MULTI-PERIOD DEMAND RESPONSE MANAGEMENT IN THE SMART GRID: A STACKELBERG GAME APPROACH

BY

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THESIS

Submitted in partial fulfillment of the requirements
for the degree of Master of Science in Electrical and Computer Engineering
in the Graduate College of the
University of Illinois at Urbana-Champaign, 2015

Urbana, Illinois

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ABSTRACT

This thesis studies a multi-period demand response management problem in the smart grid where multiple utility companies compete among themselves. The user-utility interactions are modeled by a noncooperative game of a Stackelberg type where the interactions among the utility companies are captured through a Nash equilibrium. It is shown that this game has a unique Stackelberg equilibrium at which the utility companies set prices to maximize their revenues (within a Nash game) while the users respond accordingly to maximize their utilities subject to their budget constraints. Closed-form expressions are provided for the corresponding strategies of the users and the utility companies. It is shown, both analytically and numerically, that the multi-period scheme, compared with the single-period one, provides more incentives for energy consumers to participate in demand response programs. A necessary and sufficient condition on the minimum budget needed for a user to participate is provided. The large population regime is also investigated and an appropriate company-to-user ratio is provided.

To my grandmother, Halyah Alshehri, who passed away during the preparation of this thesis. I will always remember your endless love and emotional support. I will always miss hugging you.

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LIST OF NOTATION

$\mathcal{N}$	Set of users
$\mathcal{K}$	Set of utility companies
$\mathcal{T}$	Set of periods
N	Number of users
K	Number of utility companies
T	Number of periods
$d_{n,k}(t)$	The demand of user $n \in \mathcal{N}$ from utility company $k \in \mathcal{K}$ at time $t \in \mathcal{T}$
$p_k(t)$	The price of energy announced by utility company $k \in \mathcal{K}$ at time $t \in \mathcal{T}$
B_n	The budget of user $n \in \mathcal{N}$
$E_n^{\min}$	The total minimum energy needed by user $n \in \mathcal{N}$ during the entire time horizon
$G_k(t)$	The power availability of utility company $k \in \mathcal{K}$ at time $t \in \mathcal{T}$

CHAPTER 1

INTRODUCTION

The smart grid is envisioned to be a reliable, secure, and self-healing power network for the 21st century that incorporates various sources of energy [1, 2]. It is a large-scale network of intelligent nodes that can communicate, operate, and interact autonomously for reliable and efficient power delivery. The accommodation of this vision requires the adoption of computer, sensing, communication, and control technologies in parallel with the electric power network [3–5]. Interestingly, the smart grid can be looked at as a large-scale control problem [6]. A general overview of the smart grid can be found in [1, 2]. For a background on systems and control challenges and opportunities in the smart grid, see [6, 7].

Demand side management (DSM) is an essential component of the smart grid as it captures important aspects of the interactions between utility companies (UCs) and consumers, including residential, commercial, industrial users and vehicles [4]. DSM is categorized into different components, with the aim of improving the energy efficiency in both the short-term and the long-term, see [4] and [8], respectively. These components can be technical, such as using advanced metering infrastructure to improve the reliability and efficiency of the grid [4], or social through agreements between consumers and energy providers [5]. Examples of DSM programs include energy efficiency, time-of-use pricing, and demand response [8]. Energy efficiency is about the long-term solutions to improve the overall efficiency; one example is replacing the air conditioning system of a building with a more efficient one. Through time-of-use pricing, companies can agree with the users that different prices are to be charged at different periods; these prices are agreed upon beforehand by both parties. A tutorial on the demand-side view of electricity markets can be found in [9].

Demand response management (DRM) is the response of consumers' demands to price signals from the utility companies [10–12]. DRM allows companies to manage the consumers' demands, either directly or indirectly, through incentive-based programs [11–13]. One of the main goals of demand response management is to provide more incentives for consumers' participation in order to minimize the peak-to-average consumption ratio. A comprehensive survey on the pricing methods and optimization algorithms for demand response programs can be found in [14]. For an overview of the methodologies and the challenges of load/price forecasting and managing demand response in the smart grid, see [15]. It is worth mentioning that using the framework of game theory, the idea of load adaptive pricing goes back decades [16].

Game theory is a powerful mathematical tool that can lead to effective multi-person decision making [17–20]. Due to the nature of the smart grid which features different entities with conflicting objectives, applying game-theoretic methods can improve their reliability and efficiency [4, 21–42]. In this thesis, we concentrate on multi-period demand response management.

Game theory can be classified into two main categories: cooperative games and noncooperative games [17, 18]. A game is cooperative if it allows the participants to cooperate in order to achieve a better outcome as a group. A game is called noncooperative if each player takes the decisions that are in favor of his/her self-interest, while these decisions are usually conflicting with the ones made by other players. In this thesis, we focus on noncooperative games because of their relevance to the nature of demand response management. A noncooperative game can be either static or dynamic. In static games, the actions taken by the players are independent of the time and information, but in dynamic games, the order in which decisions are made and the information players have access to can influence the outcome.

A static N-person noncooperative game consists of three components: players set, action sets, and utility functions. Each player has an action set that he/she wishes to maximize his/her utility function over. One key point is that the utility function of each player depends not only on his/her actions, but also on the decisions made by other players. An equilibrium concept that is suitable for such games is the Nash equilibrium. At the equilibrium point, given the decisions made by other players, a player cannot benefit by deviating from his/her action. Moreover, the Nash equilibrium does not necessarily always exist, and one may have to introduce some conditions on the utility function and/or action sets to ensure existence, or expand the strategy spaces to include probability distributions [17].

The Nash equilibrium solution concept is reasonable when the decision process is not dominated by any of the actions taken by any of the players. Sometimes it would be beneficial to allow for hierarchy in the decision process. In such a case, there are two types of players: leaders and followers. The leaders' decisions are more dominant, and the followers respond to the decisions taken by the leaders. This kind of hierarchal game is called a Stackelberg game, and the corresponding solution concept is called the Stackelberg equilibrium. For the Stackelberg equilibrium to exist in the standard sense and not lead to ambiguity, each follower's optimal response to the actions taken by the leaders has to be unique.[1] The leaders have the privilege of choosing how to take their actions at the beginning of the game. However, they have to take into account how the followers would respond to these actions and how each leader's decision is influenced by the decisions of the other leaders.

1.1 Literature Review

There are three main components of the smart grid in which game theory provides promising tools [4]. These are: microgrids, communications, and demand-side management. For a comprehensive survey of game-theoretic methods for these three components, we refer to [4]. A microgrid is located on the distribution side of the power network and can be looked at as a collection of energy sources that serves a particular area. An energy exchange game was developed in [22] where the players are the microgrids. Each microgrid serves a geographical area and has either a surplus of energy or a shortage of energy. In this game, microgrids form coalitions and hence a local energy exchange market when they find it beneficial to do so. The prices and the amounts of energy to be sold are determined by auction theory and the coalitions are decided through coalition formation games (they are cooperative). For smart grid communications, the authors of [23] designed a network formation game to minimize the communication delay between the smart elements and a common access point. The players in this game are the smart elements, and each one of them selects a preferred partner to forward its packets to, in order to minimize the overall delay. This game was found to have a Nash equilibrium.

There are several works where game-theoretic methods have been applied to DSM and demand response, and improved the reliability and efficiency of the grid [25–38,43]. An autonomous DSM through scheduling of appliances has been

[1] Otherwise one has to extend the notion of Stackelberg equilibrium to "robust" equilibrium where non-unique responses of followers are also accommodated [17].

implemented within a noncooperative game framework in [25]. The participants in the game are energy users who are connected to the same utility company, and the outcome of the game is the power consumption schedule of appliances that minimizes the overall energy cost. Two kinds of appliances were considered: shiftable and non-shiftable. Each user selects the schedule of appliances while meeting his/her energy needs. Users play a noncooperative game and each user aims to minimize his/her payment to the utility company. Interestingly, simulation results show that this energy scheduling game can provide solutions that are close to the solution of the peak-to-average ratio minimization problem. This is not always the case as the pricing can vary depending on the time of the day. A recent extension adds energy storage into the picture [28], where a Stackelberg game was developed between the utility company and the end-users. The leader of the game is the utility company, which aims to maximize its profits by selecting optimal prices. End-users are the followers in this hierarchical game. This work also shows that the Stackelberg game is actually a generalization of the peak-to-average ratio problem. These energy storage and scheduling works [25, 28] are important because they cover different aspects. However, each of them is somewhat restrictive. For example, the scheduling of appliances game [25] did not consider the case when the end-users are able to have energy storage. Energy scheduling and storage are usually co-related [4]. In [29], a noncooperative game to reduce peak-to-average ratio has been proposed. Users decide when to charge their batteries, so that the overall peak-to-average ratio is minimized. Accordingly, a single utility company updates the prices. A Nash equilibrium exists for this game and the computation of the equilibrium was done by a distributed algorithm. Although these works are limited to the single company case, they show how game theory can be useful and provides an appropriate framework when it comes to the multi-period consideration for demand-side management.

Plug-in hybrid electric vehicles can sometimes have a surplus of energy, and it can be more beneficial to sell energy quickly instead of holding onto it. This motivates the work of [26] in which an energy trading noncooperative game between vehicle groups and distribution grids has been studied. In [27], joint consumers discomfort and billing costs minimization within a repeated game framework has been shown to lead to an optimal DSM mechanism.

Because of the two-way communication infrastructure in the smart grid [5], there are numerous interactions between utility companies and the end-users. In order to take this fact into account, utilizing a multi-level framework such as

Stackelberg games would be very useful [28, 30–33, 35–38]. A multi-level game-theoretic framework has been developed for demand response management, where consumers choose their optimal demands in response to prices announced by different utility companies [30]. This Stackelberg game was shown to have a unique Stackelberg equilibrium at which utility companies maximize their revenues and end-users maximize their payoff functions. In this framework, utility companies were the leaders of the game and users were the followers. In [31], a Stackelberg game for demand response management in a large population regime has been proposed. In this game, the leaders are the utility companies and they aim to maximize their profits, and the followers are the end-users who wish to maximize their welfare. It was shown that a unique number of utility companies exists at which profits are being maximized. These works, though they effectively capture user-utility interactions, are limited to the single-period scenario. In the smart grid, users are expected to be able to schedule their energy consumption, store or sell surplus energy, based on their self-interest. With energy scheduling and storage, users might have flexibility on when to receive a certain amount of energy, particularly for shiftable appliances [25]. Some energy consumption can be scheduled and some cannot. For example, users might be flexible about doing the laundry, but not much flexibility is there for refrigerators.

It is worth mentioning that multi-level games for DRM have been studied in a limited context in the literature. For example, a four-stage Stackelberg game has been studied where three stages are at the leader-level (the utility retailer), and the fourth stage is at the consumer level [32]. Retailers choose the amount of energy to procure, and the sources to produce it, in addition to deciding on the price. Consumers respond to these prices through demand selection. This game is also a single-period game and it does not take into account the competition between the utility companies, but it incorporates other aspects of the decision making at the company-level. Additionally, a noncooperative Stackelberg game between plug-in electric vehicle groups such as parking lots and the smart grid was formulated in [33] and a socially optimal equilibrium has been obtained. A two-level game (a noncooperative game between multiple utility companies and an evolutionary game for the users at the lower level) has been proposed in [34], but this game is also limited to the single-period case. In [35], a Stackelberg game for three-party energy management was shown to have a unique equilibrium. The leader of the game is a shared facility controller, and the followers are energy users, and each user is equipped with distributed energy resources, with no storage capacity. The computation of the Stackelberg equilibrium used a distributed algorithm. Furthermore, a Stackelberg game between a demand response aggregator (the leader) and electricity generators (the followers) has been proposed in [37].

Motivated by the limitations of existing works above, this thesis aims to propose a multi-period-multi-company game-theoretic framework for demand response management in the smart grid. Such a generalization can consider the market competition between utility companies, along with the multi-period considerations at the user and the utility sides. Our goal is to have the generalization flexible enough so that the effects of energy scheduling and storage on a market with multiple utility companies can be studied analytically. Moreover, one of the goals of the demand response management is to minimize the peak-to-average ratio; such a goal is clearly a multi-period issue, which cannot be studied in a single-period model.

There is a need to develop an analytical multi-period, multi-utility, and a multi-level framework. By introducing multi-period inter-temporal constraints, we could study a generalization of [30] to the multi-period case. Our work differs from the work in [30] at both the user-side and the company-side. At the user-side, we have an additional minimum energy constraint that needs to be satisfied across all periods, while at the company-side, we provide an alternative computationally cheap closed-form solution for the prices. Having such a multi-period framework can make it possible to accommodate numerous extensions, such as energy scheduling and storage, and peak-to-average ratio minimization.

