ONLINE CHOOSABILITY OF GRAPHS

BY

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DISSERTATION

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Abstract

We study several problems in graph coloring. In *list coloring*, each vertex v has a set $L(v)$ of available colors and must be assigned a color from this set so that adjacent vertices receive distinct colors; such a coloring is an *L-coloring*, and we then say that G is *L-colorable*. Given a graph G and a function $f : V(G) \to \mathbb{N}$, we say that G is *f-choosable* if G is L-colorable for any list assignment L such that $|L(v)| \geq f(v)$ for all $v \in V(G)$. When $f(v) = k$ for all v and G is f-choosable, we say that G is *k-choosable*. The least k such that G is k-choosable is the *choice number*, denoted $\mathrm{ch}(G)$. We focus on an online version of this problem, which is modeled by the *Lister/Painter game*.

The game is played on a graph in which every vertex has a positive number of *tokens*. In each round, *Lister* marks a nonempty subset M of uncolored vertices, removing one token at each marked vertex. *Painter* responds by selecting a subset D of M that forms an independent set in G. A color distinct from those used on previous rounds is given to all vertices in D. Lister wins by marking a vertex that has no tokens, and Painter wins by coloring all vertices in G. When Painter has a winning strategy, we say that G is *f-paintable*. If $f(v) = k$ for all v and G is f-paintable, then we say that G is *k-paintable*. The least k such that G is k-paintable is the *paint number*, denoted $\mathring{\mathrm{ch}}(G)$.

In Chapter 2, we develop useful tools for studying the Lister/Painter game. We study the paintability of graph joins and of complete bipartite graphs. In particular, $\mathring{\mathrm{ch}}(K_{k,r}) \leq k$ if and only if $r < k^k$.

In Chapter 3, we study the Lister/Painter game with the added restriction that the proper coloring produced by Painter must also satisfy some property $\mathcal{P}$. The main result of Chapter 3 provides a general method to give a winning strategy for Painter when a strategy for the list coloring problem is already known. One example of a property $\mathcal{P}$ is that of having an *r-dynamic coloring*, where a proper coloring is *r-dynamic* if each vertex v has at least $\min\{r, d(v)\}$ distinct colors in its neighborhood. For any graph G and any r, we give upper bounds on how many tokens are necessary for Painter to produce an r-dynamic coloring of G. The upper bounds are in terms of r and the genus of a surface on which G embeds.

In Chapter 4, we study a version of the Lister/Painter game in which Painter must assign m colors to each vertex so that adjacent vertices receive disjoint color sets. We characterize the graphs in which $2m$ tokens is sufficient to produce such a coloring. We strengthen Brooks' Theorem as well as Thomassen's result that planar graphs are 5-choosable.

In Chapter 5, we study *sum-paintability.* The *sum-paint number* of a graph G, denoted $\mathrm{s}\mathring{\mathrm{c}}\mathrm{h}(G)$, is the least $\sum f(v)$ over all f such that G is f-paintable. We prove the easy upper bound: $\mathrm{s}\mathring{\mathrm{c}}\mathrm{h}(G) \leq |V(G)| + |E(G)|$. When $\mathrm{s}\mathring{\mathrm{c}}\mathrm{h}(G) = |V(G)|+|E(G)|$, we say that G is *sp-greedy.* We determine the sum-paintability of *generalized theta-graphs.* The generalized theta-graph $\Theta_{\ell_1,\dots,\ell_k}$ consists of two vertices joined by k paths of lengths $\ell_1, \dots, \ell_k$. We conjecture that outerplanar graphs are sp-greedy and prove several partial results toward this conjecture.

In Chapter 6, we study what happens when Painter is allowed to allocate tokens as Lister marks vertices. The *slow-coloring game* is played by Lister and Painter on a graph G. Lister marks a nonempty set of uncolored vertices and scores 1 point for each marked vertex. Painter colors all vertices in an independent subset of the marked vertices with a color distinct from those used previously in the game. The game ends when all vertices have been colored. The *sum-color cost* of a graph G, denoted $\mathring{\mathrm{s}}(G)$, is the maximum score Lister can guarantee in the slow-coloring game on G. We prove several general lower and upper bounds for $\mathring{\mathrm{s}}(G)$. In more detail, we study trees and prove sharp upper and lower bounds over all trees with n vertices. We give a formula to determine $\mathring{\mathrm{s}}(G)$ exactly when $\alpha(G) \leq 2$. Separately, we prove that $\mathring{\mathrm{s}}(G) = \mathrm{s}\mathring{\mathrm{c}}\mathrm{h}(G)$ if and only if G is a disjoint union of cliques. Lastly, we give lower and upper bounds on $\mathring{\mathrm{s}}(K_{r,s})$.

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Chapter 1

Introduction

Graph coloring is one of the most widely studied areas of graph theory. It originated in the middle of the 19th century when Francis Guthrie posed the Four Color Conjecture, asking if every planar map can be colored using four colors so that regions sharing a border receive different colors. Although it took more than a century to discover a proof of the Four Color Theorem, along the way graph coloring was used to solve many real-world applications dealing with scheduling and resource allocation.

In classical graph coloring, vertices are assigned colors with the goal that vertices joined by an edge receive distinct colors. Many variations on this idea have been studied, including restricting which colors are available at each vertex and putting additional requirements on the desired coloring. In this thesis, we study results on one particular variation of graph coloring called "paintability" or "online choosability". Readers may refer to Section 1.6 at the end of this chapter for elementary graph-theoretic definitions and notation.

This thesis contains results that also appear in the following joint works: [10, 11, 38, 37, 36].

1.1 Choosability and Paintability

In classical graph coloring, every color is available for use at every vertex. More generally, we suppose that each vertex v in a graph G has a set $L(v)$ (called its *list*) of available colors. We seek a proper coloring ϕ such that $\phi(v) \in L(v)$; such a coloring is an *L-coloring*, and we then say that G is *L-colorable*. Vizing [55] and independently Erdős, Rubin, and Taylor [16] studied this problem under the names *list coloring* and *choosability*. Given a graph G and a function $f : V(G) \to \mathbb{N}$, we say that G is *f-choosable* if G is L-colorable for any list assignment L such that $|L(v)| \geq f(v)$ for all $v \in V(G)$. When $f(v) = k$ for all v and G is f-choosable, we say that G is *k-choosable*. The least k such that G is k-choosable is the *choice number*, denoted $\mathrm{ch}(G)$.

Schauz [48] and independently Zhu [57] introduced an online version of choosability. Given a list assignment with colors from $\mathbb{N}$, suppose that on round i, the coloring algorithm must decide which vertices will receive color i without knowing which colors will appear later in the lists. Since colors are revealed one at a time, the coloring algorithm has less information than in the standard version of choosability; thus finding a proper coloring is more difficult. The fundamental question is "how much larger (if at all) do the lists need to be to accommodate this added difficulty?" To model the "worst-case" for the coloring algorithm responding to lists presented in this online fashion, we study the *Lister/Painter game*.

Definition 1.1.1. Suppose that G is a graph and $f : V(G) \to \mathbb{N}$. Initially, each vertex v has $f(v)$ tokens and is uncolored. In each round, *Lister* marks a nonempty subset M of uncolored vertices, removing one token at each marked vertex. *Painter* responds by selecting a subset D of M that forms an independent set in G. A color distinct from those used on previous rounds is given to all vertices in D. Lister wins by marking a vertex that has no tokens, and Painter wins by coloring all vertices in G.

We call f a *token assignment* for G. When Painter has a winning strategy, we say that G is f*-paintable*. If $f(v) = k$ for all v and G is f-paintable, then we say that G is k*-paintable*. The least k such that G is k-paintable is the *paint number*, denoted $\mathring{\mathrm{ch}}(G)$. ■

If G is not f-choosable, then G is not f-paintable, since Lister can mimic a bad list assignment L by marking in round i the set $\{v \in V(G) : i \in L(v)\}$; winning moves by Painter would form an L-coloring, which does not exist. Thus $\mathring{\mathrm{ch}}(G) \geq \mathrm{ch}(G)$. For consistency in discussing the Lister/Painter game, we say that Lister *marks* a set M and that Painter *colors* an independent subset of M. A vertex v is *rejected* on a round if it is marked by Lister but not colored by Painter.

In Chapter 2, we develop many of the basic tools used throughout this thesis. For example, if G is f-paintable, then every subgraph H of G is f-paintable when f is restricted to $V(H)$. A vertex v is f*-degenerate* when $f(v) > d(v)$. If v is f-degenerate, then G is f-paintable if and only if $G - v$ is f-paintable. For a set of vertices S, let δ_S be the characteristic function for S, defined as $\delta_S(v) = 1$ if $v \in S$ and $\delta_S(v) = 0$ otherwise. Frequently the notation for the characteristic function is χ, but we reserve χ for the chromatic number of a graph. If $f(v) = 1$, then G is f-paintable if and only if $G - v$ is $(f - \delta_{N(v)})$-paintable. Under optimal play, Lister may be assumed to mark vertices that induce a connected subgraph of G, and Painter may be assumed to always color a maximal independent subset of Lister's marked set.

Ohba [42] conjectured that $\mathrm{ch}(G) = \chi(G)$ when $|V(G)| \leq 2\chi(G) + 1$. Noel, Reed, and Wu [40] proved Ohba's Conjecture, and Huang, Wang, and Zhu [57] conjectured for the paintability version that $\mathring{\mathrm{ch}}(G) = \chi(G)$ when $|V(G)| \leq 2\chi(G)$. The *join* of graphs G and H, denoted $G \oplus H$, is obtained by adding to the disjoint union of G and H the edges $\{uv\colon\ u \in V(G), v \in V(H)\}$. In this thesis, we prove the following results toward the paintability version of Ohba's conjecture. The results in Chapter 2 also appear in joint work with Carraher, Loeb, Puleo, Tsai, and West [10].

Theorem 1.1.2. $\mathring{\mathrm{ch}}(G \diamond K_t) \le k+1$ *when* $\mathring{\mathrm{ch}}(G) \le k$ *and* $|V(G)| \le \frac{t}{t-1}k$.

Corollary 1.1.3. $\mathring{\mathrm{ch}}(K_{2*r}) = \chi(K_{2*r}) = r$.

Corollary 1.1.4. $\mathring{\mathrm{ch}}(G) = \chi(G)$ *when* $|V(G)| \le \chi(G) + 2\sqrt{\chi(G)-1}$.

Theorem 1.1.5. *For any graph G, there exists t such that* $\mathring{\mathrm{ch}}(G \diamond K_t) = \chi(G \diamond K_t)$.

Erdős, Rubin, and Taylor [16] proved that $\mathrm{ch}(K_{k,r}) \le k$ if and only if $r < k^k$. We prove a stronger result for paintability, which implies the following corollary.

Corollary 1.1.6. $\mathring{\mathrm{ch}}(K_{k,r}) \le k$ *if and only if* $r < k^k$.

Additionally, for $j \ge 1$ we give an upper bound on the least r such that $K_{k+j,r}$ is k-paintable. When $j = 1$ and k is sufficiently large, we show that $\mathrm{ch}(K_{k+1,r}) < \mathring{\mathrm{ch}}(K_{k+1,r})$ for certain values of r.

1.2 $\mathcal{P}$-Suitable Paintability

Sometimes, in addition to requiring that a coloring be proper, we want the desired coloring to satisfy some additional property $\mathcal{P}$. A coloring of G satisfying $\mathcal{P}$ is a *$\mathcal{P}$-suitable* coloring. Let G be a graph, $S \subseteq V(G)$, and ϕ be a coloring of S. If there exists a $\mathcal{P}$-suitable coloring ϕ' of $V(G)$ such that $\phi'(v) = \phi(v)$ for all $v \in S$, then we say that ϕ is *$\mathcal{P}$-extendable to G*. When property $\mathcal{P}$ is understood, we abbreviate by saying ϕ *extends* to G. Since the additional requirement placed on the coloring changes the winning conditions for Lister and Painter, we now define the *$(f, \mathcal{P})$-game*.

Definition 1.2.1. Let a graph G and token assignment f for G be given. In each round, *Lister* marks a nonempty subset M of uncolored vertices, removing one token at each marked vertex. *Painter* responds by selecting a subset D of M such that when a color distinct from those used on previous rounds is given to all vertices in D, the resulting coloring is $\mathcal{P}$-extendable to G. Painter wins by producing a $\mathcal{P}$-suitable coloring of G, and Lister wins by marking a vertex with no tokens.

When Painter has a winning strategy, we say that G is *$\mathcal{P}$-suitably f-paintable*. If $f(v) = k$ for all v and G is f-paintable, then we say that G is *$\mathcal{P}$-suitably k-paintable*. The least k such that G is $\mathcal{P}$-suitably k-paintable is the *$\mathcal{P}$-paint number*, denoted $\mathring{\mathrm{ch}}_{\mathcal{P}}(G)$. ■

When $f(v) = k$ for each v, we call this the *$(k, \mathcal{P})$-game*. We now state the analogous definitions for the choosability version.

Definition 1.2.2. Given a graph G and a list assignment L, we say that G is *$\mathcal{P}$-suitably L-colorable* if a $\mathcal{P}$-suitable coloring can be chosen from the lists. For $f\colon V(G) \to \mathbb{N}$, we say that G is *$\mathcal{P}$-suitably f-choosable* if G is $\mathcal{P}$-suitably L-colorable for any list assignment L satisfying $|L(v)| \geq f(v)$ for all $v \in V(G)$. When $f(v) = k$ for all v and G is $\mathcal{P}$-suitably f-choosable, we say that G is *$\mathcal{P}$-suitably k-choosable.* The least k such that G is $\mathcal{P}$-suitably k-choosable is the *$\mathcal{P}$-choice number*, denoted $\mathrm{ch}_{\mathcal{P}}(G)$. ■

The main result of Chapter 3 is a tool that provides a general method to prove results about $\mathcal{P}$-suitable k-paintability using existing proofs about $\mathcal{P}$-suitable k-choosability. One example of a property $\mathcal{P}$ is that of having an *r-dynamic coloring*, where a proper coloring is *r-dynamic* if each vertex v has at least $\min\{r, d(v)\}$ distinct colors in its neighborhood. Let G be a graph that embeds on a surface of genus g. Heawood [18] proved that G is $h(g)$-colorable, where $h(g) = \left\lfloor \frac{7+\sqrt{1+48g}}{2} \right\rfloor$. Heawood's proof uses the fact that nonplanar graphs are $(h(g)-1)$-degenerate, which also implies that they are $h(g)$-paintable. Chen et al. [12] strengthened Heawood's result, showing that G is 2-dynamically $h(g)$-choosable. In this thesis, we give an application of the above-mentioned method that further strengthens their result, answering a question by [35]. Let $\gamma(G)$ denote the genus of G.

Theorem 1.2.3. *If G is a nonplanar graph, then G is 2-dynamically $h(\gamma(G))$-paintable.*

1.3 g-Fold Coloring

An *m-fold coloring* of a graph G assigns a set of m colors to each vertex so that adjacent vertices have disjoint color sets. The *m-chromatic number* of G, denoted $\chi^{(m)}(G)$, is the least k such that an m-fold coloring of G can be produced by using only colors from $[k]$. The *fractional chromatic number* of G, denoted $\chi^*(G)$, is $\lim_{m\to\infty} \frac{\chi^{(m)}(G)}{m}$. There is an equivalent formulation for the fraction chromatic number using linear programming. In Chapter 4, we consider a generalization of m-fold colorings. Each vertex is to be assigned a set of $g(v)$ colors so that adjacent vertices receive disjoint sets. Such a coloring is a *g-fold coloring* of G. Under this coloring property $\mathcal{P}$, we say "(f, g)-game" instead of "$(f, \mathcal{P})$-game". A graph is *(f, g)-paintable* if Painter has a winning strategy in the (f, g)-game. Analogously, a graph is *(f, g)-choosable* if a g-fold coloring can be chosen from any list assignment L satisfying $|L(v)| \geq f(v)$ for all v. When $f(v) = a$ and $g(v) = b$ for all v and G is (f, g)-paintable or (f, g)-choosable, we say that G is *(a, b)-paintable* or *(a, b)-choosable*, respectively.

The *core* of a graph is obtained by iteratively deleting vertices of degree 1. We first give a characterization of the graphs that are 3-paint-critical, which appears in [10] and was proved independently in [47]. In this thesis, we use this result to characterize the graphs that are $(2m, m)$-paintable for all $m \geq 1$. This result also appears in joint work with Meng and Zhu [36].

Theorem 1.3.1. *For $m \geq 1$, a graph G is $(2m, m)$-paintable if and only if the core of G is K_1, an even cycle, or $K_{2,3}$.*

We also prove the following theorem about odd cycles.

Theorem 1.3.2. *C_{2t+1} is (k, m)-paintable if and only if $k \geq 2m + \lceil \frac{m}{t} \rceil$.*

A graph G is *m-degree paintable* if G is (f, m)-paintable where $f(v) = d(v)m$ for all v. A graph in which every block is an odd cycle or a clique is called a *Gallai tree*. Using the result of Tuza and Voigt [54] that G is m-degree choosable if and only if it is not a Gallai tree, we prove the following theorem.

Theorem 1.3.3. *A graph G is m-degree paintable if and only if it is not a Gallai tree.*

Let G be a connected graph other than a complete graph or an odd cycle. Brooks' Theorem [9] states that G is $\Delta(G)$-colorable. Tuza and Voigt [54] strengthened this by proving that G is $(\Delta(G)m, m)$-choosable for all m. Hladký, Kráľ, and Schauz [20] also strengthened Brooks' Theorem by proving that G is $\Delta(G)$-paintable. As a common strengthening of both results, we prove the following theorem by using Theorem 1.3.3.

Theorem 1.3.4. *For $m \geq 1$, a connected graph G is $(\Delta(G)m, m)$-paintable if and only if G is not a complete graph or an odd cycle.*

Thomassen [52] proved that planar graphs are 5-choosable. Tuza and Voigt [54] showed that planar graphs are $(5m, m)$-choosable for all m, and Schauz [48] proved that planar graphs are 5-paintable. We strengthen both improvements of Thomassen's result by proving the following theorem.

Theorem 1.3.5. *Planar graphs are $(5m, m)$-paintable for all $m \geq 1$.*

1.4 Sum-Paintability

In Chapter 5, we study *sum-paintability*, where instead of considering how large the number of tokens must be when all vertices receive the same number of tokens, we consider how large the average number of tokens must be. Isaak [23, 24] introduced this idea for *sum-choosability*, defining the *sum-choice number*, denoted $\mathrm{sch}(G)$, to be the least $\sum f(v)$ over all f such that G is f-choosable. He observed that always $\mathrm{sch}(G) \leq |V(G)| + |E(G)|$; graphs achieving equality in this bound are *sum-choice greedy*, abbreviated to *sc-greedy*. Analogously, the *sum-paint number* of a graph G, denoted $\mathring{\mathrm{sch}}(G)$, is the least $\sum f(v)$ over all f such that G is f-paintable. We show that the same upper bound holds: $\mathring{\mathrm{sch}}(G) \le |V(G)| + |E(G)|$. We use $\sigma(G)$ to denote the *greedy bound* $|V(G)|+|E(G)|$. When $\mathring{\mathrm{sch}}(G) = \sigma(G)$, we say that G is *sum-paint greedy*, which we shorten to *sp-greedy*.

One tool we use is the following.

Lemma 1.4.1. *If $G' = (G + K_n) \cup \{u\phi(u) : u \in V(K_n)\}$ for some $n \in \mathbb{N}$ and $\phi : V(K_n) \to V(G)$, then $\mathring{\mathrm{sch}}(G') = \mathring{\mathrm{sch}}(G) + \frac{n(n+3)}{2}$. Additionally, if G is sp-greedy, then G' is sp-greedy.*

Isaak [23] proved that $\mathrm{sch}(K_n \,\square\, K_2) = n^2 + \lceil \frac{5n}{3} \rceil$. Using Lemma 1.4.1, we obtain the following result.

Corollary 1.4.2. $\mathring{\mathrm{sch}}(K_n \,\square\, K_2)$ *is sp-greedy for all $n \ge 1$.*

Since $\sigma(K_n \,\square\, K_2) = n^2 + 2n$, this shows that the difference $\mathring{\mathrm{sch}}(G) - \mathrm{sch}(G)$ can be arbitrarily large.

Applying this tool again, we determine the sum-paint number of all *generalized theta-graphs*. The generalized theta-graph $\Theta_{\ell_1,\ldots,\ell_k}$ consists of two vertices joined by k paths of lengths $\ell_1, \ldots, \ell_k$. The *fan* F_n is the graph $P_n \diamondplus K_1$. Heinold [19] showed that not all outerplanar graphs are sc-greedy by proving $\mathrm{sch}(F_n) \le \sigma(F_n) - \lfloor \frac{n+1}{11} \rfloor$. For sum-paintability, in joint work with Tomlinson and Wise [38], we made the following conjecture.

Conjecture 1.4.3 ([38])**.** *Outerplanar graphs are sp-greedy.*

Toward this conjecture, we prove that several families of outerplanar graphs are sp-greedy.

Theorem 1.4.4. *If the weak dual of an outerplanar graph G is a path, then G is sp-greedy.*

Fans satisfy the hypothesis of Theorem 1.4.4 and are thus sp-greedy. This proves that $\mathring{\mathrm{sch}}(F_n) - \mathrm{sch}(F_n) = \lfloor \frac{n+1}{11} \rfloor$, which grows linearly with n. For non-outerplanar graphs, we show that two additional families are sp-greedy.

Theorem 1.4.5. *The wheel $C_n \diamondplus K_1$ is sp-greedy when $n \ge 3$.*

Theorem 1.4.6. *$K_r \diamondplus \overline{K}_s$ is sp-greedy if and only if $r \le 1$ or $s \le 3$.*

Theorems 1.4.4 and 1.4.5 also appear in joint work with Tomlinson and Wise [38].

1.5 The Slow Coloring Game

In sum-paintability, Painter must allocate tokens to vertices before the game begins. In Chapter 6, we study what happens when Painter is allowed to allocate tokens as Lister marks vertices. We introduce the *slow-coloring game*, which is played by Lister and Painter on a graph G. Lister marks a nonempty set of uncolored vertices and scores 1 point for each marked vertex. Painter colors all vertices in an independent subset of the marked vertices with a color distinct from those used previously in the game. The game ends when all vertices have been colored. The *sum-color cost* of a graph G, denoted $\mathring{s}(G)$, is the maximum score Lister can guarantee in the slow-coloring game on G.

The sum-color cost is related to several other graph parameters. The *chromatic sum* of a graph G, denoted $\Sigma(G)$, is the smallest value of $\sum_{v\in V(G)} \phi(v)$ over all proper colorings $\phi : V(G) \to \mathbb{N}$. The *Hall ratio* of a graph G, denoted $\rho(G)$, is defined to be the quantity $\max\left\{\frac{|V(H)|}{\alpha(H)} :\ H \subseteq G\right\}$.

Since Lister can guarantee using at least $\mathring{s}(G)$ tokens in the Lister/Painter game, always $\mathring{s}(G) \le \mathring{sch}(G)$. We prove that $\mathring{s}(G) \ge \Sigma(G)$ and that $\frac{|V(G)|}{2\alpha(G)} + \frac{1}{2} \le \frac{\mathring{s}(G)}{|V(G)|} \le \rho(G)$. In addition, we determine the minimum and maximum values for $\mathring{s}(G)$ when G is a tree with n vertices.

Theorem 1.5.1. *If T is a tree and $|V(T)| = n$, then*

$$n + \left\lfloor \frac{-1 + \sqrt{8n-7}}{2} \right\rfloor = \mathring{s}(K_{1,n-1}) \le \mathring{s}(T) \le \mathring{s}(P_n) = \left\lfloor \frac{3n}{2} \right\rfloor .$$

A *k-tree* is a graph that can be obtained from K_k by iteratively adding a vertex whose neighborhood is a k-clique in the existing graph. The k-tree generalization of a star is $K_k \oplus \overline{K}_{n-k}$, and the k-tree generalization of a path is P_n^k. Explicitly, P_n^k is a graph whose vertices can be labeled $v_1, \ldots, v_n$ such that $v_i v_j \in E(P_n^k)$ if and only if $0 < |i-j| \le k$. We conjecture that these achieve the extremal values of $\mathring{s}(G)$ when G is a k-tree with n vertices.

Conjecture 1.5.2. *If T is a k-tree and $|V(T)| = n$, then*

$$\mathring{s}(K_k \oplus \overline{K}_{n-k}) \ \le\ \mathring{s}(T) \ \le\ \mathring{s}(P_n^k).$$

We give a formula to determine $\mathring{s}(G)$ exactly when $\alpha(G) \le 2$. Separately, we prove that $\mathring{s}(G) = \mathring{sch}(G)$ if and only if G is a disjoint union of cliques. Lastly, we give lower and upper bounds on $\mathring{s}(K_{r,s})$.

All results in Chapter 6 come from joint work with Puleo and West [37].

1.6 Definitions and Notation

We use $[k]$ to denote the set $\{1, \ldots, k\}$ of integers. We use $\mathbb{N}$ to denote the set of positive integers. We use $\lg(x)$ for $\log_2(x)$ and $\ln(x)$ for $\log_e(x)$. For a set S, let δ_S be the characteristic function for S, defined as

$\delta_S(x) = 1$ if $x \in S$ and $\delta_S(x) = 0$ otherwise. Let f and g be nonnegative real-valued functions. If there exists a positive constant c such that $f(x) \le c \cdot g(x)$ for all sufficiently large x, then we write $f(x) = O(g(x))$. If there exists a positive constant c such that $f(x) \ge c \cdot g(x)$ for all sufficiently large x, then we write $f(x) = \Omega(g(x))$. If $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$, then we write $f(x) = o(g(x))$.

A *graph* G consists of a set $V(G)$ of *vertices* and a set $E(G)$ of *edges*. In this thesis, all graphs are assumed to be *simple*, which means that each edge is a set of two vertices, called it *endpoints*, and distinct edges are distinct pairs of vertices. When vertices u and v form an edge, we write the edge as uv. When $uv \in E(G)$, we say that u and v are *adjacent*; we also say that v is a *neighbor* of u. Two edges are *incident* if they share a common endpoint. Two graphs G and H are *isomorphic* if there exists a map $f\colon V(H) \to V(G)$ such that $uv \in E(H)$ if and only if $f(u)f(v) \in E(G)$. Such a map is called an *isomorphism*.

The *complement* of a graph G, denoted $\overline{G}$, is the graph with $V(\overline{G}) = V(G)$ such that $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$. We say that a graph H is a *subgraph* of G, written $H \subseteq G$, when $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. When S is a subset of $V(G)$ or $E(G)$, we use $G - S$ to denote the subgraph obtained by removing the elements of S from G. When $S \subseteq V(G)$, removing S also entails removing all edges incident to S. When removing a single vertex v from G, we write $G - v$ instead of $G - \{v\}$, and similarly we use $G - uv$ to denote the removal of a single edge uv. A subgraph H with $V(H) = V(G)$ is a *spanning subgraph*. A subgraph of the form $G - S$ for $S \subseteq V(G)$ is an *induced subgraph*. For $S \subseteq V(G)$, the subgraph *induced by* S, denoted $G[S]$, is $G - (V(G) - S)$.

The *neighborhood* of v, denoted $N_G(v)$ is the set of vertices adjacent to v. The *closed neighborhood* of v, denoted $N_G(v)$, is $N_G(v) \cup \{v\}$. For $S \subseteq V(G)$, we define $N_G(S)$ to be $\bigcup_{v\in S} N_G(v)$. The *degree* of a vertex v, denoted $d_G(v)$, is the number of vertices adjacent to v, which also equals $|N_G(v)|$. When the graph G is understood, we omit the subscript in each of these definitions. The *maximum degree* of a graph G, denoted $\Delta(G)$, is the maximum of $d_G(v)$ over $v \in V(G)$. Similarly, the *minimum degree* of G, denoted $\delta(G)$, is the minimum of $d_G(v)$ over $v \in V(G)$. We say that G is *r-regular* if $d_G(v) = r$ for all $v \in V(G)$. A *k-vertex* is a vertex of degree k, and a *k^--vertex* is a vertex of degree at most k. A graph is *d-degenerate* if every subgraph contains a d^--vertex.

A *clique* is a set of pairwise adjacent vertices. An *independent set* is a set S of vertices such that $G[S]$ contains no edges. An independent set S is *maximal* if no other independent set contains S. The *independence number* of G is the maximum size of an independent set in G and is denoted by $\alpha(G)$. A *coloring* is an assignment of colors to the vertices of a graph; when at most k colors are used, we call this a *k-coloring*. Given a set C of colors, a coloring $f : V(G) \to C$ is *proper* if $f(u) \ne f(v)$ whenever $uv \in E(G)$.

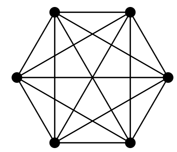

Figure 1.1: K_6

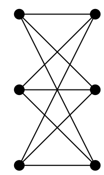

Figure 1.2: $K_{3,3}$

When G has a proper coloring using at most k colors, we say that G is *k-colorable*. The *chromatic number* of G, denoted $\chi(G)$, is the least k such that G is k-colorable.

The graph on n vertices in which every pair of vertices is adjacent is denoted by K_n and called a *complete graph* (Figure 1.1). The complement of a complete graph is an *empty* graph. A graph is *bipartite* if its vertex set can be expressed as the disjoint union of sets X and Y so that $G[X]$ and $G[Y]$ are empty graphs; the sets X and Y are called the *parts* or *partite sets* of G. The *complete bipartite graph $K_{r,s}$* is a bipartite graph with partite sets X and Y of sizes r and s, respectively, in which every $u \in X$ is adjacent to every $v \in Y$ (Figure 1.2). A graph is *k-partite* if its vertex set can be expressed as the disjoint union of k independent sets $X_1, \ldots, X_k$. The *complete multipartite graph $K_{k_1,\ldots,k_r}$* is an r-partite graph in which $uv \in E(K_{k_1,\ldots,k_r})$ if and only if u and v lie in distinct partite sets.

A *path P_n* is a graph whose vertices can be labeled $v_1, \ldots, v_n$ such that $E(P_n) = \{v_i v_{i+1} \colon i \in [n-1]\}$ (Figure 1.3); v_1 and v_n are then the *endpoints* of the path. A *cycle C_n* is a graph whose vertices can be labeled $v_0, \ldots, v_{n-1}$ such that $E(C_n) = \{v_0v_1, v_1v_2, \ldots, v_{n-2}v_{n-1}, v_{n-1}v_0\}$. If G is a path or a cycle, then the *length* of G is $|E(G)|$. If G contains no cycles, then G is a *forest*. A graph is *connected* if any two vertices form the set of endpoints of a path. A graph that is connected and has no cycles is a *tree*. If P is a path with endpoints u and v and $P \subseteq G$, then we say that P is a *u,v-path* in G. The *distance* between u and v in G, denoted $d_G(u,v)$, is the minimum length of a u,v-path in G. Given a graph G and an positive integer k, the *kth power of* G, denoted G^k, has vertex set $V(G)$, with $uv \in E(G^k)$ whenever $d_G(u,v) \le k$.

A *component* of G is a maximal connected subgraph of G. We say that G is *k-connected* if $|V(G)| > k$

Figure 1.3: P_6

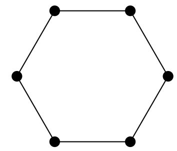

Figure 1.4: C_6

and $G - S$ is connected for any $S \subseteq V(G)$ such that $|S| < k$. A *cut-vertex* in G is a vertex v such that $G - v$ has more components than G. A *cut-edge* in G is an edge uv such that $G - uv$ has more components than G. A *block* is a maximal subgraph containing no cut-vertex. The *diameter* of a connected graph G, denoted $\mathrm{diam}(G)$, is the maximum distance between any pair of vertices.

A *leaf* is another term for a 1-vertex. The *core* of a G is the graph obtained by iteratively removing leaves. The *star* on n vertices is the tree $K_{1,n-1}$.

The *disjoint union* of graphs G and H, denoted $G + H$, has vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. The *join* of graphs G and H, denoted[1] $G \oplus H$, is obtained by adding to the disjoint union of G and H the edges $\{uv\colon\ u \in V(G), v \in V(H)\}$. An *ear* in a graph G is a subgraph that is a path on vertices $v_1, \ldots, v_n$ in which $d_G(v_i) = 2$ for $2 \leq i \leq n - 1$ or a cycle in which all but at most one vertex has degree 2. We call the latter case a *closed ear*.

A graph *embeds* on a surface if it can be drawn on that surface so that no edges intersect; a particular drawing meeting this condition is an *embedding* on the surface. The *genus* of G is the least genus among all the orientable surfaces on which G embeds. A graph is *planar* if it has an embedding in the plane. A *plane graph* is an embedding of a graph in the plane. A region on a surface is *connected* if any two points in it lie on some piecewise linear curve contained in the region. In an embedding, a *face* is a maximally connected region containing no point in the image of an edge. A *k-face* is a face bounded by a cycle of length k. A planar graph is *outerplanar* if it has a planar embedding in which all vertices lie on the unbounded face. The *weak dual* of a plane graph G has vertices corresponding to bounded faces of G; two vertices form an edge in the weak dual whenever the two corresponding faces in G have a common boundary edge. We say that a plane graph G is a *triangulation* of a plane graph H if $H \subseteq G$ and every face in G is a 3-face; we say that G is a *weak triangulation* of H if every face except the unbounded face is a 3-face.

[1] Introduced by Douglas West, this notation is consistent with the "Czech notation" introduced by Nešetřil in which the notation displays the result of the operation on K_2 and K_2. This notation evokes the additivity of the vertex sets and avoids conflicting with the proper use of "+" for disjoint union.

Chapter 2

Online Choosability

The list version of graph coloring, introduced by Vizing [55] and Erdős, Rubin, and Taylor [16], has now been studied in hundreds of papers. Instead of having the same colors available at all vertices, each vertex has a list of available colors. Since the lists at vertices could be identical, always $\chi(G) \le \mathrm{ch}(G)$.

An online version of list coloring was introduced by Zhu [57]; independently, Schauz [48] introduced an equivalent notion in a game setting. The lists are revealed in steps; one color at a time, the vertices with that color in their lists are shown or "marked". The coloring algorithm, which we call *Painter*, must choose an independent set of marked vertices to receive that color. Colored vertices will not be marked again; in essence, they are removed from the graph.

