h_0,\varrho_0}]^{k_1}_{j_1}.
\end{aligned}$$

By construction, Q_0 is left-normalized: $Q_0^\dagger Q_0 = \mathbf{1}$. The second equality can be reached systematically by the relationships laid out in §A.1.

We then reshape Q_0 into a set of row vectors, representing the first elements of our matrix product:

$$A^{i_0} \doteq \left\{ |G|^{-\frac{1}{2}} \left(\chi_{R_1}(g_1), \cdots, \chi_{R_{|Z|}}(g_1)\right), \ldots, |G|^{-\frac{1}{2}} \left(\chi_{R_1}(g_{|G|}), \cdots, \chi_{R_{|Z|}}(g_{|G|})\right) \right\}.$$

This set of vectors is left-normalized: $\sum_{i_0} A^{i_0\dagger} A^{i_0} = \mathbf{1}$. The matrix R_{ϱ_0} is then paired with the next element of the ket MPS,

$$\left|\psi^-_{h_-,\varrho_-}\right\rangle = \frac{1}{\sqrt{Z}} \sum_{i_0,\ldots,i_{\ell-1}} A^{i_0} \left(R_{h_0,\varrho_0} \mathcal{N}^{i_1}_{h_1,\varrho_1}\right) \mathcal{N}^{i_2}_{h_2,\varrho_2} \cdots \mathcal{N}^{i_{\ell-2}}_{h_{\ell-2},\varrho_{\ell-2}} \eta^{i_{\ell-1}}_{h_{\ell-1},\varrho_{\ell-1}} \left|\phi_{i_0}\cdots\phi_{i_{\ell-1}}\right\rangle,$$

to form a new matrix,

$$
\begin{aligned}
[R_{h_0,\varrho_0}\mathcal{N}^{i_1}_{h_1,\varrho_1}]^{j_1}_{j_2} \doteq |G|^{h_0}
&\begin{pmatrix} e^{\varrho_0 B_{R_1}} d_{R_1}^{-2h_0} & & \\ & \ddots & \\ & & e^{\varrho_0 B_{R_{|Z|}}} d_{R_{|Z|}}^{-2h_0} \end{pmatrix} \\
&\cdot |G|^{h_1} \begin{pmatrix} e^{\varrho_1 B_{R_1}} d_{R_1}^{-1-2h_1} \chi_{R_1}(i_1) & & \\ & \ddots & \\ & & e^{\varrho_1 B_{R_{|Z|}}} d_{R_{|Z|}}^{-1-2h_1} \chi_{R_{|Z|}}(i_1) \end{pmatrix} \\
&= |G|^{h_0+h_1} \begin{pmatrix} e^{(\varrho_0+\varrho_1) B_{R_1}} d_{R_1}^{-1-2(h_1+h_2)} \chi_{R_1}(i_1) & & \\ & \ddots & \\ & & e^{(\varrho_0+\varrho_1) B_{R_{|Z|}}} d_{R_{|Z|}}^{-1-(h_0+h_1)} \chi_{R_{|Z|}}(i_1) \end{pmatrix}.
\end{aligned}
$$

We then reshape this matrix, followed by another QR decomposition:

$$
\begin{aligned}
[R_{h_0,\varrho_0}\mathcal{N}_{h_1,\varrho_1}]^{(i_1 j_1)}_{j_2} \doteq |G|^{h_0+h_1}
&\begin{pmatrix} e^{(\varrho_0+\varrho_1) B_{R_1}} d_{R_1}^{-1-2(h_0+h_1)} \chi_{R_1}(g_1) & & & \\ & \ddots & & \\ & & e^{(\varrho_0+\varrho_1) B_{R_{|Z|}}} d_{R_{|Z|}}^{-1-2(h_0+h_1)} \chi_{R_{|Z|}}(g_1) & \\ & & \vdots & \\ e^{(\varrho_0+\varrho_1) B_{R_1}} d_{R_1}^{-1-2(h_0+h_1)} \chi_{R_1}(g_{|G|}) & & & \\ & \ddots & & \\ & & e^{(\varrho_0+\varrho_1) B_{R_{|Z|}}} d_{R_{|Z|}}^{-1-2(h_0+h_1)} \chi_{R_{|Z|}}(g_{|G|}) & \end{pmatrix} \\
= \; |G|^{-\frac{1}{2}}
&\begin{pmatrix} \chi_{R_1}(g_1) & & \\ & \ddots & \\ & & \chi_{R_{|Z|}}(g_1) \\ & \vdots & \\ \chi_{R_1}(g_{|G|}) & & \\ & \ddots & \\ & & \chi_{R_{|Z|}}(g_{|G|}) \end{pmatrix} \\
&\cdot |G|^{h_0+h_1+\frac{1}{2}} \begin{pmatrix} e^{(\varrho_0+\varrho_1) B_{R_1}} d_{R_1}^{-1-2(h_0+h_1)} & & \\ & \ddots & \\ & & e^{(\varrho_0+\varrho_1) B_{R_{|Z|}}} d_{R_{|Z|}}^{-1-2(h_0+h_1)} \end{pmatrix} \\
\equiv [Q_1]^{i_1}_{k_2} [R_{h_0+h_1,\varrho_0+\varrho_1}]^{k_2}_{j_2}.
\end{aligned}
$$

Once again, we reshape the Q matrix into a set of left-normalized matrices:

$$
A^{i_1} \doteq \left\{ |G|^{-\frac{1}{2}} \begin{pmatrix} \chi_{R_1}(g_1) & & \\ & \ddots & \\ & & \chi_{R_{|Z|}}(g_1) \end{pmatrix}, \ldots, |G|^{-\frac{1}{2}} \begin{pmatrix} \chi_{R_1}(g_{|G|}) & & \\ & \ddots & \\ & & \chi_{R_{|Z|}}(g_{|G|}) \end{pmatrix} \right\}.
$$

We can perform this process iteratively, as many times as we need. Eventually we arrive at

$$
\left| \psi^{-}_{h_-,\varrho_-} \right\rangle = \frac{1}{\sqrt{Z}} \sum_{i_0,\ldots,i_{\ell-1}} A^{i_0} A^{i_1} \cdots A^{i_{a-1}} R_{h_A,\varrho_A} \mathcal{N}^{i_a}_{h_a,\varrho_a} \cdots \mathcal{N}^{i_{\ell-2}}_{h_{\ell-2},\varrho_{\ell-2}} \eta^{i_{\ell-1}}_{h_{\ell-1},\varrho_{\ell-1}} \left| \phi_{i_0} \cdots \phi_{i_{\ell-1}} \right\rangle,
$$

where

$$R_{h_A,\varrho_A} = |G|^{h_A+\frac{a-1}{2}} \begin{pmatrix} e^{\varrho_A B_{R_1}} d_{R_1}^{-a+1-2h_A} & & \\ & \ddots & \\ & & e^{\sum_{\alpha=0}^{a-1}\varrho_\alpha B_{R_{|Z|}}} d_{R_{|Z|}}^{-a+1-2h_A} \end{pmatrix}.$$

