ROOT-THEORETIC YOUNG DIAGRAMS AND SCHUBERT CALCULUS

BY

DOMINIC NIGEL SEARLES

DISSERTATION

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Doctoral Committee:

Professor Henry Schenck, Chair
Associate Professor Alexander Yong, Director of Research
Professor Bruce Reznick
Associate Professor Thomas Nevins

Abstract

A longstanding problem in algebraic combinatorics is to find nonnegative combinatorial rules for the Schubert calculus of generalized flag varieties; that is, for the structure constants of their cohomology rings with respect to the Schubert basis.

There are several natural choices of combinatorial indexing sets for the Schubert basis classes. This thesis examines a number of Schubert calculus problems from the common lens of root-theoretic Young diagrams (RYDs).

In terms of RYDs, we present nonnegative Schubert calculus rules for the (co)adjoint varieties of classical Lie type. Using these we give polytopal descriptions of the set of nonzero Schubert structure constants for the (co)adjoint varieties where the RYDs are all planar, and suggest a connection between planarity of the RYDs and polytopality of the nonzero Schubert structure constants. This is joint work with A. Yong.

For the family of (nonmaximal) isotropic Grassmannians, we characterize the RYDs and give a bijection between RYDs and the k-strict partitions of A. Buch, A. Kresch and H. Tamvakis. We apply this bijection to show that the (co)adjoint Schubert calculus rules agree with the Pieri rules of A. Buch, A. Kresch and H. Tamvakis, which is needed for the proofs of the (co)adjoint rules.

We also use RYDs to study the Belkale-Kumar deformation of the ordinary cup product on cohomology of generalized flag varieties. This product structure was introduced by P. Belkale and S. Kumar and used to study a generalization of the Horn problem. A structure constant of the Belkale-Kumar product is either zero or equal to the corresponding Schubert structure constant, hence the Belkale-Kumar product captures a certain subset of the Schubert structure constants. We give a new formula (after that of A. Knutson and K. Purbhoo) in terms of RYDs for the Belkale-Kumar product on flag varieties of type A. We also extend this formula outside of type A to the (co)adjoint varieties of classical type.

With O. Pechenik, we introduce a new deformed product structure on the cohomology of generalized flag varieties, whose nonzero structure constants can be understood in terms of projections to smaller flag varieties. We draw comparisons with the ordinary cup product and the Belkale-Kumar product.

Dedicated to my father Barry Searles (1933–2009), and my daughter Ruby Searles (born October 2014).

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Chapter 1

Foreword

The field of Schubert calculus originated in H. Schubert's 19th-century investigation of certain problems in enumerative geometry. A classical example of such a problem is to determine how many lines intersect four given lines in general position in three-dimensional (projective) space. (The answer is 2.) One may also determine that, e.g., exactly 462 planes intersect 12 three-dimensional spaces inside six-dimensional space. Putting Schubert's techniques for solving such problems on a rigorous foundation is the subject of Hilbert's 15th problem. A survey of Schubert calculus can be found in [32].

The modern approach to these problems is in terms of intersection theory [23], an important tool in algebraic geometry. These problems may be considered in terms of *Schubert subvarieties* of generalized flag varieties. For the "four lines" problem, the set of all lines that intersect one of the given lines is a Schubert variety, and the solution to the problem is the cardinality of the intersection of the four Schubert varieties corresponding to the four given lines. To each Schubert variety is associated a *Schubert class* in the cohomology ring of the generalized flag variety, and the Schubert classes form a basis for this cohomology ring. These intersection numbers for Schubert varieties are the **Schubert structure constants** for the cohomology ring, that is, the coefficients in the expansion into Schubert classes of the cup product of two Schubert classes.

In this thesis, we study the longstanding **Schubert problem**: that of giving nonnegative combinatorial rules for the Schubert structure constants of cohomology rings of generalized flag varieties.

The geometric interpretation in terms of intersection numbers implies these structure constants are nonnegative integers. One would like to understand these integers combinatorially, in particular, to find rules in terms of combinatorial objects that are counted by these integers. Moreover, several applications of these numbers are more concerned with whether they are zero or not than their actual value, and having cancellation-free rules is an advantage in determining whether a given structure constant is zero. Rules involving cancellation already exist, e.g., [34].

The search for nonnegative combinatorial rules is a common theme in algebraic combinatorics. *Macdonald polynomials*, introduced by I.G. Macdonald [43] generalize many important families of symmetric functions, and it has been proven by M. Haiman [25] that the coefficients of the expansion of a Macdonald polynomial in the Schur basis of symmetric functions are polynomials with nonnegative integer coefficients. It is an open problem to give a combinatorial rule for these coefficients that manifests their nonnegativity. Another example is Kazhdan-Lusztig polynomials, which were introduced by D. Kazhdan-M. Lusztig [29] in order to study Hecke algebra representations. These polynomials, which also measure singularities of Schubert varieties [30], have nonnegative integer coefficients, and it is an open problem to give a nonnegative combinatorial formula for these. Another longstanding open problem is the Kronecker problem, see, e.g., [42]: that of finding a nonnegative combinatorial rule for the multiplicities in decomposition of tensor products of representations of the symmetric group.

Solutions to the Schubert problem are known in a few special cases, one of which is the family of *Grassmannians*, the most basic examples of generalized flag varieties. The Schubert structure constants for Grassmannians appear in several different areas of mathematics. For example, these numbers are the multiplicities in decomposition of tensor products of representations of the general linear group (cf. the aforementioned Kronecker problem for the symmetric group), and multiplicities in decomposition of certain induced representations of the symmetric group. They are also the structure constants for the Schur basis of the ring of symmetric functions. Nonzeroness of Grassmannian structure constants moreover governs other problems such as extensions of finite abelian groups, and eigenvalues of sums of Hermitian matrices.

In the Grassmannian case, the Schubert classes may be combinatorially represented by partitions, or by **Young diagrams**: the Ferrers diagrams of these partitions. Then the Schubert structure constants are computed by the *Littlewood-Richardson rule*, using, for example, the *jeu de taquin* algorithm of M.-P. Schützenberger [58], which we explain below.

Given a Young diagram λ, a standard Young tableau T_λ is a bijective assignment of the numbers 1 through $|\lambda|$ to the boxes of λ, such that the numbers increase along rows and down columns. Such an assignment is called a **standard filling** of λ, the numbers are called **labels**, and we say λ is the **shape** of T_λ. If one Young diagram λ is contained in another Young diagram ν, their set-theoretic difference ν/λ is called a **skew diagram**. A standard filling of a skew diagram is called a **skew tableau**.

For example, Figure 1.1 shows the Young diagram $\lambda = (4, 2, 1)$, a Young tableau of shape λ, and a skew tableau of shape ν/λ where $\nu = (6, 5, 1, 1)$. (A skew tableau contains only the boxes of ν/λ, but here we also depict the boxes of λ in the picture of the skew tableau. This will aid the explanation of the jeu de taquin algorithm.)

Consider a skew tableau of shape ν/λ. The boxes of λ will be called unlabelled. The jeu de taquin algorithm proceeds as follows. Choose a maximally south-east unlabelled box b, such that some box below

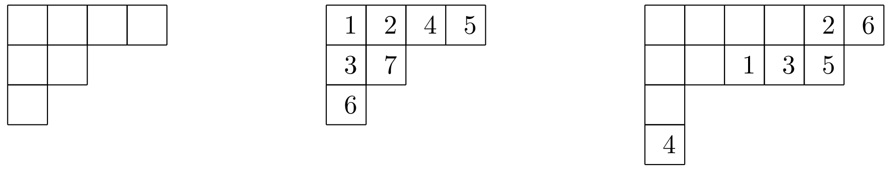


Figure 1.1: Young diagram, tableau and skew tableau.

or to the right of b is labelled. From the boxes immediately below b and immediately to the right of b, take the box b' having smallest label and move its label to b, leaving b' unlabelled. Then repeat with b', and continue in this manner until a label is removed from a box that has no labelled box below or to the right of it. Then, choose another maximally south-east unlabelled box such that some box below or to the right of it is labelled, and perform the same process. Repeat this until there are no unlabelled boxes above or to the right of any labelled box, at which point delete all unlabelled boxes. The result is a standard Young tableau with $|\nu|$-$|\lambda|$ boxes, called the **rectification** of this skew tableau.

For example, suppose $\lambda = (2)$ and $\nu = (3, 2)$. The steps of the jeu de taquin algorithm on a skew tableau of shape ν/λ are shown in Figure 1.2. In each iteration, the box on which the algorithm is operating is labelled with a bullet.

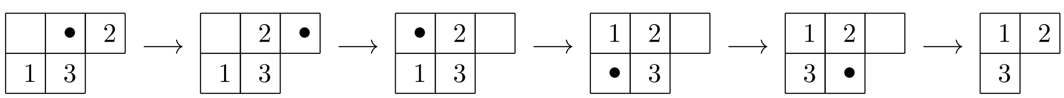


Figure 1.2: The jeu de taquin algorithm.

Jeu de taquin then provides a solution to the Schubert problem for Grassmannians as follows. The Schubert structure constant corresponding to partitions λ, μ, ν (that is, the coefficient of the Schubert basis class corresponding to ν in the product of the Schubert basis classes corresponding to λ and μ) is obtained by fixing a standard Young tableau T_μ of shape μ, and counting the number of standard fillings of ν/λ that rectify to T_μ. For example, if $\lambda = \mu = (2, 1)$ and $\nu = (3, 2, 1)$, Figure 1.3 shows a choice of standard Young tableau of shape μ and the two skew tableau of shape ν/λ that rectify to it. No other skew tableau of shape ν/λ rectifies to this choice, hence the associated structure constant is 2.

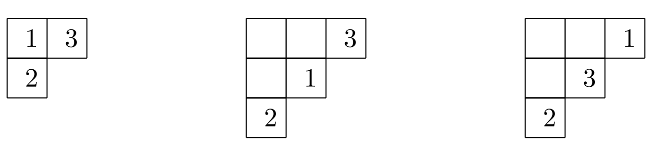


Figure 1.3: A standard Young tableau of shape μ and the two skew tableaux of shape ν/λ that rectify to it.

For the "four lines" problem, the relevant generalized flag variety is a Grassmannian whose Schubert classes are indexed by partitions contained in $(2, 2)$. The Schubert variety corresponding to all lines intersecting a given line has codimension 1 and the corresponding Schubert class is represented by the partition (1), that is, a single box. The class of a point is represented by the partition $(2, 2)$. Thus the problem is to find the coefficient of the class corresponding to $(2, 2)$ in the Schubert basis expansion of the product of four copies of the class corresponding to (1). Identifying the Schubert classes with their corresponding Young diagrams, Figure 1.4 shows the calculation in terms of Young diagrams and jeu de taquin giving the answer of two. (Partitions not contained in $(2, 2)$ are considered to be zero for this calculation, since they do not represent Schubert classes for this Grassmannian.)

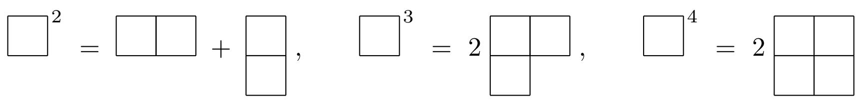


Figure 1.4: Young diagram calculation for the four lines problem.

Very few cases of the Schubert problem beyond the basic Grassmannian case have been solved. In this work, we study a uniform approach to Schubert calculus. We use a uniform combinatorial model called *root-theoretic Young diagrams* (RYDs), which generalizes the classical Young diagram model for Grassmannians.

Our hypothesis is that RYDs are a useful model for studying general patterns in Schubert combinatorics in a uniform manner. The RYD model has already encountered success in giving (uniform) rules for Schubert calculus of a family of generalized flag varieties that extend the Grassmannians [62]. To test our hypothesis for the Schubert problem, we consider another family of generalized flag varieties. Many special properties (in terms of the RYD model) of the family solved in [62] no longer hold for the family we consider, making it a useful test case.

We are able to apply the RYD model to give solutions to the Schubert problem for this test family, giving evidence for our hypothesis. These rules make use of the aforementioned jeu de taquin. As further evidence, we also establish a connection (for this test family) between the planarity of the RYDs and the existence of polytopal descriptions of nonzeroness of the Schubert structure constants. This adds to the celebrated *Horn polytope* description of nonzero Grassmannian Schubert structure constants in terms of Young diagrams [37].

To further test our hypothesis, we would like to consider disparate models and problems in Schubert calculus through the uniform lens of RYDs. Towards this end, we also study the *Belkale-Kumar product* [4] on cohomology of generalized flag varieties. This product structure, used to study a generalization of the Horn problem (see [22], [37]), captures a subset of the Schubert structure constants. We give a new jeu de taquin rule in terms of RYDs for the Belkale-Kumar product for a large class of generalized flag varieties, after the rule of [35] in terms of puzzles. The RYD model visually distinguishes the Schubert structure constants that are preserved by the Belkale-Kumar product, manifesting the relative "easiness" of the Belkale-Kumar structure constants compared to general Schubert structure constants. Moreover, we are able to extend this rule to give rules for the Belkale-Kumar product for our test family of cases for the Schubert problem. We consider this further evidence of the utility of the RYD model.

Chapter 2

Introduction

2.1 Overview

Let G be a complex reductive Lie group. Fix a Borel subgroup $B \subset G$ and parabolic subgroup $P \supset B$. The quotient space G/P is a **generalized flag variety**.

The most well-studied family of examples is the type A **flag varieties**: the G/P's where $G = GL_n$. A **(partial) flag** in $\mathbb{C}^n$ for a given sequence of integers $0 < k_1 < \ldots < k_d < n$ is a chain of subspaces $0 \subsetneq V_{k_1} \subsetneq V_{k_2} \subsetneq \ldots \subsetneq V_{k_d} \subsetneq \mathbb{C}^n$, where $\dim(V_{k_i}) = k_i$. The collection of all such flags is the **d-step partial flag variety** $Fl_{k_1,\ldots,k_d;n}$. A flag variety in the case $d = 1$ (equivalently, P is maximal) is a **Grassmannian** $Gr_k(\mathbb{C}^n)$: the set of all k-dimensional subspaces of $\mathbb{C}^n$. Another example is the family of **isotropic Grassmannians**: the G/P's where G is of type B, C or D and P is maximal. A subspace V is said to be **isotropic** with respect to a nondegenerate bilinear symmetric or skew-symmetric form Q if $Q(x, y) = 0$ for all $x, y \in V$. The type B isotropic Grassmannians are the odd orthogonal Grassmannians

$$OG(k, 2n + 1) = \{V \subset \mathbb{C}^{2n+1} : \dim(V) = k, V \text{ isotropic with respect to } Q\}$$

(where Q is symmetric on $\mathbb{C}^{2n+1}$). The (type C) Lagrangian Grassmannians $LG(k, 2n)$ and, respectively, (type D) even orthogonal Grassmannians $OG(k, 2n)$ are defined similarly: in these cases Q is skew-symmetric, respectively, symmetric on $\mathbb{C}^{2n}$.

Our primary object of study is the product structure on the cohomology ring $H^\star(G/P) = H^\star(G/P; \mathbb{Z})$; a standard reference for the following is [11]. Let W denote the Weyl group of G, W_P the associated parabolic subgroup of W, and W^P the set of minimal length coset representatives of W/W_P. A generalized flag variety decomposes into finitely many orbits of the opposite Borel subgroup B_-, indexed by the elements $w \in W^P$. These orbits are called **Schubert cells** and their closures are the **Schubert varieties** in G/P. To each Schubert variety X_w is associated a class $\sigma_w \in H^\star(G/P)$, and the collection of all these classes forms an additive basis of $H^\star(G/P)$ called the **Schubert basis**. Thus we have

$$\sigma_u \cdot \sigma_v = \sum_w c_{u,v}^w \sigma_w,$$

where $c_{u,v}^w \in \mathbb{Z}_{\geq 0}$ is a **Schubert structure constant**. There is a geometric reason for the fact the $c_{u,v}^w$ are nonnegative integers. Let w_0 denote the longest element of W, w_0^P the longest element of W_P, and let $w^\vee = w_0 w w_0^P$. By the famous Kleiman transversality theorem [31], there is an open subset $O \subset G \times G \times G$ such that for $(g_1, g_2, g_3) \in O$, the intersection $g_1 X_u \cap g_2 X_v \cap g_3 X_{w^\vee}$ is transverse. Then $c_{u,v}^w$ counts the number of points in this intersection.

While there exist algorithms to compute the structure constants $c_{u,v}^w$ for general G/P, e.g., [3] or [34], it is a longstanding problem to give manifestly nonnegative combinatorial formulas for these numbers.

Solutions to this problem are known in a few special cases. For a Grassmannian $Gr_k(\mathbb{C}^n)$, the Schubert classes may be represented by **partitions** $\lambda = (\lambda_1, \ldots, \lambda_k)$, where $\lambda_i \in \mathbb{Z}$ and $n - k \geq \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k \geq 0$, or by **Young diagrams**: the Ferrers diagrams of these partitions. Then the $c_{u,v}^w$ are computed by the *Littlewood-Richardson rule*, using, for example, the *jeu de taquin* algorithm of M.-P. Schützenberger [58]. There are other combinatorial models yielding rules for Grassmannians, e.g., the *Littelmann paths* of P. Littelmann [41] or *puzzles* of A. Knutson-T. Tao-C. Woodward [38] (see also S. Fomin's Appendix to [57] for several others).

The Grassmannians sit inside the more general family of *(co)minuscule G/P's* (see Chapter 3 for definitions). This family, which shares many properties with the Grassmannians, is of interest in representation theory and has been widely studied, see, e.g., Chapter 9 of [10]. For this family, H. Thomas-A. Yong [62] use work of R. Proctor [48] to give a root-system uniform rule using the pictures of the inversion sets of the elements of W^P.

Beyond the (co)minuscule family, rules have been discovered for the family of two-step partial flag varieties. One rule is given by I. Coskun [18] using the model of *Mondrian tableaux*, and another more recently by A. Buch-A. Kresch-K. Purbhoo-H. Tamvakis [12] using puzzles. Table 2.1 summarizes the families of G/P's for which nonnegative Schubert calculus rules have been found.

	Type A ($G = GL_n$)	Types other than A
P maximal	These are the Grassmannians; several rules for this family.	Family of (co)minuscule varieties uniformly resolved by [62].
P nonmaximal	Family of two-step flag varieties resolved by [18], [12].	No cases solved.

Table 2.1: Solved cases of the Schubert problem.

Young diagrams, puzzles and Mondrian tableaux are only some of several potential choices of combinatorial model for Schubert calculus. Others include *chains in Bruhat order* [8], and *k-strict partitions* [13]. Our analysis follows [62] in using the pictures of the inversion sets of elements of W^P. We call these pictures *root-theoretic Young diagrams* (RYDs for short). RYDs are defined for all generalized flag varieties G/P, and may be viewed as a generalization of the Young diagram model for Grassmannians. The organizing principle of our work is that RYDs are a useful combinatorial model for studying and comparing disparate problems related to Schubert calculus.

Let Φ denote the set of roots of G; a standard reference for root systems is [28]. Write $\Phi = \Phi^+ \cup \Phi^-$ to be the partition of roots into positives and negatives, and let $\Delta \subset \Phi^+$ be the base of simple roots. Every positive root β expands as a unique nonnegative combination of simple roots: $\beta = \sum_{\alpha\in\Delta} n_{\alpha\beta}\alpha$. The **Weyl group** W of G is generated by simple reflections corresponding to the roots in Δ. The Weyl group acts on Φ, permuting the roots. Specifically, for a simple root α, the generator s_α of W acts on a root β by $s_\alpha(\beta) = \beta - \frac{2(\alpha,\beta)}{(\alpha,\alpha)}\alpha$. The set Φ has a natural embedding into $\mathbb{R}^n$; here (α,β) is the ordinary dot product.

Let $\Omega_G = (\Phi^+, \prec)$ denote the canonical poset structure on Φ^+, that is, the transitive closure of the covering relation $\prec\!\!\cdot$ given by $\beta \prec\!\!\cdot\ \gamma$ if $\gamma - \beta \in \Delta$. A choice of subset $\Delta_P \subset \Delta$ identifies a parabolic subgroup $P \supset B$ of G. (For example, $|\Delta \setminus \Delta_P| = 1$ if P is maximal.) We will often write $P_{d_1,\ldots,d_k}$ for the parabolic subgroup P such that $\Delta \setminus \Delta_P = \{\alpha_{d_1},\ldots,\alpha_{d_k}\}$. The associated parabolic subgroup $W_P := W_{\Delta_P}$ is the subgroup of W generated by the simple reflections corresponding to the roots in Δ_P. Consider the subposet

$$\Lambda_{G/P} = \{\beta \in \Phi^+ : \alpha \prec \beta \text{ for some } \alpha \in \Delta \setminus \Delta_P\} \subseteq \Omega_G.$$

For $w \in W^P$, the **inversion set** of w is those positive roots sent to negative roots by the action of w on Φ. In particular, w does not invert any root of Δ_P, and every root outside $\Lambda_{G/P}$ is a positive combination of simple roots in Δ_P, so the inversion set of w is contained in $\Lambda_{G/P}$. Let λ denote the picture of the inversion set of w in $\Lambda_{G/P}$. We call λ a **root-theoretic Young diagram (RYD)** and write σ_λ in place of σ_w for the corresponding Schubert class. Let $\mathbb{Y}_{G/P}$ denote the set of RYDs for G/P.

Example 2.1.1. *The first picture in Figure 2.1 is an RYD $\lambda \in \mathbb{Y}_{GL_7/P_{3,5}}$, drawn inside Ω_{GL_7}. The subposet $\Lambda_{GL_7/P_{3,5}}$ consists of all roots above α_3 or α_5. The second picture is an RYD $\mu \in \mathbb{Y}_{SO_9/P_3}$, drawn inside Ω_{SO_9}. The subposet Λ_{SO_9/P_3} consists of all roots above α_3. In each picture, the inverted roots are colored black.*

We will make repeated use of a **regional decomposition** of $\Lambda_{G/P}$. Define an equivalence relation $\sim$ on $\Lambda_{G/P}$ by $\beta \sim \gamma$ if for every $\alpha \notin \Delta_P$, $n_{\alpha\beta} = n_{\alpha\gamma}$. This relation partitions $\Lambda_{G/P}$ into **regions**.

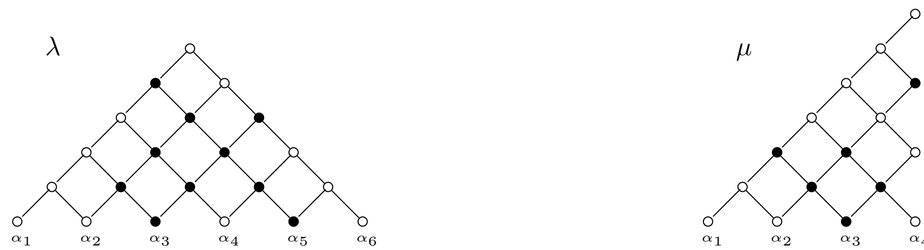


Figure 2.1: Examples of RYDs.

For a subset $S \subset \Omega_G$ and an RYD $\lambda \in \mathbb{Y}_{G/P}$, let λ_S denote the restriction of λ to the subset S and $|\lambda_S|$ the number of roots in S used by λ. A subset T of Ω_G is called a **lower order ideal** if whenever $\gamma \in T, \beta \in \Omega_G$ and $\beta \prec \gamma$, then also $\beta \in T$. The following is essentially well-known, but we provide a proof for the convenience of the reader.

Lemma 2.1.2. *Let $\lambda \in \mathbb{Y}_{G/P}$ and let R be a region of $\Lambda_{G/P}$. Then λ_R is a lower order ideal in R.*

Proof. Let $w \in W^P$ and $\beta, \gamma \in \Lambda_{G/P}$. If $\beta \sim \gamma$ and $\beta \prec\!\cdot\ \gamma$, then by definition $\gamma - \beta = \alpha \in \Delta_P$. Since $w \in W^P$ does not invert simple roots in Δ_P, we have $w(\alpha)$ is positive. Suppose that w inverts γ. Then $w(\beta) = w(\gamma - \alpha) = w(\gamma) - w(\alpha)$, which is a negative root minus a positive root, hence w also inverts β. □

Example 2.1.3. *The first picture in Figure 2.2 shows the regional decomposition of $\Lambda_{GL_7/P_{1,3,4}}$ into the six regions that are above at least one of the thicker black lines. The second picture shows the regional decomposition of $\Lambda_{SO_{10}/P_{2,3}}$; here there are five regions. An RYD is also shown in each case, illustrating Lemma 2.1.2.*

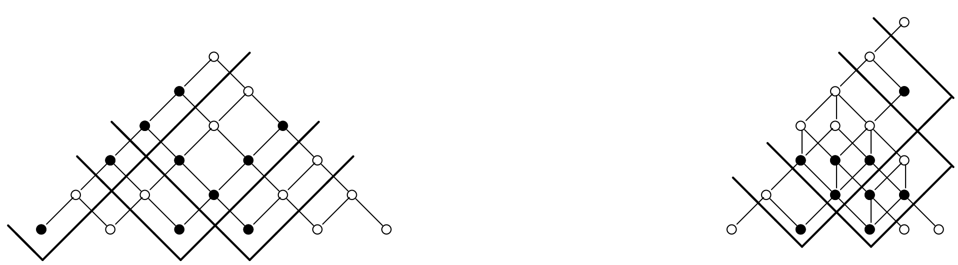


Figure 2.2: Regional decomposition of RYDs.

From the RYD perspective, Grassmannians (and more generally, cominuscule varieties) are special because for these G/P's the above root-system setup is especially graphical:

(I) $\Lambda_{G/P}$ is a planar poset;

(II) $\Lambda_{G/P}$ has only one region (so every $\lambda \in \mathbb{Y}_{G/P}$ is a lower order ideal in $\Lambda_{G/P}$);

(III) Bruhat order (closure order on Schubert cells) is containment of RYDs.

For Grassmannians, property (II) allows a natural identification of RYDs with classical Young diagrams, thus explaining the nomenclature and the sense in which RYDs generalize Young diagrams.

Example 2.1.4. *Figure 2.3 shows an RYD for the Grassmannian $Gr_3(\mathbb{C}^7)$, and the corresponding Young diagram. Here $\Lambda_{Gr_3(\mathbb{C}^7)}$ consists of all roots above α_3, as shown by the thicker black lines.*

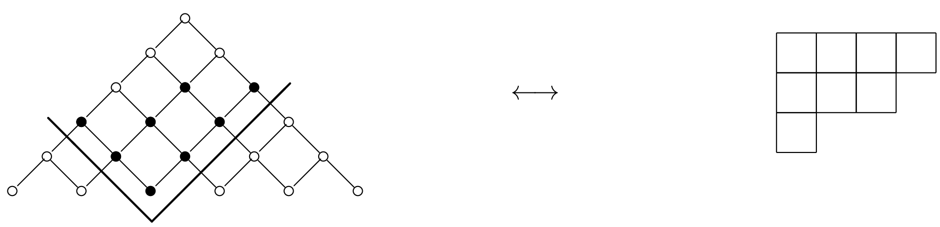


Figure 2.3: Translation between Grassmannian RYDs and Young diagrams.

2.2 Statement of results

The first direction we consider is the Schubert problem for the family of *quasi-(co)minuscule* varieties, which are of representation-theoretic interest, see, e.g., [40]. This family contains the aforementioned (co)minuscule varieties. The remaining varieties in this family are called *(co)adjoint* varieties. None of the (co)minuscule properties (I), (II) or (III) above hold in general for (co)adjoint varieties, however, the failures of these properties are quantifiably mild. It appears the degree to which these properties fail helps provide some measure of the relative "difficulty" of the Schubert calculus of (co)adjoint varieties compared to that of (co)minuscule varieties. RYDs are used by P.-E. Chaput-N. Perrin [15] to give uniform rules for a subset of the Schubert structure constants in each of the (co)adjoint varieties, generalizing [62]; this subset is exactly those structure constants that are "cominuscule-like" in the RYD sense.

In joint work with A. Yong, we use the RYD model to obtain complete Schubert calculus formulas for the classical-type (co)adjoint varieties. The formulas of main interest are those for the spaces of isotropic 2-planes: $OG(2, 2n+1)$, $LG(2, 2n)$ and $OG(2, 2n)$. Our formulas have significant, but far from complete, uniformity. To our knowledge, our formula for the type D adjoint variety $OG(2, 2n)$ is the first complete formula for any G/P where $\Lambda_{G/P}$ is nonplanar, i.e., where property (I) fails. The nonplanarity causes additional complexity in the $OG(2, 2n)$ formula. This formula also depends on the parity of n.

For the (co)adjoint varieties, we also consider another Schubert calculus problem, that of determining the set $S^{\texttt{nonzero}}(G/P)$ of $(\lambda,\mu,\nu)\in(\mathbb{Y}_{G/P})^3$ such that $c_{\lambda,\mu}^{\nu}(G/P)\neq 0$. In particular, one can ask when $S^{\texttt{nonzero}}(G/P)$ might have a *polytopal* realization.

In the case of Grassmannians, this relates to the Horn problem [27] on eigenvalues of sums of Hermitian matrices. Specifically, the eigenvalues of a Hermitian matrix may be written as a nonincreasing sequence λ of real numbers. The **Horn polytope** is the set $\mathrm{Horn}(n)\subset\mathbb{R}^{3n}$ of triples of spectra λ, μ and ν of three $n\times n$ Hermitian matrices satisfying $A+B=C$. A. Horn [27] gave a recursively-defined list of linear inequalities conjecturally describing $\mathrm{Horn}(n)$. A. Klyachko [33] proved that another list of inequalities characterized $\mathrm{Horn}(n)$; these inequalities are stated in terms of nonzeroness of Schubert structure constants of Grassmannians. A consequence of the celebrated *saturation theorem* of A. Knutson-T. Tao [37] is that $c_{\lambda,\mu}^{\nu}$ is nonzero if and only if $(\lambda,\mu,\nu)\in\mathrm{Horn}(n)$. This result, in combination with [33], proved the correctness of Horn's conjectured inequalities. Some of Horn's inequalities are implied by others and thus redundant. P. Belkale [1] found a smaller list of inequalities characterizing $\mathrm{Horn}(n)$, and in [38] this list was proved to be irredundant. Further details and background can be found in, e.g., [52] or the survey [22].

In particular, identifying a Young diagram λ with its partition in $\mathbb{Z}^k$, $S^{\texttt{nonzero}}(Gr_k(\mathbb{C}^n))$ can be viewed as the lattice points of $\mathrm{Horn}(k)$ in $\mathbb{Z}^{3k}$. More recently, K. Purbhoo-F. Sottile [50] established similar descriptions for cominuscule varieties using RYDs. Furthermore, K. Purbhoo [49] used RYDs to provide some criteria for determining zeroness and nonzeroness of structure constants $c_{u,v}^{w}$ for general G/P. These papers provide some pre-existing evidence for the utility of the RYD model.

In joint work with A. Yong, in the (co)adjoint cases we use specific drawings of $\Lambda_{G/P}$ to associate, in a type-by-type manner, a vector of row lengths to each RYD in $\mathbb{Y}_{G/P}$. These descriptions are similar to the partition description of Young diagrams used to formulate the Horn polytope, hence we call them *partition-like*. We show:

Theorem A. *For adjoint G/P, there is a polytopal realization of $S^{\texttt{nonzero}}(G/P)$ using the partition-like description of RYDs if and only if $\Lambda_{G/P}$ is planar.*

Each adjoint G/P has a coadjoint "partner" associated to the Langlands dual group $G^{\vee}$ (see Table 3.2 in Chapter 3).

Theorem B. *For adjoint G/P, let G/Q denote the coadjoint partner. Then*

$$c_{\lambda,\mu}^{\nu}(G/Q)=m(G)^{\mathrm{sh}(\lambda)+\mathrm{sh}(\mu)-\mathrm{sh}(\nu)}c_{\lambda,\mu}^{\nu}(G/P).$$

Here $m(G)$ denotes the maximum multiplicity of an edge in the Dynkin diagram of G, and sh is a statistic depending on the short roots of $\Lambda_{G/P}$ (see Chapter 3). If G is simply-laced (i.e., all roots have the same length) then $m(G) = 1$ and $G/Q = G/P$, however G/P and G/Q differ for G not simply-laced. The short roots factor uniformly extends that of [62] for the (co)minuscule family. Theorem B implies $S^{\texttt{nonzero}}(G/Q) = S^{\texttt{nonzero}}(G/P)$ (one can index the Schubert classes of G/Q by the RYDs for G/P), extending the Horn-type results of Theorem A to the coadjoint cases. Theorems A and B add to the results of [37, 50] on the nonzeroness problem within the family of quasi-(co)minuscule G/P.

Theorem A, in addition to the relative complexity of the (nonplanar) $OG(2, 2n)$ rule in comparison to the planar cases, suggests a planar/non-planar dichotomy in combinatorial Schubert calculus. While it is of course possible $S^{\texttt{nonzero}}(G/P)$ could have a polytopal realization using a different vectorial description of RYDs to our partition-like descriptions, in Example 3.2.14 we explore some other natural descriptions for $OG(2, 10)$ and find that none of these yield a polytopal realization of $S^{\texttt{nonzero}}(OG(2, 10))$. In Chapter 3, we prove our planarity results and present our formulas for and analysis of the (co)adjoint varieties.

Another approach to Schubert calculus concerns the *Belkale-Kumar product* on $H^\star(G/P)$, introduced by P. Belkale-S. Kumar [4]. They used this deformation of the ordinary cup product on $H^\star(G/P)$ to give an irredundant solution to the Horn problem in general Lie type (the type A version is the aforementioned Horn problem on eigenvalues of sums of Hermitian matrices). The irredundancy of their answer was proved by N. Ressayre [51].

A structure constant $b_{u,v}^w$ for the Belkale-Kumar product is either zero or equal to the Schubert structure constant $c_{u,v}^w$. Hence this product captures a subset of the Schubert structure constants of $H^\star(G/P)$, specifically, those corresponding to *Levi-movable* triples of Schubert varieties (see Chapter 4). In the case $G = GL_n$, the structure constants $b_{u,v}^w$ are given by a beautiful combinatorial formula of [35] in terms of puzzles.

In Chapter 4, we characterize the subsets of $\Lambda_{GL_n/P}$ that are RYDs, and we use a factorization formula of [35] to derive a new formula in terms of RYDs for the Belkale-Kumar product in type A. This RYD formula manifests the product/factorization structure of the numbers $b_{u,v}^w$ in terms of Schubert structure constants of Grassmannians. Each factor corresponds to a region of $\Lambda_{GL_n/P}$. To compute the Belkale-Kumar structure constants, we form skew RYDs, generalizing the concept of skew diagrams for Grassmannians. The calculations are then carried out using a skew Littlewood-Richardson rule, e.g., the jeu de taquin of [58]. The RYD description also provides a concrete context to explain in what sense the Belkale-Kumar product is "easier" than the cup product. Specifically, general Schubert structure constants cannot be reduced to independent computations using the regional decomposition. Example 4.2.9 in Chapter 4 exhibits concretely how the Belkale-Kumar case differs from the general problem.

For general G/P Levi-movability can be described in terms of the RYDs and the regional decomposition of $\Lambda_{G/P}$ (see Proposition 4.1.1); this is a straightforward consequence of a result of N. Ressayre-E. Richmond [53]. We use this to show that a natural extension of our rule for the Belkale-Kumar product in type A provides a formula for the Belkale-Kumar product for the (co)adjoint varieties of classical type. Our (co)adjoint Schubert calculus formulas (Chapter 3) in fact separate out the case of Schubert structure constants corresponding to triples (λ, μ, ν) that are not Levi-movable. Therefore, we obtain adjoint Belkale-Kumar rules directly from the adjoint Schubert calculus rules by setting this case equal to zero. Since this case is the most complex in the Schubert calculus rules, this adds to the type A RYD explanation of the relative "easiness" of the Belkale-Kumar structure constants.

In Chapter 5 we obtain a new deformed product structure $\star_t$ on $H^\star(G/P)$. This is joint work with O. Pechenik. We prove $\star_t$ can be understood in terms of projections of Schubert varieties to smaller flag varieties. In type A, $\star_t$ is equal to the Belkale-Kumar product. For general type G/P's where P is maximal, $\star_t$ is equal to the ordinary cup product (which is not the case for the Belkale-Kumar product). In general, $\star_t$ differs from both the Belkale-Kumar product and the cup product; we provide some examples to compare and contrast $\star_t$ with these products.

Finally, in Chapter 6 we study the RYD model in the context of the family of isotropic Grassmannians. The **maximal isotropic Grassmannians** $OG(n, 2n+1)$, $LG(n, 2n)$ and $OG(n, 2n)$ are (co)minuscule, and hence the Schubert problem has already been solved in these cases.