Accordingly, we formulate in this thesis a Stackelberg game for multi-period-multi-company demand response management. We derive solutions in closed form and find precise expressions for the maximizing demands at the users' level, and the revenue-maximizing prices for the utility companies. We also prove the existence and uniqueness of the Stackelberg equilibrium, and propose a distributed algorithm to compute it using only local information. Furthermore, we exploit the closed-form expressions to formulate a new power allocation game, study the asymptotic behavior when the number of periods is large, and find an appropriate company-to-user ratio for the large population regime. This work captures the competition between utility companies, budget limitations at the consumer-level, energy need for the entire time-horizon, and details opportunities for future directions.

1.2 Overview of Chapters

The balance of this thesis is organized as follows. The problem is formulated in chapter 2 for both energy users and utility companies. In chapter 3, the Stackelberg game is analyzed and solutions in closed form are derived. Additionally, the existence and uniqueness of the equilibrium point is shown, and a distributed algorithm is proposed to compute it using only local information. Furthermore, a new power allocation game at the companies side is proposed. In chapter 4, the asymptotic behavior is studied when the number of periods or the number of users is large. Numerical results illustrating the effectiveness of the multi-period Stackelberg framework and the distributed algorithm are included in chapter 5. We conclude the thesis in chapter 6 with a recap of main points and identification of future directions.

CHAPTER 2

PROBLEM FORMULATION

The system consists of $K \geq 1$ UCs and $N \geq 1$ end-users. We consider a finite time horizon with $T \geq 1$ periods. Let $\mathcal{K} = \{1, 2, \ldots, K\}$ be the set of UCs, $\mathcal{N} = \{1, 2, \ldots, N\}$ be the set of end-users, and $\mathcal{T} = \{1, 2, \ldots, T\}$ be the finite set of time slots. The unit of time can be an hour, a day, a week, or a month. Mathematically speaking, it does not matter which unit the set $\mathcal{T}$ represents.

We model the interaction between UCs and end-users as a Stackelberg game. Thus, the DRM problem is formulated as two optimization problems, one for the followers of the game and the other one for the leaders of the game. The followers of the game are the consumers, who can be residential, commercial, or industrial, and the leaders of the game are the UCs. On the followers (users) side, each user has a certain amount of energy demand to meet and we assume that the users have flexibility in scheduling their consumption of energy. It is also assumed that users have limited budgets. The goal of each user is to maximize her utility without exceeding the budget while meeting the minimum energy demand. On the leaders (UCs) side, the goal of each UC is to maximize its revenue while its price setting is influenced by both the competition against other UCs and the behavior of the users. In the game considered here, the UCs announce their prices for each period to the users, and then the users respond accordingly by scheduling their demands. We discuss the underlying models for both sides separately below.

2.1 User-Side

Because of energy scheduling and storage, users may have some flexibility on when to receive a certain amount of energy. In our analysis, we are concerned about the total amount of shiftable energy. For non-shiftable energy, one can add some period-specific constraints. Additionally, each user has a budget constraint that she cannot exceed for the entire time horizon.

The demand of user $n \in \mathcal{N}$ from UC $k \in \mathcal{K}$ at time $t \in \mathcal{T}$ is denoted by $d_{n,k}(t)$, and $p_k(t)$ denotes the price of energy announced by UC k at time t. Each end-user n has a budget of B_n, and a minimum amount of energy to be met, denoted by $E_n^{\min}$. The value of $E_n^{\min}$ can be thought of as the total energy needed by user n during the considered total period (for example, one day or one week). User payoff functions are increasing functions of the available energy. The utility of user n is defined as

$$U_{\text{user},n} = \gamma_n \sum_{k \in \mathcal{K}} \sum_{t \in \mathcal{T}} \ln(\zeta_n + d_{n,k}(t)) \tag{2.1}$$

where γ_n and ζ_n (typically, $\zeta_n = 1$) are user specific parameters. The logarithmic function is well known to provide a good demand response [30, 44–47]. Users aim to end up with high payoffs while meeting the threshold of minimum amount of energy and not exceeding a certain budget. To be more precise, given $B_n \geq 0$, $E_n^{\min} \geq 0$, and $p_k(t) > 0$, the user-side optimization problem is formulated as follows:

$$\begin{aligned}
\underset{\mathbf{d_{n,k}}}{\text{maximize}} \quad & U_{\text{user},n} \\
\text{subject to} \quad & \sum_{k \in \mathcal{K}} \sum_{t \in \mathcal{T}} p_k(t) d_{n,k}(t) \leq B_n && (2.2) \\
& \sum_{k \in \mathcal{K}} \sum_{t \in \mathcal{T}} d_{n,k}(t) \geq E_n^{\min} && (2.3) \\
& d_{n,k}(t) \geq 0, \quad \forall k \in \mathcal{K}, \quad \forall t \in \mathcal{T} && (2.4)
\end{aligned}$$

Note that there is no game played among the users. Each user responds to the price signals using only her local information. These price signals depend on all the demands selected by the users and hence users indirectly affect each other's decisions, that is, they are coupled through the prices picked by the companies.

2.2 Company-Side

Given the prices of other UCs, denoted by $\mathbf{p_{-k}}$, the total revenue for UC k is given by

$$U_{\text{gen},k}(p_k, \mathbf{p_{-k}}) = \sum_{t \in \mathcal{T}} p_k(t) \sum_{n \in \mathcal{N}} d_{n,k}(t) \tag{2.5}$$

Let $G_k(t)$ denote the power availability of UC k at period t. The utility-side optimization problem for each company is then described for each set of fixed prices of other companies, $\mathbf{p_{-k}}$, as follows:

$$\begin{aligned} \underset{\mathbf{p_k}}{\text{maximize}} \quad & U_{\text{gen},k}(p_k, \mathbf{p_{-k}}) \\ \text{subject to} \quad & \sum_{n \in \mathcal{N}} d_{n,k}(t) \leq G_k(t), \;\; \forall\, t \in \mathcal{T} \qquad (2.6) \\ & p_k(t) > 0, \;\; \forall\, t \in \mathcal{T} \qquad (2.7) \end{aligned}$$

The goal of each UC is to maximize its revenue and hence maximize its profit. Also, companies want to meet the consumers' demands without exceeding the maximum power availability (overloading the system can cause contingencies). Additionally, because of the market competition, the prices announced by other companies also affect the determination of the price at company k. So, company k's price selection is actually a response to what other UCs in the market have announced; this response is also constrained by the availability of power. Thus, what we have is a Nash game among the utility companies.

CHAPTER 3

THE STACKELBERG GAME

In this chapter, we first solve the optimization problems introduced in the previous chapter in closed form. Then, we study the existence and uniqueness of Nash equilibrium at the UC-level and the Stackelberg equilibrium for the entire two-level game and show their connection to the closed-form solutions. Furthermore, we devise a distributed algorithm to compute the equilibrium point using only local information. Finally, we further extend the results by formulating a power allocation game at the UC-level that aims to answer the following question: Given a number of periods T, how can each UC allocate its available power across the periods to maximize its revenue?

3.1 Followers-Side Analysis

The user-side utility function is strictly concave and the constraints are linear; see [48, 49] for details about analyzing and solving such problems. We start by relaxing the minimum energy constraint (2.3) and then find the necessary budget that makes the maximizing demands feasible. For each user $n \in \mathcal{N}$, the associated Lagrange function is given as follows:

$$\begin{aligned} L_{\text{user},n} &= \gamma_n \sum_{k\in\mathcal{K}} \sum_{t\in\mathcal{T}} \ln(\zeta_n + d_{n,k}(t)) \\ &\quad - \lambda_{n,1} \left(\sum_{k\in\mathcal{K}} \sum_{t\in\mathcal{T}} p(t)_k d_{n,k}(t) - B_n \right) \\ &\quad + \sum_{k\in\mathcal{K}} \sum_{t\in\mathcal{T}} \lambda_{n,2}(k,t) d_{n,k}(t) \end{aligned}$$

where $\lambda_{n,i}$'s are the Lagrange multipliers. For optimality, by Krush-Kuhn-Tucker necessary conditions [49], we need

$$\frac{\partial L_{user,n}}{\partial d_{n,k}(t)} = 0 \;\forall\, t \in \mathcal{T}, k \in \mathcal{K} \tag{3.1}$$

$$\lambda_{n,1}\left(\sum_{k\in\mathcal{K}}\sum_{t\in\mathcal{T}} p_k(t) d_{n,k}(t) - B_n\right) = 0 \tag{3.2}$$

$$\lambda_{n,2}(k,t) d_{n,k}(t) = 0 \;\forall\, t \in \mathcal{T}, k \in \mathcal{K} \tag{3.3}$$

$$\lambda_{n,1}, \lambda_{n,2}(k,t) \geq 0 \;\forall\, t \in \mathcal{T}, k \in \mathcal{K} \tag{3.4}$$

The above conditions are also sufficient because of the concavity of the optimization problem [48]. In the sequel, we derive the solution in closed form by considering two separate cases.

Case 1: $d_{n,k}(t) > 0, k \in \mathcal{K}, t \in \mathcal{T}$

In this case, $\lambda_{n,1} > 0$ (constraint (2.2) is active [49]), and $\lambda_{n,2}(k,t) = 0 \;\forall t \in \mathcal{T}, k \in \mathcal{K}$. Thus,

$$\frac{\partial L_{user,n}}{\partial d_{n,k}(t)} = \frac{\gamma_n}{\zeta_n + d_{n,k}(t)} - \lambda_{n,1} p_k(t) = 0$$

which implies that

$$d_{n,k}(t) = \frac{\gamma_n}{\lambda_{n,1} p_k(t)} - \zeta_n \quad \forall\, t \in \mathcal{T}, k \in \mathcal{K} \tag{3.5}$$

Plugging (3.5) into (3.2) leads to

$$\sum_{k\in\mathcal{K}}\sum_{t\in\mathcal{T}}(\frac{\gamma_n}{\lambda n, 1} - p_k(t)\zeta_n) = B_n$$

So,

$$\lambda_{n,1} = \frac{KT\gamma_n}{B_n + \sum_{k\in\mathcal{K}}\sum_{t\in\mathcal{T}} p_k(t)\zeta_n}$$

Thus,

$$d_{n,k}(t) = \frac{B_n + \sum_{k\in\mathcal{K}}\sum_{t\in\mathcal{T}} p_k(t)\zeta_n}{KTp_k(t)} - \zeta_n \quad \forall\, t \in \mathcal{T}, k \in \mathcal{K} \tag{3.6}$$

which is a generalization of the single-period case considered in [30].

Case 2: at least one $d_{n,k}(t)$ is zero

We show that the expression (3.6) also holds for

$$d_{n,k}(t) \geq 0, k \in \mathcal{K}, t \in \mathcal{T}$$

Without loss of generality, suppose that $d_{n,1}(1) = 0$ and $d_{n,e}(f) > 0$ for $e \in \mathcal{K}$, $f \in \mathcal{T}$, except when $(e, f) = (1, 1)$. Following a similar analysis as in the previous case,

$$d_{n,e}(f) = \frac{B_n + \sum_{e \in \mathcal{K}} \sum_{f \in \mathcal{T}} p_e(f)\zeta_n}{(KT-1)p_e(f)} - \zeta_n$$

Note that since $d_{n,1}(1) = 0$, we have

$$\zeta_n = \frac{B_n + \sum_{k \in \mathcal{K}} \sum_{t \in \mathcal{T}} p_k(t)\zeta_n}{KTp_1(1)}$$

and hence

$$B_n = \zeta_n(KTp_1(1) - \sum_{k \in \mathcal{K}} \sum_{t \in \mathcal{T}} p_k(t))$$

So,

$$\begin{aligned} d_{n,e}(f) &= \frac{\zeta_n(KTp_1(1) - \sum_{k \in \mathcal{K}} \sum_{t \in \mathcal{T}} p_k(t) + \sum_{e \in \mathcal{K}} \sum_{f \in \mathcal{T}} p_e(f))}{(KT-1)p_e(f)} - \zeta_n \\ &= \frac{(KT-1)\zeta_n p_1(1)}{(KT-1)p_e(f)} - \zeta_n \\ &= \frac{B_n + \sum_{k \in \mathcal{K}} \sum_{t \in \mathcal{T}} p_k(t)\zeta_n}{KTp_e(f)} - \zeta_n \end{aligned} \tag{3.7}$$

which matches the expression (3.6). By the budget constraint

$$\sum_{k \in \mathcal{K}} \sum_{t \in \mathcal{T}} p_k(t) d_{n,k}(t) \leq B_n$$

we can see that when $d_{n,k}(t) = 0 \;\forall\; k \in \mathcal{K}, t \in \mathcal{T}$, user n has a zero budget (i.e., $B_n = 0$). The announced prices cannot be infinite because of the nature of the Stackelberg equilibrium, as we discuss later. The following theorem states the necessary and sufficient condition for B_n so that the above $d_{n,k}(t)$'s are guaranteed to be feasible.

