COLORING AND CONSTRUCTING (HYPER)GRAPHS WITH RESTRICTIONS

BY

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DISSERTATION

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Abstract

We consider questions regarding the existence of graphs and hypergraphs with certain coloring properties and other structural properties.

In Chapter 2 we consider color-critical graphs that are nearly bipartite and have few edges. We prove a conjecture of Chen, Erdős, Gyárfás, and Schelp concerning the minimum number of edges in a "nearly bipartite" 4-critical graph.

In Chapter 3 we consider coloring and list-coloring graphs and hypergraphs with few edges and no small cycles. We prove two main results. If a bipartite graph has maximum average degree at most $2(k-1)$, then it is colorable from lists of size k; we prove that this is sharp, even with an additional girth requirement. Using the same approach, we also provide a simple construction of graphs with arbitrarily large girth and chromatic number (first proved to exist by Erdős).

In Chapter 4 we consider list-coloring the family of kth power graphs. Kostochka and Woodall conjectured that graph squares are chromatic-choosable, as a strengthening of the Total List Coloring Conjecture. Kim and Park disproved this stronger conjecture, and Zhu asked whether graph kth powers are chromatic-choosable for any k. We show that this is not true: we construct families of graphs based on affine planes whose choice number exceeds their chromatic number by a logarithmic factor.

In Chapter 5 we consider the existence of uniform hypergraphs with prescribed degrees and codegrees. In Section 5.2, we show that a generalization of the graphic 2-switch is insufficient to connect realizations of a given degree sequence. In Section 5.3, we consider an operation on 3-graphs related to the octahedron that preserves codegrees; this leads to an inductive definition for 2-colorable triangulations of the sphere. In Section 5.4, we discuss the notion of fractional realizations of degree sequences, in particular noting the equivalence of the existence of a realization and the existence of a fractional realization in the graph and multihypergraph cases.

In Chapter 6 we consider a question concerning poset dimension. Dorais asked for the maximum guaranteed size of a subposet with dimension at most d of an n-element poset. A lower bound of $\sqrt{dn}$ was observed by Goodwillie. We provide a sublinear upper bound.

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$[k]$	The set of the first k positive integers.
$\binom{X}{k}$	The set of all k-element subsets of X.
K_n	The complete graph on n vertices.
K_{r*s}	The complete r-partite graph with each part of size s.
$G[S]$	The subgraph of G induced by S (for $S \subseteq V(G)$).
$N_G(v)$	The neighborhood of v in G.
$\Delta(G)$	The maximum degree of G.
$\omega(G)$	The clique number of G.
$\chi(G)$	The chromatic number of G.
$\mathrm{ch}(G)$	The choice number of G.
$L(G)$	The line graph of G.
$T(G)$	The total graph of G.
B_n	The Boolean lattice of order n.
S_n	The standard example on $2n$ elements.
$\dim(P)$	The (Dushnik-Miller) dimension of the poset P.
$w(p)$	The width of P.
$\mathrm{ex}^*(P, \mathcal{F})$	The relative Turán number of the family of posets $\mathcal{F}$ relative to the poset P.
$\mathrm{ex}^*(n, \mathcal{F})$	The minimum of $\mathrm{ex}^*(P, \mathcal{F})$ over all n-element posets P.

Chapter 1

Introduction

1.1 Sparse nearly-bipartite 4-critical graphs

A (hyper)graph is *k-critical* if it has chromatic number k but every proper subgraph is $(k-1)$-colorable. The complete graph K_k is k-critical; for $k \leq 2$ these are the only k-critical graphs. The 3-critical graphs are precisely the odd cycles. For $k \geq 4$, the family of k-critical graphs is more interesting.

Deleting one edge from any k-critical graph makes it $(k-1)$-colorable. How many edges need to be deleted to make it $(k-2)$-colorable? 2-colorable?

Question 1.1.1. *How many edges must be deleted from a k-critical graph in order to make it bipartite? What is the minimum among all k-critical n-vertex graphs?*

Erdős conjectured that the answer to Question 1.1.1 should tend to infinity with n. Rödl and Tuza [58] disproved this conjecture: they proved that, for large enough n, the answer to the latter part of Question 1.1.1 is $\binom{k-1}{2}$. For $k = 4$, they proved that every 4-critical graph except for K_4 requires the deletion of at least 3 edges to become bipartite, and they constructed infinitely many graphs requiring the deletion of only 3 edges (using a modification of Mycielski's construction). We call graphs for which three edges may be deleted to become bipartite *$B + E_3$ graphs*.

Another important question about k-critical graphs concerns the number of edges.

Question 1.1.2. *How few edges may a k-critical n-vertex graph have?*

Question 1.1.2 has been studied at length, with a sequence of improvements by Dirac, Gallai, Ore, Krivelevich, Kostochka and Steibitz, and Kostochka and Yancey. For $k = 4$, Kostochka and Yancey [49] showed that every 4-critical n-vertex graph has at least $\frac{5n-2}{3}$ edges.

Chen, Erdős, Gyárfás, and Schelp [16] considered (for $k = 4$) the restriction of Question 1.1.2 to the class of nearly bipartite graphs (in the sense of Question 1.1.1). They considered the slightly more restricted family of graphs for which a matching of size three may be deleted to become bipartite, which we call *$(B + M_3)$-graphs*. They found an infinite family of examples of 4-critical n-vertex $(B + M_3)$-graphs with

$2n-3$ edges (one less edge than Rödl and Tuza's examples of generic 4-critical graphs). They conjectured that such graphs should have many more than $5n/3$ edges, and suggested that perhaps they should have at least $2n$ edges asymptotically.

We prove the conjecture of Chen et al. In fact, we confirm their suggestion, and even give an exact lower bound, stated below. Our results also work for $(B+E_3)$-graphs rather than just $(B+M_3)$-graphs. This is joint work with Kostochka and appears in [45].

Theorem 1.1.3. *Every n-vertex 4-critical $(B+E_3)$-graph has at least $2n-3$ edges.*

Our proof uses two main techniques. The first is the *potential function method*, also used by Kostochka and Yancey [48] to prove that any 4-critical n-vertex graph has at least $(5n-2)/3$ edges. The second is a connection between orientations and colorings of graphs.

1.2 Sparse (hyper)graphs with large girth and (list-)chromatic number

In Chapter 3, we construct several extremal examples for graphs and hypergraphs with large chromatic number or choice number, large girth, and few edges. All these examples are based on the construction of *augmented trees*: complete d-ary trees with additional edges joining each leaf to some ancestors.

Theorem 1.2.1. *For every $d, r, g \in \mathbb{N}$, there is an augmented d-ary tree with girth at least g having r edges from each leaf to its ancestors.*

If the maximum average degree of a graph is less than k, then it has a vertex with degree less than k, and inductively it is k-choosable. If the graph is also bipartite, then Alon and Tarsi [4] proved that maximum average degree at most $2(k-1)$ is sufficient for k-choosability. Our main application is that this is sharp, even when we require large girth.

Theorem 1.2.2. *For $g, k \in \mathbb{N}$, there is a bipartite graph G with girth at least g that is not k-choosable even though every proper subgraph has average degree at most $2(k-1)$ and G itself has just one too many edges for average degree $2(k-1)$.*

In fact, we also show that such graphs exist that fail to be L-colorable even when special restrictions are placed on the list assignment L. When the lists at adjacent vertices are disjoint, every coloring chosen from the lists is proper; we show that our graphs of Theorem 1.2.2 admit a k-list assignment in which any two adjacent lists have exactly one common color and yet no proper coloring can be chosen. For bipartite graphs, a proper coloring can be chosen from any k-lists whose union has size at most $2k-2$; we prove that this is sharp (for any girth) by constructing a bipartite graph with k-lists whose union has size $2k-1$ from which no proper coloring can be chosen.

We also provide a simple construction of t-uniform hypergraphs with arbitrarily large girth and chromatic number (Section 3.4.1). Mycielski's construction produces k-critical triangle-free graphs, but is fraught with 4-cycles. Erdős proved that there are graphs with arbitrarily large chromatic number and girth, but his proof was probabilistic. Subsequently, several explicit constructions were given for such graphs, but many of these constructions require hypergraphs with enormous uniformity, and are also often inductive. Our construction based on augmented trees does not use hypergraphs at all as input and is noninductive (though our construction of the augmented trees themselves is inductive).

1.3 Coloring graph powers from lists

A graph G is *chromatic-choosable* if $\chi(G) = \mathrm{ch}(G)$. The *kth power* of a graph G, denoted by G^k, is the graph on the same vertex set as G such that uv is an edge if and only if the distance from u to v in G is at most k. To *subdivide* an edge uv of a graph is to replace uv by a u,v-path whose internal vertices do not appear in the original graph.

An *edge-coloring* of a graph G is a function $f : E(G) \to X$ for some set X. An edge-coloring is *proper* if any two incident edges receive distinct colors. The *line graph* of G is the graph $L(G)$ with vertex set $E(G)$ and with $e, e' \in E(G)$ adjacent in $L(G)$ if and only if e, e' are incident in G. A proper edge-coloring of G is equivalent to a proper vertex coloring of $L(G)$. Several authors in the 1970s and 80s conjectured that for every graph G, $L(G)$ is chromatic-choosable.

A *total coloring* of a graph G is a function $f : V(G) \cup E(G) \to X$ for some set X. A total coloring is *proper* if every adjacent pair of vertices receive different colors, every adjacent pair of edges receive different colors, and every vertex receives a different color from its incident edges. The *total graph* of G is the graph $T(G)$ obtained from G by subdividing every edge into a path of length two then squaring the result. A proper total coloring of G is equivalent to a proper vertex coloring of $T(G)$. Borodin, Kostochka, and Woodall [13] conjectured that for every G, $T(G)$ is chromatic-choosable.

Kostochka and Woodall [47] conjectured that G^2 is chromatic-choosable for every graph G; this conjecture would imply the total coloring conjecture of the previous paragraph. Kim and Park [40] disproved this stronger conjecture, finding a family of graphs G with $\chi(G^2) \to \infty$ and $\mathrm{ch}(G^2) \geq c\chi(G^2)\log\chi(G^2)$ for some positive absolute constant c. Zhu asked whether there is any k such that G^k is chromatic-choosable for every

G. We answer this question in the negative, and in fact match Kim and Park's separation between choice number and chromatic number.

Theorem 1.3.1. *There is a positive constant c such that for every $k \in \mathbb{N}$, there is an infinite family of graphs G with $\chi(G^k)$ unbounded such that* $\mathrm{ch}(G^k) \geq c\chi(G^k) \log \chi(G^k)$.

On the other hand, upper bounds are known for the choice number of line graphs and total graphs in terms of their chromatic numbers; specifically, $\mathrm{ch}(L(G)) \leq 2\chi(L(G)) - 1$ and $\mathrm{ch}(T(G)) \leq 2\chi(T(G)) - 1$. We provide an upper bound for the choice number of kth power graphs in terms of their chromatic number.

Theorem 1.3.2. *For every graph G and every $k > 1$,* $\mathrm{ch}(G^k) < \chi(G^k)^3$. *When k is even,* $\mathrm{ch}(G^k) < \chi(G^k)^2$.

These two results are joint with Kosar, Petrickova, and Yeager and appear in [43].

1.4 Hypergraph degree sequences and codegree functions

The *degree sequence* of a graph is the list of its vertex degrees, usually written in nonincreasing order. A sequence of nonnegative integers is called *k-graphic* if it is the degree sequence of some simple k-uniform hypergraph. (If $k = 2$, we shorten the term to *graphic.*) The question of when a given sequence is the degree sequence of some (simple) graph is well understood. The same question for k-uniform hypergraphs for $k \geq 3$ is less well understood. Dewdney provided a characterization, but it does not provide an efficient algorithm.

Theorem 1.4.1 (Dewdney [19])**.** *Let π be a nonincreasing sequence of nonnegative integers, say $(d_1, \ldots, d_n)$. π is k-graphic if and only if there exists a nonincreasing sequence π' of $n-1$ nonnegative integers, say $(d_2', \ldots, d_n')$, such that*

- *π' is $(k-1)$-graphic,*
- *$\sum_{i=2}^{n} d_i' = (k-1)d_1$, and*
- *$\pi'' = (d_2 - d_2', \ldots, d_n - d_n')$ is k-graphic.*

Havel and Hakimi provided one efficient characterization of graphic sequences.

Theorem 1.4.2 (Havel [37], Hakimi [35])**.** *The nonincreasing sequence $d_1, \ldots, d_n$ is graphic if and only if the sequence $d_2 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n$ is graphic.*

The proof of Theorem 1.4.2 hinges on the notion of degree-preserving operations on graphs. In particular, a *2-switch* is an operation that deletes two edges from a graph and adds two new edges in such a way that vertex degrees are preserved. (The requirement that the added edges be *new* ensures that we deal only with simple graphs.) Fulkerson, Hoffman, and McAndrew [29] proved that the space of realizations of a graphic sequence is connected via 2-switches.

A *2-exchange* is an operation on multihypergraphs that deletes two edges and adds two edges in such a way that vertex degrees are preserved. For a 2-exchange, we do not place the restriction on the added edges that prevents multiple edges. In [8], it is proved that the space of multihypergraph realizations of a k-graphic sequence is connected via 2-exchanges. In Section 5.2, we prove that simple realizations are not always connected in this way.

Theorem 1.4.3. *For $k \geq 3$, there is a k-graphic sequence π with (at least) two simple realizations, neither of which admits a 2-switch (to a simple k-graph).*

The *codegree function* of a k-uniform hypergraph is the function that assigns to each $(k-1)$-set its degree. (Here we speak of the *degree* of any set of vertices, and use the word *codegree* specifically for the degree of a $(k-1)$-set. In the literature, some authors use the term codegree for any set of more than one vertex, reserving the term degree for single vertices.) In Section 5.3, we consider edge exchanges that preserve the codegree function of a uniform hypergraph.

For 3-uniform hypergraphs, every possible such edge exchange can be represented by a triangulation of a surface whose dual is bipartite (say the faces are colored red/blue): the vertices of the surface are labelled by vertices in a hypergraph, and the edge exchange deletes edges corresponding to red triangles in the triangulation and adds edges corresponding to blue triangles in the triangulation. The smallest such exchange corresponds to the octahedron. By taking the connected sum of several octahedra we can produce larger exchanges, each corresponding to a triangulation of the sphere with bipartite dual. We prove that every triangulation of the sphere whose dual is bipartite arises in this fashion. This result is related to earlier similar results of Pachner [55] and others, as well as to Barnette's Conjecture on the Hamiltonicity of bipartite cubic polyhedral graphs.

Given two realizations G and H, taking the connected sum of an octahedron with a surface corresponding to $G \triangle H$ acts as a local operation on G and H that yields G' and H' such that the codegree function of G' is the same as that of H' (but possibly not the same as that of G). Our result in this context says that a sequence of these operations can be applied such that the final graphs G'' and H'' are the same.

In Section 5.4, we interpret the question of graphicality of a sequence as an integer program and make some observations about the linear relaxation of that program. In particular, for graphicality questions for which TONCAS ("The Obvious Necessary Conditions are Also Sufficient") theorems are known, we show that the fractional relaxation has a feasible solution if and only if the integer program does. Since the fractional relaxation is a linear program, there are polynomial-time algorithms to check whether feasible solutions exist.

1.5 Large subposets with small dimension

The *intersection* of a family of posets (X, R_i) for $i \in I$ is the poset $(X, \bigcap_{i\in I} R_i)$. The *dimension* of a poset P is the minimum number of linear extensions whose intersection is P. In many ways, the dimension of posets is analogous to the chromatic number of graphs. The *standard example* S_n has dimension n, and S_n is the only n-dimensional poset with at most $2n$ elements. The standard examples seem to relate to poset dimension as complete graphs do to graph coloring (c.f. [10, 11]).

F. Dorais [21] asked how large a subposet of dimension at most d must every n-element poset contain. That is, what is $\min_{|P|=n} \max\{|Q|\colon\ Q \subseteq P, \dim Q \le d\}$? We denote this quantity by $\mathrm{ex}^*(n, \mathcal{D}_{d+1})$. The case $d = 1$ is trivial: a 1-dimensional poset is just a chain, so for fixed P the maximum size is just the height of P, and minimizing over P gives just 1 with P an antichain; so we will always consider $d \ge 2$.

Goodwillie [32] proved that the answer is at least $\sqrt{dn}$ by considering the width of P. For each d, we provide a sublinear upper bound on $\mathrm{ex}^*(n, \mathcal{D}_{d+1})$ by considering the lexicographic order on powers of standard examples. In particular, for $d = 2$, we prove that $\mathrm{ex}^*(n, \mathcal{D}_3) \le n^{0.8295}$ (for sufficiently large n).

1.6 Definitions and background

For a set X and a nonnegative integer k, we write $\binom{X}{k}$ for the family of all k-element subsets of X. We write $[k]$ for the set $\{1, \dots, k\}$.

A *multihypergraph* G is a pair $(V(G), E(G))$, where $V(G)$ is an arbitrary set of elements called the *vertices* of G and $E(G)$ is a multiset of subsets of $V(G)$ called the *edges* of G. If $E(G)$ is a set, then the multihypergraph is called a *hypergraph*, or *simple hypergraph* for emphasis. For a nonnegative integer k, a *k-uniform (multi)hypergraph* is a (multi)hypergraph in which every edge consists of exactly k vertices. A 2-uniform multihypergraph is a *multigraph*. A 2-uniform hypergraph is a *graph*, or *simple graph* for emphasis. A *uniform hypergraph* is a k-uniform hypergraph for some k.

In a graph, two vertices are *adjacent* if they form an edge, and two edges are *incident* if they share an endpoint. A vertex is *incident* to each edge containing it. In a graph G, the *neighborhood* of a vertex v, denoted $N_G(v)$ or just $N(v)$ when G is clear from context, is the set of vertices adjacent to v. A *matching* in a hypergraph is a set of edges that are pairwise non-incident.

In a hypergraph G, the *degree* of a set of vertices A is the number of edges of G containing A and is denoted $d_G(A)$ or just $d(A)$ when the hypergraph is clear from context. Most commonly we will discuss the degrees of single vertices, and we shorten the notation $d(\{v\})$ to just $d(v)$. In Chapter 5 we will sometimes use $\deg(v)$ rather than $d(v)$ for clarity.

In a graph, a vertex with degree 1 is a *leaf*, and a vertex that is adjacent to every other vertex is a *dominating vertex*.

A *subgraph* of a graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A subgraph H of G is *proper* if $E(H) \subsetneq E(G)$. For $X \subseteq V(G)$, the *subgraph of G induced by X*, denoted $G[X]$, is the subgraph of G with vertex set X and edge set $E(G) \cap \binom{X}{2}$.

A *coloring* of a hypergraph H is a function $\phi : V(H) \to X$ for some set X whose elements are called *colors*. A *k-coloring* of H is a coloring with $|X| = k$. A coloring of H is *proper* if no edge of H is monochromatic. We say that H is *k-colorable* if it admits a proper k-coloring. The *chromatic number* of H, denoted $\chi(H)$, is the minimum k such that H is k-colorable.

A *list assignment* for a hypergraph H is an assignment of a list $L(v)$ of *available colors* to each vertex v. Given such a list assignment, an *L-coloring* is a proper coloring ϕ of H such that $\phi(v) \in L(v)$ for every vertex v. We say that H is *L-colorable* if there is an L-coloring of H. If H is L-colorable for every list assignment such that $|L(v)| \geq k$ for every v, then we say that H is *k-choosable*. The least k such that H is k-choosable is the *choice number* or *list-chromatic number* of H, denoted $\mathrm{ch}(H)$.