We characterize the subsets of $\Lambda_{G/P}$ that are RYDs for isotropic Grassmannians. In particular, each RYD for an isotropic Grassmannian can naturally be thought of as a pair of partitions. Pairs of partitions are used in many other indexing sets for Schubert classes for these spaces, e.g., [46], [47], [61], [20], [19]. However, the pairs of partitions used in these indexing sets differ from those that arise from RYDs. For example, the pair of partitions indexing the class of a point in $H^\star(OG(2,7))$ is $((3,3)|(1,0))$ in the RYD case, but $((1,1)|(3,2))$ in [46]. A. Buch-A. Kresch-H. Tamvakis [13] combinatorially index the Schubert classes for (nonmaximal) isotropic Grassmannians by k-strict partitions. They use this model to give **Pieri rules** for the Schubert calculus of these spaces: formulas for computing the product of an arbitrary Schubert class with a Schubert class belonging to a given generating subset of $H^\star(G/P)$. Their Pieri rules follow those of [46], [47], which are stated in terms of pairs of partitions.

The proof of the (co)adjoint Schubert calculus formulas requires Pieri rules for the (co)adjoint varieties, and the most interesting classical (co)adjoint varieties belong to the family of nonmaximal isotropic Grassmannians. Therefore, we provide a reformulation of the k-strict partitions of [13] in terms of the pairs of partitions arising from the RYDs. In the (co)adjoint cases, we use this reformulation to prove the restriction to the Pieri cases of our (co)adjoint formulas agrees with the Pieri rule of [13]. In tandem with the proofs of associativity of the (co)adjoint formulas given in [60], this yields a proof of the (co)adjoint formulas.

Chapter 3

Adjoint and coadjoint Schubert calculus

3.1 (Co)adjoint preliminaries

The following is standard, see, e.g., [28]. For $u, v \in \mathbb{R}^n$ let $\langle u, v\rangle = \frac{2(u,v)}{(u,u)}$, where $(\cdot,\cdot)$ is the standard dot product on $\mathbb{R}^n$. A **root system** is a finite subset $\Phi \subset \mathbb{R}^n$ satisfying the following conditions:

1. $\mathrm{span}(\Phi) = \mathbb{R}^n$;
2. If $\beta \in \Phi$, then $\mathbb{R}\beta \cap \Phi = \{\beta, -\beta\}$;
3. If $\beta, \gamma \in \Phi$, then $\gamma - \langle \beta, \gamma\rangle\beta \in \Phi$;
4. $\langle \beta, \gamma\rangle \in \mathbb{Z}$.

A choice of **positive roots** Φ^+ is a subset of Φ such that exactly one of β, $-\beta$ is in Φ^+ for each $\beta \in \Phi$, and for any $\beta, \gamma \in \Phi^+$, if $\beta + \gamma \in \Phi$ then $\beta + \gamma \in \Phi^+$. The subset $-\Phi^+$ is called the **negative roots**. A root $\alpha \in \Phi^+$ is called a **simple root** if it cannot be written as the sum of two roots in Φ^+. The set Δ of simple roots of V is a basis of V, and every positive root is a nonnegative combination of simple roots. A root system is said to be **irreducible** if it cannot be partitioned $\Phi = \Phi_1 \cup \Phi_2$ with $(\beta, \gamma) = 0$ for all $\beta \in \Phi_1$, $\gamma \in \Phi_2$. The irreducible root systems are classified by their *Dynkin diagrams* (see Table 3.1 below), certain graphs whose vertices are the simple roots and edges drawn between non-orthogonal pairs of simple roots. Directed edges point from a longer root to a shorter root, and the multiplicity of an edge indicates the angle between two simple roots. Root systems as defined are naturally in bijection with semisimple Lie algebras. Let e_i denote the ith standard basis vector in $\mathbb{R}^n$; below we give standard embeddings in $\mathbb{R}^n$ of the irreducible root systems corresponding to the four classical types A_{n-1}, B_n, C_n, D_n.

Type A_{n-1}:

$$\Phi = \{e_i - e_j : 1 \leq i \neq j \leq n\}.$$

(Here, the root system spans the $(n-1)$-dimensional subspace of $\mathbb{R}^n$ consisting of vectors whose coordinates sum to zero.) The positive roots are those where $i < j$, and the simple roots are those where $j - i = 1$.

Type B_n:

$$\Phi = \{\pm e_i \pm e_j : 1 \le i < j \le n\} \cup \{\pm e_i : 1 \le i \le n\}.$$

The positive roots of Φ are $\{e_i \pm e_j : 1 \le i < j \le n\} \cup \{e_i : 1 \le i \le n\}$ and the simple roots are $\{e_i - e_j : j - i = 1\} \cup \{e_n\}$.

Type C_n:

$$\Phi = \{\pm e_i \pm e_j : 1 \le i < j \le n\} \cup \{\pm 2e_i : 1 \le i \le n\}.$$

The positive roots of Φ are $\{e_i \pm e_j : 1 \le i < j \le n\} \cup \{2e_i : 1 \le i \le n\}$, and the simple roots are $\{e_i - e_j : j - i = 1\} \cup \{2e_n\}$.

Type D_n:

$$\Phi = \{\pm e_i \pm e_j : 1 \le i < j \le n\}.$$

The positive roots of Φ are $\{e_i \pm e_j : 1 \le i < j \le n\}$, and the simple roots are $\{e_i - e_j : j - i = 1\} \cup \{e_{n-1} + e_n\}$.

A vector $w \in \mathbb{R}^n$ such that $\langle \beta, w\rangle \in \mathbb{Z}$ for every $\beta \in \Phi$ is called an **(abstract) weight**; if moreover $\langle \beta, w\rangle \in \mathbb{Z}_{\ge 0}$ for all $\beta \in \Phi^+$ then w is called **dominant**. To each dominant weight w is associated a finite-dimensional irreducible representation of G with highest weight w. For a representation $\rho : G \to GL(V)$ for some finite dimensional complex vector space V, G acts on $\mathbb{P}(V)$ through the projection $\pi : V \setminus \{0\} \to \mathbb{P}(V)$. If w is a highest weight vector of ρ, then $\pi(G \cdot w) \subseteq \mathbb{P}(V)$ is a homogeneous projective variety, see, e.g., [24, Section 23.3].

Weights corresponding to quasi-(co)minuscule varieties are classified as follows, cf. [16]. For $\beta \in \Phi$, let $\beta^\vee = \frac{2\beta}{(\beta,\beta)}$. A dominant weight w is **minuscule** if for every $\beta \in \Phi^+$ we have $\langle \beta^\vee, w\rangle \le 1$. Such a weight is **quasi-minuscule** if for every $\beta \in \Phi^+$ we have $\langle \beta^\vee, w\rangle \le 2$, with equality only if $\beta = w$. The quasi-minuscule weights that are not minuscule are precisely the coadjoint ones. A weight is cominuscule (respectively, adjoint and quasi-cominuscule) if it is minuscule (respectively, coadjoint and quasi-minuscule) for the dual root system.

Now, G/P is an **adjoint variety** if P is the parabolic subgroup corresponding to an adjoint weight w. Similarly, quasi-(co)minuscule weights correspond to quasi-(co)minuscule varieties. For example, in type B_4 the roots of Φ^+ are $\{e_i \pm e_j : 1 \le i < j \le 4\} \cup \{e_i : 1 \le i \le 4\}$ (where e_i is the ith standard basis vector in $\mathbb{R}^4$), and one can check the weight $(1, 1, 0, 0)$ is adjoint. The quasi-(co)minuscule spaces were studied in [40] as part of a program to extend standard monomial theory for Grassmannians to more general G/P's. The classification of adjoint G/P's is given in Table 3.1; simple roots associated to P are marked. The adjoint G/P's for the non-simply-laced types have a coadjoint "partner" different from G/P. These are given in Table 3.2.

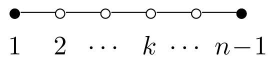
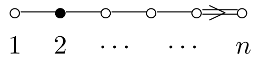
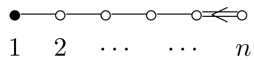
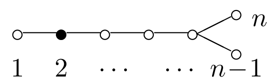
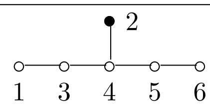
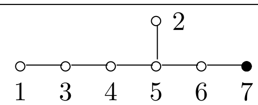
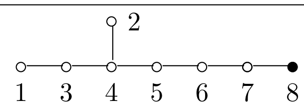
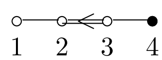
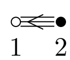
Root system	*Dynkin Diagram*	*Nomenclature (if any)*
A_{n-1}		Point-hyperplane incidence in $\mathbb{P}^{n-1}$; $Fl_{1,n-1;n}$
B_n		Odd orthogonal Grassmannian; $OG(2, 2n+1)$
$C_n, n \geq 3$		Lagrangian Grassmannian; $LG(1, 2n)$
$D_n, n \geq 4$		Even orthogonal Grassmannian; $OG(2, 2n)$
E_6		E_6/P_2
E_7		E_7/P_7
E_8		E_8/P_8
F_4		F_4/P_4
G_2		G_2/P_2

Table 3.1: Classification of adjoint G/P's.

In this chapter, we present our formulas ([60]) for the classical-type (co)adjoint varieties and the results of our computations for the exceptional types. Our formulas of main interest are those for the spaces of isotropic planes: $OG(2, 2n+1)$, $LG(2, 2n)$ and $OG(2, 2n)$. Formulas already exist for the remaining classical-type (co)adjoint varieties, but for completeness, we give formulas in our setup for those cases too. We also prove the Horn-type results of Theorems A and B. As mentioned in the introduction, Theorem B uniformly extends a short roots correspondence for (co)minuscule varieties from [62]. Theorems A and B add to Horn-type results on the (co)minuscule family, giving answers for the remaining cases of the Horn problem in the quasi-(co)minuscule family.

Adjoint variety	*Coadjoint partner*
$OG(2, 2n+1)$ (type B_n)	Lagrangian Grassmannian $LG(2, 2n)$ (type C_n)
$LG(1, 2n)$ (type C_n)	Orthogonal Grassmannian $OG(1, 2n+1)$ (type B_n)
F_4/P_4	F_4/P_1
G_2/P_2	G_2/P_1

Table 3.2: Coadjoint partners to adjoint G/P's for non-simply-laced types.

We prove Theorem A in a case-by-case manner. For the classical types, we obtain descriptions of $S^{\texttt{nonzero}}(G/P)$ directly from our formulas. For the exceptional types, we use explicit computer calculation. In each of the non-planar cases we find a "zero triple" (λ, μ, ν) that is a convex combination of some "nonzero triples", thus contradicting the existence of a polytopal realization (at least, for our partition-like descriptions of the RYDs). We prove Theorem B using explicit computer calculation in types F_4 and G_2, and using a standard result relating cohomology of SO_{2n+1}/B to cohomology of Sp_{2n}/B in the type B_n/C_n case (see Proposition 3.2.4). Our analysis in the exceptional types is made possible by rapid computation of all structure constants in these cases using the presentation of the cohomology ring in [16], their Giambelli-type formulas [17], and standard Gröbner basis techniques. Specifically, multiplying the polynomial representatives of u, v and $w^\vee$ of [17] and reducing with respect to a Gröbner basis for the ideal generated by the relations yields a multiple of the polynomial representative for the class of the point: this number is the structure constant $c_{u,v}^w$. The results of this chapter are joint work with A. Yong, and appear in [60].

In this chapter, we call the highest root of $\Lambda_{G/P}$ the **adjoint root**. If λ uses it we say λ is **on** and we write $\lambda = \langle \overline{\lambda} | \bullet \rangle$; otherwise we say λ is **off** and we write $\lambda = \langle \overline{\lambda} | \circ \rangle$, where $\overline{\lambda}$ comprises the roots of $\Lambda_{G/P} \setminus \{\text{adjoint root}\}$ used by λ.

3.2 The classical types

We will need a reusable definition. For any $\overline{\nu} = (\overline{\nu}_1, \overline{\nu}_2) \in \mathbb{Z}^2$ let $\overline{\nu}^\star = (\overline{\nu}_1 - 1, \overline{\nu}_2)$ and $\overline{\nu}_\star = (\overline{\nu}_1, \overline{\nu}_2 - 1)$. Fix λ and μ and define

$$\mathbb{A}_{\lambda,\mu}(\overline{\nu}) = \begin{cases} 0 & \text{if } \lambda \text{ and } \mu \text{ are on} \\ \sigma_{\langle \overline{\nu} | \bullet \rangle} & \text{if exactly one of } \lambda \text{ or } \mu \text{ is on} \\ \sigma_{\langle \overline{\nu} | \circ \rangle} & \text{if } |\lambda| + |\mu| \leq \frac{|\Lambda_{G/P}|-1}{2} \\ \sigma_{\langle \overline{\nu}^\star | \bullet \rangle} + \sigma_{\langle \overline{\nu}_\star | \bullet \rangle} & \text{otherwise.} \end{cases}$$

In the "otherwise" case of the definition of $\mathbb{A}_{\lambda,\mu}(\overline{\nu})$ a nonadjoint root from $\overline{\nu}$ has "jumped" to become the adjoint root. Understanding how this jumping occurs in each type is key in the (co)adjoint cases. This reflects the additional complexity coming from the failure of point (II) from the introduction.

Type A_{n-1}

The type A_{n-1} adjoint variety is $G/P = Fl_{1,n-1;n}$. This is the two-step partial flag variety $\{\langle 0 \rangle \subset F_1 \subset F_{n-1} \subset \mathbb{C}^n\}$ where F_1 and F_{n-1} have dimensions 1 and $n-1$ respectively. It has dimension $|\Lambda_{G/P}| = 2n-3$.

Rules for all two-step flag varieties have already been given by [18] and [12], however, our approach is in line with our study of other (co)adjoint cases.

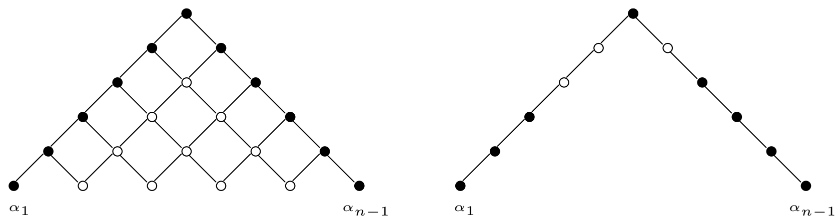


Figure 3.1: $\Lambda_{Fl_{1,n-1;n}}$, Ω_{GL_n} and an RYD (for $n = 7$).

We denote the RYDs λ by $\langle \overline{\lambda}_1, \overline{\lambda}_2 | \circ \rangle$ and $\langle \overline{\lambda}_1, \overline{\lambda}_2 | \bullet \rangle$ where $0 \leq \overline{\lambda}_1, \overline{\lambda}_2 \leq \frac{|\Lambda_{G/P}|-1}{2}$. Set $\sigma_{\langle \overline{\nu} | \bullet / \circ \rangle}$, $\sigma_{\langle \overline{\nu}^\star | \bullet \rangle}$ or $\sigma_{\langle \overline{\nu}_\star | \bullet \rangle}$ to be zero if $\overline{\nu}$, $\overline{\nu}^\star$ or $\overline{\nu}_\star$ are not in $\left[0, \frac{|\Lambda_{G/P}|-1}{2}\right] \times \left[0, \frac{|\Lambda_{G/P}|-1}{2}\right]$.

Proposition 3.2.1. *[60, Proposition 2.10]* $\sigma_\lambda \cdot \sigma_\mu = \mathbb{A}_{\lambda,\mu}(\overline{\lambda} + \overline{\mu}) \in H^\star(Fl_{1,n-1;n})$.

Example 3.2.2. *For* $n = 5$*, the rule gives* $\sigma_{\langle 2,0|\circ \rangle} \cdot \sigma_{\langle 1,2|\circ \rangle} = \mathbb{A}_{\langle 2,0|\circ \rangle, \langle 1,2|\circ \rangle}(3,2) = \sigma_{\langle 2,2|\bullet \rangle} + \sigma_{\langle 3,1|\bullet \rangle}$. *Figure 3.2 shows this computation using RYDs.*

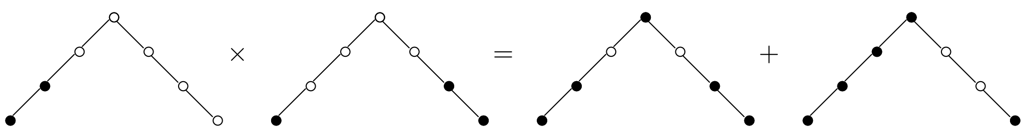


Figure 3.2: An RYD computation in $H^\star(Fl_{1,n-1;n})$ (for $n = 5$).

Now we prove our first case of Theorem A. Declare the partition-like description of RYDs in this case to identify

$$\lambda = \langle \overline{\lambda}_1, \overline{\lambda}_2 | \circ \rangle \text{ with } (\overline{\lambda}_1, \overline{\lambda}_2, 0) \in \mathbb{Z}^3 \text{ and } \lambda = \langle \overline{\lambda}_1, \overline{\lambda}_2 | \bullet \rangle \text{ with } (\overline{\lambda}_1, \overline{\lambda}_2, 1) \in \mathbb{Z}^3. \tag{3.1}$$

We describe $S^{\texttt{nonzero}}(Fl_{1,n-1;n})$ using the identification (3.1). The following is clear:

Corollary 3.2.3. *Assume* $\overline{\lambda} = (\overline{\lambda}_1, \overline{\lambda}_2), \overline{\mu} = (\overline{\mu}_1, \overline{\mu}_2), \overline{\nu} = (\overline{\nu}_1, \overline{\nu}_2) \in \mathbb{Z}^2 \cap \left[0, \frac{|\Lambda_{G/P}|-1}{2}\right] \times \left[0, \frac{|\Lambda_{G/P}|-1}{2}\right]$ *and* $\overline{\lambda}_3, \overline{\mu}_3, \overline{\nu}_3 \in \{0, 1\}$*. Then* $c^\nu_{\lambda,\mu}(Fl_{1,n-1;n}) \neq 0$ *if and only if:*

$$|\nu| = |\lambda| + |\mu|$$

$$\overline{\nu}_1 \leq \overline{\lambda}_1 + \overline{\mu}_1$$

$$\overline{\nu}_2 \leq \overline{\lambda}_2 + \overline{\mu}_2$$

$$\overline{\lambda}_3 + \overline{\mu}_3 \leq \overline{\nu}_3$$

Corollary 3.2.3 shows that neither the failure of (II) nor (III) bar a polytopal answer to the nonzeroness question.

Types B_n/C_n

For the Lie type B_n, G is the group $SO_{2n+1}(\mathbb{C})$ of orthogonal matrices with determinant 1, i.e., $\{A \in SL_{2n+1}(\mathbb{C}) : A^T A = I_{2n+1}\}$. The adjoint variety $G/P = OG(2, 2n+1)$ is the space of isotropic 2-planes with respect to a non-degenerate symmetric bilinear form on $\mathbb{C}^{2n+1}$. It has dimension $|\Lambda_{G/P}| = 4n - 5$.

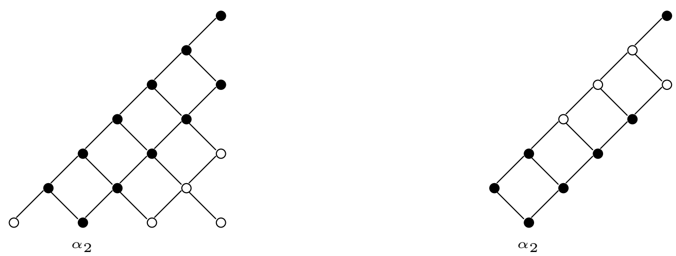


Figure 3.3: $\Lambda_{OG(2,2n+1)}$, $\Omega_{SO_{2n+1}}$ and an RYD (for $n = 4$).

The coadjoint partner to $OG(2, 2n+1)$ in the C_n root system is the variety $G/P = LG(2, 2n)$ of isotropic 2-planes with respect to a non-degenerate skew-symmetric bilinear form on $\mathbb{C}^{2n}$. Here, $G = Sp_{2n}(\mathbb{C})$ is the group of complex matrices A satisfying $A^T JA = J$, where J is the matrix $\left(\begin{smallmatrix} 0 & -I_n \\ I_n & 0 \end{smallmatrix}\right)$. As with all cases, we index the Schubert varieties for the coadjoint variety with RYDs for its adjoint partner. This is analogous to the approach of [62]. For $\nu \in \mathbb{Y}_{OG(2,2n+1)}$, define $\mathrm{sh}(\nu)$ to be the number of short roots used by ν. The short roots of $\Lambda_{OG(2,2n+1)}$ consist of the middle pair of the nonadjoint roots. We prove our first case of Theorem B:

Proposition 3.2.4. $c^{\nu}_{\lambda,\mu}(OG(2, 2n + 1)) = 2^{\mathrm{sh}(\nu)-\mathrm{sh}(\lambda)-\mathrm{sh}(\mu)} c^{\nu}_{\lambda,\mu}(LG(2, 2n))$.

Proof. The Weyl groups for SO_{2m+1} and Sp_{2m} are both isomorphic to the group of signed permutations on m letters. For a signed permutation w, let $\mathcal{B}_w \in H^\star(SO_{2m+1}/B)$ and $\mathcal{C}_w \in H^\star(SO_{2m+1}/B)$ denote the corresponding Schubert classes. Let $s(w)$ count the sign changes in w. It is well-known to experts, see, e.g., [7] that the map $\mathcal{C}_w \mapsto 2^{s(w)}\mathcal{B}_w$ embeds $H^*(Sp_{2m}/B)$ into $H^*(SO_{2m+1}/B)$. But $s(w)$ is exactly the number of short roots (in $\Lambda_{OG(2,2n+1)}$) that are in the inversion set of w. $\square$

We denote $\lambda \in \mathbb{Y}_{OG(2,2n+1)}$ by $\langle \overline{\lambda} | \bullet/\circ \rangle$, where $\overline{\lambda}$ is a partition in $2 \times \left(\frac{|\Lambda_{G/P}|-1}{2}\right)$. Say $\sigma_{\langle \overline{\nu}|\bullet/\circ\rangle}$, $\sigma_{\langle \overline{\nu}^\star|\bullet\rangle}$ or $\sigma_{\langle \overline{\nu}_\star|\bullet\rangle}$ is zero if $\overline{\nu}$, $\overline{\nu}^\star$ or $\overline{\nu}_\star$ is not a partition in $2 \times \left(\frac{|\Lambda_{G/P}|-1}{2}\right)$.

Theorem 3.2.5. *[60, Theorem 1.3]*

$$\sigma_\lambda \cdot \sigma_\mu = \sum_{\overline{\nu}\subseteq\left(\frac{|\Lambda_{G/P}|+1}{2},\frac{|\Lambda_{G/P}|-1}{2}\right)} c^{\overline{\nu}}_{\overline{\lambda},\overline{\mu}}\mathbb{A}_{\lambda,\mu}(\overline{\nu}) \in H^\star(LG(2,2n)).$$

In $H^\star(OG(2,2n+1))$, multiply each coefficient by $2^{\mathrm{sh}(\nu)-\mathrm{sh}(\lambda)-\mathrm{sh}(\mu)}$; the result is integral.

The rule for $OG(2,2n+1)$ is manifestly positive, but not manifestly integral since $2^{\mathrm{sh}(\nu)-\mathrm{sh}(\lambda)-\mathrm{sh}(\mu)} = \frac{1}{2}$ does occur. However, integrality is not difficult to show:

Proposition 3.2.6. *The rule for the $Y = OG(2,2n+1)$ case is integral.*

Proof. Integrality is obvious if $\mathrm{sh}(\lambda) = \mathrm{sh}(\mu) = 0$. If $\mathrm{sh}(\lambda)+\mathrm{sh}(\mu) > 2$, then it is easy to check $\sigma_\lambda \cdot \sigma_\mu = 0$. If $\mathrm{sh}(\lambda) = 2$ and $\mathrm{sh}(\mu) = 0$ (or vice versa) then $\lambda = \langle\overline{\lambda}|\bullet\rangle$ (respectively $\mu = \langle\overline{\mu}|\bullet\rangle$), and ν contains λ (respectively μ) and thus $\mathrm{sh}(\nu) = 2$. If $\mathrm{sh}(\lambda) = 1$ and $\mathrm{sh}(\mu) = 0$ (or vice versa), then if $\nu = \langle\overline{\nu}|\bullet\rangle$ it has at least 1 short root, while if $\nu = \langle\overline{\nu}|\circ\rangle$ it contains λ (respectively μ), and so has 1 short root.

Suppose λ and μ both have 1 short root. If either $\lambda = \langle\overline{\lambda}|\bullet\rangle$ or $\mu = \langle\overline{\mu}|\bullet\rangle$, then for dimension reasons ν has 2 short roots. If $\lambda = \langle\overline{\lambda}|\circ\rangle$ and $\mu = \langle\overline{\mu}|\circ\rangle$, then $\nu = \langle\overline{\nu}|\bullet\rangle$ for dimension reasons and thus $\mathrm{sh}(\nu) \geq 1$. Letting $M = \min\{\overline{\lambda}_1 - \overline{\lambda}_2, \overline{\mu}_1 - \overline{\mu}_2\}$, we have

$$\sigma_\lambda \cdot \sigma_\mu = \sigma_{\langle\overline{\lambda}_1+\overline{\mu}_1,\overline{\lambda}_2+\overline{\mu}_2-1|\bullet\rangle} + 2\sum_{1\leq k\leq M}\sigma_{\langle\overline{\lambda}_1+\overline{\mu}_1-k,\overline{\lambda}_2+\overline{\mu}_2+k-1|\bullet\rangle} + \sigma_{\langle\overline{\lambda}_1+\overline{\mu}_1-M-1,\overline{\lambda}_2+\overline{\mu}_2+M|\bullet\rangle}.$$

(Declare any σ_α in the above expression to be zero if $(\overline{\alpha}_1,\overline{\alpha}_2)$ is not a partition in $2\times(2n-3)$. Such $\overline{\alpha}$ will be called **illegal**.) The first term is illegal, and the last term is illegal or has 2 short roots, so we are done. ∎

Example 3.2.7. *In $H^\star(LG(2,8))$, $\sigma_{\langle 3,1|\circ\rangle}\cdot\sigma_{\langle 3,2|\circ\rangle} = 2\sigma_{\langle 5,3|\bullet\rangle}+\sigma_{\langle 4,4|\bullet\rangle}$. Figure 3.4 shows this computation using RYDs.*

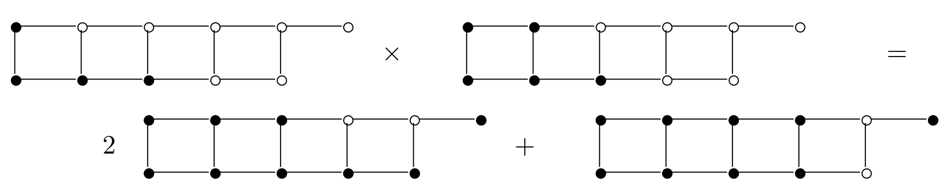


Figure 3.4: An RYD computation in $H^\star(LG(2,8))$.

Similarly, in $H^\star(OG(2,9))$, we compute $\sigma_{\langle 2,1|\circ\rangle}\cdot\sigma_{\langle 3,2|\circ\rangle} = \sigma_{\langle 5,2|\bullet\rangle}+4\sigma_{\langle 4,3|\bullet\rangle}$.

Declare the partition-like description of RYDs in this case to identify

$$\lambda = \langle \overline{\lambda}_1, \overline{\lambda}_2 | \circ \rangle \text{ with } (\overline{\lambda}_1, \overline{\lambda}_2, 0) \in \mathbb{Z}^3 \text{ and } \lambda = \langle \overline{\lambda}_1, \overline{\lambda}_2 | \bullet \rangle \text{ with } (\overline{\lambda}_1, \overline{\lambda}_2, 1) \in \mathbb{Z}^3. \tag{3.2}$$

We prove our next case of Theorem B:

Corollary 3.2.8. *Assume* $\overline{\lambda} = (\overline{\lambda}_1, \overline{\lambda}_2), \overline{\mu} = (\overline{\mu}_1, \overline{\mu}_2), \overline{\nu} = (\overline{\nu}_1, \overline{\nu}_2) \subset 2 \times \left(\frac{|\Lambda_{G/P}|-1}{2}\right)$ *are partitions and* $\overline{\lambda}_3, \overline{\mu}_3, \overline{\nu}_3 \in \{0, 1\}$. *Then* $c_{\lambda,\mu}^{\nu}(LG(2,2n)) \neq 0$ *and* $c_{\lambda,\mu}^{\nu}(OG(2,2n+1)) \neq 0$ *if and only if:*

$$\begin{aligned} |\nu| &= |\lambda| + |\mu| \\ \overline{\nu}_1 &\leq \overline{\lambda}_1 + \overline{\mu}_1 \\ \overline{\nu}_2 &\leq \overline{\lambda}_1 + \overline{\mu}_2 \\ \overline{\nu}_2 &\leq \overline{\lambda}_2 + \overline{\mu}_1 \\ \overline{\lambda}_3 + \overline{\mu}_3 &\leq \overline{\nu}_3 \end{aligned} \tag{3.3}$$

Proof. We have $c_{\lambda,\mu}^{\nu}(LG(2,2n)) \neq 0$ if and only if $c_{\lambda,\mu}^{\nu}(OG(2,2n+1)) \neq 0$. The condition $|\nu| = |\lambda| + |\mu|$ is necessary for nonzeroness, as is $\overline{\lambda}_3 + \overline{\mu}_3 \leq \overline{\nu}_3$ (by the definition of $\mathbb{A}_{\lambda,\mu}$), so assume both these conditions hold. Let (a) denote the other three inequalities. First assume $\overline{\lambda}_3 + \overline{\mu}_3 = \overline{\nu}_3$. Then by Theorem 3.2.5 and the definition of $\mathbb{A}_{\lambda,\mu}$, $c_{\lambda,\mu}^{\nu}(LG(2,2n)) \neq 0$ if and only if $c_{\overline{\lambda},\overline{\mu}}^{\overline{\nu}} \neq 0$. We have $|\overline{\lambda}| + |\overline{\mu}| = |\overline{\nu}|$, so by the Horn inequalities for a 2-row Grassmannian, $c_{\overline{\lambda},\overline{\mu}}^{\overline{\nu}} \neq 0$ if and only if the inequalities (a) hold.

Therefore assume $\overline{\lambda}_3 + \overline{\mu}_3 < \overline{\nu}_3$, i.e., $\overline{\lambda}_3 = \overline{\mu}_3 = 0$ and $\overline{\nu}_3 = 1$. Then by Theorem 3.2.5 and the definition of $\mathbb{A}_{\lambda,\mu}$, $c_{\lambda,\mu}^{\nu}(LG(2,2n)) \neq 0$ if and only if either $c_{\overline{\lambda},\overline{\mu}}^{(\overline{\nu}_1+1,\overline{\nu}_2)} \neq 0$ or $c_{\overline{\lambda},\overline{\mu}}^{(\overline{\nu}_1,\overline{\nu}_2+1)} \neq 0$. First suppose $c_{\lambda,\mu}^{\nu}(LG(2,2n)) \neq 0$. Then since we have $\overline{\nu}_1 + \overline{\nu}_2 + 1 = |\overline{\lambda}| + |\overline{\mu}|$, by the Horn inequalities either the set $(b) = \{\overline{\nu}_1 + 1 \leq \overline{\lambda}_1 + \overline{\mu}_1, \overline{\nu}_2 \leq \overline{\lambda}_1 + \overline{\mu}_2, \overline{\nu}_2 \leq \overline{\lambda}_2 + \overline{\mu}_1\}$ or $(c) = \{\overline{\nu}_1 \leq \overline{\lambda}_1 + \overline{\mu}_1, \overline{\nu}_2 + 1 \leq \overline{\lambda}_1 + \overline{\mu}_2, \overline{\nu}_2 + 1 \leq \overline{\lambda}_2 + \overline{\mu}_1\}$ holds. But if either (b) or (c) hold, then (a) holds.

Now suppose $c_{\lambda,\mu}^{\nu}(LG(2,2n)) = 0$, and also assume $\overline{\nu}_1 > \overline{\nu}_2$ (so $(\overline{\nu}_1, \overline{\nu}_2 + 1)$ is a partition). Then one of the inequalities from (b) and one from (c) must be false. If one of the latter two inequalities of (b) or the first of (c) is false, then (a) does not hold. Thus assume $\overline{\nu}_1 + 1 > \overline{\lambda}_1 + \overline{\mu}_1$, and either $\overline{\nu}_2 + 1 > \overline{\lambda}_1 + \overline{\mu}_2$ or $\overline{\nu}_2 + 1 > \overline{\lambda}_2 + \overline{\mu}_1$. Then for (a) to hold, we must have $\overline{\nu}_1 = \overline{\lambda}_1 + \overline{\mu}_1$ and either $\overline{\nu}_2 = \overline{\lambda}_1 + \overline{\mu}_2$ or $\overline{\nu}_2 = \overline{\lambda}_2 + \overline{\mu}_1$. But this contradicts $|\overline{\nu}| + 1 = |\overline{\lambda}| + |\overline{\mu}|$.

Finally suppose $c_{\lambda,\mu}^{\nu}(LG(2,2n)) = 0$, and also $\overline{\nu}_1 = \overline{\nu}_2$. Then one of the inequalities from (b) must be false. If either of the latter two inequalities of (b) is false, then (a) does not hold. Thus assume $\overline{\nu}_1 + 1 > \overline{\lambda}_1 + \overline{\mu}_1$. If (a) holds then $\overline{\nu}_1 = \overline{\lambda}_1 + \overline{\mu}_1$, and then since $\overline{\nu}_1 = \overline{\nu}_2$ all inequalities in (a) are equalities, again contradicting $|\overline{\nu}|+1=|\overline{\lambda}|+|\overline{\mu}|$. $\square$

In fact, all inequalities in Corollary 3.2.8 but the last come from those for the Horn polytope for $n=2$.

In type C_n, the adjoint variety is $G/P = LG(1,2n) \cong Gr_1(\mathbb{C}^{2n})$ and its coadjoint partner in type B_n is $OG(1,2n+1)$. In fact, $LG(1,2n)$ is minuscule as well as adjoint and $OG(1,2n+1)$ is cominuscule as well as coadjoint, so these cases are already resolved by the (co)minuscule formulas of [62]. However, for completeness we include them here, in our setup. In keeping with our conventions, we use the RYDs for the adjoint variety for both cases. Here, $\Lambda_{LG(1,2n)}$ is a chain of length $2n-1$ where the maximal element is the adjoint root, and all roots except the adjoint root are short roots. Denote the RYDs λ by $\lambda = \langle\overline{\lambda}|\bullet/\circ\rangle$, where $\overline{\lambda} \in \{1,\ldots,2n-2\}$. Since $H^\star(OG(1,2n+1))$ is generated by $\langle 1|\circ\rangle$, the following is easily obtained from the Monk-Chevalley formula for the product of a class of codimension one and an arbitrary class:

Proposition 3.2.9. *If $(\lambda,\mu,\nu) \neq (\langle\overline{\lambda}|\circ\rangle, \langle\overline{\mu}|\circ\rangle, \langle\overline{\nu}|\bullet\rangle)$ then $c^{\nu}_{\lambda,\mu}(OG(1,2n+1)) = c^{\overline{\nu}}_{\overline{\lambda},\overline{\mu}}(Gr_1(\mathbb{C}^{2n-1}))$. Otherwise, $c^{\langle\overline{\nu}|\bullet\rangle}_{\langle\overline{\lambda}|\circ\rangle,\langle\overline{\mu}|\circ\rangle}(OG(1,2n+1)) = 2\cdot c^{\overline{\nu}^\star}_{\overline{\lambda},\overline{\mu}}((Gr_1(\mathbb{C}^{2n}))$, where $\overline{\nu}^\star$ is $\overline{\nu}$ with one additional root.*

We also obtain the next case of Theorems A and B:

Fact 3.2.10. *$c^{\nu}_{\lambda,\mu}(OG(1,2n+1)) \neq 0$ and $c^{\nu}_{\lambda,\mu}(LG(1,2n)) \neq 0$ if and only if $|\nu| = |\lambda|+|\mu|$. Also*

$$c^{\nu}_{\lambda,\mu}(LG(1,2n)) = 2^{\mathrm{sh}(\nu)-\mathrm{sh}(\lambda)-\mathrm{sh}(\mu)} \cdot c^{\nu}_{\lambda,\mu}(OG(1,2n+1)).$$

Proof. This follows since $LG(1,2n)$ is isomorphic to $Gr_1(\mathbb{C}^{2n})$ and the shortroots factor is $\frac{1}{2}$ exactly when $c^{\nu}_{\lambda,\mu}(OG(1,2n+1)) = 2$ (it is equal to 1 otherwise). $\square$

Type D_n

For the Lie type D_n, G is the group $SO_{2n}(\mathbb{C})$ of orthogonal matrices that have determinant 1, that is, $\{A \in SL_{2n}(\mathbb{C}) : A^TA = I_{2n}\}$. The adjoint variety $G/P = OG(2,2n)$ is the space of isotropic 2-planes with respect to a non-degenerate symmetric bilinear form on $\mathbb{C}^{2n}$. It has dimension $|\Lambda_{G/P}| = 4n-7$.