The behavior of Lister in the Lister/Painter game models the need for Painter to win against the "worst-case" presentation of lists in online list coloring, so the problems are equivalent. (Schauz originally called the players "Mr. Paint and Mrs. Correct"; "Marker/Remover" was also used.) If G is not f-choosable, then G is not f-paintable, since Lister can mimic a bad list assignment L by marking in round i the set $\{v \in V(G)\colon\ i \in L(v)\}$; winning moves by Painter would form an L-coloring, which does not exist. In particular, $\mathring{\mathrm{ch}}(G) \ge \mathrm{ch}(G) \ge \chi(G)$. These values may all be distinct; they equal $4, 3, 2$ for $K_{4,r}$ when $12 \le r \le 18$, for $K_{5,s}$ when $9 \le s \le 12$, and for $K_{6,t}$ when $8 \le s \le 10$.

A fundamental question in paintability was whether or not $\mathring{\mathrm{ch}}(G) - \mathrm{ch}(G)$ can grow without bound. For example, let $G = K_{3,\ldots,3}$ with k parts. Kierstead [27] proved $\mathrm{ch}(G) = \lceil (4k-1)/3 \rceil$. Another proof appears in [31], where the authors also showed $\mathring{\mathrm{ch}}(G) \le 3k/2$. However, it remains unknown whether $\mathring{\mathrm{ch}}(G) > \lceil (4k-1)/3 \rceil$. Another candidate is $K_{m,m}$. Using bounds on the number of edges needed to form a non-2-choosable k-uniform hypergraph, a result of Beck [4] implies that $\mathrm{ch}(K_{m,m}) \le \lg m - (\frac{1}{3} - o(1)) \lg\lg m$. By using a result of Radhakrishnan and Srinivasan [45], Alon observed that this bound can be improved to $\mathrm{ch}(K_{m,m}) \le \lg m - (\frac{1}{2} - o(1)) \lg\lg m$. For paintability, Kim, Kwon, Liu, and Zhu [28] showed $\mathring{\mathrm{ch}}(K_{m,m}) \le \lg m$. Duraj, Gutowski, and Kozik [15] proved that $\mathring{\mathrm{ch}}(K_{m,m}) = \lg m - O(1)$. This answers the above question in the affirmative by showing that $\mathring{\mathrm{ch}}(K_{m,m}) - \mathrm{ch}(K_{m,m}) = \Omega(\lg\lg m)$.

Since $\mathrm{ch}(G) \geq \chi(G)$, a theme in the study of choosability has been to show that some earlier upper bound on $\chi(G)$ holds also for $\mathrm{ch}(G)$, thereby strengthening the earlier result. Since always $\mathring{\mathrm{ch}}(G) \geq \mathrm{ch}(G)$, paintability can be studied in a similar way. Schauz strengthened several choosability results to paintability, showing in [48] that planar graphs are 5-paintable and that bipartite graphs are $\Delta(G)$-edge-paintable. In [49] and [20], he strengthened the Alon–Tarsi Theorem [2] to the setting of paintability by a purely combinatorial proof, and from this he obtained strengthenings to paintability of various choosability consequences of the Alon–Tarsi Theorem. In the language of online choosability, Zhu [57] characterized 2-paintable graphs and proved results about 3-paintability for complete bipartite graphs.

Since $\mathrm{ch}(G) \geq \chi(G)$, it is natural to ask when $\mathrm{ch}(G) = \chi(G)$; such graphs are *chromatic-choosable*. Analogously (and more restrictively), G is *chromatic-paintable* if $\mathring{\mathrm{ch}}(G) = \chi(G)$. Ohba [42] conjectured that G is chromatic-choosable when $|V(G)| \leq 2\chi(G) + 1$; after partial results in [30, 42, 43, 46], this was proved by Noel, Reed, and Wu [40]. Various researchers (see [28]) observed that the complete multipartite graph $K_{2,\ldots,2,3}$ is chromatic-choosable but not chromatic-paintable, so the paintability analogue is slightly different.

Conjecture 2.0.1. [22] *If $|V(G)| \leq 2\chi(G)$, then G is chromatic-paintable.*

In [22], the Combinatorial Nullstellensatz was used to show that $K_{2,\ldots,2}$ and several similar graphs with $|V(G)| = 2\chi(G)$ are chromatic-paintable. For $K_{2,\ldots,2}$ this was reproved in [28] by an explicit strategy for Painter. A weaker version of Conjecture 2.0.1 would be $\mathring{\mathrm{ch}}(G) = \chi(G)$ when $|V(G)| \leq c\chi(G)$, for some $c \in (1, 2]$.

Lacking such a result, we study lower-order terms. Ohba [42] proved that G is chromatic-choosable when $|V(G)| \leq \chi(G) + \sqrt{2\chi(G)}$. We strengthen and extend this in Section 2.1, proving that G is chromatic-paintable when $|V(G)| \leq \chi(G) + 2\sqrt{\chi(G) - 1}$. We show that if G is k-paintable and $|V(G)| \leq \frac{t}{t-1}k$, then $\mathring{\mathrm{ch}}(G \mathbin{\oplus} \overline{K}_t) \leq k + 1$. The application then follows by induction on the chromatic number. We also prove for all G that $G \mathbin{\oplus} K_t$ is chromatic-paintable when t is sufficiently large; this was proved independently by Kozik, Micek, and Zhu [31] and used there to obtain a slightly weaker strengthening of Ohba's result.

In Section 2.2, the general problem of f-paintability leads to a recurrence that provides an upper bound on the smallest r such that $K_{k+j,r}$ is not k-paintable. This echoes both the elementary result by Vizing [55] that $K_{k,r}$ is k-choosable if and only if $r < k^k$ and the subsequent result by Hoffman and Johnson [21] that $K_{k+1,r}$ is k-choosable if and only if $r < k^k - (k-1)^k$. It turns out that $K_{k,r}$ is k-paintable if and only if $r < k^k$, but $K_{k+1,r}$ fails to be k-paintable when r is smaller than $k^k - (k-1)^k$ by a constant fraction.

Sections 2.1 and 2.2 are based on joint work with Carraher, Loeb, Puleo, Tsai, and West that appears in [10].

2.1 Paintability of Graph Joins

In this section we strengthen results of Ohba [42]. When f is a token assignment on G and $H \subseteq G$, we say that H is *f-paintable* when Painter has a winning strategy on H for the restriction of f to $V(H)$; that is, when each vertex of H starts with $f(v)$ tokens. The following elementary statements about the Lister/Painter game also appear in [10, 11, 57].

Proposition 2.1.1. *The following statements hold for the Lister/Painter game.*

(a) If G is f-paintable, then every subgraph H of G is f-paintable.

(b) If $f(v) > d_G(v)$, then G is f-paintable if and only if $G - v$ is f-paintable.

(c) If G is f-paintable, then there is a winning strategy for Painter that always colors a maximal independent subset of Lister's marked set.

(d) If $f(v) = 1$ for some $v \in V(G)$, then G is f-paintable if and only if $G - v$ is $(f - \delta_{N_G(v)})$-paintable.

(e) If G is not f-paintable, then Lister has a winning strategy in which every marked set except the last forms a connected subgraph of G with at least one edge.

Proof. (a) Edge deletion does not invalidate Painter moves, so we may let H be an induced subgraph. In the game on H, Painter can respond to Lister's moves as in G and win.

(b) If $G - v$ is not f-paintable, then by ignoring v in the game on G, a winning strategy for Lister on $G - v$ is also a winning strategy on G. Now suppose that $G - v$ is f-paintable. To win on G, Painter follows a winning strategy for $G - v$, coloring v when marked only if none of its neighbors are colored by that strategy. In this way at most $d_G(v)$ tokens will be used at v, so Lister cannot win.

(c) If Lister marks M and Painter, playing optimally, colors a non-maximal independent subset D of M, then by (a), Painter would still have a winning strategy by coloring a maximal independent subset D' of M that contains D.

(d) If $f(v) = 1$, then one possible move by Lister is to mark $N[v]$. Painter must color v, and G being f paintable implies that Painter wins on $G - v$, with token assignment given by $f - \delta_{N(v)}$. If $G - v$ is $(f - \delta_{N(v)})$-paintable, then in the game on G, Painter must delete v the first time it is marked. The extra token given to each neighbor of v is reserved for the round on which v is marked and colored, guaranteeing that Painter has a winning strategy in G.

(e) If, when playing optimally, Lister marks M such that $G[M]$ is not connected and Painter colors D, then Lister must be able to win in the remaining game. Let $H_1, \cdots, H_r$ be the components of $G[M]$, and let $M_i = V(H_i)$. Suppose instead that Lister marks M_1 followed by M_2 and so on for r consecutive rounds, and on each round Painter colors an independent subset D_i of M_i. Let $D' = \bigcup_{i=1}^{r} D_i$. Since there are no edges joining any vertex of H_i with any vertex of H_j for $i \neq j$, and because each D_i is an independent set, we have that D' is an independent subset of M. Since Painter may respond to M by coloring D', we have that Lister must still win on the remaining graph, even with the restriction to only mark connected sets.

Similarly, if Lister marks a single vertex v and Painter can color it, then Lister must be able to win in the remaining game. Hence Lister can win by not marking v and instead playing the rest of the game on the rest of the graph. So unless Painter is unable to color v when it is the only vertex marked (that is, when it is out of tokens), then Lister marks at least two vertices, and the marked set forms a connected subgraph of G. □

A graph is *d-degenerate* if every subgraph has a vertex of degree at most d. The importance of Proposition 2.1.1(b) is that if in the course of the game on a graph H a position arises in which a vertex has more remaining tokens than its remaining degree, then this vertex can be colored without affecting who wins. For example, d-degenerate graphs are $(d+1)$-paintable. When invoking Proposition 2.1.1(b) to discard vertices, we say "by degeneracy".

The observations in Proposition 2.1.1 lead to our tool (Theorem 2.1.2) for extending Ohba's Theorem. The special case $t = 1$, implies (when iterated) that $G \oplus K_r$ is $(k + r)$-paintable when G is k-paintable.

Theorem 2.1.2. *If G is k-paintable and $(t-1)|V(G)| \leq tk$, then $G \oplus \overline{K}_t$ is $(k+1)$-paintable.*

Proof. When $t = 1$, Painter colors the added vertex on the first round that it is marked, sacrificing at most one token at each vertex of G. Since G is k-paintable, Painter wins.

When $t > 1$, we give a more careful strategy for Painter. Let T denote the added independent set of t vertices. By Proposition 2.1.1(e), we may assume that Lister marks at least one vertex in $V(G)$ until no such vertices remain.

Let $\mathcal{S}$ be a winning strategy for Painter on G with k tokens at each vertex. Painter uses $\mathcal{S}$ until $V(G)$ is exhausted, except for one special round associated with the extra token at the vertices of G. When Lister marks M, say that the vertices of $T - M$ are *omitted*. Painter responds within $M \cap V(G)$ as specified by $\mathcal{S}$, unless condition (*) holds.

$$\text{Each vertex of } T - M \text{ has been omitted in at least } \mu \text{ rounds, where } \mu = \tfrac{k}{t-1}. \qquad (*)$$

Note that, (*) is satisfied if ever $T \subseteq M$. When (*) first occurs, Painter colors $T \cap M$. Subsequently, Painter continues to use $\mathcal{S}$ (some vertices of G may have an extra token). It suffices to show (1) condition (*) must occur before Lister can win, and (2) after the round when (*) occurs, each vertex of $T - M$ has more tokens remaining than the number of vertices remaining in G. By degeneracy, the rest of T can be ignored, and continuing to use $\mathcal{S}$ enables Painter to win.

(1): While (*) has not occurred, Lister cannot mark a vertex v of T more than k times. When v is marked and (*) has not occurred, each round that marked v has omitted at least one vertex of T that was not yet omitted μ times. Hence v has been marked fewer than $(t-1)\mu$ times. Since $(t-1)\mu \leq k$ and v is any marked vertex of T, Lister has not won.

(2): Suppose that (*) occurs when Lister marks M in round r. A vertex $v \in T - M$ still has at least $k+1-(r-\mu)$ tokens. This value exceeds $\frac{t}{t-1}k - r$. Since $d_G(v) = |V(G)|$, and at least one vertex is colored from $V(G)$ on each of these r rounds, the remaining degree of v is at most $\frac{t}{t-1}k - r$. □

Definition 2.1.3. Let G be a complete multipartite graph with t distinct sizes of parts, $k_1, \ldots, k_t$, where there are r_i parts of size k_i. Following [28], we denote G by $K_{k_1 * r_1, \ldots, k_t * r_t}$. In addition, when $r_i = 1$, we drop "$*r_i$" from the notation. ■

Erdős, Rubin, and Taylor [16] proved that K_{2*r} is chromatic-choosable, strengthened to chromatic-paintability in [28]. Theorem 2.1.2 yields this immediately using $t = 2$.

Corollary 2.1.4. *If G is chromatic-paintable, and $|V(G)| \leq 2\chi(G)$, then $G \oplus \overline{K}_2$ is chromatic-paintable. In particular, K_{2*r} is chromatic-paintable.*

An *optimal coloring* is a proper coloring of G using $\chi(G)$ colors. Theorem 2.1.2 also yields chromatic-paintability for some other complete multipartite graphs, providing partial results toward Conjecture 2.0.1. Let $\mathcal{G}_a$ denote the class of graphs having an optimal coloring in which each color class has size at most a. Corollary 2.1.4 proves Conjecture 2.0.1 for graphs in $\mathcal{G}_2$. We next prove chromatic-paintability for a subset of $\mathcal{G}_3$. By Proposition 2.1.1(a), it suffices to consider complete multipartite graphs.

Corollary 2.1.5. *$K_{1*q, 2*r, 3*s}$ is chromatic-paintable when $q \geq 1$ and $3s \leq q + 3$.*

Proof. If $K_{1*q,3*s}$ is chromatic-paintable and $q + 3s \leq 2(q+s)$, then adding independent sets of size 2 preserves chromatic-paintability, as observed above. The inequality reduces to $s \leq q$, which holds when $3s \leq q+3$ with $q \geq 1$ and s is an integer. Therefore, it suffices to show that $K_{1*q,3*s}$ is chromatic-paintable when $3s \leq q + 3$.

We start with K_{1*q} and iteratively take the join with independent sets of size 3. Consider $G \oplus \overline{K}_3$. To apply Theorem 2.1.2, we need $|V(G)| \leq \frac{3}{2}\,\mathring{\mathrm{ch}}(G)$. By induction on s, $K_{1*q,3*s}$ will be chromatic-paintable if $q + 3(s-1) \leq (3/2)(q+s-1)$, which simplifies to $3s \leq q+3$. □

Note that here the number of vertices is at most $2k+1-\frac{2}{3}q$, where k is the chromatic number. We require $q \geq 1$ because $K_{2*r,3}$ is not chromatic-paintable for $r > 1$ [28]. For balanced complete multipartite graphs K_{r*k}, less is known. Alon [1] used probabilistic methods to prove that $\mathrm{ch}(K_{k*m}) = \Theta(k \ln m)$. Kierstead, Salmon, and Wang [30] proved that $\mathrm{ch}(K_{4*s}) = \left\lceil \frac{3k-1}{2} \right\rceil$ and that $\mathring{\mathrm{ch}}(K_{4*3}) \geq 5$. Noel, West, Wu, and Zhu [41] gave constructive lower bounds on the choosability of K_{r*k}, proving that $\mathrm{ch}(K_{5*k}) \geq \left\lfloor \frac{8k}{5} \right\rfloor$ and that $\mathrm{ch}(K_{6*k}) \geq \left\lfloor \frac{5k}{3} \right\rfloor$ They also gave a simple construction showing that $\mathrm{ch}(K_{r*k}) \geq c(k-1)\frac{\log r}{\log k}$.

Ohba [42] showed that G is chromatic-choosable when $|V(G)| \leq \chi(G) + \sqrt{2\chi(G)}$. Using Theorem 2.1.2, we obtain chromatic-paintability under a weaker restriction.

Theorem 2.1.6. *If $|V(G)| \leq \chi(G) + 2\sqrt{\chi(G)-1}$, then G is chromatic-paintable.*

Proof. Let $n = |V(G)|$ and $k = \chi(G)$. A proper k-coloring of G expresses G as a subgraph of a complete k-partite graph, so we may assume that G is a complete k-partite graph. Let q be the number of parts of size 1. We prove the claim by induction on $k-q$. When $k-q=1$, G has the form $\overline{K}_t \oplus K_q$, which is chromatic-paintable by repeatedly applying Theorem 2.1.2, adding one vertex each time.

When $k-q>1$, let T be a smallest non-singleton part, with $t = |T|$, and let $G' = G - T$. It suffices to prove (1) G' is small enough for the induction hypothesis to making G' chromatic-paintable, and (2) t is small enough for Theorem 2.1.2 to apply when T is added to G'.

(1): We are given $n \leq k + 2\sqrt{k-1}$ and need $n - t \leq (k-1) + 2\sqrt{k-2}$. Thus it suffices to prove $k + 2\sqrt{k-1} - t \leq k - 1 + 2\sqrt{k-2}$. This inequality simplifies to $t - 1 \geq 2(\sqrt{k-1} - \sqrt{k-2})$. The difference of consecutive square roots is small; when $t \geq 2$ and $k \geq 3$, the inequality holds.

(2): Again we are given $n \leq k + 2\sqrt{k-1}$, but now we need $n - t \leq \frac{t}{t-1}(k-1)$. It suffices to prove $k + 2\sqrt{k-1} - t \leq \frac{t}{t-1}(k-1)$, which simplifies to $2\sqrt{k-1} \leq t - 1 + \frac{k-1}{t-1}$. The right side is minimized when $t - 1 = \sqrt{k-1}$, and there equality holds. □

The bound in Theorem 2.1.6 holds with equality when $k-1$ is a perfect square and G is the complete k-partite graph with $k-2$ parts of size 1 and two parts of size $1 + \sqrt{k-1}$.

Consider now $G \oplus K_t$. Although always $\chi(G \oplus K_t) = \chi(G) + t$, adding t vertices need not increase the paint number by t. In fact, $G \oplus K_t$ is chromatic-paintable when t is sufficiently large. Kozik, Micek, and Zhu [31] also proved this, but without an explicit bound on the value of t that suffices. We prove a technically more general statement; it applies to all graphs because always G is d-degenerate when d is the maximum degree.

Theorem 2.1.7. *Let G be a d-degenerate graph having an optimal coloring with color classes $V_1, \ldots, V_k$*

such that $|V_i| \le a$ *for* $i \in [k]$. *If* $t \ge (a+1)d$, *then* $\mathring{\mathrm{ch}}(G \Diamond K_t) = \chi(G \Diamond K_t)$.

Proof. Let π be an ordering of $V(G)$ in which each vertex has at most d earlier neighbors. Let T be the set of t added dominating vertices not in G. Since $\chi(G \Diamond K_t) = k + t$, it suffices to give a strategy for Painter to show that $G \Diamond K_t$ is $(k+t)$-paintable.

When $M \subseteq V(G)$, Painter colors an independent subset of M chosen greedily with respect to π. For $v \in V(G)$, there are at most d such rounds in which v is rejected, by the choice of π. Hence we will reserve $d+1$ tokens for such rounds, not used on rounds with $M \cap T \neq \emptyset$. When $M \cap T \neq \emptyset$, Painter will color a vertex of $M \cap T$ or a subset of $M \cap V(G)$. We must ensure that the first option is not used too often when $v \in M$; the second option causes no trouble if $d+1$ tokens are reserved for v.

Let v be a vertex of G remaining when round s begins. Let $g(v,s)$ denote the number of earlier steps on which $M \cap T \neq \emptyset$ and $v \notin M$. Let $V_{i,s} = \{v \in V_i \cap V(G_s) \colon\ g(v,s) \le d\}$. Intuitively, $V_{i,s}$ is the set of vertices in color class V_i that have not yet had $d+1$ tokens reserved for use within G. Always $V_{i,s+1} \subseteq V_{i,s}$.

When $M \cap T \neq \emptyset$, if $V_{i,s} \subseteq M$ for some i, then Painter colors $V_{i,s}$ for some such i. Otherwise, Painter colors a vertex of $M \cap T$.

Claim 1: *While* $\bigcup_i V_{i,s} \neq \emptyset$, *at most* $ad + k - 1$ *rounds with* $M \cap T \neq \emptyset$ *have been played.* On at most $k-1$ such rounds, Painter colored a nonempty set $V_{i,s}$, since a nonempty one remains. When Painter colors a vertex of $M \cap T$, some vertex of $V_{i,s}$ is not in M. Since $|V_i| \le a$, this happens at most ad times while $V_{i,s}$ remains nonempty.

Claim 2: *While* $v \in V_{i,s}$, *there remain more than* d *tokens at* v. Since $t \ge (a+1)d$, vertex v starts with at least $(a+1)d + k$ tokens. By Claim 1, at most $ad + k - 1$ rounds were played with $M \cap T \neq \emptyset$. Hence at least $d+1$ tokens remain available at v.

Claim 3: *A vertex of* T *loses at most* $k+t$ *tokens before removal.* A vertex of T loses tokens only when $M \cap T \neq \emptyset$. On such rounds, Painter colors a vertex of T or the set $V_{i,s}$ for some i. There are at most t rounds of the first type and at most k of the second type.

By Claim 3, all of T is colored. By Claim 2, $d+1$ tokens get reserved at each vertex of G. Hence the Painter strategy succeeds. □

2.2 Complete Bipartite Graphs

Vizing [55] proved that $K_{k,r}$ is k-choosable if and only if $r < k^k$. We extend this characterization to k-paintability by considering a more general f-paintability problem on $K_{k,r}$. The theorem leads to further results about the k-paintability of $K_{k+j,r}$.

Theorem 2.2.1. *Consider $K_{k,r}$ with $k \le r$, having parts X and Y with $X = x_1, \dots, x_k$. When $f(x_i) = t_i$ and $f(y) = k$ for $y \in Y$, Painter has a winning strategy if and only if $r < \prod t_i$.*

Proof. Necessity. It suffices to show that $K_{k,r}$ is not f-choosable when $r = \prod t_i$, Let $L(x_i) = U_i$ so that $|U_i| = t_i$ and $U_1, \dots, U_k$ are pairwise disjoint. Let $\{L(y)\colon y \in Y\} = U_1 \times \cdots \times U_k$. Any coloring chosen from these lists puts a color from U_i on x_i for $1 \le i \le k$, but then the vertex in Y having this set as its list cannot be properly colored.

Sufficiency. Note that $r < \prod t_i$ requires $\min t_i \ge 1$. We use induction on $\sum t_i$. When $\sum t_i = k$, we have $Y = \emptyset$, and Painter will win. For $\sum t_i > k$, consider the first marked set M. By Proposition 2.1.1(e), we may assume that M intersects both X and Y. We may also assume $|M \cap X| = 1$; otherwise, Painter colors $M \cap X$, and each remaining vertex in Y has more tokens than its degree. We may assume $M \cap X = \{x_k\}$. Let $q = |M \cap Y|$.

Case 1: $q < \prod_{i=1}^{k-1} t_i$. Painter colors v_k; there remain $k-1$ vertices in X. By degeneracy, vertices of $Y - M$ are now irrelevant. Each of the q vertices of $Y \cap M$ has $k-1$ tokens; the induction hypothesis applies.

Case 2: $q \ge \prod_{i=1}^{k-1} t_i$. Painter colors $M \cap Y$. Since $r < \prod_{i=1}^{k} t_i$, the number of vertices left in Y is less than $\prod_{i=1}^{k} t_i - \prod_{i=1}^{k-1} t_i$, which equals $(t_k - 1)\prod_{i=1}^{k-1} t_i$. Since $t_k - 1$ tokens remain on v_k, the induction hypothesis applies. □

Corollary 2.2.2. *$K_{k,r}$ is k-paintable if and only if $r < k^k$.*

Hoffman and Johnson [21] proved that $K_{k+1,r}$ fails to be k-choosable if and only if $r \ge k^k - (k-1)^k$. We will see in Corollary 2.2.8 that the least r such that $K_{k+1,r}$ is not k-paintable is smaller than this when $k \ge 4$ (also for $k = 3$, by computer search). Consider $K_{l,r}$ with partite sets X of size l and Y of size r. We present a recursive strategy for Lister on $K_{l,r}$ when the vertices in Y all have k tokens and the vertices in X have $t_1, \dots, t_l$ tokens, respectively.

Definition 2.2.3. Fix nonnegative integers k and l. Given an l-tuple t of nonnegative integers, and $S \subseteq [l]$, let t^S denote the l-tuple obtained from t by reducing by 1 the coordinates indexed by S, and let $t|_{\overline{S}}$ denote the $(l - |S|)$-tuple obtained from t by restricting t to the coordinates indexed by $[l] - S$. Define $g(k, t)$

recursively by letting

$$g(k,t) = \begin{cases} 0 & \text{if } \prod t_i = 0, \\ 1 & \text{if } k = 0, \\ \infty & \text{if } l < k, \\ \min\limits_{S\subseteq[l]} \left[g(k-1, t|_{\overline{S}}) + g(k, t^S)\right] & \text{otherwise.} \end{cases}$$

■

Proposition 2.2.4. *If $l = k$, then $g(k,t) = \prod_{i=1}^{l} t_i$.*

Proof. We use induction on $\prod t_i$. By the definition of $g(k,t)$, the claim holds when $\min t_i = 0$. Otherwise, the boundary cases imply that in the nontrivial case the minimum occurs only when $|S| = 1$. By symmetry, we may assume $S = \{l\}$. By the induction hypothesis, $g(k,t) = \prod_{i=1}^{l-1} t_i + (t_l - 1)\prod_{i=1}^{l-1} t_i = \prod_{i=1}^{l} t_i$. □

Thus the next theorem includes, within the special case $l = k$, a proof that $K_{k,r}$ is not k-paintable when $r \geq k^k$.

Theorem 2.2.5. *Consider $K_{l,r}$ with $l \leq r$, having parts X and Y with $X = x_1, \ldots, x_l$. When $f(x_i) = t_i$ and $f(y) = k$ for $y \in Y$, Lister has a winning strategy if $r \geq g(k,t)$.*

Proof. We give a recursive strategy for Lister, using induction on $\prod t_i$. If $\min t_i = 0$, then Lister wins by marking a vertex in X with no tokens, even if Y is empty.

When $\min t_i > 0$, we may assume $r = g(k,t)$ and let S be an index subset of $[l]$ that yields the minimum in the definition of $g(k,t)$. Lister marks $\{x_i\colon\ i \in S\}$ plus $g(k-1, t|_{\overline{S}})$ vertices in Y. By Proposition 2.1.1(a), Painter colors a maximal independent subset of M.

Case 1: *Painter colors $M \cap X$.* Lister continues play on $(X - M) \cup (M \cap Y)$, ignoring the vertices of $Y - M$. Each vertex of $M \cap Y$ now has $k-1$ tokens. Since $|M \cap Y| = g(k-1, t|_{\overline{S}})$, the induction hypothesis implies that Lister has a winning strategy in the remaining game.

Case 2: *Painter colors $M \cap Y$.* Lister continues play on the remaining graph, with vertices $(Y - M) \cup X$. The token-count vector on X is now t^S. Since $Y - M = g(k, t^S)$, the induction hypothesis implies that Lister has a winning strategy in the remaining game. □

When Painter colors $M \cap Y$, Lister would prefer to have marked as many vertices of X as possible to obtain the maximum reduction in tokens on X. The danger, of course, is that when $M \cap X$ is larger, Painter may then decide to color $M \cap X$.

Theorem 2.2.5 provides only an upper bound on the least r such that Lister has a winning strategy. Ignoring $Y - M$ when Painter colors $M \cap X$ limits Lister's options; possibly Lister should use these vertices. By Proposition 2.1.1(b), we may assume that $|S| \leq j + 1$. This restriction on Lister allows us to compute upper bounds for larger examples.

Example 2.2.6. One approach for Lister is to always mark $l - k + 1$ vertices in X with the most tokens. This strategy is optimal when $l = k$ (always $|M \cap X| = 1$, but it did not matter in Proposition 2.2.4 which was marked).

Consider $K_{k+1,r}$, where $l = k + 1$ and each t_i is k. If $|M \cap X| = 2$ and Painter colors $M \cap X$, then the remaining game has the form in Theorem 2.2.1. Lister can win it if $\prod_{i \notin S} t_i$ vertices remain (with $k - 1$ tokens each) in $M \cap Y$ (that is, $|M \cap Y| = g(k - 1, t|_{\overline{S}})$). By making $M \cap Y$ this size, Lister forces Painter to color $M \cap Y$ to avoid losing.

By summing instances of $\prod_{i \notin S} t_i$ as the token counts in X decrease, we can accumulate enough vertices in Y to ensure a win for Lister. Eventually some token count in X is driven to 0, and Lister wins by marking that vertex with no need for additional vertices in Y. We thus obtain a value of r such that $K_{k+1,r}$ is not k-paintable. Below we list the computation for $k \leq 4$. At each step, we list the vector t of token counts in X and the number of vertices in Y that Lister will mark and Painter will color.

$k = 2$		$k = 3$		$k = 4$	
(2,2,2)	2	(3,3,3,3)	9	(4,4,4,4,4)	64
(1,1,2)	1	(2,2,3,3)	4	(3,3,4,4,4)	36
		(2,2,2,2)	4	(3,3,3,3,4)	27
		(1,1,2,2)	1	(2,3,3,3,3)	18
		(1,1,1,1)	1	(2,2,2,3,3)	8
				(2,2,2,2,2)	8
				(1,1,2,2,2)	2
				(1,1,1,1,2)	1
Total	3	Total	19	Total	164

We conclude that $K_{3,3}$ is not 2-paintable, $K_{4,19}$ is not 3-paintable, and $K_{5,164}$ is not 4-paintable. We will show that the in general the threshold on r for $\mathring{\text{ch}}(K_{k+1,r}) > k$ is smaller than the threshold $k^k - (k-1)^k$ for $\text{ch}(K_{k+1,r}) > k$ [21] (the threshold for $\text{ch}(K_{4,r}) > 3$ is $r = 19$, but the threshold for $\text{ch}(K_{5,r}) > 4$ is $r = 175$, larger than 164).

However, Lister in fact can win even with smaller r. Exhaustive computer search of the games has shown that already $K_{4,12}$ is not 3-paintable, improving the bound $r \leq 19$ computed above ($K_{4,11}$ is 3-paintable). We have not determined the least r such that $K_{5,r}$ is not 4-paintable. However, optimizing over S to compute the recursive bound $g(4,(4,4,4,4,4))$ from Theorem 2.2.5, we find that $K_{5,126}$ is not 4-paintable, improving on $K_{5,164}$.

Marking the vertices in X with highest token counts greedily makes the product of the remaining counts smallest at each step. This heuristic is generally good, but we have found instances where it is better for Lister to mark vertices with smaller counts. We also have found instances where it is better for Lister to mark fewer than $l-k+1$ vertices in X. These anomalies suggest that determining the paintability of $K_{l,r}$ in general is very hard, and hence we only present bounds.

Proposition 2.2.7. *If t is the $(k+1)$-tuple $(k,\ldots,k)$, then $h(t) \leq \frac{k+1}{2k}(k^k-1)+k^{k-1}$, where h is the result of the recursive computation in Definition 2.2.3 when S always corresponds to two vertices in X with the most tokens.*

Proof. The iteration to compute h takes no more than $\frac{(k-1)(k+1)}{2}+1$ rounds. Each round accumulates the product of all remaining entries except the two largest, and those largest decrease by 1. To get an upper bound on the product at each round, replace all entries with their average. Summing over $0 \leq n \leq (k-1)(k+1)/2$, we have

$$h(t) \leq \sum_{n=0}^{\frac{k^2-1}{2}} \left(k - \frac{2n}{k+1}\right)^{k-1} \leq k^{k-1} + \int_0^{\frac{k^2-1}{2}} \left(k - \frac{2x}{k+1}\right)^{k-1} dx = \frac{k+1}{2k}(k^k-1)+k^{k-1}. \quad \square$$

Corollary 2.2.8. *When k is sufficiently large, $K_{k+1,r}$ is k-choosable when $r < (.62+o(1))k^k$, but $K_{k+1,r}$ is not k-paintable when $r > (.5+o(1))k^k$.*

Proof. The threshold $k^k-(k-1)^k$ for non-k-choosability [21] is asymptotic to $k^k(1-e^{-1})$. On the other hand, the bound from Proposition 2.2.7 is less than $\frac{1}{2}k^k(1+3/k)$. $\square$

Chapter 3

$\mathcal{P}$-Suitable Paintability

The classical versions of graph coloring, choosability, and paintability all require that the coloring algorithm produces a proper coloring. However, we may require that a proper coloring also fulfills some property $\mathcal{P}$. When a coloring of G satisfies $\mathcal{P}$, we call it a *$\mathcal{P}$-suitable coloring*. If there exists a $\mathcal{P}$-suitable coloring ϕ' of $V(G)$ such that $\phi'(v) = \phi(v)$ for all $v \in S$, then we say that ϕ is *$\mathcal{P}$-extendable to G*. Given a graph G, the *$\mathcal{P}$-chromatic number*, denoted $\chi_{\mathcal{P}}(G)$, is the minimum number of colors needed to produce a $\mathcal{P}$-suitable coloring of G. Similarly, the *$\mathcal{P}$-choice number*, denoted $\mathrm{ch}_{\mathcal{P}}(G)$, is the least k such that a $\mathcal{P}$-suitable coloring can be chosen from the lists in any list assignment L satisfying $|L(v)| \geq k$ for all v

Definition 3.0.1. Let a graph G and token assignment f for G be given. In each round, *Lister* marks a nonempty subset M of uncolored vertices, removing one token at each marked vertex. *Painter* responds by selecting a subset D of M such that when a color distinct from those used on previous rounds is given to all vertices in D, the resulting coloring is $\mathcal{P}$-extendable to G. Painter wins by producing a $\mathcal{P}$-suitable coloring of G, and Lister wins by marking a vertex with no tokens.