A *cycle* in a hypergraph is an alternating list of distinct vertices and edges $v_1, e_1, v_2, e_2, \ldots, v_\ell, e_\ell$ such that for every i, $v_i \in e_{i-1} \cap e_i$ (treating indices modulo ℓ). The number ℓ is the *length* of the cycle. The *girth* of a hypergraph H is the minimum length of a cycle in H.

For $u, v \in V(G)$, a *u, v-walk* in G is an alternating list of vertices and edges $v_0, e_1, v_1, e_2, \ldots, v_{\ell-1}, e_\ell, v_\ell$ such that $v_0 = u$, $v_\ell = v$, and $v_{i-1}, v_i \in e_i$ for $i \in [\ell]$. The number ℓ is the *length* of the walk. The *distance* between u and v is the minimum length of a u, v-walk (or ∞ if there is no u, v-walk) and is denoted $\mathrm{dist}(u, v)$. A graph is *connected* if, for every two vertices u and v, there is a u, v-walk. When $u \neq v$, a *u, v-path* in G is a u, v-walk with no repeated vertex.

A graph is a *forest* if it contains no cycles. A *tree* is a connected forest.

A graph is *bipartite* if it has chromatic number at most 2. Equivalently, a graph is bipartite if and only if it has no odd cycles. A graph is *k-partite* if its chromatic number is at most k. A *complete k-partite graph* is a graph with vertex set partitioned into k independent sets (called the *partite sets* or just *parts*) and an edge $\{u, v\}$ whenever u and v are in different partite sets.

A *directed graph D* (*digraph* for short) is a pair $(V(D), E(D))$, where $V(D)$ is an arbitrary set of elements called the *vertices* of D and $E(D)$ is a set of ordered pairs of $V(D)$ called the *edges* of D. For an edge (u, v) in D we often write just uv, and we say that the edge is *from u to v*. The *outdegree* of a vertex u of D is the number of edges from u to other vertices and is denoted $d^+_D(u)$ or just $d^+(u)$. A *cycle* in a digraph is an alternating list of distinct vertices and edges $v_1, e_1, v_2, e_2, \ldots, v_\ell, e_\ell$ such that for every i, $e_i = (v_i, v_{i+1})$ (treating indices modulo ℓ). The number ℓ is the *length* of the cycle. A *path* in a digraph is an alternating list of distinct vertices and edges $v_0, e_1, v_1, e_2, \ldots, v_{\ell-1}, e_\ell, v_\ell$ such that for every i, $e_i = (v_{i-1}, v_i)$. The number ℓ is the *length* of the path.

A *partially ordered set*, or *poset* for short, is a pair (X, R) where X is a set called the *ground set* and R is a reflexive, antisymmetric, transitive relation on X. When $(x, y) \in R$, we often write $x \leq_R y$ or just $x \leq y$ when the context is clear. We write $x <_R y$ if $x \leq_R y$ and $x \neq y$. If $x <_R y$ and there is no z with $x <_R z <_R y$, we say y *covers* x and write $x \lessdot y$. We say distinct elements $x, y \in X$ are *comparable* if $x \leq y$ or $y \leq x$, and say they are *incomparable* otherwise.

Given a poset (X, R), a *subposet* is a poset (Y, S) with $Y \subseteq X$ and $S = R \cap (Y^2)$. A *subrelation* is a poset (Y, S) with $Y \subseteq X$ and $S \subseteq R$. (In some texts, what we call a subposet is referred to as an *induced subposet*, and what we call a subrelation is called a subposet.) An *ideal* I in a poset is a downward-closed subposet; that is, whenever $y \in I$ and $x \leq y$, also we have $x \in I$.

A *chain* is a poset in which any two elements are comparable. The *height* of a poset is the maximum size of a subposet that is a chain. A *linear extension* of a poset (X, R) is a chain (X, S) with $R \subseteq S$. An *antichain* is a poset in which no two elements are comparable. The *width* of a poset P, denoted $w(P)$, is the maximum size of a subposet of P that is an antichain. Dilworth's Theorem states that the elements of every poset P can be covered by $w(P)$ chains in P.

The *Boolean lattice* of order n, denoted B_n, is the subset order on the set of all subsets of $[n]$. For $n \geq 3$, the *standard example* with $2n$ elements, denoted S_n, is the subposet of B_n consisting of the singleton sets and their complements.

The *lexicographic order* on k-tuples of elements of a poset P puts $(x_1, \ldots, x_k) < (y_1, \ldots, y_k)$ when $x_i < y_i$ for $i = \min\{j : x_j \neq y_j\}$.

The *cover graph* of a poset (X, R) is the graph with vertex set X and an edge xy whenver $x \lessdot y$. A *Hasse diagram* of the poset is a drawing of the cover graph in the plane such that the vertical coordinate of x is less than the vertical coordinate of y when $x <_R y$.

Chapter 2

Sparse nearly-bipartite 4-critical graphs

The results of this chapter are joint with Alexandr Kostochka and appear in [45].

2.1 Introduction

A graph G is said to be $(k+1)$-*critical* if it is $(k+1)$-chromatic but every proper subgraph G is k-colorable.

We consider $(k+1)$-critical graphs that are "nearly bipartite" in the following sense. For an integer ℓ, we say that a graph is a $(B+E_\ell)$-*graph* if it is obtained from a bipartite graph by adding ℓ edges. We say that a graph is a $(B+M_\ell)$-*graph* if it is obtained from a bipartite graph by adding a matching of size ℓ. (In [16], a $(B+M_\ell)$-graph is denoted by $B+\ell$.)

Rödl and Tuza [58] disproved a conjecture by Erdős by finding infinitely many $(k+1)$-critical $(B+E_{\binom{k}{2}})$-graphs. They also showed that this is best possible: when n is large enough, there are no n-vertex $(k+1)$-critical $(B+E_\ell)$-graphs with $\ell < \binom{k}{2}$.

Chen, Erdős, Gyárfás, and Schelp [16] strengthened Rödl and Tuza's result for $k=3$: they showed that for all sufficiently large n, there are 4-critical n-vertex $(B+M_3)$-graphs. An example is shown in Figure 2.1.

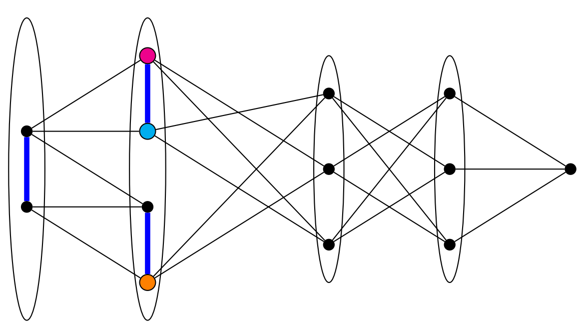


Figure 2.1: A 4-critical $(B+M_3)$-graph.

We focus on the case $k=3$ and ask how few edges such a graph may have. Chen et al. [16] provided such a graph with $2n-3$ edges when $n \geq 7$. They "suspect[ed]" that any 4-critical n-vertex $(B+M_3)$-graph has at least $2n$ edges asymptotically, and "dare[d] to conjecture only that they have significantly more than

$5n/3$ edges." (At the time it was conjectured, and it has now been proven in [48], that every 4-critical n-vertex graph has at least $(5n-2)/3$ edges.) Gyárfás renewed interest to the problem in [33]. We prove that, indeed, every such graph has at least $2n-3$ edges. Furthermore, we prove the same result for any 4-critical $(B+E_3)$-graph.

Theorem 2.1.1. *If G is a 4-critical $(B+E_3)$-graph, then $|E(G)| \geq 2|V(G)| - 3$.*

We use techniques from [48] and [49].

For $A \subseteq V(G)$, we let $G[A]$ denote the subgraph of G induced by A. When $A \cap B = \emptyset$, we let $G[A, B]$ denote the bipartite subgraph with parts A and B consisting of all edges of G having endpoints in both A and B.

2.2 Proof

For $A \subseteq V(G)$, define the *potential*, $\rho_G(A)$, to be $2|A| - |E(G[A])|$.

Theorem 2.1.1 is equivalent to the statement

$$\rho_G(V(G)) \leq 3 \textit{ for every 4-critical } (B+E_3)\textit{-graph } G.$$

We will frequently use the fact that, for $A, B \subseteq V(G)$,

$$\rho_G(A \cup B) + \rho_G(A \cap B) = \rho_G(A) + \rho_G(B) - |E(G[A-B, B-A])|.$$

Lemma 2.2.1. *Suppose $G \neq K_4$ is a 4-critical graph such that $E(G) = E(B) \cup E(S)$ where B is bipartite and $|E(S)| = 3$. Let V_1, V_2 be the bipartition of $V(B)$, indexed so that $|V_1 \cap V(S)| \geq |V_2 \cap V(S)|$. Either*

1. *$G[V_1 \cap V(S)] = K_3$, and $\rho_G(V(S)) = 3$; or*

2. *$G[V_1 \cap V(S)] = P_3$, $G[V_2 \cap V(S)] = K_2$, and $\rho_G(V(S)) \leq 3$; or*

3. *$G[V_1 \cap V(S)] = 2K_2$, $G[V_2 \cap V(S)] = K_2$, and $\rho_G(V(S)) \leq 5$.*

Proof. Observation: If there is an independent set I that intersects each edge of S, then $G - I$ is bipartite and so G is 3-colorable.

Chen et al. [16] showed that the only $B + E_2$ graph that is 4-critical is K_4. So each edge of S lies within one of V_1, V_2. If all three edges of S are in V_1, then $G[V(S)] = K_3$ by the observation. Otherwise two edges of S, say ab and cd, lie in V_1 and one edge, say xy, lies in V_2. Now $G[V_1 \cap V(S)]$ is either $2K_2$ $(a, b, c, d$

are distinct) or P_3 (say $b = c$). By the observation, x must be adjacent to both of a, b or both of c, d; by symmetry assume $xa, xb \in E(G)$. Similarly, y must be adjacent to both a, b or to both c, d; since G does not contain a K_4, we have $yc, yd \in E(G)$. These four edges together with $E(S)$ imply the given inequalities on potential. $\square$

Suppose the theorem fails. For a counterexample G, let S, V_1, V_2 be as in Lemma 2.2.1 and let

$$P(G) = \min_{V(S) \subseteq A \subseteq V(G)} \rho_G(A).$$

Among all counterexamples, choose G to have the maximum $P(G)$, and subject to this, to have the minimum number of vertices. (Note that for any counterexample H, $P(H) \leq \rho_H(V(S)) \leq 5$, so the maximum exists.) Let a, b, c, d, x, y be vertices not in $V(G)$, and let M be the matching with edges ab, cd, and xy.

Claim 2.2.2. *If $V(S) \subseteq R \subseteq V(G)$, then $\rho_G(R) \geq 4$. If also $8 < |R| < |V(G)|$, then $\rho_G(R) \geq 5$.*

Proof. If $R = V(G)$, then the claim follows from G being a counterexample. So we henceforth consider $R \subsetneq V(G)$. If the first statement of the claim fails, then there is an R such that *(i)* $\rho_G(R) = P(G) \leq 3$. If the first statement of the claim holds but the second statement fails, then there is an R such that *(ii)* $8 < |R| < |V(G)|$ and $\rho_G(R) = 4$. Fix such an R in either case.

Take any 3-coloring ϕ of $G[R]$, and construct the graph G' as follows. Let $R' = V(M) \cup \{z_2, z_3\}$, where z_2, z_3 are new vertices. Let $V(G') = (V(G) - R) \cup R'$.

Let $E(G'[V(G) - R]) = E(G[V(G) - R])$, and

$$E(G'[R']) = \{ab, ax, bx, by, xy, cd, cy, dy, z_2a, z_2c, z_3a, z_3d\}.$$

See Figure 2.2 for $G'[R']$. For each $i \in [3]$, let

$$C_i = \{v \in V(G) - R \colon\ v \text{ is adjacent to some vertex of color } i\},$$

and let $E(G'[R', V(G') - R'])$ be such that

$$\begin{aligned} N_{G'}(a) - R' &= V_2 \cap C_1, & N_{G'}(d) - R' &= V_2 \cap C_2, & N_{G'}(c) - R' &= V_2 \cap C_3, \\ N_{G'}(y) - R' &= V_1 \cap C_1, & N_{G'}(z_2) - R' &= V_1 \cap C_2, & N_{G'}(z_3) - R' &= V_1 \cap C_3, \\ N_{G'}(b) &\subseteq R', & N_{G'}(x) &\subseteq R'. \end{aligned}$$

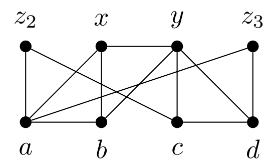


Figure 2.2: The graph $G'[R']$.

We claim that G' is not 3-colorable. If G' has a proper 3-coloring ψ, then by renaming colors as necessary, we may assume that $\psi(a) = \psi(y) = 1$, $\psi(c) = \psi(z_3) = 3$, and $\psi(d) = \psi(z_2) = 2$. Now coloring G by ϕ on R and ψ on $V(G) - R$ is a proper coloring, a contradiction. Hence there exists a 4-critical subgraph $G'' \subseteq G'$; note that $E(G'') \supset E(M)$.

For any A such that $V(M) \subseteq A \subseteq V(G')$, we have

$$|E(G'[A - R', A \cap R'])| \leq |E(G'[A - R', R'])| \leq |E(G[A - R', R])|.$$

Also, any subset of R' containing $V(M)$ has potential equal to 4. Hence for any $A \subseteq V(G'')$ containing $V(M)$,

$$\begin{aligned}
\rho_{G''}(A) &\geq \rho_{G'}(A) \\
&= \rho_{G'}(A - R') + \rho_{G'}(A \cap R') - |E(G'[A - R', A \cap R'])| \\
&\geq \rho_G(A - R') + 4 - |E(G[A - R', R])| \\
&= (4 - \rho_G(R)) + (\rho_G(A - R') + \rho_G(R) - |E(G[A - R', R])|) \\
&= (4 - \rho_G(R)) + \rho_G((A - R') \cup R).
\end{aligned}$$

Since $\rho(R) \leq 4$, we have $P(G'') \geq P(G)$. If $\rho(R) < 4$, then $P(G'') > P(G)$; if $\rho(R) = 4$ and $|R| > 8$, then $|V(G'')| < |V(G)|$. In either case, by the extremality of G, we have $\rho_{G''}(V(G'')) \leq 3$. Taking $A = V(G'')$ above, we have

$$\rho_G((V(G'') - R') \cup R) \leq \rho_{G''}(V(G'')) - 4 + \rho_G(R) \leq \rho_G(R) - 1.$$

If R satisfies *(i)*, then the set $(V(G'') - R') \cup R$ contradicts the minimality of $\rho_G(R)$. Hence the first statement of the claim holds. If R satisfies *(ii)*, then the set $(V(G'') - R') \cup R$ has potential at most 3, contradicting the first statement. Hence the second statement of the claim holds as well. □

Claim 2.2.2 and Lemma 2.2.1 imply that S is a matching, and we will henceforth assume it is M, with $V_1 \cap V(M) = \{a, b, c, d\}$ and $V_2 \cap V(M) = \{x, y\}$. Furthermore we obtain that $\rho_G(V(M)) \in \{4, 5\}$, i.e.

$|E(G[V(M)])| \in \{7, 8\}$. From the proof of Lemma 2.2.1 we see that

$$E(G[V(M)]) \supseteq \{ab, ax, bx, xy, cd, cy, dy\},$$

with equality or, up to symmetry, with the extra edge by (see Figure 2.3). We will need to consider these two cases separately. First we introduce a few lemmas that the arguments will have in common.

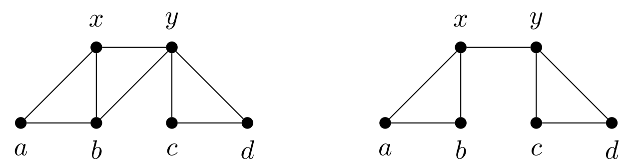


Figure 2.3: The two cases for $G[V(M)]$.

The following lemma is an old result by Hakimi [36]. A simpler version of it was used by Alon and Tarsi in [4]. For completeness, we provide a short proof using Hall's Theorem.

Lemma 2.2.3 (Theorem 4 in [36])**.** *Given a multigraph H and a function $f : V(H) \to \mathbb{N}$, one of the following holds.*

(1) There is a subset $A \subseteq V(H)$ such that $|E(H[A])| > \sum_{v \in A} f(v)$.

(2) There is an orientation of H such that for every $v \in V(H)$, $d^+(v) \le f(v)$.

Proof. Let V be the multiset consisting of $f(v)$ copies of v for each $v \in V(H)$, and let $E = E(H)$. Consider the (V, E)-bigraph B with $ve \in E(B)$ if and only if $v \in e$. A matching in B covering E yields an orientation of H satisfying (2): each edge is oriented away from its mate in the matching. So, by Hall's Theorem, if Conclusion (2) does not hold, then there is an $F \subseteq E$ with $|N_B(F)| < |F|$. But $|N_B(F)| = \sum_{v \in V(F)} f(v)$, and $|F| \le |E(H[V(F)])|$, so taking $A = V(F)$ satisfies Conclusion (1). $\square$

A *kernel* in a digraph is an independent set S such that for every $v \in V(D) - S$, there is some $s \in S$ such that $vs \in E(D)$. A digraph is called *kernel-perfect* if every induced subdigraph has a kernel.

Lemma 2.2.4 (Lemma 10 in [49])**.** *Let A be an independent set in a graph H and $B = V(H) - A$. Let D be the digraph obtained from H by replacing each edge in $H[B]$ by a 2-cycle and giving an arbitrary orientation to $H[A, B]$. Then D is kernel-perfect.*

Lemma 2.2.5 (Bondy, Boppana, and Siegel, see [4])**.** *If D is a kernel-perfect digraph and L is a list assignment such that for every $v \in V(D)$, $|L(v)| \ge 1 + d^+(v)$, then D is L-colorable.*

Now we are ready to consider the two cases.

Case 1: $\rho(V(M)) = 4$.

Let G' be obtained from G by doubling the two edges ab and cd of M. Define $f : V(G'-x) \to \mathbb{N}$ by $f|_{N(x)} \equiv 1$, $f|_{V(G')-N[x]} \equiv 2$, and apply Lemma 2.2.3 to $G'-x, f$.

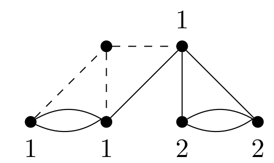


Figure 2.4: $G'[V(M)]$ with f.

If Conclusion (2) of Lemma 2.2.3 holds, then the orientation of $G'-x$ must have 2-cycles on $\{a,b\}$ and on $\{c,d\}$. Indeed, the edges ab must form a 2-cycle in order to have the outdegrees of a and b at most 1, then yb must be oriented from y, hence cy and dy must both be oriented toward y, and thus the edges cd must form a 2-cycle (see Figure 2.4). By Lemma 2.2.4, the orientation is kernel-perfect. We extend this orientation to an orientation of G' by making x a sink; the result is still kernel-perfect, and now $d^+(v) \leq 2$ for every $v \in V(G)$. By Lemma 2.2.5, G' and also G are 3-choosable, a contradiction.