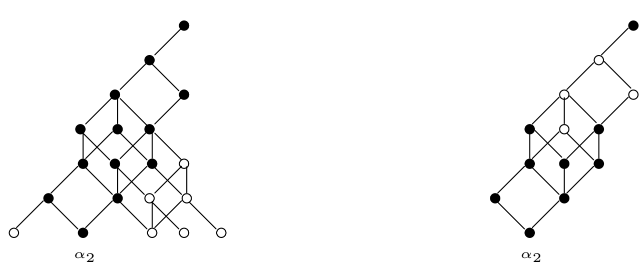


Figure 3.5: $\Lambda_{OG(2,2n)}$, $\Omega_{SO_{2n}}$ and an RYD (for $n=5$).

Here $\Lambda_{G/P}$ is not planar. An RYD $\lambda = \langle \overline{\lambda} | \bullet / \circ \rangle$ in $\Lambda_{G/P}$ is a triple $\langle \overline{\lambda}^{(1)}, \overline{\lambda}^{(2)} | \bullet / \circ \rangle$, where $\overline{\lambda}^{(1)}$ (respectively, $\overline{\lambda}^{(2)}$) is the Young diagram, in French notation, for the "bottom" (respectively, "top") $2 \times \left(\frac{|\Lambda_{G/P}|-1}{4} \right)$ rectangle, and $\bullet/\circ$ indicates if λ is on or off. See Figure 3.6 for an example.

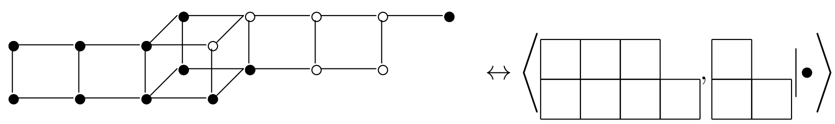


Figure 3.6: An RYD for $OG(2, 12)$ and the Young diagrams for the "bottom" and "top" rectangles.

We mainly use a different description of λ that is more convenient for comparisons with the type B/C case. Define $\pi(\overline{\lambda}) = \overline{\lambda}^{(1)} + \overline{\lambda}^{(2)} := (\overline{\lambda}_1, \overline{\lambda}_2)$, a partition inside the $2 \times \left(\frac{|\Lambda_{G/P}|-1}{2} \right)$ rectangle. Consider an auxiliary poset $\Lambda'_{OG(2,2n)}$, a "planarization" of $\Lambda_{OG(2,2n)}$ (see Figure 3.7).

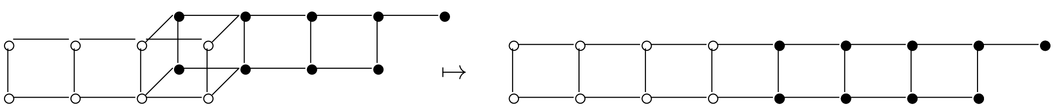


Figure 3.7: $\Lambda_{OG(2,2n)}$ and its "planarization" $\Lambda'_{OG(2,2n)}$.

In Figure 3.7, we have marked the roots of the "top layer" for emphasis.

Consider the subsets $\kappa = \langle \overline{\kappa} | \bullet / \circ \rangle$ of $\Lambda'_{OG(2,2n)}$ where $\overline{\kappa}$ is a partition contained in a $2 \times \left(\frac{|\Lambda_{G/P}|-1}{2} \right)$ rectangle and $\bullet/\circ$ indicates use of the adjoint root in $\Lambda'_{OG(2,2n)}$. Let $\mathbb{Y}'_{OG(2,2n)}$ be the set of such subsets of $\Lambda'_{OG(2,2n)}$. Extend π to a map

$$\Pi : \mathbb{Y}_{OG(2,2n)} \to \mathbb{Y}'_{OG(2,2n)}$$

by defining $\Pi(\lambda) = \langle \pi(\overline{\lambda}) | \bullet \rangle$ if λ is on, and $\Pi(\lambda) = \langle \pi(\overline{\lambda}) | \circ \rangle$ otherwise.

The map Π is either $1:1$ or $2:1$. In the former case, we identify κ and $\Pi^{-1}(\kappa)$. In the latter case, $\Pi^{-1}(\kappa) = \{\kappa^{\uparrow}, \kappa^{\downarrow}\}$ and we call κ **ambiguous**. Call $\kappa^{\uparrow}$ and $\kappa^{\downarrow}$ **charged**. If κ is on (respectively, off), let $\kappa^{\downarrow}$ be the RYD such that the second part (respectively, first part) of the Young diagram $(\pi^{-1}(\overline{\kappa}))^{(2)}$ is zero; let $\kappa^{\uparrow}$ be the other one. Thus in Example 3.2.12 below, λ is up and μ is down.

We need three more notions to state our theorem. First, for $\kappa \in \mathbb{Y}'_{OG(2,2n)}$, let $\mathrm{fsh}(\kappa)$ be the number of **fake short roots** used by κ, i.e., the number of roots in the $(n-2)$-th column used by κ. The one exception is that we need

$$\mathrm{fsh}(\langle n-2, n-2 | \circ \rangle) = 1.$$

For $\nu \in \mathbb{Y}_{OG(2,2n)}$, let $\mathrm{fsh}(\nu)$ denote $\mathrm{fsh}(\Pi(\nu))$. Second, two charged RYDs λ and μ **match** if their arrows match and are **opposite** otherwise. Third, let

$$\eta_{\lambda,\mu} = \begin{cases} 2 & \text{if } \lambda,\ \mu \text{ are charged and match and } n \text{ is even;} \\ 2 & \text{if } \lambda,\ \mu \text{ are charged and opposite and } n \text{ is odd;} \\ 1 & \text{if } \lambda \text{ or } \mu \text{ is neutral;} \\ 0 & \text{otherwise} \end{cases} \tag{3.4}$$

Say $\sigma_{\langle \overline{\nu}|\bullet/\circ\rangle}$, $\sigma_{\langle \overline{\nu}^\star|\bullet\rangle}$ or $\sigma_{\langle \overline{\nu}_\star|\bullet\rangle}$ is zero if $\overline{\nu}$, $\overline{\nu}^\star$ or $\overline{\nu}_\star$ is not a partition in $2 \times \left(\frac{|\Lambda_{G/P}|-1}{2}\right)$.

Theorem 3.2.11. *[60, Theorem 1.7] If either $\pi(\overline{\lambda})$ or $\pi(\overline{\mu})$ equals $(j,0)$ (for some $0 \leq j \leq \frac{|\Lambda_{G/P}|-1}{2}$) then the Schubert expansion of $\sigma_\lambda \cdot \sigma_\mu \in H^\star(OG(2,2n))$ is obtained by the Pieri rule of [13].*

Otherwise, compute

$$\sigma_{\Pi(\lambda)} \cdot \sigma_{\Pi(\mu)} = \sum_{\overline{\nu} \subseteq \left(\frac{|\Lambda_{G/P}|+1}{2}, \frac{|\Lambda_{G/P}|-1}{2}\right)} c^{\overline{\nu}}_{\pi(\overline{\lambda}),\pi(\overline{\mu})} \mathbb{A}_{\lambda,\mu}(\overline{\nu}). \tag{3.5}$$

(i) Replace any term σ_κ that has $\overline{\kappa}_1 = \frac{|\Lambda_{G/P}|-1}{2}$ by $\eta_{\lambda,\mu}\sigma_\kappa$

(ii) Next, replace each σ_κ by $2^{\mathrm{fsh}(\kappa)-\mathrm{fsh}(\lambda)-\mathrm{fsh}(\mu)}\sigma_\kappa$

(iii) Finally, for any ambiguous κ replace σ_κ by $\frac{1}{2}(\sigma_{\kappa^\uparrow} + \sigma_{\kappa^\downarrow})$

The result is a provably integral, and manifestly nonnegative, Schubert basis expansion, which equals $\sigma_\lambda \cdot \sigma_\mu \in H^\star(OG(2,2n))$.

For simplicity of exposition, this statement of the rule separates out the cases handled by the Pieri rule of [13]. A (more complicated) version of the $OG(2,2n)$ rule that includes these cases is given in Definition 6.5.2 of Chapter 6.

Integrality is not manifest due to (ii) and (iii), but it is proved similarly to integrality for the $OG(2,2n+1)$ rule. Rule (i) extends a parity dependency for even-dimensional quadrics, described in [62]. The point is that the "double tailed diamond" which is $\Lambda_{\mathbb{Q}^{2n-4}}$ sits as a "side" of $\Lambda_{OG(2,2n)}$. Rule (ii) is analogous to our rule for $OG(2,2n+1)$. Rule (iii) describes how to "disambiguate".

Example 3.2.12. *We wish to compute $\sigma_\lambda \cdot \sigma_\mu \in H^\star(OG(2,12))$ where $\lambda = (4,1)^\uparrow$ and $\mu = (4,2)^\downarrow$. Both of these RYDs are charged. Here $\pi(\overline{\lambda}) = (4,1)$ and $\pi(\overline{\mu}) = (4,2)$.*

The $\overline{\nu} \subseteq \left(\frac{|\Lambda_{G/P}|+1}{2}, \frac{|\Lambda_{G/P}|-1}{2}\right) = (9,8)$ such that $c^{\nu}_{\pi(\overline{\lambda}),\pi(\overline{\mu})} = 1$ are $(8,3), (7,4)$ and $(6,5)$. All other $\overline{\nu}$

have $c^{\overline{\nu}}_{\pi(\overline{\lambda}),\pi(\overline{\mu})} = 0$. *Hence,*

$$\begin{aligned}
\sigma_{\Pi(\lambda)} \cdot \sigma_{\Pi(\mu)} &= \mathbb{A}_{\lambda,\mu}(8,3) + \mathbb{A}_{\lambda,\mu}(7,4) + \mathbb{A}_{\lambda,\mu}(6,5) \\
&= (\langle 7,3|\bullet\rangle + \langle 8,2|\bullet\rangle) + (\langle 6,4|\bullet\rangle + \langle 7,3|\bullet\rangle) + (\langle 5,5|\bullet\rangle + \langle 6,4|\bullet\rangle) \\
&= \langle 8,2|\bullet\rangle + 2\langle 7,3|\bullet\rangle + 2\langle 6,4|\bullet\rangle + \langle 5,5|\bullet\rangle \\
&\mapsto 0\langle 8,2|\bullet\rangle + 2\langle 7,3|\bullet\rangle + 2\langle 6,4|\bullet\rangle + \langle 5,5|\bullet\rangle \quad \textit{(by (i) and } \eta_{\lambda,\mu} = 0\textit{)} \\
&\mapsto \langle 7,3|\bullet\rangle + 2\langle 6,4|\bullet\rangle + \langle 5,5|\bullet\rangle \quad \textit{(by (ii) and } \mathrm{fsh}(\lambda) = \mathrm{fsh}(\mu) = 1\textit{)}
\end{aligned}$$

Finally, (iii) applies to the ambiguous $\langle 6,4|\bullet\rangle$*, so:*

$$\sigma_\lambda \cdot \sigma_\mu = \langle 7,3|\bullet\rangle + (\langle 6,4|\bullet\rangle^{\uparrow} + \langle 6,4|\bullet\rangle^{\downarrow}) + \langle 5,5|\bullet\rangle.$$

Each step is nonnegative and integral, in agreement with our theorem.

We make the following identifications; cf. (3.2):

$$\Pi(\lambda) = \langle \overline{\lambda}_1, \overline{\lambda}_2|\circ\rangle \text{ with } (\overline{\lambda}_1, \overline{\lambda}_2, 0) \in \mathbb{Z}^3 \text{ and } \Pi(\lambda) = \langle \overline{\lambda}_1, \overline{\lambda}_2|\bullet\rangle \text{ with } (\overline{\lambda}_1, \overline{\lambda}_2, 1) \in \mathbb{Z}^3.$$

We can give an explicit criterion for nonzeroness:

Corollary 3.2.13. *If either* $\pi(\overline{\lambda})$ *or* $\pi(\overline{\mu})$ *equals* $(j, 0)$ *(for some* $0 \leq j \leq \frac{|\Lambda_{G/P}|-1}{2}$*) then nonzeroness of* $c^{\nu}_{\lambda,\mu}(OG(2,2n))$ *is determined by the Pieri rule of [13]).*

If $\overline{\nu}_1 = \frac{|\Lambda_{G/P}|-1}{2}$ *then* $c^{\nu}_{\lambda,\mu}(OG(2,2n)) \neq 0$ *if and only if* $\eta_{\lambda,\mu} \neq 0$ *and the inequalities (3.3) hold.*

Otherwise, assume $(\overline{\lambda}_1, \overline{\lambda}_2), (\overline{\mu}_1, \overline{\mu}_2), (\overline{\nu}_1, \overline{\nu}_2) \subset 2 \times \left(\frac{|\Lambda_{G/P}|-1}{2}\right)$ *are partitions and* $\overline{\lambda}_3, \overline{\mu}_3, \overline{\nu}_3 \in \{0, 1\}$*. Then* $c^{\nu}_{\lambda,\mu}(OG(2,2n)) \neq 0$ *if and only if the inequalities (3.3) hold.*

Proof. The first sentence of the corollary is true by definition, so we may assume we are not in that case. In the case $\overline{\nu}_1 = \frac{|\Lambda_{G/P}|-1}{2}$ if $\eta_{\lambda,\mu} = 0$ then clearly $c^{\nu}_{\lambda,\mu}(OG(2,2n)) = 0$, so we may assume that $\eta_{\lambda,\mu} \neq 0$ in this situation.

The idea is to reduce the problem to the analogous argument for Corollary 3.2.8 by running a "flattened" argument. To do this it is easiest to use a reformulation of the $OG(2,2n)$ rule given by Definition 6.5.3 in Chapter 6.

To be precise, let $\kappa = \Pi(\nu)$. Then $c^{\nu}_{\lambda,\mu}(OG(2,2n)) \neq 0$ if and only if the coefficient c of κ in the expansion $\Pi(\lambda) \diamond \Pi(\mu)$ is nonzero and applying (i), (ii) and (iii) (of Definition 6.5.3) yields a nonzero coefficient for ν (note that applying (i), (ii) and (iii) to a zero coefficient never yields a nonzero coefficient). Since we only need to consider (λ,μ,ν) where $\eta_{\lambda,\mu}\neq 0$, applying (i) multiplies c by a nonzero number. By definition, (ii) multiplies the result of (i) by a nonzero number. Finally, since we only need to worry about non-Pieri cases, if ν is charged then (iii.1) multiplies the result of (ii) by $\frac{1}{2}$ and both $c_{\lambda,\mu}^{\kappa^{\uparrow}}(OG(2,2n))$ and $c_{\lambda,\mu}^{\kappa^{\downarrow}}(OG(2,2n))$ are equal to this resulting number. Thus $c_{\lambda,\mu}^{\nu}(OG(2,2n))\neq 0$ if and only if the coefficient of κ in the expansion $\Pi(\lambda)\diamond\Pi(\mu)$ is nonzero, and then the proof is the same as that for Corollary 3.2.8. □

This gives a polytopal description of the nonzero Schubert structure constants when neither λ nor μ are Pieri classes. However, as indicated in Theorem 2.2, polytopality does not hold in general for $S^{\texttt{nonzero}}(OG(2,2n))$.

Proof of Theorem A in type D:

Identify $\lambda=\langle\overline{\lambda}^{(1)},\overline{\lambda}^{(2)}|\bullet\rangle$ with the vector $(\overline{\lambda}_1^{(1)},\overline{\lambda}_2^{(1)},\overline{\lambda}_1^{(2)},\overline{\lambda}_2^{(2)},1)\in\mathbb{Z}^5$, and $\lambda=\langle\overline{\lambda}^{(1)},\overline{\lambda}^{(2)}|\circ\rangle$ with the vector $(\overline{\lambda}_1^{(1)},\overline{\lambda}_2^{(1)},\overline{\lambda}_1^{(2)},\overline{\lambda}_2^{(2)},0)\in\mathbb{Z}^5$. Then the triple (λ,μ,ν) is a vector in $\mathbb{Z}^{15}$.

With this identification, let $\lambda=\mu=(n-2,0,0,0,0)$. First suppose $n\geq 5$ and consider

$$\nu\in\{(n-2,0,n-2,0,0),(n-2,1,n-3,0,0),(n-2,2,n-4,0,0),(n-2,3,n-5,0,0)\}.$$

This defines four collinear triples. By Theorem 3.2.11, one verifies these points alternate between being in $S^{\texttt{nonzero}}(OG(2,2n))$ and $S^{\texttt{zero}}(OG(2,2n))$ (which two are in $S^{\texttt{zero}}(OG(2,2n))$ depends on the parity of n). Thus, neither $S^{\texttt{nonzero}}(OG(2,2n))$ nor $S^{\texttt{zero}}(OG(2,2n))$ are polytopal.

If $n=4$ then $c_{\lambda,\mu}^{\nu}(OG(2,8))\neq 0$ for $\nu\in\{(2,0,2,0,0),(2,2,0,0,0)\}$ while we have $c_{\lambda,\mu}^{\nu}(OG(2,8))=0$ for $\nu=(2,1,1,0,0)$. Thus $S^{\texttt{nonzero}}(OG(2,8))$ is not polytopal. If instead $\lambda=(2,0,0,0,0)$ and $\mu=(1,0,1,0,0)$, we have $c_{\lambda,\mu}^{\nu}(OG(2,8))=0$ for $\nu\in\{(2,0,2,0,0),(2,2,0,0,0)\}$ while $c_{\lambda,\mu}^{\nu}(OG(2,8))\neq 0$ for $\nu=(2,1,1,0,0)$. Thus we see $S^{\texttt{zero}}(OG(2,8))$ is not polytopal. □

Example 3.2.14. *We now give some alternative vector descriptions of RYDs and show that polytopality is not achieved in $OG(2,10)$.*

One could choose to identify λ with the vector in $\mathbb{Z}^{2n-3}$ whose first $n-2$ coordinates are the columns of the bottom layer of λ, second $n-2$ coordinates are the columns of the top layer, and whose last coordinate is 1 if $\lambda=\langle\overline{\lambda}|\bullet\rangle$ and 0 otherwise. Consider $OG(2,10)$ and let $\lambda=(2,0,0,0,0,0,0)$, $\mu=(2,2,0,0,0,0,0)$. Then $c_{\lambda,\mu}^{\nu}(OG(2,10))=0$ for $\nu=(2,2,1,1,0,0,0)$, while $c_{\lambda,\mu}^{\nu}(OG(2,10))\neq 0$ for $\nu\in\{(2,2,2,0,0,0,0),(2,2,0,2,0,0,0)\}$.

Suppose instead we use the flattening process to identify λ with the vector in $\mathbb{Z}^4$ whose first coordinate is $\overline{\lambda}_1$, second coordinate is $\overline{\lambda}_2$, third coordinate is 1 if $\lambda=\langle\overline{\lambda}|\bullet\rangle$ and 0 otherwise, and whose fourth coordinate is 1 if λ is up, -1 if λ is down, and 0 if λ is neutral. Consider $OG(2,10)$ and let $\lambda=(3,0,0,1)$, $\mu=(3,0,0,-1)$.

Then $c^{\nu}_{\lambda,\mu}(OG(2,10)) \neq 0$ *for* $\nu \in \{(6,0,0,0),(4,2,0,0)\}$*, while* $c^{\nu}_{\lambda,\mu}(OG(2,10)) = 0$ *for* $\nu = (5,1,0,0)$.

3.3 The exceptional types

Type F_4

The adjoint variety corresponds to the fourth simple root while the coadjoint variety corresponds to the first. First, we consider the adjoint case.

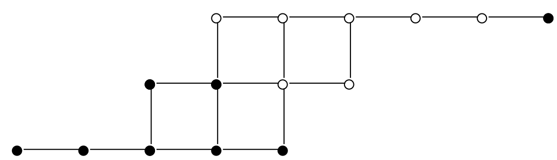


Figure 3.8: An RYD in Λ_{F_4/P_4}.

For Λ_{F_4/P_4}, the short roots consist of the third root (from the left) in the bottom row, all roots in the middle row, and the third root (from the left) in the top row. (See Figure 3.8.)

Define the partition-like description of an RYD in Λ_{F_4/P_4} by associating $\lambda = \langle \overline{\lambda} | \circ \rangle$ with the vector $(\overline{\lambda}_1, \overline{\lambda}_2, \overline{\lambda}_3, 0) \in \mathbb{Z}^4$ and $\lambda = \langle \overline{\lambda} | \bullet \rangle$ with $(\overline{\lambda}_1, \overline{\lambda}_2, \overline{\lambda}_3, 1)$. Here $\overline{\lambda}_1, \overline{\lambda}_2$ and $\overline{\lambda}_3$ are the number of roots used in the bottom, middle and top rows of Λ_{F_4/P_4} respectively. Let $\overline{\lambda}_4$ be the fourth coordinate. So for example, the displayed RYD has associated vector $(\overline{\lambda}_1, \overline{\lambda}_2, \overline{\lambda}_3, \overline{\lambda}_4) = (5,2,0,1)$ and has three short roots. The F_4 case of Theorem A is obtained by computer calculation:

Fact 3.3.1. $c^{\nu}_{\lambda,\mu}(F_4/P_4) \neq 0$ *if and only if*

$$\begin{aligned} |\nu| &= |\lambda| + |\mu| \\ \overline{\lambda}_1 + \overline{\lambda}_4 &\leq 6 - \overline{\mu}_3 - \overline{\mu}_4 \\ \overline{\lambda}_2 + \overline{\lambda}_4 &\leq 5 - \overline{\mu}_2 - \overline{\mu}_4 \\ \overline{\lambda}_3 + \overline{\lambda}_4 &\leq 6 - \overline{\mu}_1 - \overline{\mu}_4 \end{aligned}$$

$$\begin{aligned} \overline{\lambda}_i + \overline{\lambda}_4 &\leq \overline{\nu}_i + \overline{\nu}_4 \quad (\textit{for } 1 \leq i \leq 3) \\ \overline{\mu}_i + \overline{\mu}_4 &\leq \overline{\nu}_i + \overline{\nu}_4 \quad (\textit{for } 1 \leq i \leq 3) \\ \overline{\lambda}_1 + \overline{\mu}_1 - \overline{\nu}_3 &\leq 9. \end{aligned}$$

We do *not* have an isomorphism between Λ_{F_4/P_4} and Λ_{F_4/P_1}: see Figure 3.9.

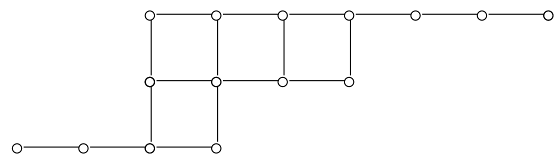


Figure 3.9: The poset Λ_{F_4/P_1}.

However, there is still a natural correspondence of $\mathbb{Y}_{F_4/P_4}$ with $\mathbb{Y}_{F_4/P_1}$: given a reduced word $s_{i_1} \cdots s_{i_\ell}$ for an element of W^{P_4}, then $s_{5-i_1} \cdots s_{5-i_\ell}$ is a reduced word of an element of W^{P_1}. If $\lambda \in \mathbb{Y}_{F_4/P_4}$ is the RYD associated to the first reduced word, we may declare it to be the RYD indexing the Schubert class of $H^\star(F_4/P_1)$ associated to the second reduced word. Thus when we write $c^\nu_{\lambda,\mu}(F_4/P_1)$ we refer to the proxy RYDs from $\mathbb{Y}_{F_4/P_4}$.

The F_4 case of Theorem B is obtained by computer calculation:

Fact 3.3.2. $c^\nu_{\lambda,\mu}(F_4/P_1) = 2^{\mathrm{sh}(\lambda)+\mathrm{sh}(\mu)-\mathrm{sh}(\nu)} c^\nu_{\lambda,\mu}(F_4/P_4)$.

Type G_2

Both the adjoint Λ_{G_2/P_2} and coadjoint Λ_{G_2/P_1} are a chain of five elements, with the maximal element being the adjoint root. Both $\mathbb{Y}_{G_2/P_2}$ and $\mathbb{Y}_{G_2/P_1}$ have six elements, one each of size k for $0 \le k \le 5$. We identify each element of $\mathbb{Y}_{G_2/P_1}$ with the element of $\mathbb{Y}_{G_2/P_2}$ having the same size, and compute using the elements of $\mathbb{Y}_{G_2/P_2}$. The short roots of Λ_{G_2/P_2} are the middle two nonadjoint roots. The G_2 case of Theorems A and B is obtained by computer calculation:

Fact 3.3.3. *If the triple $(\lambda,\mu,\nu) \neq (\langle\overline{\lambda}|\circ\rangle, \langle\overline{\mu}|\circ\rangle, \langle\overline{\nu}|\bullet\rangle)$, then $c^\nu_{\lambda,\mu}(G_2/P_1) = c^{\overline{\nu}}_{\overline{\lambda},\overline{\mu}}(Gr_1(\mathbb{C}^5))$. Otherwise, $c^{\langle\overline{\nu}|\bullet\rangle}_{\langle\overline{\lambda}|\circ\rangle,\langle\overline{\mu}|\circ\rangle}(G_2/P_1) = 2 \cdot c^{\overline{\nu}^+}_{\overline{\lambda},\overline{\mu}}(Gr_1(\mathbb{C}^6))$, where $\overline{\nu}^+$ is $\overline{\nu}$ with one additional root. Also,*

$$c^\nu_{\lambda,\mu}(G_2/P_2) = 3^{\mathrm{sh}(\nu)-\mathrm{sh}(\lambda)-\mathrm{sh}(\mu)} c^\nu_{\lambda,\mu}(G_2/P_1).$$

Moreover, $c^\nu_{\lambda,\mu}(G_2/P_2) \neq 0$ if and only if $|\nu| = |\lambda| + |\mu|$.

Type E_n series

For E_6, the adjoint variety corresponds to the second simple root, while for E_7 the adjoint variety corresponds to the first simple root. Figure 3.10 shows an example of an RYD in Λ_{E_6/P_2} and an RYD in Λ_{E_7/P_7}. In both cases, the adjoint root is the rightmost root.

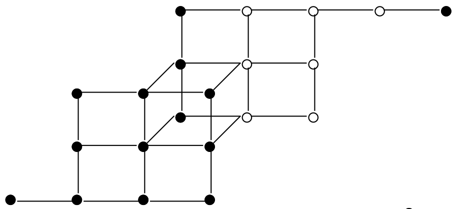
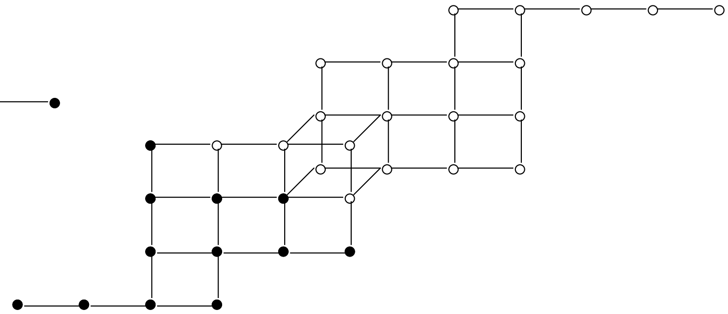
 

Figure 3.10: An RYD in Λ_{E_6/P_2} and an RYD in Λ_{E_7/P_7}.

For E_8, the adjoint variety corresponds to the eighth simple root. Figure 3.11 shows an example of an RYD in Λ_{E_8/P_8}. The adjoint root is the rightmost one.

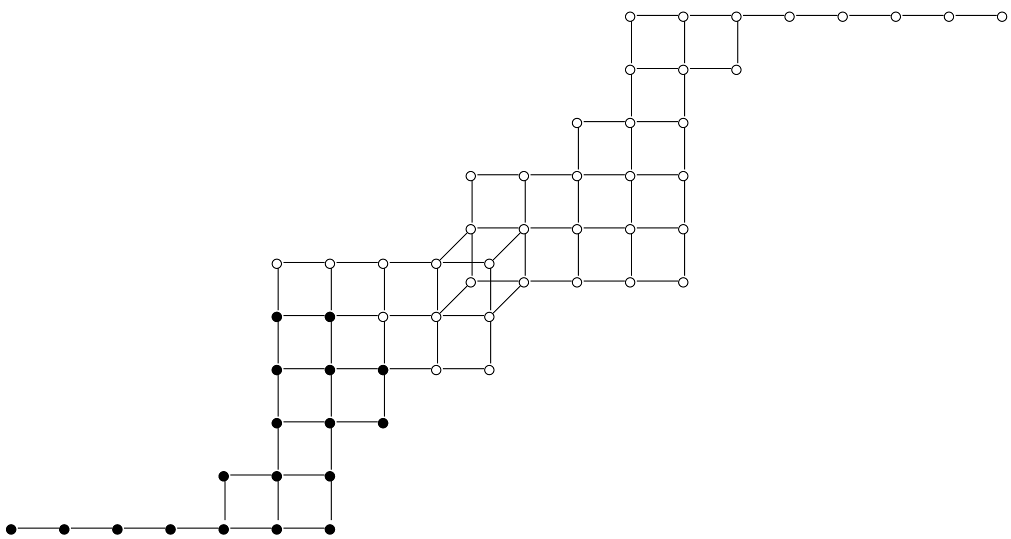


Figure 3.11: An RYD in Λ_{E_8/P_8}.

Proof of Theorem A in type E_6: Our partition-like description for $G/P = E_6/P_2$ identifies an RYD λ with a vector in $\mathbb{Z}^7$. The first three coordinates describe the number of roots used in each row on the "bottom layer" of Λ_{E_6/P_2}, the second three similarly describe the second layer, and the last coordinate indicates use of the adjoint root. For example, the displayed RYD for E_6/P_2 above is encoded as $(4,3,3,1,1,1,1)$.

Fact 3.3.4. *Let* $\lambda = (4,0,0,0,0,0,0)$ *and* $\mu = (3,2,0,1,0,0,0)$*. Then*

$$c^{\nu}_{\lambda,\mu}(E_6/P_2) \neq 0 \quad \textit{for } \nu \in \{(4,3,0,3,0,0,0),(4,3,2,1,0,0,0)\} \textit{ and}$$

$$c^{\nu}_{\lambda,\mu}(E_6/P_2) = 0 \textit{ for } \nu \in \{(4,3,1,2,0,0,0),(4,3,3,0,0,0,0)\}.$$

The four ν's above are collinear in the partition-like description. Since λ and μ are the same in each four cases, this yields four collinear triples. These triples alternate between being in $S^{\texttt{nonzero}}(E_6/P_2)$ and in $S^{\texttt{zero}}(E_6/P_2)$. This implies these embeddings of $S^{\texttt{nonzero}}(E_6/P_2)$ and $S^{\texttt{zero}}(E_6/P_2)$ are not polytopal. $\square$

Proof of Theorem A in type E_7: For $G/P = E_7/P_1$ our partition-like description identifies RYDs λ with vectors in $\mathbb{Z}^9$. The first four coordinates describe the number of roots used in each row on the "bottom layer" of Λ_{E_7/P_1}, the second four similarly describe the second layer, and the last coordinate indicates use of the adjoint root. Thus, for example the E_7/P_1 RYD above is $(4,4,3,1,0,0,0,0,0)$.

Fact 3.3.5. *Let $\lambda = \mu = (4,4,0,0,0,0,0,0,0)$. Then*

$$c^{\nu}_{\lambda,\mu}(E_7/P_1) \neq 0 \quad \textit{for } \nu \in \{(4,4,4,0,4,0,0,0,0),(4,4,4,2,2,0,0,0,0)\} \textit{ and}$$

$$c^{\nu}_{\lambda,\mu}(E_7/P_1) = 0 \textit{ for } \nu \in \{(4,4,4,1,3,0,0,0,0),(4,4,4,3,1,0,0,0,0)\}.$$

Thus the embeddings of $S^{\texttt{nonzero}}(E_7/P_1)$ and $S^{\texttt{zero}}(E_7/P_1)$ are not polytopal. $\square$

Proof of Theorem A in type E_8: For $G/P = E_8/P_8$, we identify RYDs λ with vectors in $\mathbb{Z}^{13}$. The first six coordinates describe the number of roots used in each row on the "bottom layer" of Λ_{E_8/P_8}, the second six describe the second layer, and the last coordinate indicates use of the adjoint root.

Fact 3.3.6. *Let $\nu = (7,3,3,5,5,0,5,0,0,0,0,0,0)$. Then*

$$c^{\nu}_{(1,0,0,0,0,0,0,0,0,0,0,0,0),(7,3,3,5,5,0,4,0,0,0,0,0,0)}(E_8/P_8) \neq 0,$$

$$c^{\nu}_{(5,0,0,0,0,0,0,0,0,0,0,0,0),(7,3,3,5,5,0,0,0,0,0,0,0,0)}(E_8/P_8) \neq 0,$$

and

$$c^{\nu}_{(7,2,0,0,0,0,0,0,0,0,0,0,0),(7,3,3,3,3,0,0,0,0,0,0,0,0)}(E_8/P_8) \neq 0.$$

However,

$$c^{\nu}_{(5,1,0,0,0,0,0,0,0,0,0,0,0),(7,3,3,4,4,0,1,0,0,0,0,0,0)}(E_8/P_8) = 0.$$

Note that the λ vector for the last coefficient is a convex combination of the corresponding vectors of the first three coefficients. That is:

$$(5,1,0,0,0,0,0,0,0,0,0,0,0,0) = \frac{1}{4}(1,0,0,0,0,0,0,0,0,0,0,0,0,0) + \frac{1}{4}(5,0,0,0,0,0,0,0,0,0,0,0,0,0) + \frac{1}{2}(7,2,0,0,0,0,0,0,0,0,0,0,0,0).$$

Similarly, the μ and (obviously) ν vector of the last coefficient is a convex combination of the corresponding vectors of the other coefficients, with the same parameters $\frac{1}{4}, \frac{1}{4}, \frac{1}{2}$. Therefore the convex hull of the points $S^{\texttt{nonzero}}(E_8/P_8)$ contains a point of $S^{\texttt{zero}}(E_8/P_8)$ and hence no polytopal description of $S^{\texttt{nonzero}}(E_8/P_8)$ is possible with this partition-like description.

Also, let $\lambda = (1,0,0,0,0,0,0,0,0,0,0,0,0,0)$ and $\mu = (7,3,3,5,5,1,3,0,0,0,0,0,0,0)$. Then

$$c^{\nu}_{\lambda,\mu}(E_8/P_8) = 0 \text{ for } \nu \in \{(7,3,3,5,5,3,2,0,0,0,0,0,0,0),(7,3,3,5,5,0,5,0,0,0,0,0,0,0)\} \text{ and}$$

$$c^{\nu}_{\lambda,\mu}(E_8/P_8) \neq 0 \text{ for } \nu = (7,3,3,5,5,1,4,0,0,0,0,0,0,0)$$

thus the embedding of $S^{\texttt{zero}}(E_8/P_8)$ is also not polytopal. □

Even in E_8 one can compute all vectors in $\mathbb{Z}^{39}$ that correspond to both feasible and infeasible Schubert triple intersections. With this data one can use a solver on a linear program defined by a relatively large matrix to find the vectors of the fact above. This helps automate demonstrating non-convexity for other descriptions of $S^{\texttt{nonzero}}(E_8/P_8)$.

Notice that all of the counterexamples to convexity we have given occur when $|\lambda| + |\mu| = \frac{|\Lambda_{G/P}|-1}{2}$.

Chapter 4

RYDs and the Belkale-Kumar product

4.1 Preliminaries

In this chapter, we characterize the subsets of $\Lambda_{GL_n/P}$ that are RYDs, and give an RYD formula for the Belkale-Kumar structure constants ([4]) on $H^\star(GL_n/P)$, after [35]. We extend this formula to give rules for the Belkale-Kumar product for the classical (co)adjoint varieties. The results for $G = GL_n$ appear in [59].