We then perform an identical procedure starting from the right side of the ket MPS, except we employ LQ decompositions rather than QR. The final result is

$$B^{i_\beta} \doteq \left\{ |G|^{-\frac{1}{2}} \begin{pmatrix} \chi_{R_1}(g_1) & & \\ & \ddots & \\ & & \chi_{R_{|Z|}}(g_1) \end{pmatrix}, \ldots, |G|^{-\frac{1}{2}} \begin{pmatrix} \chi_{R_1}(g_{|G|}) & & \\ & \ddots & \\ & & \chi_{R_{|Z|}}(g_{|G|}) \end{pmatrix} \right\}, \quad a \leq \beta < \ell - 1.$$

$$B^{i_{\ell-1}} \doteq \left\{ |G|^{-\frac{1}{2}} \begin{pmatrix} \chi_{R_1}(g_1) \\ \vdots \\ \chi_{R_{|Z|}}(g_1) \end{pmatrix}, \ldots, |G|^{-\frac{1}{2}} \begin{pmatrix} \chi_{R_1}(g_{|G|}) \\ \vdots \\ \chi_{R_{|Z|}}(g_{|G|}) \end{pmatrix} \right\}.$$

These sets are right-normalized: $\sum_{i_\beta} B^{i_\beta} B^{i_\beta\dagger} = \mathbf{1}$. The left over information, squeezed to the center, is

$$L_{h_B,\varrho_B} = |G|^{h_B+\frac{\ell-1-a}{2}} \begin{pmatrix} e^{\varrho_B B_{R_1}} d_{R_1}^{-\ell+1+a-2h_B} & & \\ & \ddots & \\ & & e^{\sum_{\alpha=a}^{g}\varrho_\alpha B_{R_{|Z|}}} d_{R_{|Z|}}^{-\ell+1+a-2h_B} \end{pmatrix},$$

and the MPS looks like

$$|\psi\rangle = \frac{1}{\sqrt{Z}} \sum_{i_0,\ldots,i_{\ell-1}} A^{i_0} A^{i_1} \cdots A^{i_{a-1}} R_{h_A,\varrho_A} L_{h_B,\varrho_B} B^{i_a} \cdots B^{i_{\ell-1}} B^{i_{\ell-1}} \left|\phi_{i_0} \cdots \phi_{i_{\ell-1}}\right\rangle,$$

where all the A^{i_α} are left-normalized and all the B^{i_β} are right-normalized. We then explicitly take the product $R_{h_A,\varrho_A} L_{h_B,\varrho_B}$ and decompose the (diagonal) result into a direct sum. We find

$$\left|\psi^-_{h_-,\varrho_-}\right\rangle = \sum_R s^-_R(h_-,\varrho_-) \sum_{i_0,\ldots,i_{\ell-1}} (A^{i_0} A^{i_1} \cdots A^{i_{a-1}})_R (B^{i_a} \cdots B^{i_{g-1}} B^{i_g})^R \left|\phi_{i_0} \cdots \phi_{i_g}\right\rangle,$$

with singular values

$$s^-_R \in \frac{|G|^{\frac{\ell-2}{2}+h_-}}{\sqrt{|G|^{g-1} \sum_{R'} d_{R'}^{2-2g} e^{\varrho_- B_{R'}}}} \cdot \left\{ d_{R_1}^{2-\ell-2h_-} e^{\varrho_- B_{R_1}}, \ldots, d_{R_{|Z|}}^{1-g-2h_-} e^{\varrho_- B_{R_{|Z|}}} \right\}.$$

We have absorbed into s^-_R the $\frac{1}{\sqrt{Z}}$ normalization given by (4.20).

The exact same procedure is used to isolate the singular values of the bra MPS:

$$\left\langle\psi^+_{h_+,\varrho_+}\right| = \sum_R s^+_R(h_+,\varrho_+) \sum_{i_0,\ldots,i_{\ell-1}} (\bar{A}^{i_0} \bar{A}^{i_1} \cdots \bar{A}^{i_{a-1}})_R (\bar{B}^{i_a} \cdots \bar{B}^{i_{g-1}} \bar{B}^{i_g})^R \left\langle\phi_{i_0} \cdots \phi_{i_{\ell-1}}\right|,$$

where

$$s^+_R \in \frac{|G|^{\frac{\ell-2}{2}+h_+}}{\sqrt{|G|^{g-1} \sum_{R'} d_{R'}^{2-2g} e^{\varrho_+ B_{R'}}}} \cdot \left\{ d_{R_1}^{2-\ell-2h_+} e^{\varrho_+ B_{R_1}}, \ldots, d_{R_{|Z|}}^{1-g-2h_+} e^{\varrho_+ B_{R_{|Z|}}} \right\}.$$

The product $s_R^2 \equiv s_R^- s_R^+$ is

$$s_R^2 \in \frac{1}{\sum_{R'} d_{R'}^{2-2g} e^{\varrho\, B_{R'}}} \cdot \left\{ d_{R_1}^{2-2g} e^{\varrho\, B_{R_1}}, \ldots, d_{R_{|Z|}}^{2-2g} e^{\varrho\, B_{R_{|Z|}}} \right\}.$$

The entanglement spectrum is $\left\{ E_R | E_R = -\log s_R^2 \right\}$. Finally, the entanglement entropy is

$$S_A = -\sum_R s_R^2 \log s_R^2 = -\sum_R \frac{d_R^{2-2g} e^{\varrho\, B_R}}{\sum_{R'} d_{R'}^{2-2g} e^{\varrho\, B_{R'}}} \log \left[\frac{d_R^{2-2g} e^{\varrho\, B_R}}{\sum_{R'} d_{R'}^{2-2g} e^{\varrho\, B_{R'}}} \right]. \tag{A.2}$$

Appendix B

Hamiltonian

B.1 The Hamiltonian in QTFT

In quantum mechanics, if we start with a state $|\psi(\tau)\rangle$ at Euclidean time τ, then we can write the state at a later time $\tau + \Delta\tau$ by use of the (Euclidean) time-evolution operator,

$$|\psi(\tau + \Delta\tau)\rangle = e^{-H\Delta\tau} |\psi(\tau)\rangle , \tag{B.1}$$

where H is the Hamiltonian. In a $(D+1)$-dimensional quantum field theory, we can write a similar expression in terms of the Hamiltonian density, $H = \oint_{\Sigma_\tau} d^D\vec{x}\sqrt{g_{\vec{x}}}\,\mathcal{H}(x)$. The integral is evaluated on some time-slice τ, which we assume to be closed. If the space is curved, the integral picks up a nontrivial Jacobian factor $\sqrt{g_{\vec{x}}} \equiv \sqrt{\det[g_{ij}]}$ (where i, j run over the spatial dimensions). From this perspective, (B.1) is a rather unnatural expression since it does not include any information about distances along the time dimension. A more natural expression is

$$|\psi(\tau + \Delta\tau)\rangle = T\left\{e^{-\int_\tau^{\Delta\tau} d\tau \oint d^D\vec{x}\sqrt{g(x)}\,\mathcal{H}(x)}\right\} |\psi(\tau)\rangle \equiv U(\tau + \Delta\tau, \tau)\,|\psi(\tau)\rangle . \tag{B.2}$$

This time evolution operator satisfies the necessary condition $U(\tau_2, \tau_1)U(\tau_1, \tau_0) = U(\tau_2, \tau_0)$. We can now use this to determine the form of $\mathcal{H}(x)$ for a quasitopological field theory written in the basis of boundary holonomies.