So Conclusion (1) of Lemma 2.2.3 holds. That is, there exists an $A \subseteq V(G')$ with $\rho_{G'}(A) \leq -1 + |A \cap N(x)|$. This implies that

$$\rho_G(A) \leq -1 + |A \cap N(x)| + |E(M) \cap E(G[A])|.$$

With $\epsilon_1 = \begin{cases} 1 & \text{if } A \supseteq \{a,b\}, \\ 0 & \text{otherwise} \end{cases}$ and $\epsilon_2 = \begin{cases} 1 & \text{if } A \supseteq \{c,d\}, \\ 0 & \text{otherwise,} \end{cases}$ this yields

$$\begin{aligned} \rho_G(A+x) &\leq \rho_G(A) + 2 - |A \cap N(x)| + |E(M) \cap E(G[A])| \leq 1 + \epsilon_1 + \epsilon_2 \\ \rho_G(A+x+y) &\leq 1 + (2-1) + \epsilon_1 + \epsilon_2 = 2 + \epsilon_1 + \epsilon_2 \\ \rho_G(A+V(M)) &\leq 2 + 2(4-3) - 1 = 3. \end{aligned}$$

This contradicts Claim 2.2.2.

Case 2: $\rho(V(M)) = 5$.

Let $G'_x = G - x - cd + ab$, and define $f : V(G'_x) \to \mathbb{N}$ by $f|_{N(x)\cup\{c,d\}} \equiv 1$, $f|_{V(G'_x)-(N[x]\cup\{c,d\})} \equiv 2$, and apply Lemma 2.2.3 to G'_x, f. (See Figure 2.5.)

If Conclusion (2) of Lemma 2.2.3 holds, then the orientation must havea 2-cycle on $\{a,b\}$. Extend the

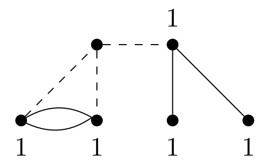


Figure 2.5: $G'_x[V(M)]$ with f.

orientation of G'_x to an orientation of $G - x + ab + cd$ by orienting the double edges cd as a 2-cycle; then Lemma 2.2.4 implies the orientation is kernel-perfect. Extend to an orientation of $G + ab + cd$ by making x a sink; the result is still kernel-perfect, and now $d^+(v) \leq 2$ for every v. By Lemma 2.2.5, $G + ab + cd$ and also G are 3-choosable, a contradiction.

Thus Conclusion (1) of Lemma 2.2.3 holds. That is, there exists an $A \subseteq V(G'_x)$ with $\rho_{G'_x}(A) \leq -1 + |A \cap (N(x) \cup \{c, d\})|$. This implies that

$$\rho_G(A + x) \leq 1 + \epsilon_1 + \epsilon_2, \tag{2.1}$$

where $\epsilon_1 = \begin{cases} 1 & \text{if } A \cap \{c, d\} \neq \emptyset, \\ 0 & \text{if } A \cap \{c, d\} = \emptyset \end{cases}$ and $\epsilon_2 = \begin{cases} 1 & \text{if } A \supseteq \{a, b\}, \\ 0 & \text{otherwise.} \end{cases}$

Adding to A in turn $\{y\}$, $\{c, d\}$, and $\{a, b\}$ each adds at most 1 to the potential. If A already intersects $\{y\}$, $\{c, d\}$, or $\{a, b\}$, then instead no potential is gained. Hence $A \cap V(M) = \emptyset$ or $A \cap V(M) = \{a, b\}$; otherwise $\rho(A + V(M)) \leq 3$, contradicting Claim 2.2.2.

Similarly, we construct the graph G'_y, and find a set B such that $\rho_G(B + y) \leq 1 + \epsilon_3$, where $\epsilon_3 = \begin{cases} 1 & \text{if } B \supseteq \{c, d\}, \\ 0 & \text{else.} \end{cases}$

We have $x \notin A \cup B$, $y \notin A \cup B$, and $B \cap V(M) = \emptyset$ or $B \cap V(M) = \{c, d\}$. So

$$\rho(A + x + B + y) + \rho(A \cap B) \leq 2 + \epsilon_2 + \epsilon_3 - |E(G[A + x - B, B + y - A])|. \tag{2.2}$$

Let $C = A \cap B$, so $C \cap V(M) = \emptyset$. If $|C| \leq 2$, then $\rho(C) \geq 0$. If $|C| > 2$, then by Claim 2.2.2 we have

$$5 \leq \rho(C + V(M)) \leq \rho(C) + \rho(V(M)) - \rho(C \cap V(M)) = \rho(C) + 5,$$

so still $\rho(C) \geq 0$. Furthermore, xy contributes to the last term of (2.2), and so by (2.2),

$$\rho(A + x + B + y) \leq 1 + \epsilon_2 + \epsilon_3.$$

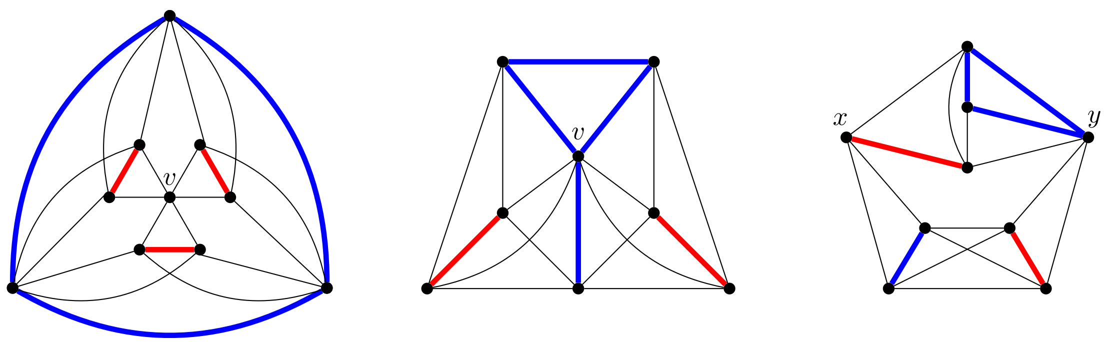


Figure 2.6: Three 5-critical $(B + E_6)$-graphs with few edges.

This implies that $\rho(A + B + V(M)) \leq 3$, which contradicts Claim 2.2.2.

The contradictions in each of these two cases imply that our counterexample G cannot exist, which completes the proof of Theorem 2.1.1. $\square$

2.3 Other k

Recall that Rödl and Tuza [58] showed that, for sufficiently large n, there are no n-vertex $(k+1)$-critical $B+E_\ell$ graphs with $\ell < \binom{k}{2}$. They also provided, for infinitely many n, an n-vertex $(k+1)$-critical $(B+E_{\binom{k}{2}})$-graph with only $(k-1)n-\binom{k}{2}+1$ edges. For large n, how few edges may an n-vertex $(k+1)$-critical $(B+E_{\binom{k}{2}})$-graph have?

For $k = 3$, Chen et al. improved over Rödl and Tuza's examples by one edge, finding infinitely many $(B + E_3)$-graphs with $2n - 3$ edges; our main result is that this is in fact the correct minimum number of edges.

For $k = 4$, the examples of Rödl and Tuza have $3n - 5$ edges. We know of only a few graphs with fewer edges. Three such examples are shown in Figure 2.6; the removal of the bolded edges makes each graph bipartite. The leftmost graph is obtained by gluing a copy of K_4 to each edge of a K_3 then adding a vertex v adjacent to all the resulting vertices of degree 3; it has $3n-6$ edges. The middle graph is obtained from the Moser Spindle by adding a dominating vertex v. It was kindly shown to us by a referee. It also has $3n - 6$ edges. The rightmost graph is obtained from the union of $K_5 - xy$ and K_4 by adding edges joining two vertices of the K_4 to x and the other two vertices of the K_4 to y. It has only $3n - 8$ edges. It is interesting whether for infinitely many n there exist 5-critical n-vertex $(B + E_6)$-graphs with fewer than $3n - 5$ edges.

Chapter 3

Sparse (hyper)graphs with large girth and (list-)chromatic number

The results of this chapter are joint with Noga Alon, Alexandr Kostochka, Douglas West, and Xuding Zhu and are based on [3].

3.1 Introduction

As discussed in Chapter 2 (specifically Lemmas 2.2.3 and 2.2.5), any bipartite graph with average degree at most 4 in every subgraph is 3-choosable. The proofs in Chapter 2 would be shorter if this statement could be strengthened by weakening the average degree condition.

If every subgraph of a graph has average degree less than k, then it has a vertex with degree less than k, and inductively it is k-choosable.

For bipartite graphs, one can guarantee k-choosability with average degree up to $2(k-1)$. Using (an early version of) the Combinatorial Nullstellensatz [2], Alon and Tarsi [4] proved Theorem 3.1.1 below, which implied the conjecture of [27] that planar bipartite graphs are 3-choosable. As mentioned in [4], another route to the result was subsequently noted by Bondy, Boppana, and Siegel, as follows. A *kernel* of a digraph is an independent set S containing a successor of every vertex outside S. If a graph G has an orientation D with maximum outdegree less than k, and every induced subdigraph of D has a kernel, then inductively G is k-choosable. Richardson [57] proved that every digraph with no odd cycle has a kernel. Hakimi [36] proved that G has an orientation with maximum outdegree at most $k-1$ when all induced subgraphs have average degree at most $2(k-1)$.

Theorem 3.1.1 ([4])**.** *If G is a bipartite graph such that every subgraph has average degree at most $2(k-1)$, then G is k-choosable.*

We show that Theorem 3.1.1 is sharp in a strong sense: we construct non-k-choosable bipartite graphs G such that after deleting any edge from G, all subgraphs of the remaining graph have average degree at most $2(k-1)$. Thus our graphs are $(k+1)$-*choice-critical* (i.e., the graph fails to be k-choosable, but every proper subgraph is k-choosable). Furthermore, such examples exist with arbitrarily large girth. We prove the following theorem.

Theorem 3.1.2. *For $g, k \in \mathbb{N}$, there is a bipartite graph G with girth at least g that is not k-choosable even though every proper subgraph has average degree at most $2(k-1)$.*

To prove this, we consider a new problem. Let an *r-augmented tree* be a graph consisting of a rooted tree (called the *underlying tree*) plus edges from each leaf to r of its ancestors (called *augmenting edges*). A *complete d-ary tree of height m* is a rooted tree whose internal vertices have d children and whose leaves have distance m from the root. For $d, r, g \in \mathbb{N}$, let a *(d, r, g)-graph* be a bipartite r-augmented complete d-ary tree with girth at least g.

Theorem 3.1.3. *For $d, r, g \in \mathbb{N}$, there exists a (d, r, g)-graph.*

In Section 3.2 we give a warmup, proving a version of Theorem 3.1.2 without the girth restriction. In Section 3.3 we prove Theorem 3.1.3, and in Section 3.4 we give several applications.

In Section 3.4.1 we present a simple construction of t-uniform hypergraphs with arbitrarily large girth and chromatic number, for all t. For $t = 2$, Erdős [23] used the probabilistic method to prove existence; see also [26, 44] for subsequent work. Explicit constructions followed in [51, 53, 41]. These are inductive and, except for the last one, use hypergraphs with large edges. Using (d, r, g)-graphs (built inductively), our construction is non-inductive and does not involve hypergraphs with larger edges. Moreover, the same method provides explicit high girth hypergraphs of any uniformity based on (d, r, g)-graphs, without using hypergraphs (besides those constructed) in the process.

We prove Theorem 3.1.2 in Section 3.4.2. Stronger versions involving restricted list assignments are proved in Section 3.4.3. For example, when the lists at adjacent vertices are disjoint, every coloring chosen from the lists is proper. We extend the analysis of the graph constructed for Theorem 3.1.2 by constructing a k-list assignment in which any two adjacent lists have exactly one common color and yet no proper coloring can be chosen.

One can also restrict list assignments by bounding the size of the union of the lists. For bipartite graphs, a proper coloring can be chosen from any k-lists whose union has size at most $2k - 2$. We prove that this is sharp (for any girth) by constructing a bipartite graph with k-lists whose union has size $2k - 1$ from which no proper coloring can be chosen.

Finally, in Section 3.5 we discuss the height of the trees used in Theorem 3.1.3. For fixed $d \geq 2$ and $r \geq 1$, we show that the height must grow extremely rapidly in terms of the girth.

3.2 Warmup

To help motivate the proof of Theorem 3.1.2, we begin with the version without the girth restriction.

Theorem 3.2.1. *For $k \in \mathbb{N}$, there is a bipartite graph G that is not k-choosable even though every proper subgraph has average degree at most $2(k-1)$.*

Proof. We proceed by induction on k. For $k = 1$, K_2 suffices. So suppose $k \geq 2$ and G_{k-1} has the desired properties. Let (A, B) be the bipartition of G_{k-1}, and let L' be a list assignment with $|L'(x)| = k - 1$ for each $x \in V(G_{k-1})$ such that G_{k-1} is not L'-colorable (and none of the lists intersects $[k]$).

Construct G_k as follows. Consider a star $K_{1,k}$ with center v and leaves $w_1, \dots, w_k$. For each leaf w_i, introduce $k-1$ copies of G_{k-1}, denoted $G_{k-1}^{i,j}$ for $j \in [k]-\{i\}$. Let $x_{i,j}$ denote the copy of vertex $x \in V(G_{k-1})$ in $G_{k-1}^{i,j}$. For each copy $x_{i,j}$ of $x \in A$, add an edge joining $x_{i,j}$ to w_i. For each copy $y_{i,j}$ of $y \in B$, add an edge joining y to v. Clearly G_k is bipartite.

Consider the list assignment L defined as follows. First let $L(v) = [k]$ and $L(w_i) = [k]$. For a copy $x_{i,j}$ of $x \in A$, let $L(x_{i,j}) = L'(x) \cup \{i\}$, and for a copy $y_{i,j}$ of $y \in B$, let $L(y_{i,j}) = L'(y) \cup \{j\}$. If there is an L-coloring f of G_k, then let $i = f(v)$ and $j = f(w_i)$; note that $i \neq j$. In $G_{k-1}^{i,j}$, one color has been forbidden from each list; specifically, the remaining available list at each vertex is precisely that of L'. So G_k is not L-colorable and hence is not k-choosable.

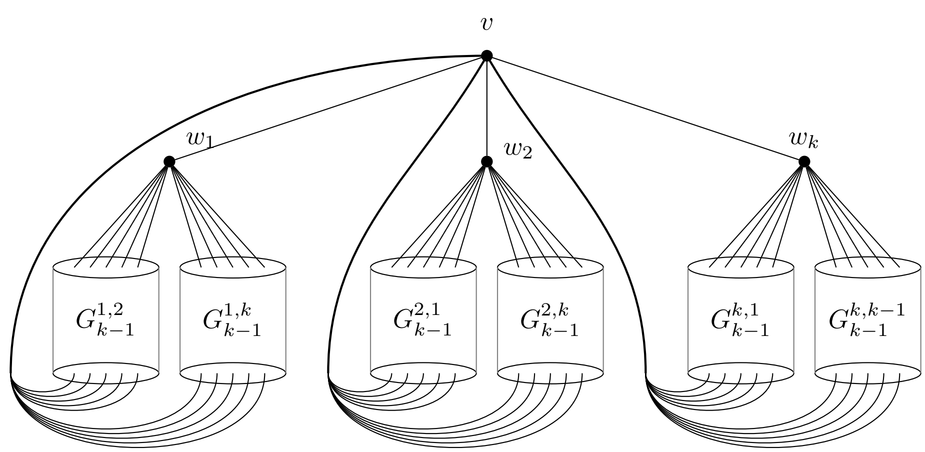


Figure 3.1: The graph G_k of Theorem 3.2.1.

To prove the claim about average degree of subgraphs, we give an orientation to G_k such that every vertex has outdegree $k-1$ except for a designated root vertex with outdegree k, and such that every vertex is reachable from the root. (In any proper subgraph, either the root has outdegree at most $k-1$ and the average degree is at most $2(k-1)$, or some other vertex has outdegree at most $k-2$ and again the average degree is at most $2(k-1)$.) The desired orientation of K_2 is obvious. Given such an orientation of G_{k-1}, use that orientation on each copy of G_{k-1} in G_k. Orient the edges of the star away from the center, orient the edges from leaves toward the roots of each copy of G_{k-1}, and orient the remaining edges from the copies of G_{k-1} toward the star. The center of the star is the root of G_k. $\square$

This construction has many 4-cycles. Using taller k-ary trees instead of stars in this construction, and adding the edges from copies of G_{k-1} to varying heights on the tree, we may fairly easily get such a graph with girth 6; this motivates our definition of augmented trees.

3.3 Augmented trees

Recall that a (d, r, g)-graph is a bipartite r-augmented complete d-ary tree with girth at least g.

If there is a (d, r, g)-graph, then let $m(d, r, g)$ denote the least height of the underlying tree in such a graph (otherwise, let $m(d, r, g) = \infty$). Theorem 3.1.3 is the statement that $m(d, r, g)$ is finite for all $d, r, g \in \mathbb{N}$. We prove this by double induction, using the following three lemmas.

Lemma 3.3.1. *For $d, r \in \mathbb{N}$, we have $m(d, r, 4) = 2r + 1$.*

Lemma 3.3.2. *For $g, d \in \mathbb{N}$ with g at least 4 and even, $m(d, 1, g + 2) \le g + m(d, d^g, g)$.*

Lemma 3.3.3. *With d, r, g as above, $m(d, r + 1, g) \le m_1 + g - 1 + m_2$, where $m_1 = 2\left\lfloor \frac{m(d,r,g)}{2} \right\rfloor + 1$ and $m_2 = m(d^{m_1+g}, 1, g)$.*

These three lemmas imply the finiteness of $m(d, r, g)$ for all $d, r, g \in \mathbb{N}$ with g even and at least 4. Letting $P(r, g)$ denote the claim that $m(d, r, g)$ is finite for all d, we prove $P(r, g)$ by induction on g. As the base step, $P(r, 4)$ holds for all r by Lemma 3.3.1. If $P(r, g)$ holds for all r, then we prove $P(r, g+2)$ by induction on r: first $P(1, g+2)$ holds by Lemma 3.3.2 (using the truth of $P(r, g)$ for all r), and then $P(r+1, g+2)$ follows from $P(r, g+2)$ by Lemma 3.3.3 (since $P(1, g+2)$ also holds). This completes the proof of Theorem 3.1.3.

It remains to prove the three lemmas. Lemma 3.3.1 is trivial: just make each leaf adjacent to its r non-parent ancestors at odd distance from it in the tree.

Proof of Lemma 3.3.2. Let G' with underlying tree T' be a (d, d^2, g)-graph with height $m(d, d^2, g)$. Replace each leaf v of T' with a complete d-ary tree T_v of height 2 rooted at v. Replace the augmenting edges from v to its ancestors by letting the d^2 lower endpoints be the leaves of T_v instead of v. This produces a 1-augmented complete d-ary tree G of height $2 + m(d, d^2, g)$. Since each augmenting edge has had its lower endpoint moved two levels down, G is bipartite.

If G has a cycle C of length at most g, then C must contain an augmenting edge, say xy, with y being a leaf in the underlying tree T of G. Let v be the leaf in T' such that y is in T_v. Since $d_G(y) = 2$, the cycle C contains the edge yy' of T_v incident with y. Contracting the added subtrees of height 2 into leaves of T' contracts C to a closed walk C' in G' of length less than g. Since C' traverses edge vx only once, the remaining walk from x to v along C' contains a path that with vx completes a cycle of G' having length less than g, a contradiction. Thus G has no cycles of length less than $g + 2$. $\square$

Proof of Lemma 3.3.3. Fix r. Assuming for all d and g that $m(d, r, g)$ and $m(d, 1, g)$ are finite, let $m_1 = 2\lfloor m(d, 1, g)/2\rfloor + 1$ and $m_2 = m(d^{m_1}, r, g)$. Note that m_1 is the least odd integer that is at least $m(d, 1, g)$. We construct the desired graph G from two graphs G_1 and G_2.