Fix a generalized flag variety G/P and let $\lambda, \mu, \nu \in \mathbb{Y}_{G/P}$. We write a definition of the **Belkale-Kumar product** in RYD language, cf. the definitions in [4], [21]. For each $\alpha \notin \Delta_P$, introduce a complex variable t_α. For a positive root β, define a monomial

$$t^\beta = \prod_{\alpha \notin \Delta_P} t_\alpha^{n_{\alpha\beta}}$$

and let

$$F_\lambda(t) = \prod_{\beta \in \lambda} t^\beta.$$

Then define

$$\sigma_\lambda \odot_t \sigma_\mu = \sum_\nu \frac{F_\nu(t)}{F_\lambda(t)F_\mu(t)} c_{\lambda,\mu}^\nu \sigma_\nu.$$

Then the Belkale-Kumar product $\odot_0$ is obtained by evaluating each t_α to 0.

The Belkale-Kumar product has a geometric interpretation. Suppose $u, v, w \in W^P$ with $c_{u,v}^w \neq 0$. The parabolic subgroup P may be written as a semidirect product of its unipotent radical and a reductive subgroup L called the **Levi subgroup** of P. The triple (u, v, w) is said to be **Levi-movable** ([4]) if the intersection $(l_1 \cdot u^{-1}X_u \cap l_2 \cdot v^{-1}X_v \cap l_3 \cdot (w^\vee)^{-1}X_{w^\vee})$ is finite and transverse at eP for generic $l_1, l_2, l_3 \in L$. Then the structure constants $b_{u,v}^w$ of the Belkale-Kumar product are

$$b_{u,v}^w = \begin{cases} c_{u,v}^w & \text{if } (u,v,w) \text{ is Levi-movable} \\ 0 & \text{otherwise.} \end{cases}$$

The Belkale-Kumar product was used in [4] to study eigencones of compact Lie groups, generalizing the type A Horn problem. The inequalities in [4] characterizing eigencones in general type were shown to be irredundant in [51]. More details can be found in [4]; we also learned much of the background from [55].

Let $\lambda, \mu, \nu \in \mathbb{Y}_{G/P}$ be the RYDs associated to $u, v, w \in W^P$. The following general-type combinatorial characterization of Levi-movability is a straightforward consequence of [53, Proposition 2.4]. We will give an independent proof.

Proposition 4.1.1. *The triple (λ, μ, ν) is Levi-movable if and only if $c_{\lambda,\mu}^{\nu} \neq 0$ and for every region R of $\Lambda_{G/P}$, the RYDs satisfy $|\lambda_R| + |\mu_R| = |\nu_R|$.*

Example 4.1.2. *Let $G/P = SO_{10}/P_{2,3}$. A triple (λ, μ, ν) of RYDs, along with the regional decomposition for $\Lambda_{SO_{10}/P_{2,3}}$ is shown in Figure 4.1. These RYDs satisfy $c_{\lambda,\mu}^{\nu} = 1$. By Proposition 4.1.1, the triple (λ, μ, ν) is Levi-movable.*

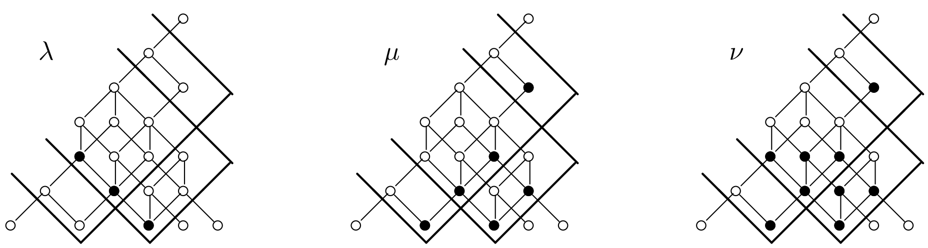


Figure 4.1: A Levi-movable triple of RYDs for $SO_{10}/P_{2,3}$.

Example 4.1.3. *Let $G/P = SO_{10}/P_{2,3}$. A triple (λ, μ, ν) of RYDs, along with the regional decomposition for $\Lambda_{SO_{10}/P_{2,3}}$ is shown in Figure 4.2. Although $c_{\lambda,\mu}^{\nu} \neq 0$, by Proposition 4.1.1 the triple (λ, μ, ν) is not Levi-movable, and thus $b_{\lambda,\mu}^{\nu} = 0$.*

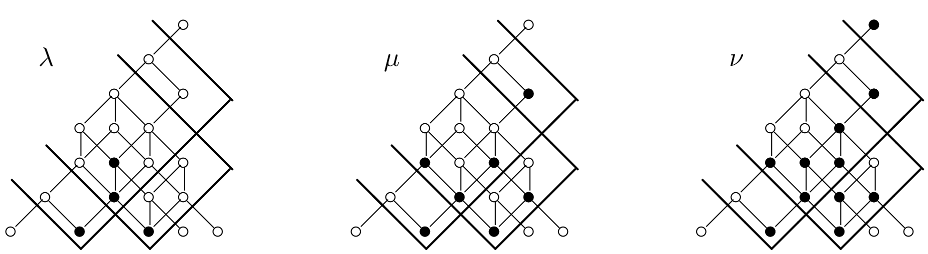


Figure 4.2: A triple of RYDs for $SO_{10}/P_{2,3}$ that is not Levi-movable.

Proof of Proposition 4.1.1. By definition, if $c_{\lambda,\mu}^{\nu} \neq 0$ then (λ, μ, ν) is Levi-movable exactly when the ratio $\frac{F_\nu(t)}{F_\lambda(t)F_\mu(t)}$ is equal to 1. Thus the "$\Leftarrow$" direction of Proposition 4.1.1 is obvious.

For the “ $\Rightarrow''$ direction, we will first need the following lemma. An **upper order ideal** in a poset Q is a set S such that if $x \prec y$ in Q and $x \in S$ then also $y \in S$.

Lemma 4.1.4. *Let Q be a poset. Suppose Q has $+$ and $-$ tokens stacked on its vertices subject to the following three conditions:*

(I) An equal number of $+$ and $-$ tokens is used,

(II) on each upper order ideal of Q there are at least as many $+$ tokens as $-$ tokens, and

(III) no vertex of Q has both types of token.

Then the tokens can be paired off in such a way that each $-$ token is paired with a $+$ token that is strictly above it in Q.

Proof. A *transversal* for a family F of finite sets is a set H and bijection $f : H \to F$ such that for each $h \in H$, h is an element of the set $f(h)$. Hall's Marriage Theorem [26] states

Theorem 4.1.5. *[26] A family F has a transversal if and only if for every subcollection $X \subset F$, we have $|X| \leq |\bigcup_{A \in X} A|$.*

For a given $-$ token in Q, let A_- denote the set of $+$'s appearing above it. Let S be any subset of the $-$ tokens. If $|S| > |\cup_{-\in S} A_-|$, then the upper order ideal generated by the vertices of S containing $-$'s would violate (II). Hence $|S| \leq |\cup_{-\in S} A_-|$ for all S, and the lemma follows from Hall's Marriage Theorem. □

Now suppose (λ, μ, ν) is Levi-movable. For each region R of $\Lambda_{G/P}$, let t^R denote the monomial t^β associated to a root β in R (by definition, t^β only depends on R). Construct a poset $\tilde{P}$ by letting the vertices of $\tilde{P}$ be the regions of $\Lambda_{G/P}$, where a vertex R covers a vertex R' if $\frac{t^R}{t^{R'}} = t_\alpha$ for some $\alpha \notin \Delta_P$.

For each vertex R, consider the value of $(\lambda, \mu, \nu)_R := |\nu_R| - |\lambda_R| - |\mu_R|$. Assign $(\lambda, \mu, \nu)_R$ tokens to each vertex R, with the sign of the tokens matching the sign of $(\lambda, \mu, \nu)_R$. Since (λ, μ, ν) is Levi-movable, we have $c_{\lambda,\mu}^{\nu} \neq 0$. We check this token apportionment satisfies the hypotheses of Lemma 4.1.4. Condition (III) is immediate, while (I) follows from $c_{\lambda,\mu}^{\nu} \neq 0$. Condition (II) follows from $c_{\lambda,\mu}^{\nu} \neq 0$ and the following theorem of K. Purbhoo (stated in RYD language):

Theorem 4.1.6. *[49, Theorem 2.1] Let $\lambda, \mu, \nu \in \mathbb{Y}_{G/P}$ and suppose S is an upper order ideal in Ω_G. If $|\lambda_S| + |\mu_S| > |\nu_S|$, then $c_{\lambda,\mu}^{\nu} = 0$.*

Thus by Lemma 4.1.4, the tokens can be paired off so that each $-$ token is paired with a $+$ token that is strictly above it in $\tilde{P}$. For a given pair $(+, -)$, let R_+ denote the region containing the $+$ and R_- the region containing the $-$. By definition,

$$\frac{F_\nu(t)}{F_\lambda(t)F_\mu(t)} = \prod_{\text{pairs }(+,-)} \frac{t^{R_+}}{t^{R_-}}.$$

Since R_+ is strictly above R_- in $\tilde{P}$, $\frac{d(R_+)}{d(R_-)}$ is a nonconstant monomial for each pair $(+,-)$. Thus if there are any tokens on $\tilde{P}$, we have $\frac{F_\nu(t)}{F_\lambda(t)F_\mu(t)} \neq 1$, contradicting Levi-movability. Hence there are no tokens on $\tilde{P}$, i.e., $|\lambda_R| + |\mu_R| = |\nu_R|$ for every region R of $\Lambda_{G/P}$. □

The following result of [4] can be easily seen from the RYD picture:

Proposition 4.1.7. *[4] For cominuscule G/P, the Belkale-Kumar product coincides with the ordinary cup product.*

Proof. By Proposition 4.1.1, the Belkale-Kumar product may be rewritten in terms of RYDs:

$$b^\nu_{\lambda,\mu} = \begin{cases} c^\nu_{\lambda,\mu} & \text{if } |\lambda_R| + |\mu_R| = |\nu_R| \text{ on all regions } R \text{ of } \Lambda_{G/P} \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 4.1.7 is then immediate since the cominuscule $\Lambda_{G/P}$ are exactly those $\Lambda_{G/P}$ with only one region. □

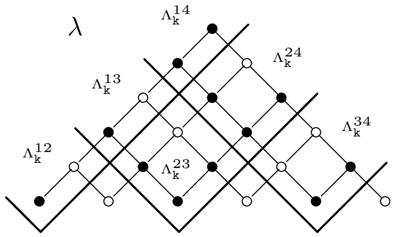
4.2 The Belkale-Kumar product for GL_n/P

Fix a set $\mathtt{k} = \{k_1, \ldots, k_{d-1}\}$ of integers satisfying $0 < k_1 < \ldots < k_{d-1} < n$. This set gives rise to the parabolic subgroup P defined by $\Delta \setminus \Delta_P = \{\alpha_{k_1}, \ldots, \alpha_{k_d}\}$. Let $GL_n/P = F_\mathtt{k} := Fl_{k_1,\ldots,k_{d-1};\mathbb{C}^n}$ denote the $(d-1)$-step partial flag variety in $\mathbb{C}^n$ associated to $\mathtt{k}$. The type A_{n-1} Weyl group is isomorphic to the symmetric group S_n. Write $w \in S_n$ in one-line notation (as in Example 4.2.1 below). We say w has a **descent** at position i if $w(i) > w(i+1)$. Concretely, the Schubert classes for $H^\star(F_\mathtt{k})$ are indexed by the set $S_n^\mathtt{k}$ which consists of the elements of S_n that have descents only in positions $k_1, \ldots, k_{d-1}$. Let $\mathbb{Y}_\mathtt{k}$ denote the set of RYDs for $F_\mathtt{k}$.

There are $\binom{d}{2}$ regions in the regional decomposition of $\Lambda_{F_\mathtt{k}}$. Let (a,b) denote the root $e_a - e_b \in \Omega_{GL_n}$. Let I_i denote the interval $[k_{i-1}+1, k_i]$ for $1 \leq i \leq d$, where we set $k_0 = 0$ and $k_d = n$. Then to each pair i, j with $1 \leq i < j \leq d$, we associate the region $\Lambda^{ij}_\mathtt{k} := I_i \times I_j \subset \Lambda_{F_\mathtt{k}}$.

Example 4.2.1. *Let $F_\mathtt{k} = Fl_{1,3,5;\mathbb{C}^7}$ and let $\lambda, \mu \in \mathbb{Y}_\mathtt{k}$ be associated to, respectively,* 5371624, $3462715 \in S_7^\mathtt{k}$*. Figure 4.3 shows λ and μ inside the regional decomposition of $\Lambda_{F_\mathtt{k}}$.*

Call S a $\mathtt{k}$**-diagram** if the roots in S form a lower order ideal in each region $\Lambda^{ij}_\mathtt{k}$, and also S satisfies a



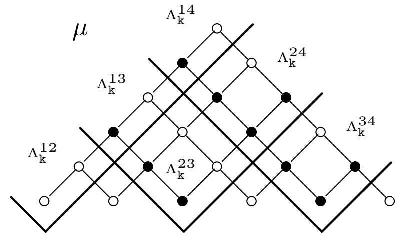


Figure 4.3: RYDs for $Fl_{1,3,5;\mathbb{C}^7}$, with the regional decomposition labelled.

hook condition: a root α must be in S (respectively, must not be in S) if more than half of the roots in Ω_{GL_n} diagonally south-east and south-west of α are in S (respectively, not in S). Let $\tilde{\mathbb{Y}}_{\mathtt{k}}$ denote the set of all $\mathtt{k}$-diagrams. We are not aware of any reference for the following:

Proposition 4.2.2. $\mathbb{Y}_{\mathtt{k}} = \tilde{\mathbb{Y}}_{\mathtt{k}}$.

Proof. Let C denote the set of all nonnegative integer vectors $c = (c_1, \ldots, c_{n-1})$ satisfying $c_j \leq n - j$. Let $C_{\mathtt{k}} \subset C$ denote the set of $c \in C$ such that for $1 \leq j < n$, $c_j > c_{j+1}$ only if j and $j+1$ are not in the same interval I_i (we set $c_n = 0$). For any permutation $w \in S_n$, its **code** is defined to be the vector $c_w \in C$ such that $(c_w)_i$ is the number of positions j satisfying $i < j$ and $w(i) > w(j)$. For example, if $n = 7$ and $\mathtt{k} = \{1, 3, 5, 6\}$ then $w = 5361742 \in S_7^{\mathtt{k}}$ has code $c_w = (4, 2, 3, 0, 2, 1) \in C_{\mathtt{k}}$.

Claim 4.2.3. *The map that takes $w \in S_n^{\mathtt{k}}$ to its code c_w is a bijection $S_n^{\mathtt{k}} \to C_{\mathtt{k}}$.*

Proof. It is well known (see, e.g., [44]) that this map is a bijection $S_n \to C$. Let $w \in S_n^{\mathtt{k}}$. Then by definition $w(j) < w(j+1)$ whenever $j, j+1$ are in the same interval I_i. Thus any entry in positions $j+2, \ldots, n$ that is smaller than $w(j)$ is also smaller than $w(j+1)$, implying $(c_w)_j \leq (c_w)_{j+1}$ and $c_w \in C_{\mathtt{k}}$. Conversely, if $c \in C_{\mathtt{k}}$ then $c_j \leq c_{j+1}$ whenever $j, j+1$ are in the same interval I_i. Then the number of entries of the corresponding permutation w after the jth position that are smaller than $w(j)$, is at most the number of entries of w after the $(j+1)$th position that are smaller than $w(j+1)$. This forces $w(j) < w(j+1)$, so $w \in S_n^{\mathtt{k}}$. $\square$

Given a subset $S \subset \Omega_{GL_n}$, define a nonnegative integer vector $h_S = (h_1, \ldots h_{n-1})$ by letting h_j be the number of roots of the form $(j, b) = e_j - e_b$ in S.

Claim 4.2.4. *The map that takes a $\mathtt{k}$-diagram θ to h_θ is an injection $\tilde{\mathbb{Y}}_{\mathtt{k}} \to C_{\mathtt{k}}$.*

Proof. By definition, $h_j \leq n - j$. The condition that the roots in θ form a lower order ideal in each region forces $h_j > h_{j+1}$ only if j and $j+1$ are not in the same interval I_i. So $h_\theta \in C_{\mathtt{k}}$.

To show injectivity, we will show that given $c \in C$, there is a unique $S \subset \Omega_{GL_n}$ satisfying both $h_S = c$ and the hook condition. We construct S by coloring a root of Ω_{GL_n} black if it is in S, and white if it is not in S. If $c_{n-1} = 0$ then we must color the root $(n-1, n)$ white, and if $c_{n-1} = 1$ we must color it black. Now proceed inductively. Fix $j < n-1$ and suppose all roots of the form (a, b) with $a > j$ have been colored white or black. Use the following procedure to color roots of the form (j, b) black one-by-one until h_j such roots have been colored black, at which point terminate the procedure and color all remaining such roots white:

> If there exists a root of the form (j, b) such that exactly half of the roots diagonally south-east and south-west of it are colored black, then color the highest such root black. Otherwise, color the lowest root of the form (j, b) black.

It is clear that each coloring of a root in the above procedure is forced by the hook condition. Therefore, since the elements of $\tilde{\mathbb{Y}}_{\mathtt{k}}$ satisfy the hook condition, the map $\tilde{\mathbb{Y}}_{\mathtt{k}} \to C_{\mathtt{k}}$ is injective. $\square$

Example 4.2.5. *Suppose* $c = (4, 2, 3, 0, 2, 1)$. *Then the unique* S *satisfying* $h_S = c$ *and the hook condition is shown in Figure 4.4, with the roots in* S *labelled according to the order in which they were colored black by the procedure of Claim 4.2.4.*

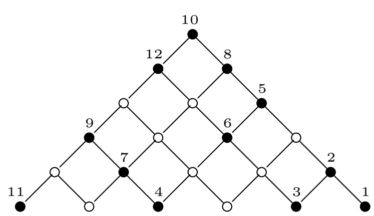


Figure 4.4: The procedure of Claim 4.2.4.

Claim 4.2.6. $\mathbb{Y}_{\mathtt{k}} \subseteq \tilde{\mathbb{Y}}_{\mathtt{k}}$.

Proof. We already know from Lemma 2.1.2 that λ is a lower order ideal in each region. It remains to show λ satisfies the hook condition. Consider any root $(a, b) \in \Omega_{GL_n}$. The hook associated to (a, b) is all roots (a, l) for $a < l < b$ and all roots (j, b) for $a < j < b$. If more than half of these are inverted by w, then there exists an m with $a < m < b$ such that $w(a) > w(m)$ and $w(m) > w(b)$, hence w must invert (a, b). Similarly, if fewer than half of the roots in the hook are inverted, then w cannot invert (a, b). Thus λ satisfies the hook condition. $\square$

Composing the injection from Claim 4.2.4 with the bijection of Claim 4.2.3 yields an injection $\tilde{\mathbb{Y}}_{\mathtt{k}} \to S_n^{\mathtt{k}}$. By definition $\mathbb{Y}_{\mathtt{k}}$ is in bijection with $S_n^{\mathtt{k}}$, thus we have an injection $\tilde{\mathbb{Y}}_{\mathtt{k}} \to \mathbb{Y}_{\mathtt{k}}$. By Claim 4.2.6, $\mathbb{Y}_{\mathtt{k}} \subseteq \tilde{\mathbb{Y}}_{\mathtt{k}}$, so $\mathbb{Y}_{\mathtt{k}} = \tilde{\mathbb{Y}}_{\mathtt{k}}$. $\square$

We now give an formula in terms of RYDs for the Belkale-Kumar structure constants for $H^\star(F_{\mathtt{k}})$. Our formula uses the jeu de taquin introduced in [58]. The following setup in terms of root posets is similar to that employed in [62]. Given a subset S of $\Lambda_{\mathtt{k}}^{ij}$, define a partial labelling T_S of $\Lambda_{\mathtt{k}}^{ij}$ by bijectively assigning each root in S a number from $\{1,\ldots,|S|\}$, subject to the condition that a root α receives a smaller number than a root α' whenever $\alpha \prec \alpha'$. Roots in $\Lambda_{\mathtt{k}}^{ij}$ that have no label will be called **unlabelled**. Let $\lambda,\mu,\nu \in \mathbb{Y}_{\mathtt{k}}$. Let ν/λ denote the set-theoretic difference of ν and λ, and call ν/λ a **skew** RYD.

Starting with a given labelling $T_{\nu_{ij}/\lambda_{ij}}$ of $\Lambda_{\mathtt{k}}^{ij}$, choose an unlabelled root α of $\Lambda_{\mathtt{k}}^{ij}$ which is maximal subject to the condition that some labelled root is above it. Among the labelled roots covering α, choose the root α' having the smallest label. Move its label to α, leaving α' unlabelled. Then find the labelled root covering α' with smallest label, and move its label to α'. Continue in this manner until a label is moved from a root that has no labelled root above it. Then, choose an unlabelled root of $\Lambda_{\mathtt{k}}^{ij}$, maximal such that some labelled root is above it and perform the same process. Repeat until there is no unlabelled root below a labelled root. Let $\mathtt{jdt}(T_{\nu_{ij}/\lambda_{ij}})$ denote the resulting partial labelling of $\Lambda_{\mathtt{k}}^{ij}$.

Fix a choice of labelling $T_{\mu_{ij}}$ of each $\Lambda_{\mathtt{k}}^{ij}$. Let $e_{\lambda_{ij},\mu_{ij}}^{\nu_{ij}}$ denote the number of labellings $T_{\nu_{ij}/\lambda_{ij}}$ of each $\Lambda_{\mathtt{k}}^{ij}$ satisfying $\mathtt{jdt}(T_{\nu_{ij}/\lambda_{ij}}) = T_{\mu_{ij}}$. Then the Belkale-Kumar coefficient $b_{\lambda,\mu}^{\nu}(F_{\mathtt{k}})$ is computed by taking the skew RYD ν/λ, performing the jeu de taquin algorithm independently on each region of $\Lambda_{F_{\mathtt{k}}}$, and multiplying the resulting numbers $e_{\lambda_{ij},\mu_{ij}}^{\nu_{ij}}$. In other words:

Theorem 4.2.7.

$$b_{\lambda,\mu}^{\nu}(F_{\mathtt{k}}) = \prod_{\text{regions } \Lambda_{\mathtt{k}}^{ij}} e_{\lambda_{ij},\mu_{ij}}^{\nu_{ij}}.$$

Example 4.2.8. *Let $n=7$ and $\mathtt{k}=\{3,6\}$. Then $F_{\mathtt{k}} = Fl_{3,6;\mathbb{C}^7}$, and* 1362475, 1462573, 3572461 $\in S_7^{\mathtt{k}}$. *Let (respectively) λ,μ,ν be the corresponding RYDs. Figure 4.5 shows a choice of labellings $\{T_{\mu_{ij}}\}$ of each $\Lambda_{\mathtt{k}}^{ij}$, followed by the two labellings $\{T_{\nu_{ij}/\lambda_{ij}}\}$ of each $\Lambda_{\mathtt{k}}^{ij}$ such that $\mathtt{jdt}(T_{\nu_{ij}/\lambda_{ij}}) = T_{\mu_{ij}}$ in each region $\Lambda_{\mathtt{k}}^{ij}$.*

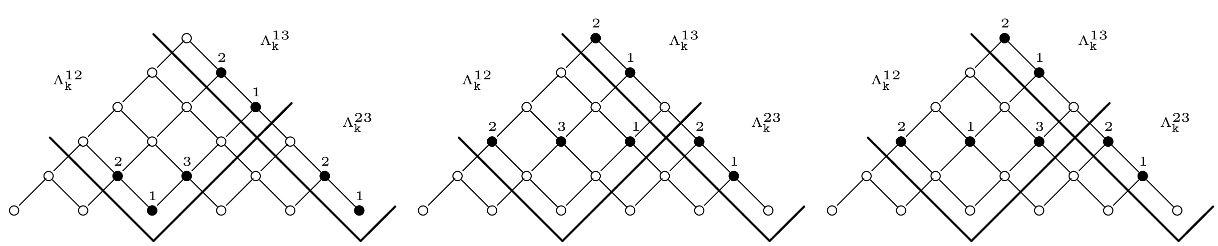


Figure 4.5: A labelling corresponding to μ and the two labellings corresponding to ν/λ that rectify to it.

The jeu de taquin algorithm yields $e^{\nu_{12}}_{\lambda_{12},\mu_{12}} = 2$, $e^{\nu_{13}}_{\lambda_{13},\mu_{13}} = 1$, $e^{\nu_{23}}_{\lambda_{23},\mu_{23}} = 1$, *hence*

$$b^{\nu}_{\lambda,\mu}(Fl_{3,6;\mathbb{C}^7}) = 2 \cdot 1 \cdot 1 = 2.$$

In contrast, for general Schubert structure constants not covered by Theorem 4.2.7 the regions are not independent. For example, let $n = 5$ and $\mathtt{k} = \{2, 4\}$.

Example 4.2.9. $\sigma_{12453} \cdot \sigma_{34125} = \sigma_{35142} + \sigma_{34251} + \sigma_{45123} \in H^\star(Fl_{2,4;\mathbb{C}^5})$. *Figure 4.6 shows this calculation using RYDs.*

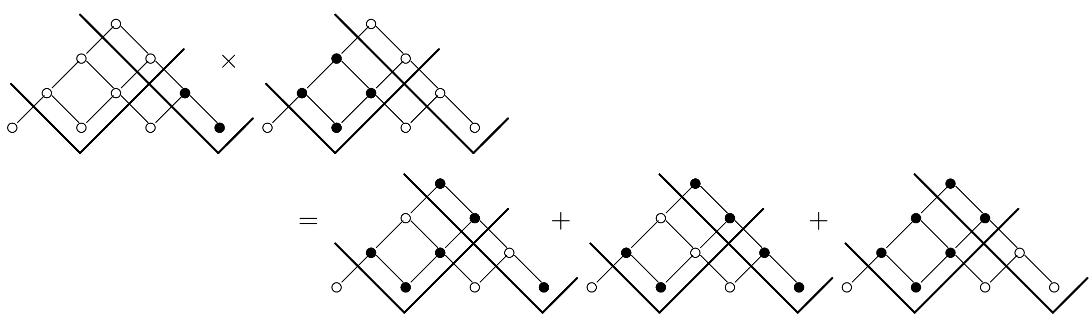


Figure 4.6: An RYD computation in $H^\star(Fl_{2,4;\mathbb{C}^5})$.

The RYDs for 12453 and 34125 use no roots from $\Lambda^{13}_{\mathtt{k}}$, but the RYDs for $35142, 34251$ and 45123 all use roots from this region. In particular, by Theorem 4.2.7 this immediately implies $\sigma_{12453} \odot_0 \sigma_{34125} = 0$.

Example 4.2.10. *For purposes of comparison, we compute the example of [35, Figure 2] in terms of RYDs. Let* $n = 5$ *and* $\mathtt{k} = \{2, 4\}$*; we use Theorem 4.2.7 to compute the structure constant* $c^{\nu}_{\lambda,\mu} = c^{35241}_{34152,13254}$. *Figure 4.7 shows the only possible set of labellings* $\{T_{\mu_{ij}}\}$ *of each* $\Lambda^{ij}_{\mathtt{k}}$*, and the only possible set of labellings* $\{T_{\nu_{ij}/\lambda_{ij}}\}$ *of each* $\Lambda^{ij}_{\mathtt{k}}$.

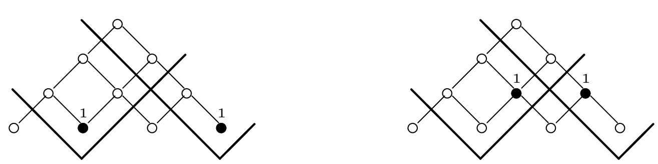


Figure 4.7: Labellings computing an example of [35].

Since $\mathtt{jdt}(T_{\nu_{ij}/\lambda_{ij}}) = T_{\mu_{ij}}$ *in each region* $\Lambda^{ij}_{\mathtt{k}}$*, we have* $b^{\nu}_{\lambda,\mu}(Fl_{2,4;\mathbb{C}^5}) = 1$.

Proof of Theorem 4.2.7. Let $r_i = |I_i| = k_i - k_{i-1}$. We now follow [35]. Let $G^{\mathtt{k}}_n$ denote the set of n-letter words τ from the alphabet $\{1, \ldots, d\}$, such that the letter i is used r_i times in τ. Then the Schubert varieties of $F_{\mathtt{k}}$ are indexed by the elements of $G_n^{\mathtt{k}}$. Define a map $f : G_n^{\mathtt{k}} \to S_n^{\mathtt{k}}$ by letting $f(\tau)$ be the permutation, in one-line notation, obtained by writing down the positions of the ones in order, then the positions of the twos in order, etc. For example, if $\mathtt{k} = \{3,5,6\}$ and $\tau = 2431121 \in G_7^{\mathtt{k}}$ then $f(\tau) = 4571632 \in S_7^{\mathtt{k}}$. This is a bijection, and the Schubert variety of $F_{\mathtt{k}}$ indexed by τ is equal to the Schubert variety indexed by $f(\tau)$. Given i, j with $1 \le i < j \le d$, let $D_{ij}(\tau)$ be the word obtained by deleting all letters of τ that are not i or j. Then $D_{ij}(\tau)$ indexes a Schubert variety in the Grassmannian $Gr_{r_i}(\mathbb{C}^{r_i+r_j})$.

Theorem 4.2.11. *[35, Theorem 3] Let $\tau, \pi, \rho \in G_n^{\mathtt{k}}$. Then*

$$b^{\rho}_{\tau,\pi}(F_{\mathtt{k}}) = \prod_{1 \le i < j \le d} c^{D_{ij}(\rho)}_{D_{ij}(\tau), D_{ij}(\pi)}(Gr_{r_i}(\mathbb{C}^{r_i+r_j})).$$

Now let $w \in S_n^{\mathtt{k}}$. Define $D'_{ij}(w)$ to be the permutation on $[1, \ldots, r_i + r_j]$ whose entries are in the same relative order as the entries of the word obtained by deleting all entries of w except those in I_i or I_j. For example, let $n = 7$, $\mathtt{k} = \{2,5\}$, and $w = 2614537 \in S_7^{\mathtt{k}}$. Then $D'_{13}(w) = 1324$, since deleting all entries of w except those in I_1 or I_3 yields 2637, which is in the same relative order as 1324. This process is the same as in [55, Definition 1], where it is noted this is also the flattening function of [9].

By definition, $D'_{ij}(w) \in S^{\{r_i\}}_{r_i+r_j}$. So $D'_{ij}(w)$ indexes a Schubert variety in the Grassmannian $Gr_{r_i}(\mathbb{C}^{r_i+r_j})$, and the RYD corresponding to $D'_{ij}(w)$ has only a single region inside $\Omega_{GL_{r_i+r_j}}$. We will denote this region Λ_{r_i, r_i+r_j}. Note that Λ_{r_i, r_i+r_j} is the subposet of $\Omega_{GL_{r_i+r_j}}$ consisting of all roots above the r_i'th simple root $e_{r_i} - e_{r_i+1}$.

Example 4.2.12. *Let $n = 7$ and $\mathtt{k} = \{2,5\}$. Then $r_1 = 2$, $r_2 = 3$ and $r_3 = 2$. Let $w = 2614537 \in S_7^{\mathtt{k}}$ and λ the corresponding RYD. Figure 4.8 shows λ and the RYDs for, respectively, $D'_{12}(w) = 25134 \in S_5^{\{2\}}$, $D'_{13}(w) = 1324 \in S_4^{\{2\}}$ and $D'_{23}(w) = 13425 \in S_5^{\{3\}}$.*

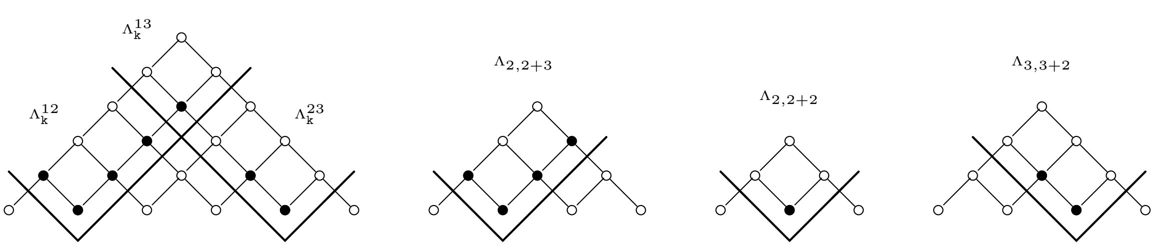


Figure 4.8: The RYD for $w = 2614537$ and the RYDs for each $D'_{ij}(w)$, $1 \le i < j \le 3$.

The following is clear from the definitions:

Lemma 4.2.13. *Let $\tau \in G_n^{\mathtt{k}}$. Then $D'_{ij}(f(\tau)) = f(D_{ij}(\tau))$.*

Now, let $\lambda, \mu, \nu \in \mathbb{Y}_{\mathtt{k}}$, respectively corresponding to permutations $u, v, w \in S_n^{\mathtt{k}}$. By Theorem 4.2.11 and Lemma 4.2.13, we have

$$b_{\lambda,\mu}^{\nu}(F_{\mathtt{k}}) = b_{u,v}^{w}(F_{\mathtt{k}}) = \prod_{1 \le i < j \le d} c_{D'_{ij}(u), D'_{ij}(v)}^{D'_{ij}(w)}(Gr_{r_i}(\mathbb{C}^{r_i + r_j})).$$

Straightforwardly, $\Lambda_{\mathtt{k}}^{ij} \subset \Omega_{GL_n}$ is isomorphic (as a poset) to Λ_{r_i, r_i+r_j}, and the roots in $\Lambda_{\mathtt{k}}^{ij}$ inverted by w correspond to the roots of Λ_{r_i, r_i+r_j} inverted by $D'_{ij}(w)$ (as depicted in Example 4.2.12). Jeu de taquin is known to compute the Schubert structure constants for Grassmannians (see [48] and [62] for this root-theoretic setting). Therefore, we have $c_{D'_{ij}(u), D'_{ij}(v)}^{D'_{ij}(w)}(Gr_{r_i}(\mathbb{C}^{r_i + r_j})) = e_{\lambda_{ij}, \mu_{ij}}^{\nu_{ij}}$. $\square$

4.3 The Belkale-Kumar product for classical-type (co)adjoint varieties

We obtain combinatorial rules for the Belkale-Kumar product in the classical-type (co)adjoint cases. For any $\overline{\nu} = (\overline{\nu}_1, \overline{\nu}_2) \in \mathbb{Z}^2$ let $\overline{\nu}^{\star} = (\overline{\nu}_1 - 1, \overline{\nu}_2)$ and $\overline{\nu}_{\star} = (\overline{\nu}_1, \overline{\nu}_2 - 1)$. Fix RYDs λ, μ for a classical-type adjoint variety and define an operator, cf. the operator $\mathbb{A}_{\lambda,\mu}$ from Chapter 3 for the (co)adjoint formulas:

$$\mathbb{B}_{\lambda,\mu}(\overline{\nu}) = \begin{cases} \sigma_{\langle \overline{\nu} | \bullet \rangle} & \text{if exactly one of } \lambda \text{ or } \mu \text{ is on} \\ \sigma_{\langle \overline{\nu} | \circ \rangle} & \text{if } |\lambda| + |\mu| \le \frac{|\Lambda_{G/P}| - 1}{2} \\ 0 & \text{otherwise.} \end{cases}$$

that is, replacing the fourth case of $\mathbb{A}_{\lambda,\mu}$ with 0.

Theorem 4.3.1. *For classical-type adjoint varieties, the Belkale-Kumar product is given by replacing $\mathbb{A}_{\lambda,\mu}$ with $\mathbb{B}_{\lambda,\mu}$ in the Schubert calculus formulas of Proposition 3.2.1, Theorem 3.2.5, Theorem 3.2.11.*

Proof. This follows by combining Proposition 4.1.1 with the RYD formulas of Proposition 3.2.1, Theorem 3.2.5, and Theorem 3.2.11 for Schubert calculus of classical adjoint varieties. $\square$

For the type A adjoint variety $Fl_{1,n-1;n}$, Theorem 4.3.1 recovers the adjoint case of Theorem 4.2.7. For the other types, these formulas extend Theorem 4.2.7 in the sense that they may also be expressed as a product of structure constants on independently-considered regions.

The remaining classical adjoint variety is $LG(1, 2n)$. In this case, the rule for the Belkale-Kumar product is obtained in the same manner; set to zero all Schubert structure constants $c_{\lambda,\mu}^{\nu}$ where ν uses the adjoint root and λ, μ do not. The coadjoint (non-adjoint) partner of $LG(1, 2n)$ is $OG(1, 2n+1)$, but this variety is cominuscule and so the Belkale-Kumar product is equal to the cup product in this case (Proposition 4.1.7).

