SPLINES ON POLYTOPAL COMPLEXES

BY

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DISSERTATION

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Abstract

This thesis concerns the algebra $C^r(\mathcal{P})$ of C^r piecewise polynomial functions (splines) over a subdivision by convex polytopes $\mathcal{P}$ of a domain $\Omega \subset \mathbb{R}^n$. Interest in this algebra arises in a wide variety of contexts, ranging from approximation theory and computer-aided geometric design to equivariant cohomology and GKM theory. A primary goal in approximation theory is to construct bases of the vector space $C^r_d(\mathcal{P})$ of splines of degree at most d on $\mathcal{P}$, although even computing the dimension of this space proves to be challenging. From the perspective of GKM theory it is more important to have a good description of the generators of $C^r(\mathcal{P})$ as an algebra; one would especially like to know the multiplication table for these generators (the case $r = 0$ is of particular interest). For certain choices of $\mathcal{P}$ and r there are beautiful answers to these questions, but in most cases the answers are still out of reach.

In the late 1980s Billera formulated an approach to spline theory using the tools of commutative algebra, homological algebra, and algebraic geometry [9], but focused primarily on the simplicial case. This thesis details a number of results that can be obtained using this algebraic perspective, particularly for splines over subdivisions by convex polytopes.

The first three chapters of the thesis are devoted to introducing splines and providing some background material. In Chapter 1 we give a brief history of spline theory. In Chapter 2 we record results from commutative algebra which we will use, mostly without proof. In Chapter 3 we set up the algebraic approach to spline theory, along with our choice of notation which differs slightly from the literature.

In Chapter 4 we investigate the algebraic structure of continous splines over a central polytopal complex (equivalently a fan) in $\mathbb{R}^3$. We give an example of such a fan where the link of the central vertex is homeomorphic to a 2-ball, and yet the C^0 splines on this fan are not free as an algebra over the underlying polynomial ring in three variables, providing a negative answer to a question of Schenck [48, Question 3.3]. This is interesting for several reasons. First, this is very different behavior from the case of simplicial fans, where the ring of continuous splines is always free if the link of the central vertex is homeomorphic to a disk. Second, from the perspective of GKM theory and toric geometry, it means that the multiplication tables of generators will be

much more complicated. In the remainder of the chapter we investigate criteria that may be used to detect freeness of continuous splines (or lack thereof).

From the perspective of approximation theory, it is important to have a basis for the vector space $C^r_d(\mathcal{P})$ of splines of degree at most d which is 'locally supported' in some reasonable sense. For simplicial complexes, such a basis consists of splines which are supported on the union of simplices surrounding a single vertex. Such bases are well known in the case of planar triangulations for $d \geq 3r+2$ [32, 33]. In Chapter 5 we show that there is an analogue of locally-supported bases over polyhedral partitions, in the sense that, for $d \gg 0$, there is a basis for $C^r_d(\mathcal{P})$ consisting of splines which are supported on certain 'local' sub-partitions. A homological approach is particularly useful for describing what these sub-partitions must look like; we call them 'lattice complexes' due to their connection with the intersection lattice of a certain hyperplane arrangement. These build on work of Rose [45, 46] on dual graphs.

It is well-known that the dimension of the vector space $C^r_d(\mathcal{P})$ agrees with a polynomial in d for $d \gg 0$. In commutative algebra this polynomial is in fact the Hilbert polynomial of the graded algebra $C^r(\widehat{\mathcal{P}})$ of splines on the cone $\widehat{\mathcal{P}}$ over $\mathcal{P}$. In Chapter 7 we provide computations for Hilbert polynomials of the algebra $C^\alpha(\Sigma)$ of mixed splines over a fan $\Sigma \subset \mathbb{R}^3$, giving an extension of the computations in [28, 37, 38, 48]. We also give a description of the fourth coefficient of the Hilbert polynomial of $HP(C^\alpha(\Sigma))$ where $\Sigma = \widehat{\Delta}$ is the cone over a simplicial complex $\Delta \subset \mathbb{R}^3$. We use this to re-derive a result of Alfeld-Schumaker-Whiteley on the generic dimension of C^1 tetrahedral splines for $d \gg 0$ [7] and indicate via example how this description may be used to give the fourth coefficient in particular non-generic configurations. These computations are possible via a careful analysis of associated primes of the spline complex $\mathcal{R}/\mathcal{J}$ introduced by Schenck-Stillman in [50] as a refinement of a complex first introduced by Billera [9].

Once the Hilbert polynomials which give the dimension of the spaces $C^r_d(\mathcal{P})$ for $d \gg 0$ are known, one would like to know how large d must be in order for this polynomial to give the correct dimension of the vector space $C^r_d(\mathcal{P})$. Indeed the formulas are useless in practice without knowing when they give the correct answer. In the case of a planar triangulation, Hong and Ibrahim-Schumaker have shown that if $d \geq 3r+2$ then the Hilbert polynomial of $C^r(\widehat{\mathcal{P}})$ gives the correct dimension of $C^r_d(\mathcal{P})$ [32, 33]. In the language of commutative algebra and algebraic geometry, this question is equivalent to asking about the *Castelnuovo-Mumford regularity* of the graded algebra $C^r(\widehat{\mathcal{P}})$. In Chapter 6, we provide bounds on the regularity of the algebra $C^\alpha(\Sigma)$ of mixed splines over a polyhedral fan $\Sigma \subset \mathbb{R}^3$. Our bounds recover the $3r+2$ bound in the simplicial case. The proof of these bounds rests on the homological flexibility of regularity, similar in philosophy to the Gruson-Lazarsfeld-Peskine theorem bounding the regularity of curves in projective space (see [23, Chapter 5]).

To my advisor, Hal Schenck.

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Table of Contents

List of Symbols

R	Polynomial ring $k[x_1, \ldots, x_n]$ over a field k (almost exclusively, $k = \mathbb{R}$)
S	Polynomial ring $k[x_0, \ldots, x_n]$, used in the case of graded modules
$\mathcal{P}$	Polytopal complex
Σ	Polyhedral fan
$\widehat{\mathcal{P}}$	Cone over $\mathcal{P}$
$\mathcal{P}(\Sigma)$	Polytopal complex obtained from Σ by taking ray generators of cones
$\mathrm{lk}(\Sigma)$	Subcomplex of $\mathcal{P}(\Sigma)$ consisting of those faces not containing the origin
X	Polytopal complex or polyhedral fan
∂X	Boundary complex of X
$\mathrm{st}_X(\gamma)$	Star of a face $\gamma \in X$
X_i	i-faces of X
X_i^0	Interior i-faces of X
$f_i(X)$	Number of i-faces of X
$f_i^0(X)$	Number of interior i-faces of X
$C^\alpha(X)$	Ring of splines over X with mixed smoothness parameters α across codimension one faces
$C^r(X)$	Ring of splines over X with uniform smoothness parameter r (no vanishing on ∂X)
$C^\alpha_d(X)$	Vector space of mixed splines over X of degree at most d
$C^\alpha(X)_d$	Vector space of mixed splines over X of degree exactly d
$C^\alpha_Q(X)$	Ring of mixed splines over X vanishing outside of a subcomplex $Q \subset X$
X^{-1}	Minimal subcomplex of X containing codimension one faces τ with $\alpha(\tau) = -1$

$X^{\geq 0}$	$X \setminus X^{-1}$
$\mathcal{A}(X, X')$	Arrangement of hyperplanes which are affine spans of codimension one faces of X not contained in a subcomplex X'
X^c_W	Subcomplex of X consisting of faces whose affine span does not contain a chosen $W \in \mathcal{A}(X, X^{-1})$
$X_{W,\sigma}$	Lattice complex associated to $W \in \mathcal{A}(X, X^{-1})$ containing the facet $\sigma \in X$
X_W	Disjoint union of nontrivial lattice complexes $X_{W,\sigma}$ associated to $W \in \mathcal{A}(X, X^{-1})$
$LS^{\alpha,k}(X)$	Subalgebra of splines supported on complexes X_W with $\mathrm{codim}(W) \leq k$
$\mathcal{R}/\mathcal{J}[X, X']$	The spline complex of X relative to a subcomplex X'
$HF(M, d)$	Hilbert function of a graded module M
$HP(M, d)$	Hilbert polynomial of a graded module M
$\dim(M)$	Krull dimension of M
$\mathrm{codim}(M)$	Codimension of M
$\wp(M)$	Postulation number of a graded module M
$\mathrm{pd}(M)$	Projective dimension of a graded module M
$\mathrm{reg}(M)$	Castelnuovo-Mumford regularity of a graded module M

Chapter 1

Introduction

1.1 A Brief History of Splines

This thesis concerns the algebra $C^r(\mathcal{P})$ of piecewise polynomial functions, continuously differentiable of order r, defined over a subdivision $\mathcal{P}$ of an n-manifold (with boundary) in $\mathbb{R}^n$ by convex polytopes. Such functions are called *splines*. Splines are widely used in approximation theory; the one-dimensional case where the subdivision consists of consecutive intervals on the real line and its natural extension to tensor product surfaces is especially well-studied [17] and dates back at least to Schoenberg [53, 54].

Strang [57] attributes the current formulation of multivariate spline spaces to Courant, who introduced the so-called Courant functions on a planar simplicial subdivision in [16]. These are particular instances of continuous splines which we will encounter again in Chapter 4. As noted by Billera [10], these provide an exact relationship to face rings of simplicial complexes. With these functions Courant pioneered the *finite element* method, which has become a standard tool for approximating solutions to certain partial differential equations (see [58]). Splines are currently used in a wide variety of other applications: computer aided geometric design (CAGD) [24], structural topology and rigidity theory [60], and isogeometric analysis [15], to name a few.

More recently splines have made an appearance in geometry. The gluing conditions for piecing together the equivariant cohomology ring of an algebraic variety with a torus action (described by GKM theory) essentially provide the local data for building splines [29]. A striking example of this correspondence is the result of Payne that the ring $C^0(\Sigma)$ of continuous splines on a fan Σ is precisely the integral equivariant Chow ring of the corresponding toric variety [44].

This thesis will primarily be concerned with the question of finding the dimension of the vector space $C^r_d(\mathcal{P})$ of splines of degree at most d on a polytopal subdivision $\mathcal{P}$ of a topological n-manifold with boundary in $\mathbb{R}^n$. Strang [57] first raised this question in the early 1970s for triangulations $\Delta \subset \mathbb{R}^2$. In the same paper he conjectured a formula for the dimension of the space $C^1_d(\Delta)$ which was an early research focus. Remarkably, this problem is still open for $d = 3$! Since then there have been a number of papers published on this subject, computing or bounding the dimension of $C^r_d(\mathcal{P})$ with respect to the combinatorial and geometric data of the complex $\mathcal{P}$. We will sketch the development of this problem to the current day and afterward explain where the results of this thesis fit in to the story. This is in no way a comprehensive picture - we only mention those results which pertain in some way to the subject matter of this thesis.

The formula conjectured by Strang for the space $C^1_d(\Delta)$ was proved by Morgan and Scott [41] for $d \geq 5$ by explicit construction of a locally supported basis. Such a basis consists of splines which are supported on a *cell*, which is the set of triangles surrounding a single interior vertex (also called the *star* of a vertex). Alfeld-Schumaker [2] extended Strang's formula to a formula for $\dim_{\mathbb{R}} C^r_d(\Delta)$ where $d \geq 4r + 1$. Their methods involved Bernstein-Bezier techniques. Using Bezier nets and the associated minimal determining sets they provide upper and lower bounds on the dimension of the spline space $C^r_d(\Delta)$ which agree for $d \geq 4r + 1$. They later improved this result, showing in [3] that the formula obtained in [2] is correct for $d \geq 3r + 1$. Hong [32] and Ibrahim-Schumaker [33] contributed to this explicitly constructed locally supported bases for the space $C^r_d(\Delta)$ when $d \geq 3r + 2$. The latter paper also considers subspaces of $C^r_d(\Delta)$ consisting of so-called *super splines* which have a higher order of differentiability imposed across specified interior vertices of the triangulation Δ.

Paralleling this story in two dimensions are dimension bounds for tetrahedral decompositions $\Delta \subset \mathbb{R}^3$. Alfeld-Schumaker-Sirvent [6] showed that locally supported bases for $C^r_d(\Delta)$ exist for $d > 8r$. Alfeld-Schumaker-Whiteley [7] computed the dimension of $C^1_d(\Delta)$ for $d \geq 8$. Alfeld-Schumaker computed general bounds on the dimension of the spline space $C^r_d(\Delta)$ [5]. A lower bound for $C^r_d(\Delta)$ was also computed by Lau [35].

We now turn to the algebraic approach to spline theory, which was pioneered in the late 1980s by Billera and is the subject of this thesis. In the late 1980s, Billera introduced techniques from homological and commutative algebra to the theory of splines, giving an entirely algebraic approach to computing the dimension of the spline space $C^r_d(\Delta)$ [9]. He built a chain complex associated to $C^r_d(\Delta)$ and used it (and a result of Whiteley [62]) to settle Strang's original question on the dimension of $C^1_d(\Delta)$ for $d \geq 2$ and Δ a generic embedding of a triangulated 2-manifold.

Another important (and natural) shift in perspective was provided in [11], where Billera-Rose considered all the spaces $C^r_d(\Delta)$ at once via a coning construction. Let $\mathbb{R}^n$ have coordinates $x_1, \cdots, x_n$, $\mathbb{R}^{n+1}$ have coordinates $x_0, x_1, \cdots, x_n$, and embed Δ in the hyperplane $x_0 = 1$ of $\mathbb{R}^{n+1}$ by sending $(x_1, \cdots, x_n)$ to $(1, x_1, \cdots, x_n)$. Define $\widehat{\Delta} \subset \mathbb{R}^{n+1}$ to be the join of the origin $\mathbf{0} \in \mathbb{R}^{n+1}$ with the image of Δ in $\mathbb{R}^{n+1}$. In [11], Billera-Rose show that $C^r(\widehat{\Delta})$ is a graded algebra over the polynomial ring $S = \mathbb{R}[x_0, \ldots, x_d]$. Moreover, this construction turns the filtration of $C^r(\Delta)$ by the spaces $C^r_d(\Delta)$ into the graded pieces $C^r(\widehat{\Delta})_d$

of $C^r(\widehat{\Delta})$. Hence the Hilbert series of $C^r(\widehat{\Delta})$ satisfies

$$HS(C^r(\widehat{\Delta}), t) = \sum_{k \geq 0} \dim_{\mathbb{R}} C^r_d(\Delta) t^d = \frac{P(t)}{(1-t)^{n+1}},$$

where $P(t) \in \mathbb{Z}[t]$. Billera-Rose also constructed a matrix whose kernel is $C^r(\widehat{\Delta})$ and used this to compute $P(1)$ and $P'(1)$. The matrix of Billera-Rose is a fundamental tool for studying splines from the algebraic perspective.

Subsequently this algebraic approach has been refined and extended by a number of authors. An incomplete list includes Billera [10], Billera-Rose [11], Rose [45, 46], Schenck-Stillman [50, 49], Schenck [47, 48], Geramita-Schenck [28], Yuzvinsky [63], McDonald-Schenck [38], Tohaneanu [59, 40], Shan [56], and Mourrain-Villamizar [42, 43].

In [50] Schenck-Stillman modified the chain complex introduced by Billera in [9] and used the modified complex to compute the Hilbert polynomial $HP(C^r(\widehat{\Delta}), d)$ for a triangulation $\Delta \subset \mathbb{R}^2$ in [49], recovering the dimension formula of Alfeld-Schumaker for $d \gg 0$. The correctness of this formula for $d \geq 3r+1$ is not shown, although recently Mourrain-Villamizar [42] gave a short homological argument that this formula is correct for $d \geq 4r+1$. In this paper Mourrain-Villamizar also provided bounds on the dimension of the spline space $C^r_d(\Delta)$ for $\Delta \subset \mathbb{R}^2$, improving on the bounds due to Schumaker [55] and Lai-Schumaker [34]. The modification of Billera's original chain complex by Schenck-Stillman elucidated a fruitful connection with fat point ideals via Macaulay inverse systems. In [28] Geramita-Schenck exploited this connection to compute the dimension of the space $C^\alpha_d(\Delta)$ of mixed planar splines with smoothness parameters given by α, for $d \gg 0$. This formula was extended by McDonald-Schenck to polytopal subdivisions $\mathcal{P} \subset \mathbb{R}^2$ in [38].

We mention some results from the algebraic perspective in three dimensions. In [56], Shan used the connection to fat points to give a lower bound on the spline space $C^2_d(\Delta)$, where $\Delta \subset \mathbb{R}^3$ is a simplicial cell, or the star of a vertex. In [43], also using the connection to fat points, Mourrain-Villamizar gave bounds on the dimension of the space $C^r_d(\Delta)$ for any tetrahedral decomposition $\Delta \subset \mathbb{R}^3$. This latter paper represented an improvement in bounds due to Alfeld-Schumaker [5] and Lau [35].

Finally, a question which is much related to the dimension question is when $C^r(\widehat{\mathcal{P}})$ is a free module over the underlying polynomial ring S, where $\mathcal{P} \subset \mathbb{R}^n$ and $S = \mathbb{R}[x_0, \ldots, x_n]$. In the case $r = 0$ Billera resolved these questions when $\mathcal{P} = \Delta$ is simplicial in [12] by showing that $C^0(\widehat{\Delta}) \cong A_\Delta$, where A_Δ is the Stanley-Reisner ring (or *face ring*) of Δ. We will discuss this in more detail at the beginning of Chapter 4.

In the case where $\mathcal{P} = \Delta$ is simplicial Schenck has shown via a spectral sequence that when $C^r(\widehat{\Delta})$ is free, the Hilbert series $HS(C^r(\widehat{\Delta}), t)$, and hence $\dim_{\mathbb{R}} C^r_d(\Delta)$, is completely determined by local data [47], just as in the $r = 0$

case. He also gave equivalent conditions (in terms of the vanishing of certain homology modules) for the freeness of $C^r(\widehat{\Delta})$. These can be extended to sufficient conditions for $C^r(\widehat{\mathcal{P}})$ to be free (the key ideas of this are present in [48]), however the criteria thus obtained are not necessary. Criteria for determining the projective dimension of $C^r(\widehat{\Delta})$ are provided by Yuzvinsky [63] in terms of the homology modules of a Cech complex of sheaves defined on a certain poset related to the polytopal complex $\mathcal{P}$. The method provided by Yuzvinsky requires some dexterity with posets, nevertheless it handles the complications that arise using a polytopal subdivision quite nicely. The notion of a *lattice complex* which we introduce in Chapter 5 is a slight modification of the equivalence relation he describes in [63].

1.2 Summary of Thesis Results

Chapters 2 and 3 are devoted to establishing some background in commutative algebra and introducing splines from the algebraic perspective. The thesis work begins in earnest in Chapter 4, where we investigate the algebra $C^0(\widehat{\mathcal{P}})$ of continuous splines on the cone over a polytopal complex $\mathcal{P} \subset \mathbb{R}^2$. When $\mathcal{P}$ is nonsimplicial, $C^0(\widehat{\mathcal{P}})$ is much more subtle than the simplicial case which was characterized by Billera in [10, 12]. In [48], particularly Example 1.1, Schenck shows that there are contributions of the geometry of $\mathcal{P}$ to the dimension of $C^0_1(\mathcal{P})$, the space of piecewise linear functions defined over $\mathcal{P}$. The main contribution of chapter 4 is to provide a simple example in which *freeness* of $C^0(\widehat{\mathcal{P}})$, and not just its Hilbert series, is subtly linked to the geometry of $\mathcal{P}$. We also investigate how the space of piecewise linear functions defined on $\mathcal{P}$ interacts with the freeness of $C^0(\widehat{\mathcal{P}})$. Part of the content of this chapter appears in the article *Shellability and Freeness of Continuous Splines* [18].

In Chapter 5 we introduce the notion of *lattice-supported* splines over a polytopal complex $\mathcal{P} \subset \mathbb{R}^n$. These build on work of Rose [45, 46] on dual graphs and arise naturally from the structure of localization of the module $C^r(\mathcal{P})$ at a prime $P \subset R = \mathbb{R}[x_1, \ldots, x_n]$. They are also a logical extension to polytopal complexes of *star-supported* or *locally supported* splines over simplicial complexes that appear in the literature [4, 6, 32, 33]. Using these we construct subalgebras $LS^{r,k}(\mathcal{P}) \subset C^r(\mathcal{P})$ which satisfy the crucial localization property $LS^{r,k}(\mathcal{P})_P = C^r(\mathcal{P})_P$ for all primes $P \subset R = \mathbb{R}[x_1, \ldots, x_n]$ of codimension $\leq k$ (see Theorem 6.2.4). This leads to the result (Theorem 5.4.6) that a so-called *lattice-supported* basis for $C^r_d(\mathcal{P})$ exists for $d \gg 0$. As a consequence we derive that a star-supported basis for $C^r_d(\Delta)$ exists for Δ simplicial when $d \gg 0$. When $\Delta \subset \mathbb{R}^2$ such bases have been explicitly constructed in [32, 33] for $d \geq 3r + 2$; for $\Delta \subset \mathbb{R}^3$ these bases have been constructed for $d > 8r$ [6]. The content of this chapter appears in the article *Lattice-supported splines on polytopal complexes* [20].

In Chapter 7 we analyze the associated primes of homology modules of the chain complex $\mathcal{R}/\mathcal{J}$ introduced by Schenck-Stillman [50]. In particular, we show that all such primes are linear and give an explicit description of the corresponding linear subspaces. We give two applications to computations of dimensions. The first is a computation of the third coefficient of the Hilbert polynomial of $C^\alpha(\Sigma)$, including cases where vanishing is imposed along arbitrary codimension one faces of the boundary of Σ, extending computations in [28, 37, 38, 48]. The second is a description of the fourth coefficient of the Hilbert polynomial of $HP(C^\alpha(\Sigma), d)$ for $\Sigma = \widehat{\Delta}$, where $\Delta \subset \mathbb{R}^3$ a simplicial complex. We use this to re-derive the result of Alfeld-Schumaker-Whiteley on the generic dimension of C^1 tetrahedral splines for $d \gg 0$ [7] and indicate via an example how this description may be used to give the fourth coefficient in particular nongeneric configurations. The contents of this chapter have been submitted for publication in a paper titled *Associated Primes of Spline Complexes* [19].

When $\mathcal{P} = \Delta$ is planar and simplicial, Schenck-Stiller apply sheaf theoretic techniques in [51] to bound the Castelnuove-Mumford regularity of $C^r(\widehat{\Delta})$. Since $C^r(\widehat{\Delta})$ is reflexive, the associated sheaf is a vector bundle on $\mathbb{P}^2$ if $n = 2$. Schenck and Stiller use results of Elencwajg and Forster on the vanishing of higher sheaf cohomologies of vector bundles to bound the regularity of $C^r(\widehat{\Delta})$ [51, Theorem 2.2]. In Chapter 6, we take a different approach to bounding regularity. The key idea is that to bound the regularity of a reflexive module such as $C^r(\widehat{\mathcal{P}})$, one can use a rather crude approximation to $C^r(\widehat{\mathcal{P}})$. The philosophy is similar to that of the Gruson-Lazarsfeld-Peskine theorem bounding the regularity of curves in projective space (see [23, Chapter 5]). For $\mathcal{P} \subset \mathbb{R}^n$, we show that the regularity of the subalgebra $LS^{r,k}(\widehat{\mathcal{P}})$ of lattice-supported splines bounds the regularity of $C^r(\widehat{\mathcal{P}})$ when the projective dimension of $C^r(\widehat{\mathcal{P}})$ is at most k. For $k = 0, 1$ we then bound the regularity of $LS^{r,k}(\widehat{\mathcal{P}})$. This computation reduces to bounding the regularity of the module of splines with support on a facet (in case $k = 0$) and the regularity of the module of splines with support on two adjacent facets (in case $k = 1$). These yield corresponding regularity bounds when $C^r(\widehat{\mathcal{P}})$ is free and when the projective dimension of $C^r(\widehat{\mathcal{P}})$ is at most 1. Since the module $C^r(\widehat{\mathcal{P}})$ is reflexive and hence has projective dimension at most $n - 1$, the latter bound applies for any planar polytopal complex $\mathcal{P} \subset \mathbb{R}^2$. The regularity bounds are obtained for mixed orders of smoothness across interior and boundary faces of codimension one. For simplicial complexes $\Delta \subset \mathbb{R}^2$ more precise regularity estimates are computed; these specialize to *reg* $C^r(\widehat{\Delta}) \leq 3(r + 1)$ in the case of uniform smoothness. This implies that the Alfeld-Schumaker formula for $\dim_{\mathbb{R}} C^r_d(\widehat{\Delta})$ for planar simplicial splines holds for $d \geq 3r + 2$, recovering the bound obtained by Hong [32] and Ibrahim-Schumaker [33] via construction of a locally supported basis. The contents of this chapter have been submitted for publication in a paper titled *Regularity of Mixed Spline Spaces* [21].

Chapter 2

Assorted Results from Commutative Algebra

We assume familiarity with basic commutative algebra throughout this thesis. In this chapter we record some of the main ideas which we will use freely. Throughout this thesis we will use $R = k[x_1, \dots, x_n], S = k[x_0, \dots, x_n]$ to denote the polynomial rings in n and $n+1$ variables over a field k (we will always take $k = \mathbb{R}$). We will use S in the graded case and R in the ungraded case, which we now explain.

2.1 Graded Rings and Modules and their Hilbert Functions

The *degree* of a polynomial $f \in S$, denoted $\deg(f)$, is the largest degree of a monomial of f. If all monomials of f have the same degree, then f is called *homogeneous*. The polynomial ring S is naturally graded by degree, where S_j is the vector space of homogeneous polynomials of degree j. An S-module M is (nonnegatively) graded if $M = \bigoplus_{i \geq 0} M_i$ where each M_i is an $\mathbb{R}$-vector space and the multiplication map satisfies

$$S_j \times M_i \to M_{i+j}.$$

This definition guarantees that M_i is an $S_0 = k$ vector space for every i. Suppose M is a *finitely generated* graded S-module; that is, there exist finitely many homogeneous elements $m_1, \dots, m_s$ in M so that, for any $m \in M, m = \sum f_i m_i$ for some choice of polynomials $f_1, \dots, f_k \in S$. This condition guarantees that $\dim M_d$ is finite for every d. If M is finitely generated, the *Hilbert function* of M in degree d is the function $HF(M, d) = \dim M_d$. If t is an indeterminate, then the series

$$HS(M,t) = \sum_{d=0}^{\infty} HF(M,d)t^d$$

is called the *Hilbert series* of M. It is a standard result in commutative algebra that

$$HS(M,t) = \frac{P(t)}{(1-t)^{n+1}},$$

where $P(t) \in \mathbb{Z}[t]$. From this expression for $HS(M,t)$ it may be derived that $HF(M,d)$ is an *eventually polynomial function*. That is, $HF(M,d)$ agrees with a polynomial function in d for every integer $d \gg 0$. This polynomial is called the *Hilbert polynomial* of M and we will denote it by $HP(M,d)$. The largest integer d for which $HF(M,d) \neq HP(M,d)$ is called the *postulation number* of M, denoted by $\wp(M)$. The degree of $HP(M,d)$ as a polynomial in d is one less than the *Krull dimension* of M (indeed this is one way the Krull dimension of M may be defined). The *codimension* of M is $\dim S - \dim M = n+1-\dim M$, where $n+1$ is the number of variables of S.

Remark 2.1.1. Hilbert functions of finitely generated graded modules over polynomial rings are sums of binomial coefficients (see [23, Corollary 1.10]). We will use the following convention for binomial coefficients:

$$\binom{A}{B} = \begin{cases} 0 & \text{if } B > A \text{ or } B < 0 \\ \binom{A}{B} & \text{if } 0 \leq B \leq A \end{cases}$$

Example 2.1.2. The Hilbert function, Hilbert polynomial, and Hilbert series of $S = k[x_0, \ldots, x_n]$ are

$$\begin{aligned} HF(S,d) &= \binom{n+d}{n} \\ HP(S,d) &= \frac{(d+n)(d+n-1)\ldots(d+1)}{n!} \\ HS(S,t) &= \frac{1}{(1-t)^{n+1}} \end{aligned}$$

A good introduction to the Hilbert function, Hilbert series, and Hilbert polynomial is [8, Chapter 11]. A more explicit introduction to the Hilbert polynomial, by way of syzygies, may be found in [23, Chapter 4].

2.2 Associated Primes

In this section we temporarily take R to denote any commutative ring. Given a subset $U \subset M$ of elements of M, the *annihilator of U in R*, denoted $\operatorname{ann}_R(U)$, is defined by

$$\operatorname{ann}_R(U) := \{r \in R | r \cdot u = 0 \text{ for every } u \in U\}.$$

Definition 2.2.1. If M is an R-module, a prime $P \subset R$ is *associated* to M if there is an injection

$$R/P \xrightarrow{\cdot m} M$$

given by multiplication by some $m \in M$. Equivalently, $P = \operatorname{ann}_R(m) = \{r \in R | rm = 0\}$.

The set of associated primes of M is denoted by $\operatorname{Ass}_R(M)$. If R is graded and M is a graded R-module, then the $\operatorname{Ass}_R(M)$ consists of homogeneous prime ideals.

We collect in the next proposition some results about associated primes which we shall need in Chapter 7. The first two can be found in [36, §6]. The third is a special case of a general theorem [36, Theorem 23.2] which describes behavior of associated primes under flat extensions. Since we only need a particular case, we give a proof here.

Proposition 2.2.2. *Let R be a commutative ring, and $M, M_1, \ldots, M_k$ R-modules.*

1. $P \in Ass_R(M) \iff PR_P \in Ass_{R_P}(M_P)$
2. $Ass(\bigoplus_{i=1}^k M_i) = \bigcup_{i=1}^k Ass(M_i)$
3. *Suppose $S = R[x_1, \ldots, x_k]$. Then*

$$Ass_S(M \otimes_R S) = \{PS | P \in Ass_R(M)\}.$$

Proof of (3). The general case follows directly from the case $S = R[x]$ by induction on k. So we prove

$$\text{Ass}_S(M \otimes_R S) = \{PS | P \in \text{Ass}_R(M)\}$$

when $S = R[x]$. First, PS is prime because $S/PS = R/P[x]$ is a domain. Now, suppose $P \in \text{Ass}_R(M)$. Then there is an injection

$$R/P \hookrightarrow M.$$

Since S is a *flat* extension of R, tensoring with S is exact. Tensoring the above injection with S hence yields an injection

$$S/PS \hookrightarrow M \otimes_R S,$$

So $PS \in \text{Ass}_S(M \otimes_R S)$. Now suppose $Q \in \text{Ass}_S(M \otimes_R S)$. Via the identification $M \otimes_R S = M[x]$, $Q = \text{ann}_S(f)$ for some $f = \sum_i m_i x^i$. First we show that $P = Q \cap S = \text{ann}_R(f) \in \text{Ass}_R(M)$. We need to show that $P = \text{ann}_R(m)$ for some $m \in M$. $rf = 0$ for some $r \in R$ iff $rm_i = 0$ for every i. Hence $P = \cap_{i=0}^k \text{ann}_R(m_i)$. But P is prime, so we must have $P = \text{ann}_R(m_i)$ for some i. Hence $P \in \text{Ass}_R(M)$. Now we show that $Q = PS$. Suppose $g = \sum_{j=0}^k a_j x^j \in Q$, with $a_j \in R$. We need to show that $a_j \in P$ for $j = 0, \ldots, k$. Allowing some a_j, m_i to be zero, we may assume that $f = \sum_{i=0}^k m_i x^i$. Expanding fg gives

$$\sum_c \left(\sum_{i+j=c} r_j m_i \right) x^c.$$

Setting $fg = 0$ yields the equations

$$\sum_{i+j=c} r_j m_i = 0$$

for every c. To establish that $a_0 \in P$, note first that $a_0 m_0 = 0 \implies a_0 \in \mathrm{ann}_R(m_0)$. Now multiply the equation

$$a_0 m_1 + a_1 m_0 = 0$$

by a_0 to obtain $a_0^2 m_1 = 0$, so $a_0^2 \in \mathrm{ann}_R(m_1)$. Continuing in this way, we see that $a_0^{k+1} \in \cap_{j=0}^{k} \mathrm{ann}_R(m_j) = P$. But P is prime, so $a_0 \in P$ and all terms involving a_0 in the above equations drop out. Now repeat this process with each successive a_j, yielding $g \in PS$. □

2.3 Syzygies of Modules

If M is finitely generated R-module (not necessarily graded), then any choice of a set of generators $m_1, \dots, m_s$ yields a surjective homomorphism

$$R^s \xrightarrow{\phi} M \to 0$$

by $\phi(r_1, \dots, r_k) = \sum_{i=1}^{k} r_i m_i$. Since R is Noetherian, $\ker(\phi) = K \subset R^s$ is finitely generated and we can choose a finite set $s_1, \dots, s_l$ of generators for K. This yields a surjection ϕ_1 of R^l onto $K \subset R^s$, giving a *presentation* of M

$$R^l \xrightarrow{\phi_1} R^s \xrightarrow{\phi} M \to 0.$$

This process may be continued indefinitely, yielding a *free resolution* of M:

$$\cdots F_k \xrightarrow{\phi_k} F_{k-1} \xrightarrow{\phi_{k-1}} \cdots \xrightarrow{\phi_1} F_0 \xrightarrow{\phi} M \to 0.$$

A module of the form $\ker(\phi_i)$ in such a resolution is called an ith syzygy module of M. A fundamental result, called the *Hilbert Syzygy Theorem*, is that $\ker(\phi_{n-1})$ is free (where n is the number of variables in R) no matter what resolution we choose, hence we can stop the free resolution after n steps. This can be proved concretely using Gröbner bases (see [22, Chapter 15]) or in a more general context using homological methods (see [22, Chapter 20]). Generally speaking, if $F_\bullet \to M$,

$$F_\bullet = \ 0 \to F_\delta \xrightarrow{\phi_\delta} F_{\delta-1} \xrightarrow{\phi_{\delta-1}} \cdots \xrightarrow{\phi_1} F_0,$$

is a free resolution of M of minimal length δ, then δ is called the *projective dimension* of M and is denoted $\mathrm{pd}(M)$.

Now suppose M is a *graded* S-module and let $m = (x_0, \dots, x_n) \subset S$ be the homogeneous maximal ideal of S. The S-module M/mM is a finite dimensional $k = S/mS$-vector space satisfying that any set of elements $m_1, \ldots, m_s$ of M mapping to a basis of M/mM is a set of *minimal generators* of M (this is the graded version of Nakayama's lemma). It follows that in any construction of a free resolution of M, we may choose a *minimal* set of generators of the kernels $\ker(\phi_i)$. This process yields a *minimal* free resolution $F_\bullet \to M$ of length $\delta = \mathrm{pd}(M)$, unique up to graded isomorphism (see [22, Chapter 20]). This resolution is characterized by the property that the entries of any matrix representing the differentials $\phi.$ in $F_\bullet$ are contained in the homogeneous maximal ideal $(x_1, \ldots, x_n)$.

A basic fact is that syzygy modules of graded modules are themselves graded. To make the differentials in a minimal free resolution of M have degree 0, the convention is to shift the grading on the free summands in the resolution to reflect the degrees of generators of the syzygy modules. This is done as follows. For an integer a, let $S(a)$ denote the polynomial ring with grading shifted by a, so $S(a)_d \cong S_{a+d}$. Then if $m_1, \ldots, m_s$ are minimal generators of M with degrees $a_{0,1}, a_{0,2}, \ldots, a_{0,s}$, the map

$$\bigoplus_{j=0}^{s} S(-a_{0j}) \xrightarrow{phi} M$$

is now a map of degree zero (degree k elements get sent to degree k elements). Doing this with the entire free resolution of M yields $F_i = \bigoplus_j S(-a_{ij})$. In a computer program such as Macaulay2, the numbers a_{ij} are typically recorded as follows. Let β_{ij} be the number of minimal generators of F_i in degree j. Then the numbers β_{ij} are recorded in a *betti diagram* whose i, j entry is the number of generators of F_i of degree $j + i$ (see [23, Chapter 1]).

2.4 Depth and the Cohen-Macaulay Condition

Definition 2.4.1. Given a graded S-module M, an M-sequence is a sequence $\{f_1, \ldots, f_k\} \subset S$ satisfying

1. $\sum f_i M \neq M$
2. f_1 is a non-zerodivisor on M
3. f_i is a non-zerodivisor on $M/(\sum_{i=1}^{i-1} f_i M)$ for $i = 2, \ldots, k$

Definition 2.4.2. Given an ideal $I \subset S$ and a graded S-module M, the *depth of I on M*, denoted $\mathrm{depth}(I, M)$, is the length of a maximal M-sequence contained in I. If $I = m$, the homogeneous maximal ideal of S, then $\mathrm{depth}(m, M)$ is denoted $\mathrm{depth}(M)$.

We will use the following result of Auslander and Buchsbaum to move back and forth between the notions of depth and projective dimension. A proof may be found in [22, § 19.3].

Theorem 2.4.3 (Auslander-Buchsbaum)**.** *Let M be a graded $S = k[x_0, \ldots, x_n]$-module. Then*

$$depth(M) + pd(M) = depth(S).$$

Since $\dim(S) = depth(S) = n + 1$*, this may be restated as*

$$depth(M) + pd(M) = n + 1$$

Observe that, according to Theorem 2.4.3, $\mathrm{pd}(M) \leq n + 1$, which is the Hilbert syzygy theorem. The following property of S-modules is widely used, and is especially important for us in Chapter 4.

Definition 2.4.4. A graded S-module M is *Cohen-Macaulay* if $\mathrm{depth}(M) = \dim M$.

As a corollary of Theorem 2.4.3, we obtain an equivalent characterization of Cohen-Macaulay modules over polynomial rings.

Corollary 2.4.5. *A graded module M over $S = k[x_0, \ldots, x_n]$ is Cohen-Macaulay iff $pd(M) = codim(M)$.*

2.5 Ext and Local Cohomology

One of the benefits of homological algebra is that it gives straightforward characterizations of notions in commutative algebra, such as depth, at the expense of some extra machinery. Since we will have occasion to use a few homological tools, we introduce two of them here. Appendix 3 of [22] is a good reference for the homological tools we use.

Our first homological tool is the Ext functor. If M is a graded S-module, then $\mathrm{Hom}_S(M, \_)$ is a covariant left-exact functor, meaning that if $0 \to A \to B \to C \to 0$ is a short exact sequence of S-modules, then

$$0 \to Hom_S(M, A) \to Hom_S(M, B) \to Hom_S(M, C)$$

is exact. Hence it has *right derived functors* which are denoted $\mathrm{Ext}^i_S(M, \_)$. More concretely, $\mathrm{Ext}^i_S(M, N)$ may be computed by taking a free resolution $F_\bullet$ of M, applying $\mathrm{Hom}(\_, N)$ to this resolution, and then taking homology. Derived functors of left exact functors give a way of continuing the left exact sequence above, namely, if $0 \to A \to B \to C \to 0$ is a short exact sequence, then we obtain a long exact sequence

$$0 \longrightarrow Hom_S(M,A) \longrightarrow Hom_S(M,B) \longrightarrow Hom_S(M,C) \longrightarrow$$

$$\longrightarrow Ext^1_S(M,A) \longrightarrow Ext^1_S(M,B) \longrightarrow Ext^1_S(M,C) \longrightarrow$$

$$\longrightarrow Ext^2_S(M,A) \longrightarrow Ext^2_S(M,B) \longrightarrow \cdots$$

One of the uses of *Ext* is its ability to detect depth.

Proposition 2.5.1. *Let M, N be finitely generated graded $S = k[x_0, \ldots, x_n]$-modules. Then*

1. *If $i <$ depth$(ann(M), N)$ or $i > pd(M)$ then $Ext^i_S(M,N) = 0$*
2. *If $i =$ depth$(ann(M), N)$ or $i = pd(M)$, then $Ext^i_S(M,N) \neq 0$*

Our second homological tool is local cohomology (see [23, Appendix 1] for a short introduction). Let Q be an ideal of S and M a graded S-module. The Q-torsion functor $H^0_Q(\_)$, where

$$H^0_Q(M) = \{x \in M | Q^j x = 0 \text{ for some } j \geq 0\},$$

is left-exact. As such it has right derived functors which are denoted $H^i_Q(\_)$. We will only be concerned with the case $Q = m$, where $m = (x_0, \ldots, x_n)$ is the graded maximal ideal of S. Just as with all right derived functors, there is a long exact sequence in local cohomology associated to any short exact sequence of S-modules $0 \to A \to B \to C \to 0$, which begins

$$0 \longrightarrow H^0_m(A) \longrightarrow H^0_m(B) \longrightarrow H^0_m(C) \longrightarrow$$

$$\longrightarrow H^1_m(A) \longrightarrow H^1_m(B) \longrightarrow H^1_m(C) \cdots$$

Local cohomology is also useful for detecting depth.

Proposition 2.5.2. *[23, Proposition A1.16] Let M be a finitely generated graded $S = k[x_0, \ldots, x_n]$-module. Then*

1. *If $i <$ depth(M) or $i > \dim M$ then $H^i_m(M) = 0$.*
2. *If $i =$ depth(M) or $i = \dim M$ then $H^i_m(M) \neq 0$.*
3. *There is an integer d (depending on M) such that $H^i_m(M)_e = 0$ for all $e \geq d$.*

Ext and local cohomology at the maximal ideal are linked through *local duality*.

Theorem 2.5.3. *(Local Duality [23, Theorem A1.9]) Let $S = k[x_0, \ldots, x_n]$, $m = (x_0, \ldots, x_n)$ the homogeneous maximal ideal, and M a finitely generated graded S-module. Then*

$$(H^i_m(M))_d = (Ext^{n+1-i}_S(M, S(-n-1))_{n+1-d})^\vee,$$

where $^\vee$ denotes the dual as a k-vector space.

Local duality and the Auslander-Buchsbaum formula (Theorem 2.4.3) allow us to go back and forth between the statements of Proposition 2.5.1 and Proposition 2.5.2. Another important use of local cohomology is that it encodes the difference between the Hilbert function of a module and its Hilbert polynomial. This is the module version of the Euler characteristic of a sheaf.

Proposition 2.5.4. *[23, Corollary A1.15] Let M be a finitely generated graded $S = k[x_0, \ldots, x_n]$ module. Then for every $d \in \mathbb{Z}$,*

$$HP(M, d) = HF(M, d) - \sum_{i=0}^{\dim M} (-1)^i \dim_k H^i_m(M)_d$$

2.6 Castelnuovo-Mumford Regularity of Graded Modules

In this section we briefly summarize the notion of Castelnuovo-Mumford regularity, which will be important for us in Chapters 5 and 6. The book [23] is an excellent introduction to the graded approach we take here (mainly Chapters 1 and 4).

Definition 2.6.1. Let M be a graded S module and $F_\bullet \to M$ the minimal free resolution of M, with $F_i \cong \bigoplus_j S(-a_{ij})$. The Castelnuovo-Mumford regularity of M, denoted *reg* (M), is defined by

$$\textit{reg}\ (M) = \max_{i,j} \{a_{i,j} - i\}.$$

Remark 2.6.2. Note that, according to this definition, *reg* (M) bounds the minimal degree of generators of M as an S-module.

From Definition 2.6.1 one derives the following theorem. Recall that $\wp(M)$, the postulation number of M, is the largest integer d such that $HP(M, d) \neq HF(M, d)$.

Theorem 2.6.3. *[23, Theorem 4.2] Let M be a finitely generated graded module over S. Then*

1. *$HF(M, d) = HP(M, d)$ for $d \geq$ reg $(M) + pd(M) - n$. Equivalently, $\wp(M) \leq$ reg $(M) + pd(M) - n - 1$.*

2. If M is a Cohen-Macaulay module, the bound in (1) is sharp.

Another characterization of regularity is obtained via local cohomology (via local duality one could alternatively use *Ext*: see [22, § 20.5]).

Theorem 2.6.4 (Theorem 4.3 of [23])**.** *Let $m \subset S$ be the maximal ideal of S and M a graded S-module. Then*

$$\text{reg }(M) = \max_i(\max_e\{e | H^i_m(M)_e \neq 0\} + i)$$

Remark 2.6.5. Part three of Proposition 2.5.2 guarantees that Theorem 2.6.4 makes sense.

The benefit of this description of regularity is that it interacts well with short exact sequences. For instance, the following result is a straightforward application of Theorem 2.6.4 and the long exact sequence in local cohomology described in § 2.5.

Proposition 2.6.6. *[23, Corollary 20.19] Let $0 \to A \to B \to C \to 0$ be a graded exact sequence of finitely generated S modules. Then*

1. $\text{reg }(A) \leq \max\{\text{reg }(B), \text{reg }(C) + 1\}$
2. $\text{reg }(B) \leq \max\{\text{reg }(A), \text{reg }(C)\}$
3. $\text{reg }(C) \leq \max\{\text{reg }(A) - 1, \text{reg }(B)\}$

Proposition 2.6.6 can be extended to bound the regularity of a module appearing in an exact sequence of any length by breaking the exact sequence into short exact pieces. We will use the following corollary to Proposition 2.6.6.

Corollary 2.6.7. *Let $m \geq 0$ and*

$$0 \to C_m \to C_{m-1} \to \ldots \to C_0 \to M \to 0$$

an exact sequence of S-modules. Then

$$\text{reg }(M) \leq \max_i\{\text{reg }(C_i) - i\}$$

The following proposition is one of the ingredients used in the proof of the Gruson-Lazarsfeld-Peskine theorem on bounding the regularity of curves in projective space [23, Proposition 5.5]. It is the main tool we will use for bounding regularity of spline modules in Chapter 6.

Proposition 2.6.8. *Let M be an S-module and $N \subset M$ a submodule of M with dim$(M/N) <$ depth(M), or equivalently codim$(M/N) >$ pd(M). Then* $\text{reg }(M) \leq \text{reg }(N)$.

Proof. We prove *reg* $(M) \leq$ *reg* (N) if $\dim(M/N) < \text{depth}(M)$. The equivalence of the statements $\dim(M/N) < \text{depth}(M)$ and $\text{codim}(M/N) > \text{pd}(M)$

follows directly from Theorem 2.4.3. Set $d = \text{depth}(M)$. By [23, Proposition A1.16], $H^i_m(M) = 0$ for $i < d$ and $H^i_m(M/N) = 0$ for $i > \dim(M/N)$. The long exact sequence in local cohomology resulting from the short exact sequence

$$0 \to N \to M \to M/N \to 0$$

yields a surjection $H^d_m(N) \twoheadrightarrow H^d_m(M)$ and isomorphisms $H^i_m(N) \cong H^i_m(M)$ for $i > d$. Since $H^i_m(M) = 0$ for $i < d$, Theorem 2.6.4 yields *reg* $(N) \geq$ *reg* (M). □

Chapter 3

Preliminaries On Splines

In this chapter we introduce the main players of the thesis, namely polytopal complexes in $\mathbb{R}^n$ and splines of uniform or mixed smoothness over these complexes.

3.1 Convex Polytopes and Polyhedral Cones

Let $V \cong \mathbb{R}^n$ be an n-dimensional vector space. A set $Y \subset V$ is *convex* if, for any $v_1, v_2 \in Y$, the line segment joining v_1 and v_2 is also in Y. More precisely, Y is convex if, for any $v_1, v_2 \in Y$, $tv_1 + (1-t)v_2 \in Y$ for $0 \le t \le 1$. Given any subset $Y \subset V$, the *convex hull* of Y, denoted $\text{conv}(Y)$, is the intersection of all convex sets containing Y. It is straightforward to see that arbitrary intersections of convex sets are convex (the empty set is also considered convex). Hence $\text{conv}(Y)$ is convex. One of the simplest convex sets is a *simplex*. In a sense, simplices are the building blocks of convex sets.

Definition 3.1.1. A *simplex* $\sigma \subset V$ is the convex hull of finitely many affinely independent points of V, called the *vertices* of σ. The *dimension* of σ is the smallest dimension of an affine space of V containing σ.

It will be useful to have an alternate characterization of a simplex. Given two sets $X, Y \subset V$, the *join* of X and Y (denoted $\text{join}(X, Y)$) is the union of all line segments with one endpoint in X and the other in Y.