For G_1 we use a $(d, 1, g)$-graph having height m_1. If $m(d, 1, g)$ is odd, then $m_1 = m(d, 1, g)$ and we use a shortest $(d, 1, g)$-graph. If $m(d, 1, g)$ is even, then $m_1 = m(d, 1, g) + 1$, and we form G_1 from d copies of a shortest $(d, 1, g)$-graph by adding a new root having the roots of those graphs as children.

For G_2, let $d' = d^{m_1}$, and consider a (d', r, g)-graph H having height m_2. Let G_2 be an induced subgraph of H formed by starting from the root of the underlying tree of H and keeping only d children of each included vertex, except that all d' children are kept at the last level. Thus G_2 has an underlying tree T' of height m_2, and deleting the $d^{m_2-1}d'$ leaves of T' yields a complete d-ary tree of height $m_2 - 1$. All ancestors in H of a leaf of T' appear in T', so each leaf of T' has r ancestors as neighbors in G_2.

Now we construct G from G_1 and G_2. In G_2, let $S(u)$ be the star consisting of a vertex u at level $m_2 - 1$ and its d' leaf children. Replace each $S(u)$ with a copy $G_1(u)$ of the graph G_1, so that the d' leaves in G_1 each become one of the leaves in $S(u)$, inheriting the r augmenting edges that were incident to that leaf in G_2. We call the augmenting edges obtained from G_2 in this way *long edges*; the augmenting edges in $G_1(u)$ are *short edges*.

The underlying tree in our construction thus has two parts. The top part is the tree T' for G_2 without its bottom level; it has height $m_2 - 1$. The bottom part, with height m_1, consists of copies of G_1. Each leaf has one incident short edge from G_1 and r incident long edges inherited from G_2. Thus G is an $(r+1)$-augmented complete d-ary tree of height $m_1 + m_2 - 1$. When replacing one of the r augmenting edges from a leaf of G_2 by a long edge, the difference in the heights of the endpoints increases by $m_1 - 1$. Since m_1 is odd, this change is even, so G is bipartite.

A cycle C in G that contains no long edges is a cycle in a copy of G_1 and hence has length at least g. When C contains a long edge, contracting a subtree $G_1(u)$ into a star $S(u)$ contracts C to a closed walk C' in G_2 using an augmenting edge e. Since leaves of $G_1(u)$ correspond bijectively to leaves of $S(u)$, the edge e is not repeated in C'. Hence the other walk in C' joining its endpoints contains a path that completes a cycle with e. Since this is a cycle in G_2 and has length at least g, also C has length at least g.

This completes the proof of Lemma 3.3.3 and Theorem 3.1.3. $\square$

3.4 Applications

In a complete k-ary tree, a *full path* is a path from the root to a leaf. A $[k]$*-coloring* is a k-coloring using the colors in $[k]$.

Definition 3.4.1. Given an ordering of the children at each internal vertex, the vertices of a complete k-ary tree with height m correspond naturally to the strings of length at most m from the alphabet $[k]$. Define an edge-coloring ϕ by letting the color of each edge from parent x to child y be the index of y in the ordering of the children of x (note that ϕ is not a proper coloring). For a $[k]$-coloring f of the vertices of T, a full path P is an f*-path* if the color of each non-leaf vertex on P equals the color of the edge to its child on P.

Whenever f is a $[k]$-coloring of a complete k-ary tree, there is a unique f-path: just start from the root and repeatedly follow the descending edge whose color matches the color of the current vertex. Similarly, every full path is an f-path for some $[k]$-coloring f.

3.4.1 Large chromatic number and girth

As mentioned in the introduction, there exist t-uniform hypergraphs with large chromatic number and girth. Our (d, r, g)-graphs provide a remarkably simple such construction. It has the benefits of being non-recursive (once (d, r, g)-graphs are constructed), and not involving hypergraphs as inputs to the construction. Thus unlike the earlier constructions which use hypergraphs to provide high girth graphs, the method described here constructs high girth graphs and hypergraphs using only graphs.

Theorem 3.4.2 ([23, 26, 51, 53, 41, 44])**.** *For $k, g, t \in \mathbb{N}$, there is a t-uniform hypergraph with girth at least g and chromatic number larger than k.*

Proof. Let G be a $(k, (t-1)k+1, 2g)$-graph with underlying tree T having leaf set L. Let $V' = V(T) - L$. For $v \in L$, consider the full path P ending at v. Among the $(t-1)k+1$ neighbors of v via augmenting edges, the pigeonhole principle yields a set of t neighbors of v whose descending edges along P have the same color; let e_v be such a set of vertices in V'. Let H be the t-uniform hypergraph with vertex set V' and edge set $\{e_v \colon v \in L\}$.

Any $[k]$-coloring f of V' yields a unique f-path in T, ending at some leaf v. As a coloring of H, this makes the edge e_v monochromatic. Hence H has no proper k-coloring.

Let C be a shortest cycle in H, with edges $e_1, \dots, e_l$ in order and vertex x_i chosen from $e_{i-1} \cap e_i$ (subscripts modulo l). Since C is a shortest cycle, $x_1, \dots, x_l$ are distinct. Each edge of H consists of neighbors of a single leaf of T via augmenting edges; let v_i be the common leaf neighbor of the vertices in e_i. Form C' in G by replacing each edge e_i of C by the copy of P_3 in G having endpoints x_{i-1} and x_i and midpoint v_i. Since for each leaf of T we formed exactly one edge in H, the leaves $v_1, \dots, v_l$ are distinct. Hence C' is a cycle, and its length is twice that of C. By the choice of G as a $(d, k, 2g)$-graph, H has girth at least g. □

The hypergraph H in Theorem 3.4.2 satisfies $|E(H)| = |L| = k^h$ and $|V(H)| = |V'| = \frac{k^h-1}{k-1}$, where $h = m(k, (t-1)k+1, 2g)$. Hence $|E(H)| = (k-1)|V(H)| + 1$. However, H may have (and actually does have) dense subgraphs. For $t = 2$, we provide a different construction, inductive, of sparse graphs with large girth and chromatic number. A graph G is sparse when it has a small value of the *maximum average degree*, defined to be $\max_{H \subseteq G} \frac{\sum_{v \in V(H)} d_H(v)}{|V(H)|}$. Our construction has asymptotically lowest average degree even in the broader class of triangle-free graphs. This follows from the lower bound by Kostochka and Stiebitz [46]: every k-chromatic triangle-free graph has maximum average degree at least $2k - o(k)$.

Definition 3.4.3. Let G be a (d, r, g)-graph with a specified ordering of the d children at each non-leaf vertex of the underlying tree T. The corresponding *reduced (d, r, g)-graph* H is obtained from G as follows: given the coloring ϕ of $E(G)$ from Definition 3.4.1, form H from G by deleting at each non-root internal vertex v of T the subtree under the descending edge whose color under ϕ is the same as the color of the edge to the parent of v. Each non-leaf vertex of $H \cap T$ has degree d in T, and ϕ is a proper edge-coloring of $H \cap T$.

The reduced (d, r, g)-graph with underlying tree T associated with the edge-coloring ϕ as in Definition 3.4.3 still has a unique f-path for any proper $[d]$-coloring f of T.

Theorem 3.4.4. *For $k, g \in \mathbb{N}$, there is a graph with girth at least g that is not k-colorable and has maximum average degree at most $2(k-1)$.*

Proof. For fixed g, we construct such a graph J_k by induction on k. For the basis step, let J_2 be an odd cycle of length at least g. Given J_{k-1}, let $r = |V(J_{k-1})|$.

Let H be a reduced $(k, (r-1)k+1, g)$-graph, with underlying tree T and edge-coloring ϕ. For each leaf v of T, consider the full path P ending at v. By the pigeonhole principle, some r neighbors of v in H (via augmenting edges) have the same color on their descending edges along P. Keep the augmenting edges from v to one such set and delete the other augmenting edges. The resulting graph H' is a reduced (k, r, g)-graph.

Next replace each leaf v of H' with a copy of J_{k-1}; each vertex in the copy for v inherits exactly one augmenting edge of H' from v. This is the graph J_k. The edge to v in T disappears; vertices at the level just before the leaves no longer have edges to children.

Any proper $[k]$-coloring f of $V(T)$ yields a unique f-path; it ends at some leaf v. Because it is an f-path, the colors on the vertices match the colors on the descending edges. Let Q be the copy of J_{k-1} corresponding to v in J_k. By the construction of J_k, there is a fixed color c that appears on the neighbor in $V(T)$ of each vertex in Q. Since J_{k-1} is not $(k-1)$-colorable, we cannot complete a proper k-coloring of J_k.

A cycle in one copy of J_{k-1} has length at least g. For any other cycle C in J_k, contracting each copy of J_{k-1} to a single vertex yields a closed walk C' in H' using some augmenting edge. Since each vertex in a copy of J_{k-1} inherits only one augmenting edge, each augmenting edge is used only once in C'. Hence as in the proof of Lemma 3.3.3, C' contains a cycle in H'. This cycle has length at least g, so C has length at least g.

For the maximum average degree, consider a subgraph F, and let $F' = F - V(T)$. Being contained in copies of J_{k-1}, the graph F' has average degree at most $2(k-2)$. Augmenting edges add at most 1 to the degree of each vertex of F' and hence at most 2 to the degree-sum in F for each vertex in F'. Working upward in T, each added vertex in F adds at most $k-1$ downward edges, which contributes at most $2(k-1)$ to the degree-sum. The root may add k downward edges, but the lowest vertex added from T adds fewer than $k-1$. Thus the degree-sum is at most $2(k-1)$ per vertex of F. □

3.4.2 Choosability

A modification of the construction in Theorem 3.4.4 yields non-k-choosable bipartite graphs that are as sparse as can be. As noted in Theorem 3.1.1, every bipartite graph with maximum average degree at most $2(k-1)$ is k-choosable. Hence the graphs we construct in Theorem 3.1.2 with just one extra edge are $(k+1)$-choice-critical.

It is well known (since [27]) that a bipartite graph consisting of two even cycles sharing one vertex is not 2-choosable; indeed, it is 3-choice-critical.

Theorem 3.1.2. For $k \geq 2$ and $g \geq 4$, there is a bipartite graph G_k with girth at least g that is not k-choosable even though every proper subgraph has average degree at most $2(k-1)$.

Proof. We proceed by induction on k for even g. To count edges in subgraphs, we will orient G_k and count edges by their tails. The orientation gives each vertex outdegree $k-1$ except a designated root vertex, which has outdegree k, and every vertex will be reachable from the root. Thus G_k will have $(k-1)\,|V(G_k)|+1$ edges, and every proper subgraph will have smaller outdegree at some vertex and thus have average degree at most $2(k-1)$.

Let G_2 be the graph consisting of two g-cycles sharing one vertex, which is the root. Orient G_2 consistently along each of the two cycles. The desired properties hold.

For $k \geq 3$, suppose that G_{k-1} has all the desired properties. Let $r = |V(G_{k-1})| - 1$, and let H' be a reduced $(k, r, 2g)$-graph, with underlying tree T. We modify the bipartite graph H' slightly to guarantee that G_k will be bipartite. Let (A, B) be the bipartition of G_{k-1}, with A containing the root, and let $a = |A| - 1$ and $b = |B|$. Each leaf v in T has $a + b$ incident augmenting edges. Let $A(v)$ denote some set of a of these edges. For the remaining b augmenting edges incident to v, move their endpoints in the tree one step closer to v along the full path to v. Let $B(v)$ denote this new set of b augmenting edges at v. Let H be the resulting graph; H is a reduced (k, r, g)-graph except for not being bipartite.

Form G_k from H by adding a copy of G_{k-1} for each leaf v of T, merging v with the root of G_{k-1}, with each vertex of A in the copy of G_{k-1} (other than the root) inheriting one edge of $A(v)$ and each vertex of B in the copy of G_{k-1} inheriting one edge of $B(v)$. Since the vertices of B have odd distance from v in G_{k-1}, this guarantees that G_k is bipartite.

Designate the root of T as the root of G_k. Orient the edges of T away from the root, keep the orientation guaranteed by the induction hypothesis on the copies of G_{k-1}, and orient the augmenting edges away from the copies of G_{k-1}. Because H' is a reduced $(k, r, 2g)$-graph, every vertex has outdegree $k - 1$ except that the root has outdegree k.

A cycle in a copy of G_{k-1} has length at least g. Let C be a cycle in G_k that is not in G_{k-1}. Contracting each copy of G_{k-1} in G to a single vertex turns C into a closed walk C' in H. Since each vertex in a copy of G_{k-1} has only one augmenting edge, C' contains a cycle in H. This cycle has length at least g, so C has length at least g.

Let L' be an assignment of lists of size $k - 1$ to G_{k-1} such that G_{k-1} is not L'-colorable and none of these lists intersects $[k]$. Form a list assignment L for G_k as follows. Put $L(x) = [k]$ for each non-leaf vertex x in $V(T)$. For each leaf $v \in V(T)$ and each vertex w of $V(G_{k-1})$, let w_v denote the copy of w in the copy of G_{k-1} at v. Let P be the full path in T ending at v. Let $L(w_v) = L'(w) \cup \{c\}$, where c is the color on the edge of P descending from the neighbor of w_v in $V(P)$. In particular, when w is the root, the added color is the color on the edge of T reaching v.

Let f be a coloring of G_k with $f(u) \in L(u)$ for $u \in V(G_k)$. If f is proper on T, then since $f(x) \in [k]$ for $x \in V(T)$, there is a unique f-path P in T. In the copy of G_{k-1} for the leaf v at the end of P, the color c that was added to each list is now forbidden in a proper coloring, leaving the list $L'(w)$ at w_v. By the choice of L', a proper coloring cannot be completed from these lists. $\square$

3.4.3 Restricted list colorings

As described in the introduction, we now strengthen Theorem 3.1.2 by proving non-choosability results for restricted list assignments. We consider both restrictions on the intersections of adjacent lists and restrictions on the size of the union of the lists.

Every graph is L-colorable (by choosing arbitrarily) when adjacent vertices have disjoint lists, but L-colorability may fail when adjacent lists are almost disjoint. List coloring with intersection constraints on adjacent lists has been studied by Kratochvíl, Tuza, and Voigt [50] and by Füredi, Kostochka, and Kumbhat [30]. We next strengthen Theorem 3.1.2 by showing that our graph G_k fails to be L-colorable for a particular k-list assignment L such that $|L(u) \cap L(v)| = 1$ for every edge uv.

Theorem 3.4.5. *Fix $g \in \mathbb{N}$ with $g \equiv 4 \pmod 6$. For $k \geq 2$, the bipartite graph G_k with girth at least g constructed in Theorem 3.1.2 admits a k-list assignment L such that G_k is not L-colorable despite satisfying $|L(u) \cap L(v)| = 1$ for all $uv \in E(G_k)$.*

Proof. For $k = 2$, let u be the common vertex of the two cycles in G_2. Set $L(u) = \{1, 2\}$. On each of the two cycles, the number of remaining vertices is a multiple of 3. Along one cycle, rotate through the lists $\{1, 3\}, \{3, 4\}, \{4, 1\}$. This forces color 1 onto a neighbor of u. On the other cycle substitute 2 for 1, forcing color 2 onto a neighbor of u. Now u cannot be colored. Adjacent lists share one color.

For $k \geq 3$, let T be the underlying tree in G_k. Color the edges of T by distinct colors. For a non-leaf vertex x in T, let $L(x)$ be the set of colors on the edges incident to x; thus lists adjacent via edges of T have one common color.

By the induction hypothesis, there is a $(k-1)$-list assignment L' on G_{k-1} such that G_{k-1} is not L'-colorable. For each leaf $v \in V(T)$, let L'_v be a copy of this assignment indexing the colors by v, so that the colors used for the copy G' of G_{k-1} at v will not be used anywhere else. For each vertex w of $V(G_{k-1})$ other than the root, let w_v denote the copy of w in G'. Let P be the full path in T ending at v. Let x be the neighbor of w_v in $V(P)$, and let c_x be the color of the edge in P descending from x along P. Let $L(w_v) = L'_v(w) \cup \{c_x\}$. Let $L(v) = L'_v(v) \cup \{c_v\}$, where c_v is the color of the edge incident to v in T.

For any proper coloring f of T chosen from these lists, there is a unique full path Q such that the color of each non-leaf vertex is the color of the edge to its child on Q, constructed from the root: that is, an f-path. Let v be the leaf reached by Q. The parent of v has been given color c_v, so that color cannot be used at v. Similarly, for each other vertex in the copy of G_{k-1} at v, the added color in its list has been used on its neighbor in T. Finding an L-coloring of G_k thus requires finding an L'-coloring of G_{k-1}, which does not exist. $\square$

Perhaps surprisingly, for bipartite graphs larger intersections than in Theorem 3.4.5 also guarantee L-colorability, giving the sharpness of Theorem 3.4.5 in another way.

Proposition 3.4.6. *If G is a bipartite graph, and L is a list assignment such that any two adjacent lists have at least two common elements (the lists may have any sizes at least 2), then G is L-colorable.*

Proof. Let X and Y be the parts of G, and index the colors in $\bigcup_{v\in V(G)} L(v)$ as $c_1, \dots, c_t$. Color each vertex of X with the highest-indexed color in its list and each vertex of Y with the lowest-indexed color in its list. If two adjacent vertices receive the same color, then it is the only common color in their lists, a contradiction. Hence the coloring is proper. $\square$

When G is j-colorable but not k-choosable, one may ask how large the union U of the lists must be in a k-list assignment L such that G is not L-colorable. Trivially $|U| > j$ is needed. In fact, one needs somewhat more, which reduces to $2k-1$ when $j = 2$.

Proposition 3.4.7. *Let G be a j-colorable graph, with $j \le k$. If L is a k-list assignment on G such that $|\bigcup_{v\in V(G)} L(v)| \le \frac{j(k-1)}{j-1}$, then G is L-colorable. Furthermore, the bound is sharp.*

Proof. Let f be a proper j-coloring of G. Let $U = \bigcup_{v\in V(G)} L(v)$. Split U into disjoint sets $U_1, \dots, U_j$, with the smallest having size $\lfloor |U|/j \rfloor$. Since $|U| \le \frac{j(k-1)}{j-1}$, the largest $j-1$ of the sets together have size at most $k-1$. (Note that $\left\lfloor \frac{j(k-1)}{j-1} \right\rfloor - \left\lfloor \frac{k-1}{j-1} \right\rfloor = k-1$, and when $|U| < \left\lfloor \frac{j(k-1)}{j-1} \right\rfloor$ the conclusion becomes easier.) Thus each k-list $L(v)$ intersects each U_i. Hence each vertex v can choose a color from $L(v) \cap U_{f(v)}$. Such a coloring is proper.