Theorem 1 *For every user $n \in \mathcal{N}$, $d_{n,k}(t)$ given by (3.6) is feasible if and only if*

$$B_n \geq \max\{f_{n,1}, f_{n,2}\}$$

where

$$\begin{aligned} f_{n,1} &= \zeta_n(KTp_k(t) - \sum_{k\in\mathcal{K}}\sum_{t\in\mathcal{T}} p_k(t)) \quad \forall k \in \mathcal{K}, t \in \mathcal{T} \\ f_{n,2} &= \frac{E_n^{min} + \zeta_n KT}{\sum_{k\in\mathcal{K}}\sum_{t\in\mathcal{T}} \frac{1}{KTp_k(t)}} - \zeta_n \sum_{k\in\mathcal{K}}\sum_{t\in\mathcal{T}} p_k(t) \end{aligned}$$

Proof: Suppose first that $B_n \geq f_{n,1}$. Then,

$$B_n \geq \zeta_n(KTp_k(t) - \sum_{k\in\mathcal{K}}\sum_{t\in\mathcal{T}} p_k(t)) \quad \forall k \in \mathcal{K},\ t \in \mathcal{T}$$

With little re-arrangements, we have

$$\frac{B_n + \zeta_n \sum_{k\in\mathcal{K}}\sum_{t\in\mathcal{T}} p_k(t)}{KTp_k(t)} - \zeta_n \geq 0 \quad \forall k \in \mathcal{K},\ t \in \mathcal{T}$$

Thus, the inequality $B_n \geq f_{n,1}$ implies that the non-negativity condition is satisfied. Next suppose that $B_n \geq f_{n,2}$. Then,

$$B_n \geq \frac{E_n^{min} + \zeta_n KT}{\sum_{k\in\mathcal{K}}\sum_{t\in\mathcal{T}} \frac{1}{KTp_k(t)}} - \zeta_n \sum_{k\in\mathcal{K}}\sum_{t\in\mathcal{T}} p_k(t)$$

Thus,

$$B_n + \zeta_n \sum_{k\in\mathcal{K}}\sum_{t\in\mathcal{T}} p_k(t) \geq \frac{E_n^{min} + \zeta_n KT}{\sum_{k\in\mathcal{K}}\sum_{t\in\mathcal{T}} \frac{1}{KTp_k(t)}}$$

from which we have

$$\sum_{k\in\mathcal{K}}\sum_{t\in\mathcal{T}} \frac{B_n + \zeta_n \sum_{k\in\mathcal{K}}\sum_{t\in\mathcal{T}} p_k(t)}{KTp_k(t)} - \sum_{k\in\mathcal{K}}\sum_{t\in\mathcal{T}} \zeta_n \geq E_n^{min}$$

This leads to

$$\sum_{k\in\mathcal{K}}\sum_{t\in\mathcal{T}} d_{n,k}(t) \geq E_n^{min}$$

With this, the inequality $B_n \geq f_{n,2}$ implies that the minimum energy need is satisfied. Combining both conditions, we conclude that the condition

$$B_n \geq \max\{f_{n,1}, f_{n,2}\}$$

guarantees that the maximizing demand in (3.6) is feasible.

Now suppose that

$$\frac{B_n + \zeta_n \sum_{k\in\mathcal{K}} \sum_{t\in\mathcal{T}} p_k(t)}{KTp_k(t)} - \zeta_n \geq 0 \quad \forall k \in \mathcal{K},\ t \in \mathcal{T}$$

It follows that

$$B_n \geq \zeta_n(KTp_k(t) - \sum_{k\in\mathcal{K}} \sum_{t\in\mathcal{T}} p_k(t)) \quad \forall k \in \mathcal{K},\ t \in \mathcal{T}$$

which implies that $B_n \geq f_{n,1}$. Finally, suppose that

$$\begin{aligned} \sum_{k\in\mathcal{K}} \sum_{t\in\mathcal{T}} d_{n,k}(t) &= \sum_{k\in\mathcal{K}} \sum_{t\in\mathcal{T}} (\frac{B_n + \zeta_n \sum_{k\in\mathcal{K}} \sum_{t\in\mathcal{T}} p_k(t)}{KTp_k(t)} - \zeta_n) \\ &\geq E_n^{min} \end{aligned}$$

Then,

$$\sum_{k\in\mathcal{K}} \sum_{t\in\mathcal{T}} \frac{B_n + \zeta_n \sum_{k\in\mathcal{K}} \sum_{t\in\mathcal{T}} p_k(t)}{KTp_k(t)} \geq E_n^{min} + KT\zeta_n$$

By re-arranging, we have

$$B_n \geq \frac{E_n^{min} + \zeta_n KT}{\sum_{k\in\mathcal{K}} \sum_{t\in\mathcal{T}} \frac{1}{KTp_k(t)}} - \zeta_n \sum_{k\in\mathcal{K}} \sum_{t\in\mathcal{T}} p_k(t) = f_{n,2}$$

from which it follows that $B_n \geq \max\{f_{n,1}, f_{n,2}\}$ when the demand is feasible. ■

3.2 Leaders-Side Analysis

Given the prices set by the other companies and subject to the power availability constraint (2.6), each UC (leader) aims to determine its most profitable prices. At the leaders level, there is a noncooperative game in which each UC chooses its optimal prices in response to the prices set by the other UCs. The revenue function of company k is an increasing function of the consumers' demands $\sum_{n\in\mathcal{N}} d_{n,k}(t)$. Thus, the optimality is reached when equality holds in (2.6). We apply the solutions derived in the users-side analysis (which was a function of the prices) here and show that in the case when we have equality in (2.6), the corresponding prices constitute the best response of UC k subject to the prices set by the other UCs.

With equality in (2.6) and relation (3.6), there holds

$$\begin{aligned}&\frac{\sum_{n\in\mathcal{N}} B_n + \sum_{n\in\mathcal{N}} \zeta_n \sum_{k\in\mathcal{K}} \sum_{t\in\mathcal{T}} p_k(t)}{KTp_k(t)}\\&= \sum_{n\in\mathcal{N}} \zeta_n + G_k(t), \quad \forall\, t \in \mathcal{T}\end{aligned}$$

Let $Z = \sum_{n\in\mathcal{N}} \zeta_n$ and $B = \sum_{n\in\mathcal{N}} B_n$. Then, for each company $k \in \mathcal{K}$,

$$B + Z \sum_{k\in\mathcal{K}} \sum_{t\in\mathcal{T}} p_k(t) = KTp_k(t)(G_k(t) + Z), \quad \forall\, t \in \mathcal{T}$$

Note that the double summation includes $p_k(t)$ and all the other prices. Thus,

$$\begin{aligned}B + Z \sum_{e\in\mathcal{K}} \sum_{h\in\mathcal{T}} p_e(h) &= KTp_k(t)(G_k(t) + Z) - p_k(t)Z,\\ &\forall\, t \in \mathcal{T},\ \forall\, k \in \mathcal{K},\ (e,h) \neq (k,t)\end{aligned} \tag{3.8}$$

Note that the equations in (3.8) can be written in the form of a larger dimensional linear equation

$$AP = Y$$

where A is a $KT \times KT$ matrix whose diagonal entries are $KT(G_k(t) + Z) - Z$, $k \in \mathcal{K}$, $t \in \mathcal{T}$, and off-diagonal entries are all equal to $-Z$, P is a vector in $\mathbb{R}^{KT}$ stacking $p_k(t)$, $k \in \mathcal{K}$, $t \in \mathcal{T}$, and Y a vector in $\mathbb{R}^{KT}$ whose entries all equal B. Specifically,

$$A =$$

$$\begin{pmatrix} KT(G_1(1)+Z)-Z & -Z & \dots & -Z \\ -Z & KT(G_1(2)+Z)-Z & \dots & -Z \\ \vdots & \ddots & & \\ -Z & \dots & -Z & KT(G_K(T)+Z)-Z \end{pmatrix}$$

and

$$\begin{aligned} P &= \begin{pmatrix} p_1(1) & \cdots & p_1(T) & p_2(1) & \cdots & p_2(T) & \cdots & p_K(T) \end{pmatrix}^T \\ Y &= \begin{pmatrix} B & B & \cdots & B \end{pmatrix}^T \end{aligned}$$

The following theorem shows that matrix A is invertible and the revenue-maximizing prices are positive and unique.

Theorem 2 *The following statements are true.*

(i) The matrix A is nonsingular and the prices announced by company $k \in \mathcal{K}$ at time $t \in \mathcal{T}$ are uniquely given by

$$p_k(t) = \frac{B}{G_k(t) + Z}\left(\frac{1}{KT - \sum_{k\in\mathcal{K}}\sum_{t\in\mathcal{T}}\frac{Z}{G_k(t)+Z}}\right) \tag{3.9}$$

(ii) The above prices are always positive.

(iii) The price given by (i) constitutes the best response of company k to the prices set by the other companies.

Proof:

(i) The matrix A can be represented as

$$A = \begin{pmatrix} KT(G_1(1)+Z) & 0 & \dots & 0 \\ 0 & KT(G_1(2)+Z) & \dots & 0 \\ \vdots & \ddots & & \\ 0 & \dots & 0 & KT(G_K(T)+Z) \end{pmatrix} + \begin{pmatrix} -Z \\ -Z \\ \vdots \\ -Z \end{pmatrix}\begin{pmatrix} 1 \dots 1 \end{pmatrix} := \hat{A} + uv^T$$

Note that $\hat{A}$ is invertible (diagonal matrix with nonzero diagonal elements). By the Sherman-Morrison formula [50], if $\hat{A}$ is invertible and $1+v^T\hat{A}^{-1}u \neq 0$, then

$$(\hat{A} + uv^T)^{-1} = \hat{A}^{-1} - \frac{\hat{A}^{-1}uv^T\hat{A}^{-1}}{1 + v^T\hat{A}^{-1}u}$$

Note that

$$1 + v^T \hat{A}^{-1} u = 1 -$$

$$\begin{pmatrix} 1 \ldots 1 \end{pmatrix} \begin{pmatrix} \frac{1}{KT(G_1(1)+Z)} & 0 & \ldots & 0 \\ 0 & \frac{1}{KT(G_1(2)+Z)} & \ldots & 0 \\ \vdots & \ddots & & \\ 0 & \ldots & 0 & \frac{1}{KT(G_K(T)+Z)} \end{pmatrix} \begin{pmatrix} Z \\ Z \\ \vdots \\ Z \end{pmatrix}$$

$$= 1 - \frac{1}{KT} \sum_{k \in \mathcal{K}} \sum_{t \in \mathcal{T}} \frac{Z}{G_k(t) + Z}$$

$G_k(t)$'s are nonnegative and for each participating company k at least one $G_k(t)$ is positive. Also, $Z > 0$ and hence the overall value of the summation is less than KT, and this clearly leads to $1 + v^T \hat{A}^{-1} u \neq 0$. Note that

$$\hat{A}^{-1} u v^T \hat{A}^{-1} = \frac{-Z}{(KT)^2} \times$$

$$\begin{pmatrix} \frac{1}{(G_1(1)+Z)^2} & \ldots & \ldots & \frac{1}{(G_1(1)+Z)(G_K(T)+Z)} \\ \frac{1}{(G_1(1)+Z)(G_1(2)+Z)} & \frac{1}{(G_1(2)+Z)^2} & \ldots & \frac{1}{(G_1(2)+Z)(G_K(T)+Z)} \\ \vdots & \ddots & & \\ \frac{1}{(G_1(1)+Z)(G_K(T)+Z)} & \ldots & \ldots & \frac{1}{(G_K(T)+Z)^2} \end{pmatrix}$$

Thus,

$$A^{-1} = \begin{pmatrix} \frac{1}{KT(G_1(1)+Z)} & 0 & \ldots & 0 \\ 0 & \frac{1}{KT(G_1(2)+Z)} & \ldots & 0 \\ \vdots & \ddots & & \\ 0 & \ldots & 0 & \frac{1}{KT(G_K(T)+Z)} \end{pmatrix}$$

$$+ \frac{Z}{(KT)^2 - KT \sum_{k \in \mathcal{K}} \sum_{t \in \mathcal{T}} \frac{Z}{G_k(t)+Z}} \times$$

$$\begin{pmatrix} \frac{1}{(G_1(1)+Z)^2} & \ldots & \ldots & \frac{1}{(G_1(1)+Z)(G_K(T)+Z)} \\ \frac{1}{(G_1(1)+Z)(G_1(2)+Z)} & \frac{1}{(G_1(2)+Z)^2} & \ldots & \frac{1}{(G_1(2)+Z)(G_K(T)+Z)} \\ \vdots & \ddots & & \\ \frac{1}{(G_1(1)+Z)(G_K(T)+Z)} & \ldots & \ldots & \frac{1}{(G_K(T)+Z)^2} \end{pmatrix}$$

Since $P = A^{-1}Y$, the price selection is uniquely given by

$$p_k(t) = \frac{B}{KT(G_k(t)+Z)}\left(1 + \frac{\sum_{k\in\mathcal{K}}\sum_{t\in\mathcal{T}} \frac{Z}{G_k(t)+Z}}{KT - \sum_{k\in\mathcal{K}}\sum_{t\in\mathcal{T}} \frac{Z}{G_k(t)+Z}}\right) \tag{3.10}$$

which simplifies to the expression (3.9).