Proposition 3.1.2. *Suppose $\sigma = \Delta_k \subset V$ is a simplex of dimension k with vertices $\{v_0, \dots, v_k\}$. Let Δ_{k-1} be the simplex formed by taking the convex hull of $v_1, \dots, v_k$. Then $\sigma = join(v_0, \Delta_{k-1})$.*

Proof. By the definition of convexity, any convex set containing $\{v_0, \dots, v_k\}$ must contain every line segment between v_0 and a point of Δ_{k-1}. Hence $\text{join}(v_0, \Delta_{k-1}) \subset \sigma = \Delta_k$. For the opposite inclusion, we must show that $\text{join}(v_0, \Delta_{k-1})$ is convex. Take $p, q \in \text{join}(v_0, \Delta_{k-1})$. Then there are points $p', q' \in \Delta_{k-1}$ and parameters $0 \le s, t \le 1$ such that $x = (1-t)p' + tv_0, y = (1-s)q' + sv_0$. If z is on the line segment between p and q, then there is a parameter $0 \le u \le 1$ so that $z = (1-u)p + uq$. Restrict to the two-dimensional plane passing through v_0, p, q, p', q', z, and change coordinates so that $v_0 = (0,0), p' = (1,0), q' = (0,1)$. Hence p lies on the nonnegative x-axis,

q on the nonnegative y-axis, and z on the line segment between them. The statement is that z lies on a line segment between the origin and a point on the line segment between p and q. To do this, we need to verify that z is inside the triangle formed by the origin, p, and q. It is readily checked that the x and y coordinates of z satisfy the three inequalities $x \geq 0$, $y \geq 0$, $x + y \leq 1$. □

For any (possibly infinite) set $\{v_\alpha\}_{\alpha \in I}$ of vectors in V with index set I, the sum

$$\sum_{\alpha \in I} \lambda_\alpha v_\alpha$$

is called a *convex combination* if

- $\lambda_\alpha \geq 0$ for all $\alpha \in I$
- $\lambda_\alpha = 0$ for all but finitely many $\alpha \in I$
- $\sum\limits_{\alpha \in I} \lambda_\alpha = 1$

Lemma 3.1.3. *Let $Y \subset V$ be any set. Then the set of all points obtained as convex combinations of points of Y is a convex set.*

Proof. Let $Y = \{y_\alpha\}$. Suppose we have two convex combinations $v_1 = \sum\limits_\alpha \lambda_\alpha y_\alpha$, $v_2 = \sum\limits_\alpha \mu_\alpha y_\alpha$. Let $0 \leq t \leq 1$. Then $(1-t)v_1 + tv_2 = \sum\limits_\alpha ((1-t)\lambda_\alpha + t\mu_\alpha)\, y_\alpha$ is a convex combination of points of Y. □

Lemma 3.1.4. *A simplex σ is the set of all points obtained as convex combinations of its vertices.*

Proof. The set of all convex combinations of vertices of σ is a convex set by Lemma 3.1.3, hence contains σ, which is by definition the intersection of all convex sets containing the vertices of σ. To show that σ is contained in the set of all convex combinations of its vertices, we proceed by induction on $k = \dim \sigma$. This is clearly true if $k = 0$. If $k > 0$, write $\sigma = \Delta_k = \operatorname{conv}(v_0, \ldots, v_k)$, $\Delta_{k-1} = \operatorname{conv}(v_1, \ldots, v_k)$. By Proposition 3.1.2, $\Delta_k = \operatorname{join}(v_0, \Delta_{k-1})$. Hence if $p \in \sigma$, then there is a parameter $0 \leq s \leq 1$ and a point $q \in \Delta_{k-1}$ so that $p = (1-s)v_0 + sq$. By induction, $q = \sum_{i=1}^k a_i v_i$, where $\sum a_i = 1$. But then $p = (1-s)v_0 + s\sum_{i=1}^k a_i v_i$, where $(1-s) + s(\sum a_i) = 1 - s + s = 1$, hence p is a convex combination of the v_i. □

Now, given $u \in W = \widehat{V}$ (the dual vector space), let $H_{u,a}$ be the hyperplane $\{v \in V | \langle u, v\rangle = a\}$, where $\langle , \rangle$ is the natural pairing and $a \in \mathbb{R}$. A closed *half-space* in V is a set of the form

$$H^+_{u,a} = \{v \in V | \langle u, v\rangle \geq a\},$$

where $u \in W, a \in \mathbb{R}$. We give three alternate characterizations of $\operatorname{conv}(Y)$.

Proposition 3.1.5. *Let $Y \subset V$ be any set. The following are equivalent characterizations of conv(Y).*

1. *The set of all points obtained as convex combinations of points of Y.*

2. *The union of all simplices with vertices in Y.*

3. *If conv(Y) is closed, the intersection of all closed half-spaces containing Y.*

Proof. By Lemma 3.1.3, conv(Y) is a subset of the points obtained as convex combinations of points of Y. We need to know that the set of convex combinations of points of Y is contained in the union of all simplices with vertices in Y. This is, in essence, the content of *Caratheodory's Theorem*, which says that a convex combination of points in Y can be written as a convex combination of affinely independent points of Y. By Lemma 3.1.4, a convex combination of affinely independent points lies in the simplex which is the convex hull of those points. To complete the equivalence of (1) and (2) with conv(Y), note that any simplex with vertices in Y is contained in every convex set which contains Y.

(3) Clearly a closed half-space is convex, hence the intersection of all closed half-spaces containing Y is convex, so contains conv(Y). To show the opposite inclusion, it suffices to show that there is a hyperplane which separates any point from a closed convex set which does not contain that point. This is true in more generality - the *hyperplane separation theorem* says that a closed convex set can be separated from a compact convex set by a hyperplane. This can be proved for a point as follows. Let C be a closed convex set and $p \notin C$ a point. Then the distance function $d(x, p)$ on V is a continuous function which attains a minimum on C since C is closed. Take a point $q \in C$ of minimum distance d from p, and let u be the vector from p to q. Choose any point x on the interior of the line segment from p to q and set $a = \langle u, x\rangle$. Then $\langle u, q\rangle > \langle u, x\rangle$. For suppose there is some $y \in C \cap H_{u,a} \neq \emptyset$. Then the line segment joining q and y is in C since C is convex. The vector v from y to q must have an angle with u less than 90 degrees by our choice of u. However, this implies that the line segment from q to y passes inside the sphere of radius d centered at p, contradicting the choice of q. Hence no such y can exist and $C \subset H^+_{u,a}$, $p \notin H^+_{u,a}$. □

Definition 3.1.6. A *convex polytope* is the convex hull of a finite set of points in V.

Suppose σ is a convex polytope. The *dimension* of σ is the smallest dimension of an affine space of V containing σ. A hyperplane $H_{u,a}$ is a *supporting hyperplane* of σ if $H_{u,a} \cap \sigma \neq \emptyset$ and $\sigma \subset H^+_{u,a}$. A *face* of σ is either σ or the intersection of σ with a supporting hyperplane. Note that a face of σ is also a polytope. The *face lattice* of σ is the set of all faces of σ ordered with respect to inclusion. A *facet* of σ is a face of dimension one less than the dimension of σ. A *vertex* of σ is a face of dimension zero. We state the following lemma without proof.

Lemma 3.1.7. *Let $\sigma \subset \mathbb{R}^n$ be an n-dimensional convex polytope with vertices $\{v_1, \ldots, v_k\}$ and facets $\{F_1, \ldots, F_m\}$. Suppose $F_i = H_{u_i,a_i} \cap \sigma$, where $\sigma \subset H^+_{u_i,a}$. Then*

1. $\sigma = conv(v_1, \ldots, v_k)$
2. $\sigma = \bigcap_{i=1}^{m} H^+_{u_i,a_i}$

If Y is a finite set of points, or more generally a compact set, then conv(Y) is closed and hence is the intersection of all closed half-spaces containing Y by Proposition 3.1.5. Lemma 3.1.7 says we can pick a finite number of these to describe a polytope. An algorithmic procedure for going back and forth between the dual descriptions in Lemma 3.1.7 is described in [64].

The 'homogeneous' analogue of a polytope is a convex polyhedral cone, which also has a pair of dual descriptions. First, a set $Y \subset V$ is called a *cone* if $y \in Y \implies \lambda y \in Y$ for all nonnegative real numbers λ (we will only consider cones which contain the origin). Y is a *convex cone* if it is a cone which is convex. If Y is any set, the *conical hull* of Y, denoted cone(Y), is the intersection of all convex cones containing Y. An arbitrary intersection of convex cones is a convex cone, so cone(Y) is itself a convex cone.

For any (possibly infinite) set $\{v_\alpha\}_{\alpha \in I}$ of vectors in V with index set I, the sum $\sum_{\alpha \in I} \lambda_\alpha v_\alpha$ is called a *conical combination* if

- $\lambda_\alpha \geq 0$ for all $\alpha \in I$
- $\lambda_\alpha = 0$ for all but finitely many $\alpha \in I$

A closed half-space $H^+_{u,a}$ is called *linear* if $a = 0$. Clearly $H^+_{u,0}$ is a convex cone. The proof of the following proposition is similar to that of Proposition 3.1.5 and is omitted.

Proposition 3.1.8. *Let $Y \subset V$ be any set. Then*

1. *cone(Y) is the set of all points obtained as conical combinations of points of Y.*
2. *If cone(Y) is closed, then it is the intersection of all closed linear half-spaces containing Y.*

Definition 3.1.9. A *convex polyhedral cone* is the conical hull of a finite set of points in V.

Suppose $\sigma \subset V$ is a convex polyhedral cone. The *dimension* of σ is the smallest dimension of a linear space of V containing σ. A hyperplane $H_{u,0}$ is a *supporting hyperplane* of σ if $H_{u,0} \cap \sigma \neq \emptyset$ and $\sigma \subset H^+_{u,0}$. A *face* of σ is either σ or the intersection of σ with a supporting hyperplane. Note that a face of σ is itself a cone. The *face lattice* of σ is the set of all faces of σ ordered with respect to inclusion. Faces of σ with dimension one are called *rays*. The *lineality space* of σ is the largest linear space contained in σ. For instance, the lineality space of $H^+_{u,0}$ is the hyperplane $H_{u,0}$. If the lineality space of σ is just the origin, then σ is *pointed*.

Lemma 3.1.10. *Let $\sigma \subset \mathbb{R}^n$ be an n-dimensional convex polyhedral cone with facets $\{F_1, \dots, F_m\}$. Suppose $F_i = H_{u_i,0} \cap \sigma$, where $\sigma \subset H^+_{u_i,0}$. Then*

1. *If σ is pointed, then σ is the conical hull of its rays.*
2. $\sigma = \bigcap_{i=1}^{m} H^+_{u_i,0}$

If Y is a finite set of points, or more generally a compact set not containing the origin, then cone(Y) is closed and hence is the intersection of all closed linear half-spaces containing Y by Proposition 3.1.8. Lemma 3.1.10 says we can pick a finite number of these to describe a convex polyhedral cone. An algorithmic procedure for going back and forth between the dual descriptions in Lemma 3.1.7 is described in [64].

3.2 Polytopal Complexes and Polyhedral Fans

Definition 3.2.1. Let $V \cong \mathbb{R}^n$ be an n-dimensional vector space. Suppose $X \subset V$ is a collection of convex polyhedral cones or a collection of convex polytopes satisfying the following two properties.

1. If $\gamma \in X$ then all faces of γ are in X.
2. $\gamma_1 \cap \gamma_2 \in \mathcal{P}$ is a face of both γ_1, γ_2, for all $\gamma_1, \gamma_2 \in X$.

A maximal face of X under inclusion is a *facet*. The *dimension* of X is the maximum dimension of a facet.

If X is a collection of *polytopes* satisfying these two properties then X is called a *polytopal complex* and will be denoted $\mathcal{P}$. If X is a collection of *convex polyhedral cones* satisfying these two properties then X is called a *fan* and will be denoted Σ.

Remark 3.2.2. Polytopal complexes are discussed at length in [64].

Let $X \subset \mathbb{R}^n$ be a polytopal complex or a polyhedral fan. We will denote by $|X|$ the underlying space of X. In the case that X is a polytopal complex all of whose faces are simplices, X is a *simplicial complex* and will be denoted by Δ. Δ will only be used to denote simplicial complexes.

Given $\gamma \in X$ a face, the *star* of γ in X, denoted $\mathrm{st}_X(\gamma)$, is defined by

$$\mathrm{st}_X(\gamma) := \{\psi \in X | \exists \sigma \in X, \psi \in \sigma, \gamma \in \sigma\}.$$

This is the smallest subcomplex/subfan of X which contains all faces which contain γ. Since every face of X contains the empty face, $\mathrm{st}_X(\emptyset) = X$. If the ambient complex/fan X is understood we will write $\mathrm{st}(\gamma)$ in place of $\mathrm{st}_X(\gamma)$.

We denote by $G(X)$ the *dual graph* of X. $G(X)$ has a vertex for every n-dimensional facet and two vertices are joined by an edge iff the corresponding facets σ and σ' satisfy $\sigma \cap \sigma' \in X_{n-1}$.

Definition 3.2.3. Fix a polytopal complex/polyhedral fan $X \subset \mathbb{R}^n$ of dimension n. Then X is

1. *Pure* if every facet $\sigma \in X$ has dimension n.
2. *Non-branching* if every codimension one face $\tau \in X_{n-1}$ is contained in at most two facets.
3. *Hereditary* if the dual graph $G(\mathrm{st}_X(\psi))$ of the star of every face ψ is connected.

In the definition of a *pseudomanifold* one assumes that X satisfies (1) and (2) and is *strongly connected*, which means that the dual graph $G(X)$ is connected. This is implied by the hereditary condition, since the star of the empty face is X. The hereditary condition is equivalent to requiring that both X and the star of every one of its faces is a pseudomanifold. If $n = 2$, the conditions in Definition 3.2.3 are equivalent to X being a subdivision of a topological 2-manifold with boundary; if $n > 2$ the conditions of Defintion 3.2.3 is strictly weaker than $\mathcal{P}$ being a subdivision of a topological n-manifold with boundary.

Remark 3.2.4. Unless otherwise specified, we will assume throughout this thesis that the underlying polytopal complexes/polyhedral fans satisfy the three conditions of Definition 3.2.3.

Given a hereditary pseudomanifold X, which is a polytopal complex or polyhedral fan, the boundary complex ∂X is the subcomplex of X consisting of all faces which are contained in a codimension one face τ so that τ is only contained in a single facet. A face $\gamma \in X$ is called *interior* if it is not a boundary face. A polytopal complex $\mathcal{P}$ is called *central* if every interior face of $\mathcal{P}$ contains the origin.

We will denote by $X_d, X_d^0, f_d(X), f_d^0(X)$ the set of d-faces of X, the set of interior d-faces of X, the number of d-faces of X, and the number of interior d-faces of X, respectively. We will also use these notations in case X is a manifold with a cell complex structure satisfying the three conditions of Definition 3.2.3. Chapter 0 of [31] is a good introduction to cell complexes; we only use the most basic properties.

Given a polytopal complex $\mathcal{P} \subset \mathbb{R}^n$, we build a polyhedral fan $\widehat{\mathcal{P}}$, called the *cone over* $\mathcal{P}$ or *homogenization* of $\mathcal{P}$, as follows. Let $\mathbb{R}^n$ have coordinates $x_1, \ldots, x_n$ and $\mathbb{R}^{n+1}$ have coordinates $x_0, \ldots, x_n$. Then set $i : \mathbb{R}^n \to \mathbb{R}^{n+1}$ to be the inclusion $i(x_1, \ldots, x_n) = (1, x_1, \ldots, x_n)$. The cone $\widehat{\mathcal{P}} \subset \mathbb{R}^{n+1}$ over $\mathcal{P}$ is the fan with cones $\mathrm{cone}(i(\gamma))$ for $\gamma \in \mathcal{P}$. We can go the other direction as well. We define two cell complexes which we will associate to a polyhedral fan.

Definition 3.2.5. Let $\Sigma \subset \mathbb{R}^{n+1}$ be a fan, $\sigma \in \Sigma$ a cone, $\mathbb{S}^n$ the unit n-sphere in $\mathbb{R}^{n+1}$, and $\mathbb{B}^{n+1}$ the unit $(n+1)$-ball in $\mathbb{R}^{n+1}$.

- $\text{lk}(\sigma) := \sigma \cap \mathbb{S}^n$
- $\mathcal{P}(\sigma) := \sigma \cap \mathbb{B}^{n+1}$
- $\text{lk}(\Sigma) := |\Sigma| \cap \mathbb{S}^n$
- $\mathcal{P}(\Sigma) := |\Sigma| \cap \mathbb{B}^{n+1}$

If $\Sigma = \widehat{\mathcal{P}}$, then $\text{lk}(\Sigma)$ is homeomorphic to $\mathcal{P}$. If $\text{lk}(\Sigma)$ is all of $\mathbb{S}^n$, then Σ is called *complete*. Both $\text{lk}(\Sigma)$ and $\mathcal{P}(\Sigma)$ have the natural structure of cell complexes, with cells $\{\text{lk}(\gamma)|\gamma \in \Sigma\}$ and $\{\mathcal{P}(\gamma)|\gamma \in \Sigma\}$, respectively. We will only care about $\mathcal{P}(\Sigma)$ and $\text{lk}(\Sigma)$ up to homeomorphism, hence a figure labelled $\mathcal{P}(\Sigma)$ or $\text{lk}(\Sigma)$ may only be homeomorphic to it.

Remark 3.2.6. Given a fan Σ, one could ask whether there exists a central polytopal complex $\mathcal{P}$ so that $\text{cone}(\sigma)$ is a facet of Σ for every facet $\sigma \in \mathcal{P}$. Such a $\mathcal{P}$ would be homeomorphic to $\mathcal{P}(\Sigma)$. If this is possible, then the facet normals to $\mathcal{P}$ would give rise to a piecewise linear function on Σ. However, there exist fans Σ (in three or more dimensions) with no piecewise linear function. Hence we must be content with $\mathcal{P}(\Sigma)$ having the structure of a cell complex.

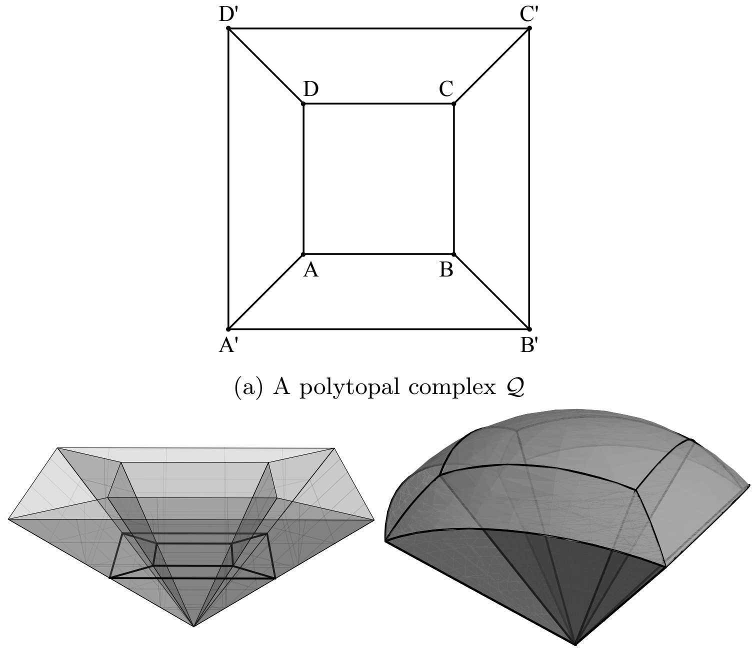
Example 3.2.7. The polytopal complex $\mathcal{Q}$ in Figure 3.1a has vertices $A = (-1,-1)$, $B = (1,-1), C = (1,1), D = (-1,1), A' = (-2,-2), B' = (2,-2), C' = (2,2), D' = (-2,2)$. It has 5 facets and 12 edges. Figure 3.1a shows the cone $\widehat{\mathcal{Q}}$ over $\mathcal{Q}$, and Figure 3.1c shows the cell complex $\mathcal{P}(\widehat{\mathcal{Q}})$. The complex $\text{lk}(\widehat{\mathcal{Q}})$ is the set of all faces of $\mathcal{P}(\widehat{\mathcal{Q}})$ that don't contain the origin. In Figure 3.1c this is the upper hull of the complex; note that $\text{lk}(\widehat{\mathcal{Q}})$ is homeomorphic to the original complex $\mathcal{Q}$.

3.3 Splines and the Spline Complex

Given a polytopal complex or polyhedral fan $X \subset \mathbb{R}^n$, we assign integers $\alpha(\tau) \geq -1$, called *smoothness parameters*, to every codimension one face $\tau \in X_{n-1}$ so that $\alpha(\tau) \geq 0$ for every $\tau \in X^0_{n-1}$. We denote by X^{-1} the subcomplex of ∂X whose faces are contained in a codimension one face τ of Σ so that $\alpha(\tau) = -1$. We also set $X^{\geq 0}_d = X_d \setminus X^{-1}_d$. The interaction of the pair (X, X^{-1}) will be crucial.

Remark 3.3.1. Let $r \geq 0$ be an integer. The most well studied case is when $\alpha(\tau) = r$ for every interior codimension one face $\tau \in X$ and $\alpha(\tau) = -1$ for every codimension one face in ∂X. In this case $X^{-1} = \partial\Sigma$. We will refer to this as the case of *uniform smoothness* and call r the *smoothness parameter*.

Let $X \subset \mathbb{R}^n$ be a polytopal complex or polyhedral fan and α a list of smoothness parameters. Every codimension one face τ has a unique affine span



(a) A polytopal complex $\mathcal{Q}$

(b) The fan $\widehat{\mathcal{Q}}$ over $\mathcal{Q}$

(c) The cell complex $\mathcal{P}(\widehat{\mathcal{Q}})$

Figure 3.1

$\text{aff}(\tau) \subset \mathbb{R}^n$. Let $l_\tau \in R = \mathbb{R}[x_1, \ldots, x_n]$ be a choice of generator for the ideal $I(\tau)$ of polynomials which vanish on τ (equivalently vanish on its affine span).

Definition 3.3.2. The algebra of splines $C^\alpha(X)$ is the subalgebra of tuples

$$\{F = (F_\sigma)_{\sigma \in X_n}\} \subset \bigoplus_{\sigma \in X_n} R$$

satisfying

1. $l_\tau^{\alpha(\tau)+1} | (F_{\sigma_1} - F_{\sigma_2})$ for every pair of facets σ_1, σ_2 with $\sigma_1 \cap \sigma_2 = \tau \in X_{n-1}$.
2. $l_\tau^{\alpha(\tau)+1} | F_\sigma$ for every $\tau \in \sigma \cap \partial X$, provided this is nonempty.

$C_d^\alpha(X)$ denotes the vector space

$$C_d^\alpha(X) := \{(F_\sigma) \in C^\alpha(X) | \deg(F_\sigma) \leq d\}$$

Remark 3.3.3. In the case of uniform smoothness with smoothness parameter r, $C^\alpha(X)$ is denoted by $C^r(X)$.

Remark 3.3.4. Since $C^\alpha(X) \subset R^{f_n(X)}$, we can add and multiply splines $F, G \in C^\alpha(X)$ componentwise within $R^{f_n(\mathcal{P})}$. $C^\alpha(X)$ is closed under these operations since pointwise addition and multiplication do not disturb the divisibility conditions of Definition 3.3.2. In particular, $C^\alpha(X)$ is an R-module. $C^\alpha(X)$ is an R-algebra provided $\mathbf{1} \in C^\alpha(X)$, where $\mathbf{1}$ is the unit of $R^{f_n(X)}$. This is true iff

$\alpha(\tau) = -1$ for every $\tau \in \partial X$, in other words no vanishing is imposed on the boundary of X.

Remark 3.3.5. Definition 3.3.2 follows the definition of Geramita-Schenck [28]. $C^r(X)$ is usually defined as follows. For $U \subset \mathbb{R}^n$, let $C^r(U)$ denote the set of functions $F : U \to \mathbb{R}$ continuously differentiable of order r. For $F : |X| \to \mathbb{R}$ a function and $\sigma \in X_n$, let F_σ denote the restriction of F to σ. Then

$$C^r(X) := \{F \in C^r(|X|)|F_\sigma \in R \text{ for every } \sigma \in X_n\}$$

If X satisfies the conditions of Definition 3.2.3, Billera-Rose show [11, Corollary 1.3] that the global C^r condition can be expressed as a differentiability condition across internal faces of codimension one, recovering Definition 3.3.2.

The following lemma, due to Billera-Rose, provides the means for computing $C^\alpha(\mathcal{P})$.

Lemma 3.3.6. *Let $X \subset \mathbb{R}^n$ be a polytopal complex or polyhedral fan, and α a list of smoothness parameters. Then $C^\alpha(X)$ is (isomorphic to) the kernel of the map*

$$\phi : R^{f_n(X)} \oplus \left(\bigoplus_{\tau \in X_{n-1}} R \right) \to R^{f_n(X)},$$

where ϕ is the matrix

$$\left(\begin{array}{c|ccc} & l_{\tau_1}^{\alpha(\tau_1)+1} & & \\ \delta_n & & \ddots & \\ & & & l_{\tau_k}^{\alpha(\tau_k)+1} \end{array} \right),$$

$k = |X_{n-1}^{\geq 0}|$ and the matrix δ_n is the top dimensional cellular boundary map of X relative to X^{-1}.

Proof. This is an expression of the divisibility conditions in Definition 3.3.2 in the form of a matrix. □

Example 3.3.7. Consider the polytopal complex $\mathcal{Q}$ from Example 3.2.7, and label the facets and edges as in Figure 3.2.

Letting l_i be the affine form corresponding to τ_i, we have

$$\begin{array}{llll} l_1 = x+1 & l_2 = y-1 & l_3 = x-1 & l_4 = y+1 \\ l_5 = x+y & l_6 = x-y & l_7 = x+y & l_8 = x-y \\ l_9 = x+2 & l_{10} = y-2 & l_{11} = x-2 & l_{12} = y+2 \end{array}$$

The top dimensional map $\partial_2(F_i) = \sum_{\tau_j \in \partial F_i} \pm\tau_j$ is determined by any choice of signs so that each edge τ_i appears in the two facets which contain it with opposite signs. Then the matrix ϕ of Lemma 3.3.6 for $\mathcal{Q}$ is shown in Table 3.1. We label the columns by the corresponding edges and faces for clarity. The rows

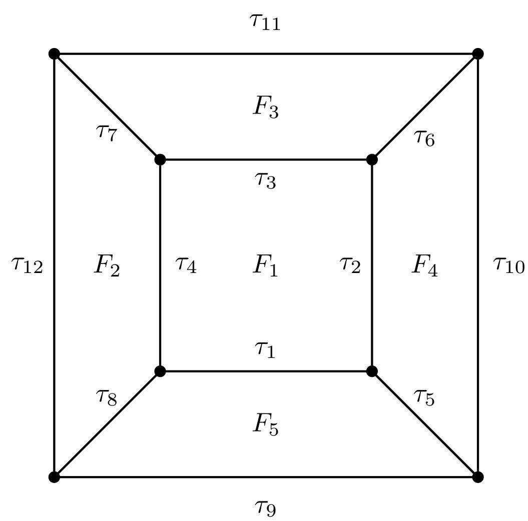


Figure 3.2: $\mathcal{Q}$

correspond to edges and are ordered accordingly. If any of the edges $\tau_9, \tau_{10}, \tau_{11}$, or τ_{12} have corresponding smoothness parameter -1, then the row and column corresponding to that edge are eliminated from the matrix.

F_1	F_2	F_3	F_4	F_5	τ_1	τ_2	τ_3	τ_4	τ_5	τ_6	τ_7	τ_8	τ_9	τ_{10}	τ_{11}	τ_{12}
1	0	0	0	-1	$l_1^{\alpha_1}$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	-1	0	0	$l_2^{\alpha_2}$	0	0	0	0	0	0	0	0	0	0
1	0	-1	0	0	0	0	$l_3^{\alpha_3}$	0	0	0	0	0	0	0	0	0
1	-1	0	0	0	0	0	0	$l_4^{\alpha_4}$	0	0	0	0	0	0	0	0
0	0	0	1	-1	0	0	0	0	$l_5^{\alpha_5}$	0	0	0	0	0	0	0
0	0	1	-1	0	0	0	0	0	0	$l_6^{\alpha_6}$	0	0	0	0	0	0
0	1	-1	0	0	0	0	0	0	0	0	$l_7^{\alpha_7}$	0	0	0	0	0
0	1	0	0	-1	0	0	0	0	0	0	0	$l_8^{\alpha_8}$	0	0	0	0
0	0	0	0	1	0	0	0	0	0	0	0	0	$l_9^{\alpha_9}$	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0	0	$l_{10}^{\alpha_{10}}$	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	$l_{11}^{\alpha_{11}}$	0
0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$l_{12}^{\alpha_{12}}$

Table 3.1: Spline Matrix for $C^\alpha(\mathcal{Q})$

As long as we impose no vanishing along the boundary of $\mathcal{P}(\Sigma)$ (which is $\mathrm{lk}(\Sigma)$), it makes sense to define splines on $\mathcal{P}(\Sigma)$.

Definition 3.3.8. Let $\mathcal{P} \subset \mathbb{R}^n$ be a polytopal complex with smoothness parameters α and $\Sigma \subset \mathbb{R}^{n+1}$ a polyhedral fan with smoothness parameters β. Then

1. $C^\alpha(\widehat{\mathcal{P}})$ denotes the ring of splines on $\widehat{\mathcal{P}}$ with smoothness parameters $\alpha(\widehat{\tau}) = \alpha(\tau)$ for every codimension one face $\tau \in \mathcal{P}_{n-1}$

2. $C^\beta(\mathcal{P}(\Sigma))$ denotes the ring of splines on $\mathcal{P}(\Sigma)$ with smoothness parameters $\beta(\tau) = \beta(\mathrm{cone}(\tau))$ for every codimension one cell $\tau \in \mathcal{P}(\Sigma)_n$ so that $\mathrm{cone}(\tau) \in \Sigma_n$ and $\beta(\tau) = -1$ for every codimension one cell $\tau \in \mathrm{lk}(\Sigma)$.

Corollary 3.3.9. *Let $\Sigma \subset \mathbb{R}^{n+1}$ be a fan. Then $C^\alpha(\Sigma) = C^\alpha(\mathcal{P}(\Sigma))$.*

Example 3.3.10. With conventions as in Definition 3.3.8, the matrix whose kernel is $C^\alpha(\widehat{\mathcal{Q}})$, where $\mathcal{Q}$ is the complex from Example 3.2.7, is found by introducing a z-variable and homogenizing the entries of the matrix from Example 3.3.7 with respect to z.

Proposition 3.3.11. *Suppose $X \subset \mathbb{R}^n$ is a polyhedral fan or a central polytopal complex. Define*

$$C^\alpha(X)_d := \{(F_\sigma) \in C^\alpha(X) | degF_\sigma = d \text{ for all } \sigma \in X_n\}.$$

Then $C^\alpha(X)$ decomposes as a direct sum

$$C^\alpha(X) = \bigoplus_{d=0}^{\infty} C^\alpha(X)_d.$$

Proof. The entries in the spline matrix ϕ for X (given in Lemma 3.3.6) are homogeneous since the forms l_τ for $\tau \in X_{n-1}$ are linear (the affine span of each τ passes through the origin). The kernel of a map given by a matrix with homogeneous entries is graded, hence we have the result. □

The following proposition, due to Billera and Rose, is crucial for using graded commutative algebra to find $\dim C^\alpha_d(\mathcal{P})$ for a polytopal complex $\mathcal{P}$.

Proposition 3.3.12. *[11, Theorem 2.6] $C^\alpha_d(\mathcal{P}) \cong C^\alpha(\widehat{\mathcal{P}})_d$.*

Proof. If $f \in R = \mathbb{R}[x_1, \ldots, x_n]$ is a polynomial with $\deg(f) \leq d$, then let $f^h(x_0, \ldots, x_n) = x_0^d f(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0})$. The map $\phi^h : C^\alpha_d(\mathcal{P}) \to C^\alpha(\widehat{\mathcal{P}})_d$ is given by $\phi^h((F_\sigma)) = (F_\sigma^h)$, where $(F_\sigma) \in C^\alpha_d(\mathcal{P})$. This is an $\mathbb{R}$-linear map with inverse given by setting $x_0 = 1$. Check that ϕ_h is injective, ϕ_h^{-1} is surjective, and the maps respect the divisibility conditions from Definition 3.3.2. □

For the rest of this section we switch to exclusively using fans. By Corollary 3.3.9, one could also use central polytopal complexes. We will do this in Chapter 6.

Following Billera in [9], we extend the top dimensional boundary map in 3.3.6 to a complex taking into account information of all relevant lower dimensional faces. It will be useful to do this for an arbitrary pair of fans (Σ, Σ'), where $\Sigma' \subset \Sigma$ is a subfan.

Definition 3.3.13. Let $\Sigma \subset \mathbb{R}^{n+1}$ be a fan with smoothness parameters α, $\Sigma' \subset \Sigma$ a subfan, and set $S = \mathbb{R}[x_0, \ldots, x_n]$. Define the complex $\mathcal{R}[\Sigma, \Sigma']$ with the following modules in homological degree i for $i = 0, \ldots, n+1$.

$$\begin{aligned}\mathcal{R}[\Sigma, \Sigma']_i &= \bigoplus_{\gamma \in (\Sigma_i \setminus \Sigma'_i)} S \\ &= \bigoplus_{\gamma \in (\mathcal{P}(\Sigma)_i \setminus (\mathcal{P}(\Sigma')_i \cup \mathrm{lk}(\Sigma)_i)} S,\end{aligned}$$

where the differential $\delta_i : \mathcal{R}[\Sigma, \Sigma']_i \to \mathcal{R}[\Sigma, \Sigma']_{i-1}$ is the cellular differential of the relative chain complex of the pair $(\mathcal{P}(\Sigma), \mathrm{lk}(\Sigma) \cup \mathcal{P}(\Sigma'))$ with coefficients in S.

Given a fan Σ with smoothness parameters α, we associate ideals to its faces as follows. For a codimension one face $\tau \in \Sigma_n^{\geq 0}$, set $J(\tau) = \langle l_\tau^{\alpha(\tau)+1} \rangle$. For any non-facet $\gamma \in \Sigma$,

$$J(\gamma) := \sum_{\gamma \in \tau \in \Sigma_n^{\geq 0}} J(\tau);$$

if $\sigma \in \Sigma_{n+1}$, set

$$J(\sigma) := 0.$$

Definition 3.3.14. Let $\Sigma' \subset \Sigma$ be a subfan of an $(n+1)$-dimensional fan $\Sigma \subset \mathbb{R}^{n+1}$ with smoothness parameters α, and set $S = \mathbb{R}[x_0, \ldots, x_n]$. Define complexes $\mathcal{J}[\Sigma, \Sigma'], \mathcal{R}/\mathcal{J}[\Sigma, \Sigma']$ with the following modules in homological degree i for $i = 0, \ldots, n+1$.

$$\begin{aligned} \mathcal{J}[\Sigma, \Sigma']_i &= \bigoplus_{\gamma \in (\Sigma_i \setminus \Sigma_i')} J(\gamma) \\ \mathcal{R}/\mathcal{J}[\Sigma, \Sigma']_i &= \bigoplus_{\gamma \in (\Sigma_i \setminus \Sigma_i')} S/J(\gamma). \end{aligned}$$

The differentials of $\mathcal{J}[\Sigma, \Sigma'], \mathcal{R}/\mathcal{J}[\Sigma, \Sigma']$ are obtained by restricting and quotienting the differential of $\mathcal{R}[\Sigma, \Sigma']$.

Lemma 3.3.15. *Let $\Sigma \subset \mathbb{R}^{n+1}$ be a pure, hereditary, $(n+1)$-dimensional fan with smoothness parameters α, and let $\mathcal{R}/\mathcal{J}[\Sigma, \Sigma^{-1}]$ be as in Definition 3.3.14. Then*

$$H_{n+1}(\mathcal{R}/\mathcal{J}[\Sigma, \Sigma^{-1}]) = C^\alpha(\Sigma).$$

Proof. This is equivalent to the statement

$$C^\alpha(\Sigma) = \ker(S^{f_{n+1}(\Sigma)} \xrightarrow{\bar{\delta}_{n+1}} \bigoplus_{\tau \in \Sigma_n^{\geq 0}} S/J(\tau)),$$

where $\delta_{n+1} : S^{f_{n+1}(\Sigma)} \to \bigoplus_{\tau \in \Sigma_n^{\geq 0}} S$ is the top dimensional cellular boundary map of $\mathcal{P}(\Sigma)$ relative to $\mathcal{P}(\Sigma^{-1}) \cup \mathrm{lk}(\Sigma)$. This follows from Lemma 3.3.6; or it can be seen directly since it is another way to state the divisibility conditions from Definition 3.3.2. Explicitly, a tuple $(F_\sigma)_{\sigma \in \Sigma_{n+1}}$ is sent by $\bar{\delta}$ to the tuple $(F_{\sigma_1} - F_{\sigma_2})_\tau$ mod $J(\tau)$, where $\tau \in \Sigma_n$ is the codimension one face along which σ_1, σ_2 intersect. This is 0 iff $F_{\sigma_1} - F_{\sigma_2} \in J(\tau)$, i.e. iff $l_\tau^{\alpha(\tau)+1} | F_{\sigma_1} - F_{\sigma_2}$. If $\tau \in \partial\mathcal{P}$, then there is only one facet, say σ, containing τ and $F_\sigma \equiv 0$ mod $J(\tau)$ iff $l_\tau^{\alpha(\tau)+1} | F_\sigma$. ☐

Remark 3.3.16. There is a tautological short exact sequence of complexes

$$0 \to \mathcal{J}[\Sigma, \Sigma'] \to \mathcal{R}[\Sigma, \Sigma'] \to \mathcal{R}/\mathcal{J}[\Sigma, \Sigma'] \to 0$$

We will frequently use this exact sequence of complexes in proofs.

We spend the rest of the section investigating the homology of $\mathcal{R}[\Sigma,\Sigma']$. The complex $\mathcal{R}[\Sigma,\Sigma']$ is defined so that

$$H_i(\mathcal{R}[\Sigma,\Sigma']) = H_i(\mathcal{P}(\Sigma),\mathcal{P}(\Sigma')\cup \mathrm{lk}(\Sigma);S)$$

where the homology group on the right is the cellular homology of $\mathcal{P}(\Sigma)$ relative to $\mathcal{P}(\Sigma')\cup \mathrm{lk}(\Sigma)$ with coefficients in S. This agrees with the so-called *Borel-Moore* homology of the fan Σ relative to the subfan Σ'. The homology of this complex is described in more detail in the following proposition.

Proposition 3.3.17. *Let Σ be an $(n+1)$-dimensional abstract fan, with $n \geq 1$, $\Sigma' \neq \mathbf{0} \subset \Sigma$ a subfan (where $\mathbf{0}$ is the cone vertex), possibly empty, and $\mathcal{R}[\Sigma,\Sigma']$ as defined above.*

$$H_i(\mathcal{R}[\Sigma,\Sigma']) \cong \begin{cases} 0 & \text{if } i=0,1 \\ \widetilde{H}_{i-1}(\mathcal{P}(\Sigma')\cup lk(\Sigma);S) \cong H_{i-1}(lk(\Sigma),lk(\Sigma');S) & \text{if } i\geq 2. \end{cases}$$

Remark 3.3.18. If $\Sigma = \widehat{\mathcal{P}}$, then $\mathrm{lk}(\Sigma)$ is homeomorphic to $\mathcal{P}$ and $\mathrm{lk}(\Sigma^{-1})$ is homeomorphic to $\mathcal{P}^{-1}$.

Proof. We use the identification $H_i(\mathcal{R}[\Sigma,\Sigma']) \cong H_i(\mathcal{P}(\Sigma),\mathcal{P}(\Sigma')\cup \mathrm{lk}(\Sigma);S)$. Consider the long exact sequence of the pair in singular homology corresponding to the inclusion $\mathcal{P}(\Sigma')\cup \mathrm{lk}(\Sigma) \hookrightarrow \mathcal{P}(\Sigma)$, with coefficients in S:

$$\cdots \to H_i(\mathcal{P}(\Sigma)) \to H_i(\mathcal{P}(\Sigma),\mathcal{P}(\Sigma')\cup \mathrm{lk}(\Sigma)) \to H_{i-1}(\mathcal{P}(\Sigma')\cup \mathrm{lk}(\Sigma)) \to \cdots$$

$\mathcal{P}(\Sigma)$ is contractible, so

$$H_i(\mathcal{P}(\Sigma)) = \begin{cases} S & \text{if } i=0 \\ 0 & \text{otherwise,} \end{cases}$$

The map $H_0(\mathcal{P}(\Sigma')\cup \mathrm{lk}(\Sigma)) \to H_0(\mathcal{P}(\Sigma)) = S$ is surjective, hence
$H_0(\mathcal{P}(\Sigma),\mathcal{P}(\Sigma')\cup \mathrm{lk}(\Sigma)) = 0$ and we have a short exact sequence

$$0 \to H_1(\mathcal{P}(\Sigma),\mathcal{P}(\Sigma')\cup \mathrm{lk}(\Sigma)) \to H_0(\mathcal{P}(\Sigma')\cup \mathrm{lk}(\Sigma)) \to S \to 0,$$

Hence $H_1(\mathcal{P}(\Sigma),\mathcal{P}(\Sigma')\cup \mathrm{lk}(\Sigma)) = 0$ if $\mathcal{P}(\Sigma')\cup \mathrm{lk}(\Sigma)$ is connected. Since Σ is hereditary, $\mathrm{lk}(\Sigma)$ is connected. So if $\Sigma' = \emptyset$, $\mathcal{P}(\Sigma')\cup \mathrm{lk}(\Sigma) = \mathrm{lk}(\Sigma)$ is connected. If $\Sigma' \neq \emptyset$, then $\mathcal{P}(\Sigma')$ is connected since $\mathbf{0}$ is contained in every face. Furthermore $\mathcal{P}(\Sigma')\cap \mathrm{lk}(\Sigma) \neq \emptyset$ since every face of Σ other than $\mathbf{0}$ intersects nontrivially with $\mathrm{lk}(\Sigma)$, and we assumed $\Sigma' \neq \mathbf{0}$. So $\mathcal{P}(\Sigma')\cup \mathrm{lk}(\Sigma)$ is connected and the conclusion follows.

The isomorphisms

$$H_i(\mathcal{P}(\Sigma),\mathcal{P}(\Sigma')\cup \mathrm{lk}(\Sigma)) \cong H_{i-1}(\mathcal{P}(\Sigma')\cup \mathrm{lk}(\Sigma))$$

for $i \geq 2$ are immediate from the long exact sequence of the pair. Finally, the isomorphism

$$\widetilde{H}_j(\mathcal{P}(\Sigma') \cup \mathrm{lk}(\Sigma)) \cong H_j(\mathrm{lk}(\Sigma), \mathrm{lk}(\Sigma'))$$

is a consequence of excision and the long exact sequence of the pair $(\mathcal{P}(\Sigma') \cup \mathrm{lk}(\Sigma), \mathcal{P}(\Sigma'))$. The key observation is that $\mathcal{P}(\Sigma') \cup \mathrm{lk}(\Sigma)$ is the *mapping cone* of the inclusion $\mathrm{lk}(\Sigma') \hookrightarrow \mathrm{lk}(\Sigma)$. That is, topologically, $\mathcal{P}(\Sigma') \cup \mathrm{lk}(\Sigma)$ may be identified with the space

$$\mathrm{lk}(\Sigma) \cup (\mathrm{lk}(\Sigma') \times I)/\sim,$$

where $I = [0, 1]$ is the unit interval, all points of the form $(x, 0)$ are identified as a single point, and $(x, 1)$ is identified with the image of x in $\mathrm{lk}(\Sigma)$. A more detailed discussion may be found in [31, p. 125]. □

Example 3.3.19. Let $\Sigma = \widehat{\mathcal{Q}}$ as in Figure 3.1b, with uniform smoothness parameters $\alpha(\tau) = r$ on interior codimension one faces and $\alpha(\tau) = -1$ on boundary codimension one faces. Then $\Sigma^{-1} = \partial\Sigma$. The complex $\mathcal{R}[\Sigma, \Sigma^{-1}]$ is nonzero in homological degrees $1, 2$, and 3. It has the form

$$S^5 \to S^8 \to S^4 \to 0,$$

where $S = \mathbb{R}[x, y, z]$ is the polynomial ring in three variables. By definition, $H_*(\mathcal{R}[\Sigma, \partial\Sigma])$ computes the homology of the complex $\mathcal{P}(\Sigma)$ relative to $\mathcal{P}(\partial\Sigma) \cup \mathrm{lk}(\Sigma) = \partial\mathcal{P}(\Sigma)$ with coefficients in $S = \mathbb{R}[x, y, z]$. From Figure 3.1c it is clear that this is equivalent to computing $H_*(D^3, \mathbb{S}^2; S)$, the homology of a 3-disk relative to its boundary with coefficients in S. By excision, $H_*(D^3, \mathbb{S}^2; S) \cong \widetilde{H}_*(D^3/\mathbb{S}^2; S) = \widetilde{H}_*(\mathbb{S}^3; S)$. Hence $H_i(\mathcal{R}[\Sigma, \partial\Sigma]) = 0$ except when $i = 3$, when $H_3(\mathcal{R}[\Sigma, \partial\Sigma]) = S$.

Equivalently, using Proposition 3.3.17, we see that $H_0(\mathcal{R}[\Sigma, \partial\Sigma]) = 0$ for $i = 0, 1$. We have $\mathrm{lk}(\Sigma)$ is homeomorphic to $\mathcal{Q}$ and $\mathrm{lk}(\partial\Sigma)$ is homeomorphic to $\partial\mathcal{Q}$. It is clear that the homology of $\mathcal{Q}$ relative to its boundary gives the homology of a 2-sphere. Shifting the homological dimensions up, we again arrive at the homology of the complex $\mathcal{R}[\Sigma, \partial\Sigma]$.

Example 3.3.20. Again let $\Sigma = \widehat{\mathcal{Q}}$ be as in Figure 3.1b, but suppose we impose vanishing on the entire boundary, so that $\Sigma^{-1} = \emptyset$. Then the complex $\mathcal{R}[\Sigma, \Sigma^{-1}] = \mathcal{R}[\Sigma]$ has the form

$$S^5 \to S^{12} \to S^8 \to S,$$

where the final S corresponds to the cone vertex. $H_*(\mathcal{R}) \cong H_*(\Sigma, \mathcal{P}(\Sigma^{-1}) \cup \mathrm{lk}(\Sigma); S) = H_*(\Sigma, \mathrm{lk}(\Sigma); S)$. Since $\mathrm{lk}(\Sigma)$ is contractible, this is the same as the reduced homology of a point. We conclude that $H_i(\mathcal{R}[\Sigma])$ vanishes for all i.

Equivalently, using Proposition 3.3.17, we see that $H_i(\mathcal{R}[\Sigma, \partial\Sigma]) = 0$ for

$i = 0, 1$. We have $\mathrm{lk}(\Sigma)$ is homeomorphic to $\mathcal{Q}$. The homology of $\mathcal{Q}$ gives the homology of a 2-disk. Hence $H_i(\mathcal{R}[\Sigma]) = H_{i-1}(\mathcal{Q}; S) = 0$ for $i = 2, 3$. Note that $H_1(\mathcal{R}[\Sigma]) = 0$ while $H_0(\mathcal{Q}; S) = S$.

3.4 Homology Computations

In this section we restrict to the case of uniform smoothness r, where $\Sigma^{-1} = \partial\Sigma$. We collect several results on the homologies of $\mathcal{J}[\Sigma, \partial\Sigma]$ and $\mathcal{R}/\mathcal{J}[\Sigma, \partial\Sigma]$ which we will use in upcoming chapters.

Lemma 3.4.1. *Let $\Sigma \subset \mathbb{R}^{n+1}$ be a fan such that $lk(\Sigma)$ is homeomorphic to the unit n-ball $\mathbb{B}^n$, $lk(\partial\Sigma)$ is homeomorphic to the unit $(n-1)$-sphere $\mathbb{S}^{n-1}$, and let $S = \mathbb{R}[x_0, \ldots, x_n]$. Then we have the isomorphism (non-canonical)*

$$\begin{aligned} C^r(\Sigma) \cong &\quad S \oplus H_n(\mathcal{J}[\Sigma, \partial\Sigma]) \\ H_i(\mathcal{R}/\mathcal{J}[\Sigma, \partial\Sigma]) \cong &\quad H_{i-1}(\mathcal{J}[\Sigma, \partial\Sigma]) \end{aligned}$$

for every $i = 1, \ldots, n$. In particular, this holds for $\Sigma = \widehat{\mathcal{P}}$, where $\mathcal{P} \subset \mathbb{R}^n$ is a polytopal complex homeomorphic to $\mathbb{B}^n$ and $\partial\mathcal{P}$ is homeomorphic to $\mathbb{S}^{n-1}$.