We start by writing the MPS with quasitopological Boltzmann weights decomposed into topological Boltzmann weights and area dependent exponentials:

$$\begin{aligned}|\psi(\tau)\rangle = &\left(\eta_{i'_0 j_1}\left[e^{\varrho_0(\tau)B}\right]^{i'_0}_{\ i_0}\left|\phi_0^{i_0}\right\rangle\right)\otimes\left(\mathcal{N}_{i'_1}^{\ j_1}{}_{j_2}\left[e^{\varrho_1(\tau)B}\right]^{i'_1}_{\ i_1}\left|\phi_1^{i_1}\right\rangle\right)\otimes\cdots \\ &\cdots\otimes\left(\mathcal{N}_{i'_{\ell-2}}^{\ j_{\ell-2}}{}_{j_{\ell-1}}\left[e^{\varrho_{g-1}(\tau)B}\right]^{i'_{\ell-2}}_{\ i_{\ell-2}}\left|\phi_{\ell-2}^{i_{\ell-2}}\right\rangle\right)\otimes\left(\eta_{i'_{\ell-1}}^{\ j_{\ell-1}}\left[e^{\varrho_{\ell-1}(\tau)B}\right]^{i'_{\ell-1}}_{\ i_{\ell-1}}\left|\phi_{\ell-1}^{i_{\ell-1}}\right\rangle\right),\end{aligned}$$

where ℓ is the the number of boundaries in time-slice τ. We call this the ℓ-sector. We compare this expression to what we expect a time evolved state to look like. For the moment, we ignore the possibility of splitting and joining of tubes (i.e. there will be no spatial topology changes via time evolution). Under this restriction, time evolution simply extends boundary tubes, increasing their

areas:

$$U(\tau+\Delta\tau,\tau)\,|\psi(\tau)\rangle = \left(\eta_{i_0'' j_1}\left[e^{\varrho_0(\tau)B}\right]^{i_0''}_{i_0'}\left[e^{\Delta\varrho_0(\tau)B}\right]^{i_0'}_{i_0}|\phi_{i_0}\rangle\right)$$
$$\otimes\left(\mathcal{N}_{i_1''\ j_2}^{\ j_1}\left[e^{\varrho_1(\tau)B}\right]^{i_1''}_{i_1'}\left[e^{\Delta\varrho_1(\tau)B}\right]^{i_1'}_{i_1}|\phi_{i_1}\rangle\right)\otimes\cdots$$
$$\cdots\otimes\left(\mathcal{N}_{i_{\ell-2}''\ j_{\ell-1}}^{\ j_{\ell-2}}\left[e^{\varrho_{\ell-2}(\tau)B}\right]^{i_{\ell-2}''}_{i_{\ell-2}'}\left[e^{\Delta\varrho_{\ell-2}(\tau)B}\right]^{i_{\ell-2}'}_{i_{\ell-2}}|\phi_{i_{\ell-2}}\rangle\right)$$
$$\otimes\left(\eta_{i_{\ell-1}''}^{\ j_{\ell-1}}\left[e^{\varrho_{\ell-1}(\tau)B}\right]^{i_{\ell-1}''}_{i_{\ell-1}'}\left[e^{\Delta\varrho_{\ell-1}(\tau)B}\right]^{i_{\ell-1}'}_{i_{\ell-1}}|\phi_{i_{\ell-1}}\rangle\right).$$

Since $\Delta\varrho_\alpha(\tau)=\int_\tau^{\Delta\tau}d\tau\oint_\alpha d\vec{x}\sqrt{g(\vec{x},\tau)}$, it is now easy to guess the form of the Hamiltonian density

$$\mathcal{H} = -\sum_{\alpha=0}^{\ell-1}\sum_{i_0,\ldots,i_{\ell-1}}\sum_{i_0',\ldots,i_{\ell-1}'}\delta^{i_0'}_{i_0}\cdots\delta^{i_{\alpha-1}'}_{i_{\alpha-1}}B^{i_\alpha'}_{i_\alpha}\delta^{i_{\alpha+1}'}_{i_{\alpha+1}}\cdots\delta^{i_{\ell-1}'}_{i_{\ell-1}}\left|\phi_{i_0}\cdots\phi_{i_{\ell-1}}\right\rangle\left\langle\phi_{i_0'}\cdots\phi_{i_{\ell-1}'}\right|$$
$$\equiv -\sum_{\alpha=0}^{\ell-1}\sum_{i_\alpha,i_\alpha'}B^{i_\alpha'}_{i_\alpha}\left|\phi_{i_\alpha}\right\rangle\left\langle\phi_{i_\alpha'}\right|, \tag{B.3}$$

(The second equality is just convenient notation. It must be unpackaged for use in careful calculations.) Since B is a constant matrix, we can trivially integrate the Hamiltonian density over a spatial time-slice. This yields

$$\hat{H} = -\sum_{\alpha=0}^{\ell-1}\left(L_\alpha(\tau)\sum_{i_\alpha,i_\alpha'}B^{i_\alpha'}_{i_\alpha}\left|\phi_{i_\alpha}\right\rangle\left\langle\phi_{i_\alpha'}\right|\right), \tag{B.4}$$

where $L_\alpha(\tau)$ is the length of boundary α in time-slice τ. The energy of a particular state $|\psi(\tau)\rangle$, living on time-slice τ, is then

$$E_{\psi(\tau)} = -\sum_{\alpha=0}^{\ell-1}L_\alpha(\tau)\sum_{i_\alpha,i_\alpha'}\langle\psi(\tau)|\,B^{i_\alpha'}_{i_\alpha}\,|\phi_{i_\alpha}\rangle\,\langle\phi_{i_\alpha'}|\psi(\tau)\rangle \tag{B.5}$$

The energy depends on the choice of time-slice only via the lengths of the boundary loops, $\{L_\alpha\}$. As we will show by explicit example, the energy does not depend on the number of handles in the lower/upper surfaces, $h_\mp$.

B.2 The groundstate energies of generalized YM_2

As an example, we now carry out the calculation of the groundstate energy for genearlized YM_2.