For sharpness, consider a universe U of colors, and let G be a complete j-partite graph with $\binom{|U|}{k}$ vertices in each part. Assign lists by letting L give each k-subset of U as a list to one vertex in each part. In an L-coloring, each color can be chosen in only one part. Since a color must be chosen from every vertex, on each part at least $|U| - (k-1)$ colors must be chosen. Hence $j(|U| - k + 1)$ colors must be chosen. Thus L-colorability requires $j(|U| - k + 1) \le |U|$, which is precisely the inequality $|U| \le \frac{j(k-1)}{j-1}$. $\square$

The sharpness examples in Proposition 3.4.7 are very dense and have small cycles. The special case $j = 2$ states that a bipartite graph is L-colorable when L is a k-list assignment with $|\bigcup_{v\in V(G)} L(v)| \le 2k-2$. This condition forces any two lists to have at least two common elements, so Proposition 3.4.6 is stronger than Proposition 3.4.7 for the case $j = 2$. Nevertheless, we show next that Proposition 3.4.7 remains sharp when $j = 2$ even for sparse graphs with large girth having just one extra edge beyond where Theorem 3.1.1 applies.

Theorem 3.4.8. *Fix $k, g \in \mathbb{N}$ with g even and $k \geq 2$. There is a bipartite graph H_k and a k-list assignment L on H_k such that H_k is not L-colorable, even though $|\bigcup_{v \in V(H_k)} L(v)| = 2k - 1$ and H_k has girth at least g with each proper subgraph having average degree at most $2(k - 1)$.*

Proof. We use induction on k. For $k = 2$, let H_2 be G_2, the graph consisting of two g-cycles sharing one vertex u. Set $L(u) = \{1, 2\}$. On one cycle, use lists $\{1, 3\}$ and $\{1, 2\}$ on the neighbors of u and $\{2, 3\}$ on the rest of the cycle. Since the number of copies of $\{2, 3\}$ is odd, color 1 must be chosen on a neighbor of u. Interchanging 1 and 2 yields the lists on the other cycle, forcing a neighbor of u to have color 2. Now u cannot be colored. The union of the lists has three colors.

For $k \geq 3$, let $r = |V(H_{k-1})| - 1$, and let $a + 1$ be the number of vertices of H_{k-1} in the partite set containing the root; note that $a < r$. We construct H_k with a list assignment L. Consider a reduced $(k, (r - 1)k, 2g)$-graph with underlying tree T and corresponding proper $[k]$-edge-coloring of T. The root of T will be the root of H_k.

For each leaf v of T, proceed as follows. Let P be the full path to v in T. Since v has more than $(a - 1)k$ augmenting edges, by the pigeonhole principle there are a such edges for which the edge along P descending from the neighbor of v has the same color; call it c. Move the other endpoints of all $(r - 1)k - a$ other augmenting edges at v one step closer to v along P, as in the proof of Theorem 3.1.2. Since $(r - 1)k - a > (r - a - 1)k$, by the pigeonhole principle there are $r - a$ of these remaining edges for which the edge along P descending from the neighbor of v has the same color; call it c'. Discard all augmenting edges not chosen in these two steps. After doing this for each leaf v of T, the result is a reduced (k, r, g)-graph except for not being bipartite.

For each leaf v of T, add a copy H'_v of H_{k-1}, merging its root with v and letting each non-root vertex inherit one of the augmenting edges at v, with the vertices in the part opposite v inheriting the $r - a$ edges whose other endpoints were moved closer to v. Let H_k be the resulting graph; it is bipartite, and the density bound for its subgraphs is computed as for G_k in Theorem 3.1.2. Arguing as for G_k also shows that H_k has girth at least g.

Next we produce the list assignment L. Assign list $[k]$ to each non-leaf vertex of T. By the induction hypothesis, for each leaf v of T there is a $(k - 1)$-list assignment L'_v on H' whose lists are contained in a $(2k - 3)$-set. For this $(2k - 3)$-set use $[2k - 1] - \{c, c'\}$, discarding any additional color if $c' = c$. Also, let c_v be the color of the edge reaching v in T. Since $2k - 3 > k - 1$ when $k > 2$, we may permute the colors within L'_v to ensure that L' does not assign color c_v to v.

To define lists, form $L(v)$ by adding c_v to the list given by L'_v to the root. For $w \in V(H_{k-1})$ other than the root, let w_v be the copy of w in H'_v. Set $L(w_v) = L'_v(w) \cup \{c\}$ if w is in the same partite set as the root of H_{k-1}, and otherwise set $L(w_v) = L'_v(w) \cup \{c'\}$.

It remains to show that H_k is not L-colorable. Let f be a proper coloring chosen from L. Since the list on each non-leaf vertex of T is $[k]$ and the coloring is proper, there is a unique f-path Q leading to a particular leaf v. Since the color of each non-leaf vertex on Q agrees with the color on the edge descending from it along Q, the color added to the list of each vertex w_v in the copy of H_{k-1} at v has been used on its neighbor in T and is now forbidden from use on w_v. Finding an L-coloring of H_k thus requires finding an L'-coloring of H_{k-1}, which does not exist. $\square$

3.5 The height of the trees in Theorem 3.1.3

The underlying trees in our construction of (d, r, g)-graphs are astoundingly tall; their height in terms of g is a version of the Ackermann function. Here we show that even for $r = 1$ and $d = 2$, they must be very tall. In the discussion below all logarithms are in base 2.

Theorem 3.5.1. *If G is a $(2, 1, g)$-graph with height m, then $g \le (4 + o(1)) \log(\log^* m)$.*

Proof. For simplicity, we omit floor and ceiling signs; they are not crucial.

For $g \in \mathbb{N}$, let $q = 2^{g/4-2}$. Let $k_{-1} = -1$, $k_0 = g - 1$, and for $0 \le i < r$ set

$$k_{i+1} = 2^{(k_i - g/2+4)/2} + k_i.$$

This yields $g \approx 4 \log(\log^* k_q)$. Let G be a 1-augmented binary tree of height m, and let g be the least integer such that $k_q \ge m$. We will find in G a cycle of length at most g.

Define integer intervals $I_0, \ldots, I_q$ by $I_j = [m - k_j, m - k_{j-1} - 1]$ (deleting any negative elements). These intervals group the levels in T. The number of levels in I_j is at most $k_j - k_{j-1}$, the value of which is roughly a tower of height j. However, since we only choose g so that $k_q \ge m$, the least j with $k_j \ge m$ may be less than q, so the intervals toward the end of the list may be empty.

Let the *mate* of a leaf of T be the other endpoint of its augmenting edge in G. Let the *type* of the leaf be j if the level of its mate lies in I_j. We may assume that no leaf has type 0, since otherwise G has a cycle of length at most g. With each leaf having type in the integer interval $[1, q]$, some type is assigned to at least $1/q$ of the leaves of G. Fix such a type t.

By averaging, for some vertex u at level $m - k_{t-1} - 1$ at least $1/q$ of the leaves under u have type t. Let C denote the set of all leaves of type t under u. Let v be the ancestor of u at level $m - k_t$ (or level 0 if $m < k_t$). For each leaf $x \in C$, the mate of x is on the u, v-path P in T. Note that $|V(P)| \le k_t - k_{t-1} = 2^{(k_{t-1} - g/2+4)/2}$.

The vertex u has $2^{k_{t-1}-g/4+2}$ descendants at level $m-(g/4-1)$; call this set D. The subtree rooted at any $y \in D$ has $2q$ leaves. Call y *full* if at least two leaves of T under y belong to C. Let $\beta|D|$ be the number of full vertices in D. The number of leaves under u is $2q|D|$. Allowing all leaves under full vertices of D and at most one leaf under non-full vertices, the number of leaves in C under u is at most $(2q\beta+1)|D|$. The fraction of leaves under u in C is thus at most $\beta+\frac{1}{2q}$, but by the choice of u it is at least $1/q$. Thus $\beta \geq \frac{1}{2q}$.

Hence at least $2^{k_{t-1}-g/2+3}$ vertices of D are full. Under each full vertex of D some two leaves v and v' have mates in P. If v and v' have the same mate x, then x completes a cycle of length at most $2+2(g/4-1)<g$ with the path joining v and v' in T. Otherwise, each full vertex of D has two leaves under it whose mates are distinct vertices of P. Since the number of full vertices of D exceeds $\binom{|V(P)|}{2}$, by the pigeonhole principle some two vertices $y, y' \in D$ yield the same pair $x, x' \in V(P)$ of mates of two leaves under them. The paths joining those leaves in the subtrees under y and y' and the edges from those leaves to x and x' form a cycle of length at most $2(g/4-1)+2(g/4-1)+4$, which equals g. $\square$

Chapter 4

Coloring graph powers from lists

The results of this chapter are joint with Nicholas Kosar, Sarka Petrickova, and Elyse Yeager and appear in [43].

4.1 Introduction

A graph is *chromatic-choosable* if $\mathrm{ch}(G) = \chi(G)$. The *kth power* of a graph G, denoted by G^k, is the graph on the same vertex set as G such that uv is an edge if and only if the distance from u to v in G is at most k. To *subdivide* an edge uv of a graph is to replace uv by a u,v-path whose internal vertices do not appear in the original graph.

Perhaps the most important open question in list-coloring is the List (Edge) Coloring Conjecture. It first appeared in print in a 1985 paper of Bollobás and Harris [12], but it seems (see [38] and [34]) to have been formulated also by Albertson, Collins, and Gupta (motivated by Erdős, Rubin, and Taylor [27]), as well as by Vizing as early as 1975.

Conjecture 4.1.1 (List Edge-Coloring Conjecture (LECC))**.** *$L(G)$ is chromatic-choosable for every graph G.*

A *total coloring* of a graph G (with colors from a set X) is a function $f : V(G) \cup E(G) \to X$; a total coloring is *proper* if every adjacent pair of vertices receive different colors, every adjacent pair of edges receive different colors, and every pair of a vertex and incident edge receive different colors. The *total graph* of G, denoted $T(G)$, is the square of the graph obtained by subdividing each edge of G into a path of length two. A proper total coloring of G is equivalent to a proper coloring of $T(G)$.

Conjecture 4.1.2 (List Total Coloring Conjecture (LTCC) [13])**.** *$T(G)$ is chromatic-choosable for every graph G.*

Since $T(G)$ is the square of a graph, a stronger conjecture is that the square of any graph is chromatic-choosable.

Conjecture 4.1.3 (List Square Coloring Conjecture (LSCC) [47]). *G^2 is chromatic-choosable for every graph G.*

However, the LSCC was disproved by Kim and Park [40], who constructed a family of graphs G with $\chi(G^2)$ unbounded and $\mathrm{ch}(G^2) \geq c\chi(G^2)\log\chi(G^2)$. Xuding Zhu asked whether there is any k such that all kth powers are chromatic-choosable. We give a negative answer to Zhu's question, with a bound on $\mathrm{ch}(G^k)$ that matches that of Kim and Park for $k = 2$.

Theorem 4.3.4. There is a positive constant c such that for every $k \in \mathbb{N}$, there is an infinite family of graphs G with $\chi(G^k)$ unbounded and

$$\mathrm{ch}(G^k) \geq c\chi(G^k)\log\chi(G^k).$$

Kim, Kwon, and Park arrived at a similar result in [39]. They found, for each k, an infinite family of graphs G whose kth powers satisfy $\mathrm{ch}(G^k) \geq \frac{10}{9}\chi(G^k) - 1$.

Letting $f_k(m) = \max\{\mathrm{ch}(G^k)\colon\ \chi(G^k) = m\}$, Theorem 4.3.4 says that $f_k(m) \geq cm\log m$.

For upper bounds, it is not hard to see that $\mathrm{ch}(L(G)) \leq 2\chi(L(G)) - 1$ and $\mathrm{ch}(T(G)) \leq 2\chi(T(G)) - 1$ for every graph G. Kwon (see [54]) observed that $\mathrm{ch}(G^2) \leq \chi(G^2)^2$ for any G; that is, $f_2(m) < m^2$. We extend this observation to larger powers in Section 4.4.

Theorem 4.4.1. Let $k > 1$. If k is even, then $f_k(m) < m^2$. If k is odd, then $f_k(m) < m^3$.

Question 4.1.4. *What is the correct order of magnitude of $f_k(m)$? Does it depend on k?*

4.2 Construction

The example of Kim and Park [40] for $k = 2$ is based on complete sets of mutually orthogonal latin squares. We will use this structure to find examples for all k, but we find the language of affine planes to be more convenient. An affine plane is a finite geometry; it consists of a set of *points* $\mathcal{P}$ and a set of *lines* $\mathcal{L}$, and is subject to a set of axioms. In particular, we have the following properties (see [18] for instance) for some integer n (called the *order* of the affine plane):

- Each line is a set of n points.
- For each pair of points, there is a unique line containing them.
- The set of lines admits a partition $L_0, L_1, \ldots, L_n$, where each L_i is called a *parallel class* of lines, such that

– Two lines in the same parallel class do not intersect.

– Two lines in different parallel classes intersect in exactly one point.

- Such a plane exists whenever n is a (positive) power of a prime. (Figure 4.1 shows the four parallel classes of lines in the affine plane of order 3.)

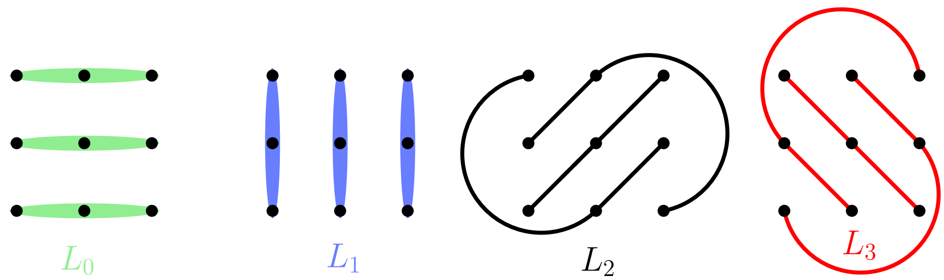


Figure 4.1: The four parallel classes of lines in the affine plane of order 3.

Fix a prime power n and an affine plane $(\mathcal{P}, \mathcal{L})$ of order n.

Form the bipartite graph H with parts $\mathcal{P}$ and $B = \mathcal{L} - L_0$, with $p\ell \in E(H)$ for $p \in \mathcal{P}$ and $\ell \in B$ if and only if $p \in \ell$. Let $a_1, \ldots, a_n$ denote the lines of L_0. Consider the refinement $\mathcal{V}'$ of the bipartition of H obtained by partitioning $\mathcal{P}$ into $a_1, \ldots, a_n$ and B into $L_1, \ldots, L_n$. Note that $H[a_i, L_j]$ is a matching for each i and j. In Figure 4.2, the graph H is shown with $n = 3$. Edges are drawn differently according to which parallel class their line-endpoint belongs to, and the parts of $\mathcal{V}'$ are indicated. Note that by removing the lines of L_0, we have point-line duality as in a projective plane.

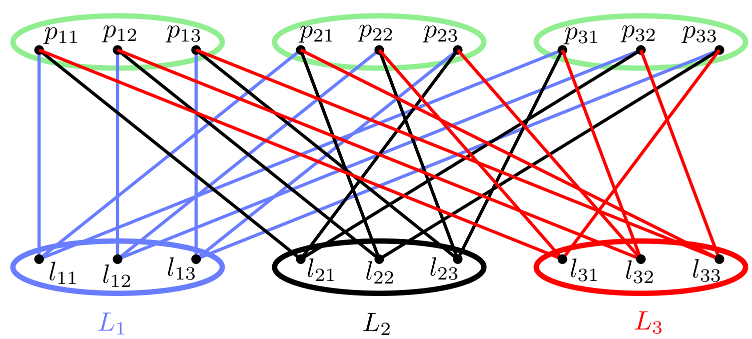


Figure 4.2: The graph H, here with $n = 3$.

The graph H (or rather, the family of graphs obtained by varying n over prime powers) proves the case $k = 3$ of Theorem 4.3.4. In fact, H^3 is complete multipartite with parts $\mathcal{V}'$, which we show now as a warmup to the main proof of the theorem. Consider two points from different lines of L_0; they lie on a common line that is not in L_0, and this line is a common neighbor in H of the two points. Consider two lines from different parallel classes; they intersect in some point, and this point is a common neighbor in H of the two lines. Consider a point p and a line ℓ. Possibly p is on ℓ and they are adjacent already in H. Otherwise, consider any line a of L_0 that does not contain p, and let q be the unique point of $a \cap \ell$. Note that p and q are on some common line b not in L_0. The path $pbqa$ in H shows that $a\ell$ is an edge of H^3. Consider two lines in the same parallel class (but not L_0). They do not share a common point, and hence they have no common neighbor in H; since H is bipartite, the distance between them is at least 4. Consider two points on the same line of L_0; they do not have a common neighbor in H, and again since H is bipartite they are not adjacent in H^3.

The graph obtained by subdividing every edge of H into a path of length m proves Theorem 4.3.4 for $k = 4m-1$, and so whenever $k \equiv 3 \pmod 4$ we are done. To prove the theorem for other k, we need to modify the graph a bit more.

Let $k \geq 2$. Subdivide the edges of H into paths: edges incident to L_1 are subdivided into paths of length k, while edges not incident to L_1 are subdivided into paths of length $k+1$. For an edge $p\ell \in E(H)$, denote the vertices along the subdivision path as $p = (p\ell)_0, (p\ell)_1, (p\ell)_2, \ldots$. If $\ell \in L_1$, then $(p\ell)_k = \ell$, and if $\ell \notin L_1$, then $(p\ell)_{k+1} = \ell$. For a vertex $(p\ell)_i$, say its *level* is i, its *point* is p, and its *line* is ℓ (levels are well-defined, and points and lines of vertices of degree 2 are well-defined). Form the graph G by, for each $\ell \in \bigcup_{2 \leq i \leq n} L_i$, adding edges to make the neighborhood of ℓ a clique and then deleting ℓ. For each $i, j \in [n]$ and $m \in \{0, \ldots, k\}$, let $V_{i,j,m} = \{(p\ell)_m \colon\ p\ell \in E(H), p \in a_i, \ell \in L_j\}$; then $\{V_{i,j,m} \colon\ i,j \in [n], m \in \{0, \ldots, k\}\}$ is a partition of $V(G)$ into sets of size n, which we call $\mathcal{V}$. In Figure 4.3, the graph G is shown. Again we use $n = 3$, and here the parts of $\mathcal{V}$ are indicated.

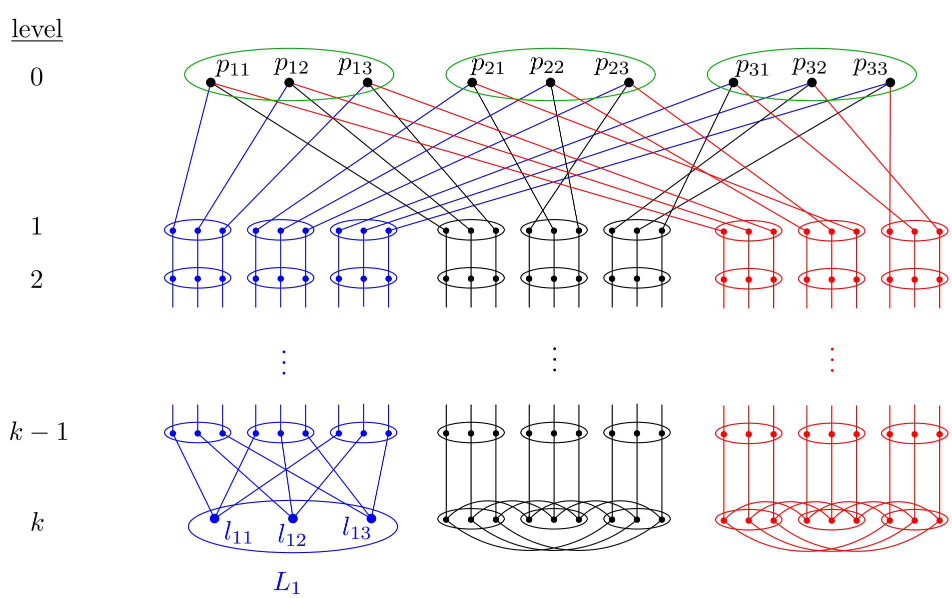


Figure 4.3: The graph G when $n = 3$.