When Painter has a winning strategy, we say that G is *$\mathcal{P}$-suitably f-paintable.* If $f(v) = k$ for all v and G is f-paintable, then we say that G is *$\mathcal{P}$-suitably k-paintable.* The least k such that G is $\mathcal{P}$-suitably k-paintable is the *$\mathcal{P}$-paint number*, denoted $\mathring{\mathrm{ch}}_{\mathcal{P}}(G)$. ■

The *$\mathcal{P}$-paint number*, denoted $\mathring{\mathrm{ch}}_{\mathcal{P}}(G)$, is the least k such that Painter has a winning strategy in the the $(k, \mathcal{P})$-game. When $\chi_{\mathcal{P}}(G), \mathrm{ch}_{\mathcal{P}}(G)$, and $\mathring{\mathrm{ch}}_{\mathcal{P}}(G)$ are defined for a graph G, the natural inequalities hold.

Proposition 3.0.2. *Given a coloring property $\mathcal{P}$ and a graph G, if $\chi_{\mathcal{P}}(G), \mathrm{ch}_{\mathcal{P}}(G)$, and $\mathring{\mathrm{ch}}_{\mathcal{P}}(G)$ are defined, then $\chi_{\mathcal{P}}(G) \leq \mathrm{ch}_{\mathcal{P}}(G) \leq \mathring{\mathrm{ch}}_{\mathcal{P}}(G)$.*

Proof. Let $k = \mathrm{ch}_{\mathcal{P}}(G)$. Let L be the list assignment where $L(v) = [k]$ for all v, and note that G is $\mathcal{P}$-suitably L-colorable since it is $\mathcal{P}$-suitably k-choosable. Any $\mathcal{P}$-suitable L-coloring is also a $\mathcal{P}$-suitable k-coloring, so $\chi_{\mathcal{P}}(G) \leq \mathrm{ch}_{\mathcal{P}}(G)$.

Now let $k = \mathring{\mathrm{ch}}_{\mathcal{P}}(G)$, and let L be a list assignment to G satisfying $|L(v)| \geq k$ for all v. Suppose that Lister marks on round i the set $\{v : i \in L(v)\}$. Since G is $\mathcal{P}$-suitably k-paintable, Painter can win against this Lister strategy. The final coloring produced by Painter is a $\mathcal{P}$-suitable L-coloring of G, so $\mathrm{ch}_{\mathcal{P}}(G) \leq \mathring{\mathrm{ch}}_{\mathcal{P}}(G)$. □

A coloring that is $\mathcal{P}$-extendable to a graph G is helpful because it means that if enough colors (or list elements or tokens) are available, then it is possible to produce a $\mathcal{P}$-suitable coloring of G by starting with the partial $\mathcal{P}$-extendable coloring. But how do we find $\mathcal{P}$-extendable colorings? One way is to find a graph G' that has the following property:

Let G and G' be graphs satisfying $V(G') \subseteq V(G)$. We say that G' is *fully $\mathcal{P}$-extendable to G* if every $\mathcal{P}$-suitable coloring of G' is $\mathcal{P}$-extendable to G when the coloring is viewed as a partial coloring of G. It is not necessary for G' to be a subgraph of G. In fact, Example 3.1.3 shows one instance where it is necessary for G' to contain edges that are not in $E(G)$.

When G' is fully $\mathcal{P}$-extendable to G, a winning strategy for Painter on G' may be combined with a strategy on $G - G'$ to produce a $\mathcal{P}$-suitable coloring of G. To state this process more formally, we give the following definition. Recall that when f is a token assignment to G, and we say "G' is f-paintable" where $V(G') \subseteq V(G)$, we are only considering the restriction of f to $V(G')$.

Definition 3.0.3. Let G' be a graph that is fully $\mathcal{P}$-extendable to G. For any marked set M in the $(f, \mathcal{P})$-game on G, if Painter's response D contains a winning response D' to the marked set $M \cap V(G')$ in the $(f, \mathcal{P})$-game on G', then we say that Painter plays on G according to a *G'-first strategy*. ■

If Painter wins the $(f, \mathcal{P})$-game on G' when G' is fully $\mathcal{P}$-extendable to G, then by using a G'-first strategy $\mathcal{S}$, Painter is always winning in the auxiliary $(f, \mathcal{P})$-game being played on G', regardless of what is happening on $G - G'$. Let $T = V(G) - V(G')$. By controlling how many times $\mathcal{S}$ will reject vertices in T, we can give conditions for $\mathcal{S}$ to be a winning strategy for Painter on G. Recall that a vertex v is rejected on a round if it is marked by Lister but not colored by Painter.

Remark 3.0.4. Given a property $\mathcal{P}$ and graphs G and G', where G' is both $\mathcal{P}$-suitably f-paintable and fully $\mathcal{P}$-extendable to G, let $T = V(G) - V(G')$. If Painter has a G'-first strategy on G such that each $v \in T$ is rejected at most $f(v)$ times, then G is $\mathcal{P}$-suitably f-paintable. □

Remark 3.0.4 holds because it describes the conditions for an inductive strategy to succeed for Painter. By relaxing the "f-paintable" condition on G' in Remark 3.0.4 to "f-choosable" or "k-colorable," we obtain as corollaries the conclusions that G is $\mathcal{P}$-suitably f-choosable or $\mathcal{P}$-suitably k-colorable, respectively. Thus Remark 3.0.4 serves as a general tool for proving upper bounds on $\mathcal{P}$-chromatic, $\mathcal{P}$-choice, and $\mathcal{P}$-paint numbers.

A *configuration* in a graph is a set of vertices that satisfies some specified condition, for example, a condition on the degrees or adjacencies of the vertices in the configuration. We say that a configuration in a graph is *reducible* for a graph property if it cannot occur in a minimal graph failing that property. A family $\mathcal{G}$ of graphs is *hereditary* if it is closed under induced subgraphs. Explicitly, if $G \in \mathcal{G}$, then $H \in \mathcal{G}$ for every induced subgraph H of G. A set of configurations $\mathcal{F}$ is *unavoidable* by $\mathcal{G}$ if every $G \in \mathcal{G}$ contains some member of $\mathcal{F}$. Remark 3.0.4 is particularly useful for proving reducibility arguments. In the following example, we consider the case when the desired coloring only needs to be proper, which allows us to consider "k-paintability" instead of "$\mathcal{P}$-suitable k-paintability".

Example 3.0.5. Let $\mathcal{G}$ be a hereditary family of graphs. Suppose we wish to show that G is $\mathcal{P}$-suitably k-paintable for every $G \in \mathcal{G}$. By the definition of "reducible", it suffices to show that every $G \in \mathcal{G}$ contains a configuration that is reducible for $\mathcal{P}$-suitable k-paintability. Let $\mathcal{F}$ be a set of unavoidable configurations for $\mathcal{G}$.

Let F be a configuration in $\mathcal{F}$, and let G be a graph that contains F. To show that F is reducible, let G' be fully $\mathcal{P}$-extendable to G and have vertex set $V(G) - F$. If G is a graph with fewest vertices in $\mathcal{G}$ failing to be $\mathcal{P}$-suitably k-paintable, then G' is $\mathcal{P}$-suitably k-paintable. To apply Remark 3.0.4, it suffices to give a G'-first strategy $\mathcal{S}$ that rejects each $v \in F$ fewer than k times. Finding such a strategy implies that F is reducible for k-paintability.

Given a hereditary family $\mathcal{G}$, the process descried in Example 3.0.5 is often called "finding an unavoidable set of reducible configurations". This process frequently shows up in the Discharging Method (our terminology follows [13]).

More concretely, consider the following application. Proposition 3.0.6 can be proved using Proposition 2.1.1(b), but the proof below shows how Remark 3.0.4 can be applied to yield the same result. Recall that a k-vertex is a vertex of degree k, that a k^--vertex is a vertex of degree at most k, and that a k-face is a face of length k.

Proposition 3.0.6. *If G is d-degenerate, then G is $(d+1)$-paintable.*

Proof. Let $v \in V(G)$ be a d^--vertex. To show reducibility of d^--vertices for $(d+1)$-paintability, we let $G' = G - v$ and assume that G' is $(d+1)$-paintable. If Painter plays according to a G'-first strategy and only rejects v when one of its neighbors is being colored, then v is rejected at most d times, never more than once for each neighbor. Remark 3.0.4 then implies that G is $(d+1)$-paintable.

Since every d-degenerate graph contains a d^--vertex, "having a vertex with degree at most d" is an unavoidable reducible configuration for d-degenerate graphs. Therefore, no minimal d-degenerate graph fails to be $(d+1)$-paintable. $\square$

3.1 r-Dynamic Paintability

One example of a coloring property $\mathcal{P}$ is *r-dynamic coloring.* A proper coloring of a graph G is *r-dynamic* when each vertex v has at least $\min\{r, d(v)\}$ distinct colors used on $N(v)$. These colorings facilitate information sharing; if each color represents a different set of information, then r-dynamic colorings ensure that a d-vertex has access to at least $\min\{r, d\}$ sets of information. Note that 1-dynamic coloring is the same as proper coloring.

When the coloring property $\mathcal{P}$ is r-dynamic coloring, the $\mathcal{P}$-chromatic, $\mathcal{P}$-choice, and $\mathcal{P}$-paint numbers are referred to as the *r-dynamic chromatic*, *r-dynamic choice*, and *r-dynamic paint numbers*, denoted $\chi_r(G)$, $\mathrm{ch}_r(G)$, and $\mathring{\mathrm{ch}}_r(G)$, respectively. Also, we say that G' is *r-dynamically extendable to G* if every r-dynamic coloring of G' extends to G. By Proposition 3.0.2, always $\chi_r(G) \le \mathrm{ch}_r(G) \le \mathring{\mathrm{ch}}_r(G)$ for all r.

Montgomery [39] introduced 2-dynamic coloring and the generalization to r-dynamic coloring. When $r \ge \Delta(G)$, an r-dynamic coloring of G repeats no colors on $N[v]$ for any v. In this case, any r-dynamic coloring of G is a proper coloring of G^2. Also any s-dynamic coloring is automatically an r-dynamic coloring when $r \le s$, so we have the following inequalities:

$$\begin{aligned}
\chi(G) &= \chi_1(G) \le \chi_2(G) \le \cdots \le \chi_{\Delta(G)}(G) = \cdots = \chi(G^2),\\
\mathrm{ch}(G) &= \mathrm{ch}_1(G) \le \mathrm{ch}_2(G) \le \cdots \le \mathrm{ch}_{\Delta(G)}(G) = \cdots = \mathrm{ch}(G^2),\\
\mathring{\mathrm{ch}}(G) &= \mathring{\mathrm{ch}}_1(G) \le \mathring{\mathrm{ch}}_2(G) \le \cdots \le \mathring{\mathrm{ch}}_{\Delta(G)}(G) = \cdots = \mathring{\mathrm{ch}}(G^2).
\end{aligned} \tag{1}$$

Thus r-dynamic colorings bridge the gap between coloring a graph and coloring its square.

A *claw* is the graph $K_{1,3}$, and a graph is *claw-free* if it has no induced copy of $K_{1,3}$. Li [34] proved that if G is a claw-free graph with $\delta(G) \ge 5$, then $\chi_3(G) = \chi(G)$. Theorem 3.1.1 strengthens Li's result by showing that it holds also for 3-dynamic choosability and 3-dynamic paintability, as well as generalizing it for $K_{1,r}$-free graphs with $r \ge 3$.

Theorem 3.1.1. *Let G be a $K_{1,r}$-free graph with $\delta(G) \ge (r-1)^2+1$. Then $\chi_r(G) = \chi(G)$, $\mathrm{ch}_r(G) = \mathrm{ch}(G)$, and $\mathring{\mathrm{ch}}_r(G) = \mathring{\mathrm{ch}}(G)$.*

Proof. We show that *every* proper coloring of G is also an r-dynamic coloring of G, which proves $\chi_r(G) = \chi(G)$ since always $\chi_r(G) \ge \chi(G)$. Similarly, this proves equality for the choosability and paintability versions of the parameter.

Let ϕ be a proper coloring of G. Fix any vertex v. Let H be the subgraph induced by $N(v)$. Since G is $K_{1,r}$-free, $\alpha(H) \le r-1$. Thus $\chi(H) \ge \left\lceil \frac{|V(H)|}{\alpha(H)} \right\rceil \ge \left\lceil \frac{(r-1)^2+1}{r-1} \right\rceil \ge r$. Therefore, the neighborhood of v must contain at least r distinct colors, which shows that ϕ is an r-dynamic coloring of G.

For the same reason, $\mathrm{ch}_r(G) = \mathrm{ch}(G)$ and $\mathring{\mathrm{ch}}_r(G) = \mathring{\mathrm{ch}}(G)$. □

We now show that the bound on $\delta(G)$ is sharp for the chromatic number.

Proposition 3.1.2. *There exists a $K_{1,r}$-free graph with $\delta(G) = (r-1)^2$ and $\chi_r(G) > \chi(G)$.*

Proof. Let $H_1 = K_1 + (r-2)K_{r-1}$, $H_2 = (r-1)K_{r-1}$, and $G = H_1 \oplus H_2$. Since $\alpha(G) < r-1$, we have that G is $K_{1,r}$-free. Let v be the vertex of G corresponding to the isolated vertex H_1. Note that v is also the unique vertex in G with degree $(r-1)^2$, which is $\delta(G)$. For $i \in [2]$, we have $K_{r-1} \subseteq H_i$, so $\chi(H_i) \ge r-1$. Also, in any proper coloring of G, no color can be used on both $V(H_1)$ and $V(H_2)$. We can properly color G by assigning colors $1, \ldots, r-1$ to $V(H_1)$ and $r, \ldots, 2r-2$ to $V(H_2)$. Thus $\chi(G) = 2(r-1)$.

We now show that $\chi_r(G) > 2(r-1)$. For an r-dynamic coloring, $N(v)$ must receive at least r distinct colors. Thus Y must use at least r colors. Since X must receive at least $r-1$ colors in a proper coloring, this requires at least $2r-1$ colors for an r-dynamic coloring of G. □

In the proof of Theorem 3.2.5, we will apply of Remark 3.0.4 to 2-dynamic paintability. When no restriction is placed on the number of available colors and $V(G') \subseteq V(G)$, every proper coloring of G' extends to a proper coloring of G. However, when studying r-dynamic coloring for $r > 1$, this is not necessarily the case.

Example 3.1.3. Let $r > 1$ and let G be a graph. Suppose that $x \in V(G)$ has degree 2 with $N(x) = \{y, z\}$ and that $yz \notin E(G)$. Let $G' = G - x$. Any r-dynamic coloring of G' that gives y and z the same color does not extend to an r-dynamic coloring of G since $N(x)$ will not contain at least 2 colors.

We overcome this problem by letting $G' = G - x + yz$. Any r-dynamic coloring of G' give y and z different colors, so $N(x)$ always receives two colors. However, while an r-dynamic coloring of G' gives $N_{G'}(w)$ at least $\min\{r, d(w)\}$ colors for $w \in \{y, z\}$, it may be the case that $N_G(w) - v$ receives only $\min\{r, d(w)\} - 1$ colors for $w \in \{y, z\}$ since each of y and z loses the other as a neighbor when moving from G' back go G. By forcing x to avoid $\min\{r, d(w)\} - 1$ colors used on $N_G(w) - v$ for $w \in \{y, z\}$, we can ensure that an r-dynamic coloring of G' extends to an r-dynamic coloring of G.

When G' is r-dynamically extendable to G, the previous example shows that while $V(G') \subseteq V(G)$, it it not necessarily true that $E(G') \subseteq E(G)$.

3.2 Heawood Bound on 2-Dynamic Paintability

Kim, Lee, and Park [29] proved that planar graphs are 2-dynamically 5-choosable. Given a planar graph G, their proof involves constructing G' by adding edges to G, maintaining planarity, so that there is an edge in the neighborhood of every vertex. Every proper coloring of G' is then a 2-dynamic coloring of G. Using that G' is 5-choosable [52], they conclude that G is 2-dynamically 5-choosable. Since G' is also 5-paintable [48], their result can be stated more strongly: planar graphs are 2-dynamically 5-paintable.

In joint work with Loeb, Reiniger, and Wise [35], we used the Discharging Method to find a set of unavoidable configurations for graphs with genus at most 1. We then applied Remark 3.0.4 to show that each configuration is reducible for 3-dynamic 10-paintability.

Theorem 3.2.1 ([35])**.** *If a graph G embeds in the torus, then $\mathring{\mathrm{ch}}_3(G) \leq 10$.*

Using the discharging method we proved results for larger values of r and for surfaces of higher genus. Let $\gamma(G)$ denote the minimum genus of a surface on which G embeds.

Theorem 3.2.2 ([35])**.** *Let G be a graph, and let $g = \gamma(G)$.*

1. *If $g \leq 2$ and $r \geq 2g + 11$, then $\mathring{\mathrm{ch}}_r(G) \leq (g + 5)(r + 1) + 3$.*

2. *If $g \geq 3$ and $r \geq 4g + 7$, then $\mathring{\mathrm{ch}}_r(G) \leq (2g + 2)(r + 1) + 3$.*

Furthermore, if $r < 2g + 11$ when $g \leq 2$, then $\mathring{\mathrm{ch}}_r(G) \leq (g + 5)(2g + 12) + 3$. Similarly, if $r < 4g + 7$ when $g \geq 3$, then $\mathring{\mathrm{ch}}_r(G) \leq (2g + 2)(4g + 8) + 3$.

When $\gamma(G) = 1$, the Petersen graph (Figure 3.1) shows that Theorem 3.2.1 can be sharp; the ten vertices must receive distinct colors in any 3-dynamic coloring. For $\gamma(G) = 0$, we found a planar graph with seven vertices (Figure 3.2) whose vertices must receive distinct colors in any 3-dynamic coloring. However, it is not known whether a planar graph G satisfying $\chi_3(G) > 7$ exists. Theorem 3.2.2 shows that for fixed genus g, the r-dynamic paint number grows at most linearly in g.

For a nonnegative integer g, the *Heawood bound* is defined as

$$h(g) = \left\lfloor \frac{7 + \sqrt{1 + 48g}}{2} \right\rfloor .$$

Heawood [18] proved that for $g > 0$, graphs of (orientable) genus g are $(h(g) - 1)$-degenerate and hence $h(g)$-colorable. Because $(k - 1)$-degenerate graphs are k-paintable, any nonplanar graph with genus at most g is $h(g)$-paintable. Chen et al. [12] proved that such a graph is 2-dynamically $h(g)$-choosable.

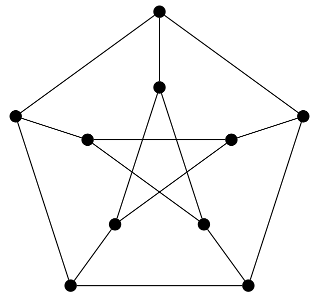

Figure 3.1: Petersen Graph

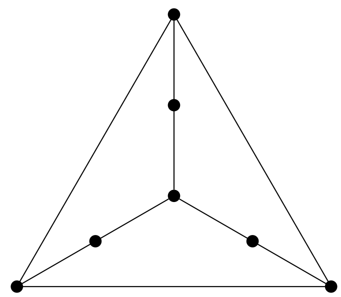

Figure 3.2: A graph that is not 3-dynamically 6-colorable.

Using methods in [12] we prove that any nonplanar graph G is 2-dynamically $h(\gamma(G))$-paintable. We first give a family of configurations that are reducible for 2-dynamic $h(g)$-paintability.

Theorem 3.2.3. *If G is a graph satisfying $0 < \gamma(G) \leq g$, then the following configurations are reducible for 2-dynamic $h(g)$-paintability.*

1. *An $(h(g)-3)^-$-vertex.*
2. *An $(h(g)-1)^-$-vertex adjacent to an $(h(g)-2)^-$-vertex.*
3. *Adjacent vertices u and v satisfying $d(u) = d(v) = h(g)-1$ such that $G[N(v)]$ contains an edge.*
4. *An $(h(g)-1)^-$-vertex that lies on a 3-face.*
5. *An $(h(g)-2)^-$-vertex that lies on a 4-face.*

Chen et al. [12] showed that Configuration 3 is reducible for 2-dynamic $h(g)$-choosability without the condition that $G[N(v)]$ contains an edge. They go on to prove that this configuration and the other configurations listed in Theorem 3.2.3 form an unavoidable set of reducible configurations, which implies that G is 2-dynamically $h(g)$-choosable.

Proof. We first give an overview of the proofs that each configuration is reducible for 2-dynamic $h(g)$-paintability. Painter defines a graph G', plays according to a G'-first strategy, and creates a 2-dynamic coloring of G by rejecting vertices of $G - G'$ fewer than $h(g)$ times. The G'-first strategy ensures that the coloring of $V(G')$ is proper. In Configurations 1 and 5, G' contains an edge xy that may not be present in G, and the endpoints of the added edge are adjacent to the vertex $v \in V(G) - V(G')$. If $N_G(x)$ is an independent set, then the G'-first strategy may give the same color to all of $N_G(x) - v$ and a different color to y so that $N_{G'}(x)$ has at least two colors. To obtain a 2-dynamic coloring of G, we arbitrarily choose vertices $x' \in N_G(x) - v$ and $y' \in N_G(y) - v$ and require that v avoids the colors used on x' and y'. We now verify that the this strategy works to show that each configuration is reducible.

Configuration 1: *v is a vertex such that $d(v) \le h(g) - 3$.* If $d(v) = 1$, then let $G' = G - v$, let u be the neighbor of v, and let u' be a neighbor of u other than v. Painter rejects v only when u or u' is colored, so Painter rejects v at most two times.

When $d(v) \ge 2$ with $x, y \in N(v)$, let $G' = G - v + xy$, and let x' and y' be neighbors of x and y, respectively, other than v. Painter rejects v only when a vertex of $N(v) \cup \{x', y'\}$ is colored. Thus Painter rejects v at most $d(v) + 2$ times.

Configuration 2: *v_1v_2 is an edge such that $d(v_1) < h(g)$ and $d(v_2) < h(g) - 1$.* Let $G' = G - \{v_1, v_2\}$. For $i \in \{1, 2\}$, let v_i' be a neighbor of v_i other than v_{2-i}. Painter rejects v_i when a vertex of $N(v_i) \cup \{v_{2-i}'\}$ is colored. Thus Painter rejects v_i fewer than $h(g)$ times.

Configuration 3: *v_1v_2 is an edge such that $d(v_1) = d(v_2) = h(g) - 1$ and $G[N(v_1)]$ contains an edge.* Let $G' = G - \{v_1, v_2\}$, and let v_2' be a neighbor of v_2 other than v_1. Painter rejects v_1 when a vertex of $(N(v_1) - v_2) \cup \{v_2'\}$ is colored. Painter rejects v_2 when a vertex of $N(v_2)$ is colored. Thus for $i \in \{1, 2\}$, Painter rejects v_i fewer than $h(g)$ times.

Configuration 4: *vuw forms a 3-face such that $d(v) < h(g)$.* Let $G' = G - v$. Painter rejects v when a vertex of $N(v)$ is colored. Thus Painter rejects v at most $d(v)$ times.

Configuration 5: *$vuwx$ forms a 4-face such that $d(v) < h(g) - 1$.* Let $G' = G - v + ux$. Painter rejects v when a vertex of $N(v) \cup \{w\}$ is colored. Thus Painter rejects v at most $d(v) + 1$ times. $\square$

To complete the proof of Theorem 3.2.5, we show that the set configurations in Theorem 3.2.3 is unavoidable by graphs with genus at most g. For a face f, denote the *length* of f by $\ell(f)$. For an edge uv, let f_{uv} and f'_{uv} denote the two faces that have uv on their boundaries. define $\phi(uv) = 1 - \frac{1}{d(u)} - \frac{1}{d(v)} - \frac{1}{\ell(f_{uv})} - \frac{1}{\ell(f'_{uv})}$. Using Euler's Formula, it follows that $\sum_{e \in E(G)} \phi(e) = 2\gamma(G) - 2$. For completeness, we include a proof by Ore [44], who attributes it to Lebesgue.

Lemma 3.2.4 ([44])**.** *$\sum_{e \in E(G)} \phi(e) = 2\gamma(G) - 2$.*

Proof. Fix an embedding of G on a surface of genus $\gamma(G)$. Let $F(G)$ be the set of faces in this embedding. Euler's Formula states that $|V(G)| - |E(G)| + |F(G)| = 2 - 2\gamma(G)$. We wish to show that $\sum_{e\in E(G)} \phi(e) = |E(G)| - |V(G)| - |F(G)|$. Clearly, $|E(G)| - |V(G)| - |F(G)| = \sum_{e\in E(G)} 1 - \sum_{v\in V(G)} 1 - \sum_{f\in F(G)} 1$. We now rewrite the second two summations to be indexed by edges instead of vertices and faces, respectively.

Claim 1: $\sum_{v\in V(G)} 1 = \sum_{uv\in E(G)} \frac{1}{d(u)} + \frac{1}{d(v)}$. For a vertex $v \in V(G)$, its $d(v)$ incident edges each contribute $\frac{1}{d(v)}$ to the sum. Thus each vertex receives a total contribution of 1 in the sum.

Claim 2: $\sum_{f\in F(G)} 1 = \sum_{uv\in E(G)} \frac{1}{\ell(f_{uv})} + \frac{1}{\ell(f'_{uv})})$. The $\ell(f)$ edges bounding a face f each contribute $\frac{1}{\ell(f)}$ to the sum. Thus each face receives a total contribution of 1 in the sum.

Therefore, $|E(G)| - |V(G)| - |F(G)| = \sum_{uv\in E(G)} 1 - \frac{1}{d(u)} - \frac{1}{d(v)} - \frac{1}{\ell(f_{uv})} - \frac{1}{\ell(f'_{uv})} = \sum_{uv\in E(G)} \phi(uv)$. □

Using Theorem 3.2.3 and Lemma 3.2.4, we prove the main result of this section.

Theorem 3.2.5. *If G is a nonplanar graph, then $\mathring{\mathrm{ch}}_2(G) \le h(\gamma(G))$.*

Proof. Let $g = \gamma(G)$ and $k = h(g)$. Since $g > 0$, we have $k \ge 7$. In this proof, "reducible" means "reducible for 2-dynamic k-paintability". If $|V(G)| \le k$, then G is 2-dynamically k-paintable by giving distinct colors to vertices, so we may assume $|V(G)| > k$. Toward a contradiction, we assume that G does not have any configuration listed in Theorem 3.2.3.

Claim 1: *If $uv \in E(G)$ lies on a 3-face, then $\phi(uv) \ge \frac{k-6}{3k}$.* Configuration 4 is reducible, so $d(u) \ge k$ and $d(v) \ge k$. Since every face has length at least 3, $\phi(uv) \ge 1 - \frac{1}{k} - \frac{1}{k} - \frac{1}{3} - \frac{1}{3} = \frac{k-6}{3k}$.

Claim 2: *If $uv \in E(G)$ lies on a 4-face (and not a 3-face), then $\phi(uv) \ge \frac{k-5}{2(k-1)}$.* Configuration 5 is reducible, so $d(u) \ge k-1$ and $d(v) \ge k-1$. In this case, both faces containing uv have length at least 4, so $\phi(uv) \ge 1 - \frac{1}{k-1} - \frac{1}{k-1} - \frac{1}{4} - \frac{1}{4} = \frac{k-5}{2(k-1)}$.

Claim 3: *If $uv \in E(G)$ is not contained in a 3- or 4-face, then $\phi(uv) \ge \frac{3k^2-18k+21}{5(k-1)(k-2)}$.* Configurations 1–3 are reducible, so we may assume $d(u) \ge k-2$ and $d(v) \ge k-1$. In this case, both faces containing uv have length at least 5, so $\phi(uv) \ge 1 - \frac{1}{k-2} - \frac{1}{k-1} - \frac{1}{5} - \frac{1}{5} = \frac{3k^2-19k+21}{5(k-1)(k-2)}$.

Note that $\frac{k-6}{3k} < \frac{k-5}{2(k-1)} < \frac{3k^2-18k+21}{5(k-1)(k-2)}$, since $k \ge 7$. Thus $\phi(uv) \ge \frac{k-6}{3k}$ for all $uv \in E(G)$.

Claim 4: $|E(G)| \ge \frac{1}{2}(k+3)(k-2)$. If $\delta(G) \ge k$, then since $|V(G)| \ge k+1$, we have $|E(G)| = \frac{1}{2}\sum_{v\in V(G)} d(v) \ge \frac{1}{2}k(k+1) > \frac{1}{2}(k+3)(k-2)$. If $\delta(G) < k$, then consider a vertex v with $d(v) = \delta(G)$. Because Configuration 1 is reducible, we know that $\delta(G) \ge k-2$. Since Configuration 2 is reducible, every neighbor of v has degree at least $k-1$.

If also every neighbor of v has degree at least k, then $|E(G)| \ge \frac{1}{2}(|V(G)|\delta(G) + \delta(G)(k-\delta(G)) \ge \frac{1}{2}(k+3)(k-2)$. If instead v has a neighbor u with $d(u) = k-1$, then there are no edges in $N(v)$ or $N(u)$, because Configuration 3 is reducible. Now, $|V(G)| \geq |\{u,v\}| + |N(u) \cup N(v)| + |N(N(u) \cup N(v)) - \{u,v\}| \geq 2 + (k-2) + (\delta(G) - 2) \geq 2k - 4$. Thus $|E(G)| \geq \frac{1}{2}(2k-4)(k-2) > \frac{1}{2}(k+3)(k-2)$.

To obtain the desired contradiction, we now show $\sum \phi(e) > 2g-2$. If $\delta(G) \geq k$, then G has at least $k+1$ vertices of degree at least k, so $|E(G)| \geq \frac{1}{2}k(k+1)$. Thus $\sum \phi(e) \geq \frac{1}{2}k(k+1)\frac{k-6}{3k} = 2g-2+\frac{1}{6}\sqrt{1+48g}+\frac{1}{6} > 2g-2$.

Suppose $\delta(G) = k-1$, and let v be an $(k-1)$-vertex. Configuration 4 is reducible, so v does not lie on a 3-face. Thus Claim 2 implies that $\phi(uv) \geq \frac{k-5}{2(k-1)}$ for every $u \in N(v)$. Hence,

$$\begin{aligned}
\sum \phi(e) &\geq |E(G)|\frac{k-6}{3k} + d(v)\left(\frac{k-5}{2(k-1)} - \frac{k-6}{3k}\right) \\
&\geq \frac{1}{2}(k+3)(k-2)\frac{k-6}{3k} + (k-1)\left(\frac{k-5}{2(k-1)} - \frac{k-6}{3k}\right) \\
&= \frac{1}{6}(k^2-4k-13) + \frac{4}{k} \\
&= 2g-2+\frac{1}{12}(3\sqrt{1+48g}-5) + \frac{4}{k} \\
&> 2g-2.
\end{aligned}$$

Suppose $\delta(G) = k-2$, and let v be an $(k-2)$-vertex. Configuration 5 is reducible, so v does not lie on a 4-face. Thus Claim 3 implies that $\phi(uv) \geq \frac{3k^2-19k+21}{5(k-1)(k-2)}$ for every $u \in N(v)$. Hence,

$$\begin{aligned}
\sum \phi(e) &\geq |E(G)|\frac{k-6}{3k} + d(v)\left(\frac{3k^2-19k+21}{5(k-1)(k-2)} - \frac{k-6}{3k}\right) \\
&\geq \frac{1}{2}(k+3)(k-2)\frac{k-6}{3k} + (k-2)\left(\frac{3k^2-19k+21}{5(k-1)(k-2)} - \frac{k-6}{3k}\right) \\
&= \frac{1}{30}(5k^2-17k-76) + \frac{3k-2}{k(k-1)} \\
&> \frac{1}{30}(5k^2-17k-76) \\
&= 2g-2+\frac{1}{30}(9\sqrt{1+48g}-13) \\
&> 2g-2.
\end{aligned}$$

In each case, we have shown that $\sum \phi(e) > 2g-2$, which contradicts that $\sum \phi(e)$. $\square$

Chapter 4

(f, g)-Paintability

Let G be a graph, and let $g : V(G) \to \mathbb{N}$. A *g-fold coloring* of G assigns to each vertex v a set of $g(v)$ distinct colors such that adjacent vertices have disjoint sets of colors. When $g(v) = m$ for all v, we call this an *m-fold coloring*. When all colors come from $[k]$, we call this a *g-fold k-coloring* and say that G is *(k, g)-colorable*. When G is (k, g)-colorable and $g(v) = m$ for all v, we say that G is *(k, m)-colorable*. An ordinary proper k-coloring is also a 1-fold k-coloring.

Let G be a graph, and let $f : V(G) \to \mathbb{N}$ and $g : V(G) \to \mathbb{N}$. A *g-fold L-coloring* of G is a g-fold coloring ϕ of G such that $\phi(v) \subseteq L(v)$ for each vertex v. A graph G is called *(f, g)-choosable* if there is a g-fold L-coloring for any list assignment L such that $|L(v)| \geq f(v)$ for all $v \in V(G)$. When $f(v) = k$ and $g(v) = m$ for all $v \in V(G)$ and G is (f, g)-choosable, we say that G is *(k, m)-choosable*.

The *fractional chromatic number* $\chi^*(G)$ of G is defined as

$$\chi^*(G) = \inf \left\{ \frac{k}{m} : G \text{ is } (k, m)\text{-colorable} \right\}.$$

Let G be a graph and define $\mathcal{I}(G)$ to be the set of independent sets in G and $\mathcal{I}(G, v)$ to be the set of independent sets in G that contain the vertex v. We now give an equivalent [50] definition of the fractional chromatic number as a solution to the following linear program.

$$\chi^*(G) = \min \sum_{I \in \mathcal{I}(G)} x_I, \text{ subject to } \sum_{I \in \mathcal{I}(G,v)} x_I \geq 1 \text{ for all } v \in V(G).$$

Always, $\omega(G) \leq \frac{|V(G)|}{\alpha(G)} \leq \chi^*(G) \leq \chi(G)$, where $\omega(G)$ is the number of vertices in a largest clique in G.

The *fractional choice number* $\mathrm{ch}^*(G)$ is defined as

$$\mathrm{ch}^*(G) = \inf \left\{ \frac{k}{m} : G \text{ is } (k, m)\text{-choosable} \right\}.$$

In each of these definitions, the "inf" can be replaced by "min" [3]. The *m-choice number* of G, denoted

$\mathrm{ch}^{(m)}(G)$, is the least k such that G is (k,m)-choosable. The *m-chromatic number* of G, denoted $\chi^{(m)}(G)$, is the least k such that G is (k,m)-colorable.