Proof. If $\mathrm{lk}(\Sigma)$ is homeomorphic to $\mathbb{B}^n$ and $\mathrm{lk}(\partial\Sigma)$ is homeomorphic to $\mathbb{S}^{n-1}$, then the homologies of $\mathcal{R}[\Sigma, \partial\Sigma]$ match up with the reduced homology of an $(n+1)$-sphere by Proposition 3.3.17. Hence the only non-vanishing homology is $H_{n+1}(\mathcal{R}[\Sigma, \partial\Sigma]) \cong S$. The long exact sequence in homology coming from the short exact sequence of complexes

$$0 \to \mathcal{J}[\Sigma, \partial\Sigma] \to \mathcal{R}[\Sigma, \partial\Sigma] \to \mathcal{R}/\mathcal{J}[\Sigma, \partial\Sigma] \to 0$$

yields $H_i(\mathcal{R}/\mathcal{J}[\Sigma, \partial\Sigma]) \cong H_{i-1}(\mathcal{J}[\Sigma, \partial\Sigma])$ for $i = 1, \ldots, n$ and the short exact sequence

$$0 \to S \to C^r(\Sigma) \to H_n(\mathcal{J}[\Sigma, \partial\Sigma]) \to 0.$$

The inclusion of S is the inclusion of *trivial* splines (given by the same polynomial on each facet) into $C^r(\widehat{\mathcal{P}}) = H_{n+1}(\mathcal{R}/\mathcal{J})$. This is a split inclusion, with the splitting $C^r(\widehat{\mathcal{P}}) \to S$ given (noncanonically) by restriction $F \to F_\sigma$ for any choice of facet σ. □

Lemma 3.4.2. *Let $\Sigma \subset \mathbb{R}^{n+1}$ be a fan and $S = \mathbb{R}[x_0, \ldots, x_n]$. Let*

$$\phi : S^{f_{n+1}(\Sigma)} \oplus \left(\bigoplus_{\tau \in \Sigma_n^0} S(-r-1) \right) \to S^{f_n(\Sigma)}$$

be the map from Lemma 3.3.6 whose kernel is $C^r(\Sigma)$. If M is the cokernel of ϕ, then it naturally fits into the complex $\mathcal{R}/\mathcal{J}[\Sigma, \partial\Sigma]$ by replacing the first two *modules, i.e.*

$$0 \to M \to \bigoplus_{\gamma \in \Sigma_{n-1}} \frac{S}{J(\gamma)} \to \cdots$$

Moreover, the two leftmost homology modules of this new complex are $H_n(\mathcal{R}/\mathcal{J}[\Sigma, \partial\Sigma])$ *and* $H_{n-1}(\mathcal{R}/\mathcal{J}[\Sigma, \partial\Sigma])$*. In particular, if* $\Sigma \subset \mathbb{R}^3$ *is a non-complete fan, then there is a short exact sequence*

$$0 \to H_2(\mathcal{R}/\mathcal{J}[\Sigma, \partial\Sigma] \to M \to \bigoplus_{v \in \Sigma_1^0} \frac{S}{J(v)} \to 0$$

Proof. Consider the commutative diagram below whose top row is the same as in Lemma 3.3.6 and whose bottom row is $\mathcal{R}/\mathcal{J}[\Sigma, \partial\Sigma]$.

$$\begin{array}{ccccc} S^{f_{n+1}(\Sigma)} \oplus \left(\bigoplus_{\tau \in \Sigma_n^0} S(-r-1)\right) & \xrightarrow{\phi} & S^{f_n(\Sigma)} & & \\ \downarrow & & \downarrow \pi & & \\ S^{f_{n+1}(\Sigma)} & \xrightarrow{\delta_{n+1}} & \bigoplus_{\tau \in \Sigma_n^0} \frac{S}{J(\tau)} & \xrightarrow{\delta_n} & \bigoplus_{\gamma \in \Sigma_{n-1}^0} \frac{S}{J(\gamma)} \cdots \end{array}$$

The cokernel of ϕ is precisely the same as the cokernel of δ_{n+1}. Hence $M \cong \bigoplus_{\tau \in \Sigma_n^0} \frac{S}{J(\tau)}/\mathrm{im}(\delta_{n+1})$ and δ_n descends to a map from M to $\bigoplus_{\gamma \in \Sigma_{n-1}^0} \frac{S}{J(\gamma)}$. Since $H_n(\mathcal{R}/\mathcal{J}[\Sigma, \partial\Sigma]) = \ker(\delta_n)/\mathrm{im}(\delta_{n+1})$, the lemma is proved. □

We give presentations for the homology module $H_1(\mathcal{R}/\mathcal{J}[\Sigma, \partial\Sigma])$ when $\dim \Sigma = 3$. There are two cases, depending on whether Σ is complete or not.

Lemma 3.4.3. *[50, Lemma 3.8] Let* $\Sigma \subset \mathbb{R}^3$ *be a hereditary, non-complete fan. A codimension one face* $\tau \in \Sigma_2$ *is totally interior if* $\tau \cap \partial\Sigma = \emptyset$*. Define* $K^r \subset \bigoplus_{\tau \in \Sigma_2} S(-r-1)e_\tau$ *to be the submodule generated by*

$$\{e_\tau | \tau \text{ not totally interior}\}$$

and

$$\{\sum_{v \in \tau} a_\tau e_\tau | v \in \Sigma_1, \sum a_\tau l_\tau^{r+1} = 0\}.$$

The S*-module* $H_1(\mathcal{J}[\Sigma, \partial\Sigma])$ *is given by generators and relations by*

$$0 \to K^r \to \bigoplus_{\tau \in \Sigma_2^0} Se_\tau \to H_1(\mathcal{J}[\Sigma, \partial\Sigma]) \to 0.$$

Proof. Let $K_v^r \subset \bigoplus_{v \in \tau} S(-r-1)e_\tau$ be the module of relations around the ray $v \in \Sigma_1$, namely

$$K_v^r = \{\sum_{v \in \tau} a_\tau e_\tau | \sum a_\tau l_\tau^{r+1} = 0\}.$$

Set up the following diagram with exact columns, whose first row is the complex $J[\Sigma, \partial\Sigma]$.

$$\begin{array}{ccc}
\bigoplus_{\tau\in\Sigma_2^0} J(\tau) & \longrightarrow & \bigoplus_{v\in\Sigma_1^0} J(v) \\
\uparrow & & \uparrow \\
\bigoplus_{\tau\in\Sigma_2^0} S(-r-1) & \longrightarrow & \bigoplus_{v\in\Sigma_1^0} \bigoplus_{\tau\in\Sigma_2^0, v\in\tau} S(-r-1) \\
\uparrow & & \uparrow \\
0 & \longrightarrow & \bigoplus_{v\in\Sigma_1^0} K_v^r
\end{array}$$

The inclusion on the middle row has the effect (in the cokernel) of gluing together copies of $S(-r-1)$ that correspond to codimension one faces which are incident on two rays in Σ_1 and killing copies of $S(-r-1)$ which correspond to not totally interior codimension one faces. Hence the cokernel is $\bigoplus_{\tau\in\Sigma_2^{00}} S(-r-1)$, where Σ_2^{00} denotes totally interior codimension one faces. The tail end of the exact sequence coming from the snake lemma yields that $H_1(\mathcal{J}[\Sigma])$ is $\bigoplus_{\tau\in\Sigma_2^{00}} S(-r-1)$ modulo the image of $\bigoplus_{v\in\Sigma_1} K_v^r$ in $\bigoplus_{\tau\in\Sigma_2^{00}} S(-r-1)$. This is isomorphic to $\bigoplus_{\tau\in\Sigma_2^0} S(-r-1)$ modulo K^r, so we are done. □

Lemma 3.4.4. *Let $\Sigma \subset \mathbb{R}^3$ be a hereditary, complete fan. Define $K^r \subset \bigoplus_{\tau\in\Sigma_2} S(-r-1)e_\tau$ by*

$$K^r = \{\sum_{v\in\tau} a_\tau e_\tau | v \in \Sigma_1, \sum a_\tau l_\tau^{r+1} = 0\}.$$

Also define $V^r \subset \bigoplus_{\tau\in\Sigma_2} S(-r-1)$ by

$$V^r = \{\sum_{\tau\in\Sigma_2} a_\tau e_\tau | \sum a_\tau l_\tau^{r+1} = 0\}.$$

Then $K^r \subset V^r$ and $H_1(\mathcal{J}[\Sigma]) \cong V^r/K^r$ as S-modules.

Proof. Let $K_v^r \subset \bigoplus_{v\in\tau} S(-r-1)e_\tau$ be as in the proof of Lemma 3.4.4. Furthermore, let $J(\nu)$ be the ideal of the central vertex of Σ. Set up the following diagram with exact columns, whose first row is the complex $J[\Sigma]$.

$$
\begin{array}{ccccc}
0 & & 0 & & 0 \\
\uparrow & & \uparrow & & \uparrow \\
\bigoplus_{\tau\in\Sigma_2} J(\tau) & \longrightarrow & \bigoplus_{v\in\Sigma_1} J(v) & \longrightarrow & J(\nu) \\
\uparrow & & \uparrow & & \uparrow \\
\bigoplus_{\tau\in\Sigma_2} S(-r-1) & \longrightarrow & \bigoplus_{\substack{v\in\Sigma_1,\tau\in\Sigma_2 \\ v\in\tau}} S(-r-1) & \longrightarrow & \bigoplus_{\tau\in\Sigma_2} S(-r-1) \\
\uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & \bigoplus_{v\in\Sigma_1} K_v^r & \xrightarrow{\iota} & V^r \\
 & & \uparrow & & \uparrow \\
 & & 0 & & 0
\end{array}
$$

The middle row is exact - this follows from the proof of Lemma 3.4.4. Now the tail end of the long exact sequence in homology yields $H_1(\mathcal{J}[\Sigma]) \cong \text{coker}(\iota)$. The image of $\bigoplus_{v\in\Sigma_1} K_v^r$ under ι is precisely K^r, so we are done. $\square$

Chapter 4

Continuous Splines

In [48, Question 3.3] Schenck raises the following question: If $\mathcal{P} \subset \mathbb{R}^n$ is a shellable polytopal complex, is $C^0(\widehat{\mathcal{P}})$ a free module over the polynomial ring $S = \mathbb{R}[x_0, \ldots, x_n]$? One of our main results in this chapter is Example 4.2.1, which answers this question in the negative. In fact, this example establishes the following theorem.

Theorem 4.0.5. *For a pure, shellable, d-dimensional polytopal complex $\mathcal{P} \subset \mathbb{R}^n$ with $n \geq 2$, freeness of $C^0(\widehat{\mathcal{P}})$ as an S-module depends on the embedding of $\mathcal{P}$ in $\mathbb{R}^n$.*

In § 4.4 we also obtain in Theorem 4.4.3 (1) a nonfreeness criterion for $C^0(\widehat{\mathcal{P}})$, where $\mathcal{P} \subset \mathbb{R}^2$ is an arbitrary polytopal subdivision of a disk and (2) freeness criteria for $C^0(\widehat{\mathcal{P}})$, where $\mathcal{P} \subset \mathbb{R}^2$ is a *generic* polytopal subdivision of a disk. These criteria are in the spirit of [50, Theorem 4.3].

4.1 Preliminaries

Let Δ be an abstract simplicial complex on the vertex set $V = \{1, \ldots, k\}$ and let $H = k[y_1, \ldots, y_k]$. The Stanley-Reisner ideal of Δ is defined by

$$I_\Delta = \langle \prod_{i \in \sigma} y_i | \sigma \subset V \text{ a non-face } \rangle$$

The ring $A_\Delta = H/I_\Delta$ algebraically encodes the combinatorial data of the simplicial complex Δ.

Now let $\Delta \subset \mathbb{R}^n$ be a pure embedded n-dimensional simplicial complex with vertices $\Delta_0 = \{v_1, \ldots, v_k\}$. There are unique piecewise linear functions $X_i : \Delta \to \mathbb{R}$ determined by $X_i(v_j) = \delta_{ij}$, where δ_{ij} is the Kronecker-delta function. These are the Courant functions alluded to in § 1.1. In [12, Theorem 4.2], Billera shows that there is an isomorphism of rings

$$A_\Delta \xrightarrow{\Phi} C^0(\widehat{\Delta})$$

given by $\Phi(y_i) = X_i$. This has the rather remarkable consequence that the

Sections 4.2 and 4.3 of this chapter appear in [18], *Shellability and Freeness of Continuous Splines*, J. Pure Appl. Algebra 216 (2012) 2519-2523. Reprinted with permission.

ring $C^0(\widehat{\Delta})$, in spite of being defined on an embedded simplicial complex, has no geometric content whatsoever. This correspondence also leads to complete answers for the dimensions of the vector spaces $C^r_d(\Delta)$; the Hilbert series of $C^0(\widehat{\Delta})$ is entirely determined by the f-vector of Δ ([10, Corollary 3.17]).

Furthermore, via the Auslander-Buchsbaum formula, $C^0(\widehat{\Delta})$ is free as an $S = \mathbb{R}[x_0, \ldots, x_n]$ module if and only if $C^0(\widehat{\Delta})$ is Cohen-Macaulay as an S-module. Since $C^0(\widehat{\Delta})$ is finite over S, this is equivalent to $C^0(\widehat{\Delta})$ being Cohen-Macaulay as a *ring*. Hence $C^0(\widehat{\Delta})$ is free if and only if A_Δ is Cohen-Macaulay. This is a well-studied property of simplicial complexes. For instance, Reisner's Criterion [39, Theorem 5.53] states that A_Δ being Cohen-Macaulay can be detected from the topology of vertex links. Another way to verify that A_Δ is Cohen-Macaulay is to check that Δ is *shellable* [39, Theorem 13.45], that is to say, that it can be built up sequentially by its facets in a nice way. In the next section we see that the corresponding statement fails for polytopal complexes.

Definition 4.1.1. [64, §8.1] A *shelling* of a pure n-dimensional polytopal complex $\mathcal{P}$ is a linear ordering $P_1, P_2, \ldots P_s$ of the facets of $\mathcal{P}$ such that either $\mathcal{P}$ is zero dimensional or the following conditions are satisfied:

1. The boundary complex ∂P_1 of the first facet P_1 has a shelling.

2. For $1 < j \leq s$ the intersection of the facet P_j with the previous facets is nonempty and is a beginning segment of a shelling of the $(k-1)$-dimensional boundary complex of P_j.

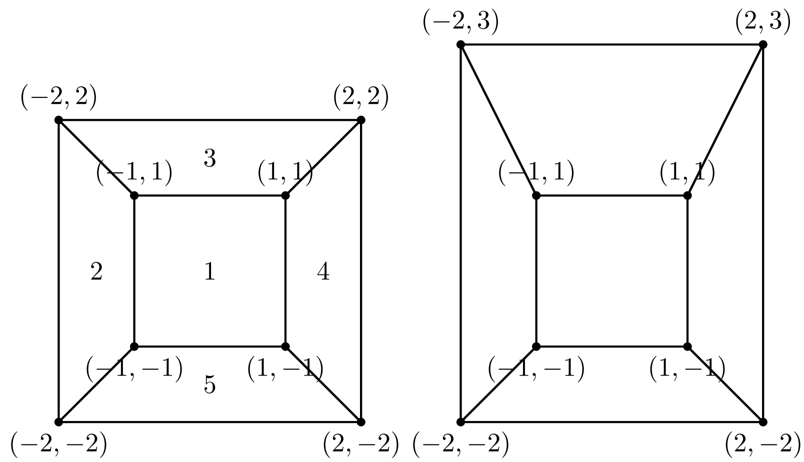
A complex $\mathcal{P}$ is *shellable* if it has a shelling.

4.2 The Counterexample

When Δ is simplicial the isomorphism $A_\Delta \cong C^0(\widehat{\Delta})$ implies that $C^0(\widehat{\Delta})$ is a combinatorial object. This is not true for polytopal complexes: [48, Example 1.1] displays a pair of pure, two dimensional, combinatorially equivalent polytopal complexes $P_1, P_2 \subset \mathbb{R}^2$ such that $\dim_{\mathbb{R}} C^0_1(P_1) \neq \dim_{\mathbb{R}} C^0_1(P_2)$. These complexes are shellable and $C^0(\widehat{P_1})$ and $C^0(\widehat{P_2})$ are both free.

In [63], Yuzvinsky exhibits combinatorially equivalent polytopal complexes $P, Q \subset \mathbb{R}^2$, both pure of dimension two and homotopic to a circle, so that $C^0(\widehat{P})$ is free while $C^0(\widehat{Q})$ is not. However, pure d-complexes with nontrivial singular homology in dimension $< d$ are not shellable. Example 4.2.1 below shows that even for shellable polytopal complexes $\mathcal{P} \subset \mathbb{R}^2$, freeness of $C^0(\widehat{\mathcal{P}})$ may depend on the embedding.

Example 4.2.1. In Figure 4.1 the two complexes $\mathcal{Q}$,$\mathcal{Q}' \subset \mathbb{R}^2$ are two combinatorially equivalent embeddings of a two dimensional complex with $f_2 = 5$, $f^0_1 = 8$, and $f^0_0 = 4$. These are in fact Schlegel diagrams for a cube, obtained by projecting from a vertex just above one of the square faces. In $\mathcal{Q}$ the projection vertex is above the centroid of the square face and in $\mathcal{Q}'$ the projection vertex is not above the centroid. The numbering on the facets of $\mathcal{Q}$ gives a shelling order.



(a) $\mathcal{Q}$ (b) $\mathcal{Q}'$

Figure 4.1

Using the description $C^0(\widehat{\mathcal{P}}) = \ker\phi$ of Lemma 3.3.6 and projecting onto $S^{f_2} = S^5$ we find explicit generators for $C^0(\widehat{\mathcal{Q}}_1)$ and $C^0(\widehat{\mathcal{Q}}_2)$ as submodules of S^5. A free basis for $C^0(\widehat{\mathcal{Q}}_1)$ is given by the columns of the following matrix. Each row corresponds to a facet of $\mathcal{Q}_1$ listed in the same order as the shelling above.

$$\begin{pmatrix} 1 & y+z & (y+z)(x-z) & 0 & 0 \\ 1 & y-x & 2z(x-y) & (x+z)(x-y) & 0 \\ 1 & 2y & 2z(x-y) & (x-y)(z-y) & (x+y)(x-y)(y-z) \\ 1 & x+y & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Now let $l_1 = y - 2x + z$, $l_2 = x + y$, $l_3 = y - x$, $l_4 = y + 2x + z$, $l_5 = y - z$, $l_6 = x - z$, $l_7 = y + z$, and $l_8 = x + z$ (l_i corresponds to the edge e_i in $\mathcal{Q}_2$ above). A minimal set of generators for $C^0(\widehat{\mathcal{Q}}_2)$ is given by the columns of the matrix below.

$$\begin{pmatrix} 1 & 2l_7 & -4l_6l_7 & 0 & 2l_6l_7^2 & 2l_6l_7l_8 \\ 1 & 2l_3 & -8xl_3 & 0 & -4zl_3l_7 & 0 \\ 1 & 3y+z & (3y+z)l_1 & -l_1l_4l_5 & -z(3y+z)l_1 & -zl_1l_4 \\ 1 & 2l_2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

If an S-module M is free on 6 generators it is readily seen that the leading coefficient of the Hilbert polynomial $HP(M, k)$ is 3. However it follows from [11, Theorem 4.5] that the leading coefficient of $HP(C^0(\widehat{\mathcal{Q}}_2), k)$ is $f/2 = 5/2$.

Hence $C^0(\widehat{\mathcal{Q}}_2)$ is not free. In Theorem 4.3.3 we prove the claims in this example without the use of a computer algebra system.

Let $\mathcal{P}^n$ denote the nth iterate of the coning construction over $\mathcal{P}$, starting with $\mathcal{P}^1 = \widehat{\mathcal{P}}$. The complexes $\mathcal{Q}_1^n, \mathcal{Q}_2^n \subset \mathbb{R}^{n+2}$ for $n \geq 1$ are combinatorially equivalent and shellable since coning preserves shellability. If $C^0(\mathcal{P})$ is graded, Theorem 6.3 of [12] shows that freeness of $C^0(\widehat{\mathcal{P}})$ is equivalent to freeness of $C^0(\mathcal{P})$. Since both $C^0(\widehat{\mathcal{Q}}_1)$ and $C^0(\widehat{\mathcal{Q}}_2)$ are graded, it follows that $C^0(\mathcal{Q}_1^n)$ is free and $C^0(\mathcal{Q}_2^n)$ is not free for $n \geq 1$. This establishes Theorem 4.0.5.

4.3 Non-freeness and Ext

We now indicate more precisely where the nonfreeness of $C^0(\widehat{\mathcal{Q}}')$ arises. Note that by the exact sequence of Lemma 3.3.6, $C^0(\widehat{\mathcal{P}})$ is a second syzygy module over $S = \mathbb{R}[x, y, z]$ hence can have projective dimension at most 1 by the Hilbert Syzygy Theorem. So $C^0(\widehat{\mathcal{P}})$ is free iff $Ext^1_S(C^0(\widehat{\mathcal{P}}), S) = 0$. We refer the reader to § 2.5 for the basics on Ext modules.

Our strategy is to express $Ext^1_S(C^0(\widehat{\mathcal{P}}), S)$ as an Ext^3 of a simpler module with an explicit presentation given in [50]. This analysis is then applied to the complex $\mathcal{Q}_2$ from Example 4.2.1 to show that $Ext^1_S(C^0(\widehat{\mathcal{Q}}'), S) \neq 0$ and hence $C^0(\widehat{\mathcal{Q}}')$ is not free. To do this we use the spline matrix and the spline complex from Chapter 3.

Write $\mathcal{J}, \mathcal{R}, \mathcal{R}/\mathcal{J}$ for the complexes $\mathcal{J}[\widehat{\mathcal{P}}, \partial\widehat{\mathcal{P}}], \mathcal{R}[\widehat{\mathcal{P}}, \partial\widehat{\mathcal{P}}], \mathcal{R}/\mathcal{J}[\widehat{\mathcal{P}}, \partial\widehat{\mathcal{P}}]$.

Proposition 4.3.1. *Suppose that $\mathcal{P} \subset \mathbb{R}^2$ satisfies that $|\mathcal{P}|$ is homeomorphic to a disk. Then*

1. $C^0(\widehat{\mathcal{P}}) \cong S \oplus H_2(\mathcal{J})$
2. $H_2(\mathcal{J}/\mathcal{R}) \cong H_1(\mathcal{J})$
3. $Ext^1_S(C^0(\widehat{\mathcal{P}}), S) \cong Ext^3_S(H_1(\mathcal{J}), S)$

Proof. (1) and (2) follow from Lemma 3.4.1. For (1), we have the short exact sequence

$$0 \to S \to H_2(\mathcal{R}/\mathcal{J}) \to H_1(\mathcal{J}) \to 0$$

coming from the beginning of the long exact sequence for $0 \to \mathcal{J} \to \mathcal{R} \to \mathcal{R}/\mathcal{J} \to 0$.

For (3), we establish that $Ext^1_S(C^0(\widehat{\mathcal{P}}), S) \cong Ext^3_S(H_2(\mathcal{R}/\mathcal{J}), S)$. Then we are done by (2). Let $M = \text{coker}\phi$ where ϕ is the matrix from Lemma 3.3.6. Since $C^0(\widehat{\mathcal{P}})$ is a second syzygy module for M, $Ext^3_S(M, S) = Ext^1_S(C^0(\widehat{\mathcal{P}}), S)$. The following short exact sequence relates M and $H_2(\mathcal{R}/\mathcal{J})$ by Lemma 3.4.2:

$$0 \to H_2(\mathcal{R}/\mathcal{J}) \to M \to \bigoplus_{v \in \widehat{\mathcal{P}}_1^0} S/J(v) \to 0.$$

From the long exact sequence in Ext and the fact that $Ext^3_S(S/J(v), S) = 0$ we obtain $Ext^3_S(M, S) \cong Ext^3_S(H_2(\mathcal{R}/\mathcal{J}), S)$. □

We make the presentation from Lemma 3.4.4 for $H_2(\mathcal{R}/\mathcal{J})$ more explicit in the case $r = 0$. If d edges are incident at a vertex v, the modules $K^0_v \cong \{\sum_{v\in\tau} a_\tau e_\tau | \sum a_\tau l_\tau = 0\} \subset \bigoplus_{v\in\tau} S(-r-1)e_\tau$ are generated by $d-2$ relations of degree zero and a single relation of degree one. Let η denote the number of $\tau \in \mathcal{P}^0_1$ which are not totally interior.

Corollary 4.3.2. *Order the edges of $\mathcal{P}$ so that those which are not totally interior occur last. The S-module $H_1(\mathcal{J})$ has presentation*

$$S^h \xrightarrow{N} S^{e^0} \to H_1(\mathcal{J}) \to 0,$$

where h is the number of columns of N and N has block decomposition

$$N = \begin{pmatrix} B_0 & \Big| & B_1 & \Big| & M \end{pmatrix}.$$

The columns of B_0 run over syzygies of degree zero and the columns of B_1 run over syzygies of degree one at each $v \in \mathcal{P}^0_0$, and $M = \begin{pmatrix} 0 \\ id_\eta \end{pmatrix}$, where id_η is the $\eta \times \eta$ identity matrix. Pruning out id_η gives the more compact presentation

$$S^{h-\eta} \xrightarrow{N'} S^{e^0-\eta} \to H_1(\mathcal{J}) \to 0$$

where $N' = (B'_0|B'_1)$ is obtained from N by deleting M and the rows corresponding to e_τ with τ not totally interior.

We illustrate this corollary by applying it to a generic embedding $\mathcal{Q}_g \subset \mathbb{R}^2$ of $\mathcal{Q}$ (preserving convexity), where $\mathcal{Q}$ is the complex from Example 4.2.1. Referring

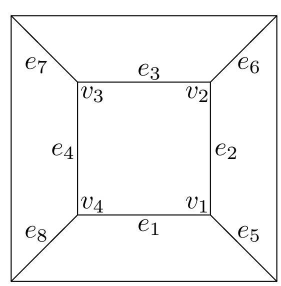


Figure 4.2: $\mathcal{Q}_g$

to Figure 4.2, let l_i be the linear form vanishing on $\widehat{e_i}$. For each interior vertex v, K^0_v is generated by a degree zero syzygy and a degree one syzygy. Let the degree zero syzygies at v_1, v_2, v_3, and v_4 respectively be $a_1e_1+b_1e_2+c_1e_5, a_2e_2+b_2e_3+c_2e_6, a_3e_3+b_3e_4+c_3e_7$, and $a_4e_4+b_4e_1+c_4e_8$ for some constants $a_i, b_i, c_i \in \mathbb{R}$. Similarly choose linear syzygies $l_5e_1-l_1e_5, l_6e_2-l_2e_6, l_7e_3-l_3e_7$, and $l_8e_4-l_4e_8$

at each interior vertex. Using $id_\eta = id_4$ to eliminate the rows corresponding to the not totally interior edges e_5, e_6, e_7, e_8 gives the presentation below, where the ith row of N' corresponds to e_i and the ith columns of B_0' and B_1' correspond to v_i.

$$S^8 \xrightarrow{N'} S^4 \to H_1(\mathcal{J}[\widehat{\mathcal{Q}}_g, \partial\widehat{\mathcal{Q}}_g]) \to 0$$

$$N' = (B_0'|B_1') = \left(\begin{array}{cccc|cccc} a_1 & 0 & 0 & b_4 & l_5 & 0 & 0 & 0 \\ b_1 & a_2 & 0 & 0 & 0 & l_6 & 0 & 0 \\ 0 & b_2 & a_3 & 0 & 0 & 0 & l_7 & 0 \\ 0 & 0 & b_3 & a_4 & 0 & 0 & 0 & l_8 \end{array}\right)$$

We use this matrix to establish the following theorem.

Theorem 4.3.3. *Let $\mathcal{Q}, \mathcal{Q}'$ be the polytopal complexes from Example 4.2.1. Then*

1. *$C^0(\widehat{\mathcal{Q}})$ is free.*
2. *$C^0(\widehat{\mathcal{Q}}')$ is not free.*
3. *If $\mathcal{Q}_g$ is a generic embedding of $\mathcal{Q}$ (preserving convexity), then $C^0(\widehat{\mathcal{Q}}_g)$ is free.*

Proof. We use the matrix $N' = (B_0'|B_1')$ defined above, which presents $H_1(\mathcal{J}[\widehat{\mathcal{Q}}, \partial\widehat{\mathcal{Q}}])$ for generic embeddings of $\mathcal{Q}$. Clearly rank $(B_0') \geq 3$ since a_i, b_i, c_i are all nonzero (otherwise the l_i would not define distinct lines). If rank$(B_0') = 4$ then $H_1(\mathcal{J}) = 0$. If rank$(B_0') = 3$ then the presentation above can be simplified to $S^4 \xrightarrow{N''} S \to H_1(\mathcal{J}) \to 0$, where $N'' = \begin{pmatrix} l_5 & l_6 & l_7 & l_8 \end{pmatrix}$. So in the rank 3 case $H_1(\mathcal{J}) \cong S/(l_5, l_6, l_7, l_8)$. It is easy to check that both $\mathcal{Q}$ and $\mathcal{Q}'$ from Example 4.2.1 fall into the rank 3 case (this is because $\mathcal{Q}$ and $\mathcal{Q}'$ both have a single nontrivial piecewise linear function).

(1) In the case of $\mathcal{Q}$, $l_5 = l_7 = x + y$ and $l_6 = l_8 = x - y$. Hence $H_1(\mathcal{J}[\widehat{\mathcal{Q}}, \partial\widehat{\mathcal{Q}}]) \cong S/(x, y)$. But $Ext^3_S(S/(x, y), S) = 0$ so $C^0(\widehat{\mathcal{Q}})$ is free by Proposition 4.3.1 since $Ext^i_S(C^0(\widehat{\mathcal{Q}}), S) = 0$ for $i > 1$.

(2) In the case of $\mathcal{Q}'$, $l_5 = y+x, l_6 = y-2x+z, l_7 = y+2x-z$, and $l_8 = y-x$. So $H_1(\mathcal{J}[\widehat{\mathcal{Q}}', \partial\widehat{\mathcal{Q}}']) \cong S/(x, y, z)$ and $Ext^3_S(S/(x, y, z), S) = \mathbb{R} \neq 0$. Proposition 4.3.1 then implies $C^0(\widehat{\mathcal{Q}}')$ is not free.

(3) If det$B_0' \neq 0$, then $H_1(\mathcal{J}) = 0$ and $C^0(\widehat{\mathcal{Q}})$ is free by Proposition 4.3.1. This is the appropriate notion of *generic* for this particular complex. □

Theorem 4.3.3 confirms the claims in Example 4.2.1 and gives a more suggestive reason for the nonfreeness of $C^0(\widehat{\mathcal{Q}}')$.

4.4 Non-freeness Conditions via Local Cohomology

In this section we use local cohomology to present a more general analysis; Example 4.4.4 shows that we can use this analysis to recover parts (2) and (3) of Theorem 4.3.3. The main point of using local cohomology is the following theorem (essentially the same as [50, Theorem 4.1]). We refer the reader to § 2.5 for a rough sketch of the basics of local cohomology.

Theorem 4.4.1. *$C^0(\widehat{\mathcal{P}})$ is free as an $S = \mathbb{R}[x, y, z]$ module iff $H^0_m(H_1(\mathcal{J})) = 0$, where $m = (x, y, z)$ is the homogeneous maximal ideal.*

Proof. $C^0(\widehat{\mathcal{P}})$ is free iff $Ext^1_S(C^0(\widehat{\mathcal{P}}), S) = 0$. By part (3) of Proposition 4.3.1, $Ext^1_S(C^0(\widehat{\mathcal{P}}), S) \cong Ext^3(H_1(\mathcal{J}), S)$. Now the result follows by local duality [23, Theorem A1.9]. □

In order to give the more general analysis promised, we first need to review the computation of the Hilbert polynomial $HP(C^0(\widehat{\mathcal{P}}), d)$, which can be found in [48]. The computation in full generality is also given in Chapter 7, but we do not need this level of generality. We assume $\mathcal{P}$ is a polytopal subdivision of a topological disk, to eliminate any contributions from the topology of $\mathcal{P}$. From an Euler characteristic computation we have

$$HP(C^0(\widehat{\mathcal{P}}), d) = \sum_{i=0}^{2} (-1)^i HP\left(\bigoplus_{\gamma \in \widehat{\mathcal{P}}^0_{3-i}} \frac{S}{J(\gamma)}, d \right) + HP(H_2(\mathcal{R}/\mathcal{J}), d).$$

It is straightforward to show that

$$\sum_{i=0}^{3} (-1)^i HP\left(\bigoplus_{\gamma \in \widehat{\mathcal{P}}^0_{3-i}} \frac{S}{J(\gamma)}, d \right) = f_2 \binom{d+2}{2} - f^0_1 \binom{d+1}{1} + f^0_0. \qquad (4.1)$$

If $\Delta \subset \mathbb{R}^2$ is a triangulation of a topological disk and $r = 0$, then $H_2(\mathcal{R}/\mathcal{J}) = 0$. This is not true for polytopal complexes; however by Theorem 7.2.4 (see also [48, Lemma 2.3]), $H_2(\mathcal{R}/\mathcal{J})$ has codimension ≥ 2 and is supported at certain points in $\mathbb{P}^2_{\mathbb{R}}$ (actually $\mathbb{A}^2_{\mathbb{R}}$ as we will see). It follows that $HP(H_2(\mathcal{R}/\mathcal{J}), d)$ is a constant, and this constant is given by counting the points in $\mathbb{P}^2_{\mathbb{R}}$ at which $H_2(\mathcal{R}/\mathcal{J})$ is supported, possibly with some multiplicities. To count these points of support we review some results and notation from [48].

Let $\mathcal{P} \subset \mathbb{R}^2$ be a polytopal complex, and ξ a point which lies at the intersection of at least 2 affine spans of interior edges $\tau \in \mathcal{P}^0_1$. Associate a graph $G_\xi(\mathcal{P})$ to ξ as follows: the vertices of $G_\xi(\mathcal{P})$ correspond to facets $\sigma \subset \mathcal{P}_2$ having an edge τ so that $\xi \in \text{aff}(\tau)$. Two vertices v_1, v_2 are connected in $G_\xi(\mathcal{P})$ iff the corresponding facets σ_1, σ_2 intersect in an edge τ so that $\xi \in \text{aff}(\tau)$. The connected components of $G_\xi(\mathcal{P})$ are homotopic to segments or circles [38, Corollary 3.9]. Each connected component $G^i_\xi(\mathcal{P})$ of $G_\xi(\mathcal{P})$ is the dual graph of a unique subcomplex $\mathcal{P}^i_\xi$. Let $\mathcal{P}_\xi$ be the disjoint union of the $\mathcal{P}^i_\xi$. We generalize this construction in Chapter 5 and a more general version of the graph above is found in Chapter 7.

Let $I(\xi)$ be the ideal of the point ξ. It is a fact that the localization $H_2(\mathcal{R}/\mathcal{J})_{I(\xi)}$ is nonzero iff $G_\xi(\mathcal{P})$ has a loop ([48, Theorem 2.6]). Let $\alpha_\xi(\mathcal{P})$ be the number of cycles of $G_\xi(\mathcal{P})$ which do not correspond to loops around vertices. Set $\alpha_1(\mathcal{P}) = \sum_\xi \alpha_\xi(\mathcal{P})$, where the sum runs over points ξ which appear in the intersection lattice of affine spans of edges. We have the following results from [48].

Theorem 4.4.2. *Let $\mathcal{P} \subset \mathbb{R}^2$ be a hereditary, pure, 2-dimensional polytopal complex, and $\alpha_1(\mathcal{P})$ be as above. Then*

1. $HP(H_2(\mathcal{R}/\mathcal{J})) = \alpha_1(\mathcal{P})$
2. $HP(C^0(\widehat{\mathcal{P}}), d) = f_2\binom{d+2}{2} - f_1^0\binom{d+1}{1} + f_0^0 + \alpha_1(\mathcal{P})$

Proof. (1) is a special case of [48, Corollary 2.7]. (2) is [48, Corollary 2.9]. □

The following theorem is a polytopal analogue (for $r = 0$) of [50, Theorem 4.3].

Theorem 4.4.3. *Let $\mathcal{P} \subset \mathbb{R}^2$ be a polytopal subdivision of a topological disk in $\mathbb{R}^2$, with $\alpha_1(\mathcal{P})$ as above. Then*

1. *If $2f_0^0 - f_1^0 + (\dim_{\mathbb{R}} C^0_1(\mathcal{P}) - 3) > \alpha_1(\mathcal{P})$, then $C^0(\widehat{\mathcal{P}})$ is not free.*
2. *If $\alpha_1(\mathcal{P}) = 0$, then $C^0(\widehat{\mathcal{P}})$ is free iff $2f_0^0 - f_1^0 + (\dim_{\mathbb{R}} C^0_1(\mathcal{P}) - 3) = 0$.*

Proof. By Proposition 2.5.4,

$$HP(H_1(\mathcal{J}), d) = \dim H_1(\mathcal{J})_d - \dim H^0_m(H_1(\mathcal{J}))_d + \dim H^1_m(H_1(\mathcal{J}))_d \quad (4.2)$$

Recall we have the tautological exact sequence

$$0 \to H_2(\mathcal{J}) \to \bigoplus_{\tau \in \widehat{\mathcal{P}}^0_2} J(\tau) \to \bigoplus_{v \in \widehat{\mathcal{P}}^0_1} J(v) \to H_1(\mathcal{J}) \to 0.$$

Also, by part (1) of Proposition 4.3.1, $S \oplus H_2(\mathcal{J}) \cong C^0(\widehat{\mathcal{P}})$. So taking the tautological exact sequence above in degree 1 yields

$$\dim H_2(\mathcal{J})_1 = 2f_0^0 - f_1^0 + (\dim C^0_1(\widehat{\mathcal{P}}) - 3).$$

Putting this together with equation (4.2) in degree $d = 1$ yields

$$\begin{aligned}\alpha_1(\mathcal{P}) = 2f_0^0 - f_1^0 &+ (\dim C^0_1(\widehat{\mathcal{P}}) - 3) \\ &- \dim H^0_m(H_1(\mathcal{J}))_1 + \dim H^1_m(H_1(\mathcal{J}))_1 \quad (4.3)\end{aligned}$$

(1) By Theorem 4.4.1, we must show that if $2f_0^0 - f_1^0 + (\dim_{\mathbb{R}} C_1^0(\mathcal{P}) - 3) > \alpha_1(\mathcal{P})$, then $H_m^0(H_1(\mathcal{J})) \neq 0$. This follows directly from equation (4.3). (2) If $\alpha_1(\mathcal{P}) = 0$ then $\dim H_1(\mathcal{J}) = 0$, hence $H_m^0(H_1(\mathcal{J})) = H_1(\mathcal{J})$ and $H_m^1(H_1(\mathcal{J})) = 0$. Equation (4.3) becomes

$$2f_0^0 - f_1^0 + (\dim C_1^0(\widehat{\mathcal{P}}) - 3) = \dim H_1(\mathcal{J})_1$$

By Theorem 4.4.1, since $H_1(\mathcal{J})$ has finite length, $C^0(\widehat{\mathcal{P}})$ is free iff $H_1(\mathcal{J}) = 0$. By the presentation in Lemma 3.4.4, $H_1(\mathcal{J})$ is generated in degree 1, so $H_1(\mathcal{J}) = 0$ iff $H_1(\mathcal{J})_1 = 0$. Now the result follows from the above equation. □

Example 4.4.4. We revisit the polytopal complexes $\mathcal{Q}$ and $\mathcal{Q}'$ from Example 4.2.1. These both have $f_0^0 = 4$ and $f_1^0 = 8$. There is at least one nontrivial piecewise linear function on both $\mathcal{Q}$ and $\mathcal{Q}'$; the graph of this function is a deformed version of the cube and is shown for $\mathcal{Q}'$ in Figure 4.3a. It is easy to see that $H_1(\mathcal{J}[\widehat{\mathcal{Q}}, \partial\widehat{\mathcal{Q}}])$ is supported at only the origin, and $\alpha_1(\mathcal{Q}) = 1$. The cycle corresponding to this point of support for $H_1(\mathcal{J}[\widehat{\mathcal{Q}}, \partial\widehat{\mathcal{Q}}])$ is shown in Figure 4.3b. Hence

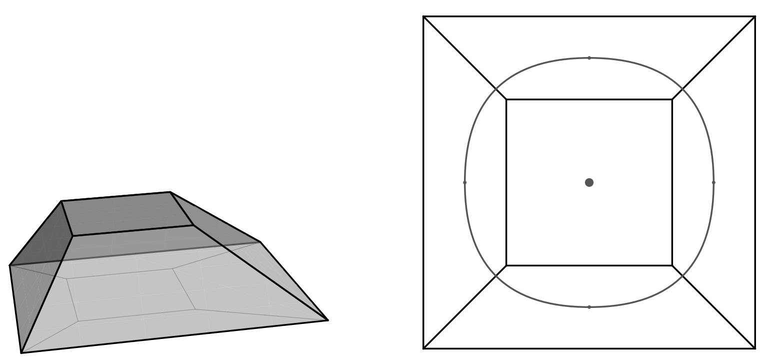
$$2f_0^0 - f_1^0 + (\dim_{\mathbb{R}} C_1^0(\mathcal{Q}) - 3) = 0 + 1 = \alpha_1(\mathcal{Q}).$$

Theorem 4.4.3 does *not* apply in this situation, however Theorem 4.3.3 shows that $C^0(\mathcal{Q})$ is free. For $\mathcal{Q}'$ we have

$$2f_0^0 - f_1^0 + (\dim_{\mathbb{R}} C_1^0(\mathcal{Q}') - 3) = 0 + 1 > 0 = \alpha_1(\mathcal{Q}'),$$

hence $C^0(\mathcal{Q}')$ is *not* free by Theorem 4.4.3.

In fact,a little work shows that generic embeddings $\mathcal{Q}_g$ (preserving convexity) of $\mathcal{Q}$ have a nontrivial piecewise linear function iff the rank of the matrix B_0' from Theorem 4.3.3 is 3. Moreover, the linear forms corresponding to interior edges of $\mathcal{Q}_g$ which meet the boundary span the entire maximal ideal $m = (x, y, z)$ iff $\alpha_1(\mathcal{Q}_g) = 0$. These observations, coupled with Theorem 4.4.3, recover part (3) of Theorem 4.3.3.



(a) PL Function on $\mathcal{Q}'$ (b) Support of $H_1(\mathcal{J}[\widehat{\mathcal{Q}}, \partial\widehat{\mathcal{Q}}])$

Figure 4.3

Example 4.4.5. Consider the polytopal complex $\mathcal{T}$ in Figure 4.4b, which is the Schlegel diagram of the truncated cube in Figure 4.4a. $\mathcal{T}$ is the image of the truncated cube under a projection centered at a point just above the centroid of one of the octagonal faces. It has $f_0^0 = 16, f_1^0 = 28, f_2 = 13$. Hence $2f_0^0 - f_1^0 = 4$. There is at least one nontrivial piecewise linear function on $\mathcal{T}$; the graph of this function is a deformed version of the truncated cube and is shown in Figure 4.4c. This is in fact the only nontrivial piecewise linear function on $\mathcal{T}$. So $\dim C_1^0(\mathcal{T}) - 3 = 1$. We also have $\alpha_1(\mathcal{T}) = 5$; the points of support of $H_1(\mathcal{J}[\mathcal{T}, \partial\mathcal{T}])$ and their corresponding cycles are shown in Figure 4.4d. Hence

$$2f_0^0 - f_1^0 + (\dim_{\mathbb{R}} C_1^0(\mathcal{T}) - 3) = 4 + 1 = \alpha_1(\mathcal{T}).$$

Theorem 4.4.3 does *not* apply to this situation. However, a computation in Macaulay2 yields that $C^0(\widehat{\mathcal{T}})$ is in fact free.

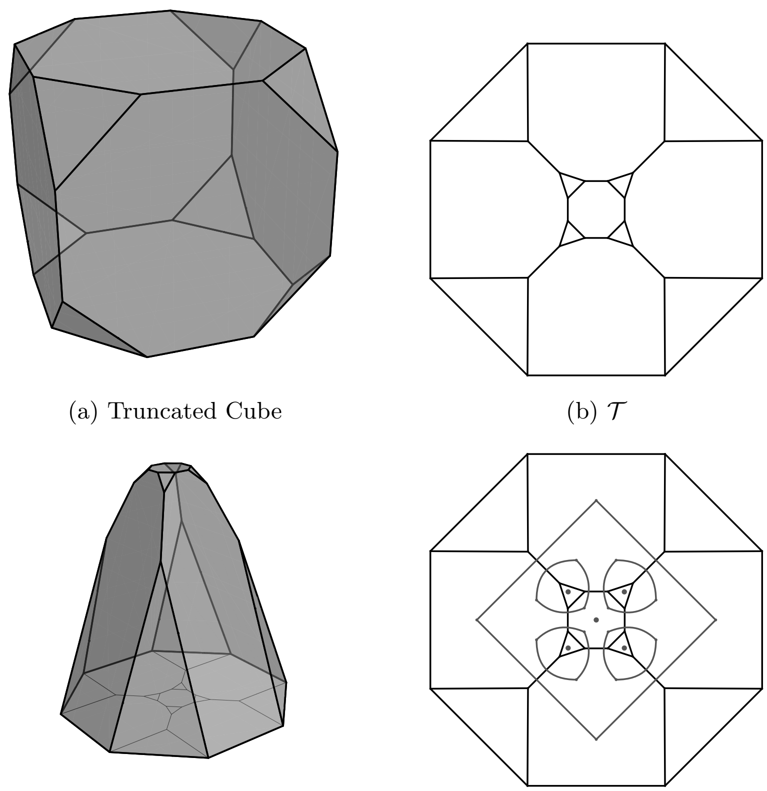
Now let $\mathcal{T}'$ be a polytopal complex obtained by a slight perturbation of a single interior vertex of $\mathcal{T}$, placing it outside of the affine span of interior edges of $\mathcal{T}$. This perturbation destroys two of the cycles which contribute to $\alpha_1(\mathcal{T}')$, as well as the nontrivial piecewise linear function. Hence

$$2f_0^0 - f_1^0 + (\dim_{\mathbb{R}} C_1^0(\widehat{\mathcal{T}}') - 3) \geq 4 > 3 = \alpha_1(\mathcal{T}')$$

and $C^0(\widehat{\mathcal{T}}')$ is *not* free by Theorem 4.4.3. For generic perturbations $\mathcal{T}_g$ of $\mathcal{T}$, $\alpha_1(\mathcal{T}_g) = 0$ and $C^0(\widehat{\mathcal{T}_g})$ is again *not* free by Theorem 4.4.3.

Remark 4.4.6. It is not difficult to prove extensions of Theorem 4.4.3 for $C^r(\mathcal{P})$, where $r > 0$. However the statement is much simpler for $r = 0$.

Remark 4.4.7. In [50, Corollary 4.5], Schenck-Stillman build on results of Billera-Whiteley to show that $C^1(\widehat{\Delta})$ is free for $\Delta \subset \mathbb{R}^2$ a generically embedded triangulation of a 2-ball. Example 4.4.5 shows that this is not even true in the continuous case for generic embeddings (preserving convexity) of a decomposition of a 2-ball by convex polytopes.