(ii) Suppose that to the contrary, $p_k(t) \leq 0$, this leads to

$$\frac{B}{KT(G_k(t)+Z)}\left(1 + \frac{\sum_{k\in\mathcal{K}}\sum_{t\in\mathcal{T}} \frac{Z}{G_k(t)+Z}}{KT - \sum_{k\in\mathcal{K}}\sum_{t\in\mathcal{T}} \frac{Z}{G_k(t)+Z}}\right) \leq 0$$

for some t and k. Note that since $\frac{B}{KT(G_k(t)+Z)}$ is non-negative, there holds

$$1 + \frac{\sum_{k\in\mathcal{K}}\sum_{t\in\mathcal{T}} \frac{Z}{G_k(t)+Z}}{KT - \sum_{k\in\mathcal{K}}\sum_{t\in\mathcal{T}} \frac{Z}{G_k(t)+Z}} \leq 0$$

This implies that

$$\sum_{k\in\mathcal{K}}\sum_{t\in\mathcal{T}} \frac{Z}{G_k(t)+Z} \leq -(KT - \sum_{k\in\mathcal{K}}\sum_{t\in\mathcal{T}} \frac{Z}{G_k(t)+Z})$$

and hence $KT \leq 0$. But $K \geq 1$ and $T \geq 1$. Thus, this is a contradiction. Therefore, we conclude that $p_k(t) > 0 \quad \forall t \in \mathcal{T}, k \in \mathcal{K}$.

(iii) Given the prices announced by other UCs, suppose that a UC k announces a price of $p'_k(t) = p_k(t) + \epsilon$ at a fixed time t. If $\epsilon < 0$, the company will decrease its price. But, the total demand from all users from UC k cannot exceed the power availability (recall that the solution in (i) happens when all the available power is being sold). Since the revenue at time t is the total demand multiplied by the price, UC k does not achieve a higher revenue when $\epsilon < 0$. Now suppose that $\epsilon > 0$; this leads to

$$\begin{aligned}
&U_{gen,k}(p'_k(t), \mathbf{p_{-k}(t)}) - U_{gen,k}(p_k(t), \mathbf{p_{-k}(t)}) \\
&= (p_k(t)+\epsilon)\frac{B + Z\sum_{t\in\mathcal{T}} p_k(t) + Z\epsilon}{KT(p_k(t)+\epsilon)} \\
&- (p_k(t)+\epsilon)Z - p_k(t)\frac{B + Z\sum_{t\in\mathcal{T}} p_k(t)}{KTp_k(t)} + Zp_k(t) \\
&= -Z\epsilon\frac{KT-1}{KT}
\end{aligned}$$

But $KT \geq 1$ and $Z \geq 1$. Thus, if $\epsilon > 0$, the UC does not achieve a higher revenue. Therefore, for every period t, company k does not benefit from deviating from (3.9). Since this applies to every period, it applies for the entire time horizon because of the linearity of the revenue function (it is a linear combination of the demands multiplied by the prices).

■

In practice, due to production costs and market regulations, $p_k(t)$ cannot be outside the range of some lower and upper bounds $[p_k^{\min}(t), p_k^{\max}(t)]$ for all $t \in \mathcal{T}$ and $k \in \mathcal{K}$, as in [30]. If $p_k(t) < p_k^{\min}(t)$, then $p_k(t)$ is set to $p_k^{\min}(t)$, and similarly for the upper-bound, if $p_k(t) > p_k^{\max}(t)$, then we set $p_k(t) = p_k^{\max}(t)$. The expression (3.9) will still hold for the other prices because of the concavity of the problem.

Remark 1 *Using the expression (2.5), it can be verified that the revenue function for UC k is*

$$
\begin{aligned}
U_{\text{gen},k}(p_k, \mathbf{p_{-k}}) &= \sum_{t\in\mathcal{T}} p_k(t) \sum_{n\in\mathcal{N}} d_{n,k}(t) \\
&= \sum_{t\in\mathcal{T}} p_k(t)\Big(\frac{\sum_{n\in\mathcal{N}} B_n + \sum_{n\in\mathcal{N}} \zeta_n \sum_{k\in\mathcal{K}} \sum_{t\in\mathcal{T}} p_k(t)}{KTp_k(t)} - \sum_{n\in\mathcal{N}} \zeta_n\Big) \\
&= \sum_{t\in\mathcal{T}} \frac{B + Z\sum_{k\in\mathcal{K}} \sum_{t\in\mathcal{T}} p_k(t)}{KT} - Z\sum_{t\in\mathcal{T}} p_k(t) \\
&= \frac{B}{K} + \frac{Z(\sum_{k\in\mathcal{K}} \sum_{t\in\mathcal{T}} p_k(t) - K\sum_{t\in\mathcal{T}} p_k(t))}{K}
\end{aligned} \tag{3.11}
$$

We can see that the revenue function of each company depends on the price selections of other companies and thus it is appropriate to use the notion of Nash equilibrium.

Remark 2 *Since $Z = \sum_{n\in\mathcal{N}} \zeta_n$ and ζ_n is typically 1, the value of Z typically equals N. In this case, by (3.9), we observe that for any given $G_k(t)$'s,*

$$p_k(t)(G_k(t) + Z) = p_k(t)(G_k(t) + N)$$

which is a constant for all $t \in \mathcal{T}$ and $k \in \mathcal{K}$. Thus, the power availability is inversely proportional to the prices.

Remark 3 *Theorem 2 provides a computationally cheap expression for the prices. Since $p_k(t)$ can be directly computed using (3.9), there is no need to numerically compute A^{-1} or $|A|$. This enables us to deal with a large number of periods or UCs, without worrying about computational complexity.*

3.3 Existence and Uniqueness of the Stackelberg Equilibrium

Companies play a noncooperative game at the market level and announce their prices to the consumers. As discussed above, consumers optimally respond to these prices and the response of user $n \in \mathcal{N}$ is uniquely given by (3.6). We assume that the participants of the game at the followers level have a sufficient budget to participate (derived in Theorem 1). Denote the strategy space of user (follower in the game) $n \in \mathcal{N}$ by $\mathcal{F}_n$ and the strategy space of all users by $\mathcal{F} = \mathcal{F}_1 \times \cdots \times \mathcal{F}_N$. Denote the strategy space of UC (a leader in the game) $k \in \mathcal{K}$ at $t \in \mathcal{T}$ by $\mathcal{L}_{k,t} := [p_k^{\min}(t), p_k^{\max}(t)]$. Note that $p_k(t) \in \mathcal{L}_{k,t}$ for all $t \in \mathcal{T}$ and $k \in \mathcal{K}$. The strategy space of UC k for the entire time horizon is $\mathcal{L}_k = \mathcal{L}_{k,1} \times \cdots \times \mathcal{L}_{k,T}$ and the composite strategy space of all companies is $\mathcal{L} = \mathcal{L}_1 \times \cdots \times \mathcal{L}_K$. Before stating our main theorem, we need the following game-theoretic concepts from [17].

The vector of prices $\mathbf{p}^* \in \mathcal{L}$ constitutes a *Nash equilibrium* for the price selection game at the UCs-level if

$$U_{\text{gen},k}(p_k^*, \mathbf{p}_{-\mathbf{k}}^*) \geq U_{\text{gen},k}(p_k, \mathbf{p}_{-\mathbf{k}}{}^*), \;\; \forall\, p_k \in \mathcal{L}_k$$

For given price selections $(p_1, \ldots, p_K) \in \mathcal{L}_1 \times \cdots \times \mathcal{L}_K$, the optimal response from all users is

$$\mathbf{d}^*(\mathbf{p}) = \{d_1^*(\mathbf{p}), d_2^*(\mathbf{p}), \ldots, d_N^*(\mathbf{p})\}$$

where for each $n \in \mathcal{N}$, $d_n^*(\mathbf{p})$ is the unique maximizer for $U_{\text{user},n}(d_n, \mathbf{p})$ over $d_n \in \mathcal{F}_n$. For the game considered here, the *Stackelberg equilibrium* is defined as $(\mathbf{d}^*(\mathbf{p}), \mathbf{p}^*)$.

In general, in Stackelberg games, the response from the followers has to be unique for the equilibrium to be well defined [17]. In the game here, due to market competition, leaders aim to choose their prices in the most profitable way while taking into account what other leaders are doing. We capture the competition on the leaders' level through a Nash game. We note that in the parlance of dynamic game theory [17], we are dealing here with open-loop information structures, with the corresponding equilibrium at the utilities level being open-loop Nash equilibrium. Therefore, this is a one-shot game at which all the prices for the all periods are announced at the beginning of the game, and the followers respond to these prices by solving their local optimization problems. We have the following theorem:

Theorem 3 *The following statements are true.*

(i) There exists a unique (open-loop) Nash equilibrium for the noncooperative game at the leaders' level.

(ii) There exists a unique (open-loop) Stackelberg equilibrium.

(iii) The maximizing demands given by (3.6) and the revenue-maximizing prices given in Theorem 2 constitute the (open-loop) Stackelberg equilibrium for the demand response management game.

Proof:

(i) Note that the set $\mathcal{L}_{k,t} := [p_k^{min}(t), p_k^{max}(t)] \in \mathbb{R}$ is compact (closed and bounded) $\forall\, t \in \mathcal{T}, k \in \mathcal{K}$. Thus, $\mathcal{L} = \mathcal{L}_1 \times \cdots \times \mathcal{L}_K$ is a compact subset of $\mathbb{R}^{KT}$. Furthermore,

$$\frac{\partial^2 U_{gen,k}(p_k(t), \mathbf{p_{-k}})}{\partial p_k^2(t)} = 0 \quad \forall\, t \in \mathcal{T}, k \in \mathcal{K}$$

So, the revenue function is concave in $p_k(t)$. Hence, there exists a Nash equilibrium for the noncooperative game at the leaders level [17]. By Theorem 2, $p_k(t)$ that constitutes the best response of company $k \in \mathcal{K}$ to prices set by other companies is uniquely given by (3.9). Thus, the Nash equilibrium is unique.

(ii) From (i), a unique Nash equilibrium exists at which the maximizing prices are announced to the consumers. Since the maximizing demands were uniquely given by (3.6), then there exists a unique Stackelberg equilibrium.

(ii) Immediately follows from Theorem 2 and parts (i)-(ii).

■

Algorithm 1 Distributed algorithm for computing the prices with local information

1: Arbitrarily choose $p_k^{(1)}(t),\ \forall t \in \mathcal{T},\ \ \forall k \in \mathcal{K}$
2: Repeat for $i = 2, 3, \ldots$
3: For each user $n \in \mathcal{N}$, compute $d_{n,k}^{(i)}(t)$ from $k \in \mathcal{K}$ at $t \in \mathcal{T}$ by (3.6), then update utility companies with demand signals
4: For each un-updated price $p_k^{(i+1)}(t)$ announced by $k \in \mathcal{K}$ at $t \in \mathcal{T}$, use (3.15)
5: If $p_k^{(i+1)}(t) \neq p_k^{(i)}(t)$, update users and go to 3
6: Else, send a no-change signal to users and go to 4
7: If $p_k^{(i+1)}(t) = p_k^{(i)}(t)\ \forall t \in \mathcal{T},\ \ \forall k \in \mathcal{K}$, stop
8: Else, go to 2

3.4 Distributed Algorithm

It can be seen from (3.6) that in the computation of user n's optimal demand selection, no information from other users is needed, and user n's local information would suffice for optimal response (recall that there is no game played among the users). However, the closed-form solution for $p_k(t)$ given by (3.9) requires each company to know consumers' budgets and the power availability of all the other companies. Utility companies might not want to share such information with each other. To avoid such a conflict, we propose a distributed algorithm that allows companies to compute their revenue-maximizing prices using only local information and show that this algorithm converges to the optimal prices given by (3.9). This algorithm, combined with utility-maximizing demands given by (3.6), leads to the computation of the equilibrium point of the open-loop Stackelberg game with only local information at both the company level and the user level.