In the paintability setting, we now consider a special case of the $(f,\mathcal{P})$-game. The following definition is the same as the definition of the $(f,\mathcal{P})$-game (Definition 3.0.1) where $\mathcal{P}$ is the property of having a g-fold coloring, except that "uncolored vertices" is interpreted to mean "vertices that have received fewer than $g(v)$ colors". For clarity, we restate the game definition in its entirety for this special case.

Definition 4.0.1. Let G be a graph, and let $f,g : V(G) \to \mathbb{N}$. The *$(f,g)$-game* on G is played by two players: Lister and Painter. Initially, each vertex v has $f(v)$ tokens and no colors. Lister marks a nonempty set M of vertices such that each $v \in M$ has been colored fewer than $g(v)$ times by Painter; Lister also removes a token from each marked vertex. Painter responds by selecting a subset D of M that forms an independent set in G. A color distinct from those used on the previous rounds is given to all vertices in D. Painter wins by producing a g-fold coloring of G, and Lister wins by marking a vertex with no tokens.

A graph G is *(f,g)-paintable* when Painter has a winning strategy in the (f,g)-game on G. When $f(v)=k$ and $g(v)=m$ for all $v \in V(G)$ and G is (f,g)-paintable, we say that G is *(k,m)-paintable.* The *fractional paint number* $\mathring{\mathrm{ch}}^*(G)$ of G is defined as

$$\mathring{\mathrm{ch}}^*(G) = \inf\left\{\frac{k}{m} : G \text{ is } (k,m)\text{-paintable}\right\}.$$

The *m-paint number* of G, denoted $\mathring{\mathrm{ch}}^{(m)}(G)$, is the least k such that G is (k,m)-paintable. ■

In this notation, $(k,1)$-paintability is the same as k-paintability. Gutowski [17] proved that $\mathring{\mathrm{ch}}^*(G) = \mathrm{ch}^*(G) = \chi^*(G)$ for any graph G. However, there exist graphs where the infimum in the definition of $\mathring{\mathrm{ch}}^*(G)$ cannot be attained, and hence the infimum cannot be replaced by the minimum [17].

If G is a nonempty bipartite graph, then $\mathring{\mathrm{ch}}^*(G) = \mathrm{ch}^*(G) = \chi^*(G) = \chi(G) = 2$. Since $\mathrm{ch}^*(G) = \min\left\{\frac{k}{m} : G \text{ is } (k,m)\text{-choosable}\right\}$, we know that for some integer m, G is $(2m,m)$-choosable. Since we cannot, in general, replace the "infimum" in the definition of $\mathring{\mathrm{ch}}^*(G)$ with "minimum", knowing that $\mathring{\mathrm{ch}}^*(G) = 2$ does not in itself imply that G is $(2m,m)$-paintable for some integer m. A natural question is which graphs are $(2m,m)$-paintable for some integer m.

Erdős, Rubin, and Taylor [16] conjectured that if a graph G is k-choosable, then for any positive integer m, G is (km,m)-choosable. More generally, it was conjectured that if G is (a,b)-choosable, then for any positive integer m, G is (am,bm)-choosable. In joint work with Meng and Zhu [36], we proposed the online version of this conjecture.

Conjecture 4.0.2. *If G is (a,b)-paintable, then G is (am,bm)-paintable for any $m \in \mathbb{Z}^+$.*

It was asked in [57] (Question 24) whether k-paintable graphs are (km,m)-paintable for any positive integer m. Our main result in [36] is the following theorem, which answers this question in the affirmative for $k=2$. Surprisingly, the converse is also true in this case.

Theorem 4.0.3. *For every graph G and every positive integer m, G is $(2m,m)$-paintable if and only if G is 2-paintable.*

In Sections 4.2 and 4.3, we prove Theorem 4.0.3. Section 4.4 contains the proof that planar graphs are $(5m,m)$-paintable for all $m \geq 1$. We consider the m-chromatic, m-choice, and m-paint numbers of odd cycles in Section 4.1.

Thomassen [52] proved that planar graphs are 5-choosable. Tuza and Voigt [54] showed that planar graphs are $(5m,m)$-choosable for all m, and Schauz [48] proved that planar graphs are 5-paintable. We strengthen both improvements of Thomassen's result in Section 4.4 by proving that planar graphs are (gm,m)-paintable for $m \geq 1$.

Let G be a connected graph other than a complete graph or an odd cycle. Brooks' Theorem [9] states that G is $\Delta(G)$-colorable. Tuza and Voigt [54] strengthened this by proving that G is $(\Delta(G)m,m)$-choosable for all m. Hladký, Kráľ, and Schauz [20] also strengthened Brooks' Theorem by proving that G is $\Delta(G)$-paintable. In Section 4.5, we prove that G is $(\Delta(G)m,m)$-paintable for all $m \geq 1$.

In the (f,g)-game, we say that v is an *(a,b)-vertex* if $f(v)=a$ and $g(v)=b$. We now prove several easy results that generalize parts of Proposition 2.1.1. Proposition 4.0.4 and Corollary 4.0.6 can be viewed as Proposition 2.1.1(a) being applied to a single vertex or edge in the (f,g)-game.

Proposition 4.0.4. *If a graph G is (f,g)-paintable, then*

- *$f(v) \geq g(v)$ for all $v \in V(G)$,*
- *$\max\{f(u),f(v)\} \geq g(u)+g(v)$ for all $uv \in E(G)$.*

Proof. If $f(v) < g(v)$, then Lister wins by marking $\{v\}$ until it has no more tokens, but still needs to be colored. If $uv \in E(G)$ and $\max\{f(u),f(v)\} < g(u)+g(v)$, then Lister wins by marking $\{u,v\}$ as long as $f(u), f(v), g(u)$, and $g(v)$ are nonnegative. Each round, $g(u)+g(v)$ decreases by at most 1, so some vertex still needs to be colored after losing all of its tokens. □

Definition 4.0.5. We say that a vertex v is *forced* when $f(v) = g(v)$, and an edge uv is *tight* when $\max\{f(u), f(v)\} = g(u) + g(v)$. For $uv \in E(G)$, we say that the ordered pair (u, v) is *strictly tight* when $f(u) = g(u) + g(v)$ and $f(v) < g(u) + g(v)$. ■

Note that if uv is tight, but not strictly tight, we have $f(u) = f(v) = g(u) + g(v)$. Also, observe that in the $(2m, m)$-game, every edge begins the game as a tight edge that is not strictly tight. We use the terminology in Definition 4.0.5 to make assumptions about how Painter must respond to certain sets marked by Lister.

Corollary 4.0.6. *If Painter plays according to a winning strategy and the marked set contains a forced vertex v, then Painter colors v; if the marked set contains u and v where uv is a tight edge, then Painter colors one of u and v; if the marked set contains u and not v where (u, v) is a strictly tight pair, then Painter colors u.*

To make studying the (f, g)-game more efficient, we make the following observation about Painter's responses on bipartite graphs, which follows from Corollary 4.0.6.

Corollary 4.0.7. *Let G be bipartite. Assume that in the (f, g)-game on G, the set of tight edges induces a connected spanning subgraph of G. If Lister marks $V(G)$, then Painter must color all vertices in one of the partite sets.*

We will use Corollary 4.0.7 to reduce the number of cases that we must consider in later proofs.

Remark 4.0.8. Let G be bipartite with parts A and B. In the (f, g)-game on G, let every edge be tight but not strictly tight. Thus $f(a) = f(b) = g(a) + g(b)$ for every edge ab with $a \in A$ and $b \in B$. In particular, f is constant on $V(G)$ and g is constant on each of A and B.

In order to use induction on $\sum g(v)$, there are two base cases. We have $f \equiv r + 1$ for some $r \in \mathbb{N}$ and either (1) $g|_A \equiv 1$ and $g|_B \equiv r$, or (2) $g|_A \equiv r$ and $g|_B \equiv 1$.

Suppose that Lister has a winning strategy for both base cases, and consider the case when $\min\{g(v)\} > 1$. If Lister begins by marking $V(G)$, then Painter must color A or B by Corollary 4.0.7. If Lister wins the $(f - \delta_{V(G)}, g - \delta_A)$-game and the $(f - \delta_{V(G)}, g - \delta_B)$-game, then it will follow that G is not (f, g)-paintable. □

We later use Remark 4.0.8 in the proofs of Theorems 4.2.11 and 4.2.14. The next result was first proved by Zhu [57].

Proposition 4.0.9. *If some $v \in V(G)$ is forced, then G is (f,g)-paintable if and only if $G - v$ is (f', g')-paintable, where*

$$f'(w) = \begin{cases} f(w) - g(v), & \text{if } w \in N_G(v), \\ f(w), & \text{otherwise,} \end{cases}$$

and $g'(w) = g(w)$ for all $w \in V(G - v)$.

Proof. If G is (f,g)-paintable and Lister marks $N[v]$ for $g(v)$ consecutive rounds, then Corollary 4.0.6 implies Painter colors v each time. After these moves, each vertex w of $G - v$ is an $(f'(w), g'(w))$-vertex. If G is (f,g)-paintable, then $G - v$ is (f', g')-paintable, since Painter had no other possible responses against this Lister strategy. If $G - v$ is (f', g')-paintable, then in G, Painter "reserves" $g(v)$ tokens at each neighbor of v. Anytime v is marked, Painter colors v and allots one of the reserved tokens for each $u \in N(v)$. Since this happens at most $g(v)$ times and $G - v$ is (f', g')-paintable, Painter has a winning strategy in G. $\square$

Proposition 4.0.9 generalizes Proposition 2.1.1(d). For a subset X of $V(G)$, let $f(X) = \sum_{v \in X} f(v)$ and $g(X) = \sum_{v \in X} g(v)$. We say a vertex v is (f,g)-*degenerate* if $f(v) \geq g(N[v])$. The following generalizes the degeneracy result from Proposition 2.1.1(b). Recall that when $G' \subseteq G$ and we say "G' is f-paintable" where f is a token assignment to G, we are only considering the restriction of f to $V(G')$.

Proposition 4.0.10. *If v is (f,g)-degenerate, then G is (f,g)-paintable if and only if $G - v$ is (f,g)-paintable.*

Proof. If G is (f,g)-paintable, then Painter has a winning strategy on every subgraph of G. If $G - v$ is (f,g)-paintable, then Painter wins in G by following a winning strategy $\mathcal{S}$ for $G - v$ and coloring v when it is marked and none of its neighbors are colored by $\mathcal{S}$. Thus v is rejected at most $g(N_G(v))$ times, leaving enough tokens for v to be colored $g(v)$ times. $\square$

The following proposition follows as in the proof of Proposition 2.1.1(c), generalizing the result. In our later proofs, we implicitly assume that Painter responds by coloring maximal independent subsets of marked sets.

Proposition 4.0.11. *If G is (f,g)-paintable, then Painter has a winning strategy in which on each round, the colored vertices form a maximal independent subset of the marked sets.*

In Section 4.2, we frequently arrive at a position in the (f,g)-game that contains a copy of C_4 with a particular token assignment. We show that Lister wins in this situation.

Lemma 4.0.12. *Let G be a 4-cycle with vertices v_0, v_1, v_2, v_3, in order. In the (f,g)-game on G, if all the edges of G are tight and (v_1, v_0) and (v_3, v_0) are strictly tight pairs, then G is not (f,g)-paintable.*

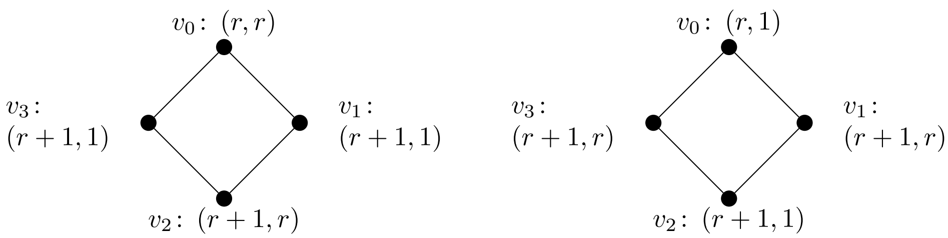

Figure 4.1: Two possible applications of Lemma 4.0.12

Proof. We use induction on the total number $\sum f(v_i)$ of tokens. First suppose that v_0. By Proposition 4.0.9, it suffices to show that $G - v_0$ is not (f', g')-paintable, where $f'(v_i) = f(v_i) - g(v_0) = g(v_i)$ for $i \in \{1, 3\}$, $f'(v_2) = f(v_2)$, and g' is the restriction of g to $\{v_1, v_2, v_3\}$. However, with respect to (f', g'), both v_1 and v_3 are forced. Lister wins the game by applying Proposition 4.0.9 again.

We may now assume that v_0 is not forced. Since (v_1, v_0) and (v_3, v_0) are strictly tight pairs, meaning, $1 \le f(v_0) - g(v_0) < g(v_i)$ for $i \in \{1, 3\}$, we conclude that $g(v_1), g(v_3) \ge 2$.

Lister marks $\{v_2, v_3\}$. By Corollary 4.0.6, Painter colors v_3. Either (v_1, v_2) is now strictly tight, or it was previously strictly tight and Lister now wins by repeatedly playing $\{v_1, v_2\}$. Next Lister marks $\{v_2, v_1\}$. By Corollary 4.0.6 again, Painter colors v_1; now (v_3, v_2) is strictly tight and (v_1, v_2) remains strictly tight. Since $g(v_i) > 0$ for all v_i and $\sum f(v_i)$ is smaller, the induction hypothesis implies that Lister wins. $\square$

In Figure 4.1, each ordered pair for a vertex v represents $(f(v), g(v))$. In each case, Lister has a winning strategy to show that the corresponding C_4 is not (f, g)-paintable.

4.1 Odd Cycles

Odd cycles are not 2-colorable, and $\chi^*(C_{2k+1}) = 2 + \frac{1}{k}$. Alon, Tuza, and Voigt [3] showed that C_{2k+1} is $(2r+1, r)$-choosable when $1 \le r \le k$. In this section, we determine the m-paint number of odd cycles, further showing that $\chi^{(m)}(C_{2k+1}) = \mathrm{ch}^{(m)}(C_{2k+1}) = \mathring{\mathrm{ch}}^{(m)}(C_{2k+1}) = 2m + \lceil \frac{m}{k} \rceil$.

Proposition 4.1.1. *For $k \ge 1$, $\chi^{(m)}(C_{2k+1}) \ge 2m + \lceil m/k \rceil$.*

Proof. Each color can be used on at most k vertices. Since each vertex must receive m colors, we have $\chi^{(m)}(C_{2k+1}) \cdot k \ge (2k+1)m$. $\square$

In fact, equality holds, but we prove a stronger result.

Theorem 4.1.2. $\mathring{\mathrm{ch}}^{(m)}(C_{2k+1}) = 2m + \lceil m/k \rceil$ *for all* $k, m \geq 1$.

Proof. Proposition 4.1.1 implies that it suffices to show that $\mathring{\mathrm{ch}}^{(m)}(C_{2k+1}) \leq 2m + \lceil m/k \rceil$.

Give the cycle a consistent orientation, and label the vertices $v_0, \ldots, v_{2k}$. For the entire proof, all indices are taken modulo $2k+1$. Let M be the set Lister marks. If $|M| < 2k+1$, then the graph induced by the marked set is a linear forest. Painter colors vertices greedily along each path starting at the tail.

If $|M| = 2k+1$, then we keep track of how many times moves of this type have occurred in the game. If this is the ith move of this type, then Painter colors $\{v_i, v_{i+2}, \ldots, v_{i+2k-2}\}$ (indices taken modulo $2k+1$). There are exactly $2k+1$ distinct independent sets of size k for C_{2k+1}. In this strategy, Painter balances which of these independent sets is colored by cycling through all possible choices.

Suppose Lister can win against this particular Painter strategy when each vertex has $2m + \lceil m/k \rceil$ tokens. If Lister can win, then there is some earliest point in the game when a vertex v_i has been rejected $m + \lceil m/k \rceil + 1$ times. On a round when v_i is rejected, Painter's strategy implies that $v_{i-1} \in M$. Since v_{i-1} is colored at most m times among the $m + \lceil \frac{m}{k} \rceil + 1$ rounds, there must be at least $\lceil m/k \rceil + 1$ rounds where both v_i and v_{i-1} are rejected, which only occurs in rounds where all vertices are marked. Furthermore, it only occurs for $\frac{1}{2k+1}$ of the rounds in which all vertices are marked. Thus the number of rounds where all vertices are marked is at least $\lceil m/k \rceil (2k+1) + 1$, which is greater than $2m + \lceil m/k \rceil$. □

Corollary 4.1.3. *The* m*-chromatic,* m*-choice, and* m*-paint numbers* C_{2k+1} *all equal* $2m + \lceil m/k \rceil$.

4.2 Non-$(2m, m)$-paintable graphs

In order to characterize the graphs that are $(2m, m)$-paintable for each $m \geq 1$, we prove in this section that if a graph is not 2-paintable, then it is not $(2m, m)$-paintable for any $m \geq 1$.

A graph is *k-choice-critical* if $\mathrm{ch}(G) = k$ but $\mathrm{ch}(G - e) < k$ for all $e \in E(G)$. Similarly, a graph is *k-paint-critical* if $\mathring{\mathrm{ch}}(G) = k$ but $\mathring{\mathrm{ch}}(G - e) < k$ for all $e \in E(G)$. Voigt in [56] characterized 3-choice-critical graphs, using the characterization of 2-choosable graphs. Using an analogous characterization of 2-paintable graphs [57], Carraher et al. [10] adapted the methods of Voigt to characterize 3-paint-critical graphs. This characterization was proved independently by Riasat and Schauz [47].

Recall that the *core* of a connected graph G is the graph obtained from G by successively deleting vertices of degree 1; it is unique up to isomorphism. The following theorem characterizes 2-paintable graphs.

Theorem 4.2.1 ([57])**.** *A connected graph G is 2-paintable if and only if the core of G is K_1, an even cycle, or $K_{2,3}$.*

The difference between 2-choosability and 2-paintability is that a graph whose core is $\Theta_{2,2,2k}$ with $k > 1$ is 2-choosable but not 2-paintable.

Theorem 4.2.2 ([10, 47])**.** *A graph is 3-paint-critical if and only if it is one of the following*

1. *An odd cycle.*
2. *Two edge-disjoint even cycles connected by a path (of length at least* 0*).*
3. $\Theta_{2r,2s,2t}$ *with* $r > 1, s \geq 1, t \geq 1$.
4. $\Theta_{2r+1,2s+1,2t+1}$ *with* $r \geq 1, s \geq 1, t \geq 0$.
5. $\Theta_{2,2,2,2}$.

Proof. It suffices to consider connected graphs. Let $\mathcal{G}$ be the family of graphs listed above. Graphs in $\mathcal{G}$ are not 2-paintable, since they not the cores of graphs listed by Theorem 4.2.1. They are 2-degenerate, however, so they are 3-paintable. To show that they are 3-paint-critical, it suffices to check that deleting any edge from any graph in $\mathcal{G}$ yields a graph whose core is named in Theorem 4.2.1. Deleting an edge from C_{2k+1} yields K_1. Deleting an edge from two even cycles joined by a path yields one even cycle or two disjoint even cycles. Deleting an edge from a theta-graph consisting of three paths whose lengths have the same parity yields an even cycle. Deleting an edge from $\Theta_{2,2,2,2}$ yields $\Theta_{2,2,2}$.

It remains to show that every 3-paint-critical graph G is in $\mathcal{G}$. Note first that G is connected. Also G has no leaves (vertices of degree 1) by degeneracy. Since $G - e$ is 2-paintable whenever $e \in E(G)$, each component of its core is K_1, an even cycle, or $\Theta_{2,2,2}$ (by Theorem 4.2.1). Since G is connected, the core of $G - e$ has at most two components.

Suppose first that the core of $G - e$ has two components. If either is K_1, then G has a vertex of degree 1, already excluded. Otherwise, each component has a cycle, and they are connected in G by a path through e. Now G has a proper non-2-paintable subgraph (and hence is not 3-critical) unless G itself consists of two even cycles connected by a path.

We may therefore assume that the core of $G - e$ is connected. Hence e is not a cut-edge of G, and G has a cycle C through e. If G has no other cycle, then G is a unicyclic graph with minimum degree at least 2 and hence is a cycle (in fact an odd cycle if not 2-paintable).

Hence G has a cycle other than C. If G has a cycle that shares no edges with C, then since G is connected there is a path connecting them. As argued earlier, G now properly contains a graph in $\mathcal{G}$ or belongs to $\mathcal{G}$.

Finally, suppose that G has a cycle sharing an edge with C. Now G contains a theta-graph consisting of three paths joining two vertices. If any two of their lengths have opposite parity, then G properly contains an odd cycle. Otherwise, the three lengths have the same parity. Now G properly contains a theta-graph in

$\mathcal{G}$ unless all three paths have length 2. If $G = \Theta_{2,2,2}$, then G is 2-paintable. Hence $\Theta_{2,2,2}$ occurs as a proper subgraph G' of G.

Since G is 2-edge-connected, we can grow G from any 2-edge-connected subgraph by iteratively adding ears or closed ears. Consider growing G from G' in this way. The possibilities for the first such addition are shown in Figure 4.2.

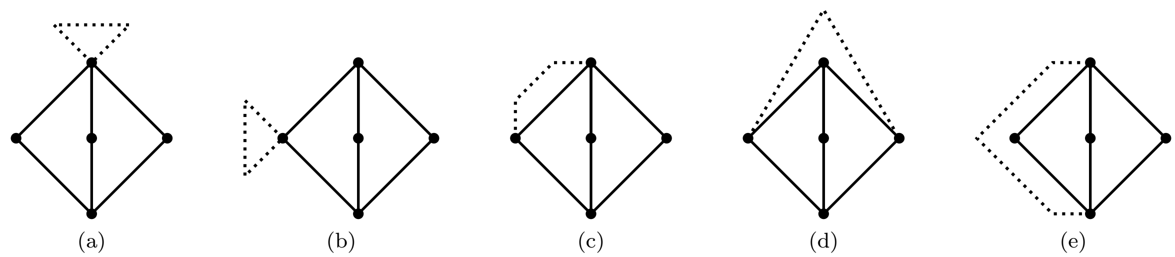

Figure 4.2: Ways to add an ear or closed ear to $\Theta_{2,2,2}$

If a cycle is added or an added path forms a cycle with one edge of G' (cases (a), (b), and (c) in Figure 4.2), then the cycle has odd length or yields edge-disjoint even cycles.

If the added path connects the two high-degree vertices of G' (case (e)), then it forms an odd cycle or $\Theta_{2,2,2,2}$ (which lies in $\mathcal{G}$) or a graph containing $\Theta_{2r,2,2}$ with $r > 1$ (again in $\mathcal{G}$).

Finally, if the added path connects two low-degree vertices of G', then it forms an odd cycle with two edges of G' or forms a theta-graph of the form $\Theta_{2r+1,3,1}$ with $r \geq 1$ having a high-degree vertex at an endpoint of the added path. $\square$

If G is not 2-paintable, then it must contain a 3-paint-critical subgraph. To show that non-2-paintable graphs are not $(2m, m)$-paintable, it suffices to show that each 3-paint-critical graph is not $(2m, m)$-paintable. Since Theorem 4.2.2 describes an infinite family of graphs, we wish to reduce the family of 3-paint-critical graphs a finite number of graphs that we must consider. To achieve this, we present several definitions and small results that allow us to conclude that replacing an edge with a longer path preserves non-$(2m, m)$-paintability.

Definition 4.2.3. Let U be a set of vertices in a graph H, and fix $a, b \in \mathbb{N}$ with $a \geq b$. We say that (H, U) is an (a, b)-*gadget* if H is (a, b)-colorable, and in any (a, b)-coloring of H, all vertices in U are colored by the same b-set. $\blacksquare$

For example, consider P_{2r+1} with $r \geq 0$, where the vertices are labeled $v_1, \ldots, v_{2r+1}$ along the path. Proposition 4.0.10 implies that P_{2r+1} is $(2m, m)$-paintable for all $m \geq 1$, so it is also $(2m, m)$-colorable.

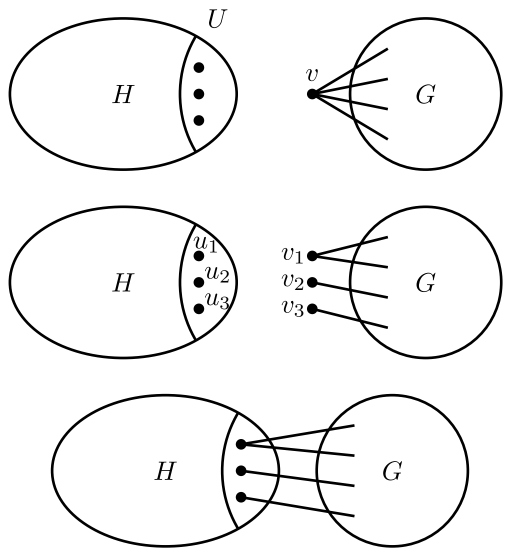

Figure 4.3: Forming an (H, U)-augmentation of G (Definition 4.2.4)

Additionally, any m-fold $2m$-coloring P_{2r+1} gives the same set of m colors to both v_1 and v_{2r+1}. Thus $(P_{2r+1}, \{v_1, v_{2r+1}\})$ is a $(2m, m)$-gadget for all $m \geq 1$.

Definition 4.2.4. Let G and H be graphs with $v \in V(G)$ and $U \subseteq V(H)$. If G' is obtained from $G + H$ by splitting v into $|U|$ copies arbitrarily partitioning the edges incident to v among those copies, and identifying the $|U|$ copies of v with the vertices of U in H, then G' is an (H, U)-*augmentation* of G. ■

We use the (a, b)-gadget from Definition 4.2.3 and the augmentation from Definition 4.2.4 to build many non-(a, b)-paintable graphs G' from a single non-(a, b)-paintable graph G. The following lemma makes this idea precise by showing how "non-(a, b)-paintability" can be preserved when augmenting G to form G'.

Lemma 4.2.5. *If G is not (a, b)-paintable, (H, U) is an (a, b)-gadget, and G' is an (H, U)-augmentation of G, then G' is not (a, b)-paintable.*

Proof. Since G is not (a, b)-paintable, Lister has a winning strategy $\mathcal{S}$. Each round, Lister obtains a marked set $M \subseteq V(G)$ according to $\mathcal{S}$. If $v \in M$, then in G', Lister marks $V(H) \cup (M - v)$; otherwise Lister marks M as a subset of $V(G')$. Let D be the set that Painter colors. If $0 < |D \cap U| < |U|$, then Lister marks $V(H)$ in every remaining round. Since every (a, b)-coloring of H assigns vertices in U the same set of b colors, Painter will not be able to color each vertex of H by a set of b colors. So Lister wins the game. If $D \cap U = \emptyset$, then Lister views Painter's response as $D - V(H)$ in the game on G. If $D \cap U = U$, then Lister views Painter's response as $D \cup \{v\} - V(H)$ in the game on G. Thus Lister can continue using strategy $\mathcal{S}$ and eventually wins the game. □

Corollary 4.2.6. *Let uv be an edge in a graph G. If G is not $(2m, m)$-paintable, then the graph obtained by replacing uv with a path of odd length is not $(2m, m)$-paintable.*

Proof. Assume G' is obtained from G by replacing uv with a path of length $2r + 1$. Let H be a path of length $2r$ with vertices $v_1, v_2, \ldots, v_{2r+1}$ in order along the path. Note that $(H, \{v_1, v_{2r+1}\})$ is a $(2m, m)$-gadget, and we obtain G' as in Definition 4.2.4 by taking the disjoint union of G and H, splitting u into two vertices u_1 and u_2, and letting u_1 be adjacent to v and u_2 be adjacent to the rest of $N_G(u)$; finally, u_1 is identified with v_1 and u_2 with v_{2r+1}. Using this (H, U)-augmentation of G, Lemma 4.2.5 implies that G' is not $(2m, m)$-paintable. $\square$

We now devote the rest of the section to proving the following theorem.

Theorem 4.2.7. *If G is not 2-paintable, then G is not $(2m, m)$-paintable for any m.*

It suffices by Corollary 4.2.6 to show that the seven graphs in Figure 4.4 are not $(2m, m)$-paintable for any positive integer m.

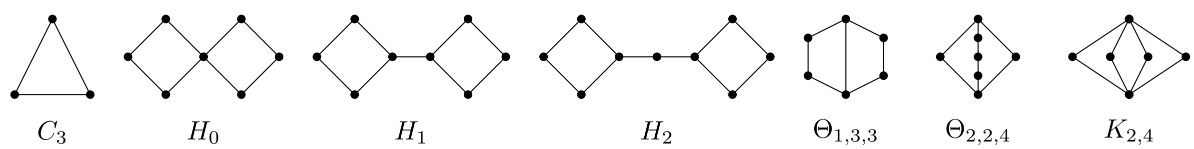

Figure 4.4: Family of graphs for Theorem 4.2.7

Theorem 4.1.2 shows that C_3 is not $(2m, m)$-paintable for any $m \geq 1$. We now reduce this family further by applying Lemma 4.2.5 to H_0.

Proposition 4.2.8. *For $m \in \mathbb{Z}^+$, if H_0 is non-$(2m, m)$-paintable, then H_2 is non-$(2m, m)$-paintable.*

Proof. Let $G = H_0$, let u be the vertex of degree 4, and suppose that G is not $(2m, m)$-paintable. Let $H = P_3$ and $V(H) = \{v_1, v_2, v_3\}$ with v_1, v_3 as the endpoints. Note that $(H, \{v_1, v_3\})$ is a $(2m, m)$-gadget. We split u into u_1, u_2 and partition the edges incident to u so that u_1 is incident to the two edges in the copy of C_4 on the left and u_2 is incident to the two edges in the copy of C_4 on the right. Identifying u_1 with v_1 and u_2 with v_3 yields the graph H_2. Therefore, H_2 is an $(H, \{v_1, v_3\})$-augmentation of H_0, and by Lemma 4.2.5, H_2 is not $(2m, m)$-paintable. $\square$

It remains to show that each of $K_{2,4}, H_0, H_1, \Theta_{2,2,4}$, and $\Theta_{1,3,3}$ is not $(2m, m)$-paintable for any $m \in \mathbb{Z}^+$. All these graphs are bipartite. We use A and B to denote the two parts. Vertices in A are named $a_1, \ldots, a_{|A|}$, and vertices in B are $b_1, \ldots, b_{|B|}$. Recall that for a set S, $\delta_S(v) = 1$ if $v \in S$ and $\delta_S(v) = 0$ otherwise.

Theorem 4.2.9. *Let $G = K_{2,4}$, and let f and g be positive on every vertex. If every edge of G is tight, then G is not (f, g)-paintable.*

Proof. We use induction on the total number of tokens. Let $A = \{a_1, a_2\}$ be the set of vertices of degree 4, and let $B = \{b_1, \ldots, b_4\}$ be the set of vertices of degree 2. Lister first marks $\{a_1, b_1, b_2\}$. Painter must color a_1, for otherwise, (b_3, a_1) and (b_4, a_1) would be strictly tight pairs and Lister wins in the 4-cycle $[a_1, b_3, a_2, b_4]$ by Lemma 4.0.12. If $g(a_1) = 1$, then after the first move, b_1 and b_2 become forced vertices, and it is easy to check that Lister wins on the subgraph induced by $\{a_2, b_1, b_2\}$. Assume $g(a_1) \geq 2$. Next Lister marks $\{a_2, b_3, b_4\}$, and Painter must color a_2 by Corollary 4.0.6 since (a_2, b_1) is strictly tight. By symmetry, we may assume $g(a_2) \geq 2$. Now $g' = g - \delta_{\{a_1,a_2\}}$ is positive on every vertex v and every edge is tight, so Lister wins by the induction hypothesis. □

Note that in the $(2m, m)$-game, every edge begins the game as tight, so we have the following corollary.

Corollary 4.2.10. *$K_{2,4}$ is not $(2m, m)$-paintable for any positive integer m.*

Theorem 4.2.11. *Let $G = H_0$, and let f and g be positive on every vertex. If every edge of G is tight, then G is not (f, g)-paintable.*

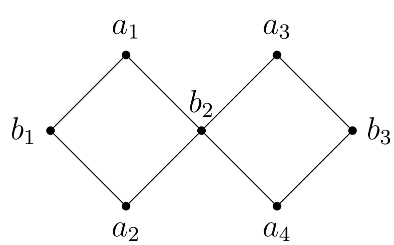

Figure 4.5: H_0 with vertices labeled

Proof. We use induction on the total number of tokens. Let $A = \{a_1, a_2, a_3, a_4\}$. If there is a forced vertex x, then let $[x, y, z, w]$ be a 4-cycle in G containing x. We know that (w, x) and (y, x) are strictly tight pairs. It follows from Lemma 4.0.12 that Lister has a winning strategy. Thus we assume that there is no forced vertex.

Suppose there is a strictly tight pair (x, y). Since y is not forced, $g(y) < f(y) < g(y) + g(x)$, and thus $g(x) \geq 2$. Also, since no vertex is forced, $f(v) \geq 2$ for all v. Lister marks $N_G[x] - \{y\}$, and by Corollary 4.0.6, Painter colors x. After this move, f and g are still positive on every vertex. By the induction hypothesis, Lister has a winning strategy. We may therefore assume that there is no strictly tight pair. By Remark 4.0.8, it suffices to consider the following two cases.

Case 1: *A consists only of $(r+1,1)$-vertices.* Lister marks $\{a_3, b_3\}$. If Painter colors a_3, then b_3 becomes a forced vertex. Applying Corollary 4.0.6 to the forced vertex b_3 results in a_4 becoming a $(1,1)$-vertex. Lister now marks $\{a_4, b_2\}$, Painter must color a_4, and Lister wins by Lemma 4.0.12.

If Painter colors b_3, then Lister marks $V(G) \setminus \{a_3, b_3\}$. Since (b_2, a_3) is a strictly tight pair, by Corollary 4.0.6, Painter colors $\{b_1, b_2\}$. Now all the edges of G are tight, and f and g are positive on every vertex, so Lister wins by the induction hypothesis.

Case 2: *A consists only of $(r+1,r)$-vertices.* Lister marks $\{a_3, b_3\}$. If Painter colors a_3, then Lister marks $V(G)\setminus\{a_3, b_3\}$. Since (a_4, b_3) is now a strictly tight pair, by Corollary 4.0.6, Painter colors $\{a_1, a_2, a_4\}$. Now all the edges of G are tight, and f and g are positive on every vertex, so Lister wins by the induction hypothesis.