(a) Truncated Cube

(b) $\mathcal{T}$

(c) PL Function on $\mathcal{T}$

(d) Support of $H_1(\mathcal{J}[\mathcal{T}, \partial\mathcal{T}])$

Figure 4.4

Chapter 5

Lattice-Supported Splines

For a polytopal complex $\mathcal{P} \subset \mathbb{R}^n$, dual graphs are used to study the spline module $C^r(\mathcal{P})$ in [45, 46]. In [38] dual graphs are used to analyze associated primes of the complex $\mathcal{R}/\mathcal{J}$. In this chapter we use dual graphs associated to certain linear subspaces $W \subset \mathbb{R}^n$ (a slight generalization of the construction in [38, Definition 3.6]) to build certain subcomplexes $\mathcal{P}_W \subset \mathcal{P}$. These subcomplexes reduce to unions of stars of faces if $\mathcal{P} = \Delta$ is simplicial, and we use them in § 5.3 to understand the localization $C^r(\mathcal{P})_P$ with respect to a prime $P \subset \mathbb{R}[x_1, \ldots, x_n]$. Then in § 5.4 we use the subcomplexes $\mathcal{P}_W$ to construct submodules $LS^{r,k}(\mathcal{P}) \subset \mathcal{P}$ with the property that $LS^{r,k}(\mathcal{P})_P = C^r(\mathcal{P})_P$ for all primes $P \subset R$ of codimension at most k (Theorem 5.4.3). As a consequence we derive Theorem 5.4.6, that $C^r_d(\mathcal{P})$ has a *lattice-supported* basis for $d \gg 0$. Such a basis is supported on complexes of the form $\mathcal{P}_W$ and is in some sense the best one can do to find a *locally supported basis* for $C^r_d(\mathcal{P})$. We begin by illustrating with a couple of examples.

5.1 Example of locally supported splines

Consider the two dimensional polytopal complex $\mathcal{Q}$ in Figure 5.1 with 5 faces, 8 interior edges, and 4 interior vertices.

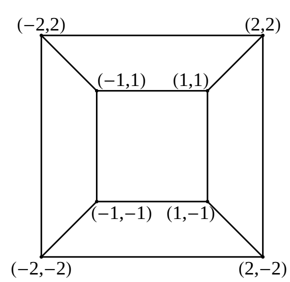


Figure 5.1: $\mathcal{Q}$

It is readily verifiable that the constant function $\mathbf{1} \in C^0(\mathcal{Q})$ cannot be written as a sum of splines which are supported on the stars of the 4 interior vertices, i.e. splines which restrict to 0 outside of the shaded regions in Figure 5.2. How-

The content of this chapter appears in [20], *Lattice-Supported Splines on Polytopal Complexes*, Adv. in Appl. Math. 55 (2014) 1-21. Reprinted with permission.

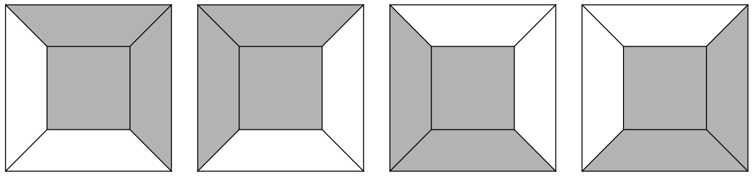


Figure 5.2: Stars of Interior Vertices of $\mathcal{Q}$

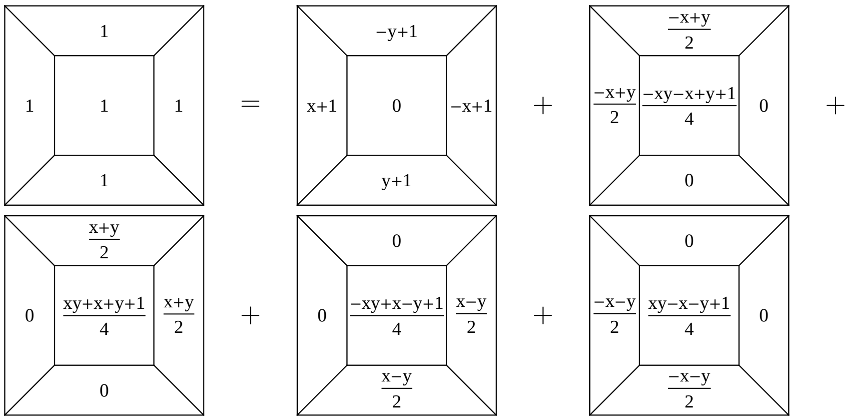


Figure 5.3: A 'local' decomposition of $\mathbf{1} \in C^2_0(\mathcal{Q})$

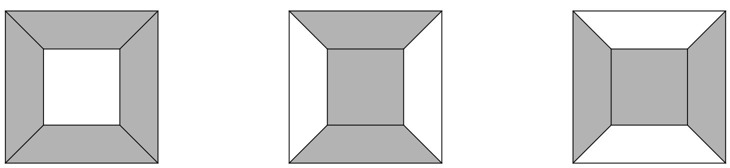


Figure 5.4: Non-Star Lattice Complexes of $\mathcal{Q}$

ever, a decomposition of $\mathbf{1} \in C^0_2(\mathcal{Q})$ computed in Macaulay2 [30] is shown in Figure 5.3.

The support of the first spline in the sum depicted in Figure 5.3 is the annular subcomplex in Figure 5.4. According to Theorem 5.4.6, $C^r_d(\mathcal{Q})$ has a basis of splines supported on either the star of an interior vertex or one of the shaded complexes in Figure 5.4, for $d \gg 0$ (See Example 5.4.9). Such complexes are an example of *lattice complexes*, explained in §5.2.

It is the presence of such an annular subcomplex in Figure 5.4 that contributes to the constant term of the dimension formula $\dim_{\mathbb{R}} C^r_d(\mathcal{Q})$ for $d \gg 0$ provided in [38]. So the lattice complexes described in §5.2 encode subtle interactions between the geometry and combinatorics of $\mathcal{P}$ that are manifested in $C^r(\mathcal{P})$.

If we disturb the symmetry of $\mathcal{Q}$ slightly to get $\mathcal{Q}'$ as in Figure 5.5, then the affine spans of the four edges connecting the inner and outer squares do not all intersect at the same point. By Theorem 5.4.6, $C^r_d(\mathcal{Q}')$ has a basis of splines with support in either the star of an interior vertex or one of the shaded complexes in Figure 5.6, for $d \gg 0$ (see Example 5.4.9).

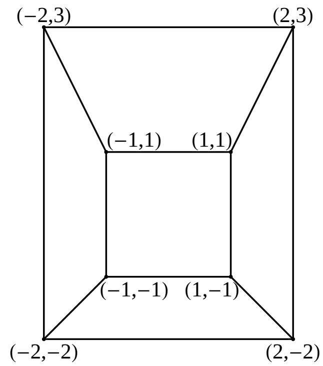


Figure 5.5: $\mathcal{Q}'$

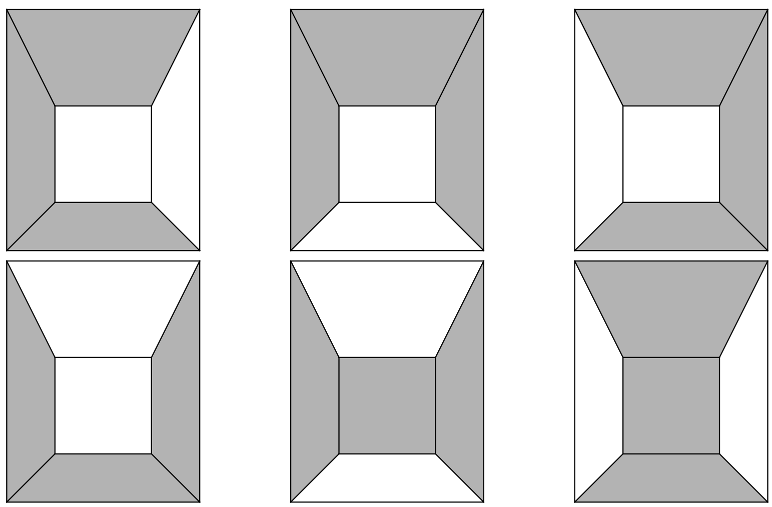


Figure 5.6: Non-Star Lattice Complexes of $\mathcal{Q}'$

5.2 Lattice Complexes

Throughout this chapter, $\mathcal{P} \subset \mathbb{R}^n$ is assumed to be a pure, n-dimensional, hereditary polytopal complex.

Let $R = \mathbb{R}[x_1, \ldots, x_n]$ be the polynomial ring in n variables. For a subset $S \subset \mathbb{R}^n$, let $I(S) \subset R$ denote the ideal of polynomials vanishing on S. If $\tau \in \mathcal{P}_{n-1}$ then l_τ denotes any linear form generating the principal ideal $I(\tau)$.

Recall In what follows we use a subgraph $G_J(\mathcal{P})$ of $G(\mathcal{P})$ determined by an ideal $J \subset R$. This is a slight generalization of a graph used by McDonald and Schenck in [38] and builds on the dual graphs of Rose [45, 46].

Definition 5.2.1. Let $\mathcal{P} \subset \mathbb{R}^n$ be a polytopal complex, and $J \subset R$ an ideal. The vertices of the graph $G_J(\mathcal{P})$ correspond to facets of $\mathcal{P}$ having a codimension one face τ such that $l_\tau \in J$. Two vertices corresponding to facets σ_1 and σ_2 which intersect along the edge τ are connected in $G_J(\mathcal{P})$ if $l_\tau \in J$.

Let $G_J(\mathcal{P})$ be the union of k connected components $G^1_J(\mathcal{P}), \ldots, G^k_J(\mathcal{P})$. There is a unique subcomplex $\mathcal{P}^i_J$ of $\mathcal{P}$ whose dual graph is $G^i_J(\mathcal{P})$.

Definition 5.2.2. With notation as above, define $\mathcal{P}_J$ to be the **disjoint** union of $\mathcal{P}_J^1, \ldots, \mathcal{P}_J^k$.

We call the $\mathcal{P}_J^i$ the connected components, or simply components, of $\mathcal{P}_J$, although two components may share codimension one faces within $\mathcal{P}$ as will be apparent in Example 5.2.5. There are only finitely many distinct complexes $\mathcal{P}_J$ associated to $J \subset R$. They are in bijection with the nontrivial elements of the intersection poset of a certain hyperplane arrangement, which we now describe.

Recall that a hyperplane arrangement $\mathcal{H} \subset \mathbb{R}^n$ is a set $\mathcal{H} = \{H_1, \ldots, H_k\}$ of hyperplanes. The *intersection poset* $L(\mathcal{H})$ of $\mathcal{H}$ includes the whole space, the hyperplanes H_i, and nonempty intersections of these hyperplanes (called *flats*) ordered with respect to reverse inclusion. $L(\mathcal{H})$ is a *ranked* poset with rank function $\mathrm{rk}(W) = \mathrm{codim}(W)$ for $W \in L(\mathcal{H})$. $L(\mathcal{H})$ is a *meet* semilattice and is a lattice iff $\mathcal{H}$ is *central*, that is, iff $\cap_i H_i \neq \emptyset$.

Definition 5.2.3. Let $\mathcal{P} \subset \mathbb{R}^n$ be a polytopal complex.

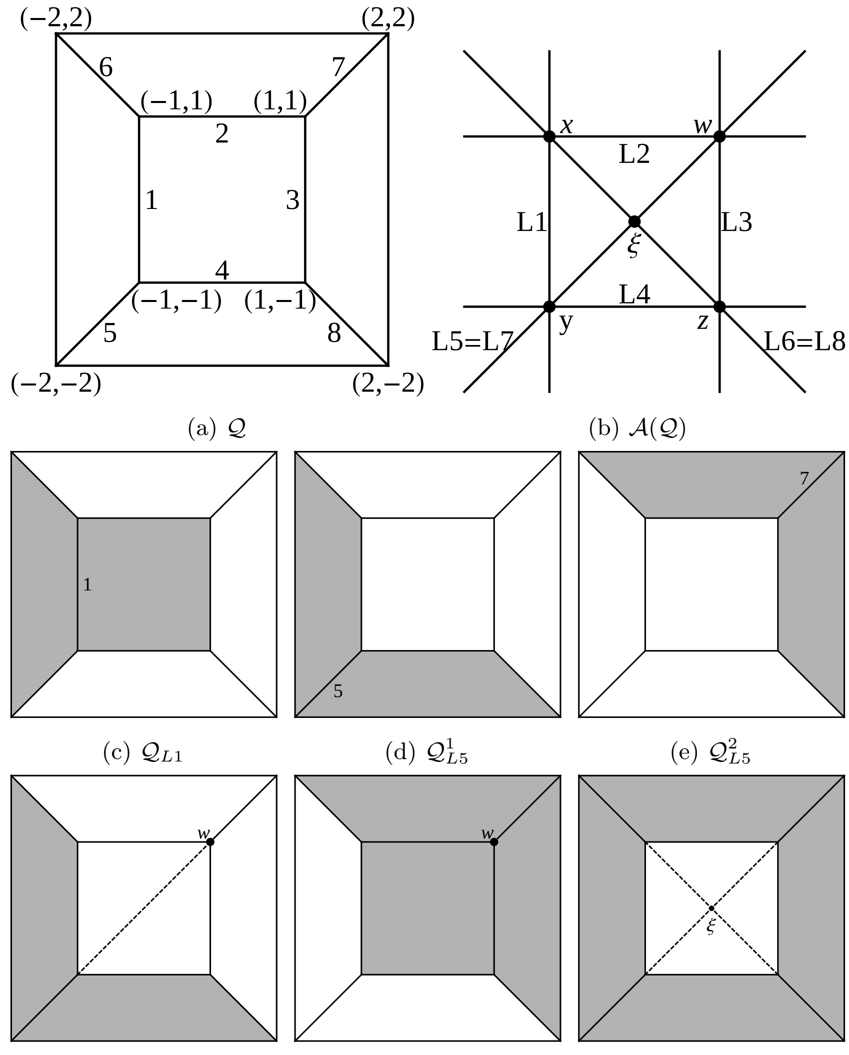
1. For $\tau \in \mathcal{P}$ a face, $\mathrm{aff}(\tau)$ denotes the linear (or affine) span of τ.
2. $\mathcal{A}(\mathcal{P})$ denotes the hyperplane arrangement $\cup_{\tau \in \mathcal{P}^0_{n-1}} \mathrm{aff}(\tau)$.
3. $L_{\mathcal{P}}$ denotes the intersection semi-lattice $L(\mathcal{A}(\mathcal{P}))$ of $\mathcal{A}(\mathcal{P})$.

Lemma 5.2.4. *For every ideal $J \subset R$, there is a unique $W \in L_{\mathcal{P}}$ so that $\mathcal{P}_J = \mathcal{P}_{I(W)}$. Furthermore, the ideal $I(W)$ is minimal with respect to $\mathcal{P}_{I(W)} = \mathcal{P}_J$.*

Proof. Set $W = \cap_{\tau \in (\mathcal{P}_J)^0_{n-1}} \mathrm{aff}(\tau)$. Clearly $\mathcal{P}_{I(W)} = \mathcal{P}_J$. To prove minimality, let Q be any ideal satisfying $\mathcal{P}_Q = \mathcal{P}_J$. Then all codim 1 faces $\tau \in (\mathcal{P}_Q)^0_{n-1}$ satisfy $l_\tau \in Q$. Since $(\mathcal{P}_Q)^0_{n-1} = (\mathcal{P}_J)^0_{n-1}$, $I(W) \subset Q$. To show uniqueness, assume $V \in L_{\mathcal{P}}$ and $\mathcal{P}_{I(V)} = \mathcal{P}_J$. By minimality of $I(W)$, $I(W) \subset I(V)$, implying $V \subset W$. If $V \subsetneq W$, then there is some $\tau \in \mathcal{P}^0_{n-1} \setminus (\mathcal{P}_J)^0_{n-1}$ so that $V \subset \mathrm{aff}(\tau)$. But then τ is an interior edge of $\mathcal{P}_{I(V)}$ that is not an interior edge of $\mathcal{P}_J$, a contradiction. So $V = W$. □

For brevity, henceforth we write $G_S(\mathcal{P})$ and $\mathcal{P}_S$ to denote $G_{I(S)}(\mathcal{P})$ and $\mathcal{P}_{I(S)}$ for $S \subset R^n$.

Example 5.2.5. The planar polytopal complex $\mathcal{Q}$ from the introduction is shown in Figure 5.7, along with its associated line arrangement $\mathcal{A}(\mathcal{Q})$ and a representative sample of the complexes $\mathcal{Q}_W$ for $W \neq \emptyset \in L_{\mathcal{Q}}$. We label the interior edges of $\mathcal{Q}$ by $1, \ldots, 8$ and denote their affine spans by $L1, \ldots, L8$. For each complex $\mathcal{Q}_W$ the facets are shaded and the corresponding flat W is labelled. If $\mathcal{Q}_W$ is the disjoint union of several subcomplexes $\mathcal{Q}_W^i$, we display these subcomplexes separately.



(a) $\mathcal{Q}$ (b) $\mathcal{A}(\mathcal{Q})$

(c) $\mathcal{Q}_{L1}$ (d) $\mathcal{Q}^1_{L5}$ (e) $\mathcal{Q}^2_{L5}$

(f) $\mathcal{Q}^1_w$ (g) $\mathcal{Q}^2_w$ (h) $\mathcal{Q}_\xi$

Figure 5.7: Lattice Complexes of Example 5.2.5

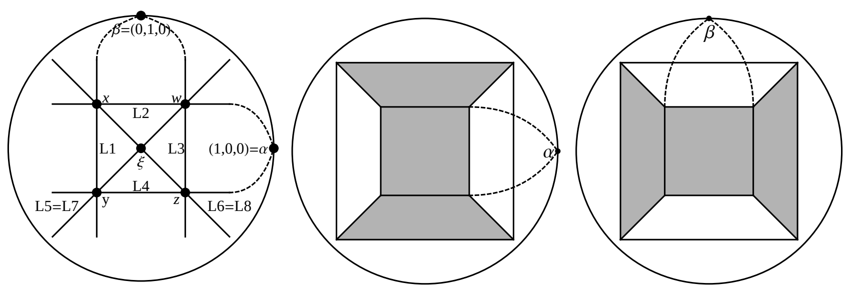
5.2.1 The central case and homogenization

The case where $\mathcal{P}$ is a central complex, i.e. $\mathcal{A}(\mathcal{P})$ is a central arrangement, is of particular interest for splines since then $C^r(\mathcal{P})$ is a graded R-algebra. All the hyperplanes of $\mathcal{A}(\mathcal{P})$ pass through the origin (perhaps after a coordinate change), so we can remove the origin and consider the projective arrangement $\mathbb{P}\mathcal{A}(\mathcal{P}) \subset \mathbb{P}^{n-1}_{\mathbb{R}}$ obtained by quotienting under the action of $\mathbb{R}^*$ by scalar multiplication. The intersection poset $L(\mathbb{P}\mathcal{A}(\mathcal{P}))$ is identical to $L_{\mathcal{P}}$ except it may not contain the maximal flat of $L_{\mathcal{P}}$ (if that flat was the origin).

One of the most important central complexes for splines is the *homogenization* $\widehat{\mathcal{P}} \subset \mathbb{R}^{n+1}$ of a polytopal complex $\mathcal{P} \subset \mathbb{R}^n$. $\widehat{\mathcal{P}} \subset \mathbb{R}^{n+1}$ is constructed by taking the join of $i(\mathcal{P})$ with the origin in $\mathbb{R}^{n+1}$, where $i : \mathbb{R}^n \to \mathbb{R}^{n+1}$ is defined by $i(a_1, \ldots, a_n) = (1, a_1, \ldots, a_n)$. $C^r(\widehat{\mathcal{P}})$ is a graded algebra whose dth graded piece $C^r(\widehat{\mathcal{P}})_d$ is a vector space isomorphic to $C^r_d(\mathcal{P})$ (see [11]). If we regard $x_0, \ldots, x_n$ as coordinate functions on $\mathbb{R}^{n+1}$, then we obtain the original complex $\mathcal{P}$ from $\widehat{\mathcal{P}}$ by setting $x_0 = 1$.

Remark: We associate a subcomplex $\mathcal{P}_W \subset \mathcal{P}$ to $W \in L_{\widehat{\mathcal{P}}}$ by slicing the complex $\widehat{\mathcal{P}}_W$ with the hyperplane $x_0 = 1$. Note that the subcomplex $\widehat{\mathcal{P}}_W$ is the cone over the subcomplex $\mathcal{P}_W$. The subcomplexes $\mathcal{P}_W \subset \mathcal{P}$ obtained this way are the same as those obtained by first embedding $\mathcal{P}$ in $\mathbb{P}^n_{\mathbb{R}}$ by adding the hyperplane at infinity, taking the arrangement of hyperplanes $\mathcal{A}(\mathcal{P})$ in $\mathbb{P}^n_{\mathbb{R}}$ (including intersections in the hyperplane at infinity), and forming the complexes $\mathcal{P}_W$ for flats W in this projective arrangement.

In Figure 5.8 we show the arrangement $\mathbb{P}\mathcal{A}(\widehat{\mathcal{Q}})$ for the complex $\mathcal{Q}$ in Example 5.2.5. The lattice $L_{\widehat{\mathcal{Q}}}$ has two rank 2 flats α and β which do not appear in $L_{\mathcal{Q}}$, corresponding to the intersections of the two pairs of parallel lines $L1, L3$ and $L2, L4$ in $\mathbb{P}^2$. The complexes $\mathcal{Q}_\alpha, \mathcal{Q}_\beta$, also depicted in $\mathbb{P}^2$, are identical up to rotation.



(a) $\mathbb{P}\mathcal{A}(\widehat{\mathcal{Q}})$ (b) $\mathcal{Q}_\alpha$ (c) $\mathcal{Q}_\beta$

Figure 5.8: Projective Lattice Complexes

5.2.2 The simplicial case

We close this section by showing that the complexes $\mathcal{P}_W$ reduce to unions of stars of faces when $\mathcal{P} = \Delta$ is a pure n-dimensional hereditary simplicial complex, as Proposition 5.2.8 shows. We use the following lemma.

Lemma 5.2.6. *Let $\Delta \subset \mathbb{R}^n$ be an $n+1$-simplex and $\sigma_1, \sigma_2 \in \Delta$. Then aff($\sigma_1$) $\cap$ aff(σ_2) $=$ aff($\sigma_1 \cap \sigma_2$). This includes the case aff(σ_1) $\cap$ aff(σ_2) $= \emptyset$, assuming aff($\emptyset$) $= \emptyset$.*

Proof. Δ is the convex hull of $n+1$ vertices $\{v_0, \ldots, v_n\}$. Let Δ_i be the convex hull of $\{v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n\}$, the codimension one face of Δ determined by leaving out vertex v_i. It has supporting hyperplane H_{v_i}, the affine hull of all vertices except v_i. A k-dimensional face τ of Δ determined by $\{v_{i_0}, \ldots, v_{i_k}\}$ has $\mathrm{aff}(\tau) = \cap_{v \in \Delta_0 \setminus \tau_0} H_v$. Now if $\sigma_1, \sigma_2 \in \Delta$,

$$\mathrm{aff}(\sigma_1) \cap \mathrm{aff}(\sigma_2) = \bigcap_{v \in (\Delta_0 \setminus (\sigma_1)_0) \cup (\Delta_0 \setminus (\sigma_2)_0)} H_v = \bigcap_{v \in \Delta_0 \setminus (\sigma_1 \cap \sigma_2)_0} H_v = \mathrm{aff}(\sigma_1 \cap \sigma_2)$$

□

Lemma 5.2.7. *Let $\Delta \subset \mathbb{R}^n$ be a pure n-dimensional hereditary simplicial complex, and $W \in \mathcal{A}(\Delta)$. Then each component of Δ_W has the form st(τ) for some $\tau \in \Delta$.*

Proof. Let $G^i_W(\Delta)$ be a connected component of $G_W(\Delta)$ and $\Delta^i_W \subset \Delta$ the corresponding complex. Let Δ^i_W have facets $\sigma_1, \dots, \sigma_k$, set $V_i = \bigcap_{\tau \in (\sigma_i)_{n-1} \cap (\Delta^i_W)^0_{n-1}} \text{aff}(\tau)$ and $V = \cap_{\tau \in (\Delta^i_W)^0_{n-1}} \text{aff}(\tau) = \cap^k_{i=1} V_i$. By applying Lemma 5.2.6 iteratively, $V_i = \text{aff}(\gamma_i)$ for some face $\gamma_i \in \sigma_i$. Now let K be a walk of length $m+1$ on the graph G^i_W with the corresponding sequence of facets and codimension 1 faces of Δ^i_W being $S = \{\sigma_{j_0}, \tau_{j_1}, \sigma_{j_1}, \dots, \tau_{j_m}, \sigma_{j_m}\}$, where $\tau_{j_i} = \sigma_{j_{i-1}} \cap \sigma_{j_i}$ is a codimension 1 face of $\sigma_{j_{i-1}}$ and σ_{j_i}. Set $\beta_0 = \gamma_{j_0}$, $\beta_c = \cap^c_{i=0} \gamma_{j_i}$. We prove $\cap^c_{i=0} V_{j_i} = \text{aff}(\beta_c)$, for $c = 0, \dots, m$ by induction. We already have $V_{j_0} = \text{aff}(\gamma_{j_0}) = \text{aff}(\beta_0)$. Assume $\cap^c_{i=0} V_{j_i} = \text{aff}(\beta_c)$. β_c is a face of γ_{j_c}, which in turn is a face of $\tau_{j_{c+1}}$, since $\tau_{j_{c+1}}$ is a codimension 1 face of σ_{j_c} such that $\text{aff}(\tau_{j_{c+1}})$ contains W. So β_c is a face of $\sigma_{j_{c+1}}$. By Lemma 5.2.6, $\text{aff}(\beta_c) \cap \text{aff}(\gamma_{j_{c+1}}) = \text{aff}(\beta_c \cap \gamma_{j_{c+1}}) = \text{aff}(\beta_{c+1})$. Putting everything together, we have $\cap^{c+1}_{i=0} V_{j_i} = (\cap^c_{i=0} V_{j_i}) \cap V_{j_{c+1}} = \text{aff}(\beta_c) \cap \text{aff}(\gamma_{c+1}) = \text{aff}(\beta_{c+1})$.

Setting $\tau = \beta_m$ and noting that $V = \cap^k_{i=1} V_i = \cap^m_{i=0} V_{j_i}$, we have $V = \text{aff}(\tau)$. By construction τ is a face of σ_i for $i = 1, \dots, k$, hence $\Delta^i_W \subset \text{st}(\tau)$. On the other hand, since $W \subset \text{aff}(\tau)$, $\text{st}(\tau) \subset \Delta_W$. Δ is hereditary, so $G(\text{st}(\tau))$ is connected and $\text{st}(\tau)$ is a component of Δ_W. Hence $\Delta^i_W = \text{st}(\tau)$. □

We indicate precisely which stars appear in Δ_W, following notation of Billera and Rose [12].

Proposition 5.2.8. *Let Δ be a pure simplicial complex and $W \in L_\Delta$. Set*

$$S(W) = \{\tau \in \Delta | W \subset \textit{aff}(\tau) \textit{ and } \tau \textit{ is minimal with respect to this property}\}.$$

Then

$$\Delta_W = \bigsqcup_{\tau \in S(W)} st(\tau)$$

Proof. Let Δ^i_W be a component of Δ_W. By Lemma 5.2.7, $\Delta^i_W = \text{st}(\tau)$ for some $\tau \in \Delta$, where we may assume τ is the intersection of all simplices appearing in $\text{st}(\tau)$. Clearly $W \subset \text{aff}(\tau)$. If τ is not minimal with respect to this property, then there is some γ a proper face of τ so that $W \subset \text{aff}(\gamma)$. But then $\text{st}(\tau) \subsetneq \text{st}(\gamma) \subset \Delta_W$, contradicting that $\text{st}(\tau)$ is a component of Δ_W. So $\tau \in S(P)$. Now suppose $\tau \in S(P)$. Clearly $\text{st}(\tau) \subset \Delta_W$ and $G(\text{st}(\tau))$ is connected since Δ is hereditary. Hence $\text{st}(\tau)$ is contained in a component Δ^i_W of Δ_W. By Lemma 5.2.7, $\Delta^i_W = \text{st}(\gamma)$ for some $\gamma \in \Delta$. We may assume τ is the intersection of the simplices contained in $\text{st}(\tau)$, implying γ is a face of τ. But $W \subset \text{aff}(\gamma)$ and $\tau \in S(W)$, so $\tau = \gamma$. □

5.3 Localization of $C^r(\mathcal{P})$

Our objective in this section is to give an explicit description of $C^r(\mathcal{P})_P$ for any prime $P \subset R$, using the complexes $\mathcal{P}_W$ defined in the previous section. Given a pure n-dimensional polytopal complex $\mathcal{P}$, the boundary complex of $\mathcal{P}$ is a pure $(n-1)$-dimensional complex denoted by $\partial\mathcal{P}$. For a pure n-dimensional subcomplex $\mathcal{Q} \subset \mathcal{P}$, not necessarily hereditary, we use the following notation.

1. $C^r_{\mathcal{Q}}(\mathcal{P}) := \{F \in C^r(\mathcal{P}) | F_\sigma = 0 \text{ for all } \sigma \in \mathcal{P}_n \setminus \mathcal{Q}_n\}$.

2. $L_{\partial\mathcal{Q}} := \prod_{\tau \in (\partial\mathcal{Q})_{n-1} \setminus (\partial\mathcal{P})_{n-1}} l_\tau$.

We can naturally view $C^r(\mathcal{Q})$ as a submodule of $\oplus_{\sigma\in\mathcal{P}_n} R$ by defining $F_\sigma = 0$ for $F \in C^r(\mathcal{Q}), \sigma \in \mathcal{P}_n \setminus \mathcal{Q}_n$. In this way $C^r(\mathcal{Q})$ and $C^r(\mathcal{P})$ are submodules of the same ambient module. We will assume this throughout the paper.

$C^r_{\mathcal{Q}}(\mathcal{P})$ satisfies $L^{r+1}_{\partial\mathcal{Q}} \cdot C^r(\mathcal{Q}) \subseteq C^r_{\mathcal{Q}}(\mathcal{P})$. This follows since $L^{r+1}_{\partial\mathcal{Q}} \cdot F$ vanishes on $\partial\mathcal{Q} \setminus \partial\mathcal{P}$ to order $r+1$ for any $F \in C^r(\mathcal{Q})$. From this we get the following observation on localization which we refer to as $(\star)$.

Observation: $C^r_{\mathcal{Q}}(\mathcal{P})_P = C^r(\mathcal{Q})_P$ for any prime P such that $L_{\partial\mathcal{Q}} \notin P$. $(\star)$

Let $\mathcal{Q}_1, \dots, \mathcal{Q}_k$ be pure n-dimensional polytopal subcomplexes of $\mathcal{P}$. Call $\mathcal{Q}_1, \dots, \mathcal{Q}_k$ a **partition** of $\mathcal{P}$ if the facets of $\mathcal{Q}_1, \dots, \mathcal{Q}_k$ partition the facets of $\mathcal{P}$.

Lemma 5.3.1. *Let $\mathcal{Q}$ and $\mathcal{O}$ be two polytopal subcomplexes which partition $\mathcal{P}$. Let P be a prime of R such that $L_{\partial\mathcal{Q}} \notin P$ and $L_{\partial\mathcal{O}} \notin P$. Then*

$$C^r(\mathcal{P})_P = C^r(\mathcal{Q})_P \oplus C^r(\mathcal{O})_P$$

as submodules of $\oplus_{\sigma\in\mathcal{P}_n} R_P$. More precisely,

$$(C^r_{\mathcal{Q}}(\mathcal{P}) + C^r_{\mathcal{O}}(\mathcal{P}))_P = C^r(\mathcal{Q})_P + C^r(\mathcal{O})_P.$$

Proof. It is clear that $C^r(\mathcal{Q}) \cap C^r(\mathcal{O}) = 0$ and $C^r_{\mathcal{Q}}(\mathcal{P}) \cap C^r_{\mathcal{O}}(\mathcal{P}) = 0$ in $\oplus_{\sigma\in\mathcal{P}_n} R$ since $\mathcal{Q}$ and $\mathcal{O}$ have no common facets. So both $C^r_{\mathcal{Q}}(\mathcal{P}) + C^r_{\mathcal{O}}(\mathcal{P})$ and $C^r(\mathcal{Q}) + C^r(\mathcal{O})$ are internal direct sums. The result follows from $(\star)$. □

Corollary 5.3.2. *Let $\mathcal{Q}_1, \dots, \mathcal{Q}_k$ be a partition of $\mathcal{P}$ into polytopal subcomplexes and $P \subset R$ a prime such that $L_{\partial\mathcal{Q}_i} \notin P$ for $i = 1, \dots, k$.*

$$C^r(\mathcal{P})_P = \bigoplus_{i=1}^{k} C^r(\mathcal{Q}_i)_P$$

as submodules of $\oplus_{\sigma\in\mathcal{P}_n} R$.

Proof. Apply Corollary 5.3.1 iteratively. □

Corollary 5.3.3. *Let P be a prime of R so that $l_\tau \notin P$ for every edge $\tau \in \mathcal{P}_1^0$. Then*

$$C^r(\mathcal{P})_P = \bigoplus_{\sigma \in \mathcal{P}_n} R_P$$

Proof. Apply Corollary 5.3.2 to the partition of $\mathcal{P}$ into individual facets. This yields

$$C^r(\mathcal{P})_P = \bigoplus_{\sigma \in \mathcal{P}_n} C^r(\sigma)_P$$

Since $C^r(\sigma) = R$, we are done. □

Now let $I \subset R$ be an ideal and $\mathcal{P}_I \subset \mathcal{P}$ be the subcomplex defined in the previous section.

Definition 5.3.4. Let $\mathcal{P}_I$ have components $\mathcal{P}_I^1, \ldots, \mathcal{P}_I^k$. Define

$$C^r(\mathcal{P}_I) := \bigoplus_{i=1}^{k} C^r(\mathcal{P}_I^i).$$

Proposition 5.3.5. *Let $P \subset R$ be a prime ideal, and $\mathcal{P} \subset \mathbb{R}^n$ a polytopal complex. Then*

$$\begin{aligned} C^r(\mathcal{P})_P &= C^r(\mathcal{P}_P)_P \oplus \bigoplus_{\sigma \in \mathcal{P}_n \setminus (\mathcal{P}_P)_n} R_P \\ &= C^r(\mathcal{P}_W)_P \oplus \bigoplus_{\sigma \in \mathcal{P}_n \setminus (\mathcal{P}_W)_n} R_P \end{aligned},$$

where W is the unique flat of $L_\mathcal{P}$ so that $\mathcal{P}_W = \mathcal{P}_P$ guaranteed by Lemma 5.2.4.

Proof. Consider the partition of $\mathcal{P}$ determined by $\mathcal{P}_P$ and $\mathcal{Q}$, where $\mathcal{Q}$ is the subcomplex of $\mathcal{P}$ generated by all facets $\sigma \in \mathcal{P}_n \setminus (\mathcal{P}_P)_n$. By the construction of $\mathcal{P}_P$, $\tau \in (\mathcal{P}_P)_{n-1}^0 \iff l_\tau \in P$. Hence if $\tau \in (\partial\mathcal{P}_P)_{n-1} \setminus (\partial\mathcal{P})_{n-1} = (\partial\mathcal{Q})_{n-1} \setminus (\partial\mathcal{P})_{n-1}$ then $l_\tau \notin P$. Applying Corollary 5.3.1 we have

$$C^r(\mathcal{P})_P = C^r(\mathcal{P}_P)_P \oplus C^r(\mathcal{Q})_P$$

Again since all $\tau \in \mathcal{P}_{n-1}^0$ such that $l_\tau \in P$ are interior codim 1 faces of $\mathcal{P}_P$, there is no $\tau \in \mathcal{Q}_{n-1}^0$ such that $l_\tau \in P$. Applying Corollary 5.3.3 to $C^r(\mathcal{Q})_P$ gives the result. □

We get the following result of Billera and Rose, used in the proof of Theorem 2.3 in [12], as a corollary of Proposition 5.3.5 and Proposition 5.2.8.

Corollary 5.3.6. *Let Δ be a simplicial complex and $W \in L_\Delta$. Define $S(W)$ as in Proposition 5.2.8. Then*

$$C^r(\Delta)_P = \bigoplus_{\tau \in S(W)} C^r(st(\tau))_P,$$

where $W \in L_\Delta$ is the unique flat so that $\Delta_P = \Delta_W$.

Note that if a facet σ is in $S(W)$ then it is a not a facet of Δ_W. Since $C^r(\sigma) = R$, the sum $\bigoplus_{\sigma\in\Delta_n\setminus(\Delta_W)_n} R_P$ appearing in Theorem 5.3.5 is implicit in the sum above.

5.3.1 Relation to results of Billera-Rose and Yuzvinsky

As an application of Proposition 5.3.5, we prove a slight variant of a result of Yuzvinsky [63] which reduces computation of the projective dimension of $C^r(\mathcal{P})$ to the central case.

In [63], Yuzvinsky introduces a poset L, different from $L_{\mathcal{P}}$, associated to a polytopal complex. He defines subcomplexes associated to each $x \in L$ and uses them to reduce the characterization of projective dimension to the graded case (Proposition 2.4). Proposition 5.3.5 is the analog for $L_{\mathcal{P}}$ of Lemma 2.3 in [63], and we use it to prove the following statement.

Theorem 5.3.7. *Let $\mathcal{P} \subset \mathbb{R}^n$ be a polytopal complex. Then*

1. *$pd(C^r(\mathcal{P})) \geq pd(C^r(\mathcal{P}_W))$ for all $W \in L_{\mathcal{P}}$.*
2. *$pd(C^r(\mathcal{P})) = \max\limits_{W\in L_{\mathcal{P}}} pd(C^r(\mathcal{P}_W))$. In particular, $C^r(\mathcal{P})$ is free iff $C^r(\mathcal{P}_W)$ is free for all nonempty $W \in L_{\mathcal{P}}$.*

Proof. We use the following two facts about projective dimension. Here M is any R-module, not necessarily graded.

(A) For any prime $P \subset R$, $\text{pd}(M) \geq \text{pd}(M_P)$.

(B) $\text{pd}(M) = \max\limits_{P\in\text{Spec } R} \text{pd}(M_P)$.

(A) is immediate from localizing any free resolution $F_\bullet$ of M. (B) follows from showing that there are primes which preserve $\text{pd}(M)$ under localization. Set $\text{pd}(M) = r$. We have $\text{Ext}^r_R(M, R) \neq 0$ and taking any prime P in its support will suffice. For such a prime, we have $\text{Ext}^r_{R_P}(M_P, R_P) \cong \text{Ext}^r_R(M, R) \otimes_R R_P \neq 0$. Hence $\text{pd}(M_P) \geq r$. Since we already have $\text{pd}(M_P) \leq r$, $\text{pd}(M_P) = r$. (1) Observe that $C^r(\mathcal{P}_W)$ is graded with respect to $I(\xi)$ for any point $\xi \in W$ since the affine span of every interior codimension 1 face of $\mathcal{P}_W$ contains W, hence also contains ξ. Choose $\xi \in W \setminus \cup_{V\subset W} V$, where $V \in L_{\mathcal{P}}$. Then $\mathcal{P}_\xi = \mathcal{P}_W$. We have

$$\text{pd}(C^r(\mathcal{P})) \geq \text{pd}(C^r(\mathcal{P})_{I(\xi)}) = \text{pd}(C^r(\mathcal{P}_\xi)_{I(\xi)}) = \text{pd}(C^r(\mathcal{P}_W)_{I(\xi)}),$$

where the first equality follows from fact (A) above and the second follows from Theorem 5.3.5 since $C^r(\mathcal{P})_{I(\xi)}$ is the direct sum of a free module and $C^r(\mathcal{P}_\xi)_{I(\xi)}$. $C^r(\mathcal{P}_W)$ is graded with respect to $I(\xi)$, so a minimal resolution of $C^r(\mathcal{P}_W)$ has differentials with entries in $I(\xi)$. It follows that this remains a minimal resolution under localization with respect to $I(\xi)$, so $\text{pd}(C^r(\mathcal{P}_W)_{I(\xi)}) = \text{pd}(C^r(\mathcal{P}_W))$, and the result follows. (2) Set $m = \max\limits_{W\in L_{\mathcal{P}}} \text{pd}(C^r(\mathcal{P}_W))$. From (1)

$\mathrm{pd}(C^r(\mathcal{P})) \geq m$. For any prime $P \subset R$ we have $\mathcal{P}_P = \mathcal{P}_W$ for some $W \in L_{\mathcal{P}}$ by Lemma 5.2.4, so

$$\mathrm{pd}(C^r(\mathcal{P})_P) = \mathrm{pd}(C^r(\mathcal{P}_W)_P) \leq \mathrm{pd}(C^r(\mathcal{P}_W)) \leq m.$$

Hence $\mathrm{pd}(C^r(\mathcal{P})) \leq m$ from fact (B) above, and $\mathrm{pd}(C^r(\mathcal{P})) = m$ as desired. □

Since the $\mathcal{P}_W$ are central complexes, this reduces computation of $\mathrm{pd}(C^r(\mathcal{P}))$ to the central case. Via Proposition 5.2.8 we obtain the following theorem of Billera and Rose as a corollary to Theorem 5.3.7. Recall an R-module M is free iff $\mathrm{pd}(M) = 0$.

Theorem 5.3.8. *[2.3 of [12]] Let $\mathcal{P} \subset \mathbb{R}^n$ be a polytopal complex. Then*

1. *If $C^r(\mathcal{P})$ is free over R then $C^r(st(\tau))$ is free over R for all $\tau \neq \emptyset \in \mathcal{P}$.*
2. *If $\mathcal{P} = \Delta$ is simplicial then the converse is also true: if $C^r(st(\sigma))$ is free for all nonempty $\sigma \in \Delta$, then $C^r(\Delta)$ is free.*

5.4 Lattice-Supported Splines

Our main application of Proposition 5.3.5 is to construct "locally supported approximations" to $C^r(\mathcal{P})$ which allow us to generalize the notion of a star-supported basis of $C^r_d(\mathcal{P})$ [4] in the simplicial case. In particular we show that a basis of $C^r_d(\mathcal{P})$ consisting of splines supported on complexes of the form $\mathcal{P}_W$ always exists for $d \gg 0$, *as long as we allow $W \in L_{\widehat{\mathcal{P}}}$*.

Recall for $\mathcal{Q}$ a pure n-dimensional connected subcomplex of $\mathcal{P} \subset \mathbb{R}^n$, we defined $C^r_{\mathcal{Q}}(\mathcal{P})$ to be the set of splines vanishing outside of $\mathcal{Q}$. Also recall that $\widehat{\mathcal{P}} \subset \mathbb{R}^{n+1}$ is the fan obtained by coning over $\mathcal{P}$ in $\mathbb{R}^{n+1}$. In the remark in subsection 5.2.1, we defined subcomplexes $\mathcal{P}_W \subset \mathcal{P}$ for $W \in L_{\widehat{\mathcal{P}}}$ by slicing the subcomplex $\widehat{\mathcal{P}}_W$ with the hyperplane $x_0 = 1$. We make the following definitions.

Definition 5.4.1. For $W \in L_{\widehat{\mathcal{P}}}$ let $\mathcal{P}_W$ have components $\mathcal{P}^1_W, \ldots, \mathcal{P}^m_W$.

1. $C^r_W(\mathcal{P}) := \bigoplus_{i=1}^{m} C^r_{\mathcal{P}^i_W}(\mathcal{P})$
2. $LS^{r,k}(\mathcal{P}) := \sum_{\substack{W \in L_{\widehat{\mathcal{P}}} \\ 0 \leq rk(W) \leq k}} C^r_W(\mathcal{P})$

$C^r_W(\mathcal{P})$ is the submodule of $C^r(\mathcal{P})$ generated by splines which are 0 outside of a component of $\mathcal{P}_W$. We say that elements of $C^r_W(\mathcal{P})$ are *supported at* W. If a spline F is supported at some $W \in L_{\widehat{\mathcal{P}}}$ then we call F *lattice-supported.* By Proposition 5.2.8, *lattice-supported* splines generalize the notion of *star-supported* splines [4] to polytopal complexes.

Remark: Results in this section which do not depend on grading (Proposition 5.4.2, Theorem 5.4.3, and Corollary 5.4.4) only require the sum in (2) to run over $W \in L_{\mathcal{P}}$, not $W \in L_{\widehat{\mathcal{P}}}$.

Proposition 5.4.2. *Let $\mathcal{P} \subset \mathbb{R}^n$ be a pure n-dimensional polytopal complex. If $P \subset R$ is a prime such that the unique $W \in L_{\mathcal{P}}$ satisfying $\mathcal{P}_P = \mathcal{P}_W$, guaranteed by Lemma 5.2.4, has* $\mathrm{rk}(W) \leq k$, *then $LS^{r,k}(\mathcal{P})_P = C^r(\mathcal{P})_P$.*

Proof. Let P satisfy the given condition. Since $\mathrm{rk}(W) \leq k$ the module $C^r_W(\mathcal{P})$ appears as a summand in $LS^{r,k}(\mathcal{P})$. Note that $C^r_{\mathbb{R}^n}(\mathcal{P}) = \sum_{\sigma \in \mathcal{P}_n} C^r_\sigma(\mathcal{P})$. Define the submodule $N(P) \subset LS^{r,k}(\mathcal{P})$ by $N(P) = C^r_W(\mathcal{P}) + \sum_{\sigma \in \mathcal{P}_n \setminus (\mathcal{P}_P)_n} C^r_\sigma(\mathcal{P})$. The support of the summands of $N(P)$ is disjoint, so it is clear that this is an internal direct sum. Also, for $\sigma \notin (\mathcal{P}_P)_n$, $C^r_\sigma(\mathcal{P})_P = R_P$. We have

$$\begin{aligned} N(P)_P &= C^r_W(\mathcal{P})_P \oplus \bigoplus_{\sigma \in \mathcal{P}_n \setminus (\mathcal{P}_P)_n} C^r_\sigma(\mathcal{P})_P \\ &= C^r(\mathcal{P}_W)_P \oplus \bigoplus_{\sigma \in \mathcal{P}_n \setminus (\mathcal{P}_P)_n} R_P \\ &= C^r(\mathcal{P})_P \end{aligned}$$

where the second equality follows from applying observation ($\star$) to the summands of $C^r_W(\mathcal{P})$ and the third equality follows from Theorem 5.3.5. Since $N(P) \subset LS^{r,k}(\mathcal{P}) \subset C^r(\mathcal{P})$ and $N(P)_P = C^r(\mathcal{P})_P$, we have $LS^{r,k}(\mathcal{P})_P = C^r(\mathcal{P})_P$. □

Theorem 5.4.3. *Let $\mathcal{P} \subset \mathbb{R}^n$ be a polytopal complex. Then $LS^{r,k}(\mathcal{P})$ fits into a short exact sequence*

$$0 \to LS^{r,k}(\mathcal{P}) \to C^r(\mathcal{P}) \to C \to 0$$

where C has codimension $\geq k+1$ and the primes in the support of C with codimension $k+1$ are contained in the set $\{I(W) | W \in L_{\mathcal{P}} \text{ and } \mathrm{rk}(W) = k+1\}$.

Recall the *support* of C is the set of primes $P \subset R$ satisfying $C_P \neq 0$.