The operator $\mathbb{B}_{\lambda,\mu}$ does *not* give a rule for $\odot_0$ for the remaining classical-type coadjoint (non-adjoint) variety $LG(2, 2n)$. The formula of Theorem 3.2.5 for Schubert calculus of $H^\star(LG(2, 2n))$ is stated in terms of the RYDs in $\Lambda_{OG(2,2n+1)}$. However, the embedding of $H^\star(Sp_{2n}/B)$ into $H^\star(SO_{2n+1}/B)$ used in Proposition 3.2.4 to relate the rule for $H^\star(OG(2, 2n+1))$ to the rule for $H^\star(LG(2, 2n))$ does not preserve Levi-movability on triples of Schubert varieties.

Example 4.3.2. *The Weyl groups of Sp_6 and SO_7 are isomorphic to the group of signed permutations on three letters; let W denote this group. Let $u = (1, -3, 2)$, $v = (2, -3, 1)$, $w = (3, -1, 2)$ in W^{P_2}. Figure 4.9 shows the associated RYDs $\lambda, \mu, \nu \subset \Lambda_{LG(2,2n)}$ and $\hat{\lambda}, \hat{\mu}, \hat{\nu} \subset \Lambda_{OG(2,2n+1)}$, with the regional decomposition in each case. Here $c^{\nu}_{\lambda,\mu} = 2$. By Proposition 4.1.1 the triple (λ, μ, ν) is Levi-movable, but the triple $(\hat{\lambda}, \hat{\mu}, \hat{\nu})$ is not.*

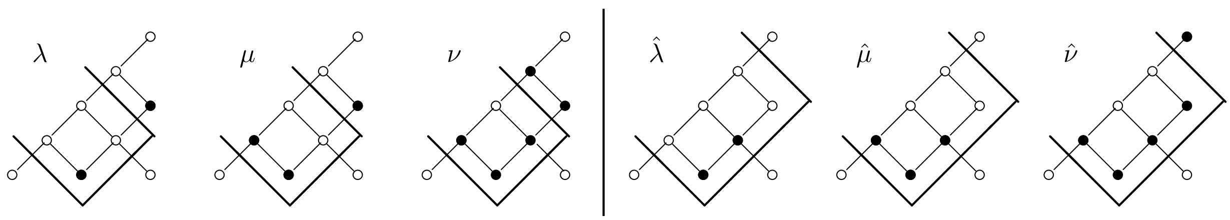


Figure 4.9: (λ, μ, ν) (type C_3) is Levi-movable, but $(\hat{\lambda}, \hat{\mu}, \hat{\nu})$ (type B_3) is not.

We present a combinatorial rule for the Belkale-Kumar product on $H^\star(LG(2, 2n))$, by using the RYDs in $\Lambda_{LG(2,2n)}$. This rule also extends Theorem 4.2.7. Let $e^{\nu}_{\lambda,\mu}$ (respectively, $\tilde{e}^{\nu}_{\lambda,\mu}$) denote the number obtained by performing jeu de taquin on the lower (respectively, upper) region of $\Lambda_{LG(2,2n)}$. In the upper region, we use the (co)minuscule jeu de taquin from [62]. Let $\mathrm{sr}(\lambda)$ denote the number of short roots *in the upper region* of $\Lambda_{LG(2,2n)}$ used by λ.

Theorem 4.3.3. $b^{\nu}_{\lambda,\mu}(LG(2, 2n)) = 2^{\mathrm{sr}(\nu)-\mathrm{sr}(\lambda)-\mathrm{sr}(\mu)} e^{\nu}_{\lambda,\mu} \tilde{e}^{\nu}_{\lambda,\mu}$.

Proof. Let (λ, μ, ν) be the RYDs representing a triple $(X_\lambda, X_\mu, X_{\nu^\vee})$ of Schubert varieties of $LG(2, 2n)$. We may assume the triple satisfies $|\lambda_R| + |\mu_R| = |\nu_R|$ on each region R of $\Lambda_{LG(2,2n)}$, since otherwise it is clear that $e^{\nu}_{\lambda,\mu} = \tilde{e}^{\nu}_{\lambda,\mu} = 0$. Let $(\hat{\lambda}, \hat{\mu}, \hat{\nu})$ be the corresponding triple of RYDs in $\Lambda_{OG(2,2n+1)}$. Then $b^{\nu}_{\lambda,\mu}$ is computed by applying the $LG(2, 2n)$ Schubert calculus rule to the triple $(\hat{\lambda}, \hat{\mu}, \hat{\nu})$. Now,

Lemma 4.3.4. *Suppose the triple (λ, μ, ν) satisfies $|\lambda_R| + |\mu_R| = |\nu_R|$ on each region R of $\Lambda_{LG(2,2n)}$. Then $c^{\nu}_{\lambda,\mu} \neq 0 \iff e^{\nu}_{\lambda,\mu} \neq 0$ and $\tilde{e}^{\nu}_{\lambda,\mu} \neq 0$.*

Proof. The upper region is a chain, so by the (co)minuscule jeu de taquin of [62], $|\lambda_R| + |\mu_R| = |\nu_R|$ implies $\tilde{e}^{\nu}_{\lambda,\mu} = 1$. It is easy to check that the Grassmannian Horn inequalities for (λ, μ, ν) on the lower region of $\Lambda_{LG(2,2n)}$ are satisfied (i.e., $e^{\nu}_{\lambda,\mu} \neq 0$) if and only if $(\hat{\lambda}, \hat{\mu}, \hat{\nu})$ satisfies the Horn-style inequalities of Corollary 3.2.8 (i.e., $c^{\nu}_{\lambda,\mu} \neq 0$). $\square$

It is a straightforward consequence of Theorem 3.2.5 that $c^{\nu}_{\lambda,\mu} \in \{0, 1, 2\}$. Now assume (λ, μ, ν) is Levi-movable (in particular, $c^{\nu}_{\lambda,\mu} \neq 0$). Then clearly $e^{\nu}_{\lambda,\mu} = \tilde{e}^{\nu}_{\lambda,\mu} = 1$. Evidently, $2^{\mathrm{sr}(\nu)-\mathrm{sr}(\lambda)-\mathrm{sr}(\mu)} = 2$ if and only if both λ, μ use one root of the upper region of $\Lambda_{LG(2,2n)}$ and ν uses two.

Lemma 4.3.5. *Suppose (λ, μ, ν) is Levi-movable. Then $c^{\nu}_{\lambda,\mu} = 2$ if and only if both λ, μ use one root of the upper region of $\Lambda_{LG(2,2n)}$ and ν uses two.*

Proof. Suppose $c^{\nu}_{\lambda,\mu} = 2$. By Theorem 3.2.5 for the Schubert calculus of $LG(2, 2n)$, this implies $\hat{\nu}$ uses the top root of $\Lambda_{OG(2,2n+1)}$, while $\hat{\lambda}$, $\hat{\mu}$ do not. Then ν uses at least two roots of the upper region of $\Lambda_{LG(2,2n)}$ and each of λ, μ uses at most one. But since (λ, μ, ν) is Levi-movable, the number of roots used by λ and μ on the upper region is equal to the number of roots used by ν, hence both λ, μ use one root of the upper region and ν uses two.

Now suppose both λ, μ use one root of the upper region and ν uses two. Then $\hat{\nu}$ uses the top root of $\Lambda_{OG(2,2n+1)}$, while $\hat{\lambda}$, $\hat{\mu}$ do not. Moreover, all three of $\hat{\lambda}$, $\hat{\mu}$, $\hat{\nu}$ use exactly one short root of $\Lambda_{OG(2,2n+1)}$. By the Schubert calculus rule for $OG(2, 2n+1)$, this implies $c^{\hat{\nu}}_{\hat{\lambda},\hat{\mu}}(OG(2, 2n+1)) = 1$. Since $2^{\mathrm{sr}(\hat{\nu})-\mathrm{sr}(\hat{\lambda})-\mathrm{sr}(\hat{\mu})} = \frac{1}{2}$ in this calculation, we have $c^{\nu}_{\lambda,\mu} = 2$. $\square$

Theorem 4.3.3 then follows immediately from Lemmas 4.3.4 and 4.3.5. $\square$

Corollary 4.3.6. *A Belkale-Kumar structure constant of $H^{\star}(LG(2, 2n))$ decomposes as a product of a Schubert structure constant of $H^{\star}(Gr_2(\mathbb{C}^{n+1}))$ and a Schubert structure constant of $H^{\star}(LG(2, 4))$.*

Proof. The number $e^{\nu}_{\lambda,\mu}$ is a Schubert structure constant of $H^{\star}(Gr_2(\mathbb{C}^{n+1}))$. By the RYD rule of [62] for $H^{\star}(LG(n, 2n))$, $2^{\mathrm{sr}(\nu)-\mathrm{sr}(\lambda)-\mathrm{sr}(\mu)}\tilde{e}^{\nu}_{\lambda,\mu}$ is a Schubert structure constant of $H^{\star}(LG(2, 4))$. $\square$

Unfortunately, Corollary 4.3.6 does not generalize in the obvious way to $LG(k, 2n)$. There are counterexamples even for G/P where P is maximal.

Example 4.3.7. *For $LG(4, 12)$, consider*

$$u = (1, 2, 5, -4, 3, 6),\ v = (1, 4, -6, -3, 2, 5),\ w = (3, 6, -4, -1, 2, 5) \in W^{P_4}.$$

We have $c^{w}_{u,v} = 8$, but $e^{w}_{u,v} = 3$, which does not divide 8.

The proof of Lemma 4.3.5 yields the following relationship between the Schubert structure constants $c^{\nu}_{\lambda,\mu}(LG(2,2n))$ where (λ,μ,ν) is Levi-movable, and Levi-movability of the corresponding triple $(\hat{\lambda},\hat{\mu},\hat{\nu})$ for $OG(2,2n+1)$.

Corollary 4.3.8. *Suppose (λ,μ,ν) is Levi-movable for $LG(2,2n)$ and let $(\hat{\lambda},\hat{\mu},\hat{\nu})$ be the corresponding triple for $OG(2,2n+1)$. Then*

(i) $c^{\nu}_{\lambda,\mu}(LG(2,2n)) = 1 \iff (\hat{\lambda},\hat{\mu},\hat{\nu})$ *is Levi-movable.*

(ii) $c^{\nu}_{\lambda,\mu}(LG(2,2n)) = 2 \iff (\hat{\lambda},\hat{\mu},\hat{\nu})$ *is not Levi-movable.*

Proof. It is a straightforward consequence of Theorem 3.2.5 that $c^{\nu}_{\lambda,\mu} \in \{0,1,2\}$. Since (λ,μ,ν) is Levi-movable for $LG(2,2n)$, $c^{\nu}_{\lambda,\mu} \in \{1,2\}$. By Lemma 4.3.5, $c^{\nu}_{\lambda,\mu} = 2$ if and only if both λ,μ use one root of the upper region of $\Lambda_{LG(2,2n)}$ and ν uses two. By the proof of Lemma 4.3.5, both λ,μ use one root of the upper region of $\Lambda_{LG(2,2n)}$ and ν uses two, if and only if $\hat{\nu}$ uses the top root of $\Lambda_{OG(2,2n+1)}$ while $\hat{\lambda}$, $\hat{\mu}$ do not. By Proposition 4.1.1, this is equivalent to $(\hat{\lambda},\hat{\mu},\hat{\nu})$ being not Levi-movable. □

Chapter 5

A new deformed product on $H^\star(G/P)$

Fix a generalized flag variety G/P. In this chapter, which is based on joint work with O. Pechenik, we obtain a new product structure on $H^\star(G/P)$. This structure constants of this product pick out triples of Schubert varieties that behave well under projections to other generalized flag varieties G/Q, where $Q \supset P$. The results stated in this chapter also appear in [45].

5.1 Definition of the product

Define

$$\overline{n_{\alpha\beta}} = \begin{cases} 1 & \text{if } n_{\alpha\beta} > 0 \\ 0 & \text{if } n_{\alpha\beta} = 0 \end{cases}$$

For each $\alpha \notin \Delta_P$, we introduce a complex variable t_α, and define

$$t^{\overline{\beta}} = \prod_{\alpha \notin \Delta_P} t_\alpha^{\overline{n_{\alpha\beta}}}.$$

For $u, v, w \in W^P$, let $\lambda, \mu, \nu \in \mathbb{Y}_{G/P}$ denote the associated RYDs. Define

$$S_\lambda(t) = \prod_{\beta \in \lambda} t^{\overline{\beta}}.$$

Then we define a product $\star_t$ on $H^\star(G/P)$ by

$$\sigma_\lambda \star_t \sigma_\mu = \sum_\nu \frac{S_\nu(t)}{S_\lambda(t) S_\mu(t)} c_{\lambda,\mu}^\nu \sigma_\nu.$$

The product $\star_t$ is obtained by replacing $n_{\alpha\beta}$ with $\overline{n_{\alpha\beta}}$ in the definition of the Belkale-Kumar product given in Chapter 4. This product can also be regarded as a special limiting case of a two-parameter deformation $\star_{t,s}$ of $H^\star(G/P)$ appearing in [45].

Claim 5.1.1. *The product $\star_t$ is commutative and associative.*

Proof. Commutativity of $\star_t$ is obvious from the definition. For associativity, compute:

$$\begin{aligned}(\sigma_\lambda \star_t \sigma_\mu) \star_t \sigma_\nu &= \sigma_\nu \star_t \sum_\gamma \frac{S_\gamma(t)}{S_\lambda(t)S_\mu(t)} c_{\lambda,\mu}^{\gamma} \sigma_\gamma \\ &= \sum_{\gamma,\rho} \frac{S_\gamma(t)}{S_\lambda(t)S_\mu(t)} \frac{S_\rho(t)}{S_\nu(t)S_\gamma(t)} c_{\lambda,\mu}^{\gamma} c_{\nu,\gamma}^{\rho} \sigma_\rho \\ &= \sum_{\gamma,\rho} \frac{S_\rho(t)}{S_\lambda(t)S_\mu(t)S_\nu(t)} c_{\lambda,\mu}^{\gamma} c_{\nu,\gamma}^{\rho} \sigma_\rho,\end{aligned}$$

while similarly

$$\sigma_\lambda \star_t (\sigma_\mu \star_t \sigma_\nu) = \sum_{\gamma,\rho} \frac{S_\rho(t)}{S_\lambda(t)S_\mu(t)S_\nu(t)} c_{\mu,\nu}^{\gamma} c_{\lambda,\gamma}^{\rho} \sigma_\rho.$$

Associativity of $\star_t$ then follows immediately from associativity of the ordinary cup product. □

For generic choices of $t_\alpha \in \mathbb{C}$, $(H^\star(G/P), \star_t)$ will be isomorphic to $H^\star(G/P)$ with the ordinary cup product. As with the Belkale-Kumar deformation $\odot_t$, most interest is in the product structure obtained by setting all t_α equal to zero. Setting any t_α equal to zero raises a well-definedness issue: if $c_{\lambda,\mu}^{\nu} \neq 0$ and the degree of t_α in $\frac{S_\nu(t)}{S_\lambda(t)S_\mu(t)}$ is negative, then the expression is not defined.

Theorem 5.1.2. *The product $\star_t$ is well-defined, that is, $\frac{S_\nu(t)}{S_\lambda(t)S_\mu(t)}$ is a polynomial whenever $c_{\lambda,\mu}^{\nu} \neq 0$.*

Proof. Let P_α denote the maximal parabolic subgroup of G associated to a given $\alpha \notin \Delta_P$. Define a projection $\pi_\alpha : G/P \to G/P_\alpha$ by $\pi_\alpha(gP) = gP_\alpha$. The following result is basic.

Claim 5.1.3. *π_α is G-equivariant, i.e., $\pi(g'gP) = g'\pi_\alpha(gP)$ for $g, g' \in G$.*

Proof. Let $g, g' \in G$. Then $\pi_\alpha(g' \cdot (gP)) = \pi_\alpha((g'g)P) = (g'g)P_\alpha = g' \cdot (gP_\alpha) = g' \cdot \pi_\alpha(gP)$. Alternatively, see page 7 of [11]. □

For $w \in W^P$, let w_α denote the minimal length coset representative of wW_{P_α}, and let ν_α be the associated RYD in $\mathbb{Y}_{G/P_\alpha}$. Then π_α maps points of X_ν to X_{ν_α}.

Claim 5.1.4. *If $c_{\lambda,\mu}^{\nu} \neq 0$, then for each α, $|\lambda_\alpha| + |\mu_\alpha| \leq |\nu_\alpha|$.*

Proof. If $|\lambda_\alpha| + |\mu_\alpha| > |\nu_\alpha|$, then $\operatorname{codim}(X_{\lambda_\alpha}) + \operatorname{codim}(X_{\mu_\alpha}) + \operatorname{codim}(X_{\nu_\alpha^\vee}) > \dim(G/P_\alpha)$, which implies generic translates of X_{λ_α}, X_{μ_α}, $X_{(\nu^\vee)_\alpha}$ must have empty intersection in G/P_α.

Since $c^{\nu}_{\lambda,\mu} \neq 0$, by [31] there is a point $gP \in (g_1 X_\lambda \cap g_2 X_\mu \cap g_3 X_{\nu^\vee}) \subseteq G/P$, for generic $(g_1, g_2, g_3) \in G^3$. Then since π_α is G-equivariant, there is a point $\pi_\alpha(gP) \in (g_1 X_{\lambda_\alpha} \cap g_2 X_{\mu_\alpha} \cap g_3 X_{(\nu^\vee)_\alpha}) \subseteq G/P_\alpha$. In particular this latter intersection is nonempty, hence we must have $|\lambda_\alpha| + |\mu_\alpha| \leq |\nu_\alpha|$. □

The degree of t_α in $S_\lambda(t)$ is exactly the number of roots used by λ in Λ_{G/P_α}, which is equal to $|\lambda_\alpha|$. Therefore, the degree of t_α in $\frac{S_\nu(t)}{S_\lambda(t) S_\mu(t)}$ is $|\nu_\alpha| - |\lambda_\alpha| - |\mu_\alpha|$.

Suppose $c^{\nu}_{\lambda,\mu} \neq 0$. Then by Claim 5.1.4, $|\nu_\alpha| - |\lambda_\alpha| - |\mu_\alpha| \geq 0$ for all α, and so $S_\lambda(t) S_\mu(t)$ divides $S_\nu(t)$ as desired. □

Corollary 5.1.5. *For $G = GL_n$, $\star_t = \odot_t$.*

Proof. For GL_n, we always have $n_{\alpha\beta} \leq 1$, so $n_{\alpha\beta} = \overline{n_{\alpha\beta}}$ and the definition of $\star_t$ is identical to the definition in Chapter 4 of $\odot_t$. □

In particular, this gives a proof that the type A Belkale-Kumar product is well-defined. We also compare $\star_t$ to the ordinary cup product:

Claim 5.1.6. *If P is maximal, then $\star_t$ is the ordinary cup product on $H^\star(G/P)$.*

Proof. Suppose P is maximal. Then there is only one variable t_α in $S_\lambda(t)$, so $\frac{S_\nu(t)}{S_\lambda(t) S_\mu(t)} = \frac{t_\alpha^{|\nu|}}{t_\alpha^{|\lambda|} t_\alpha^{|\mu|}}$. If $c^{\nu}_{\lambda,\mu} \neq 0$, then $|\lambda| + |\mu| = |\nu|$, so $\frac{S_\nu(t)}{S_\lambda(t) S_\mu(t)} = 1$. □

5.2 Geometric interpretation

For simplicity of notation, we now index Schubert varieties by $w \in W^P$ rather than RYDs, For any $Q \supset P$ and $w \in W^P$, there is a unique parabolic decomposition $w = w'w''$, where $w' \in W^Q$ and $w'' \in W^P \cap W_Q$ ([28]). (In the case $Q = P_\alpha$ is maximal, then $w' = w_\alpha$.) Fix $Q \supset P$ and suppose $c^{w^\vee}_{u,v} \neq 0$. We say that the triple $(u, v, w) \in (W^P)^3$ is **Q-factoring** if $g_1 X_{u'} \cap g_2 X_{v'} \cap g_3 X_{w'}$ is a finite (nonempty) set of points for generic $g_i \in G$, or equivalently if $g_1 X_{u''} \cap g_2 X_{v''} \cap g_3 X_{w''}$ is generically a finite (nonempty) set of points.

Let $a^w_{u,v} := \frac{S_w(0)}{S_u(0) S_v(0)} c^w_{u,v}$ denote the structure constants of the ring $(H^\star(G/P), \star_0)$. These structure constants can be interpreted geometrically as follows:

Proposition 5.2.1.

$$a^w_{u,v} = \begin{cases} c^w_{u,v} & \textit{if } (u, v, w^\vee) \textit{ is } Q\textit{-factoring for every } Q \supset P, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. This is trivial if $c_{u,v}^{w} = 0$, so assume it is positive. Suppose $(u, v, w^\vee)$ is not Q-factoring for some $Q \supset P$, and suppose $\Delta \setminus \Delta_Q = \{\alpha_1, \ldots, \alpha_k\}$. Then $\frac{F_w(t)}{F_u(t)F_v(t)}$ can be written as $\frac{(\alpha_1^{r_1} \ldots \alpha_k^{r_k})\alpha_{k+1}^{r_{k+1}} \cdots}{(\alpha_1^{s_1} \ldots \alpha_k^{s_k})\alpha_{k+1}^{r_{s+1}} \cdots}$. By the argument of Claim 5.1.4 applied to nonmaximal parabolic subgroups, $r_1 + \ldots + r_k \geq s_1 + \ldots + s_k$. If this was an equality, then the argument of Claim 5.1.4 would further imply that there is a point in the intersection of generic translates of the corresponding Schubert varieties, and moreover the sum of their codimensions is equal to $\dim(G/Q)$, so there are finitely many points in the intersection. This contradicts the non-Q-factoring assumption. Therefore, we must have the degree of t_{α_i} is positive for some i, so $\frac{S_w(0)}{S_u(0)S_v(0)} = 0$ and $a_{u,v}^{w} = 0$.

On the other hand, if $(u, v, w^\vee)$ is Q-factoring for all $Q \supset P$, then in particular it is P_α-factoring for all (maximal) $P_\alpha \supset P$. By Claim 5.1.4, the degree of t_{α_i} is nonnegative for all i, and Q-factoring implies it is equal to zero. Hence $\frac{S_w(t)}{S_u(t)S_v(t)} = 1$ and $a_{u,v}^{w} = c_{u,v}^{w}$. □

Proposition 5.2.2. *Every $a_{u,v}^{w}$ can be factorized as a product of Schubert structure constants $c_{x,y}^{z}$ on maximal parabolic quotients G/P_α.*

Proof. By [56, Theorem 1.1], the numbers $a_{u,v}^{w}$ factor as $c_{u',v'}^{w'} c_{u'',v''}^{w''}$. Iterating this factorization for every maximal $P_\alpha \supset P$, we obtain a factorization of $a_{u,v}^{w}$ as a product of Schubert structure constants $c_{x,y}^{z}$ on maximal parabolic quotients G/P_α. □

Proposition 5.2.3. *If $b_{u,v}^{w} \neq 0$, then $a_{u,v}^{w} = b_{u,v}^{w}$.*

Proof. Richmond [56] also shows that (u, v, w) is Q-factoring for each $Q \supset P$ when (u, v, w) is Levi-movable. □

Therefore $\star_0$ may be thought of as 'less-degenerate' than $\odot_0$, since a generally smaller collection of Schubert structure constants are set to 0.

We conclude with some examples comparing $\star_0$, $\odot_0$ and the ordinary cup product.

Example 5.2.4. *Let $G/P = SO_9/P_{2,4}$. Of the 8271 nonzero Schubert structure constants for $H^\star(G/P)$, 807 are nonzero for $\star_0$. Of these only 597 represent Levi-movable triples and so are nonzero for the Belkale-Kumar product $\odot_0$. An example of one of the 210 nonzero $a_{u,v}^{w}$ coefficients for a non-Levi-movable triple is $a_{1324,1\bar{2}34}^{3\bar{2}14} = 1$.*

Of the 193116 nonzero Schubert structure constants for $H^\star(G/B)$, only 2439 are nonzero for $\star_0$. Of these, 2103 arise from Levi-movable triples.

Example 5.2.5. *Let $G/P = Sp_{12}/P_4$. There are* 99105 *nonzero Schubert structure constants for $H^\star(G/P)$. Since P is maximal, these are all nonzero for $\star_0$. However only* 7962 *are nonzero for $\odot_0$.*

Chapter 6

Nonmaximal isotropic Grassmannians and Pieri formulas

6.1 Preliminaries and statement of theorems

Fix a positive integer $k < n$. Recall from the introduction that a subspace V is said to be **isotropic** with respect to a nondegenerate bilinear symmetric or skew-symmetric form Q if $Q(x, y) = 0$ for all $x, y \in V$.

The type B isotropic Grassmannians are the **odd orthogonal Grassmannians**

$$SO_{2n+1}/P_k = OG(k, 2n+1) = \{V \subset \mathbb{C}^{2n+1} : \dim(V) = k, V \text{ isotropic with respect to } Q\}$$

(where Q is symmetric on $\mathbb{C}^{2n+1}$).

The type C isotropic Grassmannians are the **Lagrangian Grassmannians**

$$Sp_{2n}/P_k = LG(k, 2n) = \{V \subset \mathbb{C}^{2n} : \dim(V) = k, V \text{ isotropic with respect to } Q\}$$

(where Q is skew-symmetric on $\mathbb{C}^{2n}$).

The type D isotropic Grassmannians are the **even orthogonal Grassmannians**

$$SO_{2n}/P_k = OG(k, 2n) = \{V \subset \mathbb{C}^{2n} : \dim(V) = k, V \text{ isotropic with respect to } Q\}$$

(where Q is symmetric on $\mathbb{C}^{2n}$).

The proof of the (co)adjoint formulas of Chapter 3 follows the strategy used in [39]. In [60], we show each (co)adjoint formula defines an associative product structure $\star$ on the free $\mathbb{Z}$-module $\mathbb{Z}[\mathbb{Y}_{G/P}]$. This yields a ring $(\mathbb{Z}[\mathbb{Y}_{G/P}], \star)$. We then wish to show that the linear map $\Psi : (\mathbb{Z}[\mathbb{Y}_{G/P}], \star) \to H^\star(G/P)$ given by sending an RYD λ to its corresponding Schubert class σ_λ extends to an isomorphism of rings. Given a Pieri rule for $H^\star(G/P)$, one may obtain the expansion $\sigma_\lambda \cdot \sigma_\mu$ in the Schubert basis by rewriting σ_λ as a polynomial in the Pieri classes and iterating the Pieri rule. Hence, it is enough to show that the Pieri cases of the (co)adjoint rules agree with a Pieri formula for $H^\star(G/P)$, that is, $\Psi(\lambda \star \mu) = \sigma_\lambda \cdot \sigma_\mu$ when σ_λ is a Pieri class and σ_μ is an arbitrary Schubert class.

In this chapter, for G/P a nonmaximal isotropic Grassmannian we characterize the subsets of $\Lambda_{G/P}$ that are RYDs, and give a bijection to the indexing set for Schubert varieties used by [13]. We then use this to prove that the Pieri cases of the (co)adjoint rules of Chapter 3 agree with the Pieri rules of [13]. This result, combined with the proofs of associativity in [60], then gives a proof of the (co)adjoint rules for $OG(2,2n+1)$, $LG(2,2n)$ and $OG(2,2n)$.

Using a similar convention to that of our study of the (co)adjoint varieties, for $LG(k,2n)$ we use the RYDs associated to $OG(k,2n+1)$ (i.e., in the type B_n root system).

Example 6.1.1. *Figure 6.1 shows two RYDs shown inside $\Omega_{SO_{11}}$. The first is an element of $\mathbb{Y}_{OG(3,11)}$, the second an element of $\mathbb{Y}_{OG(4,11)}$.*

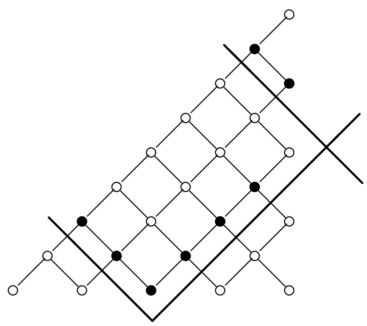


Figure 6.1: An RYD for $\mathbb{Y}_{OG(3,11)}$ and an RYD for $\mathbb{Y}_{OG(4,11)}$.

Example 6.1.2. *Figure 6.1.2 shows two RYDs shown inside $\Omega_{SO_{12}}$. The first is an element of $\mathbb{Y}_{OG(3,12)}$, and also shown is a "double-tailed diamond" from its base region (see the explanation below). The second is an element of $\mathbb{Y}_{OG(4,12)}$.*

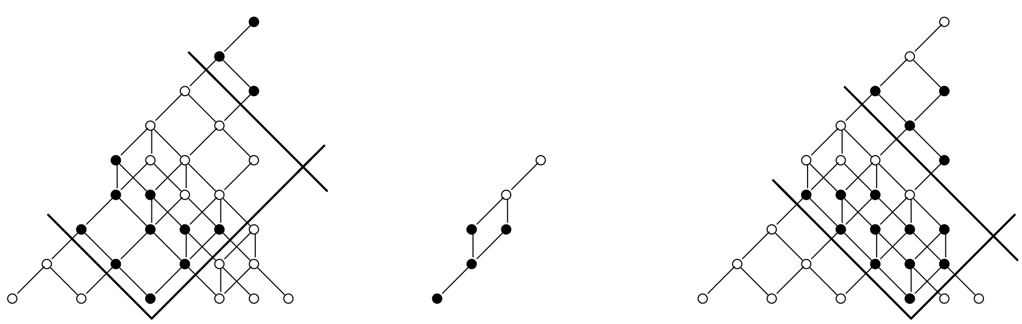


Figure 6.2: An RYD for $\mathbb{Y}_{OG(3,12)}$, a "double-tailed diamond" and an RYD for $\mathbb{Y}_{OG(4,12)}$.

We now explain the diagrams of Examples 6.1.1 and 6.1.2 above. We denote both $\Lambda_{OG(k,2n+1)} \subset \Omega_{SO_{2n+1}}$ and $\Lambda_{OG(k,2n)} \subset \Omega_{SO_{2n}}$ by Λ_k.

In each case there are two regions in the regional decomposition of Λ_k. We call these the **base region** and the **top region**. In Examples 6.1.1 and 6.1.2, the thicker black lines show Λ_k and its regional decomposition. In each type, the top region is a "staircase" $(k-1, k-2, \ldots, 0)$. In types B_n/C_n the base region is a $k \times (2n+1-2k)$ "rectangle", while in type D_n the base region consists of k "double-tailed diamonds" (following the nomenclature of [62]) each having $2n-2k$ roots.

By Lemma 2.1.2, an RYD λ for a nonmaximal isotropic Grassmannian is a lower order ideal in each of the two regions. Thus an RYD has a natural visual interpretation as a pair of partitions $(\lambda^{(1)}|\lambda^{(2)})$. Here $\lambda^{(1)}$ corresponds to the base region and $\lambda^{(2)}$ the top region. This allows us to write the RYDs in a compact way. As mentioned in the introduction, pairs of partitions are used in other indexing sets for Schubert classes for these spaces, but the pairs of partitions used in these indexing sets differ from the pairs of partitions that arise from RYDs.

We now describe the pair of partitions $(\lambda^{(1)}|\lambda^{(2)})$ associated to an RYD λ. In each type, $\lambda^{(2)}$ is a strict partition in $(k-1, k-2, \ldots, 0)$. In types B_n/C_n, $\lambda^{(1)}$ is a partition in $k \times (2n+1-2k)$. In type D_n, $\lambda^{(1)}$ is a partition in $k \times (2n-2k)$, and also if $\lambda^{(1)}_i = n-k$ for some $1 \le i \le k$ we assign a $\uparrow$ (respectively, $\downarrow$) if λ uses the root above α_{n-1} (respectively, α_n) in the ith double-tailed diamond. (In Example 6.1.2, α_n is the rightmost simple root, and α_{n-1} is the second from right, cf., Example 6.1.3 below.)

Example 6.1.3. *The RYDs of Example 6.1.1 are respectively* $((4,1,1)|(2,0,0))$ *and* $((3,2,1,0)|(2,1,0,0))$ *in the partition pair notation, and the RYDs of Example 6.1.2 are respectively* $((4,3,3)|(2,1,0))^{\uparrow}$ *and* $((4,3,3,1)|(3,1,0,0))$.

In the standard embedding of the B_n root system into $\mathbb{R}^n$ (see Chapter 3), denote the root $e_a - e_b$ by $(a,b,-)$, $e_a + e_b$ by $(a,b,+)$, and e_a by (a). Then the base region consists of all $(a,b,\pm)$ with $a \ge k > b$ and all (a) with $a \ge k$, while the top region consists of all $(a,b,+)$ with $a > b \ge k$.

Let $W^{OG(k,2n+1)} \subset W$ index the Schubert classes for $H^\star(OG(k,2n+1))$. Call a subset $S \subset \Lambda_k$ a $W^{OG(k,2n+1)}$**-diagram** if the roots in S form a lower order ideal in each region, and also satisfy a **support condition**: A root $(a,b,+)$ in the top region must be in S if S uses more than $2n+1-2k$ roots in the ath and bth rows combined, similarly, (a,b) must not be in S if S uses fewer than $2n+1-2k$ roots in the ath and bth rows combined. Let $\tilde{\mathbb{Y}}_{OG(k,2n+1)}$ denote the set of all $W^{OG(k,2n+1)}$-diagrams.

In [13], an $(n-k)$**-strict partition** is defined to be a partition γ such that $\gamma_i > \gamma_{i+1}$ whenever $\gamma_i > n-k$. The Schubert varieties of $OG(k,2n+1)$ and $LG(k,2n)$ are indexed by the set $P(n-k,n)$ of all $(n-k)$-strict partitions in a $k \times (2n-k)$ rectangle. We will prove:

Theorem 6.1.4. $\mathbb{Y}_{OG(k,2n+1)} = \tilde{\mathbb{Y}}_{OG(k,2n+1)}$.

Moreover, there is a bijection $f_k : \mathbb{Y}_{OG(k,2n+1)} \to P(n-k,n)$ *for each* $1 \le k < n$, *via*

$$f_k(\lambda) = (\lambda_i^{(1)} + \lambda_i^{(2)})_{1\le i\le k}.$$

The Schubert variety indexed by λ *is equal to the Schubert variety indexed by* $f_k(\lambda)$.

Example 6.1.5. *The RYDs of Example 6.1.1 correspond to* $(6,1,1) \in P(2,5)$ *and* $(5,3,1) \in P(1,5)$, *respectively.*

In the standard embedding of the D_n root system into $\mathbb{R}^n$ (see Chapter 3), denote the root $e_a - e_b$ by $(a,b,-)$ and $e_a + e_b$ by $(a,b,+)$. Let $W^{OG(k,2n)} \subset W$ index the Schubert classes for $H^\star(OG(k,2n))$. Call a subset $S \subset \Lambda_k$ a $W^{OG(k,2n)}$**-diagram** if the roots in S form a lower order ideal in each region, and also satisfy a **support condition** similar to that of type B_n/C_n: a root $(a,b,+)$ in the top region must be in S if S uses more than $2n-2k$ roots from the ath and bth double-tailed diamonds, similarly, $(a,b,+)$ must not be in S if S uses fewer than $2n-2k$ roots from the ath and bth double-tailed diamonds. Let $\tilde{\mathbb{Y}}_{OG(k,2n)}$ denote the set of all $W^{OG(k,2n)}$-diagrams.

In [13], the Schubert varieties of $OG(k,2n)$ are indexed by the set $\tilde{P}(n-k,n)$ of all pairs $\tilde{\gamma} = (\gamma; \mathtt{type}(\gamma))$, where γ is an $(n-k)$-strict partition in a $k \times (2n-1-k)$ rectangle, and also $\mathtt{type}(\gamma) = 0$ if no part of γ has size $n-k$ and $\mathtt{type}(\gamma) \in \{1,2\}$ otherwise. We will prove:

Theorem 6.1.6. $\mathbb{Y}_{OG(k,2n)} = \tilde{\mathbb{Y}}_{OG(k,2n)}$.

Moreover, there is a bijection $F_k : \mathbb{Y}_{OG(k,2n)} \to \tilde{P}(n-k,n)$ *for each* $1 \le k < n$, *via*

$$F_k(\lambda) = \begin{cases} ((\lambda_i^{(1)} + \lambda_i^{(2)})_{1\le i\le k}; 1) & \text{if } \lambda \text{ is assigned } \uparrow \\ ((\lambda_i^{(1)} + \lambda_i^{(2)})_{1\le i\le k}; 2) & \text{if } \lambda \text{ is assigned } \downarrow \\ ((\lambda_i^{(1)} + \lambda_i^{(2)})_{1\le i\le k}; 0) & \text{otherwise} \end{cases}$$

The Schubert variety indexed by λ *is equal to the Schubert variety indexed by* $F_k(\lambda)$.