If Painter colors b_3, then a_3 is forced. Lister marks $\{a_3, b_2\}$, and Painter must color a_3. Now all the edges of the 4-cycle $[a_1, b_2, a_2, b_1]$ are tight, and (a_1, b_2) and (a_2, b_2) are strictly tight pairs. Lister wins by Lemma 4.0.12. □

Corollary 4.2.12. *The graph H_0 is not $(2m, m)$-paintable for any positive integer m.*

Proposition 4.2.13. *Let $G = H_1$, and let f and g be positive on every vertex. If every edge of G is tight, then G is not (f, g)-paintable.*

Proof. Let u and v be the vertices of degree 3. Lister marks $\{u, v\}$, and by symmetry we may assume Painter colors u. Lemma 4.0.12 implies that Lister wins by marking on the copy of C_4 containing v. □

Theorem 4.2.14. *Let $G = \Theta_{2,2,4}$, and let f and g be positive on every vertex. If every edge of G is tight, then G is not (f, g)-paintable.*

Proof. We use induction on the total number of tokens. Suppose there exists a strictly tight pair (x, y). Lister marks $N_G[x] - \{y\}$; by Corollary 4.0.6, Painter colors x. As in Theorem 4.2.11, after this move f and g are still positive on every vertex and every edge is tight. By the induction hypothesis, Lister has a winning strategy. Thus we may assume that there is no strictly tight pair.

Let vertices be labeled as shown in Figure 4.6, and let $A = \{a_1, \ldots, a_3\}$ and $B = \{b_1, \ldots, b_4\}$.

By Remark 4.0.8, it suffices to consider the following two cases.

Case 1: *A consists only of $(r+1,1)$-vertices.* Lister marks $\{a_3, b_3\}$. If Painter colors a_3, then b_3 becomes a forced vertex. Lister marks $\{a_1, b_3\}$, Painter must color b_3, and Lister now wins on $\{a_1, a_2, b_1, b_2\}$ by Lemma 4.0.12.

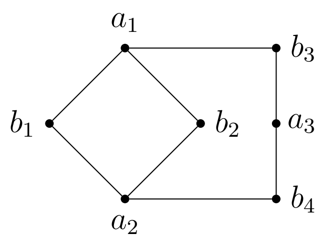

Figure 4.6: $\Theta_{2,2,4}$ with vertices labeled

If Painter colors b_3, then Lister marks $\{a_3, b_4\}$ for r rounds, after which b_4 becomes a $(1,1)$-vertex. Lister now marks $\{b_4, a_2\}$, Painter must color b_4, and Lister wins by Lemma 4.0.12.

Case 2: *A consists only of $(r+1, r)$-vertices.* Lister marks $\{a_3, b_3\}$. If Painter colors a_3, then Lister marks $\{a_1, b_1, b_2\}$. Painter colors a_1, otherwise Lister wins on $\{a_1, b_3\}$ by Proposition 4.0.4. Now Lister marks $\{a_2, b_4\}$, and Painter must color a_2, otherwise Lister wins on $\{a_2, b_1\}$. Lister wins by the induction hypothesis.

If Painter colors b_3, then a_3 becomes forced. Applying Corollary 4.0.6 with the forced vertex a_3 results in b_4 becoming a $(1,1)$-vertex. Lister marks $\{a_2, b_4\}$, and Painter must color b_4. Lister now wins by Lemma 4.0.12. □

Corollary 4.2.15. *The graph $\Theta_{2,2,4}$ is not $(2m, m)$-paintable for any positive integer m.*

Theorem 4.2.16. *Let $G = \Theta_{1,3,3}$ and f and g be positive on every vertex. If every edge of G is tight, then G is not (f, g)-paintable.*

Proof. Let u and v be adjacent vertices of degree 2. Lister repeatedly marks $\{u, v\}$. Since uv is a tight edge, eventually one vertex, say u, becomes a forced vertex. Lister then marks u and its other neighbor u'. Painter must color u, since u is forced. Lemma 4.0.12 implies that Lister wins in the remaining graph. □

Corollary 4.2.17. *The graph $\Theta_{1,3,3}$ is not $(2m, m)$-paintable for any positive integer m.*

Therefore, no 3-paint-critical graph is $(2m, m)$-paintable for any m, which implies Theorem 4.2.7.

4.3 $(2m, m)$-paintable graphs

In the $(2m, m)$-painting game, vertices of degree 1 are (f, g)-degenerate. Thus for any positive integer m, a graph G is $(2m, m)$-paintable if and only if its core is $(2m, m)$-paintable. Thus to prove that all 2-paintable graphs are $(2m, m)$-paintable for all $m \geq 1$, it suffices by Theorem 4.2.1 to show that each of K_1, C_{2n}, and

$K_{2,3}$ is $(2m, m)$-paintable for all $m \geq 1$. It is obvious that K_1 is $(2m, m)$-paintable for all $m \geq 1$. We next consider even cycles.

Theorem 4.3.1. *Even cycles are* $(2m, m)$*-paintable for all* m.

Proof. Let $G = C_{2n}$ and direct the edges of G to obtain a consistent orientation along the cycle. We give a strategy for Painter that prevents any vertex from being rejected more than m times.

If Lister marks all of $V(G)$, then Painter arbitrarily chooses one of the partite sets and colors all vertices in that set. If Lister marks a proper subset M of G, then $G[M]$ is a disjoint union of paths. Under the orientation, each path P has an endpoint u satisfying $d_P^+(u) = 1$ and an endpoint v satisfying $d_P^+(v) = 0$. For each path P, Painter begins at u and colors every other vertex along the path.

A vertex w is rejected only if its earlier neighbor w' is colored. Since this happens at most m times, no vertex is rejected too many times. □

We now introduce notation and terminology that we use in proving that $K_{2,3}$ is $(2m, m)$-paintable for all $m \geq 1$. Let $G = K_{2,3}$ have vertices labeled as shown below.

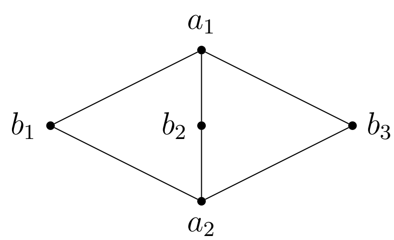

Figure 4.7: $K_{2,3}$ with vertices labeled

Let f be an assignment of tokens to $K_{2,3}$, and let g be the function specifying how many colors each vertex in $K_{2,3}$ must receive. For each edge a_ib_j of G, let

$$\begin{aligned} w_{A,f,g}(a_ib_j) &= f(a_i) - g(a_i) - g(b_j), \\ w_{B,f,g}(a_ib_j) &= f(b_j) - g(b_j) - g(a_i). \end{aligned}$$

For a set D of edges, we let $w_{A,f,g}(D) = \sum_{e \in D} w_{A,f,g}(e)$ and $w_{B,f,g}(D) = \sum_{e \in D} w_{B,f,g}(e)$. An edge set X is *special* if $|X| \geq 2$ and there is an edge $e \in X$ such that every other edge $e' \in X$ has no common endpoint with e. Observe that a special edge set X contains either two or three edges. Indeed, up to isomorphism, there are only two special edge sets $\{a_1b_1, a_2b_2\}$ and $\{a_1b_1, a_2b_2, a_2b_3\}$. We say that the pair (f, g) has Property $(\star)$ if the following two conditions hold.

(1) For each edge uv, $\max\{f(u), f(v)\} \geq g(u) + g(v)$.

(2) For any special edge set X, $w_{A,f,g}(X) \geq 0$ and $w_{B,f,g}(X) \geq 0$.

Theorem 4.3.2. *If (f, g) has Property ($\star$), then G is (f, g)-paintable.*

Proof. We use induction on the total number of tokens. Assume (f, g) has Property ($\star$). First we consider the case that there exists a forced vertex.

Assume a_1 is forced; say a_1 is an (a, a)-vertex. Let a_2 be a $(c + d, d)$-vertex, and let each b_i be a $(x_i + y_i, y_i)$-vertex. Now $w_{A,f,g}(\{a_2b_1, a_1b_2, a_1b_3\}) \geq 0$ implies $c \geq y_1 + y_2 + y_3$, so a_2 is (f, g)-degenerate. By Proposition 4.0.10, G is (f, g)-paintable if and only if $G - a_2$ is (f, g)-paintable. By (1) of Property ($\star$), $x_i \geq a$ for $i \in \{1, 2, 3\}$. Thus b_1, b_2, b_3 are all (f, g)-degenerate in $G - a_2$. Hence $G - a_2$ is (f, g)-paintable if and only if $G - \{a_2, b_1, b_2, b_3\}$ is (f, g)-paintable, which is obviously true.

Assume b_1 is forced; say b_1 is a (b, b)-vertex. Let b_j be a $(c_j + d_j, d_j)$-vertex for $j \in \{2, 3\}$, and let a_i be a $(x_i + y_i, y_i)$-vertex for $i \in \{1, 2\}$. Now $w_{B,f,g}(\{a_1b_2, a_2b_1\}) \geq 0$ implies $c_2 \geq y_1 + y_2$. and $w_{B,f,g}(\{a_1b_3, a_2b_1\}) \geq 0$ implies $c_3 \geq y_1 + y_2$. Thus both b_2 are b_3 are (f, g)-degenerate, and Proposition 4.0.10 implies that G is (f, g)-paintable if and only if $G - \{b_2, b_3\}$ is (f, g)-paintable. By (1) of Property ($\star$), $x_i \geq b$ for $i \in \{1, 2\}$. Thus in $G - \{b_2, b_3\}$, both a_1 and a_2 are (f, g)-degenerate. So $G - \{b_2, b_3\}$ is (f, g)-paintable if and only if $G - \{a_1, a_2, b_2, b_3\}$ is (f, g)-paintable, which is obviously true.

Assume there are no forced vertices. We shall prove that for any marked set M, Painter has a response D such that $(f - \delta_M, g - \delta_X)$ has Property ($\star$). By playing this strategy, eventually either some vertex will be forced, and the previous arguments imply that Painter wins, or Painter wins before any vertex is forced.

(R1) If there exist $a_i \in M$ and $b_j \notin M$ such that $w_{B,f,g}(a_ib_j) < 0$, then let $D = M \cap A$.

(R2) Else, if there exist $b_j \in M$ and $a_i \notin M$ such that $w_{A,f,g}(a_ib_j) < 0$, then let $D = M \cap B$.

(R3) Else, if $|M \cap A| \geq |M \cap B|$, then let $D = M \cap A$.

(R4) Else, let $D = M \cap B$.

Let $f' = f - \delta_M$ and $g' = g - \delta_D$.

First we show that $\max\{f'(u), f'(v)\} \geq g'(u) + g'(v)$ for any edge uv.

Suppose this is not true. Without loss of generality, assume $\max\{f'(a_1), f'(b_1)\} < g'(a_1) + g'(b_1)$. Since (f, g) has Property ($\star$), $\max\{f(u), f(v)\} \geq g(u) + g(v)$. If $f(a_1) \geq g(a_1) + g(b_1)$, then $f'(a_1) < g'(a_1) + g'(b_1)$ implies $f'(a_1) = f(a_1) - 1$ and $g'(a_1) = g(a_1), g'(b_1) = g(b_1)$. Hence $a_1 \in M, b_1 \notin M$, and $D = M \cap B$. Thus in this case (R1) is not applied, which implies $w_{B,f,g}(a_1b_1) \geq 0$. In particular, $f(b_1) \geq g(b_1) + g(a_1)$, and hence $f'(b_1) = f(b_1) \geq g'(a_1) + g'(b_1)$.

If $f(a_1) < g(a_1) + g(b_1)$, then $f(b_1) \geq g(a_1) + g(b_1)$, and $f'(b_1) < g'(a_1) + g'(b_1)$ implies $b_1 \in M, a_1 \notin M$, and $D = M \cap A$. Thus $w_{A,f,g}(a_1b_1) < 0$, and yet (R2) is not applied. This implies that (R1) is applied. Without loss of generality, we may assume that $a_2 \in M$, $b_2 \notin M$, and $w_{B,f,g}(a_2b_2) < 0$. However $w_{B,f,g}(\{a_1b_1, a_2b_2\}) \geq 0$ implies $f(b_1) \geq g(a_1)+g(b_1)+g(a_2)+g(b_2)-f(b_2) \geq g(a_1)+g(b_1)+1$. This then implies $f'(b_1) = f(b_1) - 1 \geq g(a_1) + g(b_1) \geq g'(a_1) + g'(b_1)$, a contradiction.

Next we show (2) of Property ($\star$) holds for (f', g'), meaning that for any special edge set X, $w_{A,f,g}(X) \geq 0$ and $w_{B,f,g}(X) \geq 0$.

Observe that if $D = M \cap A$, then for any edge uv, $w_{A,f,g}(uv) = w_{A,f',g'}(uv)$. Indeed, if $u \in M$, then $f'(u) = f(u) - 1$, $g'(u) = g(u) - 1$, and $g'(v) = g(v)$. So $w_{A,f,g}(uv) = w_{A,f',g'}(uv)$. If $u \notin M$, then $f'(u) = f(u)$, $g'(u) = g(u)$, and $g'(v) = g(v)$. Again $w_{A,f,g}(uv) = w_{A,f',g'}(uv)$. Similarly, if $D = M \cap B$, then $w_{B,f,g}(uv) = w_{B,f',g'}(uv)$ for any edge uv.

Case 1: *(R1) applies.* Since $D = M \cap A$, by the observation above, for any special edge set X, $w_{A,f',g'}(X) \geq w_{A,f,g}(X) \geq 0$. It remains to show that $w_{B,f',g'}(X) \geq 0$.

As (R1) applies, there is an edge, say $e = a_1b_1$, such that $w_{B,f,g}(a_1b_1) < 0$, $a_1 \in M$ and $b_1 \notin M$.

Straightforward calculation shows the following hold:

1. $w_{B,f',g'}(a_1b_1) = w_{B,f,g}(a_1b_1) + 1 \leq 0$.

2. If d is an edge incident to a_1 or incident to b_1, $w_{B,f',g'}(d) \geq w_{B,f,g}(d)$.

3. If an edge d is incident to neither a_1 nor b_1, i.e., $d \in \{a_2b_2, a_2b_3\}$, then $w_{B,f',g'}(d) \geq w_{B,f,g}(d) - 1$. However, for such an edge d, by (2) of Property ($\star$), we have $w_{B,f,g}(d) \geq -w_{B,f,g}(a_1b_1) \geq 1$, which implies that $w_{B,f',g'}(d) \geq 0$.

First we assume that $a_1b_1 \in X$. If X contains at most one of a_2b_2 and a_2b_3, then $w_{B,f',g'}(X) \geq w_{B,f,g}(X) \geq 0$ by the observations above. If X contains both a_2b_2, a_2b_3, then $X - \{a_2b_3\}$ is also special, and hence $w_{B,f',g'}(X - \{a_2b_3\}) \geq 0$. As $w_{B,f',g'}(a_2b_3) \geq 0$, we have $w_{B,f',g'}(X) = w_{B,f',g'}(X - \{a_2b_3\}) + w_{B,f',g'}(a_2b_3) \geq w_{B,f',g'}(X - \{a_2b_3\}) \geq 0$.

Next assume X does not contain a_1b_1. Then X contains at most one of the edges a_2b_2, a_2b_3 (for otherwise, X is not special). If X contains none of a_2b_2, a_2b_3, then by the observations above, $w_{B,f',g'}(X) \geq w_{B,f,g}(X) \geq 0$. Thus we may assume that $a_2b_2 \in X$ and $a_2b_3 \notin X$. If every other edge of X are non-adjacent to a_2b_2, then $X \cup \{a_1b_1\}$ is also special and $w_{B,f',g'}(X) = w_{B,f',g'}(X \cup \{a_1b_1\}) - w_{B,f',g'}(a_1b_1) \geq 0$ (as

$w_{B,f',g'}(a_1b_1) \leq 0$). Assume X contains another edge adjacent to a_2b_2. The only possible special edge set is $X = \{a_2b_1, a_2b_2, a_1b_3\}$. In this case, $w_{B,f',g'}(X) = w_{B,f',g'}(X - \{a_2b_2\}) + w_{B,f',g'}(a_2b_2) \geq w_{B,f,g}(X - \{a_2b_2\}) + w_{B,f',g'}(a_2b_2)$. Since $X - \{a_2b_2\}$ is special, we have $w_{B,f,g}(X - \{a_2b_2\}) \geq 0$. As $w_{B,f',g'}(a_2b_2) \geq 0$, we conclude that $w_{B,f',g'}(X) \geq 0$. This completes the proof of Case 1.

Case 2: *(R2) applies.* The proof of this case is the same as that of Case 1. One simply needs to interchange A and B in the subscripts and the roles of a_1 and b_1 in the marked set M.

Case 3: *(R3) applies.*

If $M \cap A = A$, then $w_{A,f',g'}(X) \geq w_{A,f,g}(X) \geq 0$ for any special edge set X. If $|M \cap A| = 2$, then $w_{B,f',g'}(e) \geq w_{B,f,g}(e)$ for every edge e. Assume that $|M \cap A| = 1$. Then $|M \cap B| \leq 1$. The case $M \cap B = \emptyset$ is trivial. Thus we may assume $M = \{a_1b_1\}$. Then $w_{B,f',g'}(a_2b_1) = w_{B,f,g}(a_2b_1) - 1$, $w_{B,f',g'}(a_1b_2) = w_{B,f,g}(a_1b_2) + 1$, $w_{B,f',g'}(a_1b_3) = w_{B,f,g}(a_1b_3) + 1$, and for every other edge e, $w_{B,f',g'}(e) = w_{B,f,g}(e)$. If a special edge set X does not contain a_2b_1, then $w_{B,f',g'}(X) \geq w_{B,f,g}(X) \geq 0$. If X contains a_2b_1, then X contains at least one of a_1b_2 and a_1b_3, In this case, we also have $w_{B,f',g'}(X) \geq w_{B,f,g}(X) \geq 0$.

Case 4: *(R4) applies.*

As (R3) does not apply, $|M \cap B| \geq 2$. If $|M \cap B| = 3$, then $w_{A,f',g'}(e) \geq w_{A,f,g}(e)$ and $w_{B,f',g'}(e) \geq w_{B,f,g}(e)$ for every edge e. Thus we may assume $M = \{a_1, b_1, b_2\}$. If X is a special edge set not containing a_1b_3, then $w_{A,f',g'}(e) \geq w_{A,f,g}(e)$ for each edge $e \in X$, and hence $w_{A,f',g'}(X) \geq w_{A,f,g}(X) \geq 0$. If $a_1b_3 \in X$, then X contains at least one of a_2b_1, a_2b_2. As in Case 3, $w_{B,f',g'}(X) \geq w_{B,f,g}(X) \geq 0$. $\square$

Corollary 4.3.3. *$K_{2,3}$ is $(2m, m)$-paintable.*

4.4 Planar Graphs

Tuza and Voigt [53] proved that planar graphs are $(5m, m)$-choosable for all $m \geq 1$. Schauz [48] proved that all planar graphs are 5-paintable. In this section, we strengthen both results by showing that planar graphs are $(5m, m)$-paintable for all $m \geq 1$. The following lemmas generalize what Schauz [48] called the Edge Lemma and the Merge Lemma. The original statements of these lemmas were for the case when $g = 1$.

Lemma 4.4.1 (Edge Lemma)**.** *If G is (f, g)-paintable and $uv \notin E(G)$, then $G \cup uv$ is (f', g)-paintable where*

$$f'(w) = \begin{cases} f(v) + f(u), & \text{if } w = v \\ f(w), & \text{otherwise} \end{cases}.$$

Proof. Consider the (f', g)-paintability game on $G \cup uv$. While all tokens are identical from the perspective of the game, we think of the tokens assigned by f as being blue, and the "extra" $f(u)$ tokens on v as being red. Let $\mathcal{S}$ be a winning strategy for Painter in the (f, g)-paintability game on G. The strategy $\mathcal{S}$ only knows about the blue tokens and does not take into consideration the red tokens on v.

Whenever Lister marks u, we sacrifice a red token on v and have Painter respond to the marked set $M - v$ according to $\mathcal{S}$. At most $f(u)$ red tokens are sacrificed on v. In rounds when u is not marked, Painter may respond according to $\mathcal{S}$ because any response in G is an independent set in $G \cup uv$. At least $f(v)$ blue tokens are available for moves of this type, so $g(v)$ colors will be assigned to v by playing according to $\mathcal{S}$. $\square$

Lemma 4.4.2 (Merge Lemma). *Let $G = G_1 \cup G_2$, and let $T = V(G_1) \cap V(G_2)$. If G_i is (f_i, g_i)-paintable and $f_2(v) = g_2(v) = g_1(v)$ for all $v \in T$, then G is (f, g)-paintable where* $f(v) = \begin{cases} f_1(v), & \text{if } v \in V(G_1) \\ f_2(v), & \text{otherwise} \end{cases}$ *and* $g(v) = \begin{cases} g_1(v), & \text{if } v \in V(G_1) \\ g_2(v), & \text{otherwise} \end{cases}$.

Proof. Because $f_2(v) = g_2(v)$ for $v \in T$ and G_2 is (f_2, g_2)-paintable, we know that $G_2[T]$ has no edges. Always $f_i(v) \geq g_i(v)$, so $f(v) = \max\{f_1(v), f_2(v)\}$ and $g(v) = g_1(v) = g_2(v)$.

We use induction on $\sum g(v)$. For the basis step, if $\sum g(v) = 0$, then G is trivially (f, g)-paintable. Now consider $\sum g(v) > 0$.

Let M be the set marked by Lister. For $i \in \{1, 2\}$, let $\mathcal{S}_i$ be a winning strategy for Painter in G_i under token assignment f_i, and let $M_i = M \cap V(G_i)$. Let D_1 be the response to M_1 in G_1 according to $\mathcal{S}_1$. In G_2, Painter responds to the marked set $(M_2 - T) \cup (D_1 \cap T)$ according to $\mathcal{S}_2$. We interpret vertices of $(M - D_1) \cap T$ as having lost a token in G_1 but not in G_2. Because $f_2(v) = g_2(v)$ for all $v \in T$, it must be the case that $(D_1 \cap T) \subseteq D_2$. Thus $D_1 \cup D_2$ is an independent set; Painter now colors $D_1 \cup D_2$.

To make use of the induction hypothesis, we define the following functions.

$$f_1'(v) = \begin{cases} f_1(v) - 1, & \text{if } v \in M \\ f_1(v), & \text{otherwise} \end{cases}$$

$$f_2'(v) = \begin{cases} f_2(v) - 1, & \text{if } M_2 - (T - D_1) \\ f_2(v), & \text{otherwise} \end{cases}$$

$$\text{For } i \in \{1, 2\},\ g_i'(v) = \begin{cases} g_i(v) - 1, & \text{if } v \in D_i \\ g_i(v), & \text{otherwise} \end{cases}$$

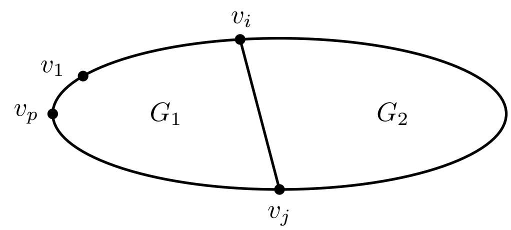

Figure 4.8: Theorem 4.4.3, Case 1: Unbounded face has a chord

Because D_1 and D_2 were chosen according to a winning strategies in G_1 and in G_2, we have that G_i is (f_i', g_i')-paintable for $i \in \{1, 2\}$ and $f_2'(v) = g_2'(v) = g_1'(v)$ for all $v \in T$. Since $M \neq \emptyset$, we may assume that $D_1 \cup D_2 \neq \emptyset$. Thus $\sum g(v)$ decreases, and the induction hypothesis implies the desired result. $\square$

We now state the main theorem of this section.

Theorem 4.4.3. *Planar graphs are $(5m, m)$-paintable for all $m \geq 1$.*

Proof. We proceed using an argument mirroring that of Schauz [48], which is modeled after Thomassen's argument in [52]. First, we restrict our attention to weak triangulations of planar graphs since adding edges only makes coloring the graph more difficult for Painter. Let G be a planar graph of order n with vertices $v_1, \ldots, v_p$ in clockwise order on the unbounded face. By induction on n, we prove a stronger result:

$$G \text{ is } (f, m)\text{-paintable when } f(v) = \begin{cases} m, & \text{if } v = v_p \\ 2m, & \text{if } v = v_1 \\ 3m, & \text{if } v = v_i \text{ for } 1 < i < p \\ 5m, & \text{otherwise} \end{cases}.$$

By Lemma 4.4.1, it suffices to show that

$$G - v_1v_p \text{ is } (f', m)\text{-paintable when } f'(v) = \begin{cases} m, & \text{if } v \in \{v_1, v_p\} \\ 3m, & \text{if } v = v_i \text{ for } 1 < i < p \\ 5m, & \text{otherwise} \end{cases}.$$

Case 1: *There is a chord v_iv_j connecting two vertices on the unbounded face.* Let G_1 be the graph induced by the vertices of the cycle containing v_1 and v_p and by the vertices on the interior of this cycle. Let G_2 have vertex set $(V(G) - V(G_1)) \cup \{v_i, v_j\}$ and edge set $E(G) - E(G')$. This setup is shown in Figure 4.8. Let $g_1(v) = m$ for all $v \in V(G_1)$, $g_2(v) = m$ for all $v \in V(G_2)$, $f_1(v) = f(v)$ for all $v \in V(G_1)$,

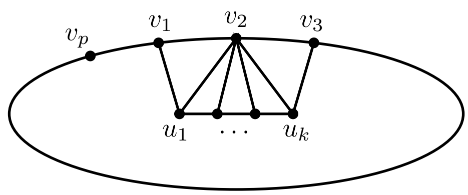

Figure 4.9: Theorem 4.4.3, Case 2: Chordless unbounded face

and $f_2(v) = \begin{cases} f(v), & \text{if } v \in V(G_2) - \{v_i, v_j\} \\ m, & \text{if } v \in \{v_i, v_j\} \end{cases}$.

By the induction hypothesis G_1 is (f_1, g_1)-paintable, and G_2 is (f_2, g_2)-paintable by first applying Lemma 4.4.1 to the edge v_iv_j and then using the induction hypothesis. Lemma 4.4.2 then implies that G is (f, g)-paintable.

Case 2: *The unbounded face is chordless.* See Figure 4.9, and consider $N(v_2)$. Since all bounded faces are triangles, there exists a path $v_1, u_1, \ldots, u_t, v_3$ through $N(v_2)$. Let $U = \{u_1, \ldots, u_t\}$, and let $G' = G - v_2$. Applying the induction hypothesis to G', we show that if each $u \in U$ is given $2m$ additional tokens, then we can extend a winning strategy for Painter on G' to a winning strategy on G.

Let $\mathcal{S}$ be a winning strategy for Painter in G'. Suppose that Lister marks a set M, and let D be Painter's response to the marked set $M - \{v_2\}$ according to $\mathcal{S}$. Note that if $v_1 \in M$, then $v_1 \in D$. If $v_2 \notin M$, then Painter colors D. If $v_2 \in M$ and $v_1 \in D$, then Painter colors D and sacrifices a token on v_2. When $v_2 \in M$, and $v_1 \notin D$, Painter obtains the response D' to the marked set $M - U$ according to $\mathcal{S}$ and colors v_2 if $v_3 \notin D'$. Each vertex of U loses at most $2m$ tokens from moves of this type. Also, v_2 is rejected at most m times because of $v_3 \in D'$. Finally, v_3 never loses tokens because of v_2. Therefore, every vertex is colored m times before it runs out of tokens. □

4.5 Brooks' Theorem

We say G is *m-degree paintable* if G is (f, m)-paintable where $f(v) = d(v)m$ for all v. When $m = 1$, we simply say "degree paintable".

Let G be a connected graph other than an odd cycle or a complete graph. Tuza and Voigt [53] proved a generalization of Brooks' Theorem for choosability by showing that G is $(\Delta(G)m, m)$-choosable for all $m \geq 1$. Hladký, Kráľ, and Schauz [20] strengthened Brooks' Theorem by proving that G is $\Delta(G)$-paintable. In this section, we strengthen both results by proving that such a G is $(\Delta(G)m, m)$-paintable.

The following lemma allows us to extend good strategies on an induced subgraph to a larger graph.

Lemma 4.5.1. *Given a graph G, if there exists an induced subgraph H that is m-degree paintable, then G is m-degree paintable for all $m \geq 1$.*

Proof. If $H = G$, there is nothing to show, so suppose $V(G) - V(H) \neq \emptyset$, and let $U = \{u_1, \ldots, u_t\} = V(G) - V(H)$. Let $\mathcal{S}$ be a winning m-degree paintability strategy for Painter on H.

Let M be the set that Lister marks. Let D be an independent subset of $M \cap U$ chosen greedily with respect to the ordering $u_1, \ldots, u_t$. According to $\mathcal{S}$, Painter obtains a response D' in H to the marked set $(M \cap V(H)) - N(D)$. We sacrifice a token on each vertex of $M \cap V(H) \cap N(D)$, and Painter colors $D \cup D'$. Note that $D \cup D'$ is an independent set because we forbid coloring any neighbors of vertices in D.

Each $v \in V(H)$ sacrifices at most m tokens for any neighbor outside of H, which guarantees that at least $d_H(v)m$ tokens are available for the strategy $\mathcal{S}$. Each $u \in U$ is rejected at most m times for each earlier neighbor, which always leaves at least m tokens available to color u when it has no more uncolored earlier neighbors. Therefore G is m-degree paintable. □

The following is a well-known structural lemma of Erdős, Rubin, and Taylor [16].

Lemma 4.5.2 ([16])**.** *If G is a 2-connected graph that is not an odd cycle or a complete graph, then G contains an induced even cycle having at most one chord.*

We now show that the induced subgraph obtained from the conclusion of Lemma 4.5.2 is m-degree paintable for all $m \geq 1$.

Lemma 4.5.3. *An even cycle with at most one chord is m-degree paintable for all $m \geq 1$.*

Proof. **Case 1:** *G is a chordless even cycle.* Zhu [57] proved that C_{2n} is $(2m, m)$-paintable for $n \geq 2, m \geq 1$.

Case 2: *G is an even cycle with exactly one chord.* Let $v_1, \ldots, v_n$ be the vertices of this cycle in clockwise order, and suppose $v_1 v_i$ is the chord (Figure 4.10). Consider the graph G' obtained from G by removing the edge $v_n v_1$. Let f' be a token assignment obtained from f by removing $2m$ tokens from v_1. By Lemma 4.4.1, if G' is (f', g)-paintable, then G is m-degree paintable. In G', we repeated apply Proposition 4.0.10 to $V(G')$ in the order $v_n, v_{n-1}, \ldots, v_1$. At each step, the vertex being colored has at least as many tokens as the number of times it and its neighbors must be colored, therefore G' is (f', g)-paintable. □

Lemmas 4.5.2 and 4.5.3 imply that every block of a graph that is not m-degree paintable must be an odd cycle or a clique. A graph in which every block is an odd cycle or a clique is called a *Gallai tree*.

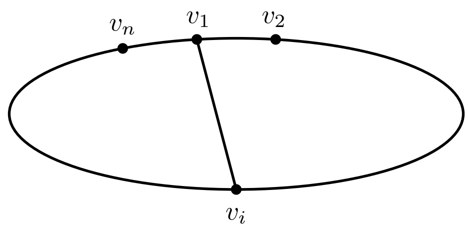

Figure 4.10: Lemma 4.5.3, Case 2: Even cycle with one chord

Theorem 4.5.4. *Given $m \geq 1$, a graph G is m-degree paintable if and only if G is not a Gallai tree.*

Proof. If G is a Gallai tree, then it is not m-degree choosable [54], and hence, not m-degree paintable.

When G is not a Gallai tree, there exists a block B that is not a complete graph or an odd cycle. By Lemma 4.5.2, B contains an induced even cycle with at most one chord. Lemma 4.5.3 implies that B is m-degree paintable. Lastly, Lemma 4.5.1 implies that G is m-degree paintable. □

We now give the main result of this section.

Theorem 4.5.5. *If G is not an odd cycle or a complete graph, then G is $(\Delta(G)m, m)$-paintable for all $m \geq 1$.*

Proof. If G is not a Gallai tree, then Theorem 4.5.4 implies $(\Delta(G)m, m)$-paintability. We may assume that G is a Gallai tree with at least two blocks. Thus G is not $\Delta(G)$-regular, and every vertex of maximum degree is a cut-vertex. In particular, G is $(\Delta(G) - 1)$-degenerate, which implies $(\Delta(G)m, m)$-paintability. □

Chapter 5

Sum-Paintability

Introduced by Isaak [23], the *sum-choosability* of a graph G, denoted $\mathrm{sch}(G)$, is the least $\sum f(v)$ over all f such that G is f-choosable. In essence, this studies how large the average list size must be to permit L-colorings rather than how large the minimum list size must be when all vertices receive lists of the same size. Isaak [23] showed that $\mathrm{sch}(K_2 \,\square\, K_n) = n^2 + \lceil \frac{5n}{3} \rceil$, and he also observed the easy upper bound $\mathrm{sch}(G) \le |V(G)| + |E(G)|$ [24].

The *sum-paintability* (or *online sum-choosability*) of a graph G, denoted $\mathring{\mathrm{sch}}(G)$, is the least $\sum f(v)$ over all functions f such that G is f-paintable. One can imagine Painter being allocated a budget of tokens to distribute to the vertices before the game begins; the sum-paintability is the smallest total that allows Painter to produce a winning distribution.

We have the natural inequality $\mathring{\mathrm{sch}}(G) \ge \mathrm{sch}(G)$. Nevertheless, $|V(G)| + |E(G)|$ is also an upper bound on sum-paintability (Proposition 5.1.1).

Letting $\sigma(G) = |V(G)| + |E(G)|$, graphs attaining the upper bound $\sigma(G)$ on sum-choosability or sum-paintability are called *sc-greedy* or *sp-greedy*, and $\sigma(G)$ is the *greedy bound*. When $\sigma(G)$ tokens are available, Painter has a "greedy" winning strategy (allocating tokens and responding to Lister) in terms of any vertex ordering. Since $\mathrm{sch}(G) \le \mathring{\mathrm{sch}}(G) \le \sigma(G)$, any graph that is sc-greedy is also sp-greedy; however, sp-greediness may be easier to prove. Graphs already known to be sc-greedy include cycles, trees [5], complete graphs [5], and graphs whose blocks are all sc-greedy [5] (generalizing [24]).