From (4.19-4.20), we know that

$$\left|\psi^-_{h_-,\varrho_-}\right\rangle = \frac{1}{\sqrt{Z}}\,|G|^{h_--1}\sum_{i_0,\ldots,i_{\ell-1}}\sum_R e^{\varrho_- B_R}d_R^{2-\ell-2h_-}\prod_{\alpha=0}^{\ell-1}\chi_R(i_\alpha)\left|\phi_{i_0}\cdots\phi_{i_{\ell-1}}\right\rangle,$$

$$\left\langle \psi^+_{h_+,\varrho_+} \right| = \frac{1}{\sqrt{Z}} |G|^{h_+-1} \sum_{i_0,\ldots,i_{\ell-1}} \langle \phi_{i_0} \cdots \phi_{i_{\ell-1}}| \sum_R e^{\varrho_+ B_R} d_R^{2-\ell-2h_+} \prod_{\alpha=0}^{\ell-1} \chi_R(i_\alpha^{-1}),$$

$$Z = |G|^{g-1} \sum_R e^{\varrho B_R} d_R^{2-2g}.$$

Furthermore, from (2.13), we have the character expansion

$$B(i,j) = \sum_R B_R \, d_R \, \chi_R(ij^{-1}).$$

We can use these relations to calculate the energy of a state $\rho = |\psi^-\rangle\langle\psi^+|$:

$$\begin{aligned}
E_{\psi^+_{h_+,\varrho_+},\psi^-_{h_-,\varrho_-}} &= -\sum_{\alpha=0}^{\ell-1} L_\alpha \sum_{i_\alpha, i'_\alpha} \left\langle \psi^+_{h_+,\varrho_+} \right| B^{i'_\alpha}_{i_\alpha} \left| \phi_{i_\alpha} \right\rangle \left\langle \phi_{i'_\alpha} \right| \left. \psi^-_{h_-,\varrho_-} \right\rangle \\
&= -\frac{1}{Z} |G|^{h_+h_- -2} \sum_{\alpha=0}^{\ell-1} L_\alpha \sum_{i_0,\ldots,i_{\ell-1}} \sum_{i'_\alpha} \Bigg\{ \sum_{R''} B_{R''} d_{R''} \chi_{R''}(i_\alpha i'^{-1}_\alpha) \\
&\qquad \sum_{R,R'} \left[e^{\varrho_+ B_R + \varrho_- B_{R'}} d_R^{2-\ell-2h_+} d_{R'}^{2-\ell-2h_-} \left(\prod_\beta^{\hat{\alpha}} \chi_R(i_\beta^{-1}) \chi_{R'}(i_\beta) \right) \chi_R(i_\alpha^{-1}) \chi_{R'}(i'_\alpha) \right] \Bigg\} \\
&= -\frac{1}{Z} |G|^g \sum_\alpha L_\alpha \sum_R B_R \, e^{\varrho B_R} \, d_R^{2-2g} \\
&= -|G| \, L \sum_R B_R \, \frac{e^{\varrho B_R} d_R^{2-2g}}{\sum_{R'} e^{\varrho B_{R'}} d_{R'}^{2-2g}},
\end{aligned} \tag{B.6}$$

where L is the total spatial length of the chosen time-slice.

Appendix C

The Migdal Formalism for YM_2 and insertion of Wilson Loops

C.1 Migdal formalism

In this appendix we present some details of the Migdal formalism discussed in §2.1. Migdal's recipe for calculating YM_2 partition functions is as follows:

1. "Triangulate" space-time Σ. More generally, tile space-time with polygons. We refer to these polygons as plaquettes. Each plaquette consists of some number of vertices and an equal number of edges. We use γ as an index for the set of all edges in Σ. The tiling may be arbitrarily crude, except it must be fine enough to encode the fundamental group $\pi_1(\Sigma)$.

2. To each edge γ, assign an element g_γ of the gauge group G. Additionally, to each plaquette P assign an area ϱ_P.

3. For each plaquette P, calculate the holonomy H_P, defined as the oriented product of group elements about plaquette P. From this, construct the following sum over all irreducible representations of G:

$$\Gamma_P(H_P, \varrho_P) = \sum_{R \in \text{irrep(G)}} d_R \, \chi_R(H_P) \, e^{-\frac{e^2 \varrho_P c_2(R)}{2}}, \tag{C.1}$$

   where d_R is the dimension of representation R, $\chi_R(g)$ is the trace over g in reprentation R, and $c_2(R)$ is the quadratic Casimir for $R(G)$.

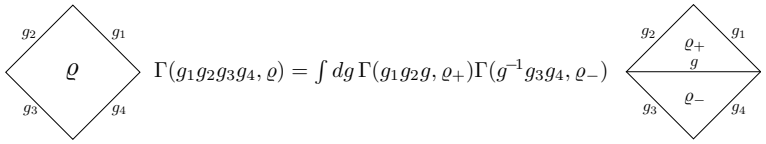
4. The partition function is computed by the following integral over all possible assignments of $\{g_\gamma\}$:

$$Z = \int \prod_{\gamma \in \Sigma} dg_\gamma \prod_{P \in \Sigma} \Gamma_P(H_P, \varrho_P). \tag{C.2}$$

   The $\Gamma_P(H_P, \varrho_P)$ can be thought of as local Boltzmann weights in a lattice path integral.[43] It is easy to show that this integral is invariant under refinement of the triangulation.[44] We

[43] In writing an integral over group elements, we are working in the langauge of continuous groups. However this analysis should work equally well for finite groups with the integral replaced by a summation.

[44] For example:



are therefore justified in taking (C.2) to be the partition function of YM_2 in the continuum (the limit in which triangles have zero area).

If we restrict ourselves to closed Riemann surfaces Σ, we can use the standard "$4g$-gon construction". Namely, if we consider a closed surface of genus g, the coarsest, and hence simplest, triangulation is a polygon with $4g$ sides, labeled $g_1 h_1 g_1^{-1} h_1^{-1} \cdots g_g h_g g_g^{-1} h_g^{-1}$, where the order of group elements indicates orientation (Fig. C.1).

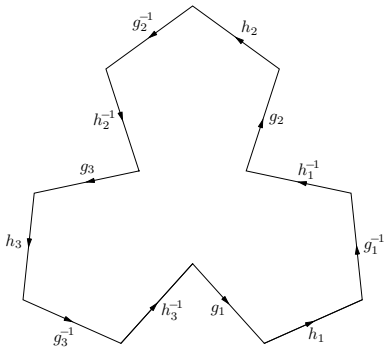


Figure C.1: Example: a one plaquette triangulation of the 3-torus.

From (C.1) and (C.2), we thus find

$$Z(\varrho, g) = \int [dg\, dh] \sum_{R \in \text{ irrep(G)}} d_R \, \chi_R \left(g_1 h_1 g_1^{-1} h_1^{-1} \cdots g_g h_g g_g^{-1} h_g^{-1} \right) \, e^{-\frac{e^2 \varrho c_2(R)}{2}},$$

where $[dg\, dh] \equiv dg_1 \cdots dg_g \, dh_1 \cdots dh_g$. By repeated use of the identities

$$\int dg \; \chi_R(a\, g\, b\, g^{-1}) = d_R^{-1} \, \chi_R(a) \, \chi_R(b), \qquad \int dg \; \chi_R(a\, g) \, \chi_{R'}(g^{-1}\, b) = d_R^{-1} \, \chi_R(a\, b) \, \delta_{RR'}, \tag{C.3}$$

along with

$$\chi_R(\mathbf{1}) = d_R, \tag{C.4}$$

we find

$$Z(\varrho, g) = \sum_{R \in \text{ irrep(G)}} d_R^{2-2g} e^{-\frac{e^2 \varrho \, c_2(R)}{2}}. \tag{C.5}$$

C.2 Wilson Loop Insertions in Gauge Theory

We now consider the insertion of Wilson loops. A Wilson loop is the trace of the holonomy around an oriented loop with a representation assigned, $W_{C(\varrho_C, R_C)} = \chi_{R_C}(H_C)$. The area enclosed by the contour C is denoted ϱ_C. It is a simple matter to include such objects in the lattice theory using the Migdal construction. We will consider various inequivalent ways of embedding Wilson loops into our geometry.[45]

[45] For the remainder of this appendix, we now work in the langauge of generalized YM_2, where the area-dependent exponentials are of the form $e^{\text{area} \times B_R}$. For the specific example of straight YM_2, simply substitute $B_R \to -\frac{e^2 c_2(R)}{2}$.