Example 6.1.7. *The RYDs of Example 6.1.2 correspond to* $((6,4,3);1) \in P(3,6)$ *and* $((7,4,3,1);0) \in P(2,6)$, *respectively.*

We will use Theorems 6.1.4 and 6.1.6 to prove agreement of our (co)adjoint Schubert calculus formulas with the Pieri rules of [13]. Specifically, if $\star$ is the product on RYDs given by our formulas for $LG(2,2n)/OG(2,2n)$, and Ψ the linear map determined by sending an RYD λ to its corresponding Schubert class σ_λ, we will show:

Theorem 6.1.8. *Suppose λ is an RYD indexing a Pieri class. Then*

(I) If $\lambda, \mu \in \mathbb{Y}_{OG(2,2n+1)}$, then $\Psi(\lambda \star \mu) = \sigma_{f_2(\lambda)} \cdot \sigma_{f_2(\mu)} \in H^\star(LG(2,2n))$

(II) If $\lambda, \mu \in \mathbb{Y}_{OG(2,2n)}$, then $\Psi(\lambda \star \mu) = \sigma_{F_2(\lambda)} \cdot \sigma_{F_2(\mu)} \in H^\star(OG(2,2n))$.

6.2 Proof of Theorem 6.1.4

Fix $k < n$. By [46], the set $W^{OG(k,2n+1)}$ consists of all signed permutations of the form

$$(y_1, y_2, \ldots, y_{k-r}, \overline{z_r}, \overline{z_{r-1}}, \ldots \overline{z_1}, v_1, v_2, \ldots v_{n-k})$$

where bars denote negative entries, $y_1 < y_2 < \ldots < y_{k-r}$, $z_r > z_{r-1} > \ldots > z_1$, $v_1 < v_2 < \ldots < v_{n-k}$ and $0 \le r \le k$.

Define a **PR shape** to be a pair of strict partitions $\alpha = (\alpha^{\mathbf{t}}, \alpha^{\mathbf{b}})$ satisfying $\alpha^{\mathbf{t}} \subseteq (n-k) \times n$, $\alpha^{\mathbf{b}} \subseteq k \times n$ and $\alpha^{\mathbf{t}}_{n-k} \ge l(\alpha^{\mathbf{b}}) + 1$. Let $PR(k,n)$ denote the set of PR shapes. Then [46] indexes the elements of $W^{OG(k,2n+1)}$ by PR shapes as follows:

Lemma 6.2.1. *[46, Lemma 1.2] $W^{OG(k,2n+1)}$ is in bijection with $PR(k,n)$ via*

$$\alpha^{\mathbf{b}}_j = n + 1 - z_j, \qquad 1 \le j \le r$$

$$\alpha^{\mathbf{t}}_s = n + 1 - v_s + |\{q : z_q < v_s\}|, \qquad 1 \le s \le n-k.$$

Let $\alpha \in PR(k,n)$. Then $\tilde{\alpha}^{\mathbf{t}} := \alpha^{\mathbf{t}} - (n-k, n-k-1, \ldots, 1)$ is a partition in $(n-k) \times k$.

Given $w \in W^{OG(k,2n+1)}$, let $Y = \{1, \ldots, k-r\}$, $Z = \{k-r+1, \ldots, k\}$ and $V = \{k+1, \ldots n\}$. Note that if $k+1-i \in Z$ then the $(k+1-i)$th entry of w is $\overline{z_i}$, while if $k+1-i \in Y$ then the $(k+1-i)$th entry of w is y_{k+1-i}.

Claim 6.2.2. *For $1 \le i \le k$, the length of the ith column of (the Ferrers diagram of) $\tilde{\alpha}^{\mathbf{t}}$ is $n-k$ if $k+1-i \in Z$, and $|\{l : y_{k+1-i} > v_l\}|$ if $k+1-i \in Y$.*

Proof. By definition, the length of the sth row of $\tilde{\alpha}^{\mathbf{t}}$ is $k + s - v_s + |\{q : z_q < v_s\}| = k - |\{t : y_t < v_s\}|$. Then if $k+1-i \in Z$, the ith column has the maximal possible length $n-k$ since $k - |\{t : y_t < v_s\}|$ is never smaller than $k - |Y|$. Now suppose $k+1-i \in Y$. Then the length of the ith column is equal to the largest s such that $y_{k+1-i} > v_s$, i.e., $|\{l : y_{k+1-i} > v_l\}|$. □

Let $(\tilde{\alpha}^{\mathbf{t}})'$ denote the conjugate partition of $\tilde{\alpha}^{\mathbf{t}}$. The bijection $PR(k,n) \to P(n-k,n)$ is given by $\alpha \mapsto (\tilde{\alpha}^{\mathbf{t}})' + \alpha^{\mathbf{b}}$ (see [13, page 46].).

Corollary 6.2.3. *$W^{OG(k,2n+1)}$ is in bijection with $P(n-k,n)$ via*

$$\gamma_i = \begin{cases} (n-k) + (n+1-z_i) & \textit{if } k+1-i \in Z \\ |\{l : y_{k+1-i} > v_l\}| & \textit{if } k+1-i \in Y. \end{cases}$$

The Schubert variety indexed by $w \in W^{OG(k,2n+1)}$ is equal to the Schubert variety indexed by the image of w in $P(n-k,n)$.

Proof. Compose the bijection $W^{OG(k,2n+1)} \to PR(k,n)$ of Lemma 6.2.1 with the bijection $PR(k,n) \to P(n-k,n)$, using Claim 6.2.2. □

Example 6.2.4. *Let $w = (2,3,7,\overline{8},\overline{4},1,5,6) \in W^{OG(5,17)}$. Then the PR shape corresponding to w is $\alpha = ((8,5,4),(5,1)) \in PR(5,8)$. Thus $\tilde{\alpha}^{\mathbf{t}} = (5,3,3)$ and $(\tilde{\alpha}^{\mathbf{t}})' = (3,3,3,1,1)$. The corresponding element of $P(3,8)$ is $\gamma = (8,4,3,1,1)$.*

The following lemma is proved by a straightforward computation of the inversion sets.

Lemma 6.2.5. $\mathbb{Y}_{OG(k,2n+1)} \subseteq \tilde{\mathbb{Y}}_{OG(k,2n+1)}$.

Lemma 6.2.6. *Let $w \in W^{OG(k,2n+1)}$ and let $\lambda \in \mathbb{Y}_{OG(k,2n+1)}$ be the corresponding RYD. Then*

$$\lambda_i^{(1)} = \begin{cases} n+1-k+|\{l : z_i < v_l\}| & \textit{if } k+1-i \in Z \\ |\{l : y_{k+1-i} > v_l\}| & \textit{if } k+1-i \in Y \end{cases}$$

and

$$\lambda_i^{(2)} = \begin{cases} |\{q : z_i < z_q\}| + |\{t : z_i < y_t\}| & \textit{if } k+1-i \in Z \\ 0 & \textit{if } k+1-i \in Y. \end{cases}$$

Proof. If $k+1-i \in Z$, then all $n-k$ roots of the form $(k+1-i, c, -)$, as well as $(k+1-i)$ in the base region are inverted by w. The roots of the form $(k+1-i, c, +)$ in the base inverted by w are exactly those where $w(k+1-i) < w(c)$, so $\lambda_i^{(1)} = n+1-k+|\{l : z_i < v_l\}|$. If $k+1-i \in Y$, then neither $(k+1-i)$ nor any root of the form $(k+1-i, c, +)$ in the base is inverted by w. The roots in the base region of the form $(k+1-i, c, -)$ inverted by w are those where $w(k+1-i) > w(c)$, so $\lambda_i^{(1)} = |\{l : y_{k+1-i} > v_l\}|$.

Let $w(a)$ denote the number in position a of w, ignoring whether that entry is barred. If $k+1-i \in Z$, then the roots of the top region of the form $(a, k+1-i, +)$ inverted by w are those where $a \in Z$, or $a \in Y$

and $w(a) > w(k+1-i)$. Thus $\lambda_i^{(2)} = |\{q : z_i < z_q\}| + |\{t : z_i < y_t\}|$. If $k+1-i \in Y$, then the roots of the top region of the form $(a, k+1-i, +)$ have $a \in Y$ also, and no such roots can be inverted by w. $\square$

Example 6.2.7. *Let $w = (2, 3, 7, \overline{8}, \overline{4}, 1, 5, 6) \in W^{OG(5,17)}$, as in Example 6.2.4. The corresponding RYD is $\lambda = ((6, 4, 3, 1, 1)|(2, 0, 0, 0, 0)) \in \mathbb{Y}_{OG(5,17)}$.*

Lemma 6.2.8. *The map f_k of Theorem 6.1.4 is an injection $\tilde{\mathbb{Y}}_{OG(k,2n+1)} \to P(n-k, n)$.*

Proof. Let $\lambda \in \tilde{\mathbb{Y}}_{OG(k,2n+1)}$. It is clear from the definition of a $W^{OG(k,2n+1)}$-diagram that $f_k(\lambda)$ is a partition in $k \times (2n-k)$. To see that it is $(n-k)$-strict, suppose for some i that $\lambda_i^{(1)} + \lambda_i^{(2)} > n-k$ and $\lambda_{i+1}^{(1)} + \lambda_{i+1}^{(2)} > n-k$. By the support condition, this implies $\lambda_i^{(1)} > n-k$ and $\lambda_{i+1}^{(1)} > n-k$. Then the support condition also implies that $\lambda_i^{(2)} > 0$, since the root $(i, i+1, +)$ must be in λ. Then since $\lambda^{(2)}$ is strict, we have $\lambda_i^{(2)} > \lambda_{i+1}^{(2)}$. Thus $\lambda_i^{(1)} + \lambda_i^{(2)} > \lambda_{i+1}^{(1)} + \lambda_{i+1}^{(2)}$, and so $f_k(\lambda) \in P(n-k, n)$.

Now suppose for a contradiction that f_k is not injective, i.e., there exist $\lambda, \mu \in \tilde{\mathbb{Y}}_{OG(k,2n+1)}$ such that $\lambda \neq \mu$ but $\lambda_i^{(1)} + \lambda_i^{(2)} = \mu_i^{(1)} + \mu_i^{(2)}$ for all $1 \leq i \leq k$. Let j largest such that $\lambda_j^{(1)} \neq \mu_j^{(1)}$ (such a j must exist), and assume without loss of generality that $\lambda_j^{(1)} > \mu_j^{(1)}$. Then by the support condition, every root in the top region of the form $(a, j, +)$ which is in μ is also in λ. So $\lambda_j^{(2)} \geq \mu_j^{(2)}$, which contradicts the assumption that $\lambda_j^{(1)} + \lambda_j^{(2)} = \mu_j^{(1)} + \mu_j^{(2)}$. $\square$

Lemma 6.2.8 gives an injection $\tilde{\mathbb{Y}}_{OG(k,2n+1)} \to P(n-k, n)$. Since Corollary 6.2.3 establishes a bijection $P(n-k, n) \to W^{OG(k,2n+1)}$, and by definition $W^{OG(k,2n+1)}$ is in bijection with $\mathbb{Y}_{OG(k,2n+1)}$, we have an injection $\tilde{\mathbb{Y}}_{OG(k,2n+1)} \to \mathbb{Y}_{OG(k,2n+1)}$. By Lemma 6.2.5, $\mathbb{Y}_{OG(k,2n+1)} \subseteq \tilde{\mathbb{Y}}_{OG(k,2n+1)}$. Thus $\mathbb{Y}_{OG(k,2n+1)} = \tilde{\mathbb{Y}}_{OG(k,2n+1)}$, and the injection $f_k : \tilde{\mathbb{Y}}_{OG(k,2n+1)} \to P(n-k, n)$ is a bijection.

It remains to show $\lambda \in \mathbb{Y}_{OG(k,2n+1)}$ indexes the same Schubert variety as $f_k(\lambda) \in P(n-k, n)$. Let $w \in W^{OG(k,2n+1)}$. Let λ be the RYD indexing the same Schubert variety as w by Lemma 6.2.6, and let γ be the element of $P(n-k, n)$ indexing the same Schubert variety as w by Corollary 6.2.3. First suppose $k+1-i \in Z$. Then by Lemma 6.2.6,

$$\lambda_i^{(1)} + \lambda_i^{(2)} = n+1-k + |\{l : z_i < v_l\}| + |\{q : z_i < z_q\}| + |\{t : z_i < y_t\}|,$$

which is equal to $n+1-k+(n-z_i)$, which is equal to γ_i by Corollary 6.2.3. Now suppose $k+1-i \in Y$. Then by Lemma 6.2.6, $\lambda_i^{(1)} + \lambda_i^{(2)} = |\{l : y_{k+1-i} > v_l\}|$, which is equal to γ_i by Corollary 6.2.3. Thus λ, $f_k(\lambda)$ index the same Schubert variety. $\square$

6.3 Proof of Theorem 6.1.6

Using the same convention as in [47], the set $W^{OG(k,2n)}$ consists of all signed permutations that have an even number of signed entries, and are of the form

$$(y_1, y_2, \ldots, y_{k-r}, \overline{z_r}, \overline{z_{r-1}}, \ldots \overline{z_1}, v_1, v_2, \ldots v_{n-k-1}, \widehat{v_{n-k}})$$

where $0 \leq r \leq k$, bars denote negative entries, $y_1 < y_2 < \ldots < y_{k-r}$, $z_r > z_{r-1} > \ldots > z_1$, $v_1 < v_2 < \ldots < v_{n-k}$, and $\widehat{v_{n-k}}$ is either v_{n-k} or $\overline{v_{n-k}}$, depending on the parity of r. Call w a permutation of **type I** if $\widehat{v_{n-k}} = v_{n-k}$, and **type II** if $\widehat{v_{n-k}} = \overline{v_{n-k}}$.

Given $w \in W^{OG(k,2n)}$, let $Y = \{1, \ldots, k-r\}$, $Z = \{k-r+1, \ldots, k\}$ and $V = \{k+1, \ldots n\}$. Note that if $k+1-i \in Z$ then the $(k+1-i)$th entry of w is $\overline{z_i}$, while if $k+1-i \in Y$ then the $(k+1-i)$th entry of w is y_{k+1-i}.

We now follow [61]. Define a **T-shape** to be a pair of partitions $\alpha = (\alpha^{\mathbf{t}}, \alpha^{\mathbf{b}})$, where $\alpha^{\mathbf{b}} \subset k \times (n-1)$ is strict, $\alpha^{\mathbf{t}} \subset (n-k) \times k$, and $\alpha^{\mathbf{t}}_{n-k} \geq l(\alpha^{\mathbf{b}})$. Let $T(k,n)$ denote the set of all T-shapes.

The notation of [61] differs from ours, specifically, the fork of the D_n Dynkin diagram consists of nodes 1 and 2 in [61] rather than $n-1$ and n. Translated into our notation, [61] defines a surjection $h : W^{OG(k,2n)} \to T(k,n)$ via:

$$\alpha^{\mathbf{t}}_i = k - v_i + i + |\{j : z_j < v_i\}|$$

$$\alpha^{\mathbf{b}}_i = n - z_i$$

For $w \in W^{OG(k,2n)}$ such that $v_{n-k} = n$, h is one-to-one. Otherwise h is two-to-one, with

$$(y_1, y_2, \ldots, y_{k-r}, \overline{n}, \overline{z_{r-1}}, \ldots \overline{z_1}, v_1, v_2, \ldots v_{n-k-1}, \widehat{v_{n-k}}) \text{ and}$$

$$(y_1, y_2, \ldots, y_{k-r}, n, \overline{z_{r-1}}, \ldots \overline{z_1}, v_1, v_2, \ldots v_{n-k-1}, \widehat{v_{n-k}})$$

mapping to the same T-shape. One of these permutations has type I, the other has type II.

Let $T'(k,n)$ be the set containing a single copy of each $\alpha \in T(k,n)$ that satisfies $|h^{-1}(\alpha)| = 1$, and two copies of each $\alpha \in T(k,n)$ that satisfies $|h^{-1}(\alpha)| = 2$, where one copy is declared to have type 1 and the other copy type 2. Define a map $h' : W^{OG(k,2n)} \to T'(k,n)$ by letting $h'(w) = h(w)$ whenever h is one-to-one, and whenever h is two-to-one let $h'(w)$ be the T-shape $h(w)$ of type 1 (respectively, type 2) if w is of type I (respectively, type II). Then h' is a bijection. Note that the definition of *type* of a T-shape used here is not the same as that used by [61].

Claim 6.3.1. *Let $w \in W^{OG(k,2n)}$ and let $h(w) = \alpha$ be the corresponding T-shape. Then for $1 \leq i \leq k$, the length of the ith column of $\alpha^{\mathbf{t}}$ is $n-k$ if $k+1-i \in Z$, and $|\{l : y_{k+1-i} > v_l\}|$ if $k+1-i \in Y$.*

Proof. Identical to the proof of Claim 6.2.2. □

Given $\alpha \in T(k,n)$, let $(\alpha^{\mathbf{t}})'$ denote the conjugate partition of $\alpha^{\mathbf{t}}$. Now we follow [13, pp 46–47]. The bijection $T'(k,n) \to \tilde{P}(n-k,n)$ is given by $\alpha \mapsto (\alpha^{\mathbf{t}})' + \alpha^{\mathbf{b}}$, where if α has type 1 (respectively, 2), its image in $\tilde{P}(n-k,n)$ has type 1 (respectively, 2).

Corollary 6.3.2. *$W^{OG(k,2n)}$ is in bijection with $\tilde{P}(n-k,n)$ via*

$$\gamma_i = \begin{cases} (n-k) + (n-z_i) & \text{if } k+1-i \in Z \\ |\{l : y_{k+1-i} > v_l\}| & \text{if } k+1-i \in Y. \end{cases}$$

where $\tilde{\gamma} = (\gamma; 0)$ if γ has no part of size $n-k$, otherwise $\tilde{\gamma} = (\gamma; 1)$ if w has type I and $\tilde{\gamma} = (\gamma; 2)$ if w has type II. The Schubert variety indexed by $w \in W^{OG(k,2n)}$ is equal to the Schubert variety indexed by the image of w in $\tilde{P}(n-k,n)$.

Proof. Compose the bijection $W^{OG(k,2n)} \to T'(k,n)$ with the bijection $T'(k,n) \to \tilde{P}(n-k,n)$, using Claim 6.3.1. It is clear that γ has a part of size $n-k$ if and only if either $z_r = n$ or $y_{k-r} = n$ in w. □

Example 6.3.3. *Let $w = (2, 4, \overline{8}, \overline{6}, \overline{1}, 3, 5, \overline{7}) \in W^{OG(5,16)}$. Then the T-shape corresponding to w is $\alpha = ((4,3,3),(7,2,0))$ (type 2). Then $(\alpha^{\mathbf{t}})' = (3,3,3,1,0)$. The corresponding element of $\tilde{P}(3,8)$ is $\tilde{\gamma} = ((10,5,3,1,0); 2)$.*

The following lemma is proved by a straightforward computation of the inversion sets.

Lemma 6.3.4. $\mathbb{Y}_{OG(k,2n)} \subseteq \tilde{\mathbb{Y}}_{OG(k,2n)}$.

Lemma 6.3.5. *Let $w \in W^{OG(k,2n)}$ and let $\lambda \in \mathbb{Y}_{OG(k,2n)}$ be the corresponding RYD. Then*

$$\lambda_i^{(1)} = \begin{cases} n-k+|\{l : z_i < v_l\}| & \text{if } k+1-i \in Z \\ |\{l : y_{k+1-i} > v_l\}| & \text{if } k+1-i \in Y, \end{cases}$$

$$\lambda_i^{(2)} = \begin{cases} |\{q : z_i < z_q\}| + |\{t : z_i < y_t\}| & \text{if } k+1-i \in Z \\ 0 & \text{if } k+1-i \in Y \end{cases}$$

and if $\lambda_i^{(1)} = n-k$ roots for some i, then λ is assigned $\uparrow$ if w is of type I and $\downarrow$ if w is of type II.

Proof. (w is of type I): If $k+1-i \in Z$, then all $n-k$ roots $(k+1-i, c, -)$ in the base are inverted by w. The roots of the form $(k+1-i, c, +)$ in the base inverted by w are exactly those where $w(k+1-i) < w(c)$, so $\lambda_i^{(1)} = n-k+|\{l : z_i < v_l\}|$. If $k+1-i \in Y$, then no roots of the form $(k+1-i, c, +)$ in the base are inverted by w. The roots in the base of the form $(k+1-i, c, -)$ inverted by w are those where $w(k+1-i) > w(c)$, so $\lambda_i^{(1)} = |\{l : y_{k+1-i} > v_l\}|$.

If $k+1-i \in Z$, then the roots of the top region of the form $(a, k+1-i, +)$ inverted by w are those where either $a \in Z$, or $a \in Y$ and $w(a) > w(k+1-i)$. Thus $\lambda_i^{(2)} = |\{q : z_i < z_q\}| + |\{t : z_i < y_t\}|$. If $k+1-i \in Y$, then the roots of the top region of the form $(a, k+1-i, +)$ have $a \in Y$ also, and no such roots can be inverted by w.

(w is of type II): If $k+1-i \in Z$, then all $n-k-1$ roots $(k+1-i, c, -)$ for $c < n$ in the base are inverted by w, and also $(k+1-i, n, +)$ is inverted by w. The number of remaining roots of the ith double-tailed diamond inverted by w is

$$|\{l < n-k : z_i < v_l\}| + \begin{cases} 1 & \text{if } z_i < v_{n-k} \\ 0 & \text{if } z_i > v_{n-k} \end{cases}$$

(the first summand is the number of $(k+1-i, c, +)$ for $c < n$ inverted, the second is whether $(k+1-i, n, -)$ is inverted). Thus $\lambda_i^{(1)} = n-k+|\{l : z_i < v_l\}|$. If $k+1-i \in Y$, then no roots of the form $(k+1-i, c, +)$ for $c < n$ in the base are inverted by w, and also $(k+1-i, n, -)$ is not inverted by w. Thus the number of roots of the ith double-tailed diamond inverted by w is

$$|\{l < n-k : y_{k+1-i} > v_l\}| + \begin{cases} 1 & \text{if } y_{k+1-i} > v_{n-k} \\ 0 & \text{if } y_{k+1-i} < v_{n-k} \end{cases}$$

(the first summand is the number of $(k+1-i, c, -)$ for $c < n$ inverted, the second is whether $(k+1-i, n, +)$ is inverted). Thus $\lambda_i^{(1)} = |\{l : y_{k+1-i} > v_l\}|$.

Since the last co-ordinate of any root of the top region is zero, it is irrelevant whether the last entry of w is barred. Hence for $\lambda_i^{(2)}$, the statement for the top region follows by the same argument as for type I permutations.

Finally, if $\lambda_i^{(1)} = n-k$ for some i, then λ uses either $(k+1-i, n, -)$ (above α_{n-1}) or $(k+1-i, n, +)$ (above α_n) but not both. If λ uses the former but not the latter then the last entry of w must be unbarred (i.e., w is of type I), and if it uses the latter but not the former then similarly w must be of type II. Thus λ is assigned $\uparrow$ (respectively, $\downarrow$) if and only if $\lambda_i^{(1)} = n-k$ for some i and w is of type I (respectively, type

II). $\square$

Example 6.3.6. *Let $w = (2, 4, \overline{8}, \overline{6}, \overline{1}, 3, 5, \overline{7}) \in W^{OG(5,16)}$, as in Example 6.3.3. The corresponding RYD is $\lambda = ((6, 4, 3, 1, 0)|(4, 1, 0, 0, 0))^{\downarrow} \in \mathbb{Y}_{OG(5,16)}$.*

Lemma 6.3.7. *The map F_k of Theorem 6.1.6 is an injection $\tilde{\mathbb{Y}}_{OG(k,2n)} \to \tilde{P}(n-k, n)$.*

Proof. Let $\lambda \in \tilde{\mathbb{Y}}_{OG(k,2n)}$. It is clear from the definition of a $W^{OG(k,2n)}$-diagram that for $\tilde{\gamma} = F_k(\lambda)$, γ is a partition in $k \times (2n-1-k)$. First we show γ is $(n-k)$-strict. Suppose for some i that $\lambda_i^{(1)} + \lambda_i^{(2)} > n-k$ and $\lambda_{i+1}^{(1)} + \lambda_{i+1}^{(2)} > n-k$. By the support condition, this implies $\lambda_i^{(1)} \geq n-k$ and $\lambda_{i+1}^{(1)} \geq n-k$. If the first inequality is strict then the support condition also implies that $\lambda_i^{(2)} > 0$ since the root $(i, i+1, +)$ must be in λ, while if it is an equality then we also have $\lambda_i^{(2)} > 0$ since $\lambda_i^{(1)} + \lambda_i^{(2)} > n-k$. Since $\lambda^{(2)}$ is a strict partition, this implies $\lambda_i^{(2)} > \lambda_{i+1}^{(2)}$, whence $\lambda_i^{(1)} + \lambda_i^{(2)} > \lambda_{i+1}^{(1)} + \lambda_{i+1}^{(2)}$.

Next, to demonstrate that F_k is well-defined, we show that $\lambda^{(1)}$ has a row of length $n-k$ if and only if γ has a row of length $n-k$. Suppose $\lambda^{(1)}$ has a row of length $n-k$, and let i be largest such that $\lambda_i^{(1)} = n-k$. Then $\lambda_l^{(1)} < n-k$ for all $l > i$, and thus by the support condition $\lambda_i^{(2)} = 0$. So $\gamma_i = n-k$. Now suppose $\lambda^{(1)}$ has no row of length $n-k$, and consider an arbitrary row $\lambda_i^{(1)}$ of $\lambda^{(1)}$. If $\lambda_i^{(1)} > n-k$ then clearly $\gamma_i > n-k$. If $\lambda_i^{(1)} < n-k$ then $\lambda_l^{(1)} < n-k$ for all $l > i$, and then by the support condition $\lambda_i^{(2)} = 0$. Hence $\gamma_i = \lambda_i^{(1)} < n-k$.

The argument that F_k is injective is then similar to that of Lemma 6.2.8. $\square$

Similarly to the type B case, it now follows from Lemmas 6.3.4, 6.3.7 and Corollary 6.3.2 that F_k is a bijection $\mathbb{Y}_{OG(k,2n)} \to \tilde{P}(n-k, n)$. It remains to show the image of λ indexes the same Schubert variety as λ.

Let $w \in W^{OG(k,2n)}$. Let λ be the RYD indexing the same Schubert variety as w by Lemma 6.3.5, and let $\tilde{\gamma} = (\gamma; \texttt{type}(\gamma))$ be the element of $\tilde{P}(n-k, n)$ indexing the same Schubert variety as w by Corollary 6.3.2. First suppose $k+1-i \in Z$. Then by Lemma 6.3.5,

$$\lambda_i^{(1)} + \lambda_i^{(2)} = n - k + |\{l : z_i < v_l\}| + |\{q : z_i < z_q\}| + |\{t : z_i < y_t\}|,$$

which is equal to $n-k+(n-z_i)$, which is equal to γ_i by Corollary 6.3.2. Now suppose $k+1-i \in Y$. By Lemma 6.3.5, $\lambda_i^{(1)} + \lambda_i^{(2)} = |\{l : y_{k+1-i} > v_l\}|$, which is equal to γ_i by Corollary 6.3.2.

By the proof of Lemma 6.3.7, either $\lambda^{(1)}, \gamma$ both have a row of length $n-k$ or both do not. If they do, then if w is of type I, λ is assigned $\uparrow$ and γ is of type 1, while if w is of type II, λ is assigned $\downarrow$ and γ is of type 2. Thus λ, $F_k(\lambda)$ index the same Schubert variety. $\square$

6.4 Proof of Theorem 6.1.8(I)

We follow [13, pg. 3-5]. The Schubert varieties of $LG(2, 2n)$ are indexed by the set $P(n-2, n)$ of $(n-2)$-strict partitions inside a $2 \times (2n-2)$ rectangle. The **Pieri classes** of [13] are those indexed by $\gamma = (p, 0) \in P(n-2, n)$. Denote these classes by σ_p.

Fix an integer $p \in [1, 2n-2]$, and suppose $\gamma, \delta \in P(n-2, n)$ with $|\delta| = |\gamma| + p$. Call a box of δ a δ-box, a box of γ a γ-box, a box of δ that is not in γ a $(\delta \setminus \gamma)$-box, and a box of γ that is not in δ a $(\gamma \setminus \delta)$-box. We say the box in row r and column c of γ is **related** to the box in row r' and column c' if $|c - (n-1)| + r = |c' - (n-1)| + r'$. Then there is a relation $\gamma \to \delta$ if δ can be obtained by removing a vertical strip from the first $n-2$ columns of γ and adding a horizontal strip to the result, such that

1. Each γ-box in the first $n-2$ columns having no δ-box below it is related to at most one $(\delta \setminus \gamma)$-box.
2. Any $(\gamma \setminus \delta)$-box and the box above it must each be related to exactly one $(\delta \setminus \gamma)$-box, and these $(\delta \setminus \gamma)$-boxes must all lie in the same row.

If $\gamma \to \delta$, let $\mathbb{A}$ be the set of $(\delta \setminus \gamma)$-boxes in columns $n-1$ through $2n-2$ which are *not* mentioned in (1) or (2). Define two boxes of $\mathbb{A}$ to be **connected** if they share at least a vertex. Then define $N(\gamma, \delta)$ to be the number of connected components of $\mathbb{A}$ that do not use a box of the $(n-1)$th column.

Then the specialization of the Pieri rule of [13, Theorem 1.1] to the coadjoint $LG(2, 2n)$ is

Theorem 6.4.1 ([13]). *(Pieri rule for LG(2,2n)) For any $\gamma \in P(n-2, n)$ and integer $p \in [1, 2n-2]$,*

$$\sigma_p \cdot \sigma_\gamma = \sum_{\delta} 2^{N(\gamma,\delta)} \sigma_\delta$$

where the sum is over all $\delta \in P(n-2, n)$ with $\gamma \to \delta$.

Now we consider the RYD model. In the coadjoint case $k = 2$, the base region is a $2 \times (2n-3)$ rectangle and the top region is a single root. We reprise the notation $\lambda = \langle \overline{\lambda} | \circ \rangle$ for RYDs from Chapter 3 in this case. Let $\lambda, \mu \in \mathbb{Y}_{LG(2,2n)}$, and let $M = \min\{\overline{\lambda}_1 - \overline{\lambda}_2, \overline{\mu}_1 - \overline{\mu}_2\}$. The following reformulates Theorem 3.2.5.

Definition 6.4.2. *Define a commutative product $\star$ on $\mathbb{Z}[\mathbb{Y}_{LG(2,2n)}]$:*

(A) If $|\langle \overline{\lambda} | \circ \rangle| + |\langle \overline{\mu} | \circ \rangle| \leq 2n-3$, then

$$\langle \overline{\lambda} | \circ \rangle \star \langle \overline{\mu} | \circ \rangle = \sum_{0 \leq k \leq M} \langle \overline{\lambda}_1 + \overline{\mu}_1 - k, \overline{\lambda}_2 + \overline{\mu}_2 + k | \circ \rangle$$

(B) If $|\langle\overline{\lambda}|\circ\rangle| + |\langle\overline{\mu}|\circ\rangle| > 2n-3$*, then*

$$\langle\overline{\lambda}|\circ\rangle \star \langle\overline{\mu}|\circ\rangle = \sum_{0\leq k\leq M} [\langle\overline{\lambda}_1+\overline{\mu}_1-k, \overline{\lambda}_2+\overline{\mu}_2+k-1|\bullet\rangle + \langle\overline{\lambda}_1+\overline{\mu}_1-k-1, \overline{\lambda}_2+\overline{\mu}_2+k|\bullet\rangle]$$

(C)

$$\langle\overline{\lambda}|\bullet\rangle \star \langle\overline{\mu}|\circ\rangle = \langle\overline{\lambda}|\circ\rangle \star \langle\overline{\mu}|\bullet\rangle = \sum_{0\leq k\leq M} \langle\overline{\lambda}_1+\overline{\mu}_1-k, \overline{\lambda}_2+\overline{\mu}_2+k|\bullet\rangle$$

(D) $\langle\overline{\lambda}|\bullet\rangle \star \langle\overline{\mu}|\bullet\rangle = 0$.

Declare any α *in the above expressions to be zero if* $(\overline{\alpha}_1, \overline{\alpha}_2)$ *is not a partition in* $2\times(2n-3)$*. Such* α *will be called* **illegal**.

The following specializes Theorem 6.1.4 to the case $k=2$. We write f instead of f_2.

Proposition 6.4.3. *The elements of* $\mathbb{Y}_{LG(2,2n)}$ *are in bijection with the elements of* $P(n-2,n)$ *via*

$$f(\lambda) = \begin{cases} (\overline{\lambda}_1, \overline{\lambda}_2) & \text{if } \lambda = \langle\overline{\lambda}|\circ\rangle \\ (\overline{\lambda}_1+1, \overline{\lambda}_2) & \text{if } \lambda = \langle\overline{\lambda}|\bullet\rangle. \end{cases}$$

Let α_p denote $\langle p, 0|\bullet/\circ\rangle \in \mathbb{Y}_{LG(2,2n)}$, and given $\lambda \in \mathbb{Y}_{LG(2,2n)}$ let γ denote $f(\lambda)$. The following restates Theorem 6.1.8(I):

Theorem 6.4.4. $f(\alpha_p \star \lambda) = f(\alpha_p)\cdot f(\lambda)$

Proof. Let $(r:c)$ denote the box in row r, column c of $2\times(2n-2)$. Let L denote the first $n-2$ columns of $2\times(2n-2)$ and R the latter n columns. Given $\gamma, \delta \in P(n-2,n)$ with $|\delta| = |\gamma|+p$, let $\mathbb{D}_1$ denote the set of $(\delta\setminus\gamma)$-boxes in row 1 of R, and $\mathbb{D}_2$ the set of $(\delta\setminus\gamma)$-boxes in row 2 of R. Let $\mathbb{D} = \mathbb{D}_1\cup\mathbb{D}_2$. By definition, both $\mathbb{D}_1$, $\mathbb{D}_2$ are connected and

Lemma 6.4.5.

$$\mathbb{D}_1 = \begin{cases} \{(1:c) : \gamma_1+1 \leq c \leq \delta_1\} & \text{if } \gamma_1 > n-2 \\ \{(1:c) : n-1 \leq c \leq \delta_1\} & \text{if } \gamma_1 \leq n-2 \end{cases}$$

and

$$\mathbb{D}_2 = \begin{cases} \{(2:c) : \gamma_2+1 \leq c \leq \delta_2\} & \text{if } \gamma_2 > n-2 \\ \{(2:c) : n-1 \leq c \leq \delta_2\} & \text{if } \gamma_2 \leq n-2. \end{cases}$$

Let γ^* denote the shape $(\gamma_1 + p + 1, \gamma_2 - 1)$. We gather some facts about which pairs γ, δ satisfy $\gamma \to \delta$.

Lemma 6.4.6. *If $\gamma \to \delta$ and $\gamma \not\subseteq \delta$, then $\delta = \gamma^*$.*

Proof. Boxes removed from γ must be a vertical strip, so at most one box can be removed from each row of γ. Since a horizontal strip of boxes must be added after removing the vertical strip, we may assume boxes are not removed from both rows and that all $(\delta \setminus \gamma)$-boxes are added in the row from which we did not remove a box. The claim follows by noting $(\gamma_1 - 1, \gamma_2 + p + 1)$ is either not a partition or has no boxes in the last n columns, violating (2). □

Lemma 6.4.7. *Suppose $|\gamma| \leq 2n - 3$ and $p + |\gamma| > 2n - 3$. If $\gamma^* \in P(n-2, n)$, then $\gamma \to \gamma^*$.*

Proof. Let $\delta = \gamma^*$. All $\mathbb{D}$-boxes are in row 1, thus (1) holds. The $(\gamma \setminus \delta)$-box $(2 : \delta_2 + 1)$ is related to $(1 : 2n - 2 - \delta_2)$ and the box $(1 : \gamma_2)$ above $(2 : \delta_2 + 1)$ is related to $(1 : 2n - 2 - \gamma_2)$. Since $\gamma_1 + 1 \leq 2n - 2 - \gamma_2 < 2n - 2 - \delta_2 \leq \delta_1$, we have $(1 : 2n - 2 - \delta_2)$ and $(1 : 2n - 2 - \gamma_2)$ are different $\mathbb{D}$-boxes. Hence (2) holds. □

Lemma 6.4.8. *If either $|\delta| \leq 2n - 3$ or $|\gamma| > 2n - 3$, then $\gamma \to \delta \Rightarrow \gamma \subseteq \delta$. In particular, δ is obtained from γ without removing any box of γ.*

Proof. Assume for a contradiction that $\gamma \to \delta$ but $\gamma \not\subseteq \delta$. Then by Lemma 6.4.6, $\delta = \gamma^*$. Suppose $|\gamma| > 2n - 3$. Then the box $(1 : \gamma_2)$ above the removed box is related to $(1 : 2n - 2 - \gamma_2)$, which is not in $\mathbb{D}$ since $\gamma_1 + 1 > 2n - 2 - \gamma_2$. This violates (2). Suppose $|\delta| \leq 2n - 3$. Then the removed box $(2 : \delta_2 + 1)$ is related to $(1 : 2n - 2 - \delta_2)$, which is not in $\mathbb{D}$ since $\delta_1 < 2n - 2 - \delta_2$. This violates (2). □

Given $\gamma \to \delta$, we will say a box of $\mathbb{D}$ is **killed** if it is mentioned in (1) or (2), i.e., if it is not in $\mathbb{A}$. We will say a connected component D of $\mathbb{D}$ is **bisected** if a box $\mathfrak{d}$ of D is killed but there exist boxes of D in both earlier and later columns than $\mathfrak{d}$, which are not killed. The following lemmas will help us in computing $N(\gamma, \delta)$.