Proof. The first statement follows from Proposition 5.4.2. Now suppose C is supported at P with codim $P = k+1$, and let $W \in L_{\mathcal{P}}$ be the unique flat so that $\mathcal{P}_P = \mathcal{P}_W$ (Lemma 5.2.4). If $\mathrm{rk}(W) \leq k$ then $LS^{r,k}(\mathcal{P})_P = C^r(\mathcal{P})_P$ by Proposition 5.4.2 and $C_P = 0$. So we must have $\mathrm{rk}(W) = k+1$ and $P = I(W)$. Hence the primes of codimension $k+1$ in the support of C are contained in $\{I(W) | W \in L_{\mathcal{P}} \text{ and } \mathrm{rk}(W) = k+1\}$. □

Setting $k = n$ in Theorem 5.4.3, we obtain

Corollary 5.4.4. *For $\mathcal{P} \subset \mathbb{R}^n$, $LS^{r,n}(\mathcal{P}) = C^r(\mathcal{P})$.*

5.4.1 Graded Results

Recall that $C^r_d(\mathcal{P}) \cong C^r(\widehat{\mathcal{P}})_d$ as $\mathbb{R}$-vector spaces by Proposition 3.3.12. The map $\phi_d : C^r_d(\mathcal{P}) \to C^r(\widehat{\mathcal{P}})_d$ in Proposition 3.3.12 is given by *homogenizing*:

$$f(x_1, \ldots, x_n) \in C^r_d(\mathcal{P}) \to x_0^d f(x_1/x_0, \ldots, x_n/x_0) \in C^r(\widehat{\mathcal{P}})_d,$$

while its inverse $\phi_d^{-1} : C^r(\widehat{\mathcal{P}})_d \to C^r_d(\mathcal{P})$ is given by

$$F(x_0, \dots, x_n) \in C^r(\widehat{\mathcal{P}})_d \to F(1, x_1, \dots, x_n) \in C^r_d(\mathcal{P}).$$

We define filtrations and gradings for $LS^{r,k}(\mathcal{P})$. For $W \in L_{\widehat{\mathcal{P}}}$, define

$$\begin{aligned} C^r_{W,d}(\mathcal{P}) &:= \quad C^r_W(\mathcal{P}) \cap C^r_d(\mathcal{P}) \\ C^r_W(\widehat{\mathcal{P}})_d &:= \quad C^r_W(\widehat{\mathcal{P}}) \cap C^r(\widehat{\mathcal{P}})_d \end{aligned}$$

Then $LS^{r,k}(\mathcal{P})$ has a filtration by vector spaces

$$LS^{r,k}_d(\mathcal{P}) := \sum_{\substack{W \in L_{\widehat{\mathcal{P}}} \\ 0 \le \mathrm{rk}(W) \le k}} C^r_{W,d}(\mathcal{P}),$$

which are subspaces of $C^r_d(\mathcal{P})$. If $\mathcal{P}$ is a central complex, $LS^{r,k}(\mathcal{P})$ is graded by the vector spaces

$$\begin{aligned} LS^{r,k}(\mathcal{P})_d &:= \quad \{F \in LS^{r,k}(\mathcal{P}) | F_\sigma \in R_d \text{ for all facets } \sigma \in \mathcal{P}_n\} \\ &= \quad \sum_{\substack{W \in L_{\widehat{\mathcal{P}}} \\ 0 \le \mathrm{rk}(W) \le k}} C^r_W(\widehat{\mathcal{P}})_d, \end{aligned}$$

which are subspaces of $C^r(\widehat{\mathcal{P}})_d$. If $\mathcal{P}$ is central, the set of subcomplexes $\mathcal{P}_W$ for $W \in L_{\widehat{\mathcal{P}}}$ is the same as the set of subcomplexes $\mathcal{P}_W$ for $W \in L_{\mathcal{P}}$ (there is a rank preserving isomorphism between $L_{\mathcal{P}}$ and $L_{\widehat{\mathcal{P}}}$ in this case). Hence we may write

$$LS^{r,k}(\mathcal{P})_d = \sum_{\substack{W \in L_{\mathcal{P}} \\ 0 \le \mathrm{rk}(W) \le k}} C^r_W(\widehat{\mathcal{P}})_d$$

Lemma 5.4.5. *$LS^{r,k}_d(\mathcal{P})$ and $LS^{r,k}(\widehat{\mathcal{P}})_d$ are isomorphic as $\mathbb{R}$-vector spaces.*

Proof. We show that the homogenization map $\phi_d : C^r_d(\mathcal{P}) \to C^r(\widehat{\mathcal{P}})_d$ restricts to an isomorphism between $LS^{r,k}_d(\mathcal{P})$ and $LS^{r,k}(\widehat{\mathcal{P}})_d$. We have

$$\begin{aligned} LS^{r,k}_d(\mathcal{P}) &:= \sum_{\substack{W \in L_{\widehat{\mathcal{P}}} \\ 0 \le \mathrm{rk}(W) \le k}} C^r_{W,d}(\mathcal{P}) \\ LS^{r,k}(\widehat{\mathcal{P}})_d &= \sum_{\substack{W \in L_{\widehat{\mathcal{P}}} \\ 0 \le \mathrm{rk}(W) \le k}} C^r_W(\widehat{\mathcal{P}})_d \end{aligned}$$

Since ϕ_d is $\mathbb{R}$-linear, it suffices to show that, given $W \in L_{\widehat{\mathcal{P}}}$, ϕ_d restricts to an isomorphism $\phi_d : C^r_{W,d}(\mathcal{P}) \to C^r_W(\widehat{\mathcal{P}})_d$. We show $\phi_d(C^r_{W,d}(\mathcal{P})) \subset C^r_W(\widehat{\mathcal{P}})_d$ and $\phi_d^{-1}(C^r_W(\widehat{\mathcal{P}})_d) \subset C^r_{W,d}(\mathcal{P})$. Suppose $f \in C^r_{W,d}(\mathcal{P})$. The support of f is by definition contained in the subcomplex $\mathcal{P}_W$, so the support of $\phi_h(f)$ is contained in the cone over $\mathcal{P}_W$, which is precisely $\widehat{\mathcal{P}}_W$. It follows that $\phi_d(f) \in C^r_W(\widehat{\mathcal{P}})$. Since $\phi_d(f) \in C^r(\widehat{\mathcal{P}})_d$, $\phi_d(f) \in (C^r_W(\widehat{\mathcal{P}}) \cap C^r(\widehat{\mathcal{P}})_d) = C^r_W(\widehat{\mathcal{P}})_d$. Now suppose $F \in C^r_W(\widehat{\mathcal{P}})_d$. The support of F is contained in the subcomplex $\widehat{\mathcal{P}}_W$, so the support of $\phi_h^{-1}(F)$ is contained in the complex obtained by slicing $\widehat{\mathcal{P}}_W$ with the hyperplane $x_0 = 1$. But this is by definition $\mathcal{P}_W$, so $\phi_d^{-1}(F) \in C^r_W(\mathcal{P})$. Since $\phi_d^{-1}(F) \in C^r_d(\mathcal{P})$, $\phi_d^{-1}(F) \in (C^r_W(\mathcal{P}) \cap C^r_d(\mathcal{P})) = C^r_{W,d}(\mathcal{P})$. □

For the following theorem, recall that the *Hilbert function* $HF(M, d)$ of a finitely generated graded module $M = \bigoplus_d M_d$ over $R = \mathbb{R}[x_1, \ldots, x_n]$ is defined by

$$HF(M, d) = \dim_{\mathbb{R}} M_d$$

and the *Hilbert polynomial* $HP(M, d)$ of M is the polynomial with which $HF(M, d)$ agrees for $d \gg 0$.

Theorem 5.4.6. *Let $\mathcal{P} \subset \mathbb{R}^n$ be a polytopal complex.*

1. *If $\mathcal{P}$ is central, the first $k+1$ coefficients of $HP(C^r(\mathcal{P}))$ and $HP(LS^{r,k}(\mathcal{P}))$ agree. In particular, $LS^{r,n-1}(\mathcal{P})_d = C^r(\mathcal{P})_d$ for $d \gg 0$.*

2. *$LS^{r,n}_d(\mathcal{P}) = C^r_d(\mathcal{P})$ for $d \gg 0$. Equivalently, $C^r_d(\mathcal{P})$ has a basis consisting of lattice-supported splines for $d \gg 0$.*

Proof. (1) Applying Theorem 5.4.3 to $LS^{r,k}(\mathcal{P})$ yields the short exact sequence

$$0 \to LS^{r,k}(\mathcal{P}) \to C^r(\mathcal{P}) \to C \to 0,$$

where codim $C \leq k + 1$. It follows that $HP(C, d)$ has degree at most $(n - 1) - (k + 1)$. On the other hand, $HP(C^r(\mathcal{P}), d)$ has degree $n - 1$. Since $HP(C^r(\mathcal{P}), d) - HP(LS^{r,k}(\mathcal{P}), d) = HP(C, d)$, the first $k + 1$ coefficents of $HP(C^r(\mathcal{P}), d)$ and $HP(LS^{r,k}(\mathcal{P}), d)$ agree. Now specialize to $k = n - 1$. Then $HP(C, d) = 0$ so $HP(C^r(\mathcal{P}), d) = HP(LS^{r,n-1}(\mathcal{P}), d)$, implying $HF(C^r(\mathcal{P}), d)$ $= HF(LS^{r,n-1}(\mathcal{P}), d)$ for $d \gg 0$. Since $LS^{r,n-1}(\mathcal{P})_d \subset C^r(\mathcal{P})_d$, we have $LS^{r,n-1}(\mathcal{P})_d = C^r(\mathcal{P})_d$ for $d \gg 0$. (2) From part (1), $LS^{r,n}(\widehat{\mathcal{P}})_d = C^r(\widehat{\mathcal{P}})_d$ for $d \gg 0$. The result follows by applying Lemma 5.4.5 to the left hand side and Proposition 3.3.12 to the right hand side. □

Example 5.4.7. We give an example to highlight the difference between Corollary 5.4.4 and Theorem 5.4.6 part (2). Take the underlying complex to be the complex $\mathcal{Q}$ from Figure 5.1. In Figure 5.3 we show a decomposition for the unit in $C^0(\mathcal{Q})$, $\mathbf{1} = \sum_{j=1}^{5} G_j$, with $G_j \in C^0_2(\mathcal{Q})$. The splines $G_1, \ldots, G_5$ have support in the subcomplexes $\mathcal{Q}_W$ for $W \in L_{\mathcal{Q}}$. Given any spline $F \in C^0_d(\mathcal{Q})$, $\sum_{j=1} G_j \cdot F$ gives a decomposition of F using lattice-supported splines in $C^0_{d+2}(\mathcal{Q})$. It follows that $C^0(\mathcal{Q}) = \sum_{W \in L_{\mathcal{Q}}} C^0_W(\mathcal{Q})$, *without* using the two complexes $\mathcal{Q}_\alpha, \mathcal{Q}_\beta$ of Figure 5.8, which correspond to intersections 'at infinity.' This is true in general; the statement of Corollary 5.4.4 can be changed to

$$\sum_{W \in L_{\mathcal{P}}} C^r_W(\mathcal{P}) = C^r(\mathcal{P}),$$

without altering the proof.

However, if we want to write every spline $F \in C^0_d(\mathcal{Q})$ as a sum of lattice-supported splines of degree *at most* d, we must use the two complexes $\mathcal{Q}_\alpha$ and $\mathcal{Q}_\beta$. For example, a computation in Macaulay2 shows that the spline $x^2 \cdot \mathbf{1} \in C^0_2(\mathcal{Q})$ is not in the vector space $\sum_{W \in L_{\mathcal{P}}} C^0_{W,2}(\mathcal{Q})$, while a decomposition for $x^2 \cdot \mathbf{1}$ in $\sum_{W \in L_{\widehat{\mathcal{P}}}} C^0_{W,2}(\mathcal{Q})$ is shown in Figure 5.9. Set $l_1 = x+1, l_2 = y-1, l_3 = x-1, l_4 = y+1, l_5 = x-y, l_6 = x+y$ below.

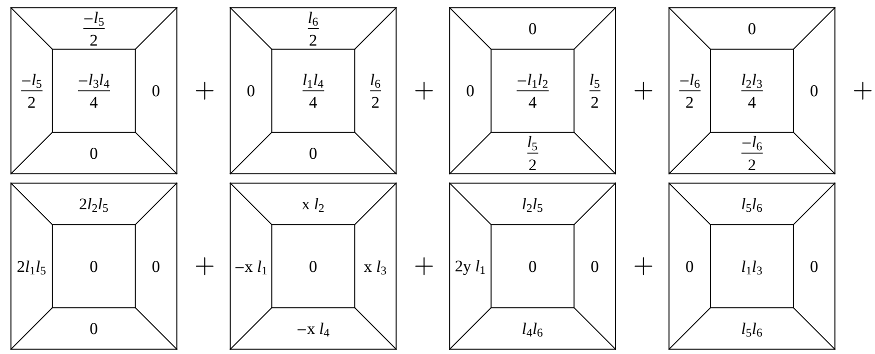


Figure 5.9: Decomposition of $x^2 \cdot \mathbf{1} \in C^0_2(\mathcal{Q})$

In [32],[33] and [6], it is shown that in the bivariate simplicial case there exists a star-supported basis for $C^r_d(\Delta)$ if $d \geq 3r+2$ and in the trivariate simplicial case for $d > 8r$. These bases are explicitly constructed using the Bernstein-Bezier method. Finding how large d must be in order to obtain a lattice-supported basis for $C^r_d(\mathcal{P})$ is a difficult question. The results in the bivariate and trivariate simplicial case suggest that the vector spaces $LS^{r,n}_d(\mathcal{P})$ and $C^r_d(\mathcal{P})$ begin to agree at a value of d which is a linear function of r. We address this in the planar polytopal case in §5 and relate this question to bounding the value of d for which known dimension formulas for $C^r_d(\mathcal{P})$ hold. Before progressing to this, we show how the poset $L_{\mathcal{P}}$ can be refined to give a cleaner description of $LS^{r,k}(\mathcal{P})$.

5.4.2 Refining posets for $LS^{r,k}(\mathcal{P})$

In this section we seek a minimal set of submodules of the form $C^r_{\mathcal{Q}}(\mathcal{P})$ which generate $LS^{r,k}(\mathcal{P})$. Observe that if $\mathcal{Q} \subseteq \mathcal{O}$ are subcomplexes of $\mathcal{P}$ then $C^r_{\mathcal{Q}}(\mathcal{P}) \subseteq C^r_{\mathcal{O}}(\mathcal{P})$. So if there is containment among subcomplexes which are the support of the summands appearing in $LS^{r,k}(\mathcal{P})$ then we may discard one of the summands. This suggests that while $L_{\mathcal{P}}$ is quite useful for describing localization of $C^r(\mathcal{P})$, there is a more useful poset to consider for understanding $LS^{r,k}(\mathcal{P})$. This is the poset $\Gamma_{\mathcal{P}}$ whose elements are subcomplexes $\mathcal{Q}$ of $\mathcal{P}$ which are a component of $\mathcal{P}_W$ for some $W \in L_{\widehat{\mathcal{P}}}$, ordered by inclusion. As we will see, $\Gamma_{\mathcal{P}}$ may be quite different from $L_{\widehat{\mathcal{P}}}$.

Define a function f_Γ on $\Gamma_{\mathcal{P}}$ by

$$f_\Gamma(\mathcal{Q}) = \begin{cases} \operatorname{codim}\left(\bigcap\limits_{\tau\in\mathcal{Q}^0_{n-1}} \operatorname{aff}(\tau)\right) & \text{if } \mathcal{Q}^0_{n-1} \neq \emptyset \\ 0 & \text{if } \mathcal{Q}^0_{n-1} = \emptyset \end{cases},$$

or equivalently

$$f_\Gamma(\mathcal{Q}) = \min\{\operatorname{rk}(V) | V \in L_{\widehat{\mathcal{P}}}, \mathcal{Q} \text{ a component of } \mathcal{P}_V\}.$$

f_Γ is increasing in the sense that if $\mathcal{O} \subsetneq \mathcal{Q}$ then $f_\Gamma(\mathcal{O}) < f_\Gamma(\mathcal{Q})$. We call $f_\Gamma(\mathcal{O})$ the Γ-rank of $\mathcal{O}$. Let $\Gamma^k_{\mathcal{P}}$ be the poset formed by $\{\mathcal{Q} \in \Gamma | f_\Gamma(\mathcal{Q}) \leq k\}$ and for a poset L let $L^{\max}$ denote the maximal elements of L. Then we have

Proposition 5.4.8. $LS^{r,k}(\mathcal{P}) = \sum\limits_{\mathcal{Q}\in\Gamma^{k,\max}_{\mathcal{P}}} C^r_{\mathcal{Q}}(\mathcal{P})$.

Proof. $C^r_W(\mathcal{P}) = \sum_{i=1}^m C^r_{\mathcal{P}^i_W}(\mathcal{P})$ by definition, where $\mathcal{P}^1_W, \ldots, \mathcal{P}^m_W$ are the components of $\mathcal{P}_W$. If $\operatorname{rk}(W) \leq k$, then $f_\Gamma(\mathcal{P}^i_W) \leq k$ for all components $\mathcal{P}^i_W$ of $\mathcal{P}_W$. Hence $\mathcal{P}^i_W \in \Gamma^k_{\mathcal{P}}$ and $\mathcal{P}^i_W \subset \mathcal{Q}$ for some $\mathcal{Q} \in \Gamma^{k,\max}_{\mathcal{P}}$. Since this holds for all components $\mathcal{P}^i_W$ of $\mathcal{P}_W$, $C^r_W(\mathcal{P}) \subset \sum_{\mathcal{Q}\in\Gamma^{k,\max}_{\mathcal{P}}} C^r_{\mathcal{Q}}(\mathcal{P})$ and we are done. □

Example 5.4.9. Let $\mathcal{Q}$ and $\mathcal{Q}'$ be as in Figures 5.1 and 5.5. Label the facets of $\mathcal{Q}$ and $\mathcal{Q}'$ by A, B, C, D, E as shown in Figure 5.10. The Hasse diagrams of $L_{\widehat{\mathcal{Q}}}$ and $\Gamma_{\mathcal{Q}}$ are shown in Figure 5.10 organized according to rank and Γ-rank, respectively. For $L_{\widehat{\mathcal{Q}}}$ we use the labels assigned to the flats in Example 5.2.5. We label the complexes in $\Gamma_{\mathcal{Q}}$ and $\Gamma_{\mathcal{Q}'}$ by listing the their facet labels. Figures 5.2, 5.4 show the complexes $\Gamma^{2,\max}_{\mathcal{Q}}$. Figure 5.6 shows the subcomplexes of $\Gamma^{2,\max}_{\mathcal{Q}'}$ which are not stars of vertices.

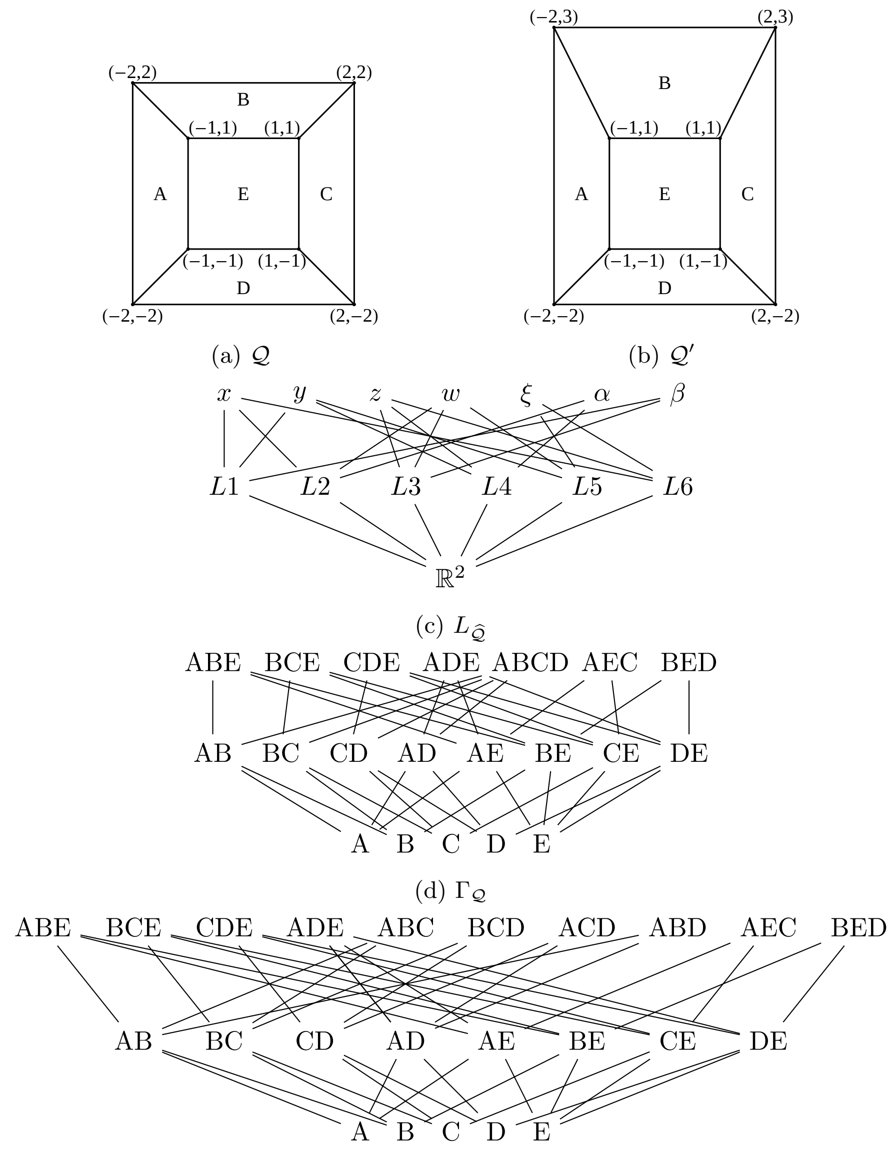
By Proposition 5.4.8 and Theorem 5.4.6, $C^r_d(\mathcal{Q})$ and $C^r_d(\mathcal{Q}')$ have a basis of splines which vanish outside of the complexes $\Gamma^{\max}_{\mathcal{Q}}$ and $\Gamma^{\max}_{\mathcal{Q}'}$, respectively, for $d \gg 0$. This proves the claims made in the Example 5.1.

Now suppose $\mathcal{P} = \Delta$ is simplicial. Proposition 5.2.8 shows that Γ_Δ is the poset of stars of faces τ so that $\operatorname{aff}(\tau)$ appears in L_Δ. Every star in $\Gamma^{k,\max}_\Delta$ is contained in the star of a face $\tau \in \Delta_{n-k}$, so we obtain the following corollary to Proposition 5.4.8.

Corollary 5.4.10. $LS^{r,k}(\Delta) := \sum_{\tau\in\Delta_{n-k}} C^r_{st(\tau)}(\Delta)$.

Setting $k = n$ and applying Theorem 5.4.6, we obtain the existence of a star-supported basis for $C^r_d(\Delta)$ in any dimension.

Corollary 5.4.11. *Let $\Delta \subset \mathbb{R}^n$ be a pure, n-dimensional, hereditary simplicial complex. Then $C^r_d(\Delta)$ has a basis consisting of splines supported on the star of a vertex for $d \gg 0$.*



(a) $\mathcal{Q}$ (b) $\mathcal{Q}'$

(c) $L_{\widehat{\mathcal{Q}}}$

(d) $\Gamma_{\mathcal{Q}}$

(e) $\Gamma_{\mathcal{Q}'}$

Figure 5.10: Example 5.4.9 - $\Gamma_{\mathcal{Q}}$ and $\Gamma_{\mathcal{Q}'}$

5.5 Lattice-Supported Splines and the McDonald-Schenck Formula

In this section we address the question of when $\dim_{\mathbb{R}} C^r_d(\mathcal{P})$ becomes polynomial, particularly in the planar case where these polynomials have been computed by Alfeld, Schumaker, McDonald, and Schenck ([3] and [38]). Rephrased, this is a question about when the Hilbert function $HF(C^r(\widehat{\mathcal{P}}), d)$ of the graded module $C^r(\widehat{\mathcal{P}})$ agrees with the Hilbert polynomial $HP(C^r(\widehat{\mathcal{P}}), d)$. We give an indication of how we may use lattice-supported splines to address this problem and also give conjectural bounds on d for when $HF(C^r(\widehat{\mathcal{P}}), d) = HP(C^r(\widehat{\mathcal{P}}), d)$ in the case $\mathcal{P} \subset \mathbb{R}^2$. A more complete picture and an answer to this question is given in Chapter 6.

There is a convenient notion for discussing when the Hilbert function $HF(M,d)$ of a graded module M over $R = \mathbb{R}[x_1,\ldots,x_n]$ agrees with the Hilbert polynomial $HP(M,d)$, namely the *regularity* of M, denoted $\operatorname{reg}(M)$, which we introduced briefly in Chapter 2. Let us recall the definition. Suppose $F_\bullet \to M$ is the graded minimal free resolution of M, with $F_i = \bigoplus_j R(-a_{ij})$. Then

$$\operatorname{reg}(M) := \max_{i,j}\{a_{ij} - i\}$$

Of particular relevance to us is Theorem 2.6.3, which states that $HF(M,d) = HP(M,d)$ for $d \geq \operatorname{reg}(M) + \delta - n$, where δ is the projective dimension of M.

We apply this theorem to $C^r(\widehat{\mathcal{P}})$. First, since $\mathcal{P}$ is hereditary, $C^r(\widehat{\mathcal{P}})$ is the kernel of a matrix by Lemma 3.3.6 and hence a second syzygy. $C^r(\widehat{\mathcal{P}})$ is a module over $S = \mathbb{R}[x_0,\ldots,x_n]$, so since $C^r(\widehat{\mathcal{P}})$ is a second syzygy, $\operatorname{pd}(C^r(\widehat{\mathcal{P}})) \leq n-1$. By Theorem 2.6.3, $HF(C^r(\widehat{\mathcal{P}}),d) = HP(C^r(\widehat{\mathcal{P}}),d)$ for $d \geq \operatorname{reg}(C^r(\widehat{\mathcal{P}})) - 1$.

According to Alfeld-Schumaker, $HF(C^r(\widehat{\Delta}),d) = HP(C^r(\widehat{\Delta}),d)$ for $d \geq 3r+1$, where $\Delta \subset \mathbb{R}^2$ is a generic simplicial complex [3]. This is implied by the regularity bound $\operatorname{reg}(C^r(\widehat{\Delta})) \leq 3r+2$. Schenck conjectures a tightening of this bound, namely $HF(C^r(\widehat{\Delta}),d) = HP(C^r(\widehat{\Delta}),d)$ for $d \geq 2r+1$ [52]. We will call this the $2r+1$ conjecture. This conjecture is equivalent to the bound $\operatorname{reg}(C^r(\widehat{\Delta})) \leq 2r+2$. Indeed, an application of Theorem 2.6.3 shows that the regularity bound implies the equality of Hilbert function and polynomial in the required degree. See [51] for a discussion of the other implication.

We now relate $\operatorname{reg}(C^r(\widehat{\mathcal{P}}))$ to $\operatorname{reg}(LS^{r,n}(\widehat{\mathcal{P}}))$. Theorem 5.4.3 provides the short exact sequence

$$0 \to LS^{r,n}(\widehat{\mathcal{P}}) \to C^r(\widehat{\mathcal{P}}) \to C \to 0,$$

where C has finite length. The following proposition identifies C as a local cohomology module of $LS^{r,n}(\widehat{\mathcal{P}})$. A brief introduction to local cohomology may be found in Chapter 2.

Proposition 5.5.1. *Let $\mathcal{P} \subset \mathbb{R}^n$ be a pure hereditary polytopal complex and C be the cokernel of the inclusion $LS^{r,n}(\widehat{\mathcal{P}}) \hookrightarrow C^r(\widehat{\mathcal{P}})$. Then*

$$C \cong H^1_m(LS^{r,n}(\widehat{\mathcal{P}})),$$

where $H^1_m(LS^{r,n}(\widehat{\mathcal{P}}))$ is the first local cohomology module of $LS^{r,n}(\widehat{\mathcal{P}})$ with respect to $m = (x_0,\ldots,x_n)$, the homogeneous maximal ideal of $S = \mathbb{R}[x_0,\ldots,x_n]$.

Proof. If M is a graded S-module, let $H^i_m(M)$ denote the ith local cohomology module of M with respect to $m = (x_0,\ldots,x_n)$, $\widetilde{M}(i)$ the associated twisted sheaf on $\mathbb{P}^n$, and $H^0(\widetilde{M}(i))$ the vector space of global sections of $\widetilde{M}(i)$. Define $\Gamma(M) = \bigoplus_i H^0(\widetilde{M}(i))$. We have the four term exact sequence (see [23] Corollary

A1.12)

$$0 \to H^0_m(M) \to M \to \Gamma(M) \to H^1_m(M) \to 0.$$

The graded modules $LS^{r,n}(\widehat{\mathcal{P}})$ and $C^r(\widehat{\mathcal{P}})$ determine the same sheaf since their localizations at nonmaximal primes agree by Theorem 5.4.3. Hence $\Gamma(LS^{r,n}(\widehat{\mathcal{P}})) = \Gamma(C^r(\widehat{\mathcal{P}}))$. Furthermore $C^r(\widehat{\mathcal{P}}) = \Gamma(C^r(\widehat{\mathcal{P}}))$. This is a consequence of the fact that $C^r(\widehat{\mathcal{P}})$ is a second syzygy. From this it follows that $Ext^i_S(C^r(\widehat{\mathcal{P}}), S) = 0$ for $i = n, n+1$ and hence $H^i_m(C^r(\widehat{\mathcal{P}})) = 0$ for $i = 0, 1$ by local duality ([23] Theorem 10.6). The four term exact sequence above then yields $C^r(\widehat{\mathcal{P}}) = \Gamma(C^r(\widehat{\mathcal{P}}))$.

Putting this all together and using the fact that $H^0_m(LS^{r,n}(\widehat{\mathcal{P}})) = 0$ since $LS^{r,n}(\widehat{\mathcal{P}})$ has no submodule of finite length, we arrive at the short exact sequence

$$0 \to LS^{r,n}(\widehat{\mathcal{P}}) \to C^r(\widehat{\mathcal{P}}) \to H^1_m(LS^{r,n}(\widehat{\mathcal{P}})) \to 0$$

So C, the cokernel of the inclusion $LS^{r,n}(\widehat{\mathcal{P}}) \hookrightarrow C^r(\widehat{\mathcal{P}})$, may be identified with $H^1_m(LS^{r,n}(\widehat{\mathcal{P}}))$. □

We record a couple of facts (see Chapter 4 or Appendix A of [23]) about regularity and local cohomology in the following lemma.

Lemma 5.5.2. *Let M be a graded S-module, $m \subset S$ the maximal homogeneous ideal.*

1. *If M has finite length, then reg $M = \max_j\{j | M_j \neq 0\}$.*
2. *$H^i_m(M)$ has finite length for every $i \geq 0$.*
3. *reg$(M) = \max_j$ reg $H^j_m(M) + j$*

Corollary 5.5.3. *Let $\mathcal{P} \subset \mathbb{R}^n$ be a pure hereditary polytopal complex. Set $t =$ reg$(LS^{r,n}(\widehat{\mathcal{P}}))$.*

1. *If $d \geq t$ then $HF(LS^{r,n}(\widehat{\mathcal{P}}), d) = HF(C^r(\widehat{\mathcal{P}}), d) = HP(C^r(\widehat{\mathcal{P}}), d)$.*
2. *reg$(C^r(\widehat{\mathcal{P}})) \leq t$ and $HF(C^r(\widehat{\mathcal{P}}), d) = HP(C^r(\widehat{\mathcal{P}}), d)$ for $d \geq t - 1$.*

Proof. From Theorem 5.4.3 we have the short exact sequence

$$0 \to LS^{r,n}(\widehat{\mathcal{P}}) \to C^r(\widehat{\mathcal{P}}) \to H^1_m(LS^{r,n}(\widehat{\mathcal{P}})) \to 0,$$

(1) If $d \geq t$ then $H^1_m(LS^{r,n}(\widehat{\mathcal{P}}))_d = 0$ by Lemma 5.5.2 and $HF(LS^{r,n}(\widehat{\mathcal{P}}), d) = HF(C^r(\widehat{\mathcal{P}}), d)$. We have $\text{pd}(LS^{r,n}(\widehat{\mathcal{P}})) \leq n$ since $LS^{r,n}(\widehat{\mathcal{P}})$ has no submodule of finite length, so Theorem 2.6.3 yields that $HF(LS^{r,n}(\widehat{\mathcal{P}}), d) = HP(LS^{r,n}(\widehat{\mathcal{P}}), d)$ for $d \geq t+n-n = t$. The result follows since $HP(LS^{r,n}(\widehat{\mathcal{P}}), d) = HP(C^r(\widehat{\mathcal{P}}), d)$
(2) If

$$0 \to A \to B \to C \to 0$$

is a short exact sequence of graded modules, then $\text{reg}(B) \leq \max\{\text{reg}(A), \text{reg}(C)\}$ (see §20.5 of [22]). This fact coupled with Proposition 5.5.1 yields $\text{reg}(C^r(\widehat{\mathcal{P}})) \leq$

t. The second statement of (2) follows from Theorem 2.6.3 and the fact that $C^r(\widehat{\mathcal{P}})$ is a second syzygy. □

In [32] and [33], star-supported bases are constructed for $C^r_d(\Delta)$, $\Delta \subset \mathbb{R}^2$ any triangulation of a disk. According to Proposition 5.5.1, this implies $H^1_m(LS^{r,2}(\widehat{\Delta}))_{3r+2} = 0$, which is compatible with (but *not necessarily* equivalent to) the statement reg $LS^{r,2}(\widehat{\Delta}) \leq 3r+2$. The following conjecture is a natural generalization of this observation.

Conjecture 5.5.4. *Let $\mathcal{P} \subset \mathbb{R}^2$ be a hereditary polytopal complex with F being the maximum length of the boundary of a polytope of $\mathcal{P}$. Then*

$$\mathit{reg}(LS^{r,2}(\widehat{\mathcal{P}})) \leq F(r+1) - 1.$$

It is important to note that Alfeld and Schumaker construct simplicial complexes $\Delta \subset \mathbb{R}^2$ so that $C^r_{3r+1}(\Delta)$ does *not* have a star-supported basis. Again via Proposition 5.5.1, this implies that (for these particular examples)
$H^1_m(LS^{r,2}(\widehat{\Delta}))_{3r+1} \neq 0$, so reg $LS^{r,2}(\widehat{\Delta}) \geq 3r+2$. So if Conjecture 5.5.4 is true, it is an optimal bound, at least in the simplicial case. Conjecture 5.5.4 would also imply, via Corollary 5.5.3, that $\dim_{\mathbb{R}} C^r(\widehat{\mathcal{P}})_d$ agrees with the McDonald-Schenck formula for $d \geq F(r+1) - 2$. We also propose the following generalization of Schenck's $2r+1$ conjecture.

Conjecture 5.5.5. *Let $\mathcal{P} \subset \mathbb{R}^2$ be a hereditary polytopal complex with F being the maximum length of the boundary of a polytope of $\mathcal{P}$. Then*

$$\mathit{reg}(C^r(\widehat{\mathcal{P}})) \leq (F-1)(r+1).$$

This conjecture would imply, via Corollary 5.5.3, that $\dim_{\mathbb{R}} C^r(\widehat{\mathcal{P}})_d$ agrees with the McDonald-Schenck formula for $d \geq (F-1)(r+1) - 1$. The simplicial case of this, Schenck's $2r+1$ conjecture, is still open. See [51] for an approach using the cohomology of sheaves on $\mathbb{P}^2$ and [40] for an example showing that this bound is tight. We give a family of examples which show that the regularity bound of Conjecture 5.5.5 cannot be lowered further. This example is generalized in Example 6.5.1.

Theorem 5.5.6. *There exists a polytopal complex $\mathcal{P} \subset \mathbb{R}^2$ having one triangular face, $n-1$ quadrilateral faces, and two $(n+1)$-gons, such that $C^r(\widehat{\mathcal{P}})$ has a minimal generator of degree $n(r+1)$ supported on a single facet. Hence $\mathit{reg}(C^r(\widehat{\mathcal{P}})) \geq n(r+1)$ and Conjecture 5.5.5 cannot be made tighter.*

Proof. Let $T_n \subset \mathbb{R}^2$ be the polytopal complex with

- $2n+1$ vertices as follows: $v_0 = w_0 = (0,0)$, $v_i = (i, i(i+1)/2)$ for $i = 1, \ldots, n$, and $w_j = (j, j(j+1)/2)$ for $j = 1, \ldots, n$
- 1 triangular face P_0 with vertices $(0,0), v_1, w_1$

- $n-1$ quadrilaterals P_i with vertices $v_i, w_i, v_{i+1}, w_{i+1}$ for $i = 1, \ldots, n-1$
- Two $(n+1)$-gons A and B with vertices $(0,0), v_1, \ldots, v_n$ and $(0,0)$, $w_1, \ldots, w_n$, respectively. (See Figure 5.11)

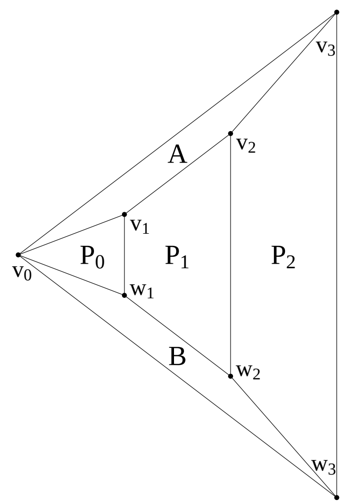


Figure 5.11: T_3

Set $S = \mathbb{R}[x,y,z]$,$R = \mathbb{R}[x,z]$. $u_k = (k+1)x - y - \binom{k+1}{2}z, h_k = x - kz, l_k = (k+1)x + y - \binom{k+1}{2}z$ are the homogenized forms defining the edges between v_k and v_{k+1}, v_k and w_k, w_k and w_{k+1}, respectively. Let $\phi_A : C^r(\widehat{T_n}) \to S$ denote the map obtained by restricting splines to the facet A and set $NT_n^r = \ker \phi_A$.

Suppose we have $G \in NT_n^r$. Then $u_i^{r+1} | G_{\widehat{P_i}}$ and $l_i^{r+1} | G_{\widehat{B}} - G_{\widehat{P_i}}$ for $i = 0, \ldots, n-1$. So $G_{\widehat{B}} \in (u_i^{r+1}, l_i^{r+1})$ for $i = 0, \ldots, n-1$ and $G_{\widehat{B}} \in \cap_{i=0}^{n-1}(u_i^{r+1}, l_i^{r+1})$ $= J_r$. Define $p_y : S \to R$ by $p_y(f(x,y,z)) = f(x,0,z)$ for $f(x,y,z) \in S$. $p_y(u_k) = p_y(l_k) = (k+1)x - \binom{k+1}{2}$, so $p_y(J_r) = I_r \subset R$ is the principal ideal $\cap_{i=0}^{n-1}(x - (k/2)z)^{r+1} = (\prod_{k=0}^{n-1}(x - (k/2)z))^{r+1}$.

So we have a graded homomorphism of S-modules $p_y \circ \phi_B : NT_n^{r+1} \to I_r$, where $\phi_B(G) = G_{\widehat{B}}$. Define $F(B) \in C^r_{\widehat{B}}(\widehat{T}_n)$ by $F(B)_{\widehat{\sigma}} = 0$ for every facet σ of T_n other than B and $F(B)_{\widehat{B}} = L_{\partial \widehat{B}}$. $(p_y \circ \phi_B)(F(B)) = (\prod_{k=0}^{n-1}(k+1)(x - (k/2)z))^{r+1}$, which is a minimal generator of the ideal I_r. It follows that $F(B)$ is a minimal generator of $C^r(\widehat{T_n})$. Since $F(B)$ has degree $n(r+1)$, we are done. $\square$

The following corollary indicates how different the polytopal case is from the simplicial, in which case $C^r(\widehat{\Delta})$ is generated as a module over $S = \mathbb{R}[x,y,z]$ in degree at most $3r+2$.

Corollary 5.5.7. *If $\mathcal{P}$ is planar polytopal complex, then $C^r(\widehat{\mathcal{P}})$ may be generated as an S-module in arbitrarily high degree.*

Chapter 6

Regularity of Mixed Spline Spaces

As we have seen, one of the fundamental questions in spline theory is to determine the dimension of the space $C^r_d(\mathcal{P})$ of splines of degree at most d, where $\mathcal{P} \subset \mathbb{R}^n$ is a polytopal complex. In the bivariate, simplicial case, these questions are studied by Alfeld-Schumaker in [2] and [3] using Bernstein-Bezier methods. A signature result in [3] is a formula for $\dim C^r_d(\Delta)$ when $d \geq 3r+1$ and $\Delta \subset \mathbb{R}^2$ is a generic simplicial complex. For $\Delta \subset \mathbb{R}^2$ simplicial and nongeneric, Hong [32] and Ibrahim-Schumaker [33] derive a formula for $\dim C^r_d(\Delta)$ when $d \geq 3r+2$ as a byproduct of constructing local bases for these spaces.

From Proposition 3.3.12, we know $C^r_d(\mathcal{P}) \cong C^r(\widehat{\mathcal{P}})_d$, the dth graded piece of the algebra $C^r(\widehat{\mathcal{P}})$ of splines on the cone $\widehat{\mathcal{P}}$ over $\mathcal{P}$. The function $HF(C^r(\widehat{\mathcal{P}}), d) = \dim_{\mathbb{R}} C^r(\widehat{\mathcal{P}})_d$ is the *Hilbert function* of $C^r(\widehat{\mathcal{P}})$ in commutative algebra, and a standard result is that the values of the Hilbert function eventually agree with the *Hilbert polynomial* $HP(C^r(\widehat{\mathcal{P}}), d)$ of $C^r(\widehat{\mathcal{P}})$. An important invariant of $C^r(\widehat{\mathcal{P}})$ is the *postulation number* $\wp(C^r(\widehat{\mathcal{P}}))$, which is the largest integer d so that $HP(C^r(\widehat{\mathcal{P}}), d) \neq HF(C^r(\widehat{\mathcal{P}}), d)$. In this terminology the Alfeld-Schumaker result above could be viewed as a computation of $HP(C^r(\widehat{\Delta}), d)$ plus the bound $\wp(C^r(\widehat{\Delta})) \leq 3r$.

The goal of this chapter is to provide upper bounds on the postulation number $\wp(C^\alpha(\mathcal{P}))$ for *central* polytopal complexes $\mathcal{P} \subset \mathbb{R}^{n+1}$, where $C^\alpha(\mathcal{P})$ is the algebra of *mixed splines* over $\mathcal{P}$. A *central* polytopal complex is one in which the intersection of all interior faces is nonempty; if $\mathcal{P}$ is central then splines on $\mathcal{P}$ are a graded algebra. *Mixed splines* are splines in which different smoothness conditions are imposed across codimension one faces.

The main reason for bounding $\wp(C^\alpha(\mathcal{P}))$ is that the Hilbert polynomial of $C^\alpha(\mathcal{P})$ has been computed in situations where there are no known bounds on $\wp(C^\alpha(\mathcal{P}))$, rendering these formulas impractical. Currently, bounds which do not make heavy restrictions on the complex $\mathcal{P}$ are known only in the simplicial case. These bounds are recorded in Table 6.1. For particular types of complexes $\mathcal{P}$, better and sometimes exact bounds are known for $\wp(C^r(\mathcal{P}))$. For brevity, we denote $\wp(C^r(\widehat{\Delta}))$ by $\wp_r$ in Table 6.1. In contrast, the Hilbert polynomial $HP(C^\alpha(\mathcal{P}), d)$ has been computed for *all* central polytopal complexes $\mathcal{P} \subset \mathbb{R}^3$. This is done in the simplicial case with mixed smoothness by Schenck-Geramita [28], in the polytopal case with uniform smoothness by

Analytic Methods

Bound	Context	Computed by
$\wp_r \leq 3r$	*generic* simplicial $\Delta \subset \mathbb{R}^2$	Alfeld-Schumaker [3]
$\wp_r \leq 3r+1$	*all* simplicial $\Delta \subset \mathbb{R}^2$	Hong [32]
		Ibrahim-Schumaker [33]
$\wp_1 \leq 3$	*all* simplicial $\Delta \subset \mathbb{R}^2$	Alfeld-Piper-Schumaker [1]
$\wp_1 \leq 7$	*generic* simplicial $\Delta \subset \mathbb{R}^3$	Alfeld-Schumaker-Whiteley [7]

Homological Methods

Bound	Context	Computed by
$\wp_r \leq 4r$	*all* simplicial $\Delta \subset \mathbb{R}^2$	Mourrain-Villamizar [42]
$\wp_1 \leq 1$	*generic* simplicial $\Delta \subset \mathbb{R}^2$	Billera [9]

Table 6.1: Bounds on $\wp_r = \wp(C^r(\widehat{\Delta}))$

Schenck-McDonald [38], and in the polytopal case with mixed smoothness and boundary conditions, in chapter 7. In this chapter we provide the first bound on $\wp(C^\alpha(\mathcal{P}))$ for all central polytopal complexes $\mathcal{P} \subset \mathbb{R}^3$. Specifically, given *smoothness parameters* $\alpha(\tau)$ associated to each codimension one face $\tau \in \mathcal{P}$, our first result is the following.

Theorem 6.6.7 Let $\mathcal{P} \subset \mathbb{R}^3$ be a central, pure, hereditary three-dimensional polytopal complex. Set

$$e(\mathcal{P}) = \max_{\tau \in \mathcal{P}_2^0} \{ \sum_{\gamma \in (\mathrm{st}(\tau))_2} (\alpha(\gamma)+1) \},$$

where $\mathrm{st}(\tau)$ denotes the star of τ and $(\mathrm{st}(\tau))_2$ denotes the 2-faces of $\mathrm{st}(\tau)$. Then

$$\wp(C^\alpha(\mathcal{P})) \leq e(\mathcal{P}) - 3.$$

In particular, $HP(C^\alpha(\mathcal{P}), d) = \dim_{\mathbb{R}} C^\alpha(\mathcal{P})_d$ for $d \geq e(\mathcal{P}) - 2$.

From an algebraic perspective, another reason for bounding $\wp(C^\alpha(\mathcal{P}))$ is that almost all existing bounds, including most in Table 6.1, have been computed using analytic techniques. There are a few instances where algebraic techniques are applied to bound $\wp(C^\alpha(\mathcal{P}))$. In [9], Billera proves $\wp(C^1(\widehat{\Delta})) \leq 1$ for *generic* simplicial complexes (this result relies on a computation of Whiteley [62]). The most general bound produced by homological techniques to date is by Mourrain-Villamizar [42]; building on work of Schenck-Stillman [50] they prove that $\wp(C^r(\widehat{\Delta})) \leq 4r$ for Δ a planar simplicial complex, recovering an earlier result of Alfeld-Schumaker [2]. Our second result is the following.

Theorem 6.7.2 Let $\Delta \subset \mathbb{R}^3$ be a central, pure, hereditary three-dimensional simplicial complex. For a 2-face $\tau \in \Delta_2^0$, set

$$M(\tau) = (\alpha(\tau)+1) + \max\{(\alpha(\gamma_1)+1) + (\alpha(\gamma_2)+1) | \gamma_1 \neq \gamma_2 \in (\mathrm{st}(\tau))_2\}.$$

Then

$$\wp(C^{\alpha}(\Delta)) \le \max_{\tau\in\Delta_2^0}\{M(\tau)\} - 2.$$

In particular, $HP(C^{\alpha}(\Delta), d) = \dim_{\mathbb{R}} C^r(\Delta)_d$ for $d \ge \max_{\tau\in\Delta_2^0}\{M(\tau)\} - 1$.

Setting $\alpha(\tau) = r$ for all $\tau \in \Delta_2^0$, we recover that $HP(C^r(\widehat{\Delta}), d) = \dim C^r(\widehat{\Delta})_d$ for $d \ge 3r + 2$. This was originally proved via constructing local bases by Hong [32] and Ibrahim-Schumaker [33], and is the best bound valid for all planar simplicial complexes recorded in Table 6.1.

A key tool we use to prove these results is the *Castelnuovo-Mumford regularity* of $C^{\alpha}(\mathcal{P})$, denoted *reg* $(C^{\alpha}(\mathcal{P}))$. The relationship between *reg* $(C^{\alpha}(\mathcal{P}))$ and $\wp(C^{\alpha}(\mathcal{P}))$ is discussed in § 2.6. This invariant is also used by Schenck-Stiller in [51]. Our particular way of using regularity is inspired by an observation used in the Gruson-Lazarsfeld-Peskine theorem bounding the regularity of curves in projective space. In the context of splines this observation is roughly that, if we are lucky, we can bound *reg* $(C^{\alpha}(\mathcal{P}))$ by the regularity of a 'bad' approximation. This statement is made precise in Proposition 2.6.8 and Theorem 6.6.2. We take as our approximation the locally-supported subalgebras of splines introduced in Chapter 5. This is an algebraic analogue of locally-supported bases used in [32, 33].

The chapter is organized as follows. We recall the construction of lattice-supported splines $LS^{\alpha,k}(\mathcal{P})$ introduced in Chapter 5 and adapt them to our situation. These will provide approximations to $C^{\alpha}(\mathcal{P})$. In § 6.3 and § 6.4 we fit lattice-supported splines into a Čech-like complex. In § 6.6 we prove our main results for bounding regularity of spline modules of low projective dimension and prove Theorem 6.6.7 bounding the regularity of $C^{\alpha}(\mathcal{P})$ where $\mathcal{P} \subset \mathbb{R}^3$ is a central polytopal complex. In § 6.7 we build on work of Tohaneanu-Minac [40] and prove the more precise regularity estimate for central simplicial complexes $\Delta \subset \mathbb{R}^3$ in Theorem 6.7.2. We close in § 6.9 with conjectured regularity bounds generalizing those derived in this chapter. The two following examples illustrate our results.