For iteration $i = 1, 2, 3, \ldots$, denote the demand from user n at time t from company k by $d_{n,k}^{(i)}(t)$, and the price announced by company k and time t by $p_k^{(i)}(t)$ (the demands are non-negative and the prices are strictly positive). In our algorithm, $p_k^{(1)}(t)$ is chosen arbitrarily for each company $k \in \mathcal{K}$ and time $t \in \mathcal{T}$. Based on the initial price selection, $d_{n,k}^{(1)}$ is computed using (3.6). Based on these demands, each company k updates its price at time t as follows:

$$p_k^{(i+1)}(t) = p_k^{(i)}(t) + \frac{\sum_{n \in \mathcal{N}} d_{n,k}^{(i)}(t) - G_k(t)}{\epsilon_{k,t}} \tag{3.12}$$

where $\epsilon_{k,t}$ is an appropriately selected positive adjustment parameter for company k at time t. When company k updates its price at time t, it transmits the price to each user $n \in \mathcal{N}$ and they change their demands accordingly. Then, the prices are sequentially updated for the other periods and for other companies as well in a similar fashion. Once all the prices converge to their optimal values, users optimally respond by (3.6) and the Stackelberg equilibrium is reached (see Theorem 3). To find an appropriate $\epsilon_{k,t}$ that leads to the convergence of the above algorithm, recall that the prices must be positive. So, the algorithm diverges whenever one of the $p_k^{(i)}(t)$'s is negative, which might happen when

$$\sum_{n \in \mathcal{N}} d_{n,k}^{(i)}(t) < G_k(t)$$

for any company $k \in \mathcal{K}$ at any time $t \in \mathcal{T}$ in iteration i. To avoid this, it suffices to require

$$p_k^{(i)}(t) > \left|\frac{\sum_{n \in \mathcal{N}} d_{n,k}^{(i)}(t) - G_k(t)}{\epsilon_{k,t}}\right|$$

whenever we have $\sum_{n \in \mathcal{N}} d_{n,k}^{(i)}(t) < G_k(t)$. This translates into requiring

$$\epsilon_{k,t} > \frac{G_k(t) - \sum_{n \in \mathcal{N}} d_{n,k}^{(i)}(t)}{p_k^{(i)}(t)}$$

for every company k at time t in iteration i. By (3.6), it suffices to have

$$\begin{aligned}
\epsilon_{k,t} &> \frac{G_k(t) - \sum_{n \in \mathcal{N}} \left(\frac{B_n + \sum_{e \in \mathcal{K}} \sum_{h \in \mathcal{T}} p_e^{(i)}(h)\zeta_n}{KTp_k^{(i)}(t)} - \zeta_n\right)}{p_k^{(i)}(t)} \\
&= \frac{G_k(t) - \left(\frac{B + Z\sum_{e \in \mathcal{K}} \sum_{h \in \mathcal{T}} p_e^{(i)}(h)}{KTp_k^{(i)}(t)} - Z\right)}{p_k^{(i)}(t)} \\
&= \frac{KTp_k^{(i)}(t)(G_k(t) + Z) - B - Z\sum_{e \in \mathcal{K}} \sum_{h \in \mathcal{T}} p_e^{(i)}(h)}{KT(p_k^{(i)}(t))^2}
\end{aligned}$$

Note that in this case we have

$$p_k^{(i+1)}(t) < p_k^{(i)}(t)$$

whenever

$$\sum_{n \in \mathcal{N}} d_{n,k}^{(i)}(t) < G_k(t)$$

and

$$p_k^{(i+1)}(t) > p_k^{(i)}(t)$$

whenever

$$\sum_{n\in\mathcal{N}} d_{n,k}^{(i)}(t) > G_k(t)$$

The algorithm terminates when the equality of the power availability constraint (2.6) is satisfied, that is,

$$\sum_{n\in\mathcal{N}} d_{n,k}^{(i)}(t) = G_k(t) \;\; \forall t \in \mathcal{T}, \;\; \forall k \in \mathcal{K}$$

Recall that in Theorem 2 the optimal price selection happens at the equality of the power availability constraint (2.6), and since the optimal response for the followers of the game given by (3.6) only requires local information, we conclude that Algorithm 1 converges to the Stackelberg equilibrium when the above inequality is satisfied using only local information at both the leaders level and the followers level.

The above bound on $\epsilon_{k,t}$ is the tightest bound, but checking it beforehand is not possible. By the above discussion, we can see that choosing

$$\epsilon_{k,t} \geq \frac{G_k(t) + Z}{p_k^{(i)}(t)} \tag{3.13}$$

would still lead to the convergence of the algorithm. By (3.13) and (3.12), the algorithm converges if the update rule satisfies

$$p_k^{(i+1)}(t) \leq p_k^{(i)}(t)(1 + \frac{\sum_{n\in\mathcal{N}} d_{n,k}^{(i)}(t) - G_k(t)}{G_k(t) + Z}) \tag{3.14}$$

Theorem 4 *For each company $k \in \mathcal{K}$ at time $t \in \mathcal{T}$ in iteration $i = 1, 2, \ldots,$ if the prices are sequentially updated such that*

$$p_k^{(i+1)}(t) = p_k^{(i)}(t)(\frac{1}{\delta} + \frac{\sum_{n\in\mathcal{N}} d_{n,k}^{(i)}(t) - G_k(t)}{G_k(t) + Z}) \tag{3.15}$$

where $\delta \geq 1$, then the distributed algorithm converges to the unique Stackelberg equilibrium.

Proof: Immediately follows by the above discussion in view of also the condition

(3.14). ∎

3.5 Formulation of a Power Allocation Game

The analysis we have in the previous sections is for the case when the power availability for company k at time t is fixed and denoted by $G_k(t)$ (see chapter 2 for complete problem formulation). In practice, each company has a certain production capacity it cannot exceed. For the entire time horizon, for each company $k \in \mathcal{K}$, denote the power availability by G_k^{total}. In this case, we have

$$\sum_{t \in \mathcal{T}} G_k(t) \leq G_k^{total}, \quad \forall\, k \in \mathcal{K} \tag{3.16}$$

Now, the question is: How can each company allocate its power so that it maximizes its revenue?

In this section, we use the closed form solutions obtained from the above analysis to formulate a noncooperative power allocation game between utility companies. Studying the existence and uniqueness of the Nash equilibrium for this power allocation game is left for future work. By (3.11), the revenue function for company k is

$$U_{\text{gen},k}(p_k, \mathbf{p_{-k}}) = \frac{B}{K} + \frac{Z \sum_{k \in \mathcal{K}} \sum_{t \in \mathcal{T}} p_k(t) - ZK \sum_{t \in \mathcal{T}} p_k(t)}{K}$$

By summing over k's, we get

$$\begin{aligned}
\sum_{k \in \mathcal{K}} U_{\text{gen},k}(p_k, \mathbf{p_{-k}}) &= \sum_{k \in \mathcal{K}} \left(\frac{B}{K} + \frac{Z \sum_{k \in \mathcal{K}} \sum_{t \in \mathcal{T}} p_k(t) - ZK \sum_{t \in \mathcal{T}} p_k(t)}{K}\right) \\
&= B + Z \sum_{k \in \mathcal{K}} \sum_{t \in \mathcal{T}} p_k(t) - Z \sum_{k \in \mathcal{K}} \sum_{t \in \mathcal{T}} p_k(t) \\
&= B
\end{aligned}$$

hence

$$\sum_{k \in \mathcal{K}} U_{\text{gen},k}(p_k, \mathbf{p_{-k}}) = \sum_{n \in \mathcal{N}} B_n \tag{3.17}$$

So, the sum of all revenues equals the sum of all budgets. This is not surprising because the utility maximizing demands given by (3.6) happen at equality of the budget constraint

$$\sum_{k\in\mathcal{K}}\sum_{t\in\mathcal{T}} p_k(t)d_{n,k}(t) \leq B_n$$

for each user $n \in \mathcal{N}$.

The results in the next chapter, along with the numerical results, show that as the number of periods increases when the total power availability is fixed, end-users always benefit in terms of their utility functions. However, utility companies in this case either gain or lose, and whenever a company k gains in terms of revenue, at least one other company must lose. Thus, our multi-period framework will always provide more incentives for consumer participation, compared to the single-period one (see [30]). But, because of the conflict of objectives that can arise between utility companies in such a framework, it would be interesting to see how each company can distribute its available power over a given number of periods T. This conflict of objectives is due to the fact that the sum of all the revenues is a constant that is equivalent to the sum of all budgets, as in (3.17).

Recall that at the Stackelberg equilibrium, we have

$$\sum_{n\in\mathcal{N}} d_{n,k}(t) = G_k(t) \;\; \forall\, t \in \mathcal{T},\; \forall\, k \in \mathcal{K} \tag{3.18}$$

For each company $k \in \mathcal{K}$, stack all the power availabilities $G_k(t)$'s into a vector G_k. Denote the power availability from other companies by $\mathbf{G_{-k}}$, and by (3.9) and (3.18), the revenue function of company k can be represented as

$$\begin{aligned} U_{\text{gen},k}(G_k, \mathbf{G_{-k}}) &= \sum_{t\in\mathcal{T}} p_k(t)G_k(t) \\ &= B\sum_{t\in\mathcal{T}} \frac{G_k(t)}{(G_k(t)+Z)(KT - \sum_{k\in\mathcal{K}}\sum_{t\in\mathcal{T}}\frac{Z}{G_k(t)+Z})} \end{aligned}$$

Denote the strategy space of company k at time t by $\mathcal{P}_{k,t}$. By (3.16), each $G_k(t)$ cannot exceed G_k^{total}, and since $G_k(t)$'s are non-negative, we can see that

$$\mathcal{P}_{k,t} := [0, G_k^{total}],\;\; \forall t \in \mathcal{T},\; \forall k \in \mathcal{K}$$

For the entire time horizon, the strategy space of company $k \in \mathcal{K}$ is

$$\mathcal{P}_k = \mathcal{P}_{k,1} \times \cdots \times \mathcal{P}_{k,T}$$

Thus, given $\mathbf{G_{-k}}$, the optimization problem for company k is as follows:

$$\begin{aligned} \underset{\mathbf{G_k} \in \mathcal{P}_\mathbf{k}}{\text{maximize}} \quad & U_{\text{gen},k}(G_k, \mathbf{G_{-k}}) \\ \text{subject to} \quad & \sum_{t \in \mathcal{T}} G_k(t) \leq G_k^{total} \end{aligned} \tag{3.19}$$

CHAPTER 4

ASYMPTOTIC BEHAVIOR

In this chapter, we study what happens when the number of periods and the number of users are large. Particularly, we study how the equilibrium prices, demands, and utilities are affected as $T, N \to \infty$.

For the remaining of this chapter, we set $\zeta_n = 1 \;\; \forall n$. Note that in this case,

$$Z = \sum_{n \in \mathcal{N}} \zeta_n = N$$

This provides more insights about the influence of the number of followers on demand selection, utility functions, and the revenue-maximizing prices, without conceptually affecting the analysis, since ζ_n is typically 1. Particularly, (3.6) becomes

$$d_{n,k}(t) = \frac{B_n + \sum_{k \in \mathcal{K}} \sum_{t \in \mathcal{T}} p_k(t)}{KTp_k(t)} - 1 \quad \forall\, t \in \mathcal{T}, k \in \mathcal{K} \tag{4.1}$$

and the price announced by company $k \in \mathcal{K}$ at time $t \in \mathcal{T}$ is

$$p_k(t) = \frac{B}{G_k(t) + N}\left(\frac{1}{KT - \sum_{k \in \mathcal{K}} \sum_{t \in \mathcal{T}} \frac{N}{G_k(t)+N}}\right) \tag{4.2}$$

Also, the utility function of user n becomes

$$U_{user,n} = \gamma_n \sum_{k \in \mathcal{K}} \sum_{t \in \mathcal{T}} \ln(1 + d_{n,k}(t)) \tag{4.3}$$

4.1 When the Number of Periods is Large

Numerical results (see chapter 5) indicate that with the same total power availability for each company, the multi-period demand response management provides more incentives for consumer participation compared to the single-period case. Particularly, the utility function of user $n \in \mathcal{N}$ increases significantly without increasing the budget, while the total power available from company $k \in \mathcal{K}$ is distributed among the periods. Such incentives are of practical importance and would encourage more consumers to participate in demand response management and hence improve the overall reliability of the grid [5]. Here, we analyze what happens as T increases when consumers' budgets and total power availabilities for each company are fixed.