Section 5.1 contains preliminary results about sum-paintability that are used in the later sections.

In Section 5.2, we construct several families of sp-greedy graphs. Our main result is a tool that takes a graph G and adds to G a clique on new vertices such that each vertex in the clique has exactly one neighbor in G. When G is sp-greedy, we show that this operation preserves sp-greediness. As a corollary, adding a vertex of degree 1 to G preserves sp-greediness. Recall that an *ear* is a path in a graph in which the two endpoints of the path may be the same, but the internal vertices all have degree 2. This tool also implies that adding an ear of length at least 3 to G preserves sp-greediness. Additionally, it shows that

$\mathring{\mathrm{sch}}(K_2 \,\square\, K_n) = \sigma(K_2 \,\square\, K_n)$, which provides an family of example graphs for which $\mathring{\mathrm{sch}}(G) - \mathrm{sch}(G)$ grows without bound.

In Section 5.3, we determine the sum-paintability of all generalized theta-graphs. Recall that the generalized theta-graph $\Theta_{\ell_1,\ldots,\ell_k}$ consists of two vertices joined by k ears of lengths $\ell_1, \ldots, \ell_k$. As a convention, we assume $\ell_1 \leq \cdots \leq \ell_k$ and that $\ell_2 > 1$ so that we avoid having a multigraph. Some graphs, such as $\Theta_{2,2,2t}$ for $t > 1$ (see Corollary 5.3.4), are sp-greedy but not sc-greedy (that is, $\mathrm{sch}(G) < \mathring{\mathrm{sch}}(G) = \sigma(G)$). Erdős, Rubin, and Taylor [16] proved that $\mathrm{ch}(\Theta_{2,2,2t}) = 2$, so $\mathrm{sch}(\Theta_{2,2,2t}) \leq 2(2t+3) < \sigma(G)$. Lastrina [33] showed that $\mathrm{sch}(\Theta_{\ell_1,\ell_2,\ell_3})$ is sc-greedy unless $\ell_1 = \ell_2 = 2$ and ℓ_3 is even.

A graph is *weakly sp-greedy* if all of its induced subgraphs are sp-greedy. While being weakly sp-greedy is a necessary condition for a graph to be sp-greedy, it is not sufficient. Studying weakly sp-greedy graphs is useful when using reducibility arguments. If a configuration of vertices and tokens is reducible for sp-greediness, then it cannot occur in a minimal graph failing to be sp-greedy. Thus if a weakly sp-greedy graph G contains a configuration that is reducible for sp-greediness, then G must be sp-greedy. In Section 5.4, we prove several such reducibility arguments. After stating a general method for proving reducibility, we characterize the values of r and s for which $K_r \mathbin{\oplus} \overline{K}_s$ is sp-greedy.

An *outerplanar* graph is a planar graph that has an embedding in which every vertex lies on the boundary of the unbounded face. *Fans* are one family of outerplanar graphs, where the fan on $n+1$ vertices, denoted F_n, is the join $P_n \mathbin{\oplus} K_1$. In his dissertation, Heinold [19] mentions that Isaak (and independently Albertson and Pelsmajer) asked whether every outerplanar graph is sc-greedy. Heinold answered this in the negative by showing that $\mathrm{sch}(F_n) \leq \sigma(F_n) - \lfloor (n+1)/11 \rfloor$. However, the question remains open for sum-paintability, and in joint work with Tomlinson and Wise [38], we make the following conjecture.

Conjecture 5.0.1 ([38]). *Every outerplanar graph is sp-greedy.*

In Section 5.5, we prove several partial results toward Conjecture 5.0.1 that also appear in [38]. The *weak dual* of a plane graph is obtained from the dual by deleting the vertex corresponding to the unbounded face. The weak dual of a connected outerplanar graph is a tree. We show that every connected outerplanar graph whose weak dual is a path is sp-greedy. This family includes fans and squares of paths; the former family provides an infinite family of graphs such that $\mathring{\mathrm{sch}}(G) - \mathrm{sch}(G)$ can be arbitrarily large, and the latter family was previously shown to be sc-greedy [19]. Using that fans are sp-greedy, we prove that $C_n \mathbin{\oplus} K_1$ is sp-greedy.

5.1 Lemmas

Odd cycles have paint number 3 but "just barely" fail to be 2-paintable, in the sense that $\mathring{\text{sch}}(C_n) = 2n$ for all n (see Corollary 5.2.7). An easy general upper bound for sum-choosability holds also for sum-paintability.

Proposition 5.1.1. *For any graph G, $\mathring{\text{sch}}(G) \leq |V(G)| + |E(G)|$.*

Proof. Given any fixed ordering π of $V(G)$, Painter allocates $1 + d^-(v)$ tokens to each vertex v, where $d^-(v)$ is the number of neighbors of v that occur earlier than v in π. Painter's strategy is greedy: for any marked set M, color the independent subset of M chosen greedily with respect to π. That is, the colored set R is the unique maximal set of vertices in M such that each vertex of M is in R if and only if it has no neighbor in M that is earlier in π.

Painter wins using this strategy, because a vertex v is marked (and not colored) at most once for each earlier neighbor. The total number of tokens is $|V(G)| + |E(G)|$; we have allocated one token for each vertex plus one token for the later endpoint of each edge. □

Berliner et al. [5] showed that the sum-choosability of any graph is determined by the sum-choosability of its blocks.

Theorem 5.1.2 ([5])**.** *If G is a graph with blocks $H_1, \dots, H_k$, then* $\text{sch}(G) = \sum_{i=1}^{k} \text{sch}(H_i) - (k-1)$.

In joint work with Carraher, Puleo, and West [11], we show that this statement also holds for sum-paintability. We need a preliminary observation.

Proposition 5.1.3. *For a graph G, let $\mathcal{S}$ be a winning strategy for Painter under an allocation of* $\mathring{\text{sch}}(G)+k$ *tokens. Given that Painter plays according to $\mathcal{S}$, for any vertex $v \in V(G)$ Lister can ensure that the number of tokens on v is reduced to at most k at the time when v is colored.*

Proof. If $\mathcal{S}$ allows Painter to guarantee that v retains $1+k$ tokens at the time when it is colored, then playing the same strategy will ensure a win for Painter even when the number of tokens at v is reduced by $1 + k$. This gives Painter a winning strategy with $\mathring{\text{sch}}(G) - 1$ tokens, a contradiction. □

Theorem 5.1.4. *If G is a graph with blocks $H_1, \dots, H_k$, then* $\mathring{\text{sch}}(G) = \sum_{i=1}^{k} \mathring{\text{sch}}(H_i) - (k-1)$.

Proof. By induction on k, we may assume that G has two blocks, H_1 and H_2, and one cut-vertex, v. We first prove $\mathring{\text{sch}}(G) \geq \mathring{\text{sch}}(H_1) + \mathring{\text{sch}}(H_2) - 1$. Let f be an allocation of tokens on G with smallest sum that enables Painter to win. Under f, let ℓ_i be the number of tokens on $H_i - v$, and let ℓ' be the number of tokens on v.

Since Painter has a winning strategy with this allocation of tokens, Painter also has a winning strategy when Lister plays only on H_i. By Proposition 5.1.3, some strategy for Lister on H_1 forces the number of tokens on v to be reduced to $\ell_1 + \ell' - \mathring{\text{sch}}(H_1) + 1$ at the end of a round before the round when v would be colored. When that value is reached, $\ell_1 + \ell_2 + \ell' - \mathring{\text{sch}}(H_1) + 1$ tokens remain on H_2.

Lister now switches to play on H_2 instead. By Proposition 5.1.3, some strategy for Lister on H_2 forces the number of tokens on v to be reduced to $\ell_1 + \ell_2 + \ell' - \mathring{\text{sch}}(H_1) - \mathring{\text{sch}}(H_2) + 1$ at the time when v is colored. This quantity must be nonnegative, so $\ell_1 + \ell_2 + \ell' \geq \mathring{\text{sch}}(H_1) + \mathring{\text{sch}}(H_2) - 1$, as desired.

For the upper bound, we give a strategy for Painter to win using $\mathring{\text{sch}}(H_1) + \mathring{\text{sch}}(H_2) - 1$ tokens. For $i \in \{1, 2\}$, let f_i be a winning allocation of $\mathring{\text{sch}}(H_i)$ tokens on H_i, with $f_i(v) = a_i$, and let $\mathcal{S}_i$ be a strategy that allows Painter to win with this allocation. Define f on $V(G)$ by $f(x) = f_i(x)$ for $x \in V(H_i) - \{v\}$ and $f(v) = a_1 + a_2 - 1$. View $f(v)$ as $(a_1 - 1) + (a_2 - 1) + 1$, reserving $a_i - 1$ tokens for v in the game on H_i, with one left over.

Given Lister's move M on G, let $M_i = M \cap V(H_i)$, and let R_i be the response for Painter dictated by $\mathcal{S}_i$. If v is in neither or both of R_1 and R_2, then Painter colors $R_1 \cup R_2$. The game continues under Painter's optimal strategy in both subgraphs independently.

If v lies in exactly one of R_1 and R_2, then by symmetry assume $v \in R_2$. Painter will not color v. One token has been lost at v since $v \in M$; we view it as a token at v associated with H_1, and we have $v \in M_1$. Painter colors R_1 from H_1; this is the move under $\mathcal{S}_1$ in the game on H_1, including the loss of the token at v. In H_2, Painter pretends that $M_2 - \{v\}$ was the marked set. Since no token associated with v in H_2 has been colored from v, responding to $M_2 - \{v\}$ according to $\mathcal{S}_2$ continues the winning strategy in H_2.

By attributing the token colored from v to the appropriate subgame, Painter can use the optimal strategies in the two subgraphs essentially independently. The number of times that tokens can be charged to v without coloring v is at most $a_1 - 1$ under $\mathcal{S}_1$ and at most $a_2 - 1$ under $\mathcal{S}_2$. Because these are winning strategies for Painter, v is marked without removal at most $a_1 + a_2 - 2$ times, and thus a token remains at v at a time when both strategies indicate that v should be colored. $\square$

The key to Painter's strategy in Theorem 5.1.4 is to break Lister's move into subsets associated with the two subgraphs. The subtlety is that how that break is made depends on how Painter's substrategies would respond after the subsets are determined.

Theorem 5.1.4 allows us to build sp-greedy graphs from smaller sp-greedy graphs, and Theorem 5.1.2 does the same for sc-greedy graphs.

Corollary 5.1.5. *A graph G is sp-greedy if and only if each of its blocks is sp-greedy, and the same holds for sc-greedy graphs.*

In any graph, the subgraph induced by a vertex of degree 1 and its neighbor is a block. Thus adding a pendant edge increases the sum-choosability and the sum-paintability by 2. This was noted for sum-choosability by Lastrina [33].

5.2 Constructing Sum-Paint Greedy Families

We say a vertex v is a *forcing vertex* when it has only one token at the start of some round. Lister makes use of a forcing vertex v by marking $N[v]$. To avoid losing, Painter must color $\{v\}$ and lose one token on each vertex of $N(v)$.

Proposition 5.2.1. *If f is an assignment of tokens to K_n with vertices $v_1, \ldots, v_n$ such that $f(v_1) \leq \cdots \leq f(v_n)$, then Painter wins if and only if $f(v_i) \geq i$ for all $1 \leq i \leq n$.*

Proof. If $f(v_i) < i$ for some i, then Lister wins by marking all uncolored vertices of $\{v_1, \ldots, v_i\}$ for i rounds; after i rounds some vertex remains with no tokens.

If $f(v_i) \geq i$ for all i, then Painter colors the earliest marked vertex in the ordering. This allocation gives to each vertex at least one more token than the number of earlier neighbors with respect to the ordering $v_1, \ldots, v_n$. Thus no vertex runs out of tokens before being colored under Painter's strategy. ☐

Lemma 5.2.2. *If $G = K_n$ and G is f-paintable and $\sum f(v) = \binom{n+1}{2} + t$, then Lister can create at least $n - t$ forcing vertices throughout the course of the game.*

Proof. Let $v_1, \ldots, v_n$ be the vertices of G such that $f(v_1) \leq \cdots \leq f(v_n)$. When Lister marks all remaining vertices each round, we show that there are at least $n - t$ forcing vertices. By Proposition 5.2.1, we may assume $f(v_i) \geq i$ for all $i \in [n]$ since G is f-paintable. There is nothing to show whenever $t \geq n$, so we may assume $t < n$. In this case, there are at least $n - t$ vertices v_i such that $f(v_i) = i$. Our goal is to show that each such index yields a distinct forcing vertex.

We use induction on n; when $n = 1$ and $t = 0$, v_1 is a forcing vertex at the start of the game. Suppose $n > 1$, and let i be the least index such that $f(v_i) = i$. As described above, Lister begins by marking $\{v_1, \ldots, v_n\}$. If Painter responds by coloring v_j with $j > i$, then Lister wins by Proposition 5.2.1 and playing on $\{v_1, \ldots, v_i\}$. Thus Painter must color some v_l with $l \leq i$. After coloring v_l, relabel the vertices to be $u_1, \ldots, u_{n-1}$, which form a copy of K_{n-1}.

Case 1: *$i = 1$ and $l = 1$.* In this case, v_1 was a forcing vertex at the start of the game. For $i \in [n-1]$, we have u_i corresponds to v_{i+1}. In particular, there are at least $(n-1) - t$ indices such that $f(u_i) = i$. By the induction hypothesis, each such index contributes one forcing vertex. Together with v_1, the original graph had at least $n - t$ forcing vertices.

Case 2: *$i > 1$ and $l = i$.* By the minimality of i and because $f(v_{i-1}) \leq f(v_i)$, we have $f(v_{i-1}) = i$. After Painter colors v_l, we relabel v_{l-1} to be u_{l-1}. Since one token was lost on v_{l-1}, we have $f(u_{l-1}) = l-1$. For all v_j with $j > i$, after relabeling v_j becomes u_{j-1}, and if $f(v_j) = j$, then $f(u_{j-1}) = j - 1$. Thus the induction hypothesis implies that at least $n-t$ forcing vertices guaranteed on the resulting K_{n-1} since there are least $n - t$ indices such that $f(u_j) = j$.

Case 3: *$i > 1$ and $l < i$.* In this case, every v_j such that $f(v_j) = j$ corresponds to u_{j-1} where $f(u_{j-1}) = j - 1$. The resulting K_{n-1} has at least $n - t$ such indices, and by the induction hypothesis, it has at least $n - t$ guaranteed forcing vertices. □

We continue with the main result in this section, which shows how to augment a graph in such a way that increases the sum-paintability by $\frac{n(n+3)}{2}$. In particular, when the original graph is sp-greedy, the larger graph is also sp-greedy.

Theorem 5.2.3. *If $G' = (G + K_n) \cup \{u\phi(u) : u \in V(K_n)\}$ for some $n \in \mathbb{N}$ and $\phi : V(K_n) \to V(G)$, then $\mathring{\mathrm{sch}}(G') = \mathring{\mathrm{sch}}(G) + \frac{n(n+3)}{2}$. Additionally, if G is sp-greedy, then G' is sp-greedy.*

Proof. Lower bound. Let f be an assignment of $\mathring{\mathrm{sch}}(G')$ tokens such that G' is f-paintable. Let $t = \sum_{v \in V(K_n)} f(v) - \mathring{\mathrm{sch}}(K_n)$. On each of the first n rounds, Lister marks all remaining vertices of K_n. If any vertex v has only one token left and is in the marked set, then on that round, Lister includes the vertex $\phi(v)$ in the marked set. Since v must be colored, $\phi(v)$ loses a token. This happens once each time Lister obtains a forcing vertex in K_n. By Lemma 5.2.2, at least $n - t$ vertices are forced in K_n, and for each forcing vertex, one token is lost in G. Since G' is f-paintable, $\sum_{v \in V(G)} f(v) \geq \mathring{\mathrm{sch}}(G) + (n - t)$. Thus $\sum f(v) \geq \mathring{\mathrm{sch}}(G) + (n - t) + (\mathring{\mathrm{sch}}(K_n) + t)$.

Upper bound. Let f be an allocation of $\mathring{\mathrm{sch}}(G)$ tokens such that G is f-paintable. Extend f to G' by letting $f(v_i) = i$ for each $v_i \in V(K_n)$, and also increase $f(\phi(v_i))$ by 1 for each $i \in [n]$. Given any marked set M, Painter responds according to a winning strategy $\mathcal{S}$ in G. If $M \cap V(K_n) \neq \emptyset$, then Painter must color the lowest index vertex, say v_i, of $V(K_n)$. Painter then forms the remaining set to be colored by responding to $M - \phi(v_i)$ in $\mathcal{S}$. Note that $\phi(v_i)$ was given one extra token to account for ignoring the vertex on this round. After this round, $\phi(v_i)$ has no neighbors in K_n and will be colored the next time $\mathcal{S}$ dictates its removal.

If $\mathring{\mathrm{sch}}(G) = \sigma(G)$, then $\mathring{\mathrm{sch}}(G') = \sigma(G) + \delta(G', K_n) = \sigma(G')$, and thus G' is sp-greedy. □

To each $v \in V(K_n)$, a single edge is added to connect the vertex to G. Thus $d_{G'}(v) = n$ for every $v \in V(K_n)$. Theorem 5.2.3 has many useful corollaries that arise from finding a k-clique in a graph in which all vertices have degree k. Corollary 5.2.4 is for $k = 1$, and Corollary 5.2.5 is for $k = 2$.

Corollary 5.2.4. *If x is a vertex of degree 1 in a graph G, then $\mathrm{sch}(G) = \mathrm{sch}(G - x) + 2$ and $\mathring{\mathrm{sch}}(G) = \mathring{\mathrm{sch}}(G - x) + 2$.*

Adding an *ear* to a graph G means adding a path whose endpoints lie in G and whose internal vertices (if any) are new vertices with degree 2. Adding a *closed ear* is adding a cycle with one vertex in G. To study $\mathring{\mathrm{sch}}$ under addition of ears, we need an observation about degeneracy that is used in [11] and in earlier papers on paintability such as [28, 31, 57]. It states that vertices with "excess" tokens are irrelevant in determining whether a graph is f-paintable.

Corollary 5.2.5. *If G' is obtained from G by adding an ear or closed ear with m edges, where $m \geq 3$, then $\mathring{\mathrm{sch}}(G') = \mathring{\mathrm{sch}}(G) + 2m - 1$.*

Proof. To construct an ear of length m where $m \geq 3$, first grow a path of length $m-3$ by successively adding vertices of degree 1. Corollary 5.2.4 implies that this increases the sum-paintability by $2(m - 3)$. □

Corollary 5.2.6. *If F is a forest, then $F \square K_n$ is sp-greedy for any $n \geq 1$.*

Proof. For each component of F, iteratively apply Corollary 5.2.3 with $G = K_n$, and let ϕ be any bijection from $V(K_n)$ to $V(G)$. □

Letting $F = K_2$, we obtain that $K_n \square K_2$ is sp-greedy for all n. Isaak [23] showed that $\mathrm{sch}(K_n \square K_2) = n^2 + \lceil 5n/3 \rceil$. Since $\mathring{\mathrm{sch}}(G) = \sigma(G) = n^2 + 2n$, this family of graphs gives an example where the difference $\mathring{\mathrm{sch}}(G) - \mathrm{sch}(G)$ can be arbitrarily large. The previous largest known difference was $\lfloor n/11 \rfloor$, witnessed by *fans* [38], the join of K_1 and P_n. Our family gives a larger difference of $\lfloor n/6 \rfloor$.

Note that the conclusion of Corollary 5.2.5 does not hold for ears of length at most 2. in fact, Corollary 5.3.1 shows that $\Theta_{2,2,2}$ is not sp-greedy. Meanwhile, using Corollaries 5.2.4 and 5.2.5, one can construct many sp-greedy graphs. In particular, we may start with K_1 to obtain and independent proof that cycles are sp-greedy.

Corollary 5.2.7. $\mathring{\mathrm{sch}}(C_n) = 2n$ *for all $n \geq 3$.*

Proof. Start with K_1, for which $\mathring{\mathrm{sch}}(K_1) = 1$. Applying Corollary 5.2.5 with a closed ear of length n yields the result. □

For another application, we may look at graphs with ear decompositions. It is well known that every 2-edge-connected graph G can be grown from any cycle in G by iteratively adding ears or closed ears.

Corollary 5.2.8. *If G is a 2-edge-connected graph having an ear decomposition in which every ear or closed ear has length at least 3, then G is sp-greedy. If G arises from a subgraph H such that $\mathring{\mathrm{sch}}(H) = \sigma(H) - t$ by such additions, then $\mathring{\mathrm{sch}}(G) = \sigma(G) - t$.*

Using these lemmas, we obtain graphs that are sp-greedy but not sc-greedy.

Corollary 5.2.9. *If $G = \Theta_{2,2,2t}$ where $t > 1$, then $\mathrm{sch}(G) < \mathring{\mathrm{sch}}(G)$.*

Proof. Since G is 2-choosable [16], $\mathrm{sch}(G) \le 2|V(G)| = 4t + 6$. Applying Corollary 5.2.5 to the ear of length $2t$ yields $\mathring{\mathrm{sch}}(G) = \mathring{\mathrm{sch}}(C_4) + 4t - 1 = 4t + 7 = \sigma(G)$, since C_4 is sp-greedy. □

Corollary 5.2.9 provides another proof of the result in [57] that $\Theta_{2,2,2t}$ is not 2-paintable for $t > 1$. As remarked in [19], all other graphs of the form $\Theta_{\ell_1,\ell_2,\ell_3}$ are sc-greedy and thus also sp-greedy.

We continue in the next section by discussion generalized theta-graphs in more detail.

5.3 Generalized Theta-Graphs

Toward determining the sum-paintability of generalized theta-graphs, note that $\Theta_{k_1,\ldots,k_r} = K_{2,r}$ when $k_1 = \cdots = k_r = 2$. Berliner et al. [5] proved that $\mathrm{sch}(K_{2,r}) = 2r + \min\{l + m\colon\ lm > r\}$. Thus $K_{2,r}$ is "far" from sc-greedy. Results in this section are from joint work with Carraher, Puleo, and West [11]. Theorem 2.2.1 implies the following corollary.

Corollary 5.3.1. $\mathrm{sch}(K_{2,r}) = \mathring{\mathrm{sch}}(K_{2,r}) = 2r + \min\{l + m\colon\ lm > r\}$ *for all r.*

Proof. Since always $\mathring{\mathrm{sch}}(G) \ge \mathrm{sch}(G)$, it suffices to find an allocation of $2r + l + m$ tokens (whenever $lm > r$) that enables Painter to win. Allocate two tokens to each vertex in the partite set of size r. Allocate l and m tokens, respectively, to the two vertices in the other partite set. By Theorem 2.2.1, Painter has a winning strategy if and only if $lm > r$. □

In terms of r, $\mathring{\mathrm{sch}}(K_{2,r}) = 2r + 1 + \lfloor \sqrt{4r + 1} \rfloor$ [19].

The *book* B_r is the graph $K_2 \Diamond \overline{K}_r$ (Figure 5.1). In this section, we study B_r as the generalized theta-graph $\Theta_{\ell_1,\ldots,\ell_{r+1}}$ with $\ell_1 = 1$ and $\ell_2 = \cdots = \ell_{r+1} = 2$. We prove that $\mathring{\mathrm{sch}}(B_r) = 2r + \min\{l + m\colon\ m(l - m) + \binom{m}{2} > r\}$.

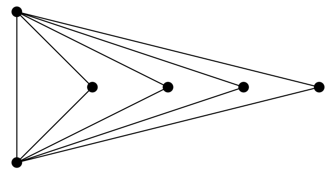

Figure 5.1: B_4

For the lower bound, we prove that $\mathrm{sch}(B_r)$ is at least this big. For the upper bound, let x and y be the two high-degree vertices, and let S be the r vertices of degree 2. We give a winning strategy for Painter under a particular allocation of tokens.

Lemma 5.3.2. $\mathrm{sch}(B_r) \geq g(r)$, *where* $g(r) = 2r + \min\{l + m\colon\ m(l-m) + \binom{m}{2} > r\}$.

Proof. First suppose that $|L(v)| = 2$ for all $v \in S$, with $|L(x)| = l$ and $|L(y)| = m$, where $l \geq m$. For $r \geq m(l-m) + \binom{m}{2}$, give the first $\binom{m}{2}$ vertices of S distinct lists from $\binom{[m]}{2}$. Give the next $m(l-m)$ vertices of S distinct lists from $([l] \setminus [m]) \times [m]$. Give any remaining vertices of S list $\{1, 2\}$. Finally, let $L(x) = [l]$ and $L(y) = [m]$. Under L, every choice of colors for x and y, which must choose distinct colors, is the list for some vertex of S, and hence an L-coloring cannot be completed.

We now prove by induction on r that if L is a list assignment on B_r such that B_r has no L-coloring, then L has sum at least $g(r)$. If $r = 1$, then $B_r = K_3$ and $\mathrm{sch}(K_3) = 6 = g(1)$. For $r > 1$, the special case treated above allows us to assume that $f(v) \neq 2$ for some $v \in S$.

If $f(v) = 1$, then construct set $L(v) = \{1\}$ and put $1 \in L(x) \cap L(y)$. Since v must receive color 1, that color cannot be chosen for x or y. If G has an L-coloring, we then must have at least $3 + \mathrm{sch}(B_{r-1})$ for the sum of the list sizes. Since $3 + g(r-1) \geq g(r)$, the induction hypothesis yields the claim.

If $f(v) \geq 3$, then considering $B_r - v$ leads to the same inequality, requiring at least $3 + \mathrm{sch}(B_{r-1})$ for the sum of the list sizes. Again the induction hypothesis applies. $\square$

For the upper bound on $\mathring{\mathrm{sch}}(B_r)$, we give a winning strategy for Painter under a token assignment with sum $g(r)$.

Theorem 5.3.3. $\mathring{\mathrm{sch}}(B_r) = g(r)$.

Proof. Theorem 5.3.2 gives the lower bound. For the upper bound, allocate $g(r)$ tokens by setting $f(v) = 2$ for $v \in S$ and letting $f(x) = l$ and $f(y) = m$, where l and m are any integers such that $m(l-m) + \binom{m}{2} > r$. We prove by induction on r that Painter has a winning strategy under any such allocation.

If $r = 0$, then $B_r = K_2$ with $\mathring{\mathrm{sch}}(K_2) = 3$, and Painter has a winning strategy when $l \geq 2$ and $m \geq 1$. Now consider $r > 0$. Let M be the set marked by Lister in the first round. We consider possible choices for

M. Let $S = V(B_r) - \{x, y\}$ and $k = |S \cap M|$.

Case 1: $x, y \notin M$. The marked set M is independent; Painter colors M. By the induction hypothesis, Painter can win in the remaining graph B_{r-k}.

Case 2: $|M \cap \{x, y\}| = 1$. Let $\{z, z'\} = \{x, y\}$, where $z \in M$ and $z' \notin M$. Let $t = f(z)$ and $t' = f(z')$. If $k < t'$, then Painter colors z, leaving a star. By Proposition 2.1.1(b), the vertices of $S - M$ are irrelevant. Let $G' = G - z$. Here $d_{G'}(z') = k < t' = f(z')$, so Proposition 2.1.1(b) applies again, reducing the problem to Painter winning on an independent set with all vertices having tokens.

If $k \geq t'$, then Painter colors $S \cap M$, which leaves the graph B_{r-k} with $t - 1$ tokens on z and t' tokens on z' (and two tokens each on the remaining vertices). If $t = l$, then the left side of the desired inequality decreases by m, but the right side decreases by at least m, since $t' = m$. If $t = m$, then the left side decreases by $l - m$, but the right side decreases by at least l, since $t' = l$. Hence in either case the condition for success is satisfied, and by the induction hypothesis Painter has a winning strategy for the remaining game.

Case 3: $x, y \in M$. If $k < \max\{l, m\} - 1$, then Painter colors a vertex of $\{x, y\}$ with $\min\{l, m\}$ tokens, leaving a star. By Proposition 2.1.1(b), the vertices of $S - M$ are irrelevant. We are left with a star having k leaves with one token at each, plus $\max\{l, m\} - 1$ tokens on the center, and Painter has a winning strategy.

If $k \geq \max\{l, m\} - 1$, then Painter colors $S \cap M$, which leaves the graph B_{r-k} with $l - 1$ and $m - 1$ tokens on x and y (and two tokens each on the remaining vertices). The left side of the desired inequality decreases by $l - 1$, but the right side decreases by at least $l - 1$. The induction hypothesis implies that Painter can win on the remaining graph B_{r-k} with the remaining token assignment. $\square$

Using Corollary 5.2.5, Corollary 5.3.1, and Theorem 5.3.3, we determine $\mathring{\mathrm{sch}}(G)$ for any generalized theta-graph G.

Corollary 5.3.4. *If G is a generalized theta-graph $\Theta_{\ell_1,\ldots,\ell_r}$, then*

$$\mathring{\mathrm{sch}}(G) = \begin{cases} \sigma(G), & \text{if } t \leq 3 \\ \mathring{\mathrm{sch}}(K_{2,t-1}) + \sum_{i=t}^{r}(2\ell_i - 1), & \text{if } \ell_1 = 2 \text{ and } t > 3 \\ \mathring{\mathrm{sch}}(B_{t-2}) + \sum_{i=t}^{r}(2\ell_i - 1), & \text{if } \ell_1 = 1 \text{ and } t > 3 \end{cases},$$

where $t = r + 1$ when $\ell_r \leq 2$ and otherwise $t = \min\{i\colon\ \ell_i > 2\}$.

Proof. First we calculate the sum-paintability of the subgraph formed by two shortest paths (when $\ell_3 > 2$) or by all paths of length at most 2 (when $\ell_3 = 2$), using $\mathring{\mathrm{sch}}(C_n) = 2n$, Corollary 5.3.1, or Theorem 5.3.3. We then apply Corollary 5.2.5 to add the remaining paths. Each remaining path of length ℓ_i contributes $2\ell_i - 1$ to the sum-paintability. $\square$

Corollary 5.3.4 implies that a generalized theta-graph G is sp-greedy if and only if $t \le 3$ or if $\ell_1 = 1$ and $t \le 5$. Although $\mathrm{sch}(K_{2,r}) = \mathring{\mathrm{sch}}(K_{2,r})$ and $\mathrm{sch}(B_r) = \mathring{\mathrm{sch}}(B_r)$, the sum-choosability and sum-paintability of generalized theta-graphs need not be equal. We noted earlier that $\mathrm{sch}(\Theta_{2,2,2t}) < \mathring{\mathrm{sch}}(\Theta_{2,2,2t}) = \sigma(\Theta_{2,2,2t})$ for $t > 1$. The reason is that Corollary 5.2.5 does not hold for sum-choosability. While the sum-choosability of generalized theta-graphs is not known, our results provide upper and lower bounds. To obtain a lower bound on $\mathrm{sch}(\Theta_{\ell_1,\dots,\ell_r})$, we instead use Corollary 5.2.4 for $\ell_i > 2$. Each path of length ℓ_i increases the sum-choosability by at least $2(\ell_i - 1)$. Since the subgraph formed by the edges from two paths of lengths ℓ_r and ℓ_{r-1} is a cycle, which is sc-greedy, we have the following corollary.

Corollary 5.3.5. *If G is a generalized theta-graph $\Theta_{\ell_1,\dots,\ell_r}$, then*

$$\mathrm{sch}(G) \ge \begin{cases} \mathrm{sch}(K_{2,t-1}) + \sum_{i=t}^{r}(2\ell_i - 2), & \textit{if } \ell_1 = 2 \textit{ and } t > 3 \\ \mathrm{sch}(B_{t-2}) + \sum_{i=t}^{r}(2\ell_i - 2), & \textit{if } \ell_1 = 1 \textit{ and } t > 3 \\ 2(\ell_r + \ell_{r-1}) + \sum_{i=1}^{r-2}(2\ell_i - 2), & \textit{otherwise} \end{cases},$$

where $t = r + 1$ when $\ell_r \le 2$ and otherwise $t = \min\{i\colon\ \ell_i > 2\}$.

5.4 Reducibility Arguments

We begin with a necessary condition for a graph G to be sp-greedy.

Proposition 5.4.1. *If G is sp-greedy and $v \in V(G)$, then $G - v$ is sp-greedy.*

Proof. If $\mathring{\mathrm{sch}}(G-v) < \sigma(G-v)$, then let f be an assignment of $\mathring{\mathrm{sch}}(G-v)$ tokens to $G-v$ under which $G-v$ is f-paintable. We show that Painter can win in G by also assigning $d_G(v)+1$ tokens to v. By assumption, Painter has some winning strategy $\mathcal{S}$ on $G-v$. In any round when Lister marks v, Painter will color v unless $\mathcal{S}$ dictates that Painter should color a neighbor of v. Since this cannot occur more than $d_G(v)$ times, a token always remains on v to start the round when it is colored. Because $\mathring{\mathrm{sch}}(G) = \mathring{\mathrm{sch}}(G-v)+(d_G(v)+1) < \sigma(G)$, we then conclude that G is not sp-greedy. □

Note that this necessary condition for G to be sp-greedy is not sufficient, since $\mathring{\mathrm{sch}}(\Theta_{2,2,2}) = 10 < \sigma(\Theta_{2,2,2}) = 11$. Thus $\Theta_{2,2,2}$ is not sp-greedy, but deleting a vertex leaves $K_{1,3}$ or C_4, both of which are sp-greedy.

Definition 5.4.2. A graph G is *weakly sp-greedy* if every proper induced subgraph of G is sp-greedy. ■

By Proposition 5.4.1, if G is sp-greedy, then G is weakly sp-greedy. This leads us to our primary tool, which we apply many times in Sections 5.4. When we assume that $\mathring{\mathrm{sch}}(G)$ tokens are allocated, we do not necessarily know the value of $\mathring{\mathrm{sch}}(G)$, but the point is that we are studying an optimal allocation.

We introduce following notation for the change in the greedy bound of G that occurs when deleting a vertex set D.