6.1 Examples

Let $\mathcal{P} \subset \mathbb{R}^2$ be a subdivision of a topological 2-disk by convex polytopes. $C^r(\mathcal{P})$ is the algebra of r-splines on $\mathcal{P}$, where $\alpha(\tau) = r$ for every interior edge and $\alpha(\tau) = -1$ for every boundary edge. By Corollary 3.14 of [38], the Hilbert polynomial of $C^r(\widehat{\mathcal{P}})$ is

$$HP(C^r(\widehat{\mathcal{P}}), d) = \quad \tfrac{f_2}{2}d^2 + \tfrac{3f_2-2(r+1)f_1^0}{2}d + f_2 + \left(\binom{r}{2} - 1\right) f_1^0 + \textstyle\sum_j c_j, \quad (6.1)$$

where f_i, f_i^0 are the number of i-faces and interior i-faces of $\mathcal{P}$, r is the smoothness parameter, and the constants c_j record the dimension of certain vector spaces coming from ideals of powers of linear forms.

Example 6.1.1. The complex $\mathcal{Q}$ in Figure 6.1 has $f_2 = 4, f_1^0 = 6, f_3^0 = 3$. It is shown in § 4 of [38] that there are 4 constants c_j in the formula (6.1), and they are all equal to the constant

$$\binom{r+2}{2} + \left\lceil \frac{r+1}{2} \right\rceil \left(r - \left\lceil \frac{r+1}{2} \right\rceil \right)$$

Hence by equation (6.1),

$$HP(C^r(\widehat{\mathcal{Q}}), d) = \quad 2d^2 - 6rd + 6\tbinom{r}{2} - 2 + 4\left(\tbinom{r+2}{2} + \left\lceil \tfrac{r+1}{2} \right\rceil \left(r - \left\lceil \tfrac{r+1}{2} \right\rceil \right)\right) \quad (6.2)$$

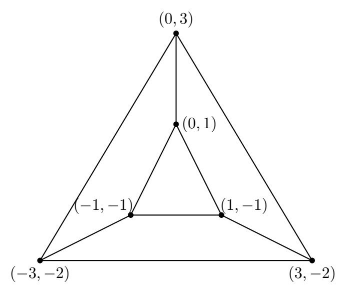


Figure 6.1: $\mathcal{Q}$

By Theorem 6.6.7, $\wp(C^r(\widehat{\mathcal{Q}})) \leq e(\mathcal{Q}) - 3$, where

$$e(\mathcal{Q}) = \max_{\tau \in \mathcal{P}_2^0} \{ \sum_{\gamma \in (\mathrm{st}(\tau))_2} (\alpha(\gamma) + 1) \}.$$

The star of each interior edge of $\mathcal{Q}$ has 5 edges which are interior. So $e(\mathcal{Q}) = 5(r+1)$ and the Hilbert function $HF(C^r(\widehat{\mathcal{Q}}), d)$ agrees with the Hilbert polynomial $HP(C^r(\widehat{\mathcal{Q}}), d)$ above for $d \geq 5(r+1) - 2$. Computations in Macaulay2 [30] suggest that $\wp(C^r(\widehat{\mathcal{Q}})) \leq 2(r+1) - 1$ (in fact the behavior is the same as Example 6.8.1 in § 6.8), indicating that there is room for improvement in Theorem 6.6.7.

In [28, Theorem 4.3], Geramita and Schenck give a formula for $HP(C^\alpha(\widehat{\Delta}), d)$, where Δ is a planar simplicial complex and α is the vector of smoothness parameters associated to codimension one faces.

Example 6.1.2. Triangulate the polytopal complex $\mathcal{Q}$ in Example 6.1.1 to obtain the simplicial complex Δ below, with $f_2 = 7, f_1^0 = 9$, and $f_0^0 = 3$. Take smoothness parameters $\alpha(\tau) = 2$ on the edges of the center triangle and $\alpha(\tau) = 3$ on the six edges which connect interior vertices to boundary vertices. In Example 4.5 of [28], Schenck and Geramita compute

$$HP(C^\alpha(\widehat{\Delta}), d) = \binom{d+2}{2} - 3\binom{d-1}{2} + 3\binom{d-2}{2} + 6\binom{d-3}{2}.$$

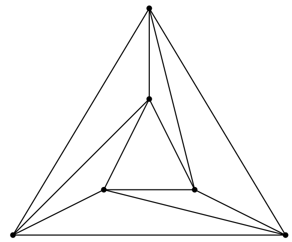


Figure 6.2: Δ

By Theorem 6.7.2, $\wp(C^\alpha(\widehat{\Delta})) \le \max\{M(\tau)|\tau \in \Delta^0_1\} - 2$, where $M(\tau) = \alpha(\tau) + 1 + \max\{\alpha(\gamma_1) + 1 + \alpha(\gamma_2) + 1 | \gamma_1 \neq \gamma_2 \in (\mathrm{st}(\tau))_1\}$. This yields $\wp(C^\alpha(\widehat{\Delta})) \le 10$, so the polynomial above gives the correct dimension of $C^\alpha_d(\Delta)$ for $d \ge 11$. Macaulay2 gives $\wp(C^\alpha(\widehat{\Delta})) = 5$, so the formula is actually correct for $d \ge 6$.

6.2 Lattice-Supported Splines

The construction of lattice-supported splines in Chapter 5 carries over directly to mixed splines; we will denote the corresponding submodules by $LS^{\alpha,k}(\mathcal{P})$. We briefly summarize the construction. For a pure n-dimensional subcomplex $\mathcal{Q} \subset \mathcal{P}$, not necessarily hereditary, define

$$C^\alpha_{\mathcal{Q}}(\mathcal{P}) := \{F \in C^\alpha(\mathcal{P}) | F_\sigma = 0 \text{ for all } \sigma \in \mathcal{P}_n \setminus \mathcal{Q}_n\}.$$

Let $\mathcal{P}^{-1} \subset \partial\mathcal{P}$ denote the set of faces of $\mathcal{P}$ which are contained in a codimension one face τ so that $\alpha(\tau) = -1$; this is a subcomplex of $\partial\mathcal{P}$.

Definition 6.2.1. Let $\mathcal{P} \subset \mathbb{R}^n$ be a polytopal complex and α a list of smoothness parameters.

1. For $\tau \in \mathcal{P}$ a face, $\mathrm{aff}(\tau)$ denotes the affine span of τ.
2. $\mathcal{A}(\mathcal{P}, \mathcal{P}^{-1})$ denotes the hyperplane arrangement $\bigcup\limits_{\substack{\tau \in \mathcal{P}_{n-1} \\ \alpha(\tau) \ge 0}} \mathrm{aff}(\tau)$.
3. $L_{\mathcal{P},\mathcal{P}^{-1}}$ denotes the intersection semi-lattice $L(\mathcal{A}(\mathcal{P}, \mathcal{P}^{-1}))$ of $\mathcal{A}(\mathcal{P}, \mathcal{P}^{-1})$.

The elements $W \in L(\mathcal{P}, \mathcal{P}^{-1})$ are called *flats*. These consist of the whole space $\mathbb{R}^n$, the hyperplanes $\{\mathrm{aff}(\tau) | \alpha(\tau) \ge 0\}$, and all nonempty intersections of these, ordered with respect to reverse inclusion. The *rank* of a flat W, denoted $\mathrm{rk}(W)$, is its codimension as a vector space.

To each flat $W \in L(\mathcal{P}, \mathcal{P}^{-1})$ we associate a *lattice complex* $\mathcal{P}_W$ as follows. Form a graph $G_W(\mathcal{P})$ whose vertices correspond to facets which have a codimension one face τ so that $W \subseteq \mathrm{aff}(\tau)$. Connect two vertices corresponding to facets σ_1, σ_2 if $\sigma_1 \cap \sigma_2$ is a codimension one face of both and $W \subseteq \mathrm{aff}(\sigma_1 \cap \sigma_2)$. Each connected component $G^i_W(\mathcal{P})$ of $G_W(\mathcal{P})$ is the dual graph of a unique subcomplex $\mathcal{P}_W^i$. The lattice complex $\mathcal{P}_W$ is defined as the disjoint union of these $\mathcal{P}_W^i$, which we call components of $\mathcal{P}_W$. Then define

$$C_W^\alpha(\mathcal{P}) := \sum_i C^\alpha_{\mathcal{P}_W^i}(\mathcal{P}),$$

the submodule generated by splines which vanish outside a component of $\mathcal{P}_W$. Then $LS^{\alpha,k}(\mathcal{P})$ is defined by

$$LS^{\alpha,k}(\mathcal{P}) := \sum_{\substack{W \in L_{\widehat{\mathcal{P}},\widehat{\mathcal{P}}^{-1}} \\ 0 \le rk(W) \le k}} C_W^\alpha(\mathcal{P}).$$

It is equivalent to let the sum in the definition of $LS^{\alpha,k}(\mathcal{P})$ run across maximal components (with respect to inclusion) occuring among lattice complexes $\mathcal{P}_W$ with the rank of W at most k. To make this more precise, let $\Gamma^k_{\mathcal{P}}$ be the poset of components of lattice complexes $\mathcal{P}_W$ with $\mathrm{rk}(W) \le k$, ordered with respect to inclusion. Let $\Gamma^{k,\max}_{\mathcal{P}}$ be the set of maximal subcomplexes appearing in $\Gamma^k_{\mathcal{P}}$. Then, just as in Proposition 5.4.8, we have

Proposition 6.2.2.

$$LS^{\alpha,k}(\mathcal{P}) = \sum_{\mathcal{Q} \in \Gamma^{k,\max}_{\mathcal{P}}} C^\alpha_{\mathcal{Q}}(\mathcal{P}).$$

Since we will use this construction primarily in the cases $k = 0$ and $k = 1$, we describe $LS^{\alpha,0}(\mathcal{P})$ and $LS^{\alpha,1}(\mathcal{P})$ precisely. If γ is a face of $\mathcal{P}$ of some dimension, we use $C^\alpha_\gamma(\mathcal{P})$ and $C^\alpha_{\mathrm{st}(\gamma)}(\mathcal{P})$ interchangeably to denote the subring of splines which vanish outside of the star of γ, as long as no confusion results. So $C^\alpha_\sigma(\mathcal{P})$ for $\sigma \in \mathcal{P}_n$ denotes the subring of splines supported on a single facet, $C^\alpha_\tau(\mathcal{P})$ for $\tau \in \mathcal{P}^0_{n-1}$ denotes the ring of splines supported on the two facets of $\mathrm{st}(\tau)$, etc.

Corollary 6.2.3. *Let $\mathcal{P} \subset \mathbb{R}^n$ be a polytopal complex. Then*

$$\begin{aligned} LS^{\alpha,0}(\mathcal{P}) &= \sum_{\sigma \in \mathcal{P}_n} C^\alpha_\sigma(\mathcal{P}) \\ LS^{\alpha,1}(\mathcal{P}) &= \sum_{\tau \in \mathcal{P}^0_{n-1}} C^\alpha_\tau(\mathcal{P}) \end{aligned}$$

Proof. For $k = 0$, the only flat $W \in L(\mathcal{P}, \mathcal{P}^{-1})$ of rank zero is the whole space $\mathbb{R}^n$. The corresponding lattice complex $\mathcal{P}_{\mathbb{R}^n}$ is the disjoint union of the facets of $\mathcal{P}$. Hence

$$LS^{\alpha,0}(\mathcal{P}) = C^\alpha_{\mathbb{R}^n}(\mathcal{P}) = \sum_{\sigma \in \mathcal{P}_n} C^\alpha_\sigma(\mathcal{P}).$$

For $k = 1$, the flats $W \in L(\mathcal{P}, \mathcal{P}^{-1})$ of rank one are precisely the hyperplanes $\mathrm{aff}(\tau)$ with $\alpha(\tau) \ge 0$, where $\tau \in \mathcal{P}_{n-1}$. The components of the lattice complex $\mathcal{P}_{\mathrm{aff}(\tau)}$ are the complexes $\mathrm{st}(\gamma)$ for all γ satisfying $\mathrm{aff}(\gamma) = \mathrm{aff}(\tau)$. If $\gamma \in \mathcal{P}^0_{n-1}$, then $\mathrm{st}(\gamma)$ consists of two facets and all their faces; otherwise $\gamma \in (\partial\mathcal{P})_{n-1}$ and $\mathrm{st}(\gamma)$ consists of a single facet of $\mathcal{P}$ and all its faces. However, as long as $\mathcal{P}$ is hereditary and has more than one facet, every facet $\sigma \in \mathcal{P}_n$ has a codimension one face γ which is interior. Hence $\sigma \subset \mathrm{st}(\gamma)$. It follows that $\Gamma_{\mathcal{P}}^{1,\max}$ consists of stars of interior codimension one faces of $\mathcal{P}$. By Proposition 6.2.2 we have

$$LS^{\alpha,1}(\mathcal{P}) = \sum_{\tau \in \mathcal{P}_{n-1}^0} C_\tau^\alpha(\mathcal{P})$$

□

Theorem 5.4.3 makes precise the sense in which $LS^{r,k}(\mathcal{P})$ is an approximation to $C^r(\mathcal{P})$. This result and its proof extend directly to mixed splines, so we state the result in this context.

Theorem 6.2.4. *Let $\mathcal{P} \subset \mathbb{R}^n$ be a polytopal complex. Then $LS^{\alpha,k}(\mathcal{P})$ fits into a short exact sequence*

$$0 \to LS^{\alpha,k}(\mathcal{P}) \to C^\alpha(\mathcal{P}) \to C \to 0$$

where C has codimension $\geq k+1$ and the primes in the support of C with codimension $k+1$ are contained in the set $\{I(W) | W \in L(\mathcal{P}, \mathcal{P}^{-1})$ and $\mathrm{rk}(W) = k+1\}$.

To use the submodules $LS^{\alpha,k}(\mathcal{P})$ effectively, it will be useful to fit $LS^{\alpha,k}(\mathcal{P})$ into a chain complex whose pieces are easier to understand. In the next section we describe such a complex.

6.3 An Intersection Complex

In this section we introduce a Čech-like complex for finite sums of submodules of a given S-module M and give a criterion for its exactness. We apply this to the submodules $LS^{\alpha,k}(\mathcal{P}) \subset C^\alpha(\mathcal{P})$ in § 6.4.

For an integer N, let $I(k)$ be the set of all subsets of size $k \geq 1$ formed from the index set $\{1, \ldots, N\}$. Thinking of $I \in I(k)$ as a k-simpex of the N-simplex Δ, we have the complex $\Delta_\bullet(S)$ with $\Delta_k(S) = \oplus_{I \in I(k)} S$ below whose homology is the simplicial homology of Δ with coefficients in S.

$$\Delta_\bullet(S): \quad S \xrightarrow{\delta_{N-1}} \bigoplus_{I \in I(N-1)} S \xrightarrow{\delta_{N-2}} \ldots \xrightarrow{\delta_k} \bigoplus_{I \in I(k)} S \xrightarrow{\delta_{k-1}} \ldots \xrightarrow{\delta_1} \bigoplus_{i=1}^{N} S \to 0$$

If $k > 0$ and $e_I \in \bigoplus_{I \in I(k)} S$ is the idempotent corresponding to $I = \{i_1, \ldots, i_k\} \in I(k)$, then

$$\delta_k(e_I) = \sum_{j=1}^{k} (-1)^{j-1} e_{I \setminus i_j}$$

It will be convenient to augment this complex with a final map $\bigoplus_{i=1}^N S \xrightarrow{\epsilon} S$ defined by $\delta_0(e_i) = 1$ for every i. We denote this augmented complex as $\Delta_\bullet^a(S)$.

The homology of $\Delta^a_\bullet(S)$ computes the *reduced* homology of Δ with coefficients in S. We extend these complexes to an S-module M by tensoring; let $\Delta_\bullet(M)$ denote $\Delta_\bullet(S) \otimes_S M$ and $\Delta^a_\bullet(M)$ denote $\Delta^a_\bullet(S) \otimes_S M$.

Now suppose $\mathcal{M} = \{M_1, \dots, M_N\}$, where each M_i is a submodule of M. For $I \subset \{1, \dots, N\}$ let M_I denote the intersection $\cap_{i \in I} M_i$. Define submodules $C_k(\mathcal{M}) = \oplus_{I \in I(k)} M_I \subset \oplus_{I \in I(k)} M = \Delta_k(M)$. Since $M_I \subset M_{I \setminus i}$ for every $i \in I$, the differential δ_k of $\Delta_\bullet(M)$ restricts to a map $\delta_k : C_k(\mathcal{M}) \to C_{k-1}(\mathcal{M})$, so $C_\bullet(\mathcal{M})$ is a subcomplex of $\Delta_\bullet(M)$. For example, if $N = 2$, $C_\bullet(\mathcal{M})$ is the complex

$$0 \to M_{12} \xrightarrow{\delta_1} M_1 \bigoplus M_2,$$

where $\delta_1(m) = (-m, m)$. Given any submodule $M' \subset M$ containing all the M_i, we may augment $C_\bullet(\mathcal{M})$ with the map

$$\bigoplus_{i=1}^{N} M_i \xrightarrow{\epsilon} M',$$

where $\epsilon(m_1, \dots, m_N) = m_1 + \dots + m_N$. We denote this augmented complex by $C^a_\bullet(\mathcal{M}, M')$.

Now consider the condition $(\star)$ on $\mathcal{M}$ given by

$$(\star) \quad M_I \cap (\textstyle\sum_{i \in T} M_i) = \sum_{i \in T}(M_I \cap M_i)$$
$$\text{for every pair of subsets } I, T \subset \{1, \dots, N\}$$

We only need to check this condition on subsets I, T with $I \cap T = \emptyset$, since if there is some $j \in I \cap T$ then $M_I \subset M_j$ and both sides are equal to M_I.

Proposition 6.3.1. *If $\mathcal{M} = \{M_1, \dots, M_N\}$ satisfies $(\star)$ then $H_i(C^a_\bullet(\mathcal{M}, M)) = 0$ for $i > 0$ and $H_0(C^a_\bullet(\mathcal{M}, M)) = M/(\sum_{i=1}^N M_i)$.*

Proof. The assertion $H_0(C^a_\bullet(\mathcal{M}, M)) = M/(\sum_{i=1}^N M_i)$ is always true, so we prove $H_i(C^a_\bullet(\mathcal{M}, M)) = 0$ for $i > 0$. We proceed by induction on the cardinality N of $\mathcal{M}$. If $N = 2$ then $C^a_\bullet(\mathcal{M}, M)$ is the complex

$$0 \to M_{12} \xrightarrow{\delta_1} M_1 \oplus M_2 \to M \to 0$$

which satisfies the conclusion of Proposition 6.3.1. Now suppose $N > 2$. Let $\mathcal{M}' = \{M_1, \dots, M_{N-1}\}$ and $\mathcal{N} = \{M_{1,N}, \dots, M_{N-1,N}\}$, where $M_{i,j} = M_i \cap M_j$. We have a short exact sequence of complexes $0 \to C^a_\bullet(\mathcal{M}', M) \to C^a_\bullet(\mathcal{M}, M) \to C^a_\bullet(\mathcal{N}, M_N)(-1) \to 0$, shown below. Here $C(i)$ denotes the complex C with shifted grading $C(i)_j = C_{i+j}$. This short exact sequence follows from the fact that $C^a_\bullet(\mathcal{M}, M)$ can be constructed as the mapping cone of the (appropriately signed) inclusion $C^a_\bullet(\mathcal{N}, M_N) \hookrightarrow C^a_\bullet(\mathcal{M}', M)$. It is also not difficult to check exactness of this sequence directly.

$$
\begin{array}{lccccccccc}
 & & & 0 & & & & 0 & & 0 \\
 & & & \downarrow & & & & \downarrow & & \downarrow \\
C^a_\bullet(\mathcal{M}', M) & 0 & \longrightarrow & M_{1,\ldots,N-1} & \ldots & \xrightarrow{\delta'_{N-1}} & & \bigoplus_{i=1}^{N-1} M_i & \xrightarrow{\delta'_1} & M \\
 & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
C^a_\bullet(\mathcal{M}, M) & M_{1,\ldots,N} & \xrightarrow{\delta_N} & \bigoplus_{I \in I(N-1)} M_I & \ldots & \xrightarrow{\delta_2} & & \bigoplus_{i=1}^{N} M_i & \xrightarrow{\delta_1} & M \\
 & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
C^a_\bullet(\mathcal{N}, M_N)(-1) & M_{1,\ldots,N} & \xrightarrow{\delta''_N} & \bigoplus_{\substack{I \in I(N-1) \\ N \in I}} M_I & \ldots & \xrightarrow{\delta''_2} & & M_N & \xrightarrow{\delta''_1} & 0 \\
 & \downarrow & & \downarrow & & & & \downarrow & & \\
 & 0 & & 0 & & & & 0 & &
\end{array}
$$

Clearly $\mathcal{M}'$ satisfies $(\star)$, inheriting the necessary conditions from the fact that $\mathcal{M}$ satisfies $(\star)$. Since $|\mathcal{M}'| = N-1$, $H_i(C^a_\bullet(\mathcal{M}', M)) = 0$ for $i > 0$ by induction. We claim $\mathcal{N}$ also satisfies $(\star)$. Interpreted for the set $\mathcal{N}$, the condition $(\star)$ is

$$
\begin{array}{ll}
(\star\star) & M_{I \cup N} \cap (\sum_{i \in T} M_{i,N}) = \sum_{i \in T} M_{I \cup N} \cap M_i \\
 & \text{for every pair of subsets } I, T \subset \{1, \ldots, N-1\}
\end{array}
$$

First note that for any subset $T \subset \{1, \ldots, N-1\}$, $\sum_{i \in T} M_{i,N} = M_N \cap (\sum_{i \in T} M_i)$ since $\mathcal{M}$ satisfies $(\star)$. So the left hand side of $(\star\star)$ is equivalent to $M_{I \cup N} \cap (\sum_{i \in T} M_i)$. Again, since $\mathcal{M}$ satisfies $(\star)$, $M_{I \cup N} \cap (\sum_{i \in T} M_i) = \sum_{i \in T} M_{I \cup N} \cap M_i$. So $\mathcal{N}$ satisfies $(\star)$. Since $|\mathcal{N}| = N-1$, $H_i(C^a_\bullet(\mathcal{N}, M_N)(-1)) = 0$ for $i > 1$ by induction. It follows from the long exact sequence in homology that

$$H_i(C^a_\bullet(\mathcal{M}, M)) = 0$$

for $i > 1$. The tail end of the long exact sequence in homology is

$$
\begin{array}{l}
0 \longrightarrow H_1(C^a_\bullet(\mathcal{M}, M)) \longrightarrow H_1(C^a_\bullet(\mathcal{N}, M_N)(-1)) \longrightarrow \\
\longrightarrow H_0(C^a_\bullet(\mathcal{M}', M)) \longrightarrow H_0(C^a_\bullet(\mathcal{M}, M)) \longrightarrow 0
\end{array}
$$

We have

$$
\begin{aligned}
H_1(C^a_\bullet(\mathcal{N}, M_N)(-1)) &= \frac{M_N}{\sum_{i=1}^{N-1} M_i \cap M_N} \\
&= \frac{M_N}{M_N \cap (\sum_{i=1}^{N-1} M_i)} \\
&= \frac{M_N + \sum_{i=1}^{N-1} M_i}{\sum_{i=1}^{N-1} M_i},
\end{aligned}
$$

where the second equality comes from the fact that $\mathcal{M}$ satisfies $(\star)$. But this is precisely the kernel of the natural surjection

$$H_0(C^a_\bullet(\mathcal{M}', M)) = \frac{M}{\sum_{i=1}^{N-1} M_i} \to \frac{M}{\sum_{i=1}^{N} M_i} = H_0(C^a_\bullet(\mathcal{M}, M)).$$

It follows that $H_1(C^a_\bullet(\mathcal{M}, M)) = 0$ and we are done. □

Corollary 6.3.2. *Let $\mathcal{M} = \{M_1, \ldots, M_N\}$ be a set of submodules of M. Then if $\mathcal{M}$ satisfies $(\star)$,*

$$C_\bullet(\mathcal{M}) \to \sum_{i=1}^{N} M_i \to 0$$

is exact.

6.4 Intersection complex for splines

In this section we apply the complex constructed in § 6.3 to the case of lattice-supported splines. Recall from § 6.2 that $\Gamma^k_{\mathcal{P}}$ is the poset of components appearing among lattice complexes of the form $\mathcal{P}_W$ with the rank of W at most k, ordered with respect to inclusion. $\Gamma^{k,\max}_{\mathcal{P}}$ is the set of maximal complexes appearing in $\Gamma^k_{\mathcal{P}}$. With this notation, Proposition 6.2.2 states

$$LS^{\alpha,k}(\mathcal{P}) := \sum_{\mathcal{O} \in \Gamma^{k,\max}_{\mathcal{P}}} C^\alpha_{\mathcal{O}}(\mathcal{P}),$$

where $C^\alpha_{\mathcal{O}}(\mathcal{P}) \subset C^\alpha(\mathcal{P})$ is the subalgebra of splines vanishing outside of $\mathcal{O}$.

Now set $\mathcal{M}_k = \{C^\alpha_{\mathcal{Q}}(\mathcal{P}) | \mathcal{Q} \in \Gamma^{k,\max}_{\mathcal{P}}\}$. $LS^{\alpha,k}(\mathcal{P})$ fits into the complex

$$C_\bullet(\mathcal{M}_k) \to LS^{\alpha,k}(\mathcal{P}).$$

If $\mathcal{Q}, \mathcal{O}$ are pure n-dimensional subcomplexes of $\mathcal{P} \subset \mathbb{R}^n$, then

$$C^\alpha_{\mathcal{Q}}(\mathcal{P}) \bigcap C^\alpha_{\mathcal{O}}(\mathcal{P}) = C^\alpha_{\mathcal{Q}\cap\mathcal{O}}(\mathcal{P}).$$

This extends to any finite intersection, hence we may write

$$C_i(\mathcal{M}_k) = \bigoplus_{\mathcal{Q}} C^\alpha_{\mathcal{Q}}(\mathcal{P}),$$

where $\mathcal{Q}$ runs across all intersections of i subcomplexes from $\Gamma^{k,max}_{\mathcal{P}}$. As will be evident below, the same subcomplex can appear multiple times as an intersection. We prove exactness of $C_\bullet(\mathcal{M}_1)$.

Proposition 6.4.1. *Let $\mathcal{M}_1 = \{C^\alpha_\tau(\mathcal{P}) | \tau \in \mathcal{P}^0_{n-1}\}$. Then the augmented complex*

$$C_\bullet(\mathcal{M}_1) \to LS^{\alpha,1}(\mathcal{P}) \to 0$$

is exact.

Proof. We show that $\mathcal{M}_1$ satisfies the condition ($\star$) from the previous section; then by Corollary 6.3.2 the proposition will be proved. First suppose given $m > 1$ codimension one faces $\tau_1, \ldots, \tau_m$. If these are all faces of a common facet σ, then $\mathrm{st}(\tau_1) \cap \ldots \cap \mathrm{st}(\tau_m) = \sigma$. Otherwise, this intersection has dimension less than n and no splines are defined on it. Hence to show ($\star$) for $\mathcal{M}_1$ amounts to showing that, given a set $T = \{\tau_1, \ldots, \tau_n\}$ of codimension one faces of $\mathcal{P}$, the following equalities hold. Keep in mind that for two subcomplexes $\mathcal{O}, \mathcal{Q}$, $C^\alpha_{\mathcal{O}}(\mathcal{P}) \cap C^\alpha_{\mathcal{Q}}(\mathcal{P}) = C^\alpha_{\mathcal{O}\cap\mathcal{Q}}(\mathcal{P})$.

1. For any facet $\sigma \in \mathcal{P}_n$,

$$C^\alpha_\sigma(\mathcal{P}) \bigcap \left(\sum_{i=1}^n C^\alpha_{\mathrm{st}(\tau_i)}(\mathcal{P}) \right) = \sum_{i=1}^n C^\alpha_{\sigma\cap\mathrm{st}(\tau_i)}(\mathcal{P}),$$

2. For any codimension one face $\tau \in \mathcal{P}^0_{n-1}$,

$$C^\alpha_{\mathrm{st}(\tau)}(\mathcal{P}) \bigcap \left(\sum_{i=1}^n C^\alpha_{\mathrm{st}(\tau_i)}(\mathcal{P}) \right) = \sum_{i=1}^n C^\alpha_{\mathrm{st}(\tau)\cap\mathrm{st}(\tau_i)}(\mathcal{P})$$

(1) If $\sigma \subset \mathrm{st}(\tau_i)$ for some $\tau_i \in T$, then both sides are equal to $C^\alpha_\sigma(\mathcal{P})$. Otherwise both sides are trivial. (2) If $\tau \in T$, then both sides are equal to $C^\alpha_\tau(\mathcal{P})$. Otherwise, set $C^\alpha_T(\mathcal{P}) = \sum_{i=1}^n C^\alpha_{\mathrm{st}(\tau_i)}(\mathcal{P})$ and let $F \in C^\alpha_{\mathrm{st}(\tau)}(\mathcal{P}) \cap C^\alpha_T(\mathcal{P})$. Since $\tau \notin T$, F must vanish along τ to order $\alpha(\tau)$. Letting σ_1, σ_2 be the two facets of $\mathrm{st}(\tau)$, we see $F|_{\sigma_i} \in C^\alpha_{\sigma_i}(\mathcal{P})$ for $i = 1, 2$. It follows that

$$C^\alpha_\tau(\mathcal{P}) \cap C^\alpha_T(\mathcal{P}) = C^\alpha_{\sigma_1}(\mathcal{P}) \cap C^\alpha_T(\mathcal{P}) + C^\alpha_{\sigma_2}(\mathcal{P}) \cap C^\alpha_T(\mathcal{P}).$$

Now by (1) the intersections $C^\alpha_{\sigma_i}(\mathcal{P}) \cap C^\alpha_T(\mathcal{P})$ distribute. □

Remark 6.4.2. It would be interesting to know if Proposition 6.4.1 holds for any $\mathcal{M}_k$, where $k > 1$.

Proposition 6.4.3. *Let $\mathcal{M}_1 = \{C^\alpha_\tau(\mathcal{P}) | \tau \in \mathcal{P}^0_{n-1}\}$, where $\mathcal{P}$ is a pure n-dimensional hereditary polytopal complex. For a facet $\sigma \in \mathcal{P}_n$, let $\partial^0(\sigma)$ denote the set of codimension one faces of σ that are interior faces of $\mathcal{P}$. Set $\delta(\mathcal{P}) = \max_{\sigma\in\mathcal{P}_n}\{|\partial^0(\sigma)|\}$. The complex $C_\bullet(\mathcal{M}_1)$ satisfies*

$$C_k(\mathcal{M}_1) = \begin{cases} \bigoplus_{\tau\in\mathcal{P}^0_{n-1}} C^\alpha_\tau(\mathcal{P}) & \text{if } k = 1 \\ \bigoplus_{|\partial^0(\sigma)|\geq k} (C^\alpha_\sigma(\mathcal{P}))^{\binom{|\partial^0(\sigma)|}{k}} & \text{if } 2 \leq k \leq \delta(\mathcal{P}) \\ 0 & \text{if } k > \delta(\mathcal{P}) \end{cases}$$

Proof. By definition $C_1(\mathcal{M}_1)$ is the direct sum of all the submodules of $\mathcal{M}_1$. In general we have

$$C_k(\mathcal{M}_1) = \bigoplus_{\mathcal{Q}} C^\alpha_{\mathcal{Q}}(\mathcal{P}),$$

where the direct sum runs over all subcomplexes $\mathcal{Q} \subset \mathcal{P}$ which are intersections of k distinct subcomplexes chosen from the set $\{\text{st}(\tau) | \tau \in \mathcal{P}^0_{n-1}\}$. If $k \geq 2$ then $\mathcal{Q}$ is the intersection of two or more stars of codimension one faces, say $\text{st}(\tau_1), \text{st}(\tau_2), \ldots, \text{st}(\tau_k)$. Hence $\mathcal{Q}$ contains at most one facet, and that facet must have $\tau_1, \ldots, \tau_k$ as faces. So if $k \geq 2$,

$$C_k(\mathcal{M}_1) = \bigoplus_{|\partial^0(\sigma)| \geq k} \left(C^\alpha_\sigma(\mathcal{P})\right)^{\binom{|\partial^0(\sigma)|}{k}},$$

where $|\partial^0(\sigma)|$ is the number of edges of σ which are interior to $\mathcal{P}$. From this we also see that $C_k(\mathcal{M}_1) = 0$ for $k > \delta(\mathcal{P})$. □

6.5 High degree generators for splines

We give a construction motivating the regularity bounds we derive in Corollary 6.6.3, Theorem 6.6.7, and Theorem 6.7.2. These results suggest that in general regularity bounds for $C^\alpha(\mathcal{P})$ might be obtained by taking the maximal sum of smoothness parameters $\alpha(\tau) + 1$ appearing in certain subcomplexes of $\mathcal{P}$. In the following example, starting with a polytope $\sigma \subset \mathbb{R}^n$, we construct a polytopal complex $\mathcal{P}$ so that $\sigma \in \mathcal{P}_n$ and $C^\alpha(\widehat{\mathcal{P}})$ ($C^\alpha(\mathcal{P})$ if $\mathcal{P}$ is central) has a minimal generator supported the facet $\widehat{\sigma}$ (σ if $\mathcal{P}$ is central). Such generators have degree $\sum_{\tau \in \sigma_{n-1}} \alpha(\tau) + 1$. Since *reg* $(C^\alpha(\mathcal{P}))$ in particular bounds the degrees of generators of $C^\alpha(\mathcal{P})$ (see Remark 2.6.2), this construction indicates that a bound on *reg* $(C^\alpha(\mathcal{P}))$ will need to be at least as large as the maximal sum of smoothness parameters over codimension one faces occurring in any facet of $\mathcal{P}$ (or at least boundary facets - see Conjecture 6.9.1). This example generalizes the construction in [20, Theorem 5.7].

For simplicity we restrict the construction to the case of uniform smoothness without imposing boundary vanishing. The generalization to arbitrary smoothness parameters should be clear.

Example 6.5.1. Suppose that $A \subset \mathbb{R}^n$ is a polytope with a codimension one face $\tau \in A_{n-1}$ so that $\partial A \setminus \tau$ is the graph of a piecewise linear function over τ. Remark 6.5.2 below shows that this can be accomplished for any polytope by a projective change of coordinates.

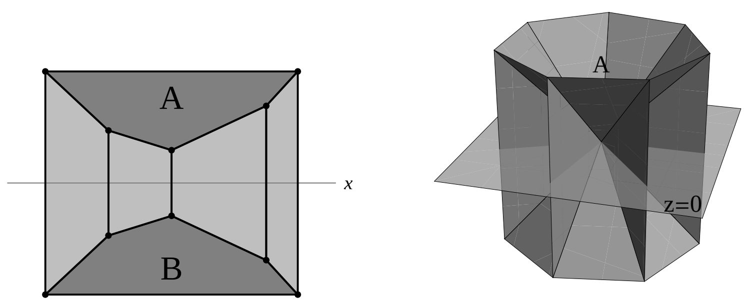
For instance this is true if A is the join of τ with the origin $\mathbf{0} \in \mathbb{R}^n$. Let l_τ be a choice of affine form vanishing on τ and let $x_1, \ldots, x_n$ be coordinates on $\mathbb{R}^n$. We further assume that

1. τ is parallel to the coordinate hyperplane $x_n = 0$
2. A lies between the hyperplanes $x_n = 0$ and $l_\tau = 0$.

3. For any two codimension one faces $\gamma_1, \gamma_2 \in A_{n-1} \setminus \tau$, affγ_1 and aff(γ_2) intersect the coordinate hyperplane $x_n = 0$ in distinct linear subspaces of codimension 2.

(1) can be obtained by rotating the original polytope, (2) and (3) can be obtained by translation. If A is the join of τ with the origin, (3) may be obtained by slight perturbations of the non-zero vertices of A (within the plane $l_\tau = 0$).

Let B be the reflection of A across the hyperplane $x_n = 0$. For a face $\gamma \in A$, let $\bar{\gamma}$ denote the corresponding face of B obtained by reflection. For $\gamma \in A_{n-1} \setminus \tau$, let $\sigma(\gamma)$ denote the polytope formed by taking the convex hull of γ and $\bar{\gamma}$. Now define $\mathcal{P}(A)$ as the polytopal complex with facets A, B and $\{\sigma(\gamma) | \gamma \neq \tau \in A_{n-1}\}$. See Figure 6.3 for examples of this construction in $\mathbb{R}^2$ and $\mathbb{R}^3$. Take the cone $\widehat{\mathcal{P}(A)} \subset \mathbb{R}^n$ over $\mathcal{P}(A)$ and consider the graded



(a) $\mathcal{P}(A)$ for $A \subset \mathbb{R}^2$ (b) $\mathcal{P}(A)$ for central $A \subset \mathbb{R}^3$

Figure 6.3

$S = \mathbb{R}[x_0, \ldots, x_n]$-module $C^r(\widehat{\mathcal{P}(A)})$. Let $\phi_B : C^r(\widehat{\mathcal{P}(A)}) \to S$ be the S-linear map obtained by restricting splines $F \in C^r(\widehat{\mathcal{P}(A)})$ to the facet $\widehat{B}$. This is a splitting of the inclusion $S \to C^r(\widehat{\mathcal{P}(A)})$ as global polynomials on $\widehat{\mathcal{P}(A)}$. Let $NT^r(\widehat{\mathcal{P}(A)})$ be the kernel of ϕ_B. Then

$$C^r(\widehat{\mathcal{P}(A)}) \cong S \oplus NT^r(\widehat{\mathcal{P}(A)}).$$

Let $S' = \mathbb{R}[x_0, \ldots, x_{n-1}]$ and, for $f \in S$, set $\overline{f} = f(x_0, \ldots, x_{n-1}, 0)$. Define an S-linear map $\phi : C^r(\widehat{\mathcal{P}(A)}) \cong S \oplus NT^r(\widehat{\mathcal{P}(A)}) \to S'$ by

$$(f, F) \to \overline{F_{\widehat{A}}},$$

where $f \in S$, $F \in NT^r(\widehat{\mathcal{P}(A)})$, and $F_{\widehat{A}}$ is the restriction of F to the facet $\widehat{A}$. Set $\Lambda(A) = \prod_{\gamma \neq \tau \in A_{n-1}} L_\gamma^{r+1}$, where $L_\gamma = l_{\widehat{\gamma}}$ is a choice of homogeneous form vanishing on $\widehat{\gamma}$. We claim that the image of ϕ is the principal ideal

$$I = \langle \overline{\Lambda(A)} \rangle.$$

ϕ is surjective since the spline $G(A)$, defined by

$$G(A)_\sigma = \begin{cases} 0 & \sigma \neq A \\ \Lambda(A) & \sigma = A, \end{cases}$$

goes to the generator of I under ϕ. To see that $\text{im}(\phi) \subset I$, let $F \in NT^r(\widehat{\mathcal{P}(A)})$. Then, since $F_{\widehat{B}} = 0$, $L_{\bar{\gamma}}^{r+1} | F_{\sigma(\gamma)}$ for every $\bar{\gamma} \neq \bar{\tau} \in B_{n-1}$. We also have $L_\gamma^{r+1} | (F_{\widehat{A}} - F_{\widehat{\sigma(\gamma)}})$ for every $\gamma \neq \tau \in A_{n-1}$. Hence $F_{\widehat{A}} \in \cap_{\gamma \neq \tau \in A_{n-1}} \langle L_\gamma^{r+1}, L_{\bar{\gamma}}^{r+1} \rangle$. But L_γ and $L_{\bar{\gamma}}$ differ at most by a scalar multiple and a sign on the variable x_n, so $\overline{L_\gamma} = \overline{L_{\bar{\gamma}}}$ and

$$\phi(F) \in \bigcap_{\gamma \neq \tau \in A_{n-1}} \langle \overline{L_\gamma^{r+1}} \rangle = \langle \prod_{\gamma \neq \tau \in A_{n-1}} \overline{L_\gamma^{r+1}} \rangle = \langle \overline{\Lambda(A)} \rangle$$

as claimed. Property (3) above is used in the first equality - this guarantees all the forms $\overline{L_\gamma}$ are distinct. It follows that the spline $G(A)$, which is supported only on the facet $\widehat{A}$ and generates splines supported on $\widehat{A}$, is a minimal generator of $C^r(\widehat{\mathcal{P}(A)})$.

If A is the join of τ with $\mathbf{0}$, then $\mathcal{P}(A)$ is central and $C^r(\mathcal{P}(A))$ is graded over the polynomial ring $R = \mathbb{R}[x_1, \ldots, x_n]$. In this case it is unnecessary to take the cone over $\mathcal{P}(A)$ above.

Remark 6.5.2. Given a convex polytope $A \subset \mathbb{R}^n \subset \mathbb{P}^n_{\mathbb{R}}$ and a choice τ of codimension one face, there is a projective change of coordinates which makes $\partial A \setminus \tau$ into the graph of a piecewise linear function over τ. If A is the join of τ with the origin $\mathbf{0} \in \mathbb{R}^n$, then this is easily done by a linear transformation. Otherwise, this can be accomplished by choosing a hyperplane $H \subset \mathbb{R}^n$ which is parallel to τ and very close to P without intersecting P. Then make a projective change of coordinates which sends H to the hyperplane at infinity (this argument is due to Sergei Ivanov). As long as H is chosen close enough to τ, this has the effect of making the face τ huge and the rest of the polytope the graph of a piecewise linear function over τ (once we restrict to affine coordinates again). Hence, given any polytope $A \subset \mathbb{R}^n$ and a choice of codimension one face $\tau \in A_{n-1}$, the construction in Example 6.5.1 allows us to build a polytopal complex $\mathcal{P}(A)$ so that $\partial^0 A = \partial A \setminus \tau$ and the generator of $C^r_{\widehat{A}}(\widehat{\mathcal{P}(A)})$ is a minimal generator of $C^r(\widehat{\mathcal{P}(A)})$.

Remark 6.5.3. The construction in Example 6.5.1 is inherently nonsimplicial. Some other construction needs to be used to obtain high degree generators in the simplicial case. In the planar simplicial case, there is an example in [59] of a planar simplicial complex Δ with minimal generator in degree $2r + 2$.

6.6 Bounding Regularity for Low Projective Dimension

In this section we combine the observations so far to bound the regularity of the spline algebra $C^\alpha(\mathcal{P})$, where $\mathcal{P} \subset \mathbb{R}^{n+1}$ is a central, pure, hereditary, $(n+1)$-dimensional polytopal complex. Recall a central complex is one in which the intersection of all interior codimension one faces is nonempty. We assume this intersection contains the origin and that $\alpha(\tau) = -1$ for every codimension one face $\tau \in \mathcal{P}_n$ so that $\mathbf{0} \notin \text{aff}(\tau)$; this makes the ring $C^\alpha(\mathcal{P})$ a graded $S = \mathbb{R}[x_0, \ldots, x_n]$-algebra with respect to the standard grading on S. The following corollary is critical to our analysis.

Corollary 6.6.1. *[12, Proposition 3.4] If $\mathcal{P}$ is a central, pure, hereditary, $(n+1)$-dimensional polytopal complex, then*

1. $pd(C^\alpha(\mathcal{P})) \leq n-1$
2. $\wp(C^\alpha(\mathcal{P})) \leq \text{reg }(C^\alpha(\mathcal{P})) - 2.$

Proof. (1) follows from Lemma 3.3.6. $C^\alpha(\mathcal{P})$ is the kernel of a map between free S-modules, so it is a second syzygy module. By the Hilbert syzygy theorem, any S-module has projective dimension at most $n+1$. Since $C^\alpha(\mathcal{P})$ is a second syzygy module, $\text{pd}(C^\alpha(\mathcal{P})) \leq n-1$. (2) follows from (1) and Theorem 2.6.3. □

Theorem 6.6.2. *Let $\mathcal{P} \subset \mathbb{R}^{n+1}$ be a pure $(n+1)$-dimensional hereditary polytopal complex which is central. Then*

$$\text{reg }(C^\alpha(\mathcal{P})) \leq \text{reg }(LS^{\alpha,n-1}(\mathcal{P}))$$

More generally, if $pd(C^\alpha(\mathcal{P})) \leq k$, then

$$\text{reg }(C^\alpha(\mathcal{P})) \leq \text{reg }(LS^{\alpha,k}(\mathcal{P}))$$

Proof. The first statement follows from the second by Corollary 6.6.1. To prove the second statement, note that by Theorem 6.2.4, the cokernel of the inclusion $LS^{\alpha,k}(\mathcal{P})$ has codimension at least $k+1$. By Proposition 2.6.8, *reg* $(C^\alpha(\mathcal{P})) \leq$ *reg* $(LS^{\alpha,k}(\mathcal{P}))$. □

To simplify the statements of later results, we introduce some additional notation. Given a pure $(n+1)$-dimensional subcomplex $\mathcal{Q} \subset \mathcal{P}$, let $\partial(\mathcal{Q})$ denote the set of n dimensional boundary faces of $\mathcal{Q}$. Define

$$\Lambda(\mathcal{Q}) = \prod_{\gamma \in (\partial(\mathcal{Q}))_n} l_\gamma^{\alpha(\gamma)+1}$$

and set

$$\lambda(\mathcal{Q}) = \deg(\Lambda(\mathcal{Q})) = \sum_{\gamma \in (\partial(\mathcal{Q}))_n} (\alpha(\gamma)+1).$$

As a first application of Theorem 6.6.2, we give a bound on the degree of generators of $C^\alpha(\mathcal{P})$ when $C^\alpha(\mathcal{P})$ is free.

Corollary 6.6.3. *Suppose $C^\alpha(\mathcal{P})$ is free and set $f(\mathcal{P}) = \max\{\lambda(\sigma)|\sigma \in \mathcal{P}_{n+1}\}$. Then $C^\alpha(\mathcal{P})$ is generated in degrees at most $f(\mathcal{P})$.*

Proof. For a free module, regularity is the maximum degree of generators (this follows from Definition 2.6.1), so we need to show *reg* $(C^\alpha(\mathcal{P})) \le f(\mathcal{P})$. $C^\alpha(\mathcal{P})$ is free iff $\mathrm{pd}(C^\alpha(\mathcal{P})) = 0$. By Theorem 6.6.2,

$$\textit{reg}\ (C^\alpha(\mathcal{P})) \le \textit{reg}\ (LS^{\alpha,0}(\mathcal{P})).$$

By Corollary 6.2.3, $LS^{\alpha,0} = \sum_{\sigma\in\mathcal{P}_{n+1}} C^\alpha_\sigma(\mathcal{P})$. Since the support of each summand is disjoint, this is a direct sum, so *reg* $(LS^{\alpha,0}(\mathcal{P})) = \max\{\textit{reg}\ (C^\alpha_\sigma(\mathcal{P}))|\sigma \in \mathcal{P}_{n+1}\}$. Also, $C^\alpha_\sigma(\mathcal{P})$ consists of splines F supported on the single facet σ. Such splines are characterized by $F|_\sigma$ being a polynomial multiple of $\Lambda(\sigma)$. It follows that $C^\alpha_\sigma(\mathcal{P}) \cong S(-\lambda(\sigma))$. Hence

$$\textit{reg}\ (LS^{\alpha,0}(\mathcal{P})) = \max\{\lambda(\sigma)|\sigma \in \mathcal{P}_{n+1}\} = f(\mathcal{P}).$$

□

We now apply Theorem 6.6.2 to the case where $C^\alpha(\mathcal{P})$ has projective dimension at most one. In particular, this includes central complexes in $\mathbb{R}^3$ by Corollary 6.6.1.