Suppose that all companies have the same total power availability and the total power availability of each company $k \in \mathcal{K}$ is equally distributed among all periods. In this case, we have

$$G_k(t) = \frac{\sum_{t \in \mathcal{T}} G_k(t)}{T}, \;\; t \in \mathcal{T}, \;\; k \in \mathcal{K} \tag{4.4}$$

and

$$\sum_{k \in \mathcal{K}} \sum_{t \in \mathcal{T}} G_k(t) = KTG_k(t) \tag{4.5}$$

which implies

$$\sum_{k \in \mathcal{K}} \sum_{t \in \mathcal{T}} \frac{N}{G_k(t) + N} = KT \frac{N}{G_k(t) + N} \tag{4.6}$$

$$\sum_{k \in \mathcal{K}} \sum_{t \in \mathcal{T}} p_k(t) = KTp_k(t) \tag{4.7}$$

By plugging-in (4.6) in (4.2), we have

$$\begin{aligned} p_k(t) &= \frac{B}{G_k(t) + N} \left(\frac{1}{KT - KT \frac{N}{G_k(t)+N}} \right) && (4.8) \\ &= \frac{B}{KTG_k(t)} && (4.9) \\ &= \frac{B}{\sum_{k \in \mathcal{K}} \sum_{t \in \mathcal{T}} G_k(t)} && (4.10) \end{aligned}$$

which is a constant since the numerator is the sum of all users' budgets and the denominator is the sum of the total power availability for all companies. Note that in this case the utility of user n simplifies to

$$U_{user,n} = \gamma_n KT \ln(1 + d_{n,k}(t)) \tag{4.11}$$

where

$$d_{n,k}(t) = \frac{B_n + \sum_{k \in \mathcal{K}} \sum_{t \in \mathcal{T}} p_k(t)}{KTp_k(t)} - 1 \tag{4.12}$$
$$= \frac{B_n + KTp_k(t)}{KTp_k(t)} - 1 \tag{4.13}$$
$$= \frac{B_n}{KTp_k(t)} \tag{4.14}$$

By plugging-in (4.14) in (4.11), we have

$$U_{user,n} = \gamma_n KT \ln(\frac{B_n}{KTp_k(t)} + 1) \tag{4.15}$$

By (4.4) and (4.9), we get

$$U_{user,n} = \gamma_n KT \ln(\frac{B_n G_k(t)}{B} + 1) \tag{4.16}$$
$$= \gamma_n KT \ln(\frac{\sum_{t \in \mathcal{T}} G_k(t) B_n / B}{T} + 1) \tag{4.17}$$

where the term $\sum_{t \in \mathcal{T}} G_k(t) B_n / B$ is positive. Thus, as T increases, the multiplicative term $\gamma_n KT$ of the logarithmic function increases at a faster rate than the decrease of $\ln(\frac{B_n \sum_{t \in \mathcal{T}} G_k(t)/B}{T} + 1)$. Hence, as T increases, the equilibrium utility of each user $n \in \mathcal{N}$ monotonically increases. Taking the limit,

$$\begin{aligned} \lim_{T \to \infty} U_{user,n}(T) &= \lim_{T \to \infty} \gamma_n KT \ln(\frac{\sum_{t \in \mathcal{T}} G_k(t) B_n / B}{T} + 1) \\ &= \lim_{T \to \infty} \gamma_n K \ln(\frac{\sum_{t \in \mathcal{T}} G_k(t) B_n / B}{T} + 1)^T \end{aligned}$$

By the continuity of the logarithmic function,

$$\begin{aligned} \lim_{T \to \infty} U_{user,n}(T) &= \gamma_n K \ln(\lim_{T \to \infty} (\frac{\sum_{t \in \mathcal{T}} G_k(t) B_n / B}{T} + 1)^T) \\ &= \gamma_n K \ln e^{\sum_{t \in \mathcal{T}} G_k(t) B_n / B} \\ &= \gamma_n \frac{K \sum_{t \in \mathcal{T}} G_k(t) B_n}{B} \end{aligned}$$

where the second equality follows from the limit definition of e. Furthermore, note that the demand $d_{n,k}(t)$ from user $n \in \mathcal{N}$ from company $k \in \mathcal{K}$ at time $t \in \mathcal{T}$ converges to zero as $T \to \infty$. We further note that the revenues are constants. To see this, recall

$$\begin{aligned} U_{gen,k}(p_k, \mathbf{p_{-k}}) &= \sum_{t \in \mathcal{T}} p_k(t) \sum_{n \in \mathcal{N}} d_{n,k}(t) \\ &= T p_k(t) \sum_{n \in \mathcal{N}} d_{n,k}(t) \\ &= T p_k(t) G_k(t) \\ &= \frac{B}{K} \end{aligned}$$

which is a constant since both the number of companies and the budgets of the users are fixed.

Remark 4 *The monotonicity of the equilibrium utilities of the users shows that increasing the number of periods and partitioning the total power among them leads to providing more incentives for consumers' participation. However, it might not be very beneficial to increase the number of periods to a very high value. Firstly, the rate of increase in terms of users' utilities gets smaller and smaller. Secondly, having a high number of periods lead to smaller demands for each period and that might violate the minimum energy need at the users' level. So, it is beneficial to increase the number of periods up to a certain point (compared to having $T = 1$), but it might not be beneficial to let T become arbitrarily large. It would be interesting to study what would be the appropriate number of periods that keeps users motivated to participate in DRM while being practical.*

Remark 5 *Note that the limit point of the utility function of user n is the proportion of his budget to the total budgets times the total power availability. So if a particular user has 1% of the sum of all the budgets, he gets 1% of the available power. Furthermore, the revenue for each company is the proportion of the sum of the budgets to the number of companies. Additionally, the demand by user n from company k at time t is the proportion of his budget to the total budgets times the total power availability at t from k.*

Remark 6 *The numerical results in the next chapter show that even when the above assumptions are relaxed, the equilibrium utilities of the consumers still increase as T increases. For example, when $T = 4$ and the power availability for each company is distributed as follows: 25%, 40%, 25%, and 10%, consumers' utility functions increase significantly, compared to the single-period case* *in [30]. So, irrespective of how the total power availability is distributed, consumers would still benefit from increasing the number of periods.*

Remark 7 *When the above assumptions are relaxed, companies are affected in different ways as T grows. Some companies gain as T increases and other companies lose, and the sum of the revenues is always a constant that is equivalent to the sum of all the budgets. This motivates us to formulate a power allocation game in section (3.5) at the companies-level to study how each company can allocate its power availability across a given number of periods. Relaxing the assumptions can also lead to varying prices as T increases.*

4.2 When the Number of Users is Large

In this section, we study the large population regime. Particularly, we analyze what happens when there is a large number of followers with T fixed. In the above results and discussions the number of users $N \geq 1$ was a given. Here, we are interested in analyzing the influence of increasing N on both users and utility companies. When the number of users increases, each additional user has some budget B_n, and since the power availability is fixed, more competition between users arises on the same amount of power and hence utility companies will increase their revenue-maximizing prices.

We start by assuming that the budget for each user $n \in \mathcal{N}$ is the same, and then increase the number of users N and see what happens as $N \to \infty$. We further assume that $G_k(t)$'s are the same for all companies at each t. Going back to (4.2), in this case we have

$$p_k(t) = \frac{\sum_{n\in\mathcal{N}} B_n}{G_k(t)+N}\left(\frac{1}{KT - \sum_{k\in\mathcal{K}}\sum_{t\in\mathcal{T}}\frac{N}{G_k(t)+N}}\right) \quad (4.18)$$

$$= \frac{NB_n}{G_k(t)+N}\left(\frac{1}{KT - KT\frac{N}{G_k(t)+N}}\right) \quad (4.19)$$

$$= \frac{NB_n}{KTG_k(t)} \in \mathcal{L}_{k,t} \quad (4.20)$$

The price company k at time t announces is the proportion of the total budget to the power availability. We can see that $p_k(t) \to \infty$ as $N \to \infty$. Denote the maximum price company k at time t can announce by p_{max}. To avoid divergence of the revenue-maximizing prices and to accommodate the large population needs, we would like to analytically find the appropriate number of companies the needs to be added to achieve this goal so that each $p_k(t)$ does not exceed p_{max}. To do this, we first analyze how the demands, users' utilities, and revenues are affected at equilibrium.

By (4.1), we can see that $\forall t \in \mathcal{T}$ and $\forall k \in \mathcal{K}$, the demand is

$$\begin{aligned} d_{n,k}(t) &= \frac{B_n + \sum_{k\in\mathcal{K}}\sum_{t\in\mathcal{T}} p_k(t)}{KTp_k(t)} - 1 && (4.21)\\ &= \frac{B_n + KTp_k(t)}{KTp_k(t)} - 1 && (4.22)\\ &= \frac{B_n}{KTp_k(t)} && (4.23)\\ &= \frac{B_n}{KT\frac{NB_n}{KTG_k(t)}} && (4.24)\\ &= \frac{G_k(t)}{N} && (4.25)\end{aligned}$$

where the fourth equality follows from (4.20). Note that $d_{n,k}(t) \to 0$ as $N \to \infty$. When the population is large and the power availability is fixed, it is not surprising that $d_{n,k}(t) \to 0$ because the portion each user can get from the available power gets smaller and smaller as N increases. Thus, a balance between the supply and demand needs to be achieved, and we do this by finding an appropriate company-to-user ratio $\frac{K}{N}$. We further note the utility function of user n will also converge to zero in the large population regime. To see this, recall (4.3) and plug-in (4.25)

$$\begin{aligned} U_{user,n} &= \gamma_n \sum_{k\in\mathcal{K}}\sum_{t\in\mathcal{T}} \ln(1 + d_{n,k}(t)) && (4.26)\\ &= \gamma_n KT \ln(1 + \frac{G_k(t)}{N}) && (4.27)\end{aligned}$$

by the continuity of the logarithmic function, we can see that

$$\begin{aligned} \lim_{N\to\infty} U_{user,n}(N) &= \lim_{N\to\infty} \gamma_n KT \ln(1 + \frac{G_k(t)}{N})\\ &= \gamma_n KT \ln(1 + \lim_{N\to\infty}\frac{G_k(t)}{N})\\ &= \gamma_n KT \ln(1)\\ &= 0\end{aligned}$$

The revenue for company k is

$$\begin{aligned} U_{gen,k}(p_k, \mathbf{p_{-k}}) &= \sum_{t\in\mathcal{T}} p_k(t) \sum_{n\in\mathcal{N}} d_{n,k}(t) & (4.28)\\ &= T p_k(t) G_k(t) & (4.29)\\ &= \frac{NB_n}{KTG_k(t)} T G_k(t) & (4.30)\\ &= \frac{NB_n}{K} & (4.31) \end{aligned}$$

where the second equality follows from the observation that

$$\sum_{n\in\mathcal{N}} d_{n,k} = G_k(t), \;\; \forall\, t \in \mathcal{T}, \; \forall\, k \in \mathcal{K} \tag{4.32}$$

at the revenue-maximizing prices. The supply-demand balance (4.32) is achieved at the Stackelberg equilibrium. Note that $U_{gen,k}$ grows as N grows. The revenue-maximizing prices balance the demand and supply in way that makes users exploit all of their available budgets (they use all the available budgets to maximize their utilities, and the sum all these budgets will eventually go to utility companies as revenues). When the population is large, the sum of the budgets $\sum_{n\in\mathcal{N}} B_n \to \infty$ as $N \to \infty$ and hence it is natural to have

$$\lim_{N\to\infty} U_{gen,k}(N) = \infty$$

Now, for a given maximum allowable price p_{max}, what is the appropriate company-to-user ratio $\frac{K}{N}$ that allows us to achieve the supply-demand balance in (4.32) with the condition

$$p_k(t) \le p_{max}, \;\; \forall\, t \in \mathcal{T}, \; \forall\, k \in \mathcal{K} \tag{4.33}$$

being satisfied? The following theorem answers this question.

Theorem 5 *Suppose*

(a.i) The power availability from company $k \in \mathcal{K}$ at $t \in \mathcal{T}$ is the same for all companies

(a.ii) All users have the same budget

Then, the following statements are true.

(i) If the number of users N is bounded by

$$N \leq \frac{p_{max} \sum_{k\in\mathcal{K}} \sum_{t\in\mathcal{T}} G_k(t)}{B_n} \tag{4.34}$$

then the revenue-maximizing $p_k(t)$'s given by (4.20) are feasible and the supply-demand balance (4.32) is satisfied

(ii) If the number of users N is larger than a specific threshold,

$$N > \frac{p_{max} \sum_{k\in\mathcal{K}} \sum_{t\in\mathcal{T}} G_k(t)}{B_n}, \tag{4.35}$$

then, the optimal company-to-user ratio that maximizes the revenues without exceeding p_{max} is

$$\frac{K}{N} = \frac{B_n}{p_{max} \sum_{t\in\mathcal{T}} G_k(t)} \tag{4.36}$$

Proof:

(i) Upon re-arrangements we can see that

$$p_k(t) = \frac{NB_n}{KTG_k(t)} = \frac{NB_n}{\sum_{k\in\mathcal{K}} \sum_{t\in\mathcal{T}} G_k(t)} \leq p_{max}, \;\; \forall\, t \in \mathcal{T}, \; \forall\, k \in \mathcal{K}$$

which implies that $p_k(t)$'s are feasible and the supply-demand balance is automatically satisfied by the Stackelberg equilibrium in Theorems 2-3.