Definition 5.4.3. For a graph G and set $D \subseteq V(G)$, define $\delta(G, D) = \sigma(G) - \sigma(G - D)$. In the case when D is single vertex v, we write $\delta(G, v)$ instead of $\delta(G, \{v\})$. ■

From Theorem 5.2.3, note that $\frac{n(n+3)}{2} = \delta(G', V(K_n))$, which implies the following corollary.

Corollary 5.4.4. *If $G' = (G + K_n) \cup \{u\phi(u) : u \in V(K_n)\}$ for some $n \in \mathbb{N}$ and $\phi : V(K_n) \to V(G)$, then $\mathring{\mathrm{sch}}(G') = \mathring{\mathrm{sch}}(G) + \delta(G', V(K_n))$.*

We now state the primary tool that is used in this section.

Lemma 5.4.5. *Let G be weakly sp-greedy, and let f be an allocation of $\mathring{\mathrm{sch}}(G)$ tokens under which Painter wins. If Lister can force a position where a set D of vertices has been deleted from G and there remain at most $\mathring{\mathrm{sch}}(G) - \delta(G, D)$ tokens on $G - D$, then G is sp-greedy.*

Proof. Since $G - D$ has at most $\mathring{\mathrm{sch}}(G) - \delta(G, D)$ tokens, we have lost at least one token for each vertex and edge that was deleted. Painter has a winning strategy under f, and $G - D$ is sp-greedy, so f has sum at least $\sigma(G - D) + \delta(G, D)$, which equals $\sigma(G)$. □

In comparing the number of tokens on G with the number remaining on $G - D$, we consider both tokens that are colored when Lister marks a vertex and extra tokens that are still available on a vertex when Painter colors it. Lemma 5.4.5 is used to show that certain configurations of vertices and tokens are reducible for sp-greediness which means that the configuration cannot occur in a minimal graph failing to be sp-greedy.

We begin with three easy applications of Lemma 5.4.5 that also appear in joint work with Tomlinson and Wise [38]. In Section 5.5, we use these corollaries to show that several families of outerplanar graphs are sp-greedy.

Corollary 5.4.6. *Let G be weakly sp-greedy with an assignment f of $\mathring{\mathrm{sch}}(G)$ tokens under which Painter wins. If $v_1, \ldots, v_k$ form a clique and $f(v_i) + k - 1 \geq d_G(v_i) + 1$ for all $1 \leq i \leq k$, then G is sp-greedy.*

Proof. We show that Lister can force the assumption of Lemma 5.4.5. Lister begins by marking $\{v_1, \ldots, v_k\}$. Since the vertices form a clique, Painter can only color a single vertex. No matter which vertex v_i Painter colors, $f(v_i)+k-1$ fewer tokens remain in $G-v_i$, and $\delta(G,v_i)=d_G(v_i)+1$. Lemma 5.4.5 then yields $\sum f(v)\ge\sigma(G)$. ☐

Corollary 5.4.6 applies when vertices of a clique have "many" tokens. The following two applications of Lemma 5.4.5 cover cases when "few" tokens are allocated.

Corollary 5.4.7. *Let G be weakly sp-greedy with an assignment f of $\mathring{\mathrm{sch}}(G)$ tokens under which Painter wins. If $f(u)=1$ for some $u\in V(G)$, then G is sp-greedy.*

Proof. Lister marks $N[u]$, and Painter must color u. There are $d_G(u)+1$ fewer tokens in $G-u$, and $\delta(G,u)=d_G(u)+1$. By Lemma 5.4.5, G is sp-greedy. ☐

Corollary 5.4.7 also follows from Proposition 2.1.1(d), but the technique of using Lemma 5.4.5 is more general and has other applications.

Corollaries 5.4.6 and 5.4.7 imply that if G is weakly sp-greedy, then $1<f(v)\le d_G(v)$ for $v\in V(G)$. In particular, if G is weakly sp-greedy, then $\delta(G)\ge 2$. Corollary 5.4.7 can be stated more generally in terms of f-paintability: If $f(u)=1$, then G is f-paintable if and only if $G-u$ is f'-paintable, where $f'(v)=f(v)-1$ if $v\in N(u)$ and $f'(v)=f(v)$ otherwise. Heinold [19] proved an analogous statement for sum-choosability.

Corollary 5.4.8. *Let G be weakly sp-greedy with an assignment f of $\mathring{\mathrm{sch}}(G)$ tokens under which Painter wins, and suppose $uv,vw,uw\in E(G)$. If $d_G(u)\le d_G(v)=3$, and $f(u)=f(v)=2$, then G is sp-greedy.*

Proof. Lister marks $\{u,v,w\}$. Painter will not color w since Lister would then win by marking $\{u,v\}$. Let $z\in\{u,v\}$ be the vertex colored by Painter. This deletes one vertex and $d_G(z)$ edges, and four tokens are lost. Hence $\delta(G,z)=1+d_G(z)\le 4$, and Lemma 5.4.5 applies. ☐

Note that the previous corollaries apply Lemma 5.4.5 with the deleted set D being a single vertex. In Lemma 5.5.5 and Theorem 5.5.10, the set D may be larger. Using Proposition 2.1.1(c), we assume that Painter always colors a maximal independent subset of the marked set, which is useful when dealing with larger deleted sets.

In Theorem 5.3.3 we determined the sum-paintability of book graphs. When $s=4$, $\mathring{\mathrm{sch}}(B_4)=14<15=\sigma(B_4)$, and since $B_4\subseteq B_s$ whenever $s\ge 4$, we have that $\mathring{\mathrm{sch}}(B_s)<\sigma(B_s)$ for all $s\ge 4$. A *generalized book graph*, also referred to as a complete split graph, is the join of a clique and an independent set. Let $G_{r,s}=K_r\oplus\overline{K}_s$; observe that $B_s=G_{2,s}$. We now use the reducibility arguments developed earlier in this section to characterize the sp-greedy generalized book graphs.

For Lemmas 5.4.9 and 5.4.10, we use the following notational conventions for $G_{r,s}$.

- $v_1, \ldots, v_r$ are the vertices of the r-clique,
- $u_1, \ldots, u_s$ are the vertices of the maximum independent set,
- f is an assignment of $\mathring{\mathrm{sch}}(G_{r,s})$ tokens such that $G_{r,s}$ is f-paintable,
- $f(v_1) \leq \cdots \leq f(v_r)$ and $f(u_1) \leq \cdots \leq f(u_s)$, and
- M is the set Lister marks on the first round of the game.

Lemma 5.4.9. *$G_{r,2}$ is sp-greedy for all r.*

Proof. We use induction on r, and the base case of $r = 0$ is trivial. Suppose that $r > 0$ and let $G = G_{r,2}$. Deleting a vertex from G leaves either $G_{r-1,2}$, which is sp-greedy by the induction hypothesis, or $G_{r,1}$, which is sp-greedy since it is a clique. Thus G is weakly sp-greedy, and we now use Lemma 5.4.5 to show that G is sp-greedy.

If $f(v_1) = 1$, then let $M = V(G)$. Painter must color v_1; there are $r + 3$ fewer tokens in $G - v_1$, and $\delta(G, v_1) = r + 2$. If $f(v_1) > 1$, then set $M = N[u_1]$. If Painter colors u_1, then at least $r + 1$ fewer tokens remain on $G - u_1$, and $\delta(G, u_1) = r + 1$. If Painter colors some v_i, then $r + f(v_i)$ fewer tokens remain on $G - v_i$, and $\delta(G, v_i) = r + 2$. Since $f(v_i) \geq 2$, in all cases Lemma 5.4.5 implies that f must sum to $\sigma(G)$. □

Lemma 5.4.10. *$G_{r,3}$ is sp-greedy for all r.*

Proof. Again, we use induction on r, and the statement is easy for $r = 0$. Suppose $r > 0$, and let $G = G_{r,3}$ and $t = f(u_3)$. Proposition 2.1.1(b) implies that we may assume $f(v) \leq d(v)$ for all v. Thus $f(u_3) \leq r$, so $t \leq r$.

Case 1: $f(v_t) < t + 1$. By Proposition 5.2.1, we may assume $f(v_t) = t$. Let $M = V(G)$. If Painter colors some v_i with $i \leq t$, then at least $r + 3$ fewer tokens remain in $G - v_i$, and $\delta(G, v_i) = r + 3$. If Painter colors some v_i with $i > t$ or $\{u_1, u_2, u_3\}$, then Lister has a winning strategy on $\{v_1, \ldots, v_t\}$ as in the previous paragraph.

Case 2: $f(v_t) = t + 1$. Let $M = V(G) - \{u_1\}$. If Painter colors a $\{u_2, u_3\}$, then Lister wins on $\{u_1, v_1, \ldots, v_t\}$ since every vertex now has at most t tokens. If Painter colors some v_i, then there are $f(v_i) + r + 1$ fewer tokens in $G - v_i$, and $\delta(G, v_i) = r + 3$. Since $f(v_i) \geq 2$, Lemma 5.4.5 applies.

Case 3: $f(v_t) > t + 1$. Let $M = \{u_3, v_t, \ldots, v_r\}$. If Painter colors u_3, then at least $r + 1$ fewer tokens remain in $G - u_3$, and $\delta(G, u_3) = r + 1$. If Painter colors some v_i, then there are $f(v_i) + r - t + 1$ fewer tokens in $G - v_i$, and $\delta(G, v_i) = r + 3$. Since $f(v_i) \geq t + 2$, Lemma 5.4.5 applies.

Thus no assignment of less than $\sigma(G)$ tokens exist, so $G_{r,3}$ is sp-greedy for all r. □

We now give the characterization of sp-greedy generalized book graphs, which follows almost immediately from Lemmas 5.4.9 and 5.4.10.

Theorem 5.4.11. *$G_{r,s}$ is sp-greedy if and only if $r \leq 1$ or $s \leq 3$.*

Proof. When $r > 2$ and $s > 3$, $G_{r,s}$ contains B_4 as an induced subgraph, and thus is not sp-greedy. When $r \leq 1$ or $s \leq 1$, $G_{r,s}$ is an empty graph, a complete graph, or a star, all of which are sp-greedy since they are sc-greedy [5]. Lemmas 5.4.9 and 5.4.10 imply that $G_{r,2}$ and $G_{r,3}$ are sp-greedy for all r, which completes the characterization. □

5.5 Outerplanar Graphs

We have conjectured that every outerplanar graph is sp-greedy (Conjecture 5.0.1). Theorem 5.1.4 implies that it suffices to consider 2-connected graphs. Results in this section also appear in joint work with Tomlinson and Wise [38].

In this section, let G be a minimal counterexample to Conjecture 5.0.1. Let f be an assignment of $\mathring{\mathrm{sch}}(G)$ tokens under which Painter wins. Let T be the weak dual of G. Since cycles are sp-greedy, we may assume $|V(T)| > 1$.

By minimality, G is weakly sp-greedy, so Lemma 5.4.5 may apply. One immediate observation from Corollary 5.4.6 is that if $xy \in E(G)$, then either $f(x) < d(x)$ or $f(y) < d(y)$. The following results further restrict the structure of G, T, and f.

Proposition 5.5.1. *If $uv \in E(G)$, then $\max\{d_G(u), d_G(v)\} > 2$.*

Proof. Since $\delta(G) \geq 2$, we suppose $d_G(u) = d_G(v) = 2$, and Corollary 5.4.6 implies that G is sp-greedy, a contradiction. □

For $t \in V(T)$, let A_t be the face in G corresponding to t.

Corollary 5.5.2. *If x is a leaf in T, then $|A_x| = 3$.*

Proof. If $|A_x| > 3$ for some leaf $x \in V(T)$, then there exist vertices $u, v \in V(A_x)$ such that $d_G(u) = d_G(v) = 2$ and $uv \in E(G)$. Apply Proposition 5.5.1. □

Theorem 5.5.3. *If x is a leaf in T with neighbor t, then $d_T(t) > 2$.*

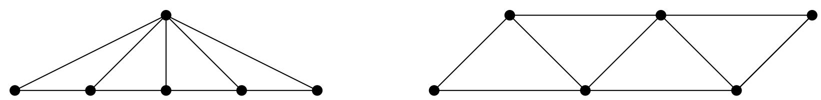

Figure 5.2: F_5 and P_6^2

Proof. Suppose $d_T(t) = 2$. By Corollary 5.5.2, $|A_x| = 3$; let $V(A_x) = \{u, v, w\}$ be labeled so that $d_G(u) = 2$, $d_G(v) = 3$, and $d_G(w) \geq 3$, which is possible since $d_T(t) = 2$. Since $1 < f(z) \leq d_G(z)$ for all $z \in V(G)$, we may assume $f(u) = 2$ and $2 \leq f(v) \leq 3$. If $f(v) = 3$, then G is sp-greedy by Corollary 5.4.6. If $f(v) = 2$, then G is sp-greedy by Corollary 5.4.8. In either case, we contradict G being a minimal counterexample. □

We say that an outerplanar graph is *pathic* if its weak dual is a path. In particular, the fan F_n is pathic, and so is the square of a path P_n^2 (Figure 5.2). Heinold [19] showed that F_n is not in general sc-greedy and that $\sigma(F_n) - \mathrm{sch}(F_n)$ can be made arbitrarily large. Hence we separate $\mathrm{sch}(F_n)$ and $\mathring{\mathrm{sch}}(F_n)$ by showing that fans are sp-greedy. In fact, we have a more general result.

Corollary 5.5.4. *Pathic graphs are sp-greedy.*

Proof. When T is a path, the neighbor of a leaf has degree at most 2. Apply Theorem 5.5.3. □

We continue by giving more properties of T for a minimal counterexample G. We say that a *leaf face* is a face in G that corresponds to a leaf in T.

Lemma 5.5.5. *If two leaf faces A_x and $A_{x'}$ each share an edge with a common face A_t, then any nontrivial path P along A_t must contain an edge in a neighboring face of A_t.*

Proof. By Corollary 5.5.2, all leaf faces in G must be triangles. Let u be the endpoint of P in A_x, and let v be the degree-2 neighbor of u. Similarly define $u', v' \in V(A_{x'})$. Note that $d_G(u) = d_G(u') = 3$. Corollaries 5.4.7 and 5.4.6 imply $f(z) = 2$ for $z \in \{u, u', v, v'\}$. Let w be the neighbor of u in P.

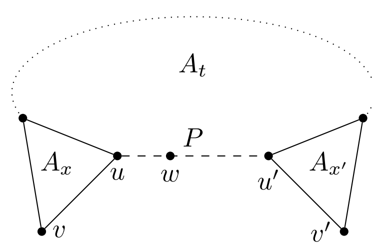

Figure 5.3: Picture for Lemma 5.5.5

Case 1: $w \neq u'$. Since $d_G(w) = 2$, Corollaries 5.4.7 and 5.4.6 imply $f(w) = 2$. Lister marks $N[u]$, and Painter must color u or v to avoid losing. If Painter colors u, then there are five fewer tokens in $G - u$, and

$\delta(G, u) = 4$. If Painter colors $\{v, w\}$, then there are six fewer tokens in $G - u$, and $\delta(G, \{v, w\}) = 6$.

Case 2: $w = u'$. Lister marks $N(u) \cup N(u')$, and Painter must color one vertex from each of the pairs $\{u, v\}$ and $\{u', v'\}$. By symmetry, we may assume Painter colors $\{u, v'\}$. There are at least seven fewer tokens in $G - \{u, v'\}$, and $\delta(G, \{u, v'\}) = 7$.

In each case, Lemma 5.4.5 implies that G is sp-greedy, contradicting that G is a minimal non-sp-greedy outerplanar graph. □

Theorem 5.5.6. *If $t \in V(T)$ has at most one non-leaf neighbor, then t has at most two leaf neighbors.*

Proof. Suppose by way of contradiction that $x, y, z \in V(T)$ are leaves of t. By Lemma 5.5.5, we may assume $A_x \cap A_y \neq \emptyset$ and $A_y \cap A_z \neq \emptyset$. Let $A_y = \{y_1, y_2, y_3\}$ where $d_G(y_3) = f(y_3) = 2$. If $f(y_1) = f(y_2) = 3$, then Corollary 5.4.6 implies that G is sp-greedy. By symmetry, we suppose that $f(y_2) = 2$. Lister marks $N[y_2]$, and Painter must color y_2 or two vertices of degree 2; otherwise, Lister wins by marking adjacent vertices with only one token on each. In either case, Lemma 5.4.5 implies that G is sp-greedy. □

We define a *semi-leaf* in a tree to be a vertex that is adjacent to at most one non-leaf vertex.

Theorem 5.5.7. *If the weak dual T of an outerplanar graph G is a double star, then G is sp-greedy.*

Proof. If a non-leaf in T does not have exactly two leaf neighbors, then G is sp-greedy 5.5.65.5.3. Thus the two non-leaves in T each have two leaf neighbors, so $|V(T)| = 6$. Let $x, y \in V(T)$ be leaves with common parent t. Let $V(A_x) = \{u, v, w\}$, and let $V(A_{x'}) = \{u', v, w'\}$. We may assume $1 < f(v) \leq d(v)$ (ref), so $f(u) = f(u') = 2$ and $f(v) \in \{2, 3\}$.

Case 1: $f(v) = 2$. Lister marks $N[v]$, and Painter cannot color w or w'; otherwise, Lister wins by marking $\{u, v\}$ or $\{u', v\}$. If Painter colors $\{u, u'\}$, then $G - \{u, u'\}$ has seven fewer tokens, and $\delta(G, \{u, u'\}) = 6$. If Painter colors v, then $G - v$ has six fewer tokens, and $\delta(G, v) = 5$.

Case 2: $f(w) = 3$. If $d(w) = 3$, then Corollary 5.4.8 applies if $f(w) = 2$, while Corollary 5.4.6 applies if $f(w) = 3$. So we may assume $d(w) > 3$, and thus $|A_t| = 3$. Because $|V(T)| = 6$, $d(w) \leq 5$. By Corollary 5.4.6, we may assume $f(w) \leq 3$, and by symmetry, we may also assume $f(w') \leq 3$. Lister marks $\{v, w, w'\}$, and if Painter colors w or w', then Lister wins by the remaining vertices of $\{u, u', v, w, w'\}$. If Painter colors v, then $G - v$ has five fewer tokens, and $\delta(G, v) = 5$. □

Corollary 5.5.8. *If $\operatorname{diam}(T) < 4$, then G is sp-greedy.*

Thus we conclude that if G is a minimal counterexample with weak dual T, then every vertex in T adjacent to exactly one non-leaf in T must be adjacent to two leaves. Also, this pair of leaves must also

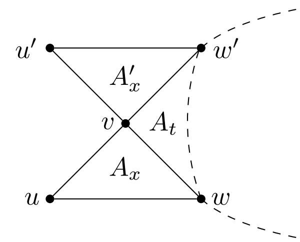

Figure 5.4: Picture for Theorem 5.5.7

correspond to two triangles in G sharing a vertex, and the diameter of T must be at least 4.

Example 5.5.9. For the graph G in Figure 5.5, $\sigma(G) = 33$. We reduce our consideration of allocations of less than 33 tokens by using Lemma 5.4.5. If any vertex receives only one token, then Corollary 5.4.7 applies. If any vertex receives more tokens than its degree, then Corollary 5.4.6 applies. Thus we may assume all vertices of degree 2 receive exactly two tokens. Also each neighbor of a vertex of degree 2 must receive fewer tokens than its degree.

If $f(z) = 2$, then Lister marks $N[z]$; Painter must color z or $\{x, y\}$, and in either case Lemma 5.4.5 implies that G is sp-greedy. Thus we may assume $f(z) = 3$. Applying Corollary 5.4.6 to the triangles $\{u, x, z\}$ and $\{v, y, z\}$ yields $f(u) \leq 3$ and $f(v) \leq 4$. By symmetry, each of these arguments holds for the corresponding neighbors of z'.

Thus the only remaining allocation of 32 tokens is shown in Figure 5.5, and we now show that Lister has a winning strategy. Lister begins by marking all vertices. If Painter could win against this initial move by coloring a maximal independent subset D, then the number of tokens lost must be more than $\delta(G, D)$, otherwise Lemma 5.4.5 would imply that G is sp-greedy, a contradiction. Thus $|V(G)| - |D| + \sum_{v \in D} f(v) < |D| + \sum_{v \in D} d(v)$, which implies $2(6 - |D|) < \sum_{v \in D}(f(v) - d(v))$. If $d(v) = 2$, then $d(v) - f(v) = 0$; if $d(v) = 4$, then $d(v) - f(v) = 1$; otherwise, $d(v) - f(v) = 2$. No matter how Painter selects D, the inequality fails. When $|D| = 6$, we have $\sum_{v \in D}(f(v) - d(v)) = 0$. When $|D| = 5$, we have $\sum_{v \in D}(f(v) - d(v)) \leq 2$. When $|D| = 4$, we have $\sum_{v \in D}(f(v) - d(v)) \leq 4$. Since all maximal independent sets have size at least 4, Lister wins on G with the allocation shown in Figure 5.5.

Using that fans are sp-greedy (Corollary 5.5.4), we prove a result about a family of non-outerplanar graphs. The *wheel* on $n + 1$ vertices, denoted W_n, is the join $C_n \oplus K_1$. Wheels contain fans as induced subgraphs, so it is natural to ask if wheels are sp-greedy. We answer this question in the affirmative.

Theorem 5.5.10. *All wheels are sp-greedy.*

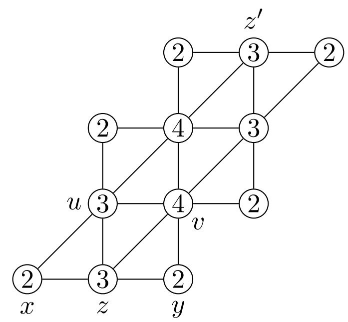

Figure 5.5: A configuration not solvable by Corollary 5.4.6

Proof. Deleting a vertex from a wheel leaves either a cycle or a fan. Cycles are sp-greedy since they are sc-greedy [5], and fans are sp-greedy by Corollary 5.5.4. Thus wheels are weakly sp-greedy, so Lemma 5.4.5 may apply.

Let W_n be a wheel with vertices $v_1, \ldots, v_n$ on the outer cycle, and let u be the dominating vertex. Let f be an allocation of $\mathring{\text{sch}}(W_n)$ tokens under which Painter wins. Since every vertex on the outer cycle of W_n has degree 3, Corollaries 5.4.7 and 5.4.6 imply that $f(v_i) \in \{2, 3\}$ for $1 \leq i \leq n$. Also, Corollaries 5.4.6 and 5.4.8 imply that the vertices with two tokens and three tokens must alternate, so it suffices to consider even n. We may assume for $1 \leq i \leq n$ that $f(v_i)$ is 2 if i is even and 3 if i is odd, as shown in Figure 5.5.10.

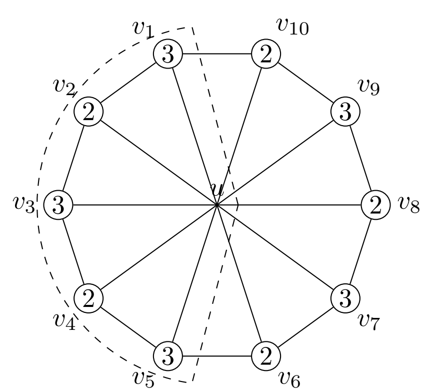

Figure 5.6: Wheel

If $n = 4$, then $f(u) \geq 3$ yields $\sum f(v) \geq 13 = \sigma(W_4)$, so we may assume $f(u) = 2$. Lister marks $V(W_4) - v_4$, and Painter must color either u or v_2; otherwise, Lister then marks $\{u, v_2\}$ and wins. Let $x \in \{u, v_2\}$ be the vertex Painter colors. In either case, $G - x$ has five fewer tokens and $\delta(G, x) \leq 5$.

For $n \geq 6$, Lister marks $\{u, v_1, \ldots, v_5\}$ in the first round; let D be the set that Painter colors. If $D = \{u\}$, then Lister wins because fewer than $\mathring{\text{sch}}(P_3)$ tokens remain on $\{v_2, v_3, v_4\}$. Since $d_G(v_i) = 3$ for all i, then $\delta(G, D) = 4|D|$. If $|D| = 3$, this only happens when $D = \{v_1, v_3, v_5\}$, and we have that $G - D$ has 12 fewer tokens. If $|D| = 2$, then always $G - D$ has at least eight fewer tokens. In each case, Lemma 5.4.5 implies that G is sp-greedy. $\square$

Wheels are both sp-greedy and planar. Other planar graphs, however, are not sp-greedy. Corollary 5.3.4 shows that not all generalized theta-graphs are sp-greedy; for example, any planar graph containing $K_2 \oplus \overline{K}_4$ is not sp-greedy. Though $K_2 \oplus \overline{K}_4$ has embeddings for which the weak dual is a multigraph with a path as the underlying graph, it is not outerplanar.

Since the difference $\mathring{\text{sch}}(B_r) - \mathring{\text{sch}}(K_{2,r})$ can be large (Theorem 5.3.3 and Corollary 5.3.1), deleting a single edge can decrease the sum-paintability by more than one. Thus it not obvious that it would be enough to consider only triangulations of outerplanar graph for Conjecture 5.0.1.

Also the 3-dimensional cube $C_4 \square C_2$ is not sp-greedy. If G contains an induced non-sp-greedy subgraph, then G is not sp-greedy. Thus many larger families of non-sp-greedy planar graphs can be formed.

Chapter 6

Sum-Color Cost

Finding a token assignment f under which G is f-paintable can be viewed as allowing Painter to distribute "coloring resources" to the vertices. We view $f(v)$ as counting tokens placed at v; marking v uses one token. In sum paintability, instead of allocating k tokens to each vertex, Painter seeks to minimize the total number of tokens.

Now suppose that Painter also can avoid placing the tokens in advance, allocating tokens to vertices as needed in response to Lister's marked set. That is, Lister scores one point for each vertex marked in each round. Eventually Painter will produce a coloring; the question is how many points Lister can score before then. Hence we call this the *slow-coloring game*. The maximum score Lister can guarantee, equal to the minimum number of tokens Painter must have available to guarantee producing a coloring, is the *sum-color cost* $\mathring{s}(G)$. Since Painter can always play as if the tokens were distributed in advance, $\mathring{s}(G) \le \mathring{sch}(G)$.

Unlike sum paintability, sum-color cost has an easily described recursive computation. The key point is that prior choices do not affect Painter's optimal strategy for coloring subsets of marked sets on the remaining subgraph.

Proposition 6.0.1. $\mathring{s}(G) = \max_{\emptyset \neq M \subseteq V(G)} \left(|M| + \min \mathring{s}(G - I)\right)$, *where the minimum is over subsets $I \subseteq M$ such that I is an independent set in G.*

In studying optimal strategies for Lister and Painter, simple observations reduce the set of moves that need to be considered.

Observation 6.0.2. *On any graph, there are optimal strategies for Lister and Painter such that Lister always marks a set M such that the induced subgraph $G[M]$ is connected, and Painter always colors a maximal independent subset of M.*

Proof. A move in which Lister marks a disconnected set M can be replaced with successive moves marking the vertex sets of the components of $G[M]$. Coloring extra vertices at no extra cost cannot hurt Painter. $\square$

Another easy observation sometimes yields a useful lower bound.

Observation 6.0.3. *If G_1 and G_2 are disjoint subgraphs of G, then $\mathring{s}(G) \geq \mathring{s}(G_1) + \mathring{s}(G_2)$.*

Proof. Lister can play an optimal strategy on G_1 while ignoring the rest and then do the same on G_2, achieving the score $\mathring{s}(G_1) + \mathring{s}(G_2)$. $\square$

One can view $\frac{\mathring{s}(G)}{|V(G)|}$ as the average cost per vertex of coloring. In Section 6.1, we prove the fairly easy but sharp general bounds

$$\frac{|V(G)|}{2\alpha(G)} + \frac{1}{2} \leq \frac{\mathring{s}(G)}{|V(G)|} \leq \max\left\{\frac{|V(H)|}{\alpha(H)} \colon\ H \subseteq G\right\}.$$

The quantity $\max\left\{\frac{|V(H)|}{\alpha(H)} \colon\ H \subseteq G\right\}$ is the *Hall ratio* $\rho(G)$, defined in [25] and explored further in [14, 26, 51]. Since always $\chi(G) \geq \rho(G)$, and almost always $\chi(G) \leq (1+\epsilon)\rho(G)$, we conclude that almost always $\frac{\mathring{s}(G)}{|V(G)|}$ is within a constant multiple of $\chi(G)$. The bounds are sharp, since equality holds in the lower bound for the complete graph K_n, its complement $\overline{K}_n$, and the complement of the matching K_{2*r}; it holds in the upper bound for $\overline{K}_n$. In the lower bound, equality does not hold for any other regular complete multipartite graph.

Our other general result, in Section 6.3, shows that equality holds in the trivial bound $\mathring{s}(G) \leq \mathring{sch}(G)$ only when every component of G is complete.

In Sections 6.4 and 6.4, we prove more difficult bounds on sum-color cost of trees. For a tree T with n vertices, the value is minimized by the star $K_{1,n-1}$ and maximized by the path P_n; that is,

$$n + \left\lfloor \frac{-1+\sqrt{8n-7}}{2} \right\rfloor \;=\; \mathring{s}(K_{1,n-1}) \;\leq\; \mathring{s}(T) \;\leq\; \mathring{s}(P_n) \;=\; \left\lfloor \frac{3n}{2} \right\rfloor.$$

A *k-tree* is a graph that can be obtained from K_k by iteratively adding a vertex whose neighborhood is a k-clique in the existing graph. We conjecture that these bounds generalize to k-trees. Recall that G^r is the rth power of G, which has vertex set $V(G)$ an where vertices are adjacent if and only the distance between them in G is at most r. The graphs $K_k \oplus \overline{K}_{n-k}$ and P_n^k are the k-tree analogues of stars and paths with n vertices. In Theorem 6.4.6, we will compute $\mathring{s}(K_k \oplus \overline{K}_{n-k})$.

Conjecture 6.0.4. *For $k \in \mathbb{N}$ and any k-tree T with n vertices,*

$$\mathring{s}(K_k \oplus \overline{K}_{n-k}) \;\leq\; \mathring{s}(T) \;\leq\; \mathring{s}(P_n^k).$$

An easy lower bound for $\mathring{s}(P_n^k)$ follows from Observation 6.0.3. Since P_n^k contains $\left\lfloor \frac{n}{k+1} \right\rfloor$ disjoint copies of

K_{k+1}, we have $\mathring{s}(P_n^k) \geq \mathring{s}\left(\left\lfloor \frac{n}{k+1} \right\rfloor K_{k+1}\right) + \mathring{s}(K_r)$, where $r \equiv n \mod (k+1)$, and we conjecture that equality holds. This formula reduces to the correct answer for $k = 1$.

6.1 General Bounds on Sum-Color Cost

In this section, we prove our general bounds on $\mathring{s}(G)$. We give a lower bound in terms of the *chromatic sum* of G, originally defined by Kubicka [32].

Definition 6.1.1. The *chromatic sum* of a graph G, written $\Sigma(G)$, is the smallest value of $\sum_{v \in V(G)} c(v)$ over all proper colorings $c\colon V(G) \to \{1, 2, 3, \ldots\}$. ■

Theorem 6.1.2. *For every graph G with n vertices,*

$$\Sigma(G) \leq \mathring{s}(G) \leq n\rho(G).$$

The lower bound is sharp when $G \in \{K_n, \overline{K}_n\}$, the upper bound when $G = \overline{K}_n$.

Proof. For the lower bound, suppose that Lister always marks the entire remaining graph. Consider any painter strategy, and let $V_1, \ldots, V_k$ be the sets removed by Painter, where V_i is removed on round i. Let $n_i = |V_i|$. At the beginning of round i, the set of remaining vertices is $V_i \cup \cdots \cup V_k$, so Painter scores $n_i + \cdots + n_k$ points on round i by marking the entire graph. It follows that the total number of points scored by Painter is $\sum_{i=1}^{n}(n_i + \cdots + n_k)$. Since each n_i appears in exactly i terms of this sum, the total score is $\sum_{i=1}^{n} in_i$. This is equal to $\sum_{v \in V(G)} c(v)$, where c is the coloring obtained by letting $c(v)$ be the unique i for which $v \in V_i$. Hence the total score is at least $\Sigma(G)$.

Equality holds for K_n because Painter colors exactly one vertex on every round, so Lister scores most by marking all vertices. Equality holds for $\overline{K}_n$ because $\mathring{s}(\overline{K}_n) = n$.

For the upper bound, let $r = \rho(G)$. Given any marked set M, the *greedy strategy* for Painter colors a largest independent set in $G[M]$. By the definition of $\rho(G)$, Painter colors at least $|M|/r$ vertices. For any Lister strategy against Painter's greedy strategy, let $m_1, \ldots, m_t$ be the sizes of the marked sets in the successive rounds. In round i Painter colors at least m_i/r vertices, so $\sum_{i=1}^{t} \frac{m_i}{r} \leq n$. Hence Lister scores at most nr, using any strategy. □

Corollary 6.1.3. *For every graph G with n vertices,*

$$\frac{n^2}{2\alpha(G)} + \frac{n}{2} \leq \mathring{s}(G).$$

Proof. Let $a = \alpha(G)$ and let $q = \lfloor n/a \rfloor$. We show that $\frac{n^2}{2a} \le \Sigma(G)$. Let c be a coloring that minimizes $\sum_{v \in V(G)} c(v)$, let V_i be the set of vertices that receive color i, and for $j \ge 0$, let $W_j = V(G) - (V_1 \cup \cdots \cup V_j)$. Since each $|V_i| \le a$, we have $|W_j| \ge n - ja$. Now

$$\sum_{v \in V(G)} c(v) = \sum_{j=0}^{\infty} |W_j| \ge \sum_{j=0}^{q} (n - ja).$$

Let $\epsilon = \frac{n}{a} - q$; note that $0 \le \epsilon < 1$. We compute

$$\begin{aligned}\Sigma(G) &\ge (q+1)n - \sum_{j=0}^{q} ja = (q+1)\left(n - \frac{qa}{2}\right) \\ &= \frac{1}{2}\left(\frac{n}{a} - \epsilon + 1\right)(n + a\epsilon) = \frac{n^2}{2a} + \frac{n}{2} + a(\epsilon - \epsilon^2) \ge \frac{n^2}{2a} + \frac{n}{2}.\end{aligned}$$

□

The *standard binomial random graph model* is the probability space $\mathbb{G}(n,p)$ in which graphs with vertex set $\{1, \ldots, n\}$ are generated by letting each vertex pair be an edge with probability p, independently. An event occurs *with high probability* if its probability in $\mathbb{G}(n,p)$ tends to 1 as $n \to \infty$.