4.3 Proof of Theorem 4.3.4

Lemma 4.3.1. *G^{4k} is multipartite with partition $\mathcal{V}$.*

Proof. For each pair of vertices in the same part of $\mathcal{V}$, we will show that any path joining those vertices has length at least $4k+1$.

Let p and q be two points in some a_i. Any p,q-path leaves p and first reaches a branch vertex of the form $(p\ell)_k$. If $\ell \in L_1$, then the path must continue from $(p\ell)_k$ to p' for some p' not on a_i; in this case, the part of the path so far described has length $2k$. If $\ell \notin L_1$, then the path must continue from $(p\ell)_k$ to $(p'\ell)_k$ for some p' not on a_i before again leaving level k, then reaches p'; in this case, the part of the path so far described has length $2k+1$. Since p' is not on a_i, p' and q are on a common line $\ell' \in \bigcup_{i=1}^n L_i$. If $\ell' \in L_1$, then the shortest path from p' to q passes through ℓ' and has length $2k$. If $\ell' \in \bigcup_{i=2}^n L_i$, then the shortest path from p' to q passes through $(p'\ell')_k$ and $(q\ell')_k$, and has length $2k+1$. Now, it cannot be that both $\ell, \ell' \in L_1$, because p' is in both ℓ and ℓ'. So our path has length at least $4k+1$.

Let $\ell_1, \ell_2 \in L_1$. Any ℓ_1, ℓ_2-path leaves ℓ_1 and first reaches a branch vertex p, and it last leaves a branch vertex q before reaching ℓ_2. If p and q are in some a_i, then since $\mathrm{dist}(p,q) \geq 4k+1$ the path from ℓ_1 to ℓ_2 would have length at least $4k+1$. Otherwise, p and q are on a common line not in L_0 or L_1, say ℓ. The shortest path between p and q contains $(p\ell)_k$ and $(q\ell)_k$ and has length $2k+1$. Hence any ℓ_1, ℓ_2-path has length at least $4k+1$.

Let $(p\ell_1)_k, (q\ell_2)_k$ be two vertices in the same part other than L_1; that is, p and q are distinct points on some a_i, and ℓ_1 and ℓ_2 are distinct lines in the same parallel line class. Consider a path joining these two vertices. The neighbors of the endpoints on this path are either in level k or $k-1$. If both neighbors are in level $k-1$, then consider the branch vertices u, v in the interior of the path that appear closest to the endpoints of the path. (That is, the part of the path between $(p\ell_1)_k$ and u consists entirely of vertices of degree 2, and the part of the path between v and $(q\ell_2)_k$ consists entirely of vertices of degree 2.) If u and v are both in level 0, then they are at distance at least $4k+1$ and so the path has length $4k+1$. So we may assume, up to symmetry, that the neighbor of $(p\ell_1)_k$ on the path is in level k. The path from $(p\ell_1)_k$ eventually leaves level k; let $(p'\ell_1)_k$ be the last vertex before the path leaves level k. Since $p, q \in a_i$ and $p, p' \in \ell_1$, we have $p' \neq q$. Hence the shortest path from $(p'\ell_1)_k$ to $(q\ell_2)_k$ has length at least $4k$, and thus the path from $(p\ell_1)_k$ to $(q\ell_2)_k$ has length at least $4k+1$.

Finally, consider two vertices of degree 2 in the same part, say x and y, and consider an x,y-path. Let u and v be the branch vertices closest to x and y (respectively) along the path. If u and v are both in level 0 or both in level k, then the part of the path joining u and v already has length at least $4k+1$. Otherwise, one of u and v is in level 0 and the other in level k. Let $x = (p\ell_1)_m$ and $y = (q\ell_2)_m$; we may assume up to symmetry that $u = p$ and $v = (q\ell_2)_k$. Since x and y are in the same part of $\mathcal{V}$, we have that p and q are distinct points in some a_i and that ℓ_1 and ℓ_2 are distinct lines in some L_j. The path must continue from x through p to some vertex in level k, say $(p\ell_3)_k$. At most one of ℓ_2 and ℓ_3 is in L_1, so the distance from $(p\ell_3)_k$ to $(q\ell_2)_k$ is at least $2k+1$; hence the length of the path is at least $4k+1$. $\square$

Lemma 4.3.2. *The subgraph of G^{4k} induced by the vertices in levels 0 through $k-1$ is complete multipartite with partition $\mathcal{V}$ restricted to those levels.*

Proof. It remains to show that for every pair of vertices in different parts of $\mathcal{V}$ (and in levels 0 through $k-1$) are joined by a path of length at most $4k$.

Consider points p and q on different lines in L_0. They are on a common line $\ell \in \bigcup_{i=1}^{n} L_i$. If $\ell \in L_1$, then there is a p,q-path through ℓ of length $2k$. If $\ell \notin L_1$, then there is a p,q-path through $(p\ell)_k$ and $(q\ell)_k$ of length $2k+1$.

Consider two vertices x and y in different parts at level i, where $1 \le i \le k-1$. Let $x = (p\ell_1)_i$ and $y = (q\ell_2)_i$. Either p and q are on different lines in L_0, or ℓ_1 and ℓ_2 are from different parallel classes. If p and q are from different lines in L_0, then let ℓ be the line containing both p and q. There is an x,y-path through p, $(p\ell)_k$, $(q\ell)_k$, and q with length at most $2i+2k+1 \le 4k-1$. If ℓ_1 and ℓ_2 are from different parallel classes, then let p' be their intersection point. There is an x,y-path through $(p\ell_1)_k$, $(p'\ell_1)_k$, p', $(p'\ell_2)_k$, and $(q\ell_2)_k$ with length at most $2(k-i)+2+2k = 4k-2i+2 \le 4k$.

Finally, consider two vertices x and y in levels i and j (respectively), where $0 \le i < j < k$. Say $x = (p\ell_1)_i$ and $y = (q\ell_2)_j$. If $\ell_2 \in L_1$, then let p' be a point in ℓ_2 such that p and p' are not on the same line of L_0, and let ℓ' be the line containing p and p'. There is an x,y-path through p, $(p\ell')_k$, $(p'\ell')_k$, p', and ℓ_2 with length at most $i+k+1+k+k+j \le 4k$. If $\ell_2 \notin L_1$, then let ℓ' be a line in L_1 containing p, and let p' be the intersection point of ℓ' and ℓ_2. There is an x,y-path through p, ℓ', p', $(p'\ell_2)_k$, and $(q\ell_2)_k$ with length at most $i+k+k+k+1+j \le 4k$. $\square$

We will use the following result of Alon.

Lemma 4.3.3 ([1]). *Let K_{r*s} denote the complete r-partite graph with each part of size s. There are two constants, d_1 and d_2, such that*

$$d_1 r \log s \le \mathrm{ch}(K_{r*s}) \le d_2 r \log s.$$

Everything is now in place to complete the proof.

Theorem 4.3.4. *There is a positive constant c such that for every $k \in \mathbb{N}$, there is an infinite family of graphs G with $\chi(G^k)$ unbounded and*

$$\mathrm{ch}(G^k) \geq c\chi(G^k)\log\chi(G^k).$$

Proof. Since G^{4k} is multipartite on kn^2+1 parts, $\chi(G^{4k}) \leq kn^2+1$, and hence $n \geq \sqrt{(\chi(G^{4k})-1)/k}$.

Since G^{4k} contains a complete multipartite subgraph with $(k-1)n^2$ parts of size n, we have from Lemma 4.3.3 that

$$\begin{aligned}
\mathrm{ch}(G^{4k}) &\geq d_1(k-1)n^2\log n \\
&\geq d_1\frac{k-1}{k}\left(\chi(G^{4k})-1\right)\log\sqrt{\frac{\chi(G^{4k})-1}{k}} \\
&= \frac{d_1}{2}\frac{k-1}{k}\left(\chi(G^{4k})-1\right)\left(\log(\chi(G^{4k})-1)-\log k\right) \\
&\geq \frac{d_1}{4}\left(\chi(G^{4k})-1\right)\left(\log(\chi(G^{4k})-1)-\log k\right).
\end{aligned}$$

Taking n large enough makes $\chi(G^{4k})$ as large as we like, and so by taking a constant c just smaller than $d_1/4$ and taking n sufficiently large we obtain

$$\mathrm{ch}(G^{4k}) \geq c\chi(G^{4k})\log\chi(G^{4k}).$$

The family $\{G^4\}$ is an infinite family of graphs whose kth powers have the desired properties. □

4.4 Upper bound

We now provide an upper bound on $\mathrm{ch}(G^k)$ in terms of $\chi(G^k)$. Recall that

$$f_k(m) = \max\{\mathrm{ch}(G^k)\colon\ \chi(G^k) = m\}.$$

Theorem 4.4.1. *Let $k > 1$. If k is even, then $f_k(m) < m^2$. If k is odd, then $f_k(m) < m^3$.*

When k is even, this follows from Kwon's observation (see [54]) that it holds for $k = 2$: Kwon proved that $f_2(m) < m^2$, i.e. $\mathrm{ch}(G^2) < \chi(G^2)^2$ for every graph G. Since $G^{2k} = (G^k)^2$, we have that $\mathrm{ch}(G^{2k}) < \chi(G^{2k})^2$, and so $f_{2k}(m) < m^2$.

When k is odd, we generalize Kwon's argument and prove the following.

Theorem 4.4.2. *If $k \geq 3$ and k is odd, then* $\mathrm{ch}(G^k) \leq \Delta(G)\,\chi(G^k)^2$ *for every graph* G.

Theorem 4.4.1 follows by noting that $\Delta(G) < \omega(G^k) \leq \chi(G^k)$ when $k > 1$.

Proof of Theorem 4.4.2. Let x be a vertex with maximum degree in G^k. Let A be the set of vertices at distance $\lceil k/2 \rceil$ from x in G. Let $B(v, r)$ denote the ball of radius r centered at v in G. Note that $\Delta(G^k) = \max\{|B(v,k)| - 1\colon\ v \in V(G)\}$ and $\omega(G^k) \geq \max\{|B(v, \lfloor k/2 \rfloor)|\colon\ v \in V(G)\}$.

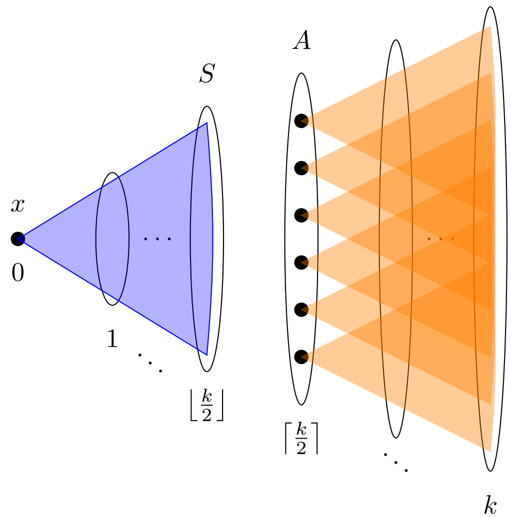


Figure 4.4: Covering a ball of radius k by balls of radius $\lfloor k/2 \rfloor$.

Since k is odd (and bigger than 1), we have

$$B(x,k) \setminus B(x, \lfloor k/2 \rfloor) \subseteq \bigcup_{y \in A} B(y, \lfloor k/2 \rfloor). \tag{4.1}$$

See Figure 4.4. Let S be the set of vertices at distance $\lfloor k/2 \rfloor$ from x in G. The set S is a clique in G^k, so $|S| \leq \omega(G^k)$. Also, A is contained in the neighborhood of S, and each vertex in S has at least one neighbor outside of A (closer to x). Hence $|A| \leq (\Delta(G) - 1)|S| \leq (\Delta(G) - 1)\omega(G^k)$. Putting everything together, we have

$$
\begin{aligned}
\operatorname{ch}(G^k) &\le 1 + \Delta(G^k) && \text{(degeneracy)} \\
&= |B(x,k)| \\
&\le |B(x,\lfloor k/2 \rfloor)| + \sum_{y\in A} |B(y,\lfloor k/2 \rfloor)| && \text{(equation (4.1))} \\
&\le (1+|A|) \max_{v\in V(G)} |B(v,\lfloor k/2 \rfloor)| && \text{(bounding terms in sum)} \\
&\le \big(1 + (\Delta(G)-1)\omega(G^k)\big)\,\omega(G^k) \\
&\le \Delta(G)\,\omega(G^k)^2 \\
&\le \Delta(G)\,\chi(G^k)^2. && \square
\end{aligned}
$$

4.5 Remark

Using constructions similar to that of Section 4.2, we found infinite families of graphs G whose kth powers are complete multipartite graphs with roughly $kn^2/4$ parts each of size n, but only when $k \not\equiv 0 \mod 4$. The construction presented here is messier and does not yield complete multipartite powers, but it proves the theorem for all values of k simultaneously.

Chapter 5

Hypergraph degree sequences and codegree functions

5.1 Introduction

The *degree sequence* of a (hyper)graph is the list of its vertex degrees, usually taken in nonincreasing order. A sequence of integers is called *k-graphic* if it is the degree sequence of some k-uniform hypergraph, and we call such a hypergraph a *realization* of the sequence. The question of when a given sequence is the degree sequence of some (simple) graph is well-understood. The same question for k-uniform hypergraphs for $k \geq 3$ is less well understood. Dewdney [19] provided a characterization, but it does not provide an efficient algorithm.

Theorem 5.1.1 (Dewdney [19]). *Let π be a nonincreasing sequence of nonnegative integers, say $(d_1, \ldots, d_n)$. π is k-graphic if and only if there exists a nonincreasing sequence π' of $n-1$ nonnegative integers, say $(d'_2, \ldots, d'_n)$, such that*

- *π' is $(k-1)$-graphic,*
- *$\displaystyle\sum_{i=2}^{n} d'_i = (k-1)d_1$, and*
- *$\pi'' = (d_2 - d'_2, \ldots, d_n - d'_n)$ is k-graphic.*

Havel and Hakimi provided one efficient characterization of graphic sequences.

Theorem 5.1.2 (Havel [37], Hakimi [35]). *The nonincreasing sequence $d_0, d_1, \ldots, d_n$ is graphic if and only if the sequence $d_1 - 1, \ldots, d_{d_0} - 1, d_{d_0+1}, \ldots, d_n$ is graphic.*

The proof hinges on the notion of degree-preserving operations on graphs. In particular, a *2-switch* is an operation that deletes two edges from a graph and adds two new edges in such a way that vertex degrees are preserved. Fulkerson, Hoffman, and McAndrew [29] proved that the space of realizations of a graphic sequence is connected via 2-switches.

In Section 5.2 we discuss analogues of the 2-switch for hypergraph degree sequences; in particular, we show that certain small families of switches are insufficient to connect realizations of a given graphic sequence.

In Section 5.3, we consider analogues of the 2-switch for hypergraph *codegree functions*: the codegree function of a k-uniform hypergraph is the function that assigns to each $(k-1)$-set its degree. The results there lead us to results in combinatorial topology.

A *triangulation* of a surface is an embedding of a multigraph in the surface such that each face is bounded by a triangle. A *strict triangulation* is a triangulation by a simple graph. (This terminology is not universal in the literature: often a strict triangulation is just called a triangulation, and other terms are used for multigraph embeddings.)

Let Σ_1 and Σ_2 be triangulations of surfaces. Suppose there is some triangulation of a disk S that appears in both Σ_1 and Σ_2; formally, let $S_1 \subseteq \Sigma_1$ and $S_2 \subseteq \Sigma_2$ with isomorphisms of triangulations $f : S \to S_1$ and $g : S \to S_2$. We define the *connected sum* of Σ_1 and Σ_2 along S_1, S_2 to be the triangulation (of a possibly new surface) obtained by deleting the interior of S_i from Σ_i for $i \in \{1, 2\}$ and gluing the resulting boundaries together according to $g \circ f^{-1}$.

Let $\Delta_b(S^2)$ denote the family of triangulations of the sphere that are bipartite. Any connected sum of two triangulations with bipartite duals is again a triangulation with bipartite dual: choose 2-colorings of Σ_1 and Σ_2 so that the colorings on S_1 and S_2 disagree; the resulting coloring of the connected sum is proper. If the two original triangulations were sphere triangulations, then so is their connected sum: deleting the interior of S_i from Σ_i results in a disk, and gluing them along their boundary circles forms a sphere. Let $\mathcal{O}$ denote the family of triangulations of the sphere defined inductively as follows: the octahedron is in $\mathcal{O}$, and if Σ is in $\mathcal{O}$, then so is any connected sum of Σ with the octahedron. We have that $\mathcal{O} \subseteq \Delta_b(S^2)$. We prove that in fact $\mathcal{O} = \Delta_b(S^2)$.

Theorem 5.1.3. *Every triangulation of the sphere with bipartite dual can be obtained from the octahedron by connected sums with octahedra. A similar result holds for strict triangulations.*

In Section 5.4, we interpret graphicality as an integer program and explore the fractional relaxation of that program. In particular, when TONCAS ("The Obvious Necessary Conditions are Also Sufficient") theorems like the Erdős-Gallai Theorem exist, we show that the feasibility of the fractional relaxation is equivalent to feasibility of the integer program.

Theorem 5.1.4 (Erdős, Gallai [25])**.** *Let $\pi : d_1 \geq \cdots \geq d_n$ be such that $\sum_{i=1}^{n} d_i$ is even. Then π is graphic if and only if, for every t,*

$$\sum_{i=1}^{t} d_i \leq 2\binom{t}{2} + \sum_{i=t+1}^{n} d_i.$$

5.2 Edge exchanges for degree sequences

The results of this section are joint with Sarah Behrens, Catherine Erbes, Michael Ferrara, Stephen Hartke, Hannah Spinoza, and Charles Tomlinson and appear in [8].

An *edge exchange* is any operation that deletes a set of edges in a k-realization of π and replaces them with another set of edges, while preserving the original vertex degrees; we allow multiple edges to arise in this operation. When i edges are removed, to preserve degrees we must add i edges, and we call this an *i-exchange*. When we require that the resulting hypergraph have no multiple edges, we call the operation an *i-switch*. The 2-switch operation has been used to prove many results about graphic sequences; for examples see [6, 15, 17, 28].

Define $\mathcal{M}_k(\pi)$ to be the set of k-uniform multihypergraphs that realize a sequence π, and let $\mathcal{S}_k(\pi) \subseteq \mathcal{M}_k(\pi)$ be the set of simple k-realizations of π. Let $\mathcal{F} \subseteq \mathcal{M}_k(\pi)$, and let $\mathcal{Q}$ be a collection of edge exchanges that is closed under reversing exchanges. Let $G(\mathcal{F}, \mathcal{Q})$ be the graph whose vertices are the elements of $\mathcal{F}$, with an edge between vertices H_1 and H_2 if and only if H_1 can be obtained from H_2 by an edge exchange in $\mathcal{Q}$. (The symmetry condition imposed on $\mathcal{Q}$ permits us to define $G(\mathcal{F}, \mathcal{Q})$ as an undirected graph.)