Lemma 6.4.9. *If $\gamma^* \in P(n-2, n)$ and $\gamma \to \gamma^*$, then $N(\gamma, \delta) = 0$.*

Proof. Let $\delta = \gamma^*$. If $\gamma_1 \geq n - 2$, all boxes of R except $(1 : n-1)$ are mentioned in (1) or (2), so $N(\gamma, \delta) = 0$. Suppose $\gamma_1 < n - 2$. Then $\mathbb{D}_2 = \emptyset$, so $\mathbb{D} = \mathbb{D}_1$. By (1), (2) it is clear the $\mathbb{D}_1$-boxes killed are the last l boxes of $\mathbb{D}_1$ for some $l > 0$, hence $\mathbb{D}_1$ is not bisected. Thus $\mathbb{A}$ is a single component containing $(1 : n-1)$, whence $N(\gamma, \delta) = 0$. □

Whenever $\gamma \to \delta$ with $\gamma \subset \delta$, define

$$S = \{(1:c) : \delta_2 + 1 \leq c \leq \gamma_1\} \cap L \qquad \text{and} \qquad T = \{(2:c) : 1 \leq c \leq \gamma_2)\} \cap L.$$

By definition, the boxes of S and T are the γ-boxes considered in (1), hence the only boxes capable of killing $\mathbb{D}$-boxes.

Lemma 6.4.10. *Let $\gamma \to \delta$ with $\gamma \subset \delta$. Suppose $(1:c) \in \mathbb{D}_1$. If $c = n-1$ then $(1:c)$ is not killed, while if $c \neq n-1$ then*

- *$(1:c)$ is killed by S if and only if $(1:c) \in S_1' = \{(1:c') : 2n-2-\gamma_1 \leq c' \leq 2n-3-\delta_2\}$*
- *$(1:c)$ is killed by T if and only if $(1:c) \in T_1' = \{(1:c') : 2n-1-\gamma_2 \leq c' \leq 2n-2\}$.*

Suppose $(2:c) \in \mathbb{D}_2$. If $c = n-1$ then $(2:c)$ is not killed, while if $c \neq n-1$ then

- *$(2:c)$ is never killed by S*
- *$(2:c)$ is killed by T if and only if $(2:c) \in T_2' = \{(2:c') : 2n-2-\gamma_2 \leq c' \leq 2n-3\}$.*

Proof. Clearly $(1:n-1)$, $(2:n-2)$ can never be killed. The existence of a $\mathbb{D}$-box in row 2 implies $\delta_2 > n-2$ and thus $S = \emptyset$, so $(2:c)$ is never killed by S and also $(2:n-1)$ can never be killed. The remaining points also follow from the definition of being related. ☐

Corollary 6.4.11. *Suppose $\gamma \to \delta$ with $\gamma \subset \delta$. Then if $(1:2n-2-\delta_2)$ is a $\mathbb{D}_1$-box, it is not killed.*

Proof. Since $2n-3-\delta_2 < 2n-2-\delta_2 < 2n-1-\gamma_2$, $(1:2n-2-\delta_2)$ is not in S_1' or T_1'. ☐

Lemma 6.4.12. *A connected component of $\mathbb{D}$ is bisected if and only if all of the following hold:*

(i) $|\gamma| \leq 2n-3$ *and* $|\delta| > 2n-3$

(ii) $\gamma \subseteq \delta$

(iii) $\gamma_1 < n-1$

(iv) $\delta_2 < \gamma_1$.

Proof. ($\Rightarrow$, by contrapositive) If (ii) does not hold, then by the proof of Lemma 6.4.9 no component of $\mathbb{D}$ is bisected, so assume (ii) holds. Then for a given component D of $\mathbb{D}$, by Lemma 6.4.10 T kills the latest l boxes of D for some $l \geq 0$ and thus does not bisect D. So only S can bisect D. If (iv) does not hold, then $S = \emptyset$ and $\mathbb{D}$ cannot be bisected. Suppose (iii) does not hold. We may assume $\mathbb{D}_2 = \emptyset$, otherwise $S = \emptyset$ and we are done. Then $\mathbb{D} = \mathbb{D}_1$, and since $2n-2-\gamma_1 \leq \gamma_1 + 1$, we have $\mathbb{D}_1 \setminus S_1'$ is connected. Finally, suppose (i) does not hold. Then either $|\gamma| > 2n-3$ or $|\delta| \leq 2n-3$. We may assume the latter three conditions hold. Then (iii) implies $|\gamma| < 2n-3$, so we must have $|\delta| \leq 2n-3$. Then $\mathbb{D} = \mathbb{D}_1$. Since $2n-3-\delta_2 \geq \delta_1$, $\mathbb{D}_1 \setminus S_1'$ is connected.

($\Leftarrow$) Suppose all four conditions hold. Then by (iii) and (iv), $\delta_2 < n-2$, so $\mathbb{D} = \mathbb{D}_1$. By (i) $|\delta| > 2n-3$, so $\delta_1 > n-1$, and since by (iii) $\gamma_1 < n-1$, we have $(1 : n-1)$ is a $\mathbb{D}_1$-box and is not killed. Next, $(1 : 2n-2-\delta_2)$ is a $\mathbb{D}_1$-box since by (i) $2n-2-\delta_2 \leq \delta_1$, and by Corollary 6.4.11 it is not killed. Finally, since by (iv) $\delta_2 < \gamma_1$ we have $n-1 < 2n-2-\gamma_1 \leq 2n-3-\delta_2 < 2n-2-\delta_2$. In particular, $S_1' \neq \emptyset$, so a $\mathbb{D}_1$-box between $(1 : n-1)$ and $(1 : 2n-2-\delta_2)$ is killed. Hence $\mathbb{D}_1$ is bisected. $\square$

Corollary 6.4.13. *If a connected component of $\mathbb{D}$ is bisected, then $N(\gamma, \delta) = 1$.*

Proof. By the proof of Lemma 6.4.12, if a connected component of $\mathbb{D}$ is bisected then $\mathbb{D} = \mathbb{D}_1$, so $\mathbb{D}_1$ is bisected. It also follows from the proof that $\mathbb{D}_1 \setminus (S_1' \cup T_1') = \mathbb{A}$ has two connected components, one of which uses $(1 : n-1)$. Thus $N(\gamma, \delta) = 1$. $\square$

Lemma 6.4.14. *If $\gamma \to \delta$ with $\gamma \subset \delta$, $|\gamma| \leq 2n-3$, $|\delta| > 2n-3$, $\gamma_1 \geq n-1$ and also $\mathbb{D}_1$ is nonempty, then not all $\mathbb{D}_1$-boxes are killed.*

Proof. Since $\delta_1 > 2n-3-\delta_2$, we have $(1 : \delta_1) \in \mathbb{D}_1 \setminus S_1'$. Since $\gamma_1 + 1 < 2n-1-\gamma_2$, we have $(1 : \gamma_1+1) \in \mathbb{D}_1 \setminus T_1'$. Thus if either S_1' or T_1' is empty, we are done. If both S_1' and T_1' are nonempty, then $(2n-2-\delta_2)$ is a $\mathbb{D}_1$-box since $2n-3-\delta_2 < 2n-2-\delta_2 < 2n-1-\gamma_2$. By Corollary 6.4.11 it is not killed. $\square$

Now we consider the RYD model.

Lemma 6.4.15. *Suppose $p \neq 2n-2$. Then a (legal) shape μ appears in the expansion $\alpha_p \star \lambda$ if and only if $f(\mu)$ appears in the expansion $\sigma_p \cdot \sigma_\gamma$.*

Proof. Let $\Delta = \{\delta \in P(n-2, n) : \gamma \subset \delta \text{ and } |\delta| = |\gamma| + p\}$. There are three cases:

($p + |\lambda| \leq 2n-3$:) By (A), the shapes in $\alpha_p \star \lambda$ are those created by adding a horizontal strip of size p to $\overline{\lambda}$. The image of the legal shapes under f are Δ. Every element of Δ satisfies (1) and (2), so $\gamma \to \delta$ for every element δ of Δ. By Lemma 6.4.8, there are no other $\delta' \in P(n-2, n)$ such that $\gamma \to \delta'$.

($|\lambda| > 2n-3$:) By (C), the shapes in $\alpha_p \star \lambda$ are those created by adding a horizontal strip of size p to $\overline{\lambda}$. If $p \leq \overline{\lambda}_1 + 1 - \overline{\lambda}_2$ the images of the legal shapes are Δ, otherwise their images are $\Delta \setminus \{(\gamma_2 + p, \gamma_1)\}$. If $p \leq \overline{\lambda}_1 + 1 - \overline{\lambda}_2$ every element of Δ satisfies (1) and (2), otherwise every element of Δ satisfies (1) and (2) except for $(\gamma_2 + p, \gamma_1)$ which fails (1). Then we are done by Lemma 6.4.8.

($|\lambda| \le 2n-3$ and $p+|\lambda| > 2n-3$:) By (B), the shapes in $\alpha_p \star \lambda$ are those created by adding a horizontal strip of size p to $\overline{\lambda}$ and then removing a box from either the first or second row (to occupy the root of the top region). The images of the legal shapes are $\Delta \cup \{\gamma^*\}$. Every element of Δ satisfies (1) and (2), and also $\gamma \to \gamma^*$ by Lemma 6.4.7. Then we are done by Lemma 6.4.6. $\square$

If $p = 2n-2$, then $\alpha_p = \langle 2n-3, 0|\bullet\rangle$ and straightforwardly $\alpha_p \star \lambda = 0$ (and thus by Lemma 6.4.15 $\sigma_p \cdot \sigma_\gamma = 0$) unless $\lambda = \langle\overline{\lambda}|\circ\rangle$ and $\overline{\lambda}_2 = 0$, i.e., $\lambda = \alpha_q$ for some $q < 2n-2$. Thus we may assume $p < 2n-2$. Then by Lemma 6.4.15 it suffices to show that for any (legal) $c \cdot \mu$ appearing in $\alpha_p \star \lambda$ we have $c = 2^{N(\gamma,\delta)}$, where $\delta = f(\mu)$. Since illegal terms do not contribute, and $f(\mu) \in P(n-2, n)$ if and only if μ is legal, we may assume the terms whose coefficients we examine below are legal.

Case 1: ($p + |\lambda| \le 2n-3$): By (A), the coefficient of each term in $\alpha_p \star \lambda$ is 1. Thus we must show the image δ of any term has $N(\gamma, \delta) = 0$. Since $|\delta| \le 2n-3$, we have $\mathbb{D} = \mathbb{D}_1$. If $\gamma_1 \ge n-1$, then since $2n-2-\gamma_1 \le \gamma_1 + 1$ and $2n-3-\delta_2 \ge \delta_1$, we have $\mathbb{D}_1 \setminus S_1' = \emptyset$, so $N(\gamma,\delta) = 0$. Suppose $\gamma_1 < n-1$. If $\mathbb{D}_1 = \emptyset$, then $N(\gamma,\delta) = 0$. Otherwise, $(1 : n-1) \in \mathbb{D}_1$ and is not killed, whence $N(\gamma,\delta) = 0$ follows since by Lemma 6.4.12, $\mathbb{D}_1$ is not bisected.

Case 2: ($|\lambda| > 2n-3$): By (C), the coefficient of each term in $\alpha_p \star \lambda$ is 1. Thus we must show the image δ of any term has $N(\gamma,\delta) = 0$. Since $2n-1-\gamma_1 \le \gamma_1 + 1$, we have $\mathbb{D}_1 \setminus T_1' = \emptyset$, so only $\mathbb{D}_2$ can contribute to $\mathbb{A}$. If $\gamma_2 \ge n-2$ then all boxes of R in row 2 except $(2 : n-1)$ are mentioned in (1), hence $N(\gamma,\delta) = 0$. Suppose $\gamma_2 < n-2$. If $\mathbb{D}_2 = \emptyset$, then $N(\gamma,\delta) = 0$. Otherwise $(2 : n-1) \in \mathbb{D}_2$ and is not killed, and then $N(\gamma,\delta) = 0$ follows since by Lemma 6.4.12, $\mathbb{D}_2$ is not bisected.

Case 3: ($|\lambda| \le 2n-3$, $p + |\lambda| > 2n-3$): Let $M = \min\{\overline{\lambda}_1 - \overline{\lambda}_2, p\}$. Then by (B), we compute

$$\alpha_p \star \lambda = \langle \overline{\lambda}_1 + p, \overline{\lambda}_2 - 1|\bullet\rangle + 2\sum_{1 \le j \le M} \langle \overline{\lambda}_1 + p - j, \overline{\lambda}_2 - 1 + j|\bullet\rangle + \langle \overline{\lambda}_1 + p - M - 1, \overline{\lambda}_2 + M|\bullet\rangle.$$

First suppose $\delta = f(\langle \overline{\lambda}_1 + p, \overline{\lambda}_2 - 1|\bullet\rangle) = \gamma^*$. Then $N(\gamma,\delta) = 0$ by Lemmas 6.4.7 and 6.4.9.

Next, suppose δ is the image of a term in the summation. If $\gamma_1 < n-1$, then since $\delta_2 < \gamma_1$ a component of $\mathbb{D}$ is bisected by Lemma 6.4.12. Thus $N(\gamma,\delta) = 1$ by Corollary 6.4.13. Therefore, suppose $\gamma_1 \ge n-1$. By Lemma 6.4.12 no component of $\mathbb{D}$ is bisected, and since $\delta_2 < \gamma_1$ we have $\mathbb{D}_1$ is not connected to $\mathbb{D}_2$. Since $\gamma_2 \le n-2$, if $\mathbb{D}_2 \neq \emptyset$ then $(2 : n-1) \in \mathbb{D}_2$ and is not killed, so $\mathbb{D}_2$ does not contribute to $N(\gamma,\delta)$. Since $\gamma_1 \ge n-1$, we have $(1 : n-1) \notin \mathbb{D}_1$, and since $\mathbb{D}_1 \neq \emptyset$, by Lemma 6.4.14 not every box of $\mathbb{D}_1$ is killed. Thus $\mathbb{D}_1$ contributes 1 to $N(\gamma,\delta)$, whence $N(\gamma,\delta) = 1$.

Finally, suppose $\delta = f(\langle \overline{\lambda}_1 + p - M - 1, \overline{\lambda}_2 + M|\bullet\rangle)$. Then either $\delta_2 = \gamma_1$ or $\mathbb{D}_1 = \emptyset$. If $\delta_2 = \gamma_1$ then

$\mathbb{D} = \mathbb{D}_1 \cup \mathbb{D}_2$ is connected, and since $\gamma_2 \leq n-2$ it uses $(2:n-1)$. By Lemma 6.4.12 $\mathbb{D}$ is not bisected, hence $N(\gamma,\delta) = 0$. Thus suppose $\mathbb{D}_1 = \emptyset$. Then if also $\mathbb{D}_2 = \emptyset$, we have $N(\gamma,\delta) = 0$. Otherwise, since $\gamma_1 \leq n-2$ we have $(2:n-1) \in \mathbb{D}_2$, and $(2:n-1)$ is not killed. Then $N(\gamma,\delta) = 0$ follows since by Lemma 6.4.12, $\mathbb{D}_2$ is not bisected. $\square$

6.5 Proof of Theorem 6.1.8(II)

We now follow [13, pg. 31-33]. The Schubert varieties of $OG(2,2n)$ are indexed by the set $\tilde{P}(n-2,n)$ of all pairs $\tilde{\gamma} = (\gamma; \mathtt{type}(\gamma))$, where γ is an element of the set $P(n-2,n)$ of all $(n-2)$-strict partitions inside a $2 \times (2n-3)$ rectangle, and also $\mathtt{type}(\gamma) = 0$ if no part of γ has size $n-2$ and $\mathtt{type}(\gamma) \in \{1,2\}$ otherwise. The **Pieri classes** of [13] are those indexed by $\tilde{\gamma}$ with $\gamma = (p,0)$. If $p \neq n-2$ then the class is denoted by σ_p. Otherwise if $\mathtt{type}(\gamma) = 1$ (respectively, $\mathtt{type}(\gamma) = 2$) the class is denoted σ_{n-2} (respectively, σ'_{n-2}).

Fix an integer $p \in [1, 2n-3]$, and suppose $\gamma, \delta \in P(n-2,n)$ with $|\delta| = |\gamma| + p$. Then the relation $\gamma \to \delta$ is defined as in the previous section, except now the box in row r and column c of γ is **related** to the box in row r' and column c' if $|c - (2n-3)/2| + r = |c' - (2n-3)/2| + r'$.

Define $\mathbb{A}$ as in the previous section. Then define $N'(\gamma,\delta)$ to be the number of connected components of $\mathbb{A}$ (respectively, one less than this number) if $p \leq n-2$ (respectively, if $p > n-2$).

Let $g(\gamma,\delta)$ be how many of the first $n-2$ columns of δ have no $(\delta \setminus \gamma)$-boxes, and let $h(\tilde{\gamma},\tilde{\delta}) = g(\gamma,\delta) + \max(\mathrm{type}(\gamma), \mathrm{type}(\delta))$. If $p \neq n-2$, set $\epsilon_{\tilde{\gamma}\tilde{\delta}} = 1$. If $p = n-2$ and $N'(\gamma,\delta) > 0$, set $\epsilon_{\tilde{\gamma}\tilde{\delta}} = \epsilon'_{\tilde{\gamma}\tilde{\delta}} = \frac{1}{2}$, while if $N'(\gamma,\delta) = 0$, define

$$\epsilon_{\tilde{\gamma}\tilde{\delta}} = \begin{cases} 1 & \text{if } h(\tilde{\gamma},\tilde{\delta}) \text{ is odd} \\ 0 & \text{otherwise} \end{cases} \qquad \textit{and} \qquad \epsilon'_{\tilde{\gamma}\tilde{\delta}} = \begin{cases} 1 & \text{if } h(\tilde{\gamma},\tilde{\delta}) \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$$

Then the specialization of the Pieri rule of [13, Theorem 3.1] to the adjoint $OG(2,2n)$ is

Theorem 6.5.1 ([13])**.** *(Pieri rule for OG(2,2n)) For any $\tilde{\gamma} \in \tilde{P}(n-2,n)$ and integer $p \in [1,2n-3]$,*

$$\sigma_p \cdot \sigma_{\tilde{\gamma}} = \sum_{\tilde{\delta}} \epsilon_{\tilde{\gamma}\tilde{\delta}} 2^{N'(\gamma,\delta)} \sigma_{\tilde{\delta}}$$

where the sum is over all $\tilde{\delta} \in \tilde{P}(n-2,n)$ with $\gamma \to \delta$ and $\mathtt{type}(\gamma) + \mathtt{type}(\delta) \neq 3$. Furthermore, the product $\sigma'_{n-2} \cdot \sigma_{\tilde{\gamma}}$ is obtained by replacing $\epsilon_{\tilde{\gamma}\tilde{\delta}}$ with $\epsilon'_{\tilde{\gamma}\tilde{\delta}}$ throughout.

Now we consider the RYD model. In the adjoint case $k = 2$, the base region consists of two double-tailed diamonds of length $2n-4$, and the top region is a single root. We reprise the notation $\lambda = \langle\overline{\lambda}|\bullet\rangle$ for RYDs from Chapter 3 in this case. If neither $\overline{\lambda}_1$ nor $\overline{\lambda}_2$ is equal to $n-2$, then λ is said to be **neutral**, otherwise λ is **charged** and is assigned a "charge" denoted $\mathrm{ch}(\lambda)$, which is either $\uparrow$ or $\downarrow$. Let $\Pi(\lambda)$ denote $\langle\overline{\lambda}_1, \overline{\lambda}_2|\bullet/\circ\rangle$, i.e., ignoring any charge. For shapes $\lambda, \mu \in \mathbb{Y}_{OG(2,2n)}$, let $M = \min\{\overline{\lambda}_1 - \overline{\lambda}_2, \overline{\mu}_1 - \overline{\mu}_2\}$. The following reformulates Theorem 3.2.11, and includes the Pieri cases.

Definition 6.5.2. *[60, Definition 5.1] For $\lambda, \mu \in \mathbb{Y}_{OG(2,2n)}$, define an expression $\Pi(\lambda) \diamond \Pi(\mu)$:*

(A) If $|\langle\overline{\lambda}|\circ\rangle| + |\langle\overline{\mu}|\circ\rangle| \leq 2n-4$, then

$$\Pi(\langle\overline{\lambda}|\circ\rangle) \diamond \Pi(\langle\overline{\mu}|\circ\rangle) = \sum_{0\leq k\leq M} \langle\overline{\lambda}_1 + \overline{\mu}_1 - k, \overline{\lambda}_2 + \overline{\mu}_2 + k|\circ\rangle$$

(B) If $|\langle\overline{\lambda}|\circ\rangle| + |\langle\overline{\mu}|\circ\rangle| > 2n-4$, then

$$\begin{aligned}\Pi(\langle\overline{\lambda}|\circ\rangle) \diamond \Pi(\langle\overline{\mu}|\circ\rangle) = \langle\overline{\lambda}_1 + \overline{\mu}_1, \overline{\lambda}_2 + \overline{\mu}_2 - 1|\bullet\rangle + 2 \sum_{1\leq k\leq M} \langle\overline{\lambda}_1 + \overline{\mu}_1 - k, \overline{\lambda}_2 + \overline{\mu}_2 + k - 1|\bullet\rangle \\ + \langle\overline{\lambda}_1 + \overline{\mu}_1 - M - 1, \overline{\lambda}_2 + \overline{\mu}_2 + M|\bullet\rangle\end{aligned}$$

(C) $\Pi(\langle\overline{\lambda}|\bullet\rangle) \diamond \Pi(\langle\overline{\mu}|\circ\rangle) = \Pi(\langle\overline{\lambda}|\circ\rangle) \diamond \Pi(\langle\overline{\mu}|\bullet\rangle) = \sum_{0\leq k\leq M} \langle\overline{\lambda}_1 + \overline{\mu}_1 - k, \overline{\lambda}_2 + \overline{\mu}_2 + k|\bullet\rangle$

(D) $\Pi(\langle\overline{\lambda}|\bullet\rangle) \diamond \Pi(\langle\overline{\mu}|\bullet\rangle) = 0$.

Declare any α in the above expressions to be zero if $(\overline{\alpha}_1, \overline{\alpha}_2)$ is not a partition in $2 \times (2n-4)$. Such α will be called **illegal**.

If λ, μ are both charged, we say they **match** if $\mathrm{ch}(\lambda) = \mathrm{ch}(\mu)$, and are **opposite** otherwise. The opposite charge to $\mathrm{ch}(\lambda)$ is denoted $\mathrm{op}(\lambda)$. Define:

$$\eta_{\lambda,\mu} = \begin{cases} 2 & \text{if } \lambda,\ \mu \text{ are charged and match and } n \text{ is even;} \\ 2 & \text{if } \lambda,\ \mu \text{ are charged and opposite and } n \text{ is odd;} \\ 1 & \text{if } \lambda \text{ or } \mu \text{ are not charged;} \\ 0 & \text{otherwise} \end{cases}$$

If a κ appearing in $\Pi(\lambda) \diamond \Pi(\mu)$ has $\overline{\kappa}_1 = n-2$ or $\overline{\kappa}_2 = n-2$, we say κ is **ambiguous**. We say $\lambda \in \mathbb{Y}_{OG(2,2n)}$ is **Pieri** if $\Pi(\lambda) = \langle j, 0|\bullet/\circ\rangle$, and **non-Pieri** otherwise.

Definition 6.5.3. *Let $\lambda, \mu \in \mathbb{Y}_{OG(2,2n)}$. Define a commutative product $\star$ on $R = \mathbb{Z}[\mathbb{Y}_{OG(2,2n)}]$:*

If $\Pi(\lambda) = \Pi(\mu) = \langle n-2, 0|\circ\rangle$, *then*

$$\lambda \star \mu = \begin{cases} \sum_{0\le k\le \frac{n-2}{2}} \langle 2n-4-2k, 2k|\circ\rangle & \textit{if } n \textit{ is even and } \lambda,\ \mu \textit{ match} \\ \sum_{0\le k\le \frac{n-4}{2}} \langle 2n-5-2k, 2k+1|\circ\rangle & \textit{if } n \textit{ is even and } \lambda,\ \mu \textit{ are opposite} \\ \sum_{0\le k\le \frac{n-3}{2}} \langle 2n-5-2k, 2k+1|\circ\rangle & \textit{if } n \textit{ is odd and } \lambda,\ \mu \textit{ match} \\ \sum_{0\le k\le \frac{n-3}{2}} \langle 2n-4-2k, 2k|\circ\rangle & \textit{if } n \textit{ is odd and } \lambda,\ \mu \textit{ are opposite} \end{cases}$$

where for the first and third cases above, the shape $\langle n-2, n-2|\circ\rangle$ *is assigned* $\mathrm{ch}(\lambda) = \mathrm{ch}(\mu)$.

Otherwise, compute $\Pi(\lambda) \diamond \Pi(\mu)$ *and*

(i) First, replace any term κ *that has* $\overline{\kappa}_1 = 2n-4$ *by* $\eta_{\lambda,\mu}\kappa$.

(ii) Next, replace each κ *by* $2^{\mathrm{fsh}(\kappa)-\mathrm{fsh}(\lambda)-\mathrm{fsh}(\mu)}\kappa$.

(iii) Lastly, "disambiguate" using one in the following complete list of possibilities:

(iii.1) (if λ, μ are both non-Pieri) *Replace any ambiguous* κ *by* $\frac{1}{2}(\kappa^{\uparrow} + \kappa^{\downarrow})$.

(iii.2) (if one of λ, μ is neutral and Pieri) *Since* $\Pi(\lambda) \diamond \Pi(\mu) = \Pi(\mu) \diamond \Pi(\lambda)$, *we may assume* λ *is Pieri. Then replace any ambiguous* κ *by* $\frac{1}{2}(\kappa^{\uparrow} + \kappa^{\downarrow})$ *if* μ *is neutral, and by* $\kappa^{\mathrm{ch}(\mu)}$ *if* μ *is charged.*

(iii.3) (if one of λ, μ is charged and Pieri, and the other is non-Pieri). *As above, we may assume* λ *is Pieri. In particular,* $\Pi(\lambda) = \langle n-2, 0|\circ\rangle$.

(iii.3a) If $\mu = \langle\overline{\mu}|\bullet\rangle$ *is neutral and* $|\overline{\mu}| = 2n-4$, *then replace the ambiguous term* $\langle 2n-4, n-2|\bullet\rangle$ *by* $\langle 2n-4, n-2|\bullet\rangle^{\mathrm{ch}(\lambda)}$ *if* $\overline{\mu}_1$ *is even and by* $\langle 2n-4, n-2|\bullet\rangle^{\mathrm{op}(\lambda)}$ *if* $\overline{\mu}_1$ *is odd.*

(iii.3b) Otherwise, replace any ambiguous κ *by* $\frac{1}{2}(\kappa^{\uparrow}+\kappa^{\downarrow})$ *if* μ *is neutral, and by* $\kappa^{\mathrm{ch}(\mu)}$ *if* μ *is charged.*

Define

$$f(\Pi(\lambda)) = \begin{cases} (\overline{\lambda}_1, \overline{\lambda}_2) \in P(n-2, n) & \text{if } \lambda = \langle\overline{\lambda}|\circ\rangle \\ (\overline{\lambda}_1 + 1, \overline{\lambda}_2) \in P(n-2, n) & \text{if } \lambda = \langle\overline{\lambda}|\bullet\rangle. \end{cases}$$

Then the following specializes Theorem 6.1.6 to the adjoint case $k = 2$, where we write F instead of F_2:

Proposition 6.5.4. *The elements of* $\mathbb{Y}_{OG(2,2n)}$ *are in bijection with the elements of* $\tilde{P}(n-2, n)$ *via*

$$F(\lambda) = \begin{cases} (f(\Pi(\lambda)); 0) & \textit{if } \lambda \textit{ is neutral} \\ (f(\Pi(\lambda)); 1) & \textit{if } \lambda \textit{ is assigned} \uparrow \\ (f(\Pi(\lambda)); 2) & \textit{if } \lambda \textit{ is assigned} \downarrow \end{cases}$$

Let α_p denote $\langle p, 0|\bullet/\circ\rangle \in \mathbb{Y}_{OG(2,2n)}$. Throughout, given $\lambda \in \mathbb{Y}_{OG(2,2n)}$ let γ denote $f(\Pi(\lambda))$ and $\tilde{\gamma}$ denote $F(\lambda)$. The following restates Theorem 6.1.8(II):

Theorem 6.5.5.

$$F(\alpha_p \star \lambda) = F(\alpha_p) \cdot F(\lambda).$$

Proof. Let $(r : c)$ denote the box in row r, column c of $2 \times (2n-3)$. Let L denote the first $n-2$ columns of $2 \times (2n-3)$ and R the latter $n-1$ columns. Given $\gamma, \delta \in P(n-2, n)$ with $|\delta| = |\gamma| + p$, recall from the previous section the definitions of $\mathbb{D}_1$, $\mathbb{D}_2$ and $\mathbb{D}$. Let γ^* denote the shape $(\gamma_1 + p + 1, \gamma_2 - 1)$. The following three lemmas are proved similarly to (respectively) Lemmas 6.4.6, 6.4.7 and 6.4.8.

Lemma 6.5.6. *If $\gamma \to \delta$ and $\gamma \not\subseteq \delta$, then $\delta = \gamma^*$.*

Lemma 6.5.7. *Suppose $|\gamma| \leq 2n-4$ and $p + |\gamma| > 2n-4$. If $\gamma^* \in P(n-2, n)$, then $\gamma \to \gamma^*$.*

Lemma 6.5.8. *If either $|\delta| \leq 2n-4$ or $|\gamma| > 2n-4$, then $\gamma \to \delta \Rightarrow \gamma \subseteq \delta$. In particular, δ is obtained from γ without removing any box of γ.*

Given $\gamma \to \delta$, recall from the previous section the definition of when a box of $\mathbb{D}$ is **killed** and when a connected component $\mathbb{D}$ is **bisected**. If also $\gamma \subset \delta$, recall the definitions of S and T.

Lemma 6.5.9. *If $\gamma^* \in P(n-2, n)$ and $\gamma \to \gamma^*$, then $N'(\gamma, \delta) = 1$ if $\gamma_1 < n-2$ and $p \leq n-2$, and $N'(\gamma, \delta) = 0$ otherwise.*

Proof. Let $\delta = \gamma^*$. If $\gamma_1 \geq n-2$, all boxes of R are mentioned in (1) or (2), so $N'(\gamma, \delta) = 0$. Suppose $\gamma_1 < n-2$. Then $\mathbb{D}_2 = \emptyset$, so $\mathbb{D} = \mathbb{D}_1$. Here $(1 : n-1)$ is a $\mathbb{D}_1$-box and is not killed. By (1), (2) it is clear the $\mathbb{D}_1$-boxes killed are the last l boxes of $\mathbb{D}_1$ for some $l > 0$, hence $\mathbb{D}_1$ is not bisected. Thus $N'(\gamma, \delta) = 0$ if $p > n-2$, and $N'(\gamma, \delta) = 1$ if $p \leq n-2$. ☐

Lemma 6.5.10. *Let $\gamma \to \delta$ with $\gamma \subset \delta$. Suppose $(1 : c) \in \mathbb{D}_1$. Then*

- $(1 : c)$ *is killed by S if and only if* $(1 : c) \in S'_1 = \{(1 : c') : 2n-3-\gamma_1 \leq c' \leq 2n-4-\delta_2\}$
- $(1 : c)$ *is killed by T if and only if* $(1 : c) \in T'_1 = \{(1 : c') : 2n-2-\gamma_2 \leq c' \leq 2n-3\}$

Suppose $(2 : c) \in \mathbb{D}_2$. *Then*

- $(2 : c)$ *is never killed by S*
- $(2 : c)$ *is killed by T if and only if* $(2 : c) \in T'_2 = \{(2 : c') : 2n-3-\gamma_2 \leq c' \leq 2n-4\}$

Proof. That $(2 : c)$ is never killed by S follows since the existence of a $\mathbb{D}$-box in row 2 implies $\delta_2 > n-2$ and thus $S = \emptyset$. The remaining points follow from the definition of being related. $\square$

Corollary 6.5.11. *Suppose $\gamma \to \delta$ with $\gamma \subset \delta$. Then if $(1 : 2n-3-\delta_2)$ is a $\mathbb{D}_1$-box, it is not killed.*

Proof. Since $2n-4-\delta_2 < 2n-3-\delta_2 < 2n-2-\gamma_2$, $(1 : 2n-3-\delta_2)$ is not in S_1' or T_1'. $\square$

Lemma 6.5.12. *A connected component of $\mathbb{D}$ is bisected if and only if all of the following hold:*

(i) $|\gamma| \leq 2n-4$ *and* $|\delta| > 2n-4$

(ii) $\gamma \subseteq \delta$

(iii) $\gamma_1 < n-2$

(iv) $\delta_2 < \gamma_1$.

Proof. Similar to the proof of Lemma 6.4.12, using Corollary 6.5.11. $\square$

Corollary 6.5.13. *If a connected component of $\mathbb{D}$ is bisected, then $\mathbb{A}$ has two connected components.*

Proof. Similarly to the proof of Lemma 6.4.12, if a connected component of $\mathbb{D}$ is bisected then $\mathbb{D} = \mathbb{D}_1$, so $\mathbb{D}_1$ is bisected. It also follows from the proof that $\mathbb{D}_1 \setminus (S_1' \cup T_1') = \mathbb{A}$ has two connected components. $\square$

Lemma 6.5.14. *If $\gamma \to \delta$ with $\gamma \subset \delta$, $|\gamma| \leq 2n-4$, $|\delta| > 2n-4$, $\gamma_1 \geq n-2$ and also $\mathbb{D}_1$ is nonempty, then not all $\mathbb{D}_1$-boxes are killed.*

Proof. Similar to the proof of Lemma 6.4.14. $\square$

Now we consider the RYD model.

Lemma 6.5.15. *Suppose $p \neq 2n-3$. Then a (legal) shape κ appears in the expansion $\Pi(\alpha_p) \diamond \Pi(\lambda)$ if and only if $\gamma \to f(\kappa)$. If also $p \neq n-2$, then a (legal) shape μ appears in the expansion $\alpha_p \star \lambda$ if and only if $F(\mu)$ appears in the expansion $\sigma_p \cdot \sigma_{\tilde{\gamma}}$.*

Proof. The first claim is proved similarly to the proof of Lemma 6.4.15. Now suppose $p \neq n-2$. Then (i) has no effect on $\Pi(\alpha_p) \diamond \Pi(\lambda)$, and (ii) multiplies every term by a nonzero coefficient. Then terms are disambiguated by (iii.2). Under F, (iii.2) translates exactly to the condition $\texttt{type}(\gamma) + \texttt{type}(\delta) \neq 3$. So the charge assignments in $\alpha_p \star \lambda$ agree with the types appearing in $\sigma_p \cdot \sigma_{\tilde{\gamma}}$. This proves the second claim. $\square$

The following lemma from [60] will be used in the proof.