Theorem 6.6.4. *Suppose $pd(C^\alpha(\mathcal{P})) \le 1$. Let $f(\mathcal{P}) = \max\{\lambda(\sigma)|\sigma \in \mathcal{P}_{n+1}\}$ and $T = \max\limits_{\tau\in\mathcal{P}^0_n}\{\mathrm{reg}\ (C^\alpha_\tau(\mathcal{P}))\}$. Then $\mathrm{reg}\ (C^\alpha(\mathcal{P})) \le \max\{f(\mathcal{P}) - 1, T\}$.*

Proof. By Corollary 6.6.2,

$$\textit{reg}\ (C^\alpha(\mathcal{P})) \le \textit{reg}\ (LS^{\alpha,1}(\mathcal{P})).$$

By Proposition 6.4.1, $LS^{\alpha,1}(\mathcal{P})$ fits into the exact sequence

$$C_\bullet(\mathcal{M}_1) \to LS^{\alpha,1}(\mathcal{P}) \to 0.$$

From Proposition 6.4.3,

$$C_k(\mathcal{M}_1) = \begin{cases} \bigoplus\limits_{\tau\in\mathcal{P}^0_n} C^\alpha_\tau(\mathcal{P}) & \text{if } k = 1 \\ \bigoplus\limits_{|\partial^0(\sigma)|\ge k} (C^\alpha_\sigma(\mathcal{P}))^{\binom{|\partial^0(\sigma)|}{k}} & \text{if } 2 \le k \le \delta(\mathcal{P}) \\ 0 & \text{if } k > \delta(\mathcal{P}) \end{cases},$$

where $\delta(\mathcal{P}) = \max_{\sigma\in\mathcal{P}_{n+1}}\{|\partial^0(\sigma)|\}$. As we saw in the proof of Corollary 6.6.3, $C^\alpha_\sigma(\mathcal{P}) \cong S(-\lambda(\sigma))$, hence

$$\textit{reg}\ (C_k(\mathcal{M}_1)) = \max\{\lambda(\sigma)|\sigma \in \mathcal{P}_{n+1}\} \le f(\mathcal{P})$$

for every k with $2 \leq k \leq \delta(\mathcal{P})$. Now the conclusion follows from Corollary 2.6.7. □

At this point we see that to obtain more precise results for projective dimension one it is necessary to understand the ring $C^\alpha_\tau(\mathcal{P})$ of splines vanishing outside the star of a codimension one face.

Proposition 6.6.5. *Let $\tau \in \mathcal{P}^0_n$ be an interior codimension one face of $\mathcal{P}$, and σ_1, σ_2 the two facets of $st(\tau)$, the star of τ. Set $L_\tau = l_\tau^{\alpha(\tau)+1}$, $L_1 = \Lambda(\sigma_1)/L_\tau, L_2 = \Lambda(\sigma_2)/L_\tau$. Define the ideal $K(\tau)$ by*

$$K(\tau) = \langle L_1, L_2, L_\tau \rangle$$

We have a graded isomorphism

$$C^\alpha_\tau(\mathcal{P}) \cong \begin{cases} S(-deg\ L_\tau - deg\ L_2) \oplus S(-deg\ L_1) & \text{if } L_1 \in \langle L_2, L_\tau\rangle \\ S(-deg\ L_\tau - deg\ L_1) \oplus S(-deg\ L_2) & \text{if } L_2 \in \langle L_1, L_\tau\rangle \\ S(-deg\ L_1 - deg\ L_2) \oplus S(-deg\ L_\tau) & \text{if } L_\tau \in \langle L_1, L_2\rangle \\ \text{syz } (K(\tau)) & \text{otherwise,} \end{cases}$$

where syz $(K(\tau))$ *is the module of syzygies on the ideal $K(\tau)$.*

Proof. Let $F \in C^\alpha_\tau(\mathcal{P})$ and set $F_1 = F|_{\sigma_1}, F_2 = F|_{\sigma_2}$. Then there are polynomials G_1, G_2, G_3 satisfying the following relations.

$$\begin{aligned} F_1 =& \quad G_1 L_1 \\ F_2 =& \quad G_2 L_2 \\ F_2 - F_1 =& \quad G_3 L_\tau \end{aligned}$$

Taking the alternating sum of the above equations yields

$$G_1 L_1 - G_2 L_2 + G_3 L_\tau = \quad 0. \tag{6.3}$$

Hence $F = (F_1, F_2)$ gives rise to a syzygy on the columns of the matrix

$$M = \begin{bmatrix} L_1 & L_2 & L_\tau \end{bmatrix}$$

Now suppose given a syzygy (G_1, G_2, G_3) on the columns of M. We obtain a spline $F \in C^\alpha_\tau(\mathcal{P})$ by setting $F_1 = G_1 L_1, F_2 = G_2 L_2$, hence $C^\alpha_\tau(\mathcal{P})$ is isomorphic to the syzygies on the columns M. If $K(\tau)$ is minimally generated by L_1, L_2, and L_τ, we obtain $C^\alpha_\tau(\mathcal{P}) \cong$ *syz* $(K(\tau))$. Otherwise we obtain the cases listed above. For instance, if $L_1 \in \langle L_2, L_\tau\rangle$, then there exist polynomials $f, g \in S$ so that $L_1 = f L_2 + g L_\tau$ and *syz* (M) is generated by

$$\begin{bmatrix} 0 \\ L_\tau \\ -L_2 \end{bmatrix}, \begin{bmatrix} 1 \\ -f \\ -g \end{bmatrix},$$

of degrees deg L_2 + deg L_τ and deg L_1, respectively. The other cases follow similarly. $\square$

Proposition 6.6.6. *Let $\mathcal{P} \subset \mathbb{R}^3$ be a central complex, and $\tau \in \mathcal{P}_2^0$ a codimension one face of $\mathcal{P}$. Define $\lambda(\tau) = \lambda(st(\tau)) + \alpha(\tau) + 1 = \sum_{\gamma \in (st(\tau))_2} \alpha(\gamma) + 1$. Then* reg $(C_\tau^\alpha(\mathcal{P})) \leq \lambda(\tau) - 1$ *unless $\alpha(\gamma) = -1$ for all $\gamma \neq \tau \in (st(\tau))$, when* reg $(C_\tau^\alpha(\mathcal{P})) = \alpha(\tau) + 1$.

Proof. Let L_1, L_2, L_τ be as defined in proposition 6.6.5. Then

$$\begin{aligned} \deg L_1 = &\left(\sum_{\gamma \in (\sigma_1)_2} (\alpha(\gamma) + 1) \right) - \alpha(\tau) - 1 \\ \deg L_1 = &\left(\sum_{\gamma \in (\sigma_2)_2} (\alpha(\gamma) + 1) \right) - \alpha(\tau) - 1 \\ \deg L_\tau = &\ \alpha(\tau) + 1, \end{aligned}$$

If the ideal $K(\tau) = \langle L_1, L_2, L_\tau \rangle$ is not minimally generated by L_1, L_2, and L_τ, then $C_\tau^\alpha(\mathcal{P})$ is free, generated in degrees indicated by Proposition 6.6.5. By that description *reg* $(C_\tau^\alpha(\mathcal{P})) \leq \lambda(\tau) - 1$ unless $\alpha(\gamma) = -1$ for all $\gamma \neq \tau \in (\mathrm{st}(\tau))$, when *reg* $(C_\tau^\alpha(\mathcal{P})) = \alpha(\tau) + 1$. So assume $K(\tau)$ is minimally generated by L_1, L_2, L_τ and $C_\tau^\alpha(\mathcal{P}) \cong$ *syz* $(K(\tau))$.

We define a submodule $N(\tau)$ of $C_\tau^\alpha(\mathcal{P})$ as follows. Let σ_1, σ_2 be the two facets of $\mathrm{st}(\tau)$ and $Se_1 + Se_2$ the free S-module on generators e_1, e_2 corresponding to σ_1, σ_2. Define $N(\tau)$ to be the submodule of $C_\tau^\alpha(\mathcal{P})$ generated by $F_1 = \Lambda(\sigma_1)e_1, F_2 = \Lambda(\sigma_2)e_2$, and $F_\tau = \Lambda(\mathrm{st}(\tau))(e_1 + e_2)$. There is a single nontrivial syzygy among F_1, F_2, F_τ given by $L_\tau F_\tau = L_2 F_1 + L_1 F_2$. So $N(\tau)$ has minimal free resolution

$$0 \longrightarrow S(-\lambda(\mathrm{st}(\tau)) - \alpha(\tau) - 1) \longrightarrow \begin{array}{c} S(-\lambda(\sigma_1)) \\ \oplus \\ S(-\lambda(\mathrm{st}(\tau))) \\ \oplus \\ S(-\lambda(\sigma_2)) \end{array}$$

From Definition 2.6.1 and the free resolution above we see that *reg* $(N(\tau)) = \lambda(\mathrm{st}(\tau)) + \alpha(\tau) = \lambda(\tau) - 1$.

Now we show $\mathrm{codim}(C_\tau^\alpha(\mathcal{P})/N(\tau)) \geq 2$. It suffices to show that $(C_\tau^\alpha(\mathcal{P}))_P = N(\tau)_P$ for every prime of codimension one. Since S is a UFD, primes of codimension one are principle, generated by a single irreducible polynomial. If $P \neq \langle l_\gamma \rangle$ for any $\gamma \in (\mathrm{st}(\tau))_2$ then

$$(C_\tau^\alpha(\mathcal{P}))_P = N(\tau)_P = S_P^2.$$

If $P = \langle l_\gamma \rangle$ for some $\gamma \in \partial^0(\mathrm{st}(\tau))$, then

$$(C^\alpha_\tau(\mathcal{P}))_P = N(\tau)_P = l_\gamma^{\alpha(\gamma)+1} S_P \oplus S_P.$$

if aff(γ) meets only one face $\gamma \in (\mathrm{st}(\tau))_2$ or

$$(C^\alpha_\tau(\mathcal{P}))_P = N(\tau)_P = l_\gamma^{\alpha(\gamma)+1} S_P \oplus l_\gamma^{\alpha(\gamma)+1} S_P$$

If aff(γ) meets both σ_1 and σ_2 in a codimension one face. If $P = \langle l_\tau \rangle$, then

$$(C^\alpha_\tau(\mathcal{P}))_P = N(\tau)_P = (C^{\alpha(\tau)}(\mathrm{st}(\tau)))_P.$$

$\mathrm{pd}(C^\alpha_\tau(\mathcal{P})) \leq 1$ follows by Corollary 6.6.1, because we assumed $\mathcal{P} \subset \mathbb{R}^3$. Since $\mathrm{codim}(C^\alpha_\tau(\mathcal{P})/N(\tau)) \geq 2$, Proposition 2.6.8 yields *reg* $(C^\alpha_\tau(\mathcal{P})) \leq$ *reg* $(N(\tau)) = \lambda(\tau) - 1$. □

Theorem 6.6.7. *Let $\mathcal{P} \subset \mathbb{R}^3$ be a pure 3-dimensional polytopal complex which is central and set $e(\mathcal{P}) = \max\{\lambda(\tau) | \tau \in \mathcal{P}^0_2\}$. Then*

1. reg $(C^\alpha(\mathcal{P})) \leq e(\mathcal{P}) - 1$
2. $\wp(C^\alpha(\mathcal{P})) \leq e(\mathcal{P}) - 3$

In particular, $HP(C^\alpha(\mathcal{P}), d) = \dim_{\mathbb{R}} C^r_d(\mathcal{P})$ for $d \geq e(\mathcal{P}) - 2$.

Proof. (1) follows by applying Theorem 6.6.4 to Proposition 6.6.6. (2) follows from (1) by Corollary 6.6.1. □

Example 6.1.1 indicates that the bound given in Theorem 6.6.7 can be far from optimal. In the next section we bound *reg* $(C^\alpha_\tau(\Delta))$ more precisely for $\Delta \subset \mathbb{R}^3$ a central simplicial complex.

6.7 Simplicial Regularity Bound

In this section we analyze the regularity of the ring of splines $C^\alpha_\tau(\Delta)$ vanishing outside the star of 2-face, for $\Delta \subset \mathbb{R}^3$ a pure three-dimensional hereditary simplicial complex which is central. Again we assume $\alpha(\tau) = -1$ for $\tau \in \Delta_2$ with $\mathbf{0} \notin \mathrm{aff}(\tau)$, so that $C^\alpha(\Delta)$ is a graded module over the polynomial ring $S = \mathbb{R}[x, y, z]$. This means that st(τ) has at most five 2-faces γ for which $\alpha(\gamma) \geq 0$ ($\alpha(\tau) \geq 0$ is required). We prove the following theorem.

Theorem 6.7.1. *Let $\tau \in \Delta^0_2$ be a 2-face. Define*

$$M(\tau) = (\alpha(\tau) + 1) + \max\{(\alpha(\gamma_1) + 1) + (\alpha(\gamma_2) + 1) | \gamma_1 \neq \gamma_2 \in (st(\tau))_2\}.$$

Then reg $(C^\alpha_\tau(\Delta)) \leq M(\tau)$.

Before proving Theorem 6.7.1 we derive a couple of corollaries.

Theorem 6.7.2. *Let $\Delta \subset \mathbb{R}^3$ be a pure 3-dimensional hereditary simplicial complex which is central. For $\tau \in \Delta_2^0$, let $M(\tau)$ be defined as in Theorem 6.7.1. Then*

1. reg $(C^\alpha(\Delta)) \leq \max\{M(\tau) | \tau \in \Delta_2^0\}$
2. $\wp(C^\alpha(\Delta)) \leq \max\{M(\tau) | \tau \in \Delta_2^0\} - 2$

In particular, $HP(C^\alpha(\Delta), d) = \dim_{\mathbb{R}} C^r(\Delta)_d$ for $d \geq \max\{M(\tau)|\tau \in \Delta_2^0\} - 1$.

Proof. (1) follows by applying Theorem 6.6.4 to Theorem 6.7.1, (2) follows by applying Theorem 2.6.3 to (1). □

Setting $\alpha(\tau) = r$ for all $\tau \in \Delta_2^0$, we obtain

Corollary 6.7.3. *Let $\Delta \subset \mathbb{R}^3$ be a pure 3-dimensional hereditary simplicial complex which is central. Then*

1. reg $(C^\alpha(\Delta)) \leq 3r + 3$
2. $\wp(C^\alpha(\Delta)) \leq 3r + 1$

In particular, $HP(C^r(\Delta), d) = \dim_{\mathbb{R}} C^r(\Delta)_d$ for $d \geq 3r + 2$.

This result was obtained in the case of $C^r(\widehat{\Delta})$, for simplicial $\Delta \subset \mathbb{R}^2$, by Hong [32] and Ibrahim and Schumaker [33] (see Table 6.1 in the introduction). Before proving Theorem 6.7.2 we set up some notation. Figure 6.4 depicts our situation. We will abuse notation and write v_i both for the corresponding edge of st(τ) and for the vector we obtain by taking positive real multiples of this edge.

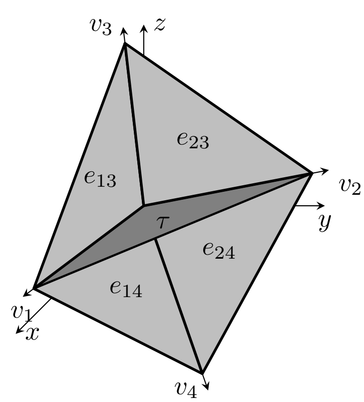


Figure 6.4: st(τ)

Let $u_1, u_2 \in S$ be the forms corresponding to the 2-faces e_{13}, e_{23}, let w_1, w_2 be the forms corresponding to the 2-faces e_{14}, e_{24}, and l_τ be the form corresponding to τ (for now do this without coordinates). Let $\alpha_\tau = \alpha(\tau) + 1, \alpha_1 = \alpha(e_{13}) + 1, \alpha_2 = \alpha(e_{23}) + 1, \beta_1 = \alpha(e_{14}) + 1, \beta_2 = \alpha(e_{24}) + 1$ be the exponents to appear on $l_\tau, u_1, u_2, w_1, w_2$ corresponding to the smoothness parameters specified by α. The following lemma is a special case of Proposition 6.6.5.

Lemma 6.7.4. *Let* $K(\tau) = (l_\tau^{\alpha_\tau}, u_1^{\alpha_1}u_2^{\alpha_2}, w_1^{\beta_1}w_2^{\beta_2})$. *Then we have a graded isomorphism*

$$C_\tau^\alpha(\Delta) \cong \begin{cases} S(-\alpha_\tau - \beta_1 - \beta_2) \oplus S(-\alpha_1 - \alpha_2) & \text{if } u_1^{\alpha_1}u_2^{\alpha_2} \in \left\langle w_1^{\beta_1}w_2^{\beta_2}, l_\tau^{\alpha_\tau} \right\rangle \\ S(-\alpha_\tau - \alpha_1 - \alpha_2) \oplus S(-\beta_1 - \beta_2) & \text{if } w_1^{\beta_1}w_2^{\beta_2} \in \langle u_1^{\alpha_1}u_2^{\alpha_2}, l_\tau^{\alpha_\tau} \rangle \\ S(-\alpha_1 - \alpha_2 - \beta_1 - \beta_2) \oplus S(-\alpha_\tau) & \text{if } l_\tau^{\alpha_\tau} \in \left\langle u_1^{\alpha_1}u_2^{\alpha_2}, w_1^{\beta_1}w_2^{\beta_2} \right\rangle \\ \text{syz } (K(\tau)) & \text{otherwise,} \end{cases}$$

where syz $(K(\tau))$ *is the module of syzygies on the ideal* $K(\tau)$.

Proof of Theorem 6.7.1. If $u_1^{\alpha_1}u_2^{\alpha_2} \in \left\langle w_1^{\beta_1}w_2^{\beta_2}, l_\tau^{\alpha_\tau} \right\rangle$ or $w_1^{\beta_1}w_2^{\beta_2} \in \langle u_1^{\alpha_1}u_2^{\alpha_2}, l_\tau^{\alpha_\tau} \rangle$ then *reg* $(C_\tau^\alpha(\Delta)) \leq M(\tau)$ is clear from Lemma 6.7.4. If $l_\tau^{\alpha_\tau} \in \left\langle u_1^{\alpha_1}u_2^{\alpha_2}, w_1^{\beta_1}w_2^{\beta_2} \right\rangle$ then $\alpha_\tau \geq \alpha_1 + \alpha_2$, $\alpha_\tau \geq \beta_1 + \beta_2$, and *reg* $(C_\tau^\alpha(\Delta)) \leq M(\tau)$ from Lemma 6.7.4. So we may assume $K(\tau)$ is minimally generated by the three given forms and $C_\tau^\alpha(\Delta) \cong$ *syz* $(K(\tau))$. In this case *reg* $(C_\tau^\alpha(\Delta)) \leq$ *reg* $(S/K(\tau)) + 2$ by two applications of Proposition 2.6.6 (equality holds but we will not need this). So it suffices to show that *reg* $(S/K(\tau)) \leq M(\tau) - 2$.

Four special cases are given by

1. $\alpha_1 = \beta_1 = 0 \implies K(\tau) = \langle l_\tau^{\alpha_\tau}, u_2^{\alpha_2}, w_2^{\beta_2} \rangle$
2. $\alpha_2 = \beta_2 = 0 \implies K(\tau) = \langle l_\tau^{\alpha_\tau}, u_1^{\alpha_1}, w_1^{\beta_1} \rangle$
3. $\alpha_1 = \beta_2 = 0 \implies K(\tau) = \langle l_\tau^{\alpha_\tau}, u_2^{\alpha_2}, w_1^{\beta_1} \rangle$
4. $\alpha_2 = \beta_1 = 0 \implies K(\tau) = \langle l_\tau^{\alpha_\tau}, u_2^{\alpha_2}, w_1^{\beta_1} \rangle$

Since $K(\tau)$ is minimally generated by the three given forms, [28, Theorem 2.7] applies in cases (1) and (2). For example, in case (1) we have

$$\begin{aligned} reg\ (S/K(\tau)) &= \left\lfloor \frac{\alpha_\tau + \alpha_2 + \beta_2 - 3}{2} \right\rfloor \\ &\leq \alpha_\tau + \alpha_2 + \beta_2 - 2 \\ &\leq M(\tau) - 2. \end{aligned}$$

A similar argument holds for case (2). In cases (3) and (4), $K(\tau)$ is a complete intersection of its generators and *reg* $(K(\tau)) \leq M(\tau) - 2$ follows from the Koszul resolution.

If at most one of $\alpha_1, \alpha_2, \beta_1, \beta_2$ vanishes we show *reg* $(S/K(\tau)) \leq M(\tau) - 2$ by fitting $S/K(\tau)$ into exact sequences and using Proposition 2.6.6. Let $Q = \langle l_\tau^{\alpha_\tau}, u_1^{\alpha_1}u_2^{\alpha_2} \rangle$. We have the short exact sequence

$$0 \to \frac{S(-\beta_1 - \beta_2)}{Q : (w_1^{\beta_1}w_2^{\beta_2})} \xrightarrow{\cdot w_1^{\beta_1}w_2^{\beta_2}} \frac{S}{Q} \to \frac{S}{K(\tau)} \to 0 \tag{6.4}$$

Q is a complete intersection with 2 generators in degrees α_τ and $\alpha_1 + \alpha_2$, so

$$reg\ (S/Q) = \alpha_\tau + \alpha_1 + \alpha_2 - 2$$

The ideal Q decomposes as $Q = \langle l_\tau^{\alpha_\tau}, u_1^{\alpha_1}\rangle \cap \langle l_\tau^{\alpha_\tau}, u_2^{\alpha_2}\rangle$. Then

$$Q : (w_1^{\beta_1} w_2^{\beta_2}) = I_1 \cap I_2,$$

where

$$\begin{aligned} I_1 = &\quad \langle l_\tau^{\alpha_\tau}, u_1^{\alpha_1}\rangle : (w_1^{\beta_1} w_2^{\beta_2}) \\ I_2 = &\quad \langle l_\tau^{\alpha_\tau}, u_2^{\alpha_2}\rangle : (w_1^{\beta_1} w_2^{\beta_2}) \end{aligned}$$

Since $(l_\tau^{\alpha_\tau}, u_1^{\beta_1})$ is (l_τ, u_1)-primary and $w_2 \notin (l_\tau, u_1)$,

$$I_1 = \langle l_\tau^{\alpha_\tau}, u_1^{\alpha_1}\rangle : (w_1^{\beta_1} w_2^{\beta_2}) = \langle l_\tau^{\alpha_\tau}, u_1^{\alpha_1}\rangle : w_1^{\beta_1}.$$

Similarly,

$$I_2 = \langle l_\tau^{\alpha_\tau}, u_2^{\alpha_2}\rangle : (w_1^{\beta_1} w_2^{\beta_2}) = \langle l_\tau^{\alpha_\tau}, u_2^{\alpha_2}\rangle : w_2^{\beta_2}.$$

From Proposition 6.7.5 below, if $I_1 \neq S$ and $I_2 \neq S$ then I_1, I_2 are complete intersections and

$$\begin{aligned} \textit{reg } (S/I_1) &\leq \alpha_\tau + \alpha_1 - \beta_1 - 2 \\ \textit{reg } (S/I_2) &\leq \alpha_\tau + \alpha_2 - \beta_2 - 2 \end{aligned}$$

We consider four final special cases before moving on to the general case.

A $w_1^{\beta_1} \in \langle l_\tau^{\alpha_\tau}, u_1^{\alpha_1}\rangle \implies I_1 = S \implies Q : (w_1^{\beta_1} w_2^{\beta_2}) = I_2$

B $w_2^{\beta_2} \in \langle l_\tau^{\alpha_\tau}, u_2^{\alpha_2}\rangle \implies I_2 = S \implies Q : (w_1^{\beta_1} w_2^{\beta_2}) = I_1$

Note that $\alpha_1 = 0$ falls under **A** and $\alpha_2 = 0$ falls under **B**. By the exact sequence (6.4) and Proposition 2.6.6 we have the corresponding bounds

A *reg* $(S/K(\tau)) \leq \max\{\alpha_\tau + \alpha_2 + \beta_1 - 3, \alpha_\tau + \alpha_1 + \alpha_2 - 2\} \leq M(\tau) - 2$

B *reg* $(S/K(\tau)) \leq \max\{\alpha_\tau + \alpha_1 + \beta_2 - 3, \alpha_\tau + \alpha_1 + \alpha_2 - 2\} \leq M(\tau) - 2$

If we use multiplication by $u_1^{\alpha_1} u_2^{\alpha_2}$ in the exact sequence (6.4) then we have the corresponding ideals $Q' = \langle l_\tau^{\alpha_\tau}, u_1^{\alpha_1} u_2^{\alpha_2}\rangle$,$I_1' = \langle l_\tau^{\alpha_\tau}, w_1^{\beta_1}\rangle : u_1^{\alpha_1}$ and $I_2' = \langle l_\tau^{\alpha_\tau}, w_2^{\beta_2}\rangle : u_2^{\alpha_2}$. We then have the analogous cases

C $u_1^{\alpha_1} \in \langle l_\tau^{\alpha_\tau}, w_1^{\beta_1}\rangle \implies I_1' = S \implies Q' : (u_1^{\alpha_1} u_2^{\alpha_2}) = I_2'$

D $u_2^{\alpha_2} \in \langle l_\tau^{\alpha_\tau}, w_2^{\beta_2}\rangle \implies I_2' = S \implies Q' : (u_1^{\alpha_1} u_2^{\alpha_2}) = I_1'$

Note that $\beta_1 = 0$ falls under **C** and $\beta_2 = 0$ falls under **D**. The corresponding bounds are

C *reg* $(S/K(\tau)) \leq \max\{\alpha_\tau + \beta_2 + \alpha_1 - 3, \alpha_\tau + \beta_1 + \beta_2 - 2\} \leq M(\tau) - 2$

D *reg* $(S/K(\tau)) \leq \max\{\alpha_\tau + \beta_1 + \alpha_2 - 3, \alpha_\tau + \beta_1 + \beta_2 - 2\} \leq M(\tau) - 2$.

We have reduced to the case where

- $w_i^{\beta_i} \notin \langle l_\tau^{\alpha_\tau}, u_i^{\alpha_i}\rangle$ (equivalently $\alpha_\tau + \alpha_i - \beta_i \geq 2$) for $i = 1, 2$
- $u_i^{\alpha_i} \notin \langle l_\tau^{\alpha_\tau}, w_i^{\beta_i}\rangle$ (equivalently $\alpha_\tau + \beta_i - \alpha_i \geq 2$) for $i = 1, 2$

- $\alpha_i \geq 1, \beta_i \geq 1$ for $i = 1, 2$ and $\alpha_\tau \geq 1$.

In particular, $u_1 \neq w_1$ implies that the vectors v_1, v_3, v_4 are linearly independent and $u_2 \neq w_2$ implies v_2, v_3, v_4 are linearly independent in Figure 6.4. It follows that we may make a change of coordinates so that v_1 points along the y-axis, v_2 points along the x-axis, and v_3 points along the z-axis. Applying appropriate scaling in the x, y, and positive z directions, we can assume that the vector defined by v_4 points in the direction of $\langle 1, 1, -1\rangle$. Under this change of coordinates, $\mathrm{st}(\tau)$ has four possible configurations, shown in Figure 6.5. The ideal $K(\tau)$ is the same for all of these. We have

$$\begin{aligned} l_\tau = &\quad z \\ u_1 = &\quad x \\ u_2 = &\quad y \\ w_1 = &\quad x + z \\ w_2 = &\quad y + z \end{aligned}$$

and

$$\begin{aligned} I_1 = &\quad \langle l_\tau^{\alpha_\tau}, u_1^{\alpha_1}\rangle : w_1^{\beta_1} = \langle x^{\alpha_1}, z^{\alpha_\tau}\rangle : (x+z)^{\beta_1} \\ I_2 = &\quad \langle l_\tau^{\alpha_\tau}, u_2^{\alpha_2}\rangle : w_2^{\beta_2} = \langle y^{\alpha_2}, z^{\alpha_\tau}\rangle : (y+z)^{\beta_2} \end{aligned}$$

By Corollary 6.7.13 in the next section, *reg* $(S/Q) =$ *reg* $(S/(I_1 \cap I_2)) \leq M(\tau) - \beta_1 - \beta_2 - 1$. By the exact sequence (6.4) and Lemma 2.6.6, the proof is complete. □

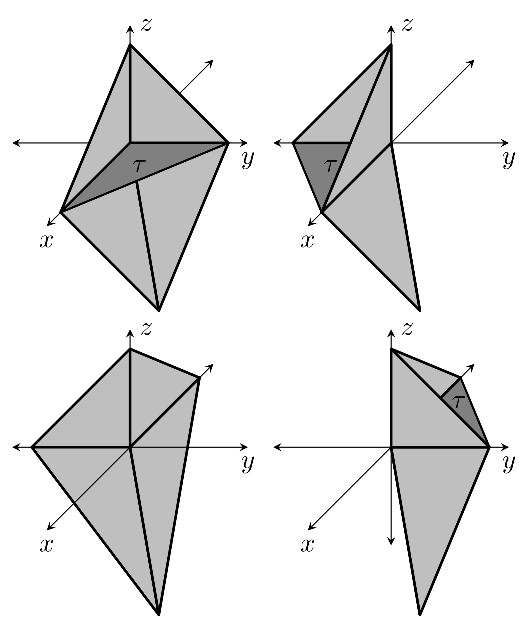


Figure 6.5: Possible configurations for generic $\mathrm{st}(\tau)$

6.7.1 Intersection of colon ideals

Let

$$I_1 = \langle x^{\alpha_1}, z^{\alpha_\tau}\rangle : (x+z)^{\beta_1}$$
$$I_2 = \langle y^{\alpha_2}, z^{\alpha_\tau}\rangle : (y+z)^{\beta_2}$$

In [40], Tohaneanu and Minac compute the Hilbert function of the ideal (up to change of coordinates)

$$\langle x^{r+1}, (x+z)^{r+1}\rangle : z^{r+1} \cap \langle y^{r+1}, (y+z)^{r+1}\rangle : z^{r+1}.$$

It is not so obvious how to apply their methods directly to the ideal $I_1 \cap I_2$. Building on their work, however, we show how to construct enough of the initial ideal (with respect to the lexicographic order) of $I_1 + I_2$ to give a fairly tight bound on the socle degree of $S/I_1 + I_2$. The methods synthesize descriptions of such ideals in terms of linear and commutative algebra.

We first compute the initial ideal of

$$I = I(p,q,r) = \langle s^p, t^q\rangle : (s+t)^r$$

in the ring $R = k[s,t]$ with standard lexicographic order. We assume $I \neq R$, so $(s+t)^r \notin \langle s^p, t^q\rangle$. This is equivalent to requiring $p+q-r \geq 2$.

Proposition 6.7.5. *Let $I = I(p,q,r) \subset R$ be as above, with $p+q-r \geq 2$. Then I is a complete intersection generated by two polynomials of*

1. *degrees $a = \min\{p, q-r\}, b = \max\{p, q-r\}$ if $p+r-q \leq 1$.*
2. *degrees $a = \min\{q, p-r\}, b = \max\{q, p-r\}$ if $q+r-p \leq 1$.*
3. *degrees*

$$a = \left\lfloor \frac{p+q-r}{2} \right\rfloor, b = \left\lceil \frac{p+q-r}{2} \right\rceil$$

if $p+r-q \geq 2$ and $q+r-p \geq 2$.

Proof. $p+r-q \leq 1$: In this case $t^q \in \langle s^p, (s+t)^r\rangle$. Let

$$t^q = fs^p + g(s+t)^r$$

for some polynomials $f, g \in R$, where g has no term divisible by s^p. It is immediate that

$$\langle s^p, t^q\rangle = \langle s^p, g(s+t)^r\rangle$$

and

$$I = \langle s^p, t^q\rangle : (s+t)^r = \langle s^p, g\rangle.$$

The polynomial g is not divisible by s since it has a term which is a constant multiple of t^{q-r}. It follows that s^p ang g are relatively prime and I is a complete intersection. Since g has degree $q-r$, (1) is proved.

$q+r-p \leq 1$: The argument is identical to the previous case.

$p+r-q \geq 2$ **and** $q+r-p \geq 2$**:** Let

$$T = T(p,q,r) = \langle s^p, t^q, (s+t)^r \rangle.$$

Since we assume $p+q-r \geq 2$ as well, T is minimally generated by the three given generators. We describe I in terms of the minimal free resolution of the ideal T. Set $a = \left\lfloor \frac{p+q-r}{2} \right\rfloor$ and $b = \left\lceil \frac{p+q-r}{2} \right\rceil$. The assumption $p+q-r \geq 2$ guarantees that $a \geq 1$. T is a codimension two Cohen-Macaulay ideal with Hilbert-Burch resolution of the form below [28, Theorem 2.7]

$$0 \to R(-a-r) \oplus R(-b-r) \xrightarrow{\phi} R(-p) \oplus R(-q) \oplus R(-r) \to T$$

where

$$\phi = \begin{pmatrix} A & D \\ B & E \\ C & F \end{pmatrix}$$

is a matrix of forms satisfying $BF-EC = s^p, AF-DC = t^q, BF-EC = (s+t)^r$. It follows that the module of syzygies on T has two generators, corresponding to the relations

$$As^p + Bt^q + C(s+t)^r = 0$$

and

$$Ds^p + Et^q + F(s+t)^r = 0.$$

In terms of the entries of the matrix ϕ we may write

$$I = (C, F)$$

where $\deg(C) = a, \deg(F) = b$, and $a+b = p+q-r$. Since $AF-DC = t^q$ and $BF-EC = (s+t)^r$, any common factor of C and F would give a common factor of t and $(s+t)$, so C and F are relatively prime. So I is a complete intersection of the required degrees. □

As an immediate corollary we have the following lemma.

Corollary 6.7.6. *With* $I = I(p,q,r)$ *as above, minimally generated by two forms of degree* $a \leq b$*, we have*

$$HF(I,d) = \binom{d+1-a}{1} + \binom{d+1-b}{1} - \binom{d+1-a-b}{1}.$$

Proof. From Proposition 6.7.5, I is a complete intersection of polynomials C, F with $\deg(C) = a, \deg(F) = b$. So I has minimal resolution of the form

$$0 \to R(-a-b) \to R(-a) \oplus R(-b) \to I \to 0.$$

The result follows from the additivity of Hilbert functions across exact sequences. ☐

Given a Hilbert function $H(I,d)$, let L_d be the vector space spanned by the $H(I,d)$ greatest monomials of degree d with respect to lex order. Then the direct sum

$$L = \bigoplus_{d=0}^{\infty} L_d$$

is an ideal, known as the *lex-segment* ideal for the Hilbert function $H(d)$ [39, Proposition 2.21]. Since two *generic* forms in $\mathbb{R}[x,y]$ of degrees $a \le b$ form a complete intersection, the ideal they generate has the same Hilbert function as I. Denote by $L(a,b)$ the corresponding lex-segment ideal. We will show that $L(a,b)$ is the initial ideal of $I(p,q,r)$.

To prove this we will use a matrix condition on the coefficients of a form f of degree d which distinguishes when $f \in I$. From Corollary 6.7.6, $I_d = R_d$ for $d \ge a+b-1$. Since $a+b = p+q-r$, this matrix condition we derive will be nontrivial for $1 \le d < p+q-r-1$. Suppose

$$f = \sum_{i+j=d} a_{i,j} s^i t^j$$

satisfies $f \in I_d$. Then by definition we have

$$f(s+t)^r \in (s^p, t^q)$$

Since the ideal on the right is a monomial ideal, $f \in I \iff$ every monomial of $f(s+t)^r$ is divisible by either s^p or t^q. Expanding $(s+t)^r$ and multiplying by f gives

$$f(s+t)^r = \sum_{i+j=d} \sum_{m+n=r} \binom{r}{m} a_{ij} s^{m+i} t^{n+j}$$

Setting $m+i=u$ and $n+j=v$ gives

$$\sum_{u+v=d+r} s^u t^v \left(\sum_{m+i=u} \binom{r}{m} a_{ij} \right).$$

$f \in I$ iff the only nonzero coefficients in this expression occur when $u \ge p$ or $v \ge q$. Since $v = d+r-u$, $v \ge q$ is equivalent to $u \le d+r-q$. So $f \in I$ iff for $u = d+r-q+1, \ldots, p-1$ we have the condition

$$\sum_{m+i=u} \binom{r}{m} a_{i,d-i} = 0.$$

Here we follow the convention that $\binom{A}{B} = 0$ when $B < 0$ or $B > A$. These fit together into the following matrix condition on the coefficients of f:

$$\begin{pmatrix} \binom{r}{d+r-q+1} & \binom{r}{d+r-q} & \binom{r}{d+r-q-1} & \cdots & \binom{r}{r-q+1} \\ \binom{r}{d+r-q+2} & \binom{r}{d+r-q+1} & \binom{r}{d+r-q} & \cdots & \binom{r}{r-q+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{r}{p} & \binom{r+1}{p-1} & \binom{r+1}{p-4} & \cdots & \binom{r}{p-d-2} \\ \binom{r}{p-1} & \binom{r}{p-2} & \binom{r}{p-3} & \cdots & \binom{r}{p-d-1} \end{pmatrix} \cdot \begin{pmatrix} a_{0,d} \\ a_{1,d-1} \\ \vdots \\ a_{d-1,1} \\ a_{d,0} \end{pmatrix} = 0.$$

Denote the $(p+q-r-d-1)\times(d+1)$ matrix on the left by $M(p,q,r,d)$. $M(p,q,r,d)$ has entries

$$M(p,q,r,d)_{i,j} = \binom{r}{d+r-q+1+i-j},$$

where $i = 0,\ldots,\min\{p-1, p+q-r-d-2\}$ and $j = 0,\ldots,d$. With this choice of indexing, column c_j of $M(p,q,r,d)$ corresponds to the coefficient $a_{j,d-j}$. The following lemma is fundamental for understanding $M(p,q,r,d)$.

Lemma 6.7.7. *Let $\mu = (\mu_0 \geq \ldots \mu_k \geq 1)$ be a partition with $k+1$ parts so that $r \geq \mu_0$. Let $N(\mu)$ be the square matrix with entries*

$$N(\mu)_{ij} = \binom{r}{\mu_j + i - j}$$

for $i = 0,\ldots,k$, $j = 0,\ldots,k$. Then $N(\mu)$ has nonzero determinant.

Proof. This observation is made in [40, § 3.1], where it is noted that determinants of such matrices play a role in the representation theory of the special linear group $SL(V)$, where V is an r-dimensional vector space. In particular, if $\lambda = \mu'$, the conjugate partition to μ, det $N(\mu)$ is the dimension of the Weyl module $\mathbb{S}_\lambda V$, which is a nontrivial irreducible representation of $SL(V)$. More explicitly, det $N(\mu) = s_\lambda(1,\ldots,1)$, where $s_\lambda(x_1,\ldots,x_r)$ is the Schur polynomial in r variables of the partition $\lambda = \mu'$. In particular, $N(\mu)$ has nonzero determinant. See [25, § 6.1] and [25, Appendix A.1] for more details. ☐

Corollary 6.7.8. *Let $M = M(p,q,r,d)$ be the $(p+q-r-d-1)\times(d+1)$ matrix defined as above, $M_{s,t}$ a nonzero entry of M, and k a nonnegative integer so that $s+k \leq p+q-r-d-1$ and $t+k \leq d+1$. Then*

1. *The $(k+1)\times(k+1)$ submatrix of M formed by the entries $\{M_{i,j} | s \leq i \leq s+k, t \leq j \leq t+k\}$ is invertible.*
2. *The rank of M is the minimum of the number of nonzero rows of M and the number of nonzero columns of M.*

Proof. (1) The submatrix of $M = M(p,q,r,d)$ above has entries

$$\binom{r}{d+r-q+1+s-t+i-j}$$

for $i = 0, \dots, k, j = 0, \dots, k$. Since we assume $M_{s,t} \neq 0$, $d + r - q + 1 + s - t \leq r$ and the first statement follows from Lemma 6.7.7 by taking $\mu_0 = \cdots = \mu_k = d + r - q + 1 + s - t$.

(2) Observe that either $M_{0,0} \neq 0$ or, if $M_{0,0} = 0$, then the first entry $M_{j,0}$ ($j > 0$) which is nonzero is equal to 1. The last entry in the first column is $\binom{r}{p-1} \geq 1$ (we assumed $p \geq 1$), so there is at least one nonzero entry in the first column of M. If the row of M with index i is nonzero, every row with index $\geq i$ is also nonzero. If the column of M with index j is zero, every column with index $\geq j$ is also zero. Now the second statement follows by taking the largest square submatrix of M whose upper left corner is the first nonzero entry of the first column of M. This is a $k \times k$ submatrix of M where k is the minimum of the number of nonzero rows of M and the number of nonzero columns of M. By the first statement, this submatrix is invertible, and from the earlier observations k must be the rank of M. □

Lemma 6.7.9. *The initial ideal of $I = I(p, q, r)$ is the lex-segment ideal $L(a, b)$, where $a \leq b$ are the degrees of the generators of I.*

Proof. For fixed degree d, let $\mathfrak{r}$ be the rank of $M(p, q, r, d)$. By definition,

$$HF(I, d) = \dim \ker M(p, q, r, d),$$

hence $HF(I, d) = d + 1 - \mathfrak{r}$. From the submatrix constructed to prove part (2) of Corollary 6.7.8, the first $\mathfrak{r}$ columns of M are linearly independent. It follows that for any column c_l of $M(p, q, r, d)$ with $\mathfrak{r} - 1 \leq l \leq d$, there is a unique (up to scaling) relation

$$\left(\sum_{i=0}^{\mathfrak{r}-1} a_{i,d-i} c_i \right) + a_{l,d-l} c_l = 0,$$

where $a_{l,d-l} \neq 0$. This gives rise to the polynomial $f = \sum_{i=0}^{p+q-r-d-2} a_{i,j} s^i t^{d-i} + a_{l,d-l} s^l t^{d-l} \in I$ with leading monomial $s^l t^{d-l}$. These monomials are the largest $d + 1 - \mathfrak{r}$ monomials of degree d with respect to lex ordering, so the result follows. □

Corollary 6.7.10. *Let $I = (s^p, t^q) : (s + t)^r$, generated in degrees a and b, with $a \leq b$. The initial ideal of I with respect to the standard lexicographic order is*

$$L(a, b) = (s^a, s^{a-1} t^{b-a+1}, s^{a-2} t^{b-a+3}, \dots, s^{a-i} t^{b-a+2i-1}, \dots, t^{a+b-1}).$$

Proof. By Lemma 6.7.9 it suffices to show that the lex-segment ideal $L(a, b)$ has the form above. The Hilbert function of a complete intersection I generated in degrees a and b is

$$HF(I, d) = \binom{d + 1 - a}{1} + \binom{d + 1 - b}{1} - \binom{d + 1 - a - b}{1}.$$

More explicitly, we have

$$HF(I,d) = \begin{cases} 0 & \text{for } 0 \le d < a \\ d+1-a & \text{for } a \le d < b \\ 2d+2-(a+b) & \text{for } b \le d \le a+b-1 \\ d+1 & \text{for } d > a+b-1 \end{cases}$$

Recall $L(a,b)_d$ is the vector space spanned by the $HF(I,d)$ greatest monomials of degree d with respect to lex order. If $a \le d < b$, then the $d+1-a$ greatest monomials are $s^d, \ldots, s^a$. These are all divisible by s^a. If $b \le d \le a+b-1$, the $2d+2-(a+b)$ greatest monomials are $\{s^{d-i}t^i | i = 0, \ldots, 2d-(a+b)+1\}$. If $i \le d-a$, $s^{d-i}t^i$ is divisible by s^a. If $d-a \le i \le 2d+1-(a+b)$, then $s^{d-i}t^i = s^{a-j}t^{d-a+j} = s^{a-j}t^{b-a+(d-b+j)}$, where $j = 1, \ldots, d-b+1$. This is divisible by $s^{a-j}t^{b-a+2j-1}$, which proves the corollary. □

Remark 6.7.11. Conca and Valla [14] parametrize of all ideals in two variables with a given initial ideal. Using this, one can show that the lex-segment ideal $L(a,b)$ is the initial ideal of any ideal generated by two *generic* forms of degree a and b. Here *generic* means there are certain polynomials in the coefficients of the forms that must not vanish (the condition is not equivalent to the two forms being relatively prime). Lemma 6.7.9 can be viewed as a proof that the ideal I, which is generated by two forms, is generic in this sense.

Proposition 6.7.12. *Set $R = k[x,y]$, $S = k[x,y,z]$ both with standard lexicographic orders. For positive integers $a \le b$, $c \le d$, let $J_1, J_2 \subset R = k[s,t]$ be ideals satisfying in$(J_1) = L(a,b)$ and in$(J_2) = L(c,d)$, respectively. Let $S = k[x,y,z]$ and define ring maps $i_1, i_2 : R \to S$ by $i_1(s) = x, i_1(t) = z$ and $i_2(s) = y, i_2(t) = z$. Set $I_1 = i_1(J_1)S$, $I_2 = i_2(J_2)S$, and $N = \max\{a+d-1, b+c-1\}$. Then*

$$(I_1 + I_2)_N = S_N$$

Proof. It suffices to show that $(\text{in}(I_1) + \text{in}(I_2))_N = S_N$. We have

$$\begin{aligned} \text{in}(I_1) = \quad & \langle x^a \rangle + \langle x^{a-i} z^{b-a+2i-1} | i = 1, \ldots, a \rangle \\ \text{in}(I_2) = \quad & \langle y^c \rangle + \langle y^{c-j} z^{d-c+2j-1} | j = 1, \ldots, c \rangle \end{aligned}$$

Let $m = x^i y^j z^k$ be a monomial of S with degree N. We claim $m \in \text{in}(I_1) + \text{in}(I_2)$. If $i \ge a$ or $j \ge c$ then $x^a | m$ or $y^c | m$ and we are done. So set $i = a-s, j = c-t$, where $1 \le s \le a, 1 \le t \le c$. If $s \le t$ then $a+c-s-t+(b-a+2s-1) = b+c-1+s-t \le N$. So $k = N-(a+c-s-t) \ge b-a+2s-1$ and $x^i y^j z^k \in \text{in}(I_1)$. If $t \le s$ then $a+c-s-t+(d-c+2t-1) = a+d-s+t-1 \le N$. So $k = N-(a+c-s-t) \ge d-c+2t-1$ and $x^i y^j z^k \in \text{in}(I_2)$. □

Corollary 6.7.13. *Let*

$$\begin{aligned} I_1 = \quad & \langle x^{\alpha_1}, z^{\alpha_\tau} \rangle : (x+z)^{\beta_1} \\ I_2 = \quad & \langle y^{\alpha_2}, z^{\alpha_\tau} \rangle : (y+z)^{\beta_2} \end{aligned}$$

where $\alpha_i + \alpha_\tau - \beta_i \geq 2$ *and* $\beta_i + \alpha_\tau - \alpha_i \geq 2$ *for* $i = 1, 2$. *Also assume* $\alpha_i \geq 1, \beta_i \geq 1$ *for* $i = 1, 2$ *and* $\alpha_\tau \geq 1$. *Let*

$$\begin{aligned} M(\tau) = \quad & \alpha_\tau + \max\{\alpha_1 + \alpha_2, \alpha_1 + \beta_1, \alpha_1 + \beta_2, \alpha_2 + \beta_1, \\ & \alpha_2 + \beta_2, \beta_1 + \beta_2, \alpha_\tau + \alpha_1, \alpha_\tau + \alpha_2, \alpha_\tau + \beta_1, \alpha_\tau + \beta_2\} \end{aligned}$$

as in the statement of Theorem 6.7.1. Then

$$\text{reg}\left(\frac{S}{I_1 \cap I_2}\right) \leq M(\tau) - \beta_1 - \beta_2 - 1$$

Proof. We use the short exact sequence

$$0 \to \frac{S}{I_1 \cap I_2} \to \frac{S}{I_1} \oplus \frac{S}{I_2} \to \frac{S}{I_1 + I_2} \to 0$$

and Proposition 2.6.6. From Proposition 6.7.5, *reg* $(S/I_1) = \alpha_1 + \alpha_\tau - \beta_1 - 2 \leq M(\tau) - \beta_1 - \beta_2 - 1$ and *reg* $(S/I_2) = \alpha_2 + \alpha_\tau - \beta_2 - 2 \leq M(\tau) - \beta_1 - \beta_2 - 1$. We show *reg* $(S/(I_1 + I_2)) \leq M(\tau) - \beta_1 - \beta_2 - 2$; then we are done by Proposition 2.6.6. Equivalently, we show $(I_1+I_2)_d = S_d$ for $d = M(\tau)-\beta_1-\beta_2-1$. Let I_1, I_2 be generated in degrees $a \leq b, c \leq d$ respectively. By Lemma 6.7.9 and Proposition 6.7.12,$(I_1 + I_2)_d = S_d$ for $d \geq \max\{a + d - 1, b + c - 1\}$. So we need to show that $\max\{a + d, b + c\} \leq M(\tau) - \beta_1 - \beta_2$. We consider 4 cases.