C.2.1 Contractible Wilson Loops

First we consider the expectation value of a contractible loop. We do this by making a slight modification to the $4g$-gon construction, adding an additional internal loop over which we trace (Fig. C.2).

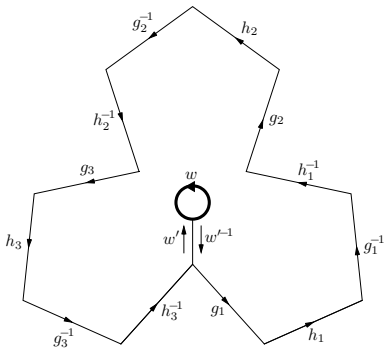


Figure C.2: Example of a 3-torus with a contractible Wilson loop (labeled by group element w and represented by a dark line). This is a two plaquette construction.

We take the internal loop to be the path over which the connection is integrated. For a Wilson loop encompassing area ϱ_C and assigned representation R_C, the expectation value is

$$\left\langle W_{C(\varrho_C,R_C)}\right\rangle_{\Sigma_{g,\varrho}} \tag{C.6}$$

$$= \sum_{R,R'} d_R d_{R'} e^{(\varrho-\varrho_C)B_R} e^{\varrho_C B_{R'}} \left\{ \int [dg\, dh] dw' dw\, \chi_R \left(\prod_{\sigma=1}^{g} g_\sigma h_\sigma g_\sigma^{-1} h_\sigma^{-1} \cdot w' w^{-1} w'^{-1} \right) \chi_{R'}(w)\, \chi_{R_C}(w) \right\}$$

$$= \sum_R d_R^{2-2g}\, e^{(\varrho-\varrho_C)B_R}\, f_{R,(\varrho_C,R_C)}, \tag{C.7}$$

where we have again made repeated use of the character identities (C.3) and (C.4). In (C.7) we have made the convenient definition

$$f_{R,(\varrho_C,R_C)} \equiv \sum_{R'} \frac{d_{R'}}{d_R}\, e^{\varrho_C B_{R'}} \int dw\, \chi_{\bar{R}}(w)\, \chi_{R'}(w)\, \chi_{R_C}(w).$$

The generalization to N non-intersecting contractible Wilson loops is straightforward. The expectation value is

$$\left\langle W_{C_1(\varrho_1,R_1)} \cdots W_{C_N(\varrho_N,R_N)}\right\rangle_{\Sigma_{g,\varrho}} = \sum_R d_R^{2-2g}\, e^{(\varrho-\varrho_C)B_R}\, f_{R,(\varrho_1,R_1)} \cdots f_{R,(\varrho_N,R_N)}. \tag{C.8}$$

C.2.2 Non-Contractible Wilson Loop

As a second example, we calculate the expectation value of a non-contractible Wilson loop, $W_{C_\alpha(R_{C_\alpha})}$, encircling a single handle (Fig. C.3).

For a non-contractible loop, we do not specify an enclosed area since there is no sensible way

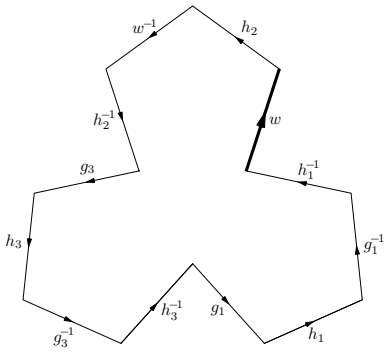


Figure C.3: Example of a 3-torus with a non-contractible Wilson loop (labeled by group element w and represented by a dark line). This is a one plaquette construction.

in which to assign one. The expectation value is

$$\begin{aligned}\left\langle W_{C_\alpha(R_{C_\alpha})}\right\rangle_{\Sigma_{g,\varrho}} &= \sum_R d_R e^{\varrho B_R}\int [dg\,dh]\,\chi_R\left(\prod_{\sigma=1}^{g} g_\sigma h_\sigma g_\sigma^{-1}h_\sigma^{-1}\right)\chi_{R_{C_\alpha}}(g_\alpha)\\ &= \sum_R d_R^{2-2g} e^{\varrho B_R} F_{R,R_{C_\alpha}},\end{aligned} \tag{C.9}$$

where we have made the definition

$$F_{R,R_{C_\alpha}} \equiv \int dg\;\chi_R(g)\,\chi_{\bar{R}}(g)\,\chi_{R_{C_\alpha}}(g).$$

C.2.3 Multiple Non-Contractible Wilson Loops

Finally, if we have N non-intersecting Wilson loops encircling N distinct handles, we find

$$\left\langle W_{C_{\alpha_1}(R_{C_{\alpha_1}})}\cdots W_{C_{\alpha_N}(R_{C_{\alpha_N}})}\right\rangle_{\Sigma_{g,\varrho}} \tag{C.10}$$

$$\begin{aligned}&= \sum_R d_R e^{\varrho B_R}\int [dg\,dh]\,\chi_R\left(\prod_{\sigma=1}^{g} g_\sigma h_\sigma g_\sigma^{-1}h_\sigma^{-1}\right)\chi_{R_{C_{\alpha_1}}}(g_{\alpha_1})\cdots\chi_{R_{C_{\alpha_N}}}(g_{\alpha_N})\\ &\equiv \sum_R d_R^{N+1-2g} e^{\varrho B_R} F_{R,\{R_m\}},\end{aligned} \tag{C.11}$$

where we have made the definition

$$F_{R,\{R_1,\ldots,R_N\}} \equiv \int dg_1\cdots dg_N\;\chi_R(g_1\ldots g_N)\;\chi_{\bar{R}}(g_1)\cdots\chi_{\bar{R}}(g_N)\;\chi_{R_1}(g_1)\cdots\chi_{R_N}(g_N).$$

C.3 Entanglement Entropy with Wilson Loops

We can now calculate entanglement entropies in the presence of Wilson loops via the replica trick. We start by dividing our surface into two regions, A and B. We consider the same A/B partitioning used in §5; namely each region, A or B, is composed of a discrete set of disjoint space-like loops.