Lemma 6.5.16. *If $\kappa = \langle \overline{\kappa}_1, \overline{\kappa}_2 | \bullet/\circ \rangle$ appears in $\Pi(\lambda) \diamond \Pi(\mu)$ then*

$$\overline{\kappa}_1 \geq \begin{cases} \max(\overline{\lambda}_1 + \overline{\mu}_2, \overline{\lambda}_2 + \overline{\mu}_1) & \text{if } \Pi(\lambda) \diamond \Pi(\mu) \text{ is described by (A) or (C)} \\ \max(\overline{\lambda}_1 + \overline{\mu}_2, \overline{\lambda}_2 + \overline{\mu}_1) - 1 & \text{if } \Pi(\lambda) \diamond \Pi(\mu) \text{ is described by (B)} \end{cases}$$

Agreement of Definition 6.5.3 with Theorem 6.5.1 when $p > n - 2$.

Suppose $p = 2n-3$. Then $\alpha_p = \langle 2n-4, 0 | \bullet \rangle$ and by Lemma 6.5.16 or by (D) $\alpha_p \star \lambda = 0$ unless $\lambda = \langle \overline{\lambda} | \circ \rangle$ and $\overline{\lambda}_2 = 0$, in which case $\alpha_p \star \lambda = \langle 2n-4, \overline{\lambda}_1 | \bullet \rangle$ (assigned $\mathrm{ch}(\lambda)$ if $\overline{\lambda}_1 = n-2$). Clearly the only δ with $\gamma \to \delta$ is $\delta = (2n-3, \overline{\lambda}_1) = f(\langle 2n-4, \overline{\lambda}_1 | \bullet \rangle)$. We have $N'(\gamma, \delta) = 0$ since $\mathbb{D} = \mathbb{D}_1 \cup \mathbb{D}_2$ is connected. Finally, if $\overline{\lambda}_1 = n-2$ then only $(\delta; \mathtt{type}(\gamma))$ appears in $\sigma_{2n-3} \cdot \sigma_{\tilde{\gamma}}$, since $\mathtt{type}(\gamma) + \mathtt{type}(\delta) \neq 3$.

Thus assume $p < 2n-3$. By Lemma 6.5.15 and since $\epsilon_{\gamma,\delta} = 1$, it suffices to show that for any (legal) $c \cdot \mu$ appearing in $\alpha_p \star \lambda$, $c = 2^{N'(\gamma,\delta)}$, where $\tilde{\delta} = F(\mu)$. As in the $LG(2, 2n)$ case, we may assume terms whose coefficients we examine below are legal.

Case 1: ($p + |\lambda| \leq 2n-4$): Then $\alpha_p \star \lambda = \sum_{0 \leq j \leq \overline{\lambda}_1 - \overline{\lambda}_2} \langle \overline{\lambda}_1 + p - j, \overline{\lambda}_2 + j | \circ \rangle$ (neutral). For the image $\tilde{\delta}$ of any term, since $|\delta| \leq 2n-4$ we have $\mathbb{D}_2 = \emptyset$ and so $\mathbb{D} = \mathbb{D}_1$. By Lemma 6.5.12 $\mathbb{D}_1$ is not bisected, so $N'(\gamma, \delta) = 0$.

Case 2: ($|\lambda| > 2n-4$): We may assume $\overline{\lambda}_2 < n-2$, since otherwise $\Pi(\alpha_p) \diamond \Pi(\lambda) = 0$ by Lemma 6.5.16. Then $\alpha_p \star \lambda = \sum_{0 \leq j \leq \overline{\lambda}_1 - \overline{\lambda}_2} \langle \overline{\lambda}_1 + p - j, \overline{\lambda}_2 + j | \bullet \rangle$ (neutral). For the image $\tilde{\delta}$ of any term, since $\gamma_1 > n-2$ and $2n - 2 - \gamma_2 \leq \gamma_1 + 1$ we have $\mathbb{D}_1 \setminus T_1' = \emptyset$. By Lemma 6.5.12 there is no bisection, thus $N'(\gamma, \delta) = 0$.

Case 3: ($|\lambda| \leq 2n-4$, $p + |\lambda| > 2n-4$): We need three subcases.

Subcase 3a: ($\overline{\lambda}_1 < n-2$): We compute

$$\alpha_p \star \lambda = \langle \overline{\lambda}_1 + p, \overline{\lambda}_2 - 1 | \bullet \rangle + 2 \sum_{1 \leq j \leq \overline{\lambda}_1 - \overline{\lambda}_2} \langle \overline{\lambda}_1 + p - j, \overline{\lambda}_2 - 1 + j | \bullet \rangle + \langle \overline{\lambda}_2 + p - 1, \overline{\lambda}_1 | \bullet \rangle \text{ (neutral)}.$$

If $\tilde{\delta} = F(\langle \overline{\lambda}_1 + p, \overline{\lambda}_2 - 1 | \bullet \rangle) = \tilde{\gamma}^*$ then $N'(\gamma, \delta) = 0$ by Lemmas 6.5.7 and 6.5.9. For the image $\tilde{\delta}$ of a term in the summation, since $\delta_2 < \gamma_1$ a component of $\mathbb{D}$ is bisected by Lemma 6.5.12. Thus $N'(\gamma, \delta) = 1$ by Corollary 6.5.13. If $\tilde{\delta} = F(\langle \overline{\lambda}_2 + p - 1, \overline{\lambda}_1 | \bullet \rangle)$, then $\delta_2 = \gamma_1 < n-2$ so $\mathbb{D} = \mathbb{D}_1$. Then $N'(\gamma, \delta) = 0$ by Lemma 6.5.12.

Subcase 3b: ($\overline{\lambda}_1 > n-2$): Let $M = \min\{\overline{\lambda}_1 - \overline{\lambda}_2, p\}$. We compute

$$\Pi(\alpha_p) \diamond \Pi(\lambda) = \langle \overline{\lambda}_1 + p, \overline{\lambda}_2 - 1 | \bullet \rangle + 2 \sum_{1 \leq j \leq M} \langle \overline{\lambda}_1 + p - j, \overline{\lambda}_2 - 1 + j | \bullet \rangle + \langle \overline{\lambda}_1 + p - M - 1, \overline{\lambda}_2 + M | \bullet \rangle.$$

The first term is illegal. Next, (ii) multiplies any term κ by $\frac{1}{2}$ if $\overline{\kappa}_2 < n-2$, and by 1 otherwise. If a κ is ambiguous, by (iii.2) it splits.

Thus for the image δ of a term in the summation, we must $N'(\gamma,\delta)=0$ if $\delta_2 \le n-2$ and $N'(\gamma,\delta)=1$ if $\delta_2 > n-2$. Assume $\delta_2 \le n-2$. Then $\mathbb{D}=\mathbb{D}_1$, and $N'(\gamma,\delta)=0$ follows from Lemma 6.5.12. Now assume $\delta_2 > n-2$. Then $\mathbb{D}=\mathbb{D}_1\cup\mathbb{D}_2$, where $\mathbb{D}_1,\mathbb{D}_2 \neq \emptyset$ and $\mathbb{D}_1$ is not connected to $\mathbb{D}_2$. Then $N'(\gamma,\delta)=1$ by Lemma 6.5.12, Lemma 6.5.14 and the fact that (since $\gamma_2 < n-2$), $(2:n-1) \in \mathbb{D}_2 \setminus T_2'$. If $\delta = f(\langle \overline{\lambda}_1+p-M-1, \overline{\lambda}_2+M|\bullet\rangle)$, we have $\mathbb{D}=\mathbb{D}_1\cup\mathbb{D}_2$ is connected. Then $N'(\gamma,\delta)=0$ by Lemma 6.5.12.

Subcase 3c: ($\overline{\lambda}_1 = n-2$): We compute

$$\alpha_p \star \lambda = \sum_{1\le j\le n-2-\overline{\lambda}_2} \langle n-2+p-j, \overline{\lambda}_2-1+j|\bullet\rangle + \langle \overline{\lambda}_2+p-1, n-2|\bullet\rangle^{\mathrm{ch}(\lambda)}.$$

For the image $\tilde{\delta}$ of each term, since $\delta_2 \le n-2$ we have $\mathbb{D}=\mathbb{D}_1$. Then $N'(\gamma,\delta)=0$ by Lemma 6.5.12.

Agreement of Definition 6.5.3 with Theorem 6.5.1 when $p < n-2$.

By Lemma 6.5.15 and since $\epsilon_{\gamma,\delta}=1$, it suffices to show that for any $c\cdot\mu$ appearing in $\alpha_p\star\lambda$, $c = 2^{N'(\gamma,\delta)}$, where $\tilde{\delta}=F(\mu)$.

Case 1: ($p+|\lambda| \le 2n-4$): There are two subcases.

Subcase 1a: ($\overline{\lambda}_1 \ge n-2$): We compute $\alpha_p\star\lambda = \sum_{0\le j\le p}\langle \overline{\lambda}_1+p-j, \overline{\lambda}_2+j|\circ\rangle$, where any term with first entry $n-2$ is assigned $\mathrm{ch}(\lambda)$. For the image $\tilde{\delta}$ of any term, since $|\delta|\le 2n-4$ we have $\delta_2\le n-2$ and $\mathbb{D}=\mathbb{D}_1$. Since $2n-3-\gamma_1 \le \gamma_1+1$ and $2n-4-\delta_2 \ge \delta_1$, we have $\mathbb{D}_1\setminus S_1' = \emptyset$, so $N'(\gamma,\delta)=0$.

Subcase 1b: ($\overline{\lambda}_1 < n-2$): We compute $\Pi(\alpha_p)\diamond\Pi(\lambda) = \sum_{0\le j\le M}\langle \overline{\lambda}_1+p-j, \overline{\lambda}_2+j|\circ\rangle$, where $M = \min\{\overline{\lambda}_1-\overline{\lambda}_2, p\}$. Now, (i) has no effect, and (ii) multiplies a term κ by 1 if $\overline{\kappa}_1 < n-2$, and by 2 otherwise. If a κ is ambiguous, it splits by (iii.2). Thus if $\delta = f(\kappa)$, we must show $N'(\gamma,\delta)=0$ if $\delta_1\le n-2$ and $N'(\gamma,\delta)=1$ if $\delta_1 > n-2$. If $\delta_1 \le n-2$ then $\mathbb{D}=\emptyset$ so $N'(\gamma,\delta)=0$. Suppose $\delta_1 > n-2$. Then since $\delta_2 < n-2$, we have $\mathbb{D}=\mathbb{D}_1$. Since $\delta_1 > n-2$ and $\gamma_1 < n-2$, we have $(1:n-1)\in\mathbb{D}_1$ and is not killed. Then $N'(\gamma,\delta)=1$ follows from Lemma 6.5.12.

Case 2: ($|\lambda| > 2n-4$): Let $M=\min\{\overline{\lambda}_1-\overline{\lambda}_2, p\}$. There are two subcases.

Subcase 2a: ($\overline{\lambda}_2 \ge n-2$): Here $\alpha_p\star\lambda = \sum_{0\le j\le M}\langle \overline{\lambda}_1+p-j, \overline{\lambda}_2+j|\bullet\rangle$, where any charged term has charge $\mathrm{ch}(\lambda)$. For the image $\tilde{\delta}$ of any term, since $\gamma_2\ge n-2$ all boxes of R except $(1:n-1)$ are mentioned in (1). Since $(1:n-1)$ is not a $\mathbb{D}$-box, we have $N'(\gamma,\delta)=0$.

Subcase 2b: ($\overline{\lambda}_2 < n-2$): Here, $\Pi(\alpha_p)\diamond\Pi(\lambda) = \sum_{0\le j\le M}\langle \overline{\lambda}_1+p-j, \overline{\lambda}_2+j|\bullet\rangle$. Then (ii) multiplies a term κ by 1 if $\overline{\kappa}_2 < n-2$, and by 2 otherwise. If a κ is ambiguous, by (iii.2) it splits. Therefore, if $\delta = f(\kappa)$, we must show that $N'(\gamma,\delta)=0$ if $\delta_2\le n-2$ and $N'(\gamma,\delta)=1$ if $\delta_2>n-2$. For any δ, since $2n-2-\gamma_2\le\gamma_1+1$ we have $\mathbb{D}_1\setminus T_1'=\emptyset$. Thus if $\delta_2\le n-2$, then $\mathbb{D}_2=\emptyset$, so $\mathbb{A}=\emptyset$ and $N'(\gamma,\delta)=0$. If $\delta_2>n-2$ then $(2:n-1)\in\mathbb{D}_2$ is not killed. Then $N'(\gamma,\delta)=1$ by Lemma 6.5.12.

Case 3: ($|\lambda|\le 2n-4$, $p+|\lambda|>2n-4$): Let $M=\min\{\overline{\lambda}_1-\overline{\lambda}_2,p\}$. There are three subcases.

Subcase 3a: ($\overline{\lambda}_1<n-2$): We compute

$$\alpha_p\star\lambda=2\langle\overline{\lambda}_1+p,\overline{\lambda}_2-1|\bullet\rangle+4\sum_{1\le j\le\overline{\lambda}_1-\overline{\lambda}_2}\langle\overline{\lambda}_1+p-j,\overline{\lambda}_2-1+j|\bullet\rangle+2\langle\overline{\lambda}_2+p-1,\overline{\lambda}_1|\bullet\rangle\text{ (neutral)}.$$

If $\tilde{\delta}=F(\langle\overline{\lambda}_1+p,\overline{\lambda}_2-1|\bullet\rangle)=\tilde{\gamma}^*$ then $N'(\gamma,\delta)=1$ by Lemmas 6.5.7 and 6.5.9. For the image $\tilde{\delta}$ of a term in the summation, since $\delta_2<\gamma_1$ a component of $\mathbb{D}$ is bisected by Lemma 6.5.12. Thus $N'(\gamma,\delta)=2$ by Corollary 6.5.13. If $\tilde{\delta}=F(\langle\overline{\lambda}_2+p-1,\overline{\lambda}_1|\bullet\rangle)$ then $\delta_2=\gamma_1<n-2$ and $\delta_1>n-2$, so $\mathbb{D}=\mathbb{D}_1$ and $(1:n-1)\in\mathbb{D}_1$ is not killed. Then $N'(\gamma,\delta)=1$ by Lemma 6.5.12.

Subcase 3b: ($\overline{\lambda}_1>n-2$): We compute

$$\Pi(\alpha_p)\diamond\Pi(\lambda)=\langle\overline{\lambda}_1+p,\overline{\lambda}_2-1|\bullet\rangle+2\sum_{1\le j\le M}\langle\overline{\lambda}_1+p-j,\overline{\lambda}_2-1+j|\bullet\rangle+\langle\overline{\lambda}_1+p-M-1,\overline{\lambda}_2+M|\bullet\rangle.$$

Then (ii) multiplies each term κ of $\Pi(\alpha_p)\diamond\Pi(\lambda)$ by 1 if $\overline{\kappa}_2<n-2$ and by 2 otherwise, after which (iii.2) splits any ambiguous κ. If $\delta=f(\langle\overline{\lambda}_1+p,\overline{\lambda}_2-1|\bullet\rangle)=\gamma^*$ then $N'(\gamma,\delta)=0$ by Lemmas 6.5.7 and 6.5.9. If $\delta=f(\langle\overline{\lambda}_1+p-M-1,\overline{\lambda}_2+M|\bullet\rangle)$ then either $\delta_2=\gamma_1$ or $\mathbb{D}_1=\emptyset$. If $\delta_2=\gamma_1$, then $N'(\gamma,\delta)=1$ follows by Lemma 6.5.12 and the fact that $(2:n-1)\in\mathbb{D}_2$ is not killed. If $\mathbb{D}_1=\emptyset$, then if $\delta_2\le n-2$ we have $\mathbb{D}_2=\emptyset$ and so $N'(\gamma,\delta)=0$, while if $\delta_2>n-2$ then by Lemma 6.5.12 and the fact that $(2:n-1)\in\mathbb{D}_2$ is not killed, we have $N'(\gamma,\delta)=1$.

For the image δ of a term in the summation, we must show $N'(\gamma,\delta)=1$ if $\delta_2\le n-2$ and $N'(\gamma,\delta)=2$ if $\delta_2>n-2$. If $\delta_2\le n-2$ then $\mathbb{D}=\mathbb{D}_1\neq\emptyset$, whence $N'(\gamma,\delta)=1$ by Lemma 6.5.12 and Lemma 6.5.14. If $\delta_2>n-2$, then since $\delta_2<\gamma_1$ we have $\mathbb{D}=\mathbb{D}_1\cup\mathbb{D}_2$, where $\mathbb{D}_1,\mathbb{D}_2\neq\emptyset$ and $\mathbb{D}_1$ is not connected to $\mathbb{D}_2$. Then $N'(\gamma,\delta)=2$ follows by Lemma 6.5.12, Lemma 6.5.14 and the fact that (since $\gamma_2<n-2$), $(2:n-1)\in\mathbb{D}_2\setminus T_2'$.

Subcase 3c: ($\overline{\lambda}_1=n-2$): We compute

$$\alpha_p\star\lambda=\langle n-2+p,\overline{\lambda}_2-1|\bullet\rangle+2\sum_{1\le j\le n-2-\overline{\lambda}_2}\langle n-2+p-j,\overline{\lambda}_2-1+j|\bullet\rangle+2\langle\overline{\lambda}_2+p-1,n-2|\bullet\rangle^{\mathrm{ch}(\lambda)}.$$

If $\tilde{\delta}=F(\langle n-2+p,\overline{\lambda}_2-1|\bullet\rangle)$ then $N'(\gamma,\delta)=0$ by Lemmas 6.5.7 and 6.5.9. The image $\tilde{\delta}$ of any other term has $\delta_2\le n-2$ and $\delta_1>n-2$, so $\mathbb{D}=\mathbb{D}_1\neq\emptyset$. Then $N'(\gamma,\delta)=1$ by Lemma 6.5.12 and Lemma 6.5.14.

Agreement of Definition 6.5.3 with Theorem 6.5.1 when $p = n-2$.

It suffices to prove this for $\sigma_{n-2} = F(\langle n-2, 0|\circ\rangle^{\uparrow})$, since the proof for $\sigma'_{n-2} = F(\langle n-2, 0|\circ\rangle^{\downarrow})$ is essentially identical.

Case 1: $(\Pi(\lambda) = \langle n-2, 0|\circ\rangle)$: We compute $\sigma_{n-2} \cdot \sigma_{\tilde{\gamma}}$. Straightforwardly, $\gamma \to \delta$ if and only if $\delta \in \{(2n-4-j, j) : 0 \le j \le n-2\}$. Then the $\tilde{\delta}$ that can appear in $\sigma_{n-2} \cdot \sigma_{\tilde{\gamma}}$ are $(\delta; 0)$ for all δ with with $\delta_2 < n-2$, and $((n-2, n-2); \texttt{type}(\gamma))$ (since $\texttt{type}(\gamma) + \texttt{type}(\delta) \neq 3$). For all such $\tilde{\delta}$ every $\mathbb{D}$-box is killed, so $N'(\gamma, \delta) = 0$. We have $g(\gamma, \delta) = n-2-\delta_2$, so $h(\gamma, \delta) = n-2-\delta_2 + \texttt{type}(\gamma)$. Thus if n is even and $\text{type}(\gamma) = 1$ or if n is odd and $\text{type}(\gamma) = 2$, we have $\epsilon_{\tilde{\gamma}, \tilde{\delta}} = 1$ for all $\tilde{\delta}$ with δ_2 even and $\epsilon_{\tilde{\gamma}, \tilde{\delta}} = 0$ for all $\tilde{\delta}$ with δ_2 odd. Likewise, if n is even and $\text{type}(\gamma) = 2$ or if n is odd and $\text{type}(\gamma) = 1$, we have $\epsilon_{\tilde{\gamma}, \tilde{\delta}} = 1$ for all $\tilde{\delta}$ with δ_2 odd and $\epsilon_{\tilde{\gamma}, \tilde{\delta}} = 0$ for all $\tilde{\delta}$ with δ_2 even. This agrees with the definition (Definition 6.5.3) of $\langle n-2, 0|\circ\rangle^{\uparrow} \star \langle n-2, 0|\circ\rangle^{\text{ch}(\lambda)}$.

In the remaining cases, we use Lemma 6.5.15. We may assume $\overline{\lambda}_2 \neq 0$, since otherwise agreement follows by previous cases.

Case 2: $(n-2+|\lambda| \le 2n-4$ and $\Pi(\lambda) \neq \langle n-2, 0|\circ\rangle)$: We compute $\langle n-2, 0|\circ\rangle^{\uparrow} \star \overline{\lambda} = \sum_{0 \le j \le \overline{\lambda}_1 - \overline{\lambda}_2} \langle n-2+\overline{\lambda}_1 - j, \overline{\lambda}_2 + j|\bullet\rangle$ (neutral, since we assume $\overline{\lambda}_2 \neq 0$). Then the images $\tilde{\delta} = (\delta; 0)$ of the terms under F are exactly the classes appearing in $\sigma_{n-2} \cdot \sigma_{\tilde{\gamma}}$. For any such $\tilde{\delta}$ we have $\gamma_1 < n-2$ and $\delta_1 > n-2$, so $\mathbb{D} = \mathbb{D}_1 \neq \emptyset$ and $(1 : n-1) \in \mathbb{D}_1$ is not killed. Then by Lemma 6.5.12 we have $N'(\gamma, \delta) = 1$, so $\epsilon_{\tilde{\gamma}, \tilde{\delta}} = \frac{1}{2}$ and $\tilde{\delta}$ has coefficient 1.

Case 3: $(|\lambda| > 2n-4)$: If $\overline{\lambda}_2 > n-2$, then $\langle n-2, 0|\circ\rangle \diamond \Pi(\lambda) = 0$ by Lemma 6.5.16. Suppose $\overline{\lambda}_2 = n-2$. Then $\langle n-2, 0|\circ\rangle^{\uparrow} \star \lambda = \frac{1}{2}\eta_{\lambda,\mu}\langle 2n-4, \overline{\lambda}_1|\bullet\rangle$, assigned $\text{ch}(\lambda)$ if $\overline{\lambda}_1 = n-2$. Let $\delta = f(\langle 2n-4, \overline{\lambda}_1|\bullet\rangle) = (2n-3, \gamma_1 - 1)$. Since $\gamma_2 = n-2$, $T'_2 = R \setminus (1 : n-1)$. Then $N'(\gamma, \delta) = 0$ since $(1 : n-1)$ is not a $\mathbb{D}$-box. Now, $g(\gamma, \delta) = n-2$ so $h(\tilde{\gamma}, \tilde{\delta}) = n-2+\texttt{type}(\gamma)$. Thus if n is even, $\epsilon_{\gamma,\delta} = 1$ if $\texttt{type}(\gamma) = 1$ and $\texttt{type}(\delta) \in \{0, 1\}$, and $\epsilon_{\gamma,\delta} = 0$ otherwise. If n is odd, $\epsilon_{\gamma,\delta} = 1$ if $\texttt{type}(\gamma) = 2$ and $\texttt{type}(\delta) \in \{0, 2\}$, and $\epsilon_{\gamma,\delta} = 0$ otherwise. This agrees with the coefficient $\frac{1}{2}\eta_{\lambda,\mu}$ of $\langle 2n-4, \overline{\lambda}_1|\bullet\rangle$, and with the charge $\text{ch}(\lambda)$ assigned if $\overline{\lambda}_1 = n-2$.

Now suppose $\overline{\lambda}_2 < n-2$. Then $\langle n-2, 0|\circ\rangle \diamond \Pi(\lambda) = \sum_{0 \le j \le M} \langle n-2+\overline{\lambda}_1 - j, \overline{\lambda}_2 + j|\bullet\rangle$. Here (i) has no effect, and since $n-2+|\overline{\lambda}| \ge 3n-6$, every (legal) term κ has $\overline{\kappa}_2 \ge n-2$, thus (ii) multiplies every term by 1. There is an ambiguous term, namely $\langle 2n-4, n-2|\bullet\rangle$, if and only if $|\lambda| = 2n-3$. Should it exist, it is disambiguated by (iii.3a). For the image δ of any term of $\langle n-2, 0|\circ\rangle \diamond \Pi(\lambda)$, since $2n-2-\gamma_2 \le \gamma_1 + 1$ we have $\mathbb{D}_1 \setminus T'_1 = \emptyset$. Then if $\delta_2 > n-2$, since $\gamma_2 < n-2$ we have $(2 : n-1) \in \mathbb{D}_2$ is not killed. So by Lemma 6.5.12, $N'(\gamma, \delta) = 1$. If $\delta = (2n-3, n-2) = f(\langle 2n-4, n-2|\bullet\rangle)$ then $\mathbb{D}_2 = \emptyset$, so $N'(\gamma, \delta) = 0$. Here $g(\gamma, \delta) = \gamma_2$, so $h(\tilde{\gamma}, \tilde{\delta}) = \gamma_2 + \texttt{type}(\delta)$. Thus $\epsilon_{\tilde{\gamma}, \tilde{\delta}} = 1$ if γ_2 is even and $\texttt{type}(\delta) = 1$ or if γ_2 is odd and $\texttt{type}(\delta) = 2$, while $\epsilon_{\gamma,\delta} = 0$ otherwise. This agrees with the disambiguation (iii.3a) of $\langle 2n-4, n-2|\bullet\rangle$.

Case 4: ($|\lambda| \leq 2n-4$, $n-2+|\lambda| > 2n-4$): There are three subcases.

Subcase 4a: ($\overline{\lambda}_1 < n-2$): We compute $\langle n-2, 0|\circ\rangle^{\uparrow} \star \lambda =$

$$\langle n-2+\overline{\lambda}_1, \overline{\lambda}_2 - 1|\bullet\rangle + 2\sum_{1\leq j\leq \overline{\lambda}_1 - \overline{\lambda}_2} \langle n-2+\overline{\lambda}_1 - j, \overline{\lambda}_2 - 1 + j|\bullet\rangle + \langle n-2+\overline{\lambda}_2 - 1, \overline{\lambda}_1|\bullet\rangle \text{ (neutral)}.$$

Then the images $\tilde{\delta} = (\delta; 0)$ of the terms under F are exactly the classes appearing in $\sigma_{n-2} \cdot \sigma_{\tilde{\gamma}}$. If $\tilde{\delta} = F(\langle n-2+\overline{\lambda}_1, \overline{\lambda}_2 - 1|\bullet\rangle) = \tilde{\gamma}^*$ then $N'(\gamma, \delta) = 1$ by Lemmas 6.5.7 and 6.5.9, hence $\epsilon_{\tilde{\gamma},\tilde{\delta}} = \frac{1}{2}$ and $\tilde{\delta}$ has coefficient 1. For the image $\tilde{\delta}$ of a term in the summation, since $\delta_2 < \gamma_1$ a component of $\mathbb{D}$ is bisected by Lemma 6.5.12, thus $N'(\gamma, \delta) = 2$ by Corollary 6.5.13. If $\tilde{\delta} = F(\langle n-2+\overline{\lambda}_2 - 1, \overline{\lambda}_1|\bullet\rangle)$ then $\delta_2 = \gamma_1 < n-2$, and since also $\delta_1 > n-2$, we have $\mathbb{D} = \mathbb{D}_1$ and $(1 : n-1) \in \mathbb{D}_1$ is not killed. Then $N'(\gamma, \delta) = 1$ by Lemma 6.5.12.

Subcase 4b: ($\overline{\lambda}_1 > n-2$): Let $M = \min\{\overline{\lambda}_1 - \overline{\lambda}_2, n-2\}$. We compute $\langle n-2, 0|\circ\rangle \diamond \Pi(\lambda) =$

$$\langle n-2+\overline{\lambda}_1, \overline{\lambda}_2 - 1|\bullet\rangle + 2\sum_{1\leq j\leq M} \langle n-2+\overline{\lambda}_1 - j, \overline{\lambda}_2 - 1 + j|\bullet\rangle + \langle n-2+\overline{\lambda}_1 - M - 1, \overline{\lambda}_2 + M|\bullet\rangle.$$

The first term is illegal. Now, (i) has no effect. Next, since $\overline{\lambda}_2 + M > n-2$, (ii) multiplies the last term by 1, while for a term κ of the summation, (ii) multiplies κ by $\frac{1}{2}$ if $\overline{\kappa}_2 < n-2$ and by 1 otherwise. Then (iii.3b) splits the ambiguous term of the summation. For the image δ of any term κ, if $\delta_2 = n-2$ we have both $(\delta; 1)$ and $(\delta; 2)$ appearing in $\sigma_{n-2} \cdot \sigma_{\tilde{\gamma}}$. This agrees with the splitting. Thus it remains to show that $N'(\gamma, \delta) = 1$ for $\delta = f(\langle n-2+\overline{\lambda}_1 - M - 1, \overline{\lambda}_2 + M|\bullet\rangle)$, while for all other δ we have $N'(\gamma, \delta) = 1$ if $\delta_2 \leq n-2$ and $N'(\gamma, \delta) = 2$ if $\delta_2 > n-2$.

Consider the image δ of a term in the summation. If $\delta_2 \leq n-2$ then $\mathbb{D} = \mathbb{D}_1 \neq \emptyset$, whence $N'(\gamma, \delta) = 1$ by Lemma 6.5.12 and Lemma 6.5.14. If $\delta_2 > n-2$, then since for any such δ we have $\delta_2 < \gamma_1$, $\mathbb{D} = \mathbb{D}_1 \cup \mathbb{D}_2$, where $\mathbb{D}_1, \mathbb{D}_2 \neq \emptyset$ and $\mathbb{D}_1$ is not connected to $\mathbb{D}_2$. Then $N'(\gamma, \delta) = 2$ follows from Lemma 6.5.12, Lemma 6.5.14 and the fact that (since $\gamma_2 < n-2$) we have $(2 : n-1) \in \mathbb{D}_2 \setminus T_1'$. If $\delta = f(\langle n-2+\overline{\lambda}_1 - M - 1, \overline{\lambda}_2 + M|\bullet\rangle)$ then either $\delta_2 = \gamma_1$ or $\mathbb{D}_1 = \emptyset$. In either case, $\mathbb{D}$ is a single connected component and $(2 : n-1) \in \mathbb{D}_2$ is not killed. Then $N'(\gamma, \delta) = 1$ follows from Lemma 6.5.12.

Subcase 4c: ($\overline{\lambda}_1 = n-2$): We compute $\langle n-2, 0|\circ\rangle^{\uparrow} \star \lambda =$

$$\frac{1}{2}\eta_{\langle n-2,0|\circ\rangle^{\uparrow},\lambda}\langle 2n-4, \overline{\lambda}_2 - 1|\bullet\rangle + \sum_{1\leq j\leq n-2-\overline{\lambda}_2} \langle 2n-4-j, \overline{\lambda}_2 - 1 + j|\bullet\rangle + \langle n-2+\overline{\lambda}_2 - 1, n-2|\bullet\rangle^{\mathrm{ch}(\lambda)}.$$

If $\delta = f(\langle 2n-4, \overline{\lambda}_2 - 1|\bullet\rangle) = \gamma^*$ then $N'(\gamma, \delta) = 0$ by Lemmas 6.5.7 and 6.5.9. Here $g(\gamma, \delta) = n-2$, so

$h(\tilde{\gamma}, \tilde{\delta}) = n - 2 + \mathtt{type}(\gamma)$. Then $\epsilon_{\gamma,\delta} = 1$ if n is even and $\mathtt{type}(\gamma) = 1$, or if n is odd and $\mathtt{type}(\gamma) = 2$, while $\epsilon_{\gamma,\delta} = 0$ otherwise. This agrees with the coefficient $\frac{1}{2}\eta_{\langle n-2,0|\circ\rangle^{\uparrow},\lambda}$ of $\langle 2n-4, \overline{\lambda}_2 - 1|\bullet\rangle$.

The F-image $\tilde{\delta} = (\delta; 0)$ of a term in the summation has $\delta_2 \leq n-2$ and $\delta_1 > n-2$, so $\mathbb{D} = \mathbb{D}_1 \neq \emptyset$. Then by Lemma 6.5.12 and Lemma 6.5.14, $N'(\gamma, \delta) = 1$. Therefore $\epsilon_{\tilde{\gamma},\tilde{\delta}} = \frac{1}{2}$, and the coefficient of $\tilde{\delta}$ is 1. For $\delta = f(\langle n-2+\overline{\lambda}_2 - 1, n-2|\bullet\rangle)$, since $\delta_2 < n-2$ and $\delta_1 > n-2$ we have $\mathbb{D} = \mathbb{D}_1 \neq \emptyset$. Then by Lemma 6.5.12 and Lemma 6.5.14, $N'(\gamma, \delta) = 1$. Therefore $\epsilon_{\tilde{\gamma},\tilde{\delta}} = \frac{1}{2}$, and the coefficient of $\tilde{\delta}$ is 1. We have only $\tilde{\delta} = (\delta; \mathtt{type}(\gamma))$ appearing in $\sigma_{n-2} \cdot \sigma_{\tilde{\gamma}}$, since $\mathtt{type}(\gamma) + \mathtt{type}(\delta) \neq 3$. This agrees with the charge assignment $\mathrm{ch}(\lambda)$. $\square$

Chapter 7

Afterword

The goal of this project was to approach the longstanding problem of finding nonnegative combinatorial rules for Schubert calculus of generalized flag varieties, using a simple, root-system uniform model (RYDs) for Schubert classes. Building on the success of this model in uniformly resolving the Schubert problem for (co)minuscule G/P's [62], we were able to use RYDs to resolve further cases (the classical (co)adjoint varieties), as well as to give a new rule for the GL_n Belkale-Kumar product.

We consider the success of the uniform RYD model in these cases to be evidence of the utility of the RYD approach to Schubert calculus, and feel it is natural to ask whether and how this model might extend to handle other cases of the Schubert problem. A next step would be to consider applying RYDs to the Schubert problem for the family of G/P's where G is of classical Lie type (other than A_n) and P is maximal. The relative easiness of the (co)adjoint cases in this family is witnessed by the relative simplicity of the two regions of the RYDs: one region is a rectangle with only two rows and the other region is a single root. For G of type B_n, C_n or D_n and P maximal, the RYDs for (non-(co)minuscule) G/P's still only have two regions, but these regions are typically much larger than those for the (co)adjoint cases. For example, the regions for $LG(k, 2n)$ are a rectangle with k rows and a staircase partition with $k-1$ rows (see the partition-pair description of RYDs in Chapter 6).

The cases where G is type B_n/C_n and $P = P_3$ excludes the third simple root seems a natural next step after the (co)adjoint cases (where $P = P_2$), and an appropriate starting point for a further application of the RYD model. In these cases, one region is a three-row rectangle and the other is a chain of three roots. It seems plausible that these regions are small and simple enough to allow one to obtain RYD rules for Schubert calculus, but also interesting enough to provide insight into how separate regions interact in the RYD model, giving potential clues as to how to proceed with applying this model to further cases of the Schubert problem.

One important challenge regarding finding how separate regions of RYDs interact is to understand the Levi-movable Schubert structure constants. Example 4.3.7 shows that, unlike for $G = GL_n$ or for the (co)adjoint cases, these structure constants are not always given by applying jeu de taquin in each region of the RYDs. Moreover, the Schubert structure constant considered in Example 4.3.7 is 8, while applying jeu de taquin in the base region gives 3. Since 3 does not divide 8, this discrepancy apparently cannot be fixed solely by multiplying by an external factor (e.g., short roots). This indicates that outside of type A_n, the regions cannot be considered independently of one another, even for Levi-movable structure constants where no roots "move" from one region to another.

It would be nice to know to what degree the planarity statements of Theorem 2.2 extend beyond the quasi-(co)minuscule family. For example, can the counterexamples to polytopality found in types D_n, E and F be extended to other cases of the Schubert problem?

It would also be useful to compare other root-system uniform models for Schubert calculus with the RYD model. One approach would be to translate the (co)adjoint RYD rules into the language of *chains in Bruhat order*, helping to build a dictionary between these two uniform models. A goal is to gain further insight as to whether/how the chains in Bruhat order model might extend to other cases of the Schubert problem. It seems plausible that increasing our understanding of either one of these models may provide clues in how to extend the other.

Another direction is to examine how the RYD model extends not just to other G/P's but to higher cohomology theories, such as K-theory or T-equivariant cohomology of a given G/P. In particular, can the RYD rules for ordinary cohomology of the classical (co)adjoint varieties be extended to rules for K-theory or T-equivariant cohomology of these varieties? No rules for (co)adjoint varieties in these cohomology theories currently exist. So far, RYDs have been used to give rules for K-theory of minuscule varieties ([63], [14]), though not for all cominuscule varieties: the K-theory of $LG(n, 2n)$ remains unsolved.

A further direction for extending the RYD model is to combine higher cohomology theories with the RYD rule for the type A_n Belkale-Kumar product (Chapter 4). For example, can we combine the K-theoretic jeu de taquin of Thomas and Yong [63] with Theorem 4.2.7 to create a K-theoretic version of the Belkale-Kumar product in type A_n? And if so, what geometric information is encoded by this new product structure?

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