1. $\beta_i + \alpha_i - \alpha_\tau \geq 2$ for $i = 1, 2$.
2. $\beta_1 + \alpha_1 - \alpha_\tau \leq 1$ and $\beta_2 + \alpha_2 - \alpha_\tau \geq 2$.
3. $\beta_2 + \alpha_2 - \alpha_\tau \geq 2$ and $\beta_2 + \alpha_2 - \alpha_\tau \leq 1$.
4. $\beta_i + \alpha_i - \alpha_\tau \leq 1$ for $i = 1, 2$.

Case 1: By Proposition 6.7.5, $b - a \leq 1$ and $d - c \leq 1$. Suppose $b < d$. Then $a \leq c$, so $b + c < d + c$ and $a + d \leq c + d$, where $c + d = \alpha_2 + \alpha_\tau - \beta_2 \leq M(\tau) - \beta_1 - \beta_2$. Similarly if $d < b$ then $b + c \leq b + a$ and $a + d < a + b$, where $a+b = \alpha_1+\alpha_\tau-\beta_1 \leq M(\tau)-\beta_1-\beta_2$. If $b = d$ then $a+d = a+b \leq M(\tau)-\beta_1-\beta_2$ and $b+c = d+c \leq M(\tau)-\beta_1-\beta_2$. Hence $\max\{a+d-1, b+c-1\} \leq M(\tau)-\beta_1-\beta_2$.

Case 2: By Proposition 6.7.5, $a = \min\{\alpha_1, \alpha_\tau - \beta_1\}$ and $b = \max\{\alpha_1, \alpha_\tau - \beta_1\}$. Since $\alpha_1 \leq \alpha_\tau - \beta_1 + 1$ by assumption, $a \leq \alpha_\tau - \beta_1$ and $b \leq \alpha_\tau - \beta_1 + 1$. By Proposition 6.7.5,

$$\begin{aligned} c = \quad & \left\lfloor \frac{\alpha_2 + \alpha_\tau - \beta_2}{2} \right\rfloor \\ d = \quad & \left\lceil \frac{\alpha_2 + \alpha_\tau - \beta_2}{2} \right\rceil. \end{aligned}$$

Hence

$$\max\{a + d, b + c\} \leq \alpha_\tau - \beta_1 + 1 + \left\lfloor \frac{\alpha_2 + \alpha_\tau - \beta_2}{2} \right\rfloor.$$

$\alpha_2 + \alpha_\tau - \beta_2 \geq 2$, so

$$\left\lfloor \frac{\alpha_2 + \alpha_\tau - \beta_2}{2} \right\rfloor \leq \alpha_2 + \alpha_\tau - \beta_2 - 1.$$

It follows that

$$\begin{aligned} \max\{a+d, b+c\} &\leq \alpha_\tau + \alpha_2 + \alpha_\tau - \beta_1 - \beta_2 \\ &\leq M(\tau) - \beta_1 - \beta_2. \end{aligned}$$

Case 3: By arguing exactly as in Case 2 we obtain

$$\begin{aligned} \max\{a+d, b+c\} &\leq \alpha_\tau + \alpha_1 + \alpha_\tau - \beta_1 - \beta_2 \\ &\leq M(\tau) - \beta_1 - \beta_2. \end{aligned}$$

Case 4: By Proposition 6.7.5, I_1 is generated in degrees $a = \min\{\alpha_1, \alpha_\tau - \beta_1\}, b = \max\{\alpha_1, \alpha_\tau - \beta_1\}$ and I_2 is generated in degrees $c = \min\{\alpha_2, \alpha_\tau - \beta_2\}, d = \max\{\alpha_2, \alpha_\tau - \beta_2\}$. We have $\alpha_1 \leq \alpha_\tau - \beta_1 + 1$ and $\alpha_2 \leq \alpha_\tau - \beta_2 + 1$ by assumption. It follows that $a \leq \alpha_\tau - \beta_1, b \leq \alpha_\tau - \beta_1 + 1$ and $c \leq \alpha_\tau - \beta_2, d \leq \alpha_\tau - \beta_2 + 1$. So

$$\begin{aligned} \max\{a+d, b+c\} &\leq \alpha_\tau + \alpha_\tau + 1 - \beta_1 - \beta_2 \\ &\leq M(\tau) - \beta_1 - \beta_2. \end{aligned}$$

The final inequality follows since we assumed $\alpha_1, \alpha_2, \beta_1, \beta_2, \alpha_\tau$ are all at least 1. □

6.8 Examples

We give several examples to illustrate both how the bounds in Theorems 6.6.7 and 6.7.2 may be used and how well they approximate the actual regularity of the spline algebra. These examples also elucidate a difference between *complete* central complexes $\mathcal{P}$ (in which the intersection of all facets of $\mathcal{P}$ is an *interior* face of $\mathcal{P}$) and central complexes which are not complete. This difference is key to Conjecture 6.9.1 in the following section.

Example 6.8.1. In this example we apply Theorem 6.6.7 to bound the regularity of $C^\alpha(\mathcal{P})$ where boundary vanishing is imposed. Consider the two dimensional polytopal complex $\mathcal{Q}$ in Figure 6.6 with five faces, eight interior edges, and four interior vertices. Impose vanishing of order r along interior codimension one edges and vanishing of order s along boundary codimension one faces. The following Hilbert polynomials are computed in Example 7.5.5. If $s = -1$, then

$$\begin{aligned} HP(C^\alpha(\widehat{\mathcal{P}}), d) = \;& \tfrac{5}{2}d^2 + \left(-8r - \tfrac{1}{2}\right) d \\ & -4 \left\lfloor \tfrac{3r}{2} \right\rfloor^2 + 12r \left\lfloor \tfrac{3r}{2} \right\rfloor - r^2 + 4r + 2 \end{aligned}$$

By Theorem 6.6.7, *reg* $C^r(\widehat{\mathcal{Q}}) \leq 6(r+1) - 1$ and $HP(C^r(\widehat{\mathcal{Q}}), d) = \dim C^\alpha_d(\mathcal{Q})$

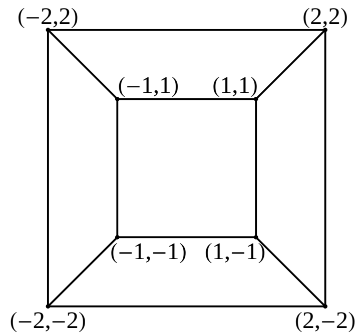


Figure 6.6: $\mathcal{Q}$

for $d \geq 6(r+1) - 2$. We compare the regularity bound $6(r+1) - 1$ with *reg* $C^r(\widehat{\mathcal{Q}})$ as computed in Macaulay2 in Table 6.2. *reg* $C^r(\widehat{\mathcal{Q}})$ appears to have alternating differences of 1 and 3 and grows roughly as $2(r+1)+1$. In fact *reg* $C^r(\widehat{\mathcal{Q}})$ appears to agree with the regularity of r-splines on the complex from Example 6.1.1.

r	0	1	2	3	4	5	6	7	8	9
$6(r+1)-1$	5	11	17	23	29	35	41	47	53	59
reg $(C^r(\widehat{\mathcal{Q}}))$	3	4	7	8	11	12	15	16	19	20

Table 6.2

Now suppose that vanishing of degree $s \geq 0$ is imposed along $\partial\mathcal{P}$. Then

$$\begin{aligned} HP(C^\alpha(\widehat{\mathcal{P}}), d) = \; & \tfrac{5}{2}d^2 + \left(-8r - 4s - \tfrac{9}{2}\right) d \\ & -3\left\lfloor \tfrac{2(r+s)}{3} \right\rfloor^2 + 4r\left\lfloor \tfrac{2(r+s)}{3} \right\rfloor + 4s\left\lfloor \tfrac{2(r+s)}{3} \right\rfloor - \left\lfloor \tfrac{2(r+s)}{3} \right\rfloor \\ & -4\left\lfloor \tfrac{r}{2} \right\rfloor^2 - 4\left\lfloor \tfrac{3r}{2} \right\rfloor^2 + 4r\left\lfloor \tfrac{r}{2} \right\rfloor + 12r\left\lfloor \tfrac{3r}{2} \right\rfloor \\ & -5r^2 + 4rs + 8r + 4s + 4. \end{aligned}$$

This formula is correct when r, s are not too small; for instance if $r = 3$ and $s = 0$, the above formula has constant term 81 while the actual constant, according to Macaulay2, is 87. By Theorem 6.6.7,

$$\textit{reg}\ (C^\alpha(\widehat{\mathcal{Q}})) \leq \max\{6(r+1) + (s+1), 5(r+1) + 2(s+1)\} - 1$$

and $HP(C^\alpha(\widehat{\mathcal{P}}), d) = \dim C^\alpha_d(\mathcal{P})$ for

$$d \geq \max\{6(r+1) + (s+1), 5(r+1) + 2(s+1)\} - 2.$$

A comparison of the bound on *reg* $(C^\alpha(\widehat{\mathcal{Q}}))$ and its actual value computed in Macau-lay2 appears in Table 6.3 for $r, s \leq 5$.

Example 6.8.2. We now give an example which has very different behavior from Example 6.5.1. Consider the two-dimensional polytopal complex $\mathcal{Q}$ formed

$\max\{6(r+1)+(s+1), 5(r+1)+2(s+1)\}-1$					
	$s=0$	$s=1$	$s=2$	$s=3$	$s=4$
$r=0$	7	8	10	12	14
$r=1$	13	14	15	17	19
$r=2$	19	20	21	22	24
$r=3$	25	26	27	28	29
$r=4$	31	32	33	34	35

reg $(C^\alpha(\widehat{\mathcal{Q}}))$					
	$s=0$	$s=1$	$s=2$	$s=3$	$s=4$
$r=0$	4	4	5	6	7
$r=1$	4	5	6	8	9
$r=2$	7	8	8	9	10
$r=3$	8	8	9	10	12
$r=4$	11	11	12	12	13

Table 6.3

by placing a regular (or almost regular) n-gon inside of a scaled copy of itself and connecting corresponding vertices by edges. $\mathcal{Q}$ has one facet with n edges and n quadrilateral facets. An example for $n = 10$ is shown in Figure 6.7. We may or may not perturb the vertices so that the affine spans of the edges between the inner and outer n-gons do not all meet at the origin. This does not appear to have much effect on regularity, although it does change the constant term of $HP(C^r(\widehat{\mathcal{Q}}), d)$.

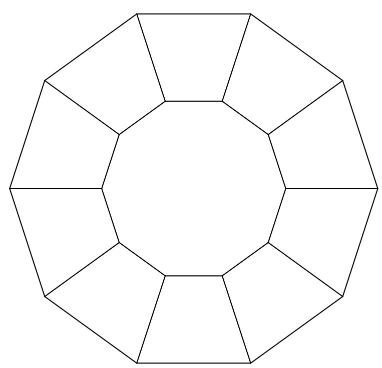


Figure 6.7

According to Theorem 6.6.7, *reg* $(C^r(\widehat{\mathcal{Q}})) \leq \max\{(r+1)(n+2), 5(r+1)\} \leq (r+1)(n+2)$ as long as $n \geq 3$. However, according to computations for $r \leq 3$ and $n \leq 10$ in Macaulay2, *reg* $(\mathcal{Q}) \leq 3(r+1)$ regardless of what value n takes. It appears that having a facet σ with many codimesion one facets may only significantly effect the regularity of $C^\alpha(\mathcal{P})$ if $\sigma \cap \partial\mathcal{P} \neq \emptyset$, as in Example 6.5.1.

Example 6.8.3. Consider a regular octahedron $\Delta \subset \mathbb{R}^3$ triangulated by placing a centrally symmetric vertex, shown in Figure 6.8. In [47, Example 5.2], Schenck shows that $C^r(\Delta)$ is free, generated in degrees $r+1, 2(r+1)$, and $3(r+1)$. Thus the regularity bound for $C^r(\Delta)$ given by Corollary 6.6.3 is tight. Computations

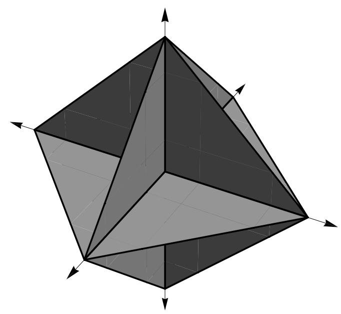


Figure 6.8: Centrally Triangulated Octahedron

in Macaulay2 suggest that the regularity of $C^r(\Delta)$ stays at $3(r+1)$ for generic perturbations of the noncentral vertices.

6.9 Regularity Conjecture

We conclude with a conjecture in the case of uniform smoothness which refines [20, Conjecture 5.6]. For $\sigma \in \mathcal{P}_n$, recall $|\partial^0(\sigma)|$ is the number of interior codimension one faces of σ.

Conjecture 6.9.1. *Let $\mathcal{P} \subset \mathbb{R}^{n+1}$ be a pure, central, hereditary n-dimensional polytopal complex. Let $F = \max\{|\partial^0(\sigma)| : \sigma \in \mathcal{P}_{n+1}\}$ and $F_\partial = \max\{|\partial^0(\sigma)| : \sigma \in \mathcal{P}_{n+1}, \sigma \cap \partial\mathcal{P} \neq \emptyset\}$.*

1. *$\mathcal{P}$ is central and complete $\implies$ reg $(C^r(\mathcal{P})) \leq$ reg $(LS^{r,n-1}(\mathcal{P})) \leq F(r+1)$.*

2. *$\mathcal{P}$ is central but not complete $\implies$ reg $(C^r(\mathcal{P})) \leq F_\partial(r+1)$.*

Furthermore, the bound is attained by free modules $C^r(\mathcal{P})$ in both cases.

Remark 6.9.2. Example 6.5.1 shows that generators can be obtained in degree $F(r+1)$ for the complete central case and degree $F_\partial(r+1)$ in the non-complete central case, so these are the lowest possible regularity bounds that we can conjecture.

Remark 6.9.3. If $C^r(\mathcal{P})$ is free, then *reg* $(C^r(\mathcal{P})) \leq F(r+1)$ by Corollary 6.6.3. Example 6.8.3, coupled with Theorem 6.7.2, shows that Conjecture 6.9.1 is true in the complete, central, three dimensional, simplicial case. If $\mathcal{P} \subset \mathbb{R}^3$ is complete, central, and non-simplicial, then Conjecture 6.9.1 should be provable using the methods of § 6.7. The difficulty is in analyzing the ideal $K(\tau)$ from Lemma 6.7.4.

Remark 6.9.4. Conjecture 6.9.1 part (2) is a natural generalization of a conjecture of Schenck [51], that *reg* $(C^r(\widehat{\Delta})) \leq 2(r+1)$ for $\Delta \subset \mathbb{R}^2$. This is a highly nontrivial conjecture in the simplicial case; it implies, for instance, that $\wp(C^1(\widehat{\Delta})) \leq 2$. To date, it is unknown whether $HP(C^1(\widehat{\Delta}), 3) = \dim C^1_3(\widehat{\Delta})$. The difficulty of this problem is in large part due to the fact that non-local geometry plays an increasingly important role in low degree [1, 4]. Since our methods hinge on using the algebras $LS^{\alpha,k}(\mathcal{P})$, which are locally supported approximations to $C^\alpha(\mathcal{P})$, our approach will not be effective in proving Conjecture 6.9.1 part (2).

Remark 6.9.5. In the non-simplicial case, Conjecture 6.9.1 part (2) appears to run contrary to the spirit of the regularity bounds we have proved in this chapter, since no account is taken of interior facets of $\mathcal{P}$, which may have many codimension one faces. It is nevertheless consistent with Example 6.5.1, where the minimal generator of high degree is supported on a boundary facet, and Example 6.8.2, where an interior facet with many codimension one faces appears to have no contribution to *reg* $(C^r(\widehat{\mathcal{P}}))$. An example of a polytopal complex $\mathcal{P}$ with a minimal generator of high degree (relative to the number of codimension one faces of boundary facets), supported on interior facets, would be quite interesting.

Chapter 7

Associated Primes of the Spline Complex

In this chapter we analyze the associated primes of homology modules of the chain complex $\mathcal{R}/\mathcal{J}$ introduced by Schenck-Stillman [50]. Working in the context of fans $\Sigma \subset \mathbb{R}^{n+1}$, we use the notation $\mathcal{R}/\mathcal{J}[\Sigma, \Sigma']$ for the spline complex, where $\Sigma' \subset \Sigma$ is a subfan. We introduced this notation in § 3.3; it is well-suited to describing the spline complexes that arise from imposing vanishing along codimension one faces of the boundary in such a way that topological contributions are made explicit. Using the notion of a lattice fan, introduced in Chapter 5, we describe localizations of the spline complex $\mathcal{R}/\mathcal{J}[\Sigma, \Sigma']$. We then prove Theorem 7.2.4, which identifies the associated primes of the homology modules of the spline complex as linear primes arising from the hyperplane arrangement of affine spans of codimension one faces, and Theorem 7.2.6, which identifies more precisely the associated primes of minimal possible codimension (this is a slight extension of [48, Theorem 2.6]).

We give two applications of these theorems to computations of dimension of the space $\dim C^{\alpha}(\Sigma)$. In Section 7.5, we derive the third coefficient of the Hilbert polynomial of the graded algebra $C^{\alpha}(\Sigma)$ of mixed splines on the polyhedral fan $\Sigma \subset \mathbb{R}^{n+1}$, where vanishing may be imposed along arbitrary codimension one faces of the boundary of Σ (Corollary 7.5.3). This result draws on two papers of Schenck, together with Geramita and McDonald, where the third coefficient is computed in the cases of simplicial mixed smoothness and polytopal uniform smoothness, respectively [28, 38, 48]; however no boundary conditions are imposed in either of these papers. The computation in Section 7.5 clarifies certain topological contributions to the third coefficient.

In Section 7.6, we describe the fourth coefficient of the Hilbert polynomial of the graded algebra $C^{\alpha}(\widehat{\Delta})$, where $\Delta \subset \mathbb{R}^3$ is a simplicial complex (Proposition 7.6.1). We use this to recover a result (for $d \gg 0$) of Alfeld, Schumaker and Whiteley on the dimension of $C^1_d(\widehat{\Delta})$ for generic $\Delta \subset \mathbb{R}^3$ [7]. In Example 7.6.5 we illustrate how Proposition 7.6.1 may be used to compute the fourth coefficient in non-generic cases.

7.1 Lattice Fans

In chapter 5 we discussed *lattice complexes* $\mathcal{P}_W \subset \mathcal{P}$ in the context of describing localization of $C^r(\mathcal{P})$. In this section we describe how this construction carries over to the context of a pair (Σ, Σ'), where $\Sigma \subset \mathbb{R}^{n+1}$ is a fan and $\Sigma' \subset \Sigma$ a subfan. In the end this will yield information about localizations of the entire complex $\mathcal{R}/\mathcal{J}[\Sigma, \Sigma']$.

Definition 7.1.1. Let $\Sigma \subset \mathbb{R}^{n+1}$ be an $(n+1)$-dimensional fan and $\Sigma' \subset \Sigma$ a subfan.

1. $\mathcal{A}(\Sigma, \Sigma')$ denotes the hyperplane arrangement $\bigcup\limits_{\tau \in \Sigma_n \setminus \Sigma'_n} \mathrm{aff}(\tau)$.
2. $L_{\Sigma,\Sigma'}$ denotes the intersection lattice $L(\mathcal{A}(\Sigma, \Sigma'))$ of $\mathcal{A}(\Sigma, \Sigma')$, ordered with respect to reverse inclusion.
3. The *support* of a face $\gamma \in \Sigma$, denoted $\mathrm{supp}(\gamma)$, is the collection of flats $W \in L_{\Sigma,\Sigma'}$ so that $W \subseteq \mathrm{aff}(\gamma)$.

Definition 7.1.2. Let Σ be an $(n+1)$-dimensional fan, $\Sigma' \subset \Sigma$ a subfan, $W \in L_{\Sigma,\Sigma'}$, and $\sigma \in \Sigma_{n+1}$.

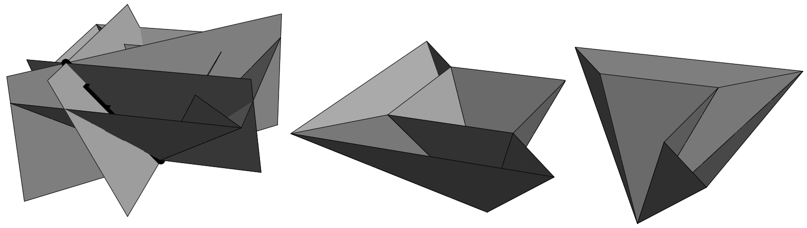
Define Σ^c_W to be the subfan of Σ consisting of all faces whose affine span does not contain W (equivalently whose support does not contain W).

Define $\Sigma_{W,\sigma} \subset \Sigma$ to be the subfan with faces $\gamma \subset \sigma' \in \Sigma_{n+1}$ so that there is a chain $\sigma = \sigma_0, \sigma_1, \ldots, \sigma_k = \sigma'$ with $\sigma_{i-1} \cap \sigma_i = \tau_i \in \Sigma_n$ and $W \subset \mathrm{aff}(\tau_i)$ for $i = 1, \ldots, k$. We call $\Sigma_{W,\sigma}$ a *lattice fan.*

Define an equivalence relation $\sim_W$ on Σ_{n+1} by $\sigma \sim_W \sigma'$ if $\sigma' \in \Sigma_{W,\sigma}$.

Definition 7.1.3. We will use the following notation

- $[\sigma]_W$: equivalence class of σ under $\sim_W$
- Υ_W : a set of distinct representatives $\sigma \in \Sigma_{n+1}$ of the equivalence classes $[\sigma]_W$
- $\Sigma_W = \bigsqcup_{\sigma \in \Upsilon_W} \Sigma_{W,\sigma}$
- $\Sigma'_{W,\sigma} = (\Sigma^c_W \cup \Sigma') \cap \Sigma_{W,\sigma}$
- $\Sigma^{-1}_{W,\sigma} = (\Sigma^c_W \cup \Sigma^{-1}) \cap \Sigma_{W,\sigma}$
- $(\Sigma^{\geq 0}_{W,\sigma})_i$ = i-faces of $\Sigma_{W,\sigma}$ not contained in $\Sigma^{-1}_{W,\sigma}$
- $(\Sigma_W, \Sigma'_W) = \sqcup_{\sigma \in \Upsilon_W} (\Sigma_{W,\sigma}, \Sigma'_{W,\sigma})$
- $\mathcal{J}[\Sigma_W, \Sigma'_W] = \bigoplus\limits_{\sigma \in \Upsilon_W} \mathcal{J}[\Sigma_{W,\sigma}, \Sigma'_{W,\sigma}]$
- $\mathcal{R}[\Sigma_W, \Sigma'_W] = \bigoplus\limits_{\sigma \in \Upsilon_W} \mathcal{R}[\Sigma_{W,\sigma}, \Sigma'_{W,\sigma}]$
- $\mathcal{R}/\mathcal{J}[\Sigma_W, \Sigma'_W] = \bigoplus\limits_{\sigma \in \Upsilon_W} \mathcal{R}/\mathcal{J}[\Sigma_{W,\sigma}, \Sigma'_{W,\sigma}]$



(a) $W = \mathrm{aff}(\widehat{v})$ (b) Σ_{W,σ_1} (c) Σ_{W,σ_2}

Figure 7.1

Remark 7.1.4. If σ has no codimension one face whose affine span contains W, then $[\sigma]_W$ consists only of σ.

Remark 7.1.5. $\Sigma_{W,\sigma}$ is the component of the lattice complex Σ_W containing the face σ.

Remark 7.1.6. The equivalence relation $\sim_W$ is similar to one used by Yuzvinsky in [63] but is different in some subtle ways. See Remark 7.3.5 following Example 7.3.4.

Remark 7.1.7. If $W \subset \mathrm{aff}(\gamma)$, where $\gamma \in \Sigma$ is a face of Σ, then $\mathrm{st}(\gamma) \subset \Sigma_{W,\sigma}$ for any facet σ with $\gamma \in \sigma$.

Example 7.1.8. Let $\mathcal{Q}$ be the polytopal complex from Example 3.2.7 and set $\Sigma = \widehat{\mathcal{Q}}$. Let $W = \mathrm{aff}(v)$, where v is an internal ray of Σ. Let σ_1 be any facet containing v and σ_2 any facet not containing v. Then W, Σ_{W,σ_1}, and Σ_{W,σ_2} are shown in Figure 7.1. Notice that Σ_W consists of two nontrivial components. Also let V be the affine span of the internal codimension one face of Σ_{W,σ_2}. Then $W \subset V$ ($V < W$ in $L_{\Sigma,\Sigma^{-1}}$) and $\Sigma_{W,\sigma_2} = \Sigma_{V,\sigma_2}$. Occasionally we will want to replace W by the minimal flat V satisfying $\Sigma_{V,\sigma_2} = \Sigma_{W,\sigma_2}$.

In the simplicial case $\Sigma_{W,\sigma}$ is always the star of a face.

Lemma 7.1.9. *Let $\Sigma \subset \mathbb{R}^{n+1}$ be a simplicial fan, $\Sigma' \subset \Sigma$ a subfan. Then $\Sigma_{W,\sigma} = st_\Sigma(\gamma)$ for some face γ with $W \subset aff(\gamma)$.*

Proof. This is the content of [20, Lemma 2.7]. □

With these notations in place the following lemma is almost immediate.

Lemma 7.1.10. *Let $\Sigma \subset \mathbb{R}^{n+1}$ be an $(n+1)$-dimensional fan, $\Sigma' \subset \Sigma$ a subfan, and $P \in spec(S)$. Set $W = \max_{V \in L_{\Sigma,\Sigma'}}\{I(V)|I(V) \subset P\}$. Then*

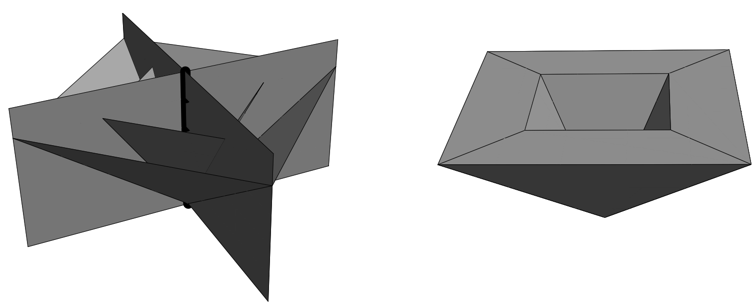
$$\begin{aligned} \mathcal{R}/\mathcal{J}[\Sigma,\Sigma']_P &= \mathcal{R}/\mathcal{J}[\Sigma,\Sigma^c_W \cup \Sigma']_P \\ &= \mathcal{R}/\mathcal{J}[\Sigma_W,\Sigma'_W]_P, \end{aligned}$$

Proof. Each module in the chain complex $\mathcal{R}/\mathcal{J}[\Sigma,\Sigma']$ is a direct sum of modules of the form $S/J(\tau)$ for $\tau \in \Sigma \setminus \Sigma'$. Under localization, all of these go to zero unless $J(\tau) \subset P$, in other words, $J(\tau) \subset I(W)$, hence $W \subset \mathrm{aff}(\tau)$ and $\tau \notin \Sigma'$.

This proves the first equality. The second equality simply rewrites the complex $\mathcal{R}/\mathcal{J}[\Sigma, \Sigma_W^c \cup \Sigma']$ as a direct sum across connected components of Σ_W, using the observation that $\Sigma \setminus (\Sigma_W^c \cup \Sigma') = \sqcup_{\sigma \in \Upsilon_W} \Sigma_{W,\sigma} \setminus \Sigma'_{W,\sigma}$. □

We do an extended computation to show how the complexes $\Sigma_{W,\sigma}$ and their topology can be used to compute certain localizations of the complex $\mathcal{R}/\mathcal{J}[\Sigma, \Sigma^{-1}]$. This is a rather long process to compute a localization which is fairly quick to do by hand, however it illustrates the general procedure.

Example 7.1.11. Let $\mathcal{Q}$ be the complex from 3.2.7 and $\Sigma = \widehat{\mathcal{Q}}$. We show in Figure 7.2a the affine spans of 4 interior codimension one faces which intersect along the z-axis, which we denote by W. In Figure 7.2b we show the cell complex $\mathcal{P}(\Sigma_{W,\sigma})$ (up to homeomorphism), where σ is any of the four facets with a codimension one face γ with $W \in \operatorname{supp}(\gamma)$. Note that the central facet is removed. In this case $\Sigma_W = \Sigma_{W,\sigma}$.



(a) W (b) $\mathcal{P}(\Sigma_{W,\sigma})$

Figure 7.2

Since the only codimension one facets whose affine spans contain W are interior, $\Sigma^{-1}_{W,\sigma} = \partial \Sigma_{W,\sigma}$ regardless of what smoothness parameters we assign.

The complex $\mathcal{R}[\Sigma_{W,\sigma}, \Sigma^{-1}_{W,\sigma}] = \mathcal{R}[\Sigma_{W,\sigma}, \partial\Sigma_{W,\sigma}]$ is concentrated in homological degrees 3 and 2 and has the form

$$S^4 \to S^4 \to 0 \to 0.$$

$H_*(\mathcal{R}[\Sigma_{W,\sigma}, \partial\Sigma_{W,\sigma}])$ computes the homology of $\mathcal{P}(\Sigma_{W,\sigma})$ relative to $\partial\mathcal{P}(\Sigma_{W,\sigma})$, so

$$H_i(\mathcal{R}[\Sigma_{W,\sigma}, \partial\Sigma_{W,\sigma}]) = H_{i-1}(\operatorname{lk}(\Sigma_{W,\sigma}), \partial \operatorname{lk}(\Sigma_{W,\sigma}))$$

for $i \geq 2$ by Proposition 3.3.17.

From Figure 7.3, which displays $\Sigma_{W,\sigma}$) and its boundary (up to homeomorphism) we see that the homology on the left-hand side is the same as the homology of a 2-sphere with 2 points identified. Thus $H_3(\mathcal{R}[\Sigma_{W,\sigma}, \partial\Sigma_{W,\sigma}) = H_2(\mathcal{R}[\Sigma_{W,\sigma}, \partial\Sigma_{W,\sigma}) = S$ while the lower two homologies vanish.

Now, via the tail end of the long exact sequence

$$0 \to \mathcal{J}[\Sigma_{W,\sigma}, \partial\Sigma_{W,\sigma}] \to \mathcal{R}[\Sigma_{W,\sigma}, \partial\Sigma_{W,\sigma}] \to \mathcal{R}/\mathcal{J}[\Sigma_{W,\sigma}, \partial\Sigma_{W,\sigma}] \to 0,$$

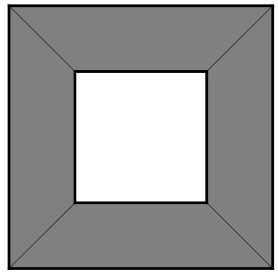


Figure 7.3: $\mathrm{lk}(\Sigma_{W,\sigma})$

we obtain that

$$H_2(\mathcal{R}/\mathcal{J}[\Sigma_W, \sigma]) = S/(\sum_{i=1}^{4} J(\tau_i)),$$

where $\tau_1, \ldots, \tau_4$ are the four interior codimension one facets of $\Sigma_{W,\sigma}$. By Lemma 7.1.10, we have shown that

$$H_2(\mathcal{R}/\mathcal{J}[\Sigma, \Sigma^{-1}])_{I(W)} = (S/(\sum_{i=1}^{4} J(\tau_i)))_{I(W)}.$$

In fact we could replace $I(W)$ by any prime P containing $I(W)$, as long is there is no other flat $V \in L_{\Sigma,\Sigma^{-1}}$, with $I(W) \subsetneq I(V)$, so that $I(V) \subset P$.

From Lemma 7.1.10 and Example 7.1.11 we see that it is useful to understand the homology of the complexes $\mathcal{R}[\Sigma_{W,\sigma}, \Sigma'_{W,\sigma}]$. To this end we introduce a variant of a graph used by Schenck [48, Definition 2.5], which also builds on dual graphs of Rose [45, 46]. This graph simplifies the computation of the homology of $\Sigma_{W,\sigma}$ in homological degree $\dim(W)+1$. In order to construct this graph we need the following easy lemma.

Lemma 7.1.12. *Suppose $\psi \subset \mathbb{R}^{n+1}$ is a convex polyhedral cone of dimension $d+2$ and $W \subset \mathit{aff}(\psi)$, where W is a linear subspace of dimension d. Then ψ has at most 2 faces $\gamma_1, \gamma_2 \in \Sigma_{d+1}$ so that $W \subset \mathit{aff}(\gamma_1)$ and $W \subset \mathit{aff}(\gamma_2)$.*

Proof. This follows from the fact that the intersection of the affine hulls of three distinct codimension one faces of a convex cone ψ cannot intersect in a codimension 2 linear space. This would require the supporting hyperplane of one of the faces to be 'between' the other two, hence this hyperplane would meet the interior of ψ, which is a contradiction. □

Definition 7.1.13. Suppose $\Sigma \subset \mathbb{R}^{n+1}$ is a pure,hereditary, $(n+1)$-dimensional fan, $\Sigma' \subset \partial\Sigma$ is a subfan, and $W \subset \mathbb{R}^{n+1}$ is a d-dimensional subspace so that $W \subset \bigcap_{\tau \in \Sigma_n \setminus \Sigma'_n} \mathrm{aff}(\tau)$.

$G_W(\Sigma, \Sigma')$ is the graph with one vertex for every face in $\Sigma_{d+1} \setminus \Sigma'_{d+1}$. Also $G_W(\Sigma, \Sigma')$ has one distinguished vertex v_b iff there is at least one face $\psi \in \Sigma_{d+2} \setminus \Sigma'_{d+2}$ having only one face γ so that $\gamma \in \Sigma_{d+1} \setminus \Sigma'_{d+1}$. Two vertices v, w

corresponding to $\gamma_v, \gamma_w \in \Sigma_{d+1} \setminus \Sigma'_{d+1}$ are connected in $G_W(\Sigma, \Sigma')$ iff there is a $\psi \in \Sigma_{d+2} \setminus \Sigma'_{d+2}$ so that γ_v, γ_w are the faces of ψ whose affine spans contain W. Connect the vertex v to the vertex v_b if the corresponding face $\gamma_v \in \Sigma_{d+1}$ is contained in a face $\psi \in \Sigma_{d+2}$ so that γ_v is the only $(d+1)$-face of ψ so that $\gamma_v \in \Sigma_{d+1} \setminus \Sigma'_{d+1}$.

Proposition 7.1.14. *Let $\Sigma \subset \mathbb{R}^{n+1}$ be a pure, hereditary, $(n+1)$-dimensional fan and $\Sigma' \subset \partial\Sigma$ a subfan. Suppose that $W \subset \mathbb{R}^{n+1}$ is a d-dimensional subspace so that $W \subset \bigcap_{\tau \in \Sigma_n \setminus \Sigma'_n} \text{aff}(\tau)$. Then*

1. *If $\Sigma_d \setminus \Sigma'_d = \emptyset$, then $H_d(\mathcal{R}[\Sigma, \Sigma']) = 0$ and*

$$H_{d+1}(\mathcal{R}[\Sigma, \Sigma']) = \begin{cases} 0 & v_b \in G_W(\Sigma, \Sigma') \\ S & \text{otherwise} \end{cases}$$

2. *If $\Sigma_d \setminus \Sigma'_d \neq \emptyset$, then $H_{d+1}(\mathcal{R}[\Sigma, \Sigma']) = H_d(\mathcal{R}[\Sigma, \Sigma']) = 0$.*

Proof. The main point of this proof is that the top cellular boundary map

$$\phi_W : \bigoplus_{e \in G_W(\Sigma, \Sigma')} S \to \bigoplus_{v \neq v_b \in G_W(\Sigma, \Sigma')} S$$

of $G_W(\Sigma, \Sigma')$ (relative to v_b, if v_b is present) is really the same (by definition!) as the cellular map

$$\delta_{d+2} : \mathcal{R}[\Sigma, \Sigma']_{d+2} \to \mathcal{R}[\Sigma, \Sigma']_{d+1}.$$

(1) Since $\mathcal{R}[\Sigma, \Sigma']_d = 0$, $H_d(\mathcal{R}[\Sigma, \Sigma']) = 0$ and $H_{d+1}(\mathcal{R}[\Sigma, \Sigma']) = \text{coker}(\phi_W) = H_0(G_W(\Sigma, \Sigma'), v_b; S)$, where v_b is understood to be the emptyset if $G_W(\Sigma, \Sigma')$ has no vertex v_b. Since $G_W(\Sigma, \Sigma')$ is connected, this proves (1).

(2) Let $\gamma \in \Sigma_d \setminus \Sigma'_d \neq \emptyset$. We claim that $\Sigma = \text{st}_\Sigma(\gamma)$. Suppose there is a facet $\sigma' \notin \text{st}_\Sigma(\gamma)$. Since Σ is hereditary, we may assume that σ' is adjacent to a $\sigma \in \text{st}_\Sigma(\gamma)$. Set $\tau = \sigma \cap \sigma'$ and $H = \text{aff}(\tau)$. $\tau \notin \Sigma'$ since τ is interior, hence $W \subset H$ and $\gamma \subset W \cap \sigma \subset H \cap \sigma$. This is a contradiction since $H \cap \sigma = \tau$ and we assumed $\gamma \notin \sigma'$. Hence $\Sigma = \text{st}_\Sigma(\gamma)$, and γ is the unique face in $\Sigma_d \setminus \Sigma'_d$. It follows that $H_d(\mathcal{R}[\Sigma, \Sigma']) = 0$ since the map

$$\delta_d : \mathcal{R}[\Sigma, \Sigma']_{d+1} = \oplus_{\gamma \in \Sigma_{d+1} \setminus \Sigma'_{d+1}} S \to S = \mathcal{R}[\Sigma, \Sigma']_d$$

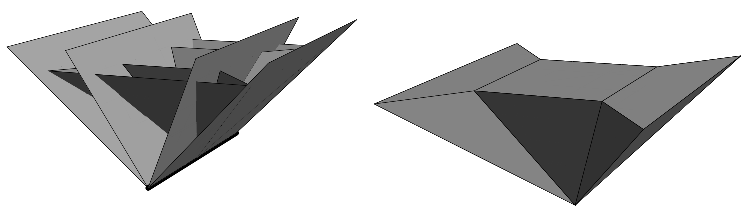
is surjective. Furthermore, in this case v_b is *not* present in $G_W(\Sigma, \Sigma')$, so $\text{coker}(\phi_W) = S = \text{coker}(\delta_{d+2})$ and $H_{d+2}(\mathcal{R}[\Sigma, \Sigma']) = 0$ as well. ☐

Example 7.1.15. Let Σ, W, and $\Sigma_{W,\sigma}$ all be as in Example 7.1.11. The graph $G_W(\Sigma_{W,\sigma}, \partial\Sigma_{W,\sigma})$ has four vertices corresponding to the four interior codimension one faces and four edges which connect these vertices into a cycle. There is no vertex v_b since all four facets having a codimension one face whose affine span contains W have precisely 2 such faces. Hence $H_2(\mathcal{R}[\Sigma_{W,\sigma}, \partial\Sigma_{W,\sigma}]) = S$ by Proposition 7.1.14, as we computed in Example 7.1.11.

Example 7.1.16. Let Σ be as in Example 7.1.11. Let $V \in L_{\Sigma,\Sigma^{-1}}$ be the x-axis, which we obtain as the intersection of the four affine spans shown in Figure 7.4a. The corresponding lattice complex $\Sigma_{V,\sigma}$, where σ is any of the three facets having a codimension one face γ so that $V \in \text{supp}(\gamma)$, is shown in Figure 7.4b. The graph $G_V(\Sigma_{V,\sigma}, \Sigma^{-1}_{V,\sigma})$ can have three or four vertices depending on whether we impose vanishing along none, one, or both of the codimension one faces of $\partial\Sigma \cap \Sigma_{V,\sigma}$. $G_V(\Sigma_{V,\sigma}, \Sigma^{-1}_{V,\sigma})$ has v_b as a vertex unless Σ^{-1} contains neither of these codimension one faces, hence this is the only case which leads to a nontrivial contribution to $H_2(\mathcal{R}/[\Sigma, \Sigma^{-1}])$. For such a choice of Σ^{-1}, Proposition 7.1.14 yields that $H_2(\mathcal{R}[\Sigma, \Sigma^{-1}]) = S$. The same arguments as in Example 7.1.11 yield that

$$H_2(\mathcal{R}/\mathcal{J}[\Sigma, \Sigma^{-1}])_{I(V)} = (S/\sum_{i=1}^{4} J(\tau_i))_{I(V)},$$

where $\tau_1, \ldots, \tau_4$ are the four codimension one faces of $\Sigma_{V,\sigma}$ whose affine spans contain V.



(a) V (b) $\mathcal{P}(\Sigma_{V,\sigma})$

Figure 7.4

7.2 Associated Primes of the Spline Complex

The primary objective of this section is to describe associated primes of the homology modules $H_i(\mathcal{R}/\mathcal{J}[\Sigma, \Sigma'])$.

Lemma 7.2.1. *Let $\Sigma \subset \mathbb{R}^{n+1}$ be a pure, hereditary, $(n+1)$-dimensional fan, α a choice of smoothness parameters, $\Sigma' \subset \Sigma$ a subfan, and $W \subset \mathbb{R}^{n+1}$ a linear subspace so that $W \subset \cap_{\tau \in \Sigma_n \setminus \Sigma'_n} \text{aff}(\tau)$. Then*

$$P \in Ass_S(H_i(\mathcal{R}/\mathcal{J}[\Sigma, \Sigma'])) \implies P \subseteq I(W)$$

Proof. Let $d = \dim W$. Let V be a complementary vector space, so $V \cap W = \mathbf{0}$ and $\dim V = n + 1 - d$, and let $\pi : \mathbb{R}^{n+1} \to V$ be the projection onto V with kernel W. Then we can view the coordinate ring $\mathbb{R}[V]$ of V in two ways. Via the inclusion $i : V \to \mathbb{R}^{n+1}$ we represent $\mathbb{R}[V]$ as the quotient $S \xrightarrow{i^*} S/I(V)$.

Via the projection $\pi : \mathbb{R}^{n+1} \to V$ with kernel W, we represent $\mathbb{R}[V] \xrightarrow{\pi^*} S$ as an inclusion, where $\mathbb{R}[V]$ is generated as a subalgebra of S by a choice of $n+1-d$ linear forms which vanish on W. Here we will regard $\mathbb{R}[V]$ as a subalgebra of S via π^*. We first prove that there is a complex $\mathcal{C}$ of $\mathbb{R}[V]$-modules so that $\mathcal{R}/\mathcal{J}[\Sigma, \Sigma'] \cong \mathcal{C} \otimes_{\mathbb{R}[V]} S$ as complexes.

Denote $J(\gamma) \cap \mathbb{R}[V]$ by $J(\pi(\gamma))$: $\pi(\gamma)$ is a cone in V and smoothness parameters can be assigned naturally to its faces so that this makes notational sense. The ideals $J(\gamma)$ for $\gamma \in \Sigma \setminus \Sigma'$ are generated by powers of linear forms contained in the subalgebra $\mathbb{R}[V]$, since every face $\gamma \in \Sigma \setminus \Sigma'$ has $W \subset \operatorname{aff}(\gamma)$. It follows that $J(\pi(\gamma)) \otimes_{\mathbb{R}[V]} S = J(\gamma)$. We hence have

$$\frac{S}{J(\gamma)} \cong \frac{\mathbb{R}[V]}{J(\pi(\gamma))} \otimes_{\mathbb{R}[V]} S.$$

This tensor decomposition respects the differential of the complex $\mathcal{R}/\mathcal{J}[\Sigma, \Sigma']$, so the desired complex $\mathcal{C}$ of $\mathbb{R}[W]$-modules is obtained by setting

$$\mathcal{C}_i = \bigoplus_{\dim \gamma = i} \frac{\mathbb{R}[V]}{J(\pi(\gamma))}$$

with cellular differential.

Now that we have found such a $\mathcal{C}$, we have $H_i(\mathcal{R}/\mathcal{J}[\Sigma, \Sigma']) = H_i(\mathcal{C} \otimes_{\mathbb{R}[V]} S)$. Since S is a *flat* $\mathbb{R}[V]$-algebra, $\otimes_{\mathbb{R}[V]} S$ is exact and hence

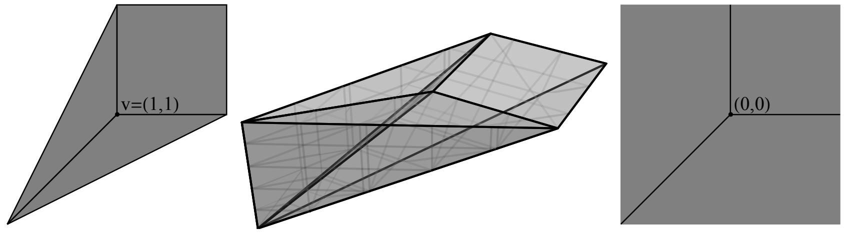
$$H_i(\mathcal{C} \otimes_{\mathbb{R}[V]} S) \cong H_i(\mathcal{C}) \otimes_{\mathbb{R}[V]} S.$$

Every associated prime of $H_i(\mathcal{C})$ is contained in the homogeneous maximal ideal $I(\pi(W)) = I(W) \cap \mathbb{R}[V]$ of $\mathbb{R}[V]$. Now the result follows from 2.2.2 part (3), since associated primes of $H_i(\mathcal{C}) \otimes_{\mathbb{R}[V]} S$ are obtained by extending associated primes of $H_i(\mathcal{C})$. □

Remark 7.2.2. There is a natural way to interpret $\mathcal{C}$ geometrically. From the proof of Lemma 7.2.1, we see that

$$\mathcal{C}_i = \bigoplus_{\dim \gamma = i} \frac{\mathbb{R}[V]}{J(\pi(\gamma))},$$

where π is the projection (with kernel W) of $\mathbb{R}^{n+1}$ onto a complementary subspace V. Hence $\mathcal{C}$ 'should' be $\mathcal{R}/\mathcal{J}[\pi(\Sigma), \pi(\Sigma) \setminus \pi(\Sigma \setminus \Sigma')]$. The reason for the choice of $\pi(\Sigma) \setminus \pi(\Sigma \setminus \Sigma')$ is that it is possible for $\pi(\tau) = \pi(\psi)$, where $\psi \in \Sigma \setminus \Sigma'$ and $\tau \in \Sigma'$. In order for this to make sense in the framework we have presented, $\pi(\Sigma)$ and $\pi(\Sigma) \setminus \pi(\Sigma \setminus \Sigma')$ both need to have the structure of fans. A priori all we know about $\pi(\Sigma)$ is that it is a union of cones (the projections of the faces of Σ), and it may be that there is no meaningful way to give this union the structure of a fan. We describe a special case where it is possible to give $\pi(\Sigma)$ and $\pi(\Sigma) \setminus \pi(\Sigma \setminus \Sigma')$ a meaningful structure, which we will use in § 7.6.



(a) $\mathcal{P}$ (b) Σ (c) $\mathcal{P}_v$

Figure 7.5: A 2-dimensional star, its cone and projection

Suppose $\mathcal{P} \subset \mathbb{R}^n$ is the star of a vertex $v \in \mathcal{P}_0$ and $\Sigma = \widehat{\mathcal{P}} \subset \mathbb{R}^{n+1}$. Let $\mathcal{P}_v$ be the fan with faces $\text{cone}(\gamma - v)$ for every $\gamma \in \mathcal{P}$ with $v \in \gamma$. This puts faces of $\mathcal{P}_v$ in a clear dimension preserving bijection with faces of $\mathcal{P}$ containing v. For each $\tau \in \mathcal{P}_{n-1}$ with $v \in \tau$, assign the smoothness parameter $\alpha(\tau)$ to the codimension one face $\text{cone}(\tau - v)$ of $\mathcal{P}_v$. See Figure 7.5 for this setup.

Lemma 7.2.3. *Let $\mathcal{P}, \Sigma, \mathcal{P}_v$ be as defined above. Set $W = \text{aff}(\widehat{v}) \subset \mathbb{R}^{n+1}$, let $V \cong \mathbb{R}^n$ be the complementary subspace defined by the vanishing of x_0, and denote by $\pi : \mathbb{R}^{n+1} \to V$ the projection with kernel W. Also let $R = \mathbb{R}[V] = \mathbb{R}[x_1, \ldots, x_n]$ and $S = \mathbb{R}[x_0, \ldots, x_n]$. Then*

1