C.3.1 Entanglement Entropy with a Contractible Wilson Loop

First let us consider the case in which our unreplicated space-time has a single contractible Wilson loop of representation R_C and area ϱ_C. When we cut and paste, we end up with an n-sheeted surface of genus $n(g-1)+1$ with n non-intersecting, contractible Wilson loops, each with representation R_C and area ϱ_C. From (C.8) we have

$$\left\langle \prod_{m=1}^{n} W_{C_m(\varrho_C,R_C)} \right\rangle_{\Sigma(n(g-1)+1,n\varrho)} = \sum_R d_R^{n(2-2g)} e^{n(\varrho-\varrho_C)B_R} f^n_{R,(\varrho_C,R_C)}$$

The replica trick entanglement entropy calculation is

$$\begin{aligned} S &= -\lim_{n\to 1} \frac{d}{dn} \frac{\left\langle \prod_{m=1}^{n} W_{C_m(\varrho_C,R_C)} \right\rangle_{\Sigma(n(g-1)+1,n\varrho)}}{\left\langle W_{C(\varrho_C,R_C)} \right\rangle^n_{\Sigma_{g,\varrho}}} \\ &= -\sum_R \frac{d_R^{2-2g} e^{(\varrho-\varrho_C)B_R} f_{R,(\varrho_C,R_C)}}{\sum_{R'} d_{R'}^{2-2g} e^{(\varrho-\varrho_C)B_{R'}} f_{R',(\varrho_C,R_C)}} \ln \left[\frac{d_R^{2-2g} e^{(\varrho-\varrho_C)B_R} f_{R,(\varrho_C,R_C)}}{\sum_{R'} d_{R'}^{2-2g} e^{(\varrho-\varrho_C)B_{R'}} f_{R',(\varrho_C,R_C)}} \right]. \end{aligned} \tag{C.12}$$

Let us make a couple of observations about this result:
1) When R_C is the trivial representation, the expression for entanglement entropy reduces back to that without a Wilson loop – *i.e.,* equation (5.1). This happens because $e^{-\varrho_C B_R} f_{R,(\varrho_C,\mathbf{1})}$ is independent of R, and hence cancels from (C.12):

$$f_{R,(\varrho_C,\mathbf{1})} = \sum_{R'} \frac{d_{R'}}{d_R} e^{\varrho_C B_{R'}} \int dg\ \chi_{\bar{R}}(g)\chi_{R'}(g) = \sum_{R'} \frac{d_{R'}}{d_R} e^{\varrho_C B_{R'}} \delta_{RR'} = |Z[G]|\ e^{\varrho_C B_R}.$$

2) In the topological limit $e \to 0$, the contractible Wilson loop again provides no contribution to the entanglement entropy. This is now true regardless of the representation R_C. This happens because, in this limit, $f^{\text{topo}}_{R,(\varrho_C,\mathbf{1})}$ is fully independent of R:

$$\begin{aligned} f^{\text{topo}}_R &= \sum_{\bar{R}} \frac{d_{R'}}{d_R} \int dg\ \chi_{\bar{R}}(g)\chi_{R'}(g)\chi_{R_C}(g) = \frac{|G|}{d_R} \sum_{R'} d_{R'} \left\langle \chi_{\bar{R}\otimes R_C}, \chi_{\bar{R}'} \right\rangle \\ &= \frac{|G|}{d_R} \sum_{R'} \mathcal{M}^{\bar{R}\otimes R_C}_{\bar{R}'} d_{\bar{R}'} = \frac{|G|}{d_R} d_{\bar{R}\otimes R_C} = |G| d_{R_C}, \end{aligned}$$

where $\mathcal{M}^{R_{\text{reducible}}}_{R_{\text{irrep}}}$ is the multiplicity of the irrep R_{irrep} in the decomposition of $R_{\text{reducible}}$. The analogues of both of these properties have been observed before in the context of 2+1-dimensional topological Chern-Simons theories [12], in which case they are understood in terms of the corresponding conformal block structures.

C.3.2 Entanglement Entropy with Non-Contractible Wilson Loop straddling regions A and B

Next we consider the case when the unreplicated space-time has a single non-contractible Wilson loop, $W_{C_\alpha(R_{C_\alpha})}$, straddling spatial regions A and B. In this case, when we cut and paste, the Wilson loop is wrapped around all n-sheets. The final pasted configuration is simply an $(n(g-1)+1)$-handled surface with a Wilson loop encircling a single handle. (In Fig. 5.2, the loop would encircle

the big handle in the middle.) From (C.9), the expectation value is

$$\left\langle W_{C_\alpha(R_{C_\alpha})} \right\rangle_{\Sigma(n(g-1)+1,n\varrho)} = \sum_{R'} d_{R'}^{n(2-2g)} e^{n\varrho B_{R'}} F_{R',R}.$$

The entanglement entropy is

$$\begin{aligned} S &= -\lim_{n\to 1} \frac{d}{dn} \frac{\left\langle W_{C_\alpha(R_{C_\alpha})} \right\rangle_{\Sigma(n(g-1)+1,n\varrho)}}{\left\langle W_{C_\alpha(R_{C_\alpha})} \right\rangle_{\Sigma_{g,\varrho}}^n} = -\lim_{n\to 1} \frac{d}{dn} \frac{\sum_{R'} d_{R'}^{n(2-2g)} e^{n\varrho B_{R'}} F_{R',R}}{\left(\sum_{R''} d_{R''}^{2-2g} e^{\varrho B_{R''}} F_{R'',R}\right)^n} \\ &= -\sum_{R'} \frac{d_{R'}^{2-2g} e^{\varrho B_{R'}} F_{R',R}}{\sum_{R''} d_{R''}^{2-2g} e^{\varrho B_{R''}} F_{R'',R}} \ln \left[\frac{d_{R'}^{2-2g} e^{\varrho B_{R'}}}{\sum_{R''} d_{R''}^{2-2g} e^{\varrho B_{R''}} F_{R'',R}} \right]. \end{aligned} \tag{C.13}$$

C.3.3 Entanglement Entropy with Non-Contractible Wilson Loop restricted to region A or region B