For a positive integer i, let $\mathcal{E}_i$ be the set consisting of all j-exchanges for all j with $j \leq i$. (The uniformity of the hypergraphs under consideration are left implicit.) Fulkerson, Hoffman, and McAndrew [29] showed that given any pair of realizations of a graphic sequence, one can be obtained from the other by a sequence of 2-switches. This result simply says that $G(\mathcal{S}_2(\pi), \mathcal{E}_2)$ is connected. Kocay and Li [42] proved a similar result for 3-uniform hypergraphs, namely that any pair of 3-uniform hypergraphs with the same degree sequence can be transformed into each other using 2-exchanges. Their proof implies that $G(\mathcal{M}_3(\pi), \mathcal{E}_3)$ is connected, but says nothing of $G(\mathcal{S}_3(\pi), \mathcal{E}_3)$ since they do not restrict to 2-switches.

In [8], we extended the result of Kocay and Li to arbitrary $k \geq 3$: If π is any sequence of nonnegative integers with a k-uniform multihypergraph realization, then $G(\mathcal{M}_k(\pi), \mathcal{E}_k)$ is connected.

Gabelman [31] gave an example of a 3-graphic sequence π with two simple realizations that cannot be transformed into each other using only 2-switches. That is, $G(\mathcal{S}_3(\pi), \mathcal{E}_2)$ is not connected. We extend his example to $k \geq 3$, which shows we cannot replace $\mathcal{M}_k$ with $\mathcal{S}_k$ in the result of the previous paragraph.

Theorem 5.2.1. *For each $k \geq 3$ there is a k-graphic sequence π such that $G(\mathcal{S}_k(\pi), \mathcal{E}_{k-1})$ is not connected. Specifically, there exist two realizations of π, neither of which admits a 2-switch (to a simple k-uniform hypergraph); i.e., these realizations are isolated vertices in $G(\mathcal{S}_k(\pi), \mathcal{E}_{k-1})$.*

Proof. Consider the following matrix A of real numbers:

$$A = \begin{bmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,k-1} & -y_1 \\ x_{2,1} & x_{2,2} & \cdots & x_{2,k-1} & -y_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{k-1,1} & x_{k-1,2} & \cdots & x_{k-1,k-1} & -y_{k-1} \\ -z_1 & -z_2 & \cdots & -z_{k-1} & w \end{bmatrix}$$

where

$$y_j = \sum_{i=1}^{k-1} x_{j,i}, \qquad z_j = \sum_{i=1}^{k-1} x_{i,j}, \qquad \text{and} \qquad w = \sum_{i,j} x_{i,j}.$$

We choose the terms $x_{i,j}$ so that if a set of k entries of the matrix sums to zero, then those entries must be from a single row or column. This can be done by choosing $\{x_{i,j} \colon i,j \in [k-1]\}$ to be linearly independent over $\mathbb{Q}$, or by choosing them to be distinct powers of some sufficiently small positive constant ϵ.

We form a hypergraph H on a set V of k^2 vertices as follows. Weight each vertex with a different entry of the matrix. (Let $\mathrm{wt}(v)$ denote the weight on the vertex v.) The edges of H are (1) the k-sets whose total weight is positive and (2) the k-sets corresponding to the rows of A. By construction of the matrix A, the only k-sets that have zero weight correspond to rows and columns. Thus the k-sets that are non-edges are precisely those with negative weight and those that correspond to columns.

The degree sequence of H is not uniquely realizable, as the k-switch that adds the k-sets corresponding to columns of A to the edge set while removing the edges corresponding to rows gives another realization. However, we next show that we cannot apply an i-switch to H for any i smaller than k. (Thus H is an isolated vertex in $G(\mathcal{S}_k(\pi), \mathcal{E}_{k-1})$.)

Note that in any edge exchange that replaces a set F_1 of edges with a set F_2 of nonedges,

$$\sum_{e \in F_1} \sum_{v \in e} \mathrm{wt}(v) = \sum_{v \in V} (\deg_{F_1}(v))\,\mathrm{wt}(v) = \sum_{v \in V} (\deg_{F_2}(v))\,\mathrm{wt}(v) = \sum_{e \in F_2} \sum_{v \in e} \mathrm{wt}(v).$$

Since edges of F_1 have nonnegative weight, the leftmost quantity is nonnegative; and since nonedges of F_2 have nonpositive weight, the rightmost quantity is nonpositive. Thus each quantity is zero. Therefore the edges of F_1 must each have zero weight and thus correspond to rows of A, and the nonedges of F_2 must each have zero weight and correspond to columns of A. But no proper subset of edges corresponding to rows can be swapped for a proper subset of nonedges corresponding to columns, because this does not maintain the degree of every vertex. Hence $|F_1| \geq k$. $\square$

This result immediately suggests the following question:

Question 5.2.2. *What is the smallest cardinality of a collection $\mathcal{Q}$ such that $G(\mathcal{S}_k(\pi), \mathcal{Q})$ is connected for every k-graphic sequence π?*

Results for graphs suggest several different possible approaches. Is there a finite collection $\mathcal{Q}$ that works? Would it be sufficient to add all possible k-switches? (I.e., is $\mathcal{E}_k$ sufficient?)

5.3 Edge exchanges for codegree functions

The results of this section are joint with Sarah Behrens and are based on [9].

5.3.1 Introduction

Recall that the *codegree function* of a k-uniform hypergraph is the function $f : \binom{V}{k-1} \to \mathbb{N}$ that assigns to each $(k-1)$-set its degree. (Again, this definition varies by author; see Section 1.4 for details.) The codegree function of hypergraphs is another possible generalization of the degree sequence of graphs. Given the codegree function, one can recover the degree of any set of at most $k-1$ vertices, and in particular the vertex degree sequence.

As discussed before, the 2-switch for ordinary graphs is an important concept. Here we seek an analogous switch (or a set of switches) for the codegree function. We consider 3-uniform hypergraphs for now.

First, a few more topological definitions. A *surface* is a compact 2-dimensional manifold. (We will not require that a surface is connected, so a disjoint union of surfaces will again be a surface.) A *pseudosurface* is obtained from a surface by identifying finitely many points of a surface together into new points. A *triangulation* of a (pseudo)surface is an embedding of a multigraph in the (pseudo)surface such that every face is bounded by a triangle of the multigraph. A *strict triangulation* is a triangulation whose embedded multigraph is a simple graph.

Suppose that two 3-uniform hypergraphs G and H have the same vertex set V, and that for every pair of vertices $u, v \in V$, $d_G(u, v) = d_H(u, v)$. (That is, G and H have the same codegree function.) Consider the hypergraph $G \ominus H$ with vertex set V, edge set $E(G) \triangle E(H)$, and with edges from G labeled by $+1$ and edges of H labeled with -1. We now show that $G \ominus H$ can be represented by a 2-colored triangulation Σ of some surface. These triangulations may be embeddings of multigraphs. We will call vertices and edges in the triangulation *nodes* and *arcs* (respectively) to distinguish them from vertices and edges of the hypergraph.

For each pair $u, v \in V$, since $d_G(u, v) = d_H(u, v)$, we have that $\{u, v\}$ appears in the same number of $+1$ edges as -1 edges in $G \ominus H$. We will first construct a pseudosurface Σ' from $G \ominus H$. The nodes of Σ'

are the vertices of G (and H). For each pair of vertices u, v and each $+1$ edge e of $G \ominus H$ containing both u and v, put an arc joining the nodes u and v, labeled by e. Then construct Σ' by adding a disk for each edge e of $G \ominus H$, gluing along the three arcs of Σ' that are labelled by e. In this way we have a surface except that perhaps some nodes of Σ' may be singular points. Let Σ be obtained from Σ' by splitting these neighorhoods and adding additional nodes as necessary.

Hence the set of all "2-colorable-triangulation switches" are sufficient to get from any realization of a codegree function to any other. We will prove in Theorem 5.3.2 that the set of "sphere switches" can be reduced to just those using an octahedron in the following sense. The octahedral operation acts locally on G and H simultaneously and produces a pair of 3-uniform hypergraphs G', H' that share the same codegree function; however, G and G' need not have the same codegree function. Theorem 5.3.2 says that if G and H have the same codegree function and $G \ominus H$ can be represented by a sphere (as above), then there is a sequence of these octahedral operations that ends with $G' = H'$.

5.3.2 Other connections

Wagner [59] proved that any two triangulations of the sphere with the same number of vertices can be transformed into one another by a sequence of *diagonal flips* (also called *diagonal transformations*, *bistellar flips*, or *Pachner moves* depending on the context). See Figure 5.1. This has been extended to any surface with several more specific results (c.f. [52]), and to arbitrary dimension [55].

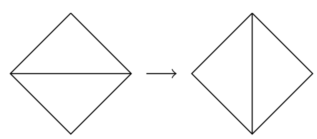


Figure 5.1: The diagonal flip.

Triangulations of the sphere with bipartite duals (aside from that with just two faces) are precisely the duals of the Barnette graphs, where the *Barnette graphs* are defined to be the bipartite cubic polyhedral graphs. Barnette conjectured that these graphs are Hamiltonian [5]; there is some hope that a sufficiently nice inductive definition of these graphs would yield an inductive proof of Barnette's Conjecture. However, the inductive construction we present seems to be insufficient for this cause.

Let Σ_1 and Σ_2 be triangulations of surfaces. Suppose there is some triangulation of a disk S that appears in both Σ_1 and Σ_2; formally, let $S_1 \subseteq \Sigma_1$ and $S_2 \subseteq \Sigma_2$ with isomorphisms of triangulations $f : S \to S_1$ and $g : S \to S_2$. We define the *connected sum* of Σ_1 and Σ_2 along S_1, S_2 to be the triangulation (of a possibly new surface) obtained by deleting the interior of S_i from Σ_i for $i \in \{1, 2\}$ and gluing the resulting boundaries together according to $g \circ f^{-1}$.

The diagonal flip of Wagner is one kind of connected sum with a tetrahedron. The Pachner moves are connected sums with the boundaries of k-dimensional simplices (appropriately defined for higher-dimensional manifolds). For brevity, we say an embedding of a graph in a surface is *2-colorable* if the dual graph is 2-colorable. That is, the faces of the embedding can be 2-colored so that two faces that share an edge have different colors.

Proposition 5.3.1. *The connected sum of 2-colorable surfaces is also 2-colorable.*

Proof. Take [2]-colorings of the faces of Σ_1 and Σ_2 such that the colorings disagree on S_1 and S_2. The resulting coloring of the sum along S_1, S_2 is proper. $\square$

In particular, taking repeated connected sums with an octahedron preserves 2-colorability. Our main theorems state that this generates all 2-colorable triangulations of the sphere.

To state our results precisely, let $\mathcal{T}$ (respectively $\mathcal{ST}$) be the graph whose nodes are the 2-colorable triangulations (respectively strict triangulations) of the sphere, with two triangulations connected by an edge if they can be obtained from one another by a connected sum with an octahedron.

Theorem 5.3.2. *$\mathcal{T}$ is connected. Equivalently, every 2-colorable triangulation of the sphere can be obtained from the empty complex by a sequence of connected sums with octahedra.*

Theorem 5.3.3. *$\mathcal{ST}$ is connected. Equivalently, every 2-colorable strict triangulation of the sphere can be obtained from the empty complex by a sequence of connected sums with octahedra in such a way that after each sum, the result is still a 2-colorable strict triangulation of the sphere.*

In Section 5.3.4 we prove Theorem 5.3.2; in Section 5.3.5 we extend this proof to strict triangulations to prove Theorem 5.3.3. We discuss some possible extensions (to other surfaces and higher dimension) in Section 5.3.7.

Inductive definitions of planar triangulations, and of subcollections such as those that are 2-colorable, were previously investigated by Batagelj [7], by Brinkmann and McKay [14], and by Drápal and Lisoněk [22]. Their constructions use slightly smaller generating gadgets, but do not act as a connected sum. In particular, in the hypergraph language of Section 5.3.1, their switches would glue different vertices together; our switches here preserve the vertex set of the hypergraphs.

5.3.3 Preliminaries

A *near-triangulation* of the plane is a graph embedded in the plane such that each bounded face is a triangle.

We start with the following simple lemma.

Lemma 5.3.4 (Parity Lemma)**.** *Consider a red/blue-colored triangulation (strict or not) of a surface. Given a disk D with boundary cycle C, let $r(D)$ and $b(D)$ be the number of red and blue (respectively) faces inside D that are incident to the edges of C. Then $r(D) \equiv b(D) \bmod 3$.*

Proof. Let H be the subgraph of G on D. This is a near-triangulation of the plane whose unbounded face has boundary cycle C. Consider the dual of H, but split the vertex corresponding to the unbounded face into two: one adjacent to the blue faces incident to C and one to the red faces incident to C. This is a bipartite graph, and all vertices except the outer two have degree 3. Since the degree-sum in each part must be equal, we must have $r(D) \equiv b(D) \bmod 3$. □

There are seven operations and their inverses arising as the result of taking a connected sum with an octahedron:

(a) swapping one face for seven faces

(b) swapping two adjacent faces for six faces

(c) swapping a path of three faces for five faces

(d) swapping a star of four faces for a new star of four faces

(e) swapping a path of four faces for a new path of four faces

(f) a trivial swap of four faces around a vertex for the same configuration

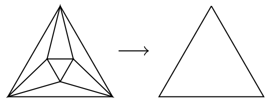
These are pictured in Figure 5.2.

5.3.4 Proof of Theorem 5.3.2

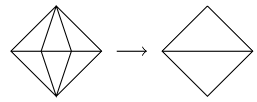
Let Σ be a 2-colorable triangulation of the sphere. Then its 1-skeleton is an Eulerian planar multigraph G. (We consider a face that self-abuts along an edge incident to a vertex of degree 1 to be self-adjacent, so vertices of degree 1 are also disallowed by our 2-colorability condition.) First, note that G has no loops. Indeed, if it did, then this 1-cycle would contradict the Parity Lemma.

Suppose Σ is a counterexample to the theorem with fewest faces. Σ is not the complex consisting of two triangular faces glued along their boundary, as this complex can be written as the sum of two octahedra along the subcomplex consisting of 7 faces (see Figure 5.2a).

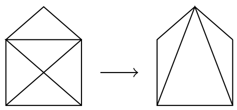
We will first show that Σ has no vertices of degree 2. Suppose vertex v has degree 2, with neighbors u_1 and u_2. Since Σ is not the triangulation with exactly two faces, there are two faces A and B with vertices



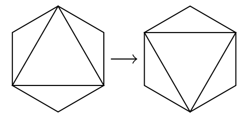
(a) The $7 \to 1$ octahedral switch.



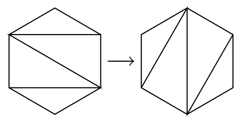
(b) The $6 \to 2$ octahedral switch.



(c) The $5 \to 3$ octahedral switch.



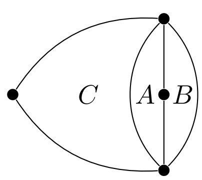
(d) The $4 \to 4$ "star" octahedral switch.



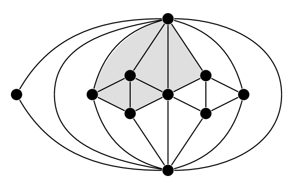
(e) The $4 \to 4$ "path" octahedral switch.

Figure 5.2: The five nontrivial connected sums with an octahedron.

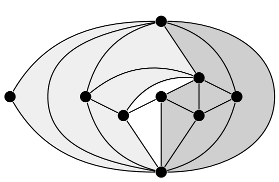
u_1, u_2, v. Let C be the face sharing the edge u_1u_2 with A, and let u_3 be the last vertex of C. In Figure 5.3, we perform a sequence of six octahedral switches that reduces the number of faces by two. First, apply the $1 \to 7$ switch to faces A and B of Figure 5.3a, resulting in Figure 5.3b. Next, apply the $5 \to 3$ switch to the shaded region of Figure 5.3b to obtain Figure 5.3c. Apply the $5 \to 3$ switch to the lightly shaded region and $6 \to 2$ switch to the dark shaded region of Figure 5.3c to obtain Figure 5.3d. Finally, apply the $7 \to 1$ switch to the region bounded by $u_1u_2u_3$ to obtain a single face, as in Figure 5.3e. This contradicts the minimality of Σ.



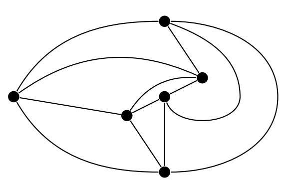
(a)



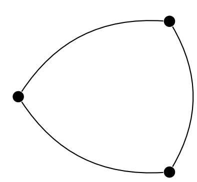
(b)



(c)



(d)



(e)

Figure 5.3: Removing vertices of degree 2.

So we may assume we have a 2-colorable triangulation of the sphere whose 1-skeleton G has minimum degree at least 4 (since it is Eulerian). But since G is a planar multigraph, the average degree is less than 6, so there must be some vertex of degree exactly 4.

Claim 5.3.5. *There is a vertex of degree 4 with four distinct neighbors.*

Proof. Suppose that v has degree four and that two consecutive edges incident to v have the same endpoints. Then these parallel edges form a bigon with the same color required on each side, contradicting the Parity Lemma. See the left image of Figure 5.4.

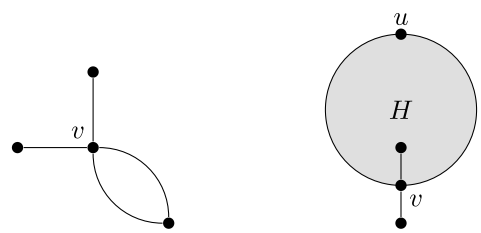


Figure 5.4: Degree-four vertices with parallel edges.

So, if the claim fails, then there is some v of degree 4 such that two nonconsecutive edges incident to v have the same other endpoint, say u. Let C be the bigon formed by the parallel edges uv. Consider the subgraph H of G induced by the vertices on and inside C. See the right image of Figure 5.4. Then H is a near-triangulation, and every vertex has even degree in H except for v and possibly u. So we have

$$\begin{aligned}
|E(H)| &= 3|V(H)| - 5 \\
\sum_{x \in V(H)} d_H(x) &= 6|V(H)| - 10 \\
\sum_{x \in V(H)-\{u,v\}} d_H(x) &= 6|V(H)| - 13 - d_H(u).
\end{aligned}$$

This implies that the average degree of vertices strictly inside C is

$$\frac{6|V(H)| - 13 - d(u)}{|V(H)| - 2} = \frac{6(|V(H)| - 2) - 1 - d(u)}{|V(H)| - 2} < 6.$$

Vertices strictly inside C have the same degree in H as in G, so none are degree 2; hence there is some vertex strictly inside C of degree 4. Since H is strictly smaller than G (we have deleted a neighbor of v at least), we may repeat this process; since G is finite, eventually we find a vertex of degree 4 with distinct neighbors. □

So let v be a vertex of degree 4 with distinct neighbors u, w, a, b in cyclic order. Since G is a triangulation, there is a quadrilateral $uwab$ whose interior contains only v. There is some other face containing the edge

ub; let c be the last vertex on this face. We perform a $5 \to 3$ octahedral switch on the disk bounded by $uwabc$ and containing v; this reduces the number of faces by two, contradicting the minimality of Σ.

5.3.5 Proof of Theorem 5.3.3

Now we prove the result for strict triangulations. Suppose Σ is a face-minimal counterexample, with underlying graph G. Note that G cannot have vertices of degree 2 (unless the triangulation is the one with exactly two faces), nor can a 4-vertex have incident edges with another common endpoint, so we save some work in this case. We have a vertex v of degree 4 in G with distinct neighbors u, w, a, b in cyclic order. Since G is a triangulation, $uwab$ is a quadrilateral. The edge aw is in a face with some vertex c other than v. If neither ac nor wc are edges of G, then we can perform the $5 \to 3$ switch on the disk bounded by $uwcab$ containing v, reducing the number of faces, a contradiction to the minimality of Σ. So we may assume that at least one of ac and wc is an edge.