(ii) Suppose

$$\frac{K}{N} < \frac{B_n}{p_{max} \sum_{t\in\mathcal{T}} G_k(t)} = \frac{B_n}{p_{max} T G_k(t)}$$

This implies

$$p_{max} < \frac{NB_n}{KTG_k(t)} = p_k(t), \;\; \forall\, t \in \mathcal{T}, \; \forall\, k \in \mathcal{K}$$

which leads to the infeasibility of revenue-maximizing prices. In this case, companies' best response is to charge p_{max} because of the concavity of their revenue-maximization problems. This implies that

$$\sum_{k\in\mathcal{K}} U_{gen,k} = \sum_{k\in\mathcal{K}} \sum_{t\in\mathcal{T}} p_{max} G_k(t) = p_{max} KTG_k(t) < NB_n = \sum_{n\in\mathcal{N}} B_n$$

which means that the sum of the revenues is strictly less than the sum of the budgets. But, the equality must hold at the Stackelberg equilibrium as in (3.17). Hence, the ratio $\frac{K}{N}$ must increase to

$$\frac{K}{N} = \frac{B_n}{p_{max} T G_k(t)} = \frac{B_n}{p_{max} \sum_{t \in \mathcal{T}} G_k(t)}$$

Finally, note that if

$$\frac{K}{N} > \frac{B_n}{p_{max} T G_k(t)} \ ,$$

then

$$p_{max} > \frac{N B_n}{K T G_k(t)} = p_k(t), \ \ \forall \, t \in \mathcal{T}, \ \forall \, k \in \mathcal{K}$$

so the revenue-maximizing prices will be feasible. However, it can be seen from (4.31) that each company $k \in \mathcal{K}$ will lose in-terms of revenue as there are more companies than necessary in the market. ■

CHAPTER 5

NUMERICAL RESULTS

there isWe conducted numerical computations capturing different scenarios. In the first case, we set $T = 1$ and compute the results with the same parameter values as in the single-period case [30]. Here, for $n \in \mathcal{N}$, we have $\zeta_n = 1$ and $\gamma_n = 1$. Additionally, $K = 3$, $N = 5$, B_1 varies from 2 to 42, $B_2 = 10$, $B_3 = 15$, $B_4 = 20$, and $B_5 = 25$. The power availabilities are $G_1(1) = 10$, $G_2(1) = 15$, and $G_3(1) = 20$. Figure 5.1 shows that the results match those in the single-period case given in [30]. This is expected since the multi-period Stackelberg game is a generalization of the single-period one.

5.1 Influence of the Number of Periods

Now, we let $T = 4$, an interpretation for which can be the following: morning, afternoon, evening, and late night. The budgets of consumers are kept the same, and as before, B_1 still varies from 2 to 42. The total power availability for each company is also kept at the same level, but distributed across the 4 periods as follows: 25%, 40%, 25%, and 10%. Figure 5.2 shows that with the same total power availability for each company, and without increasing any of the consumers' budgets, the utilities for users increase significantly (almost doubles for user 1). The total demands for the users do not change, since they match the power availability. The trend for the revenues is similar to the single-period case. One key observation is that the multi-period scheme provides more incentives for consumers' participation (shown analytically), which is quite important and a key issue [5].

Additionally, we now increase the number of users and study the behaviors with a varying T. Here, T varies from 1 to 50, $N = 50$, $K = 1$, $G_1(t) = \frac{300}{T}\ \forall\ t \in \mathcal{T}$ (so, the power availability is equally distributed among all periods). $B_{1-10} = 5$, $B_{11-20} = 10$, $B_{21-30} = 15$, $B_{31-40} = 20$, and $B_{41-50} = 25$. Figure 5.3 shows that the previous observation again holds, and increasing the number of the periods provides more incentives for users' participation. The revenue is constant since the budgets are fixed and they are exploited at the revenue-maximizing prices and utility-maximizing demands.

Finally, with the same budgets for the 50 users, we increase the number of companies and let $K = 3$. Letting $G_1(t) = \frac{300}{T}\ \forall\ t \in \mathcal{T}$, $G_2(t) = \frac{150}{T}\ \forall\ t \in \mathcal{T}$, and $G_3(t) = \frac{200}{T}\ \forall\ t \in \mathcal{T}$. Figure 5.4 shows a similar behavior for users. However, it illustrates that some companies actually lose in terms of revenues when T increases. But, the sum of revenues is always a constant (equals the sum of budgets). This illustrates the importance of the power allocation game formulated in section 3.6.

5.2 Convergence of the Distributed Algorithm

We simulate the performance of the distributed algorithm for different scenarios. In the first case the parameter values are as in the first figure ($T = 1, K = 3, N = 5$, with the following power availabilities and budgets: $G_1(1) = 10$, $G_2(1) = 15$, $G_3(1) = 20$, $B_1 = 5$, $B_2 = 10$, $B_3 = 15$, $B_4 = 20$, and $B_5 = 25$). We simulate the algorithm's convergence with $\epsilon_{k,t} = 40\ \ \forall t, \forall k$ and $\epsilon_{k,t} = 10\ \ \forall t, \forall k$. Figures 5.5 and 5.6 show that the algorithm converges to the optimal values as in [30]. When $\epsilon_{k,t}$ is smaller, the algorithm converges in a faster rate. But, there is a trade-off between the convergence and the speed of convergence. To elaborate on this, we let $\epsilon_{k,t} = 2\ \ \forall t, \forall k$ (this value violates the condition Theorem 4) and note that the algorithm diverges as in Figure 5.7. Note that for these figures, we are using the update rule (3.12).

Now, with the same values of the adjustment parameter, we increase the number of periods to $T = 4$ with a single utility company and study the convergence of the prices. The power availability is as follows: $G_1(1) = 6$, $G_1(2) = 12$, $G_1(3) = 11.25$ and $G_1(4) = 4.5$. Figures 5.8-5.10 illustrate its performance when T increases and show that there is a similar pattern.

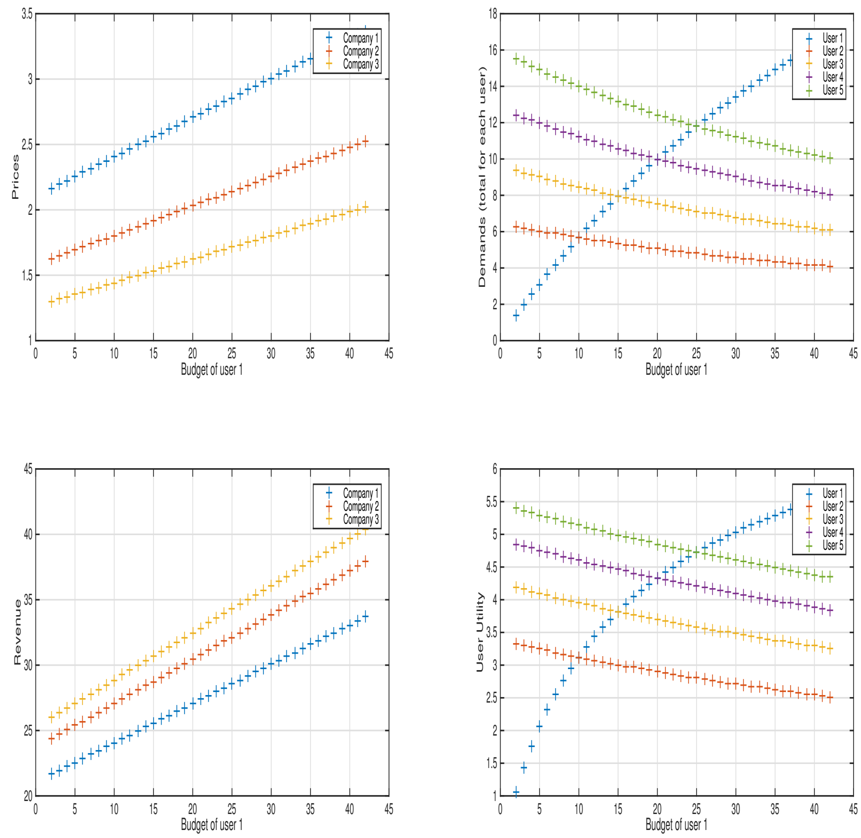

Figure 5.1: Computations with 5 users, $T = 1$, and varying B_1

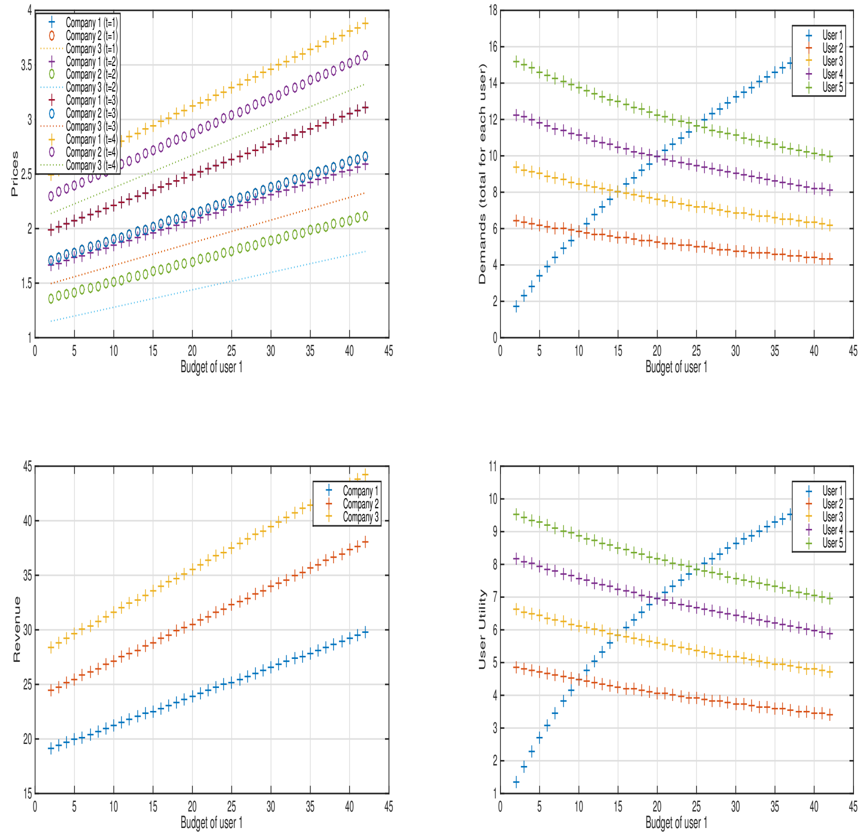

Figure 5.2: Computations with 5 users, $T = 4$, and varying B_1

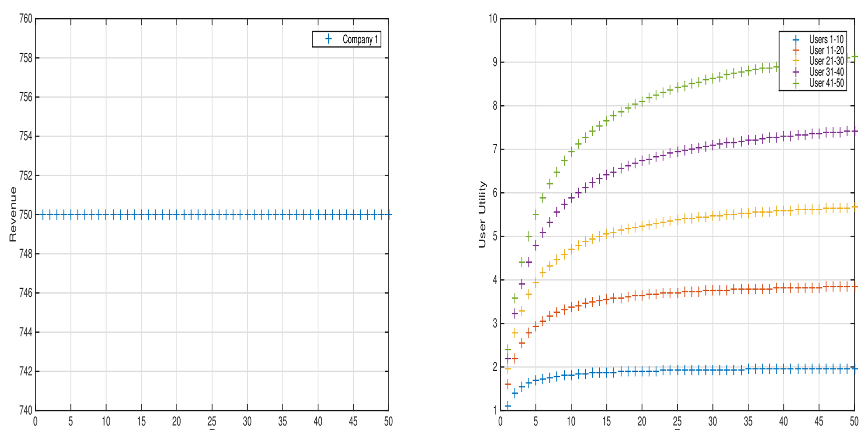

Figure 5.3: Computations for 50 users with varying T

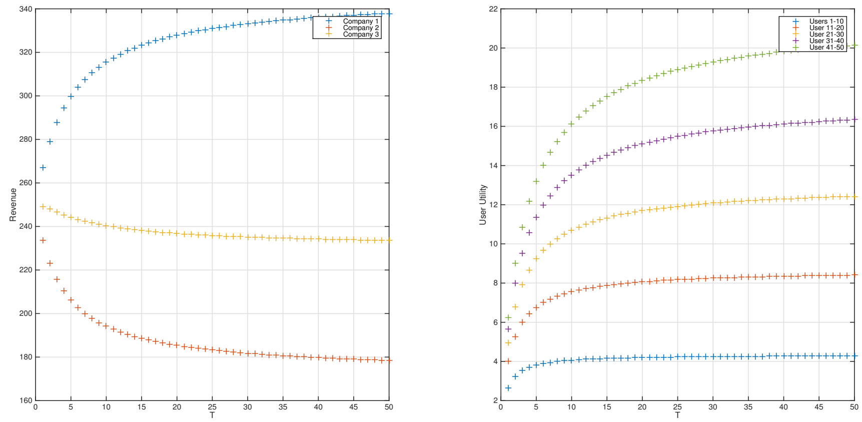

Figure 5.4: Computations for 50 users, $K = 3$, and varying T

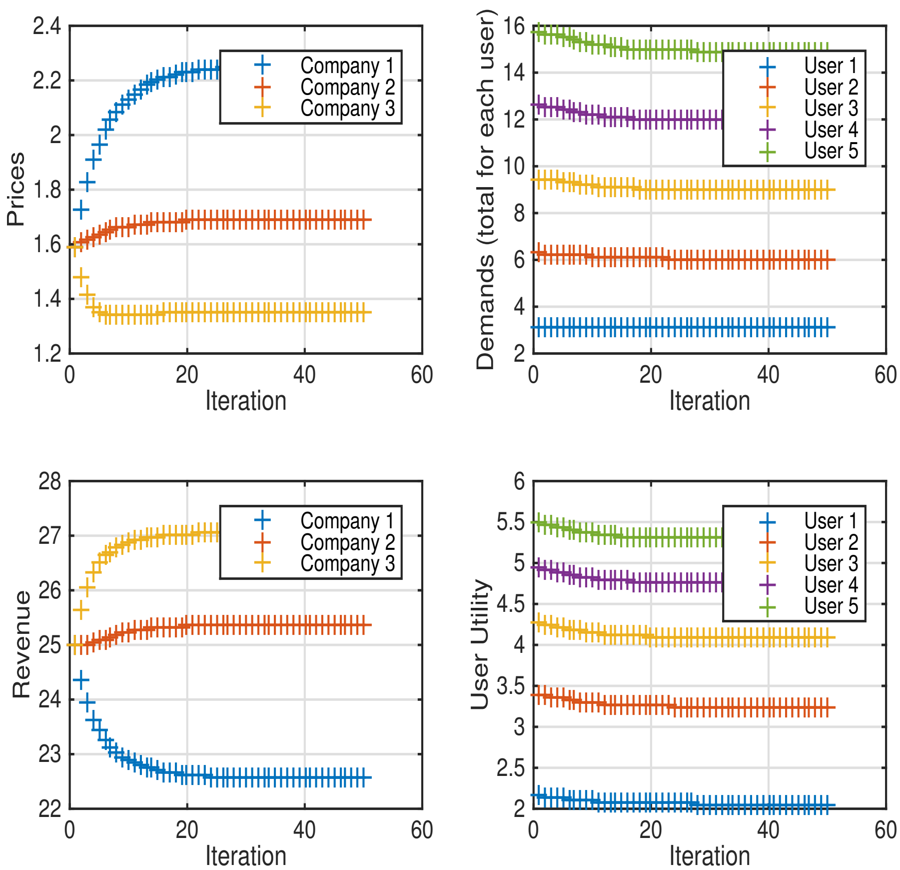

Figure 5.5: Distributed algorithm's performance with $T = 1$, $K = 3$, and $\epsilon_{k,t} = 40 \ \ \forall t, \forall k$