If we again consider a non-contractible Wilson loop, $W_{C_\alpha(R_{C_\alpha})}$, but this time straddling two disjoint subregions of spatial region A (or two disjoint subregions of B, but not both), then upon cutting and pasting, we have an $(n(g-1)+1)$-handled surface with Wilson loops encircling n distinct handles. (In Fig. 5.2, this would correspond to a loop encircling n of the small, replicated handles around the pasted geometry.) From (C.11), the expectation value for this configuration is

$$\left\langle \prod_{m=1}^{n} W_{C_{a_m}(R)} \right\rangle_{\Sigma(n(g-1)+1,n\varrho)} = \sum_{R'} d_{R'}^{n(3-2g)+1} e^{n\varrho B_{R'}} F_{R',\vec{R}},$$

where $\vec{R} = \{R, \ldots, R\}$ is the set of n copies of R. The entanglement entropy is

$$\begin{aligned} S &= -\lim_{n\to 1} \frac{d}{dn} \frac{\left\langle \prod_{m=1}^{n} W_{C_{a_m}(R)} \right\rangle_{\Sigma(n(g-1)+1,n\varrho)}}{\langle W_{C_a,R} \rangle_{\Sigma_{g,\varrho}}^n} = -\lim_{n\to 1} \frac{d}{dn} \frac{\sum_{R'} d_{R'}^{n(3-2g)-1} e^{n\varrho B_{R'}} F_{R',\vec{R}}}{\left(\sum_{R''} d_{R''}^{2-2g} e^{\varrho B_{R''}} F_{R'',R}\right)^n} \\ &= -\sum_{R'} \left\{ \frac{d_{R'}^{2-2g} e^{\varrho B_{R'}} F_{R',R}}{\sum_{R''} d_{R''}^{2-2g} e^{\varrho B_{R''}} F_{R'',R}} \ln \left[\frac{d_{R'}^{3-2g} e^{\varrho B_{R'}}}{\sum_{R''} d_{R''}^{2-2g} e^{\varrho B_{R''}} F_{R'',R}} \right] \right. \\ &\qquad \left. + \frac{d_{R'}^{2-2g} e^{\varrho B_{R'}} \, \partial_n F_{R',\vec{R}}|_{n\to 1}}{\sum_{R''} d_{R''}^{2-2g} e^{\varrho B_{R''}} F_{R'',R}} \right\}. \end{aligned} \tag{C.14}$$

References

[1] Robert G Leigh, Onkar Parrikar, and Alexander B Weiss. Holographic geometry of the renormalization group and higher spin symmetries. *Physical Review D*, 89(10):106012, 2014.

[2] Robert G Leigh, Onkar Parrikar, and Alexander B Weiss. Exact renormalization group and higher-spin holography. *Physical Review D*, 91(2):026002, 2015.

[3] Mark Srednicki. Entropy and area. *Phys. Rev. Lett.*, 71:666–669, 1993.

[4] Christoph Holzhey, Finn Larsen, and Frank Wilczek. Geometric and renormalized entropy in conformal field theory. *Nucl. Phys.*, B424:443–467, 1994.

[5] Pasquale Calabrese and John L. Cardy. Entanglement entropy and quantum field theory. *J. Stat. Mech.*, 0406:P06002, 2004.

[6] Shinsei Ryu and Tadashi Takayanagi. Holographic derivation of entanglement entropy from AdS/CFT. *Phys. Rev. Lett.*, 96:181602, 2006.

[7] Masahiro Nozaki, Shinsei Ryu, and Tadashi Takayanagi. Holographic Geometry of Entanglement Renormalization in Quantum Field Theories. *JHEP*, 1210:193, 2012.

[8] Robert C. Myers and Aninda Sinha. Seeing a c-theorem with holography. *Phys. Rev.*, D82:046006, 2010.

[9] Y. Nakagawa, A. Nakamura, S. Motoki, and V. I. Zakharov. Entanglement entropy of SU(3) Yang-Mills theory. 2009.

[10] A. Kitaev and J. Preskill. Topological Entanglement Entropy. *Phys. Rev. Lett.*, 96(11):110404, 2006.

[11] M. Levin and X.-G. Wen. Detecting Topological Order in a Ground State Wave Function. *Phys. Rev. Lett.*, 96(11):110405, 2006.

[12] Shiying Dong, Eduardo Fradkin, Robert G. Leigh, and Sean Nowling. Topological Entanglement Entropy in Chern-Simons Theories and Quantum Hall Fluids. *JHEP*, 0805:016, 2008.

[13] H.-C. Jiang, Z. Wang, and L. Balents. Identifying topological order by entanglement entropy. *Nature Physics*, 8:902, 2012.

[14] S. Depenbrock, I. P. McCulloch, and U. Schollwock. Nature of the spin liquid ground state of the s=1/2 kagome heisenberg model. *Phys. Rev. Lett.*, page 067201, 2012.

[15] Edward Witten. On quantum gauge theories in two dimensions. *Comm. Math. Phys.*, 141(1):153–209, 1991.

[16] M. Fukuma, S. Hosono, and H. Kawai. Lattice Topological Field Theory in Two Dimensions. *Comm. Math. Phys.*, 161:157–176, 1994.

[17] Bruno G. Carneiro da Cunha and Paulo Teotonio-Sobrinho. Quasitopological field theories in two-dimensions as soluble models. *Int. J. Mod. Phys.*, A13:3667–3690, 1998.

[18] G. Evenbly and G. Vidal. Tensor Network States and Geometry. *J. Stat. Phys.*, 145:891–918, 2011.

[19] U. Schollwöck. The density-matrix renormalization group in the age of matrix product states. *Ann. Phys.*, 326:96–192, 2011.

[20] M. B. Hastings. An area law for one-dimensional quantum systems. *J. Stat. Mech.*, 8:24, 2007.

[21] T. Sugihara. Matrix product representation of gauge invariant states in a Bbb Z_2 lattice gauge theory. *JHEP*, 7:22, 2005.

[22] L. Tagliacozzo and G. Vidal. Entanglement renormalization and gauge symmetry. *Phys. Rev. B*, 83(11):115127, 2011.

[23] B. Buyens, J. Haegeman, K. Van Acoleyen, H. Verschelde, and F. Verstraete. Matrix product states for gauge field theories. 2013.

[24] Alexander A. Migdal. Recursion Equations in Gauge Theories. *Sov. Phys. JETP*, 42:413, 1975.

[25] H. Li and F. D. M. Haldane. Entanglement Spectrum as a Generalization of Entanglement Entropy: Identification of Topological Order in Non-Abelian Fractional Quantum Hall Effect States. *Phys. Rev. Lett.*, 101(1):010504, 2008.

[26] Pasquale Calabrese, John Cardy, and Erik Tonni. Entanglement entropy of two disjoint intervals in conformal field theory II. *J. Stat. Mech.*, 1101:P01021, 2011.

[27] Andrey Gromov and Raul A. Santos. Entanglement Entropy in 2D Non-abelian Pure Gauge Theory. 2014.

[28] William Donnelly. Entanglement entropy and nonabelian gauge symmetry. 2014.

[29] Edward Witten. Quantum field theory and the jones polynomial. *Comm. Math. Phys.*, 121(3):351–399, 1989.

[30] Edward Witten. Topological sigma models. *Comm. Math. Phys.*, 118(3):411–449, 1988.

[31] Edward Witten. Mirror manifolds and topological field theory. 1991.

[32] P. Teotonio-Sobrinho, C. Molina, and N. Yokomizo. On two-dimensional quasitopological field theories. *Int. J. Mod. Phys.*, A24:6105–6121, 2009.

[33] Jan de Boer, Erik P. Verlinde, and Herman L. Verlinde. On the holographic renormalization group. *JHEP*, 0008:003, 2000.

[34] Jan de Boer. The Holographic renormalization group. *Fortsch.Phys.*, 49:339–358, 2001.

[35] Enrique Alvarez and Cesar Gomez. Geometric holography, the renormalization group and the c theorem. *Nucl.Phys.*, B541:441–460, 1999.