Suppose ac is an edge of G. If b has degree at least 6, then inside the disk bounded by the 3-cycle abc there is some triangular face abd. But now by planarity, ud and wd are not edges in G. Hence we may perform the $5 \to 3$ switch on the disk bounded by $uwadb$ containing v, again a contradiction.

Similarly, if wc is an edge of G and u has degree at least 6, then we obtain a contradiction.

So suppose ac is an edge of G, but b has degree 4. Then abc is a face. If wc is not an edge, then we perform an octahedral connected sum along the disk bounded by $ucaw$ containing b and v, and we get a contradiction. If wc is an edge and u has degree 4, then we perform an octahedral connected sum along the disk bounded by awc containing v, b, u, another contradiction.

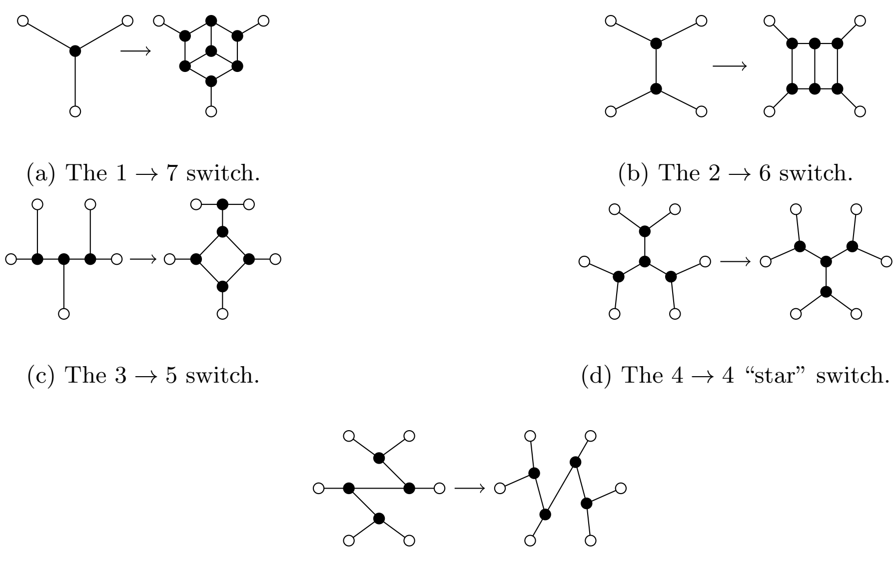
5.3.6 Dual statement

The corresponding swaps in the dual graph are shown in Figure 5.5. Hence we have the following.

Theorem 5.3.6. *Every Barnette graph can be built from the cube graph by a sequence of operations shown in Figure 5.5.*

5.3.7 Possible extensions

In the definition of connected sum, if we relax the condition that S_1 is a disk to the condition that S_1 is any disjoint union of disks, then we call the resulting operation a *sum* of Σ_1 and Σ_2 along S_1, S_2. We can introduce handles as well as nonorientability by summing with an octahedron along two disjoint faces, as



(a) The $1 \to 7$ switch.

(b) The $2 \to 6$ switch.

(c) The $3 \to 5$ switch.

(d) The $4 \to 4$ "star" switch.

(e) The $4 \to 4$ "path" switch.

Figure 5.5: The nontrivial octahedral switches, framed in the dual graph.

shown in Figure 5.6. Is it the case that any 2-colorable triangulation of any surface can be obtained as the sum of octahedra?

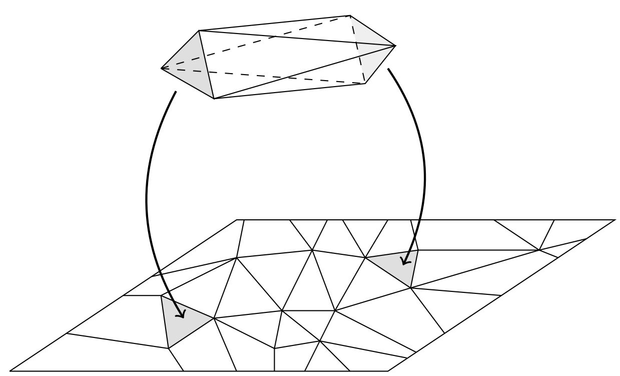


Figure 5.6: A sum with an octahedron that introduces a handle.

The definitions of 2-colorable triangulations of higher dimensional manifolds are analogous to those given here for surfaces. The smallest such triangulation of the n-sphere appears to be the cross-polytope. Can every triangulation of the n-sphere be obtained by a sequence of connected sums with the cross-polytope? It is easy to see that this holds for the 1-sphere.

5.4 Graphicality as an integer program

The question of when a given sequence is the degree sequence of some k-uniform hypergraph can be recognized as the question of whether a given integer program has a feasible solution. Let the sequence be $d_1, \ldots, d_n$. For each $A \in \binom{[n]}{k}$ introduce a variable x_A. The constraints of the desired program are

$$\begin{aligned} \sum_{\substack{A \in \binom{[n]}{k}: \\ i \in A}} x_A = d_i \qquad & \text{for each } i \in [n], \\ x_A \in \{0,1\} \quad & \text{for each } A \in \binom{[n]}{k}. \end{aligned}$$

We may relax this integer program to obtain a fractional version:

$$\begin{aligned} \sum_{\substack{A \in \binom{[n]}{k}: \\ i \in A}} x_A = d_i \qquad & \text{for each } i \in [n], \\ x_A \in [0,1] \quad & \text{for each } A \in \binom{[n]}{k}. \end{aligned}$$

We say that a sequence is *fractionally k-graphic* if this linear program is feasible.

For certain variations of this problem, the feasibility of the integer program turns out to be equivalent to the feasibility of the fractional program. (We will only consider sequences of natural numbers.) We start by showing that this is true in the classic graph case.

Proposition 5.4.1. *A sequence with even sum is graphic if and only if it is fractionally graphic.*

Proof. Integral graphicality trivially implies fractional graphicality; fractional graphicality implies that the Erdős-Gallai conditions hold, as these conditions are obvious necessary conditions that do not depend on integrality; and the Erdős-Gallai conditions imply integral graphicality (which is the content of the proof of the Erdős-Gallai theorem). $\square$

The question of when a sequence is the degree sequence of some multigraph is even easier. Even for k-uniform multigraphs, we may easily construct an integer program:

$$\begin{aligned} \sum_{\substack{A \in \binom{[n]}{k}: \\ i \in A}} x_A = d_i \qquad & \text{for each } i \in [n], \\ x_A \in \mathbb{Z}_{\geq 0} \quad & \text{for each } A \in \binom{[n]}{k}. \end{aligned}$$

and its fractional relaxation:

$$\sum_{\substack{A\in\binom{[n]}{k}:\\ i\in A}} x_A = d_i \qquad \text{for each } i \in [n],$$

$$x_A \in [0,\infty) \quad \text{for each } A \in \binom{[n]}{k}.$$

We need to introduce the *dominance order* on the set of sequences with a fixed sum. For sequences written in nonincreasing order $a : a_1, \dots, a_n$ and $b : b_1, \dots, b_n$ with $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i$, say that b *dominates* a if for every t,

$$\sum_{i=1}^t a_i \le \sum_{i=1}^t b_i.$$

Lemma 5.4.2. *The set of sequences with a fixed sum that have k-uniform multigraph realization form an ideal under the dominance order.*

Proof. It suffices to show that if b covers a in the dominance order and b has a realization, then so does a. Then there are some i, j with $i < j$ such that $b_i = a_i + 1$ and $b_j = a_i - 1$, and $b_k = a_k$ for all $k \notin \{i, j\}$. In the realization of b, there is some edge e that contains vertex i but not vertex j (since $b_i > b_j$); modify the multigraph by replacing e by $e - i + j$. This is a realization of a. ☐

For a sequence π, let $\sum \pi$ denote the sum of the entries of π.

Proposition 5.4.3. *For a sequence π with maximum entry Δ and with even sum, the following are equivalent.*

1. *π has a k-uniform multigraph realization.*

2. *π has a fractional k-uniform multigraph realization.*

3. *$\Delta \le \frac{1}{k} \sum \pi$.*

Proof. (1)⇒(2) is trivial.
(2)⇒(3): Δ is the total weight of edges that contain a vertex of maximum degree in any realization, and $\frac{1}{k}\sum \pi$ is the total weight of all edges in any realization.
(3)⇒(1): Consider the dominance order on sequences. The k-term sequence $\frac{1}{k}\sum\pi, \dots, \frac{1}{k}\sum\pi$ dominates π and clearly has a k-uniform multigraph realization, hence π has such a realization as well. ☐

For k-uniform simple hypergraphs, no TONCAS theorem is known. If the feasibility of the integer program is equivalent to the feasibility of the fractional relaxation, then there is an efficient algorithm to decide whether a sequence has a k-uniform realization. If they are not equivalent, then in any TONCAS theorem the obvious necessary conditions would have to fail to be necessary for a fractional realization, which seems to be a strange requirement (and may suggest that no such theorem should exist).

Chapter 6

Large subposets with small dimension

The results of this chapter are joint with Elyse Yeager and are based on [56].

6.1 Introduction

A *realizer* of a poset P is a set of linear extensions whose intersection is P. The *dimension* of a poset P is the minimum number of linear extensions in a realizer. Equivalently, it is the smallest n such that P is a subposet of $\mathbb{R}^n$ (under the *product order*, in which a vector x is less than a vector y if every coordinate of x is at most than the corresponding coordinate of y).

For $n \geq 3$, the *standard example* on $2n$ points, denoted S_n, is the subposet of B_n consisting of all singleton sets and their complements. The dimension of S_n is n, and S_n is the only n-dimensional poset with at most $2n$ elements.

Given a family of posets $\mathcal{F}$, let $\mathrm{ex}^*(P, \mathcal{F})$ denote the size of the largest subposet of P that does not contain any member of $\mathcal{F}$ as a subposet. Similarly, $\mathrm{ex}(P, \mathcal{F})$ is the size of the largest subposet of P that does not contain a member of $\mathcal{F}$ as a subrelation. We write $\mathrm{ex}^*(P, \{Q\})$ as simply $\mathrm{ex}^*(P, Q)$. Let $\mathrm{ex}^*(n, \mathcal{F})$ denote the minimum of $\mathrm{ex}^*(P, \mathcal{F})$ over all n-element posets P. In other words, $\mathrm{ex}^*(n, \mathcal{F})$ is the maximum k such that every n-element poset P has an $\mathcal{F}$-free subposet of size at least k. Let B_n be the boolean lattice of dimension n and A_n an antichain on n points.

Then $\mathrm{ex}^*(P, B_1)$ is just the width of P and $\mathrm{ex}^*(P, A_2)$ is the height of P. The function $\mathrm{ex}(B_n, B_2)$ is heavily studied as the maximum size of a "diamond-free" family of sets. In the literature, $\mathrm{ex}(B_n, P)$ is denoted $\mathrm{La}(n, P)$, and $\mathrm{ex}^*(B_n, P)$ is denoted $\mathrm{La}^\sharp(n, P)$ or $\mathrm{La}^*(n, P)$.

In this note we are concerned with finding large subposets of small dimension. Hence we let $\mathcal{D}_d$ denote the family of posets of dimension at least d, and ask

Question 6.1.1. *What is* $\mathrm{ex}^*(n, \mathcal{D}_{d+1})$*?*

In other words, what is the largest size of a subposet with dimension at most d we are guaranteed to find in an n-element poset? (Note that when $d = 1$, A_n shows that $\mathrm{ex}^*(n, \mathcal{D}_{d+1}) = 1$. We henceforth assume

$d > 1$.) This question was originally posed by F. Dorais [20], whose aim was to eventually understand the question for infinite posets [21]. Goodwillie [32] proved that $\mathrm{ex}^*(n, \mathcal{D}_{d+1}) \geq \sqrt{dn}$ by considering the width of P: if $w(P) \geq \sqrt{dn}$, then a maximum antichain is a large subposet of dimension 2; if $w(P) \leq \sqrt{dn}$, then by Dilworth's theorem the union of some d chains has $\geq \sqrt{dn}$ elements, and this has dimension at most d (roughly speaking, one can build a linear extension that preferentially puts elements of one chain lower than the rest; doing this for each chain in the Dilworth decomposition gives a realizer).

The *lexicographic order* on k-tuples of elements of a poset P puts $(x_1, \ldots, x_k) < (y_1, \ldots, y_k)$ when $x_i < y_i$ for $i = \min\{j : x_j \neq y_j\}$.

We provide a sublinear upper bound on $\mathrm{ex}^*(n, \mathcal{D}_{d+1})$ by considering the lexicographic order on powers of standard examples. Theorem 6.2.1 finds the extremal number for lexicographic powers, and Corollary 6.2.2 applies this to $\mathrm{ex}^*(n, \mathcal{D}_3)$. For other d, Table 6.1 provides upper bounds on $\mathrm{ex}^*(n, \mathcal{D}_{d+1})$.

6.2 Main theorem

Given a poset P and positive integer k, let P^k denote the lexicographic order on k-tuples of elements of P.

Theorem 6.2.1. *Let P be a poset, $\mathcal{F}$ a family of posets, k a positive integer, and let $n = |P|^k = |P^k|$. Then $\mathrm{ex}^*(|P|^k, \mathcal{F}) \leq \mathrm{ex}^*(P^k, \mathcal{F}) \leq n^{\log_{|P|}(\mathrm{ex}^*(P,\mathcal{F}))}$.*

Proof. Let S be a maximum $\mathcal{F}$-free subposet of P^k (so $|S| = \mathrm{ex}^*(P^k, \mathcal{F})$). For $i \leq k+1$ and each i-tuple α, let

$$S_\alpha = \{s \in S\colon\ \alpha \text{ is an initial segment of } s\},$$
$$Q(\alpha) = \{p \in P\colon\ (\alpha, p) \text{ is an initial segment of some } s \in S\}.$$

Each $Q(\alpha)$ is a subposet of S, under any of the maps that assign to $p \in P$ an element $s \in S$ with initial segment (α, p). Since S is $\mathcal{F}$-free, so is $Q(\alpha)$, hence $|Q(\alpha)| \leq \mathrm{ex}^*(P, \mathcal{F})$.

We have

$$|S_\alpha| = \sum_{p \in Q(\alpha)} \left|S_{(\alpha,p)}\right| \leq |Q(\alpha)| \cdot \max_{p \in Q(\alpha)} \left|S_{(\alpha,p)}\right| \leq \mathrm{ex}^*(P, \mathcal{F}) \cdot \max_{p \in Q(\alpha)} \left|S_{(\alpha,p)}\right|.$$

When ω is a k-tuple, S_ω is either $\{\omega\}$ or $\emptyset$. Hence we have, for an i-tuple α,

$$|S_\alpha| \leq (\mathrm{ex}^*(P, \mathcal{F}))^{k-i}.$$

In particular, when α is the 0-tuple,

$$|S| \leq (\mathrm{ex}^*(P,\mathcal{F}))^k = |P|^{\log_{|P|}(\mathrm{ex}^*(P,\mathcal{F})^k)} = n^{\log_{|P|}(\mathrm{ex}^*(P,\mathcal{F}))}. \qquad \square$$

Corollary 6.2.2. *For all sufficiently large n,* $\mathrm{ex}^*(n,\mathcal{D}_3) \leq n^{0.8295}$.

Proof. Take $P = S_m$, the standard example on $2m$ points, in the preceding theorem. It is easy to see that $\mathrm{ex}^*(S_m,\mathcal{D}_3) = m+2$. Hence the exponent on the family of posets obtained is $\log_{2m}(m+2)$, which is minimized at $m = 10$ with value approximately 0.82948. This completes the proof when n is a power of 20.

Otherwise, write $n = \sum_{i=0}^{k} \alpha_i (20)^i$, each $\alpha_i \in \{0,\dots,19\}$. Then let Q be the poset that is the disjoint union of α_i copies of S_{10}^i for each i. A maximum dimension 2 subposet of Q is precisely the union of maximum dimension 2 subposets of each S_{10}^i. So

$$\begin{aligned}
\mathrm{ex}^*(n,\mathcal{D}_3) &\leq \mathrm{ex}^*(Q,\mathcal{D}_3) \\
&= \sum_{i=0}^{k} \alpha_i \,\mathrm{ex}^*(S_{10}^i,\mathcal{D}_3) \\
&\leq \sum_{i=0}^{k} \alpha_i (20)^{0.82949i} \\
&\leq \left(\sum_{i=0}^{k} \alpha_i\right) \left(\frac{\sum_{i=0}^{k} \alpha_i (20)^i}{\sum_{i=0}^{k} \alpha_i}\right)^{0.82949} && \text{(Jensen's inequality)} \\
&= \left(\sum_{i=0}^{k} \alpha_i\right)^{1-0.82949} n^{0.82949} \\
&\leq (19(\lfloor \log_{20} n \rfloor + 1))^{0.17051} n^{0.82949} \\
&< n^{0.8295}
\end{aligned}$$

for sufficiently large n. $\square$

Essentially the same proof works for any d. We have for any m and any $\epsilon > 0$ that for sufficiently large n, $\mathrm{ex}^*(n,\mathcal{D}_{d+1}) \leq n^{\log_{2m}(m+d)+\epsilon}$. Table 6.1 shows some values of d with the minimizing m and the minimum value of the exponent (rounded to the 5th decimal place).

d	m	$\log_{2m}(m+d)$
2	10	0.82948
3	17	0.84953
4	25	0.86076
10	78	0.88663
100	1169	0.92122

Table 6.1: Values of m that minimize $\log_{2m}(m+d)$ for given d.

6.3 Remarks

There is still a rather large gap between the known lower and upper bounds for $\mathrm{ex}^*(n, \mathcal{D}_{d+1})$. Any improvement to either the lower or the upper bound would be interesting.

Given the interest in $\mathrm{ex}(B_n, B_2)$, one may be interested in $\mathrm{ex}^*(B_n, \mathcal{D}_{d+1})$ instead of $\mathrm{ex}^*(n, \mathcal{D}_{d+1})$.

Question 6.3.1. *What is* $\mathrm{ex}^*(B_n, \mathcal{D}_{d+1})$*?*

Lu and Milans (personal communication) have shown that $\mathrm{ex}^*(B_n, S_d) \leq (4d + C\sqrt{d} + \epsilon)\binom{n}{\lfloor n/2 \rfloor}$. Hence also $\mathrm{ex}^*(B_n, \mathcal{D}_d) = \Theta(\binom{n}{\lfloor n/2 \rfloor})$. For small cases, we have computed that $\mathrm{ex}^*(B_n, \mathcal{D}_3) = 1, 4, 7, 12, 20$ for $n = 1, 2, 3, 4, 5$.

In 1974, Erdős [24] posed and partially answered the following question: given an r-uniform hypergraph $G_r(n)$ on n vertices such that every m-vertex subgraph has chromatic number at most k, how large can the chromatic number of $G_r(n)$ be? Using probabilistic methods, Erdős found a lower bound for ordinary graphs when $k = 3$; that is, when every m-vertex subgraph has chromatic number at most 3. Thinking of poset dimension as analogous to graph chromatic number, we ask:

Question 6.3.2. *Given a poset P with n elements such that every m-element subposet has dimension at most d, how large can the dimension of P be?*

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