Corollary 6.1.4. *For fixed $p \in (0,1)$, there is a positive constant c such that with G sampled from $\mathbb{G}(n,p)$, the inequalities $c\chi(G) \le \frac{\mathring{s}(G)}{n} \le \chi(G)$ hold with high probability.*

Proof. It suffices to show that there is a constant c' such that $c'\chi(G) \le \frac{n}{2\alpha(G)} - \frac{1}{2}$ with high probability. By well-known results on the concentration of the clique number and the chromatic number in $\mathbb{G}(n,p)$ [6, 8, 7], there are positive constants c_1 and c_2 (depending on p) such that $\chi(G) \sim c_1 \frac{n}{\log n}$ with high probability and $\alpha(G) \sim c_2 \log n$ with high probability. The desired result follows. □

The upper and lower bounds in Theorem 6.1.2 can differ a lot for highly nonrandom graphs. For example, let $G = K_{n/2} \mathbin{\oplus} \overline{K}_{n/2}$; in Section 6.4 we show that $\mathring{s}(G)$ is approximately $\frac{1}{8}(n^2 + n^{3/2} + 6n)$. Theorem 6.1.2 gives $\frac{1}{8}(n^2 + 10n) \le \mathring{s}(G) \le \frac{1}{2}(n^2 + 2n)$.

Toward understanding the quality of the bounds, it would be helpful to understand when they hold with equality. We show that in addition to the trivial cases mentioned in Theorem 6.1.2, the lower bound is exact for the complement of a matching but for no other regular complete multipartite graphs.

Theorem 6.1.5. *Among regular complete multipartite graphs, equality holds in the lower bound in Theorem 6.1.2 only for K_n, $\overline{K}_n$, and K_{2*t}.*

Proof. Let G be the regular complete multipartite graph K_{r*t}

Since the number of vertices is tr, and $\alpha(G) = r$, the lower bound simplifies to $(rt^2 + rt)/2$, which we write as $r\binom{t+1}{2}$ (in particular, Lister achieves at least this score by applying Observation 6.0.3 to a covering of $V(G)$ by t disjoint cliques).

Consider the following Lister strategy, valid when $r > 1$ (that is, when G is not complete). In each round of the first $t-1$ rounds, Lister marks $r-1$ vertices from each part having no colored vertices. By Observation 6.0.2, Painter responds by reducing some part to one vertex, and the remaining marked parts still have no colored vertices. After $t-1$ rounds, Lister has scored $\sum_{i=2}^{t} i(r-1)$, which equals $(r-1)\binom{t+1}{2}-(r-1)$, and the uncolored graph is $\overline{K}_r \oplus K_{t-1}$. We will prove in Theorem 6.4.6 that $\mathring{s}(\overline{K}_r \oplus K_{t-1})$ differs from $r + \binom{t}{2} + (t-1)\sqrt{2r}$ by at most t. Summing the two contributions yields $\mathring{s}(G) \geq r\binom{t+1}{2} + (t-1)(\sqrt{2r}-2)$. This exceeds the lower bound when $r > 2$.

When $r = 2$, the graph at the end the first $t-1$ rounds under this strategy for Lister is $\overline{K}_2 \oplus K_{t-1}$. Theorem 6.4.6 implies that Lister earns $2 + \binom{t}{2} + (t-1)$ points on $\overline{K}_2 \oplus K_{t-1}$. Thus Lister scores at least $2\binom{t+1}{2}$ on K_{r*t}.

We claim $\mathring{s}(K_{2*t}) = 2\binom{t+1}{2}$. Lister marks all vertices, and by the same argument in Proposition 4.0.4(c), we may assume that Painter colors a maximal independent subset of the marked set. Thus the graph that remains after this round is $K_{2*(r-1)}$. Induction on r implies the desired result. $\square$

6.2 Graphs with $\alpha(G) \leq 2$

In this section, we prove the following formula for graphs with independence number at most 2.

Theorem 6.2.1. *If G is a graph with n vertices and $\alpha(G) \leq 2$, then $\mathring{s}(G) = \Sigma(G) = \binom{n-q+1}{2} + \binom{q+1}{2}$, where q is the size of a largest matching in the complement of G.*

Our proof has two parts. We first show that if $\alpha(G) \leq 2$, then $\Sigma(G) = \binom{n-q+1}{2} + \binom{q+1}{2}$. Then we show that for such G, $\mathring{s}(G) \leq \binom{n-q+1}{2} + \binom{q+1}{2}$. Since the lower bound $\Sigma(G) \leq \mathring{s}(G)$ always holds, this proves the theorem.

Lemma 6.2.2. *If $\alpha(G) \leq 2$, then $\Sigma(G) = \binom{n-q+1}{2} + \binom{q+1}{2}$, where q is the size of a largest matching in the complement of G.*

Proof. Let c be a coloring that minimizes $\sum_{v \in V(G)} c(v)$, and for each color i, let V_i be the set of vertices that receive color i. Since c is minimal, we have $|V_1| \geq \cdots \geq |V_k|$. Let $V_1, \ldots, V_p$ be the color classes of size

2. Since the color classes are disjoint independent sets, $\{V_1, \dots, V_p\}$ is a matching in the complement of G, so $p \le q$. Since $V_i = 1$ for $1 > p$, we have

$$\Sigma(G) = \sum_{v \in V(G)} c(v) = \sum_{i=1}^{k} i|V_i| = \sum_{i=1}^{p} 2i + \sum_{i=p+1}^{k} i = \sum_{i=1}^{k} i + \sum_{i=1}^{p} i = \binom{k+1}{2} + \binom{p+1}{2}.$$

Since $2p + (k-p) = n$, we have $k = n - p$, so

$$\Sigma(G) = \binom{n-p+1}{2} + \binom{p+1}{2}.$$

Since the function f defined by $f(x) = \binom{n-x+1}{2} + \binom{x}{2}$ is decreasing on $[0, n/2]$ and since $p \le q \le n/2$, we have

$$\Sigma(G) \ge \binom{n-q+1}{2} + \binom{q+1}{2}.$$

Since any maximum matching in the complement of G yields a coloring c with sum $\binom{n-q+1}{2} + \binom{q+1}{2}$, equality holds. □

Lemma 6.2.3. *If $G_{t,q}$ denotes the complete multipartite graph with q parts of size 2 and $t-q$ parts of size 1, then $\mathring{s}(G_{t,q}) \le \binom{t+1}{2} + \binom{q+1}{2}$.*

Proof. Let $f(t,q) = \binom{t+1}{2} + \binom{q+1}{2}$. We prove that $\mathring{s}(G_{t,q}) \le f(t,q)$ by induction on $t+q$. When $t+q=0$ there is nothing to prove, so assume that $\mathring{s}(G_{t',q'}) \le f(t',q')$ whenever $t'+q' < t+q$.

Let S be the set marked by Lister on the first move. For $i \in \{1,2\}$, let a_i denote the largest number of S-vertices contained in any part of size i. By marking S, Lister scores at most $(t-q)a_1 + qa_2$ points. Furthermore, we have $a_2 \in \{0,1,2\}$ and $a_1 \in \{0,1\}$, with $a_1 + a_2 > 0$.

Case 1: $a_2 = 2$. Lister scores at most $t+q$ points on the first move. Painter can delete a part of size 2, yielding the graph $G_{t-1,q-1}$. By the induction hypothesis, $\mathring{s}(G_{t-1,q-1}) \le f(t-1,q-1) = f(t,q) - t - q$. Thus, if Painter continues with optimal play in $G_{t-1,q-1}$, then Lister scores at most $f(t,q)$ points in total.

Case 2: $a_2 < 2$ *and* $a_1 = 1$. Lister scores at most t points on the first move. Painter can delete a part of size 1, yielding the graph $G_{t-1,q}$. By the induction hypothesis, $\mathring{s}(G_{t-1,q}) \le f(t-1,q) = f(t,q) - t$.

Case 3: $a_2 = 1$ *and* $a_1 = 0$. Lister scores at most q points on the first move. Painter can delete a vertex from a part of size 2, yielding the graph $G_{t,q-1}$. By the induction hypothesis, $\mathring{s}(G_{t,q-1}) \le f(t,q-1) = f(t,q) - q$. □

Corollary 6.2.4. *If G is a graph with n vertices and $\alpha(G) \le 2$, then $\mathring{s}(G) \le \binom{n-q+1}{2} + \binom{q+1}{2}$, where q is* *the size of a largest matching in the complement of G.*

Proof. Such a graph G is a subgraph of the graph $G_{t,q}$ defined in Lemma 6.2.3, with $t = n - q$. Since $G \subset H$ implies $\mathring{s}(G) \leq \mathring{s}(H)$, we have

$$\mathring{s}(G) \leq \mathring{s}(G_{n-q,q}) = \binom{n-q+1}{2} + \binom{q+1}{2}.$$

$\square$

Corollary 6.2.5. *There is a polynomial-time algorithm to determine $\mathring{s}(G)$ and $\Sigma(G)$ for the class of graphs with independence number at most 2.*

6.3 Equality in $\mathring{s}(G) \leq \mathring{sch}(G)$

For any graph G, $\mathring{s}(G) \leq \mathring{sch}(G)$. In this section, we characterize the graphs for which equality holds.

Theorem 6.3.1. $\mathring{s}(G) = \mathring{sch}(G)$ *if and only if G is a disjoint union of cliques.*

In the f-paintability game, Painter must immediately color any vertex whose tokens are exhausted. Lister can best take advantage of this when v has exactly one token by marking v and all its neighbors. Proposition 4.0.4(d) implies the following result.

Lemma 6.3.2. *Let G be a graph satisfying $\mathring{s}(G) = \mathring{sch}(G)$, and let f be an assignment of $\mathring{sch}(G)$ tokens under which G is f-paintable. Let Lister play an strategy that earns at least $\mathring{s}(G)$ points in the slow-coloring game. If Painter interprets Lister's moves as played in the f-paintability game and responds using an optimal strategy there, then the set colored by Painter always consists of vertices that began the round with one token.*

Proof. Consider the play of the game under the specified strategies. For each vertex v, let $g(v)$ denote the total number of times that v is marked. By Lister's strategy, $\mathring{s}(G) = \sum_{v \in G} g(v)$. On the other hand, $\mathring{s}(G) = \mathring{sch}(G) = \sum_{v \in G} f(v)$. Since Painter wins, $g(v) \leq f(v)$ for all v, so $g(v) = f(v)$ for all v. Since v has $f(v) - g(v) + 1$ tokens at the beginning of the round in which it is deleted, the claim follows. $\square$

Lemma 6.3.3. *Let G be a graph, and let f be assignment of $\mathring{sch}(G)$ tokens under which G is f-paintable. If there is a vertex v such that $f(v) = 1$ and $G - v$ is sp-greedy, then G is sp-greedy.*

Proof. By Proposition 4.0.4, the graph $G - v$ is f'-paintable, where $f'(w) = f(w) - 1$ for $w \in N(v)$ and $f'(w) = w$ otherwise. Since $G - v$ is sp-greedy,

$$\sum_{w \neq v} f'(w) \geq |V(G - v)| + |E(G - v)| = |V(G)| + |E(G)| - (d(v) + 1).$$

It follows that

$$\sum_{v} f(v) = f(v) + \sum_{w \neq v} f(w) = |V(G)| + |E(G)|. \qquad \square$$

Lemma 6.3.4. *If G is non-sp-greedy, then $\mathring{s}(G) < \mathring{sch}(G)$.*

Proof. Let G be a counterexample with fewest vertices, and let f be an assignment of $\mathring{sch}(G)$ tokens under which G is f-paintable.

Let M be Lister's marked set for the first round of the game. Lemma 6.3.2 implies that there is some vertex $v \in M$ such that $f(v) = 1$.

Painter responds to M by deleting vertices in two steps. Painter first deletes v. By Lemma 6.3.3, the resulting graph $G-v$ is not sp-greedy. Since G was a minimal counterexample, $\mathring{s}(G-v) < \mathring{sch}(G-v)$. On the other hand, Proposition 4.0.4 implies that $\mathring{sch}(G-v) = \mathring{sch}(G)-(d(v)+1)$. Thus $d(v)+1+\mathring{s}(G-v) < \mathring{sch}(G)$.

To complete the response to M, Painter also deletes vertices by responding to $M - N[v]$ on the sum-paintability game for $G - v$, according to the token assignment defined in Proposition 4.0.4. Regardless of what $M - N[v]$ is, Lister scores at most $d(v) + 1 + \mathring{s}(G - v)$, which is less than $\mathring{sch}(G)$. This contradicts Lister being able to guarantee a score of $\mathring{sch}(G)$. $\square$

Corollary 6.3.5. *If $\mathring{s}(G) = \mathring{sch}(G)$, then $\mathring{s}(G) = |V(G)| + |E(G)|$.*

The following theorem now completes the proof of Theorem 6.3.1.

Theorem 6.3.6. *$\mathring{s}(G) = |V(G)| + |E(G)|$ if and only if G is a disjoint union of cliques.*

Proof. We may assume that G is connected. It is clear that equality holds when G is a clique, so we show that $\mathring{s}(G) < |V(G)| + |E(G)|$ when G is not a clique.

Consider any first Marker move M in a strategy that guarantees $\mathring{s}(G)$ points. We split into two cases.

Case 1: $M = V(G)$. Since G is connected and not a clique, there exist vertices w_1, w_2, v such that $w_1v, w_2v \in E(G)$ but $w_1w_2 \notin E(G)$. Let $G' = G - \{w_1, w_2\}$, and let $M' = V(G) - (N(w_1) \cup N(w_2))$. Let D' be a Painter response to the marked set M' in an optimal Painter strategy for G'. The set $D' \cup \{w_1, w_2\}$ is an independent set. Painter deletes D' and continues play according to an optimal strategy in G'. The total number of points scored by Lister is at most $2 + |N(w_1) \cup N(w_2)| + \mathring{s}(G')$. Since $\mathring{s}(G') \le |V(G')| + |E(G')|$, we have

$$\mathring{s}(G) \le 2 + |N(w_1) \cup N(w_2)| + |V(G')| + |E(G')| = |V(G)| + |E(G')| + |N(w_1) \cup N(w_2)|.$$

Each vertex in $N(w_1) \cup N(w_2)$ is incident to at least one edge of $E(G) - E(G')$, with different vertices corresponding to different edges. Furthermore, v is incident to two edges of $E(G) - E(G')$. It follows that $|E(G')| + |N(w_1) \cup N(w_2)| \le |E(G)| - 1$, so that $\mathring{s}(G) \le |V(G)| + |E(G)| - 1$.

Case 2: $M \ne V(G)$. Since G is connected and $M \ne \emptyset$, there is an edge vw with $w \in M$ and $v \notin M$. Let $G' = G - w$ and let $M' = M - N[w]$. Let D' be an optimal Painter response to the marked set M' in G'. In G, Painter deletes $D' \cup \{w\}$ and continues play according to an optimal strategy in G'. Let $M_0 = N(w) \cap M$. We now have

$$\mathring{s}(G) \le |M_0| + 1 + \mathring{s}(G') \le |M_0| + 1 + |V(G')| + |E(G')| = |V(G)| + (|M_0| + |E(G')|).$$

Each joining w to a vertex in M is an edge of $E(G) - E(G)'$. Furthermore, vw is an edge of $E(G) - E(G')$ that is not counted in $|M_0|$. It follows that $|M_0| + |E(G')| \le |E(G)| - 1$, so that $\mathring{s}(G) \le |V(G)| + |E(G)| - 1$. $\square$

6.4 Trees

It is easy to see that Lister can score $\lfloor 3n/2 \rfloor$ on the path P_n. Lister first marks all vertices, scoring n. Since $\alpha(P_n) = \lceil n/2 \rceil$, Lister can score $\lfloor n/2 \rfloor$ more by marking all vertices that remain after Painter deletes an independent set. (Indeed, the lower bound $2n - \alpha(G)$ is the result of two rounds in the general lower bound of Theorem 6.1.2.)

We thus can prove that $\mathring{s}(T) \le \mathring{s}(P_n)$ for each tree T with n vertices by proving $\mathring{s}(T) \le \lfloor 3n/2 \rfloor$. Suppose that Lister first marks the set M, and in response Painter colors I. Let the components of $T - I$ be $T_1, \ldots, T_k$. If the claim holds for smaller trees, then optimal subsequent play by Painter will yield total score s at most $|M| + \sum_{i=1}^{k} \lfloor 3\,|V(T_i)|\,/2 \rfloor$. Letting $o(H)$ denote the number of odd components of H, we obtain

$$s \le |M| + \frac{3}{2}\,(n - |I|) - \frac{1}{2} o(T - I),$$

Hence Painter will be able to guarantee $\mathring{s}(T) \le \frac{3n}{2}$ inductively if for each $M \subseteq V(T)$ there is an independent set $I \subset M$ such that $3|I| + o(T - I) \ge 2\,|M|$. In order to prove this claim, we will need some results about trees. We begin with a definition.

Definition 6.4.1. Let M be a *connected set* in a tree T (a vertex subset such that $T[M]$ is connected). Let $T' = T - E(T[M])$; each component of T' contains one vertex of M. For $I \subset M$, let e_I be the number of components of even order in T' whose vertex of M is in I, and let o_I be the number of components of odd order in T' whose vertex of M is not in I. ■

The definitions of e_I and o_I are motivated by the next lemma. A *vertex cover* of a graph is a vertex subset containing at least one endpoint of every edge.

Lemma 6.4.2. *Let M be a connected set in a tree T. If I is a vertex cover of $T[M]$, then*

$$o(T - I) \geq e_I + o_I.$$

Proof. Group the odd components of $T - I$ into those that intersect $M - I$ and those that do not. We find o_I of the first type and at least o_E of the second type. Let $T' = T - E(T[M])$.

Since I is a vertex cover in $T[M]$, the graph $T - I$ contains no edges of $T[M]$. In particular, $T - I = T' - I$. The components of T' that do not intersect I are unaffected by the deletion of I. Thus $T - I$ has exactly o_I components that intersect $M - I$.

Now consider a component H of T' counted by e_I. The set $V(H)$ has even size and contains one vertex of I. When that vertex is deleted, the remaining vertices (an odd number) are broken into components, at least one of which must be odd. Hence there are at least e_I odd components of $T - I$ that do not intersect $M - I$. □

Lemma 6.4.3. *For any partition $\{A, B\}$ of the vertex set of a tree T, there is an independent vertex cover I of T such that*

$$3\,|I| + |A \cap I| + |B - I| \geq 2\,|V(T)|$$

Proof. Let $\{X, Y\}$ be the unique bipartition of $V(T)$ into two independent sets. Each of X and Y is an independent vertex cover of T. Using $|V(T)| = |X| + |Y| = |A| + |B|$ and $|A \cap X| + |A \cap Y| = |A|$ and $|B - X| + |B - Y| = |B|$, we obtain

$$\big(3\,|X| + |A \cap X| + |B - X|\big) + \big(3\,|Y| + |A \cap Y| + |B - Y|\big) = 3\,|V(T)| + |A| + |B| = 4\,|V(T)|$$

Thus one of $\{X, Y\}$ is an independent vertex cover satisfying the desired inequality. □

Theorem 6.4.4. *If T is a tree with n vertices, then $\mathring{s}(T) \le \lfloor 3n/2 \rfloor$.*

Proof. As remarked earlier, we use induction on n, with trivial basis. We provide an inductive strategy for Painter. Let M be the first set marked by Lister. By Observation 6.0.2, we may assume that $T[M]$ is connected. Let I be the independent subset of M colored in response by Painter. As remarked earlier, applying the induction hypothesis allows Painter to limit the score to $|M| + \frac{1}{2}(3n - 3|I| - o(T - I))$. It therefore suffices to find an independent subset I of M such that $3|I| + o(T - I) \ge 2|M|$.

Again let $T' = T - E(T[M])$; the components of T' each contain one vertex of M. Let A and B be the subsets of M whose components in T' have even order and odd order, respectively. Apply Lemma 6.4.3 to the tree $T[M]$ with the vertex partition $\{A, B\}$. We obtain I, one of the partite sets of $T[M]$, such that $3|I| + |A \cap I| + |B - I| \ge 2|M|$. By the definition of A and B, we have $|A \cap I| = e_I$ and $|B - I| = o_I$. By Lemma 6.4.2, $o(T - I) \ge |A \cap I| + |B - I|$, and the proof is complete. □

In this section, we determine $\mathring{s}(K_{1,n-1})$ and show that $\mathring{s}(T) \ge \mathring{s}(K_{1,n-1})$ for every tree T with n vertices.. More generally, we compute $\mathring{s}(\overline{K}_r \oplus K_s)$.

For $k, r \in \mathbb{N}$, let $t_k = \binom{k+1}{2}$ and $u_r = \max\{k\colon\, t_k \le r\}$. Note that $u_r = \left\lfloor \frac{-1+\sqrt{1+8r}}{2} \right\rfloor$. The numbers $\{t_q\colon\, q \in \mathbb{N}\}$ are the *triangular numbers*. Before computing $\mathring{s}(\overline{K}_r \oplus K_s)$, we need a technical lemma about u_r.

Lemma 6.4.5. *$u_{r-u_r} = u_r$ when $r+1$ is a triangular number, and otherwise $u_{r-u_r} = u_r - 1$.*

Proof. If $u_r = k$, then $t_k \le r < t_{k+1}$. Also $t_{k+1} - t_k = k+1$. Thus $r - k = t_k$ if $r + 1 = t_{k+1}$, yielding $u_{r-u_r} = u_r$. However, $t_{k-1} \le r - k < t_k$ if $t_k \le r \le t_{k+1} - 2$, yielding $u_{r-u_r} = u_r - 1$. □

Theorem 6.4.6. *For $r, s \in \mathbb{N}$,*

$$\mathring{s}(\overline{K}_r \oplus K_s) = r + \binom{s+1}{2} + su_r.$$

Proof. We use induction on $r + s$. Let $f(r,s) = r + \binom{s+1}{2} + su_r$. When r or s is 0, the claim clearly holds. For $rs > 0$, let $G = \overline{K}_r \oplus K_s$. Also let $[r] = \{1, \ldots, r\}$. Let R and S denote the sets of vertices with degree s and degree $r + s - 1$, respectively.

Since Painter can only remove one vertex of S in response, Lister should mark all of S plus some vertices of R. Painter responds by removing one vertex of S or all marked vertices of R. Applying the recursion of Proposition 6.0.1 and the induction hypothesis,

$$\mathring{s}(\overline{K}_r \oplus K_s) = \max_{k \in [r]}((k+s) + \min\{\mathring{s}(\overline{K}_{r-k} \oplus K_s), \mathring{s}(\overline{K}_r \oplus K_{s-1})\} = \max_{k \in [r]} g(k),$$

where

$$g(k) = k + s + \min\{f(r-k,s), f(r,s-1)\}.$$

By the induction hypothesis, $g(k)$ is the best result Painter can obtain when Lister marks S and k vertices of R on the first round. We compute

$$\begin{aligned} g(u_r) &= u_r + s + \min\{r - u_r + \binom{s+1}{2} + s u_{r-u_r}, r + \binom{s}{2} + (s-1)u_r\} \\ &= \min\{r + \binom{s+1}{2} + s(u_{r-u_r}+1), r + \binom{s+1}{2} + s u_r\} \\ &= \min\{f(r,s) + s(1 + u_{r-u_r} - u_r), f(r,s)\}. \end{aligned}$$

By Lemma 6.4.5, $u_r - u_{r-u_r} \in \{0,1\}$, so $g(u_r) = f(r,s)$. Furthermore, if Lister marks u_r vertices in R and all of S, then deleting a vertex of S is an optimal response for Painter.

We seek $\max_k g(k)$. Note that $g(0) = s + f(r,s-1)$, since $f(r,s) > f(r,s-1)$. While $f(r-k,s) \geq f(r,s-1)$, we have $g(k) = g(0) + k$, so in this range we maximize k. Since also $f(r-k,s)$ decreases as k increases, and $f(r-k,s)+k$ is nonincreasing, subsequently g is nonincreasing. Hence $g(k)$ is maximized by the largest k such that $f(r-k,s) \geq f(r,s-1)$. To show this is u_r, it suffices to show $f(r-u_r,s) \geq f(r,s-1)$ and $f(r-(u_r+1),s) < f(r,s-1)$.

When $r+1$ is not a triangular number, Lemma 6.4.5 yields $f(r-u_r,s) = f(r,s-1)$. Since $f(r-(u_r+1),s) < f(r-u_r,s)$, the desired value of k is u_r.

When $r+1$ is a triangular number, Lemma 6.4.5 yields $f(r-u_r,s) = f(r,s-1)+s$. Since r itself then is not a triangular number, Lemma 6.4.5 and $u_{r-1} = u_r$ yield $f(r-(u_r+1),s) = f(r-1,s-1) < f(r,s-1)$. Again the desired value is u_r. □

Setting $s=1$, we have $\mathring{s}(K_{1,n-1}) = n + u_{n-1} = n + \left\lfloor \frac{-1+\sqrt{8n-7}}{2} \right\rfloor$.

Theorem 6.4.7. *If T is a tree with n vertices, then $\mathring{s}(T) \geq \mathring{s}(K_{1,n-1}) = n + v_n$, where $v_n = u_{n-1}$.*

Proof. We use induction on n. Since $\mathring{s}(P_n) = \lfloor 3n/2 \rfloor$ and always $\lfloor n/2 \rfloor \geq v_n$, the claim holds when T is a path. Hence also the claim holds for $n \leq 4$.

The main idea is that Lister can play separately on disjoint induced subgraphs, yielding $\mathring{s}(T) \geq \mathring{s}(T_1) + \mathring{s}(T_2)$ when T_1 and T_2 are the components obtained by deleting an edge of T. If $n_i = |V(T_i)|$, then $\mathring{s}(T) \geq n + v_{n_1} + v_{n_2}$. It therefore suffices to find an edge e such that $v_{n_1} + v_{n_2} \geq v_n$. We may assume $n_1 \leq n_2$.

When $n \geq 5$, we have $v_n \leq 1 + v_{n-3}$. If T has an edge whose deletion leaves a component with two or three vertices, then $v_{n_1} = 1$, and $v_{n_1} + v_{n_2} \geq 1 + v_{n-3} \geq v_n$, as desired. Every edge not incident to a leaf has this property when $n \leq 7$.

In the remaining case, $n \geq 8$ and $n_1, n_2 \geq 4$. When $n \geq 8$, we have $v_n \leq 1 + v_{n-4}$. Let $g(x) = \frac{-1+\sqrt{8n-7}}{2}$; note that $v_n = \lfloor g(n) \rfloor$. We need $v_{n_1} + v_{n_2} \geq v_n$.

Let $p = 4$ and $q = n-4$. Since g is concave, $g(p+x) - g(p) \geq g(q) - g(q-x)$, which yields $g(p+x) + g(q-x) \geq g(p) + g(q)$. If $a + b \geq c + d$, then $\lfloor a \rfloor + \lfloor b \rfloor \geq \lfloor a + b \rfloor - 1 \geq \lfloor c + d \rfloor - 1 \geq \lfloor c \rfloor + \lfloor d \rfloor - 1$. Applying this with $(a, b, c, d) = (g(n_1), g(n_2), g(p), g(q))$, and using $v_4 = 2$, we obtain $v_{n_1} + v_{n_2} \geq v_4 + v_{n-4} - 1 = 1 + v_{n-4} \geq v_n$, as desired. □

6.5 Complete Bipartite Graphs

In this section, we give a lower and upper bounds on $\mathring{s}(K_{r,s})$. We first give a general lower bound on $\mathring{s}(K_{r,s})$, followed by a lower bound on $\mathring{s}(K_{r,r})$ that improves on the general lower bound.

Theorem 6.5.1. *Let S denote the set of s-tuples $(r_1, \ldots, r_s)$ such that $r_i \in \mathbb{N}$ for $i \in [s]$ and $\sum r_i = r$. For $r \geq s$,*

$$\mathring{s}(K_{r,s}) \geq r + s + \max_{(r_1,\ldots,r_s)\in S} \sum u_{r_i}.$$

In particular, $\mathring{s}(K_{r,s}) \geq r + s + s u_{\lfloor r/s \rfloor}$.

Proof. Lister plays on the vertex-disjoint subgraphs $K_{1,r_1}, \ldots, K_{1,r_s}$, with the center of each star lying in the partite set of size s. Since $\mathring{s}(K_{1,t}) = t + 1 + u_t$, we have

$$\mathring{s}(K_{r,s}) \geq \sum_{i=1}^{s} (r_i + 1 + u_{r_i}) = r + s + \sum_{i=1}^{s} u_{r_i}. \quad \square$$

□

Since $u_t \approx \sqrt{2t}$ for large t, the lower bound for highly unbalanced graphs is roughly $r + s + \sqrt{2rs}$.

In the special case $r = s$, Theorem 6.5.1 gives the lower bound $\mathring{s}(K_{r,r}) \geq 3r$, which Lister can achieve by repeatedly marking the entire remaining vertex set. A less trivial strategy gives an improvement in the lower bound.

Theorem 6.5.2. $\mathring{s}(K_{r,r}) \geq 3r - 1 + u_r \approx 3r + \sqrt{2r}$.

Proof. Lister starts by marking $r - 1$ vertices in each part, gaining $2(r-1)$ points. Painter may only delete vertices from one part, so after Painter's response, there is still a $K_{1,r}$-subgraph. Lister then plays optimally on $K_{1,r}$, gaining an additional $r + 1 + u_r$ points. $\square$

We now prove the following upper bound.

Theorem 6.5.3. *$\mathring{s}(K_{r,s}) \leq r + s + 2\phi\sqrt{rs}$, where $\phi = \frac{1+\sqrt{5}}{2}$.*

Proof. Let $f(r, s) = r + s + 2\phi\sqrt{rs}$. We prove that $\mathring{s}(K_{r,s}) \leq f(r, s)$ by induction on $r + s$, with trivial base case when $r + s = 0$. So assume that $r + s > 0$ and that $\mathring{s}(K_{r',s'}) \leq f(r', s')$ whenever $r' + s' < r + s$. Without loss of generality we can assume that $r \geq s$. Let R and S be the partite sets of $K_{r,s}$, with $|R| = r$ and $|S| = s$.

Painter will use the following strategy for the first move: suppose that Lister marks j vertices from R and i vertices from S, and let $j_0 = \frac{i}{\phi s}\sqrt{rs}$. Painter will remove the R-vertices if $j \geq j_0$ and otherwise remove the S-vertices. Using this strategy together with the induction hypothesis gives the inequality

$$\begin{aligned}
\mathring{s}(K_{r,s}) &\leq \max_i \Big[\max\big\{\max_{j<j_0}[i + j + \mathring{s}(K_{r,s-i})], \\
&\qquad\qquad \max_{j\geq j_0}[i + j + \mathring{s}(K_{r-j}, s)]\big\}\Big] \\
&\leq \max_i \Big[\max\big\{\max_{j<j_0}[i + j + (r + s - i + 2\phi\sqrt{r(s-i)})], \\
&\qquad\qquad \max_{j\geq j_0}[i + j + (r - j + s + 2\phi\sqrt{(r-j)s})]\big\}\Big] \\
&= \max_i \Big[\max\big\{\max_{j<j_0}[j + r + s + 2\phi\sqrt{r(s-i)})], \\
&\qquad\qquad \max_{j\geq j_0}[i + r + s + 2\phi\sqrt{(r-j)s})]\big\}\Big] \\
&\leq \max_i \Big[\max\big\{j_0 + r + s + 2\phi\sqrt{r(s-i)}, \\
&\qquad\qquad i + r + s + 2\phi\sqrt{(r-j_0)s}\big\}\Big].
\end{aligned}$$

In the last step we have used the fact that $j + r + s + 2\phi\sqrt{r(s-i)}$ is clearly increasing in j while $i + r + s + 2\phi\sqrt{(r-j)s}$ is clearly decreasing in j; hence each would be maximized at j_0 in the continuous relaxation over the relevant intervals.

As such, to prove the proposition it suffices to prove the following two claims.

(I) $i + r + s + 2\phi\sqrt{(r-j_0)s} \geq j_0 + r + s + 2\phi\sqrt{r(s-i)}$ for all $i \in \{1, \ldots, s\}$;

(II) $i + r + s + 2\phi\sqrt{(r-j_0)s} \leq r + s + 2\phi\sqrt{rs}$ for all $i \in \{1, \ldots, s\}$.

Proof of (I): The desired inequality is clearly equivalent to $i + 2\phi\sqrt{(r-j_0)s} \geq j_0 + 2\phi\sqrt{r(s-i)}$. It suffices to drop the i and prove $2\phi\sqrt{(r-j_0)s} \geq j_0 + 2\phi\sqrt{r(s-i)}$, which is equivalent to $j_0 \leq 2\phi\frac{ir - sj_0}{\sqrt{rs-sj_0}+\sqrt{rs-ir}}$.

Thus it suffices to prove that $j_0 \leq \phi\frac{ir-sj_0}{\sqrt{rs}}$. Multiplying by $\sqrt{rs}$ and using the definition of j_0, this is equivalent to $\frac{i}{\phi}r \leq \phi ir - \sqrt{rs}$. Since $r \geq s$, we have $\sqrt{rs} \leq r$ and so it suffices to show that $\frac{i}{\phi}r \leq \phi ir - ir$.

Canceling ir and multiplying through by ϕ gives the equivalent sufficient condition $1 \leq \phi^2 - \phi$, which rearranges to $0 \leq \phi^2 - \phi - 1$. Since $\phi^2 - \phi - 1 = 0$, this inequality holds, so (I) holds.

Proof of (II): The desired inequality is clearly equivalent to $i + 2\phi\sqrt{(r - j_0)s} \leq 2\phi\sqrt{rs}$, which is equivalent to $\frac{i(\sqrt{rs}+\sqrt{rs-sj_0})}{sj_0} \leq 2\phi$. As such it suffices to show that $\frac{2i\sqrt{rs}}{sj_0} \leq 2\phi$. This rearranges to $j_0 \geq \frac{i}{\phi s}\sqrt{rs}$, which is trivially true by since $j_0 = \frac{i}{\phi s}\sqrt{rs}$. Thus (II) holds. $\square$

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