[36] Emil T. Akhmedov. A Remark on the AdS / CFT correspondence and the renormalization group flow. *Phys.Lett.*, B442:152–158, 1998.

[37] Christof Schmidhuber. AdS - flows and Weyl gravity. *Nucl.Phys.*, B580:121–139, 2000.

[38] M. Henningson and K. Skenderis. The Holographic Weyl anomaly. *JHEP*, 9807:023, 1998.

[39] Vijay Balasubramanian and Per Kraus. A Stress tensor for Anti-de Sitter gravity. *Commun.Math.Phys.*, 208:413–428, 1999.

[40] Kostas Skenderis. Lecture notes on holographic renormalization. *Class.Quant.Grav.*, 19:5849–5876, 2002.

[41] Robert de Mello Koch, Antal Jevicki, Kewang Jin, and Joao P. Rodrigues. AdS_4/CFT_3 Construction from Collective Fields. *Phys.Rev.*, D83:025006, 2011.

[42] Thomas Faulkner, Hong Liu, and Mukund Rangamani. Integrating out geometry: Holographic Wilsonian RG and the membrane paradigm. *JHEP*, 1108:051, 2011.

[43] Idse Heemskerk and Joseph Polchinski. Holographic and Wilsonian Renormalization Groups. *JHEP*, 1106:031, 2011.

[44] Sung-Sik Lee. Holographic description of quantum field theory. *Nucl.Phys.*, B832:567–585, 2010.

[45] Henrique Gomes, Sean Gryb, Tim Koslowski, Flavio Mercati, and Lee Smolin. Why gravity codes the renormalization of conformal field theories. 2013.

[46] Joseph Polchinski. Renormalization and Effective Lagrangians. *Nucl.Phys.*, B231:269–295, 1984.

[47] I.R. Klebanov and A.M. Polyakov. AdS dual of the critical O(N) vector model. *Phys.Lett.*, B550:213–219, 2002.

[48] Mikhail A. Vasiliev. Higher spin gauge theories in four-dimensions, three-dimensions, and two-dimensions. *Int.J.Mod.Phys.*, D5:763–797, 1996.

[49] Mikhail A. Vasiliev. Higher spin gauge theories: Star product and AdS space. 1999.

[50] X. Bekaert, S. Cnockaert, Carlo Iazeolla, and M.A. Vasiliev. Nonlinear higher spin theories in various dimensions. 2005.

[51] Mikhail A. Vasiliev. Holography, Unfolding and Higher-Spin Theory. *J.Phys.*, A46:214013, 2013.

[52] P.A.M. Dirac. Relativistic wave equations. *Proc. Roy. Soc. Lond.*, 155(886):447–459, July 1936.

[53] Markus Fierz and Wolfgang Pauli. On relativistic wave equations for particles of arbitrary spin in an electromagnetic field. *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences*, pages 211–232, 1939.

[54] Eugene Wigner. On unitary representations of the inhomogeneous lorentz group. *Annals of mathematics*, pages 149–204, 1939.

[55] Valentine Bargmann and Eugene P Wigner. Group theoretical discussion of relativistic wave equations. *Proceedings of the National Academy of Sciences of the United States of America*, 34(5):211, 1948.

[56] C Fronsdal. On the theory of higher spin fields. *Il Nuovo Cimento (1955-1965)*, 9:416–443, 1958.

[57] Shau-Jin Chang. Lagrange formulation for systems with higher spin. *Physical Review*, 161(5):1308, 1967.

[58] LPS Singh and CR Hagen. Lagrangian formulation for arbitrary spin. i. the boson case. *Physical Review D*, 9(4):898, 1974.

[59] LPS Singh and CR Hagen. Lagrangian formulation for arbitrary spin. ii. the fermion case. *Physical Review D*, 9(4):910, 1974.

[60] Christian Fronsdal. Massless fields with integer spin. *Physical Review D*, 18(10):3624, 1978.

[61] J Fang and C Fronsdal. Massless fields with half-integral spin. *Physical Review D*, 18(10):3630, 1978.

[62] Steven Weinberg. Photons and gravitons in s-matrix theory: derivation of charge conservation and equality of gravitational and inertial mass. *Physical Review*, 135(4B):B1049, 1964.

[63] Marcus T Grisaru and HN Pendleton. Soft spin 32 fermions require gravity and supersymmetry. *Physics Letters B*, 67(3):323–326, 1977.

[64] C Aragone and S Deser. Consistency problems of hypergravity. *Physics Letters B*, 86(2):161–163, 1979.

[65] Steven Weinberg and Edward Witten. Limits on massless particles. *Physics Letters B*, 96(1):59–62, 1980.

[66] M Porrati. Universal limits on massless high-spin particles. *Physical Review D*, 78(6):065016, 2008.

[67] Juan Maldacena. The large-n limit of superconformal field theories and supergravity. *International journal of theoretical physics*, 38(4):1113–1133, 1999.

[68] E. Sezgin and P. Sundell. Massless higher spins and holography. *Nucl.Phys.*, B644:303–370, 2002.

[69] Robert G. Leigh and Anastasios C. Petkou. Holography of the N=1 higher spin theory on AdS(4). *JHEP*, 0306:011, 2003.

[70] E. Sezgin and P. Sundell. Holography in 4D (super) higher spin theories and a test via cubic scalar couplings. *JHEP*, 0507:044, 2005.

[71] Simone Giombi and Xi Yin. Higher Spin Gauge Theory and Holography: The Three-Point Functions. *JHEP*, 1009:115, 2010.

[72] Simone Giombi and Xi Yin. The Higher Spin/Vector Model Duality. *J.Phys.*, A46:214003, 2013.

[73] Michael R. Douglas, Luca Mazzucato, and Shlomo S. Razamat. Holographic dual of free field theory. *Phys.Rev.*, D83:071701, 2011.

[74] Leopoldo A. Pando Zayas and Cheng Peng. Toward a Higher-Spin Dual of Interacting Field Theories. *JHEP*, 1310:023, 2013.

[75] Ivo Sachs. Higher Spin vs. Renormalization Group Equations. 2013.

[76] Derek K. Wise. MacDowell-Mansouri gravity and Cartan geometry. *Class.Quant.Grav.*, 27:155010, 2010.

[77] Brian P. Dolan. Symplectic geometry and Hamiltonian flow of the renormalization group equation. *Int.J.Mod.Phys.*, A10:2703–2732, 1995.

[78] Nima Doroud and Lee Smolin. An Action for higher spin gauge theory in four dimensions. 2011.

[79] Nicolas Boulanger and Per Sundell. An action principle for Vasiliev's four-dimensional higher-spin gravity. *J.Phys.*, A44:495402, 2011.

[80] Nicolas Boulanger, Nicolo Colombo, and Per Sundell. A minimal BV action for Vasiliev's four-dimensional higher spin gravity. *JHEP*, 1210:043, 2012.

[81] Jean Thierry-Mieg. Geometrical reinterpretation of Faddeev-Popov ghost particles and BRS transformations. *J.Math.Phys.*, 21:2834–2838, 1980.