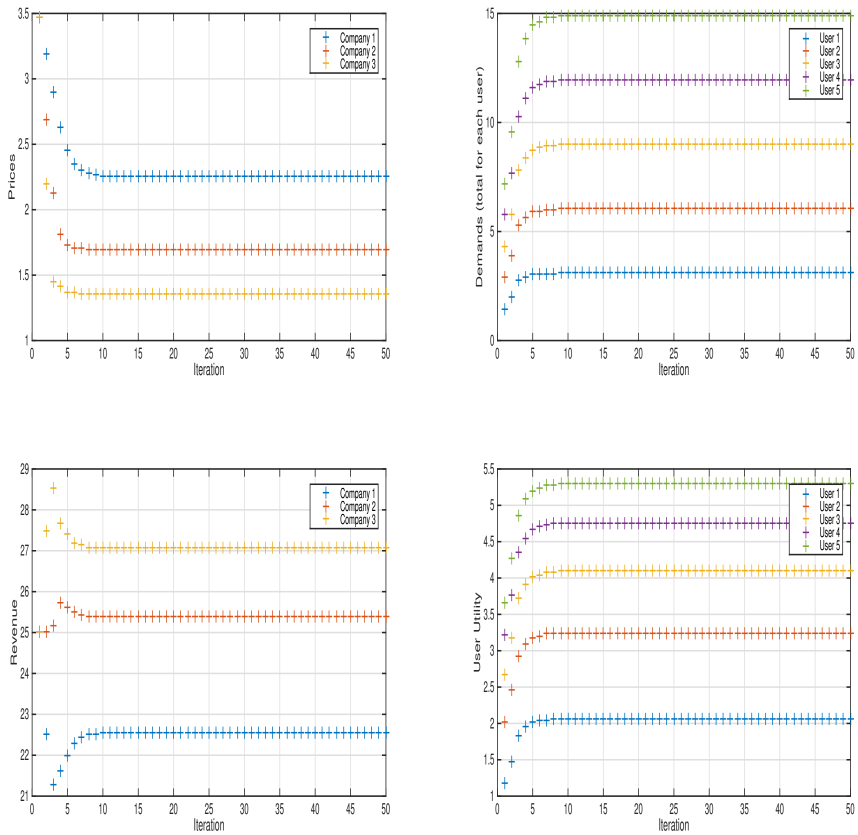

Figure 5.6: Distributed algorithm's performance with $T = 1$, $K = 3$, and $\epsilon_{k,t} = 10 \ \forall t, \forall k$

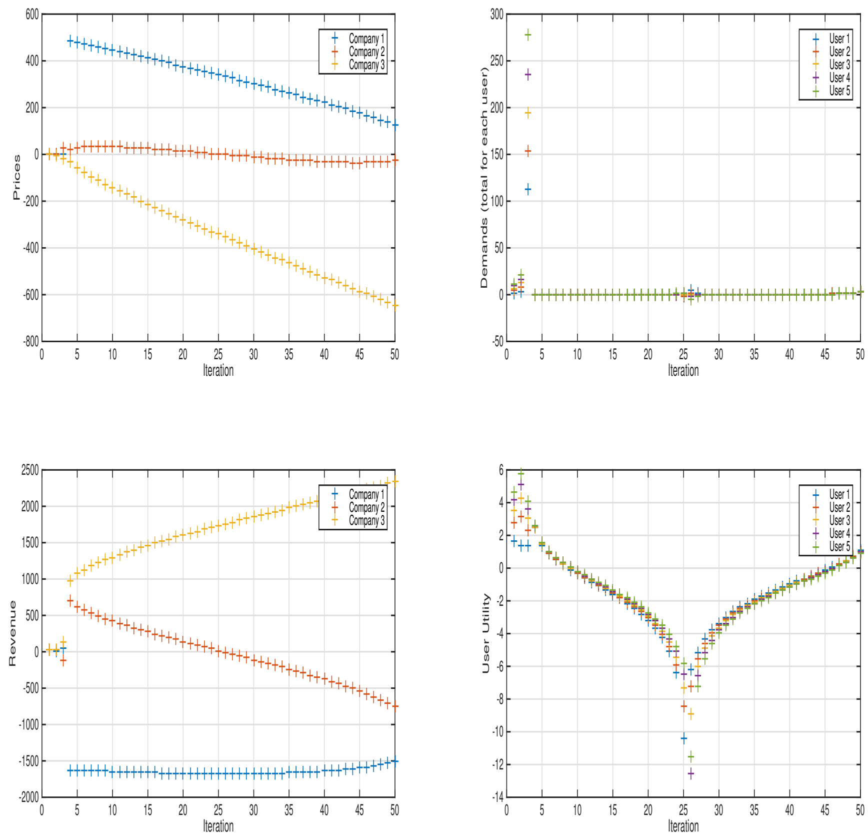

Figure 5.7: Distributed algorithm's divergence with $T = 1$, $K = 3$, and $\epsilon_{k,t} = 2 \ \forall t, \forall k$

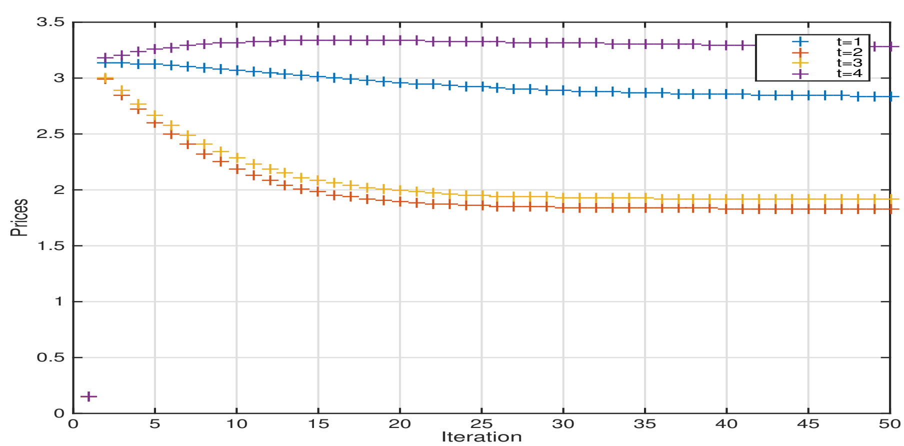

Figure 5.8: Distributed algorithm's performance with $T = 4$, $K = 1$, and $\epsilon_{1,t} = 40 \ \forall t$

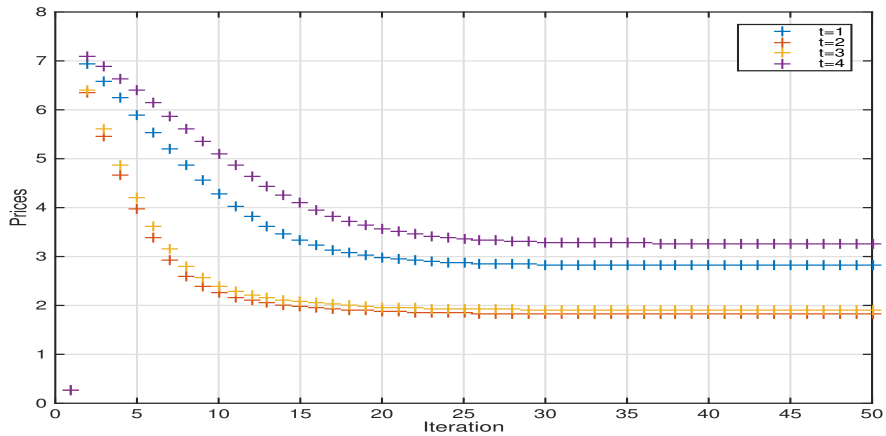

Figure 5.9: Distributed algorithm's performance with $T = 4$, $K = 1$, and $\epsilon_{1,t} = 10 \;\; \forall t$

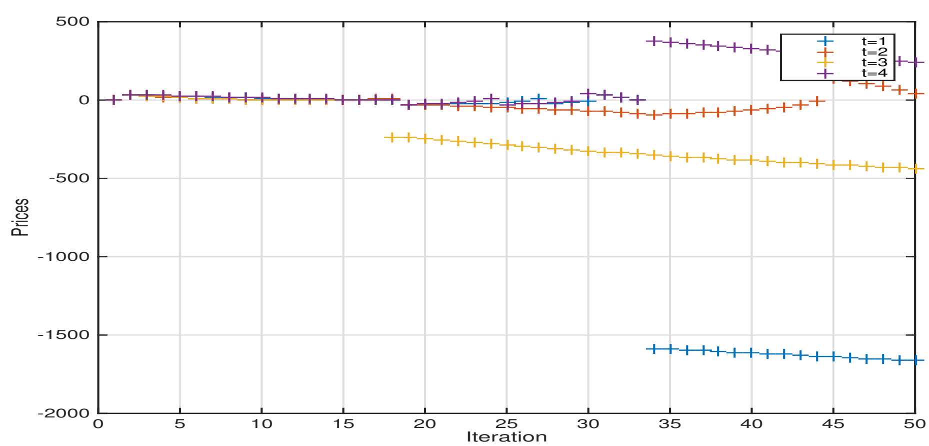

Figure 5.10: Distributed algorithm's divergence with $T = 4$, $K = 1$, and $\epsilon_{1,t} = 2 \;\; \forall t$

CHAPTER 6

CONCLUSION

In this thesis, the multi-period demand response problem through game-theoretic methods has been studied. In particular, we have developed a Stackelberg game to capture the interactions between utility companies and energy consumers. These consumers can be commercial, residential, or industrial. In this game, utility companies are the leaders and the users are the followers. The leaders play a noncooperative Nash game, which was shown to have a unique equilibrium at which companies maximize their revenues in response to price decisions made by the other companies. Then, the users optimally respond to the price selection made by utility companies by choosing their utility maximizing demands. Each user has a limited budget and a minimum amount of energy to meet for the entire time horizon. A budget condition has been derived for consumers' participation and the maximizing demands have been shown to be unique. The overall hierarchal interaction admits a unique Stackelberg equilibrium at which the revenues are maximized and the demands are utility maximizing. Furthermore, closed form solutions have been derived for the Stackelberg equilibrium, and a distributed algorithm to compute the equilibrium point using only local information was shown to converge to the Stackelberg equilibrium. A new power allocation game at the companies-level has been formulated by exploiting the closed form expressions. Moreover, the asymptotic behavior of user-utilities, demands, prices, and revenues has been analyzed as the number of periods increases. This thesis shows, both analytically and numerically, that the multi-period scheme provides more incentives for the participation of energy consumers in demand response management. In the large population regime, an appropriate company-to-user ratio was derived to maximize the revenue of each individual utility company.

There are plenty of opportunities for future work. Studying the optimal power allocation at the companies-level is one possible direction. In this thesis, it was assumed that the minimum energy need at the users-level has to be satisfied for the entire time horizon, so incorporating period-specific constraints is another possible direction for future work. Furthermore, it would be interesting to add energy scheduling and storage at the users-level and/or companies-level and study their influence on demand selections and the revenue-maximizing prices. The game developed in this thesis is static with open-loop information structures. Therefore, using tools from dynamic game theory to deal with closed-loop information structures is another direction for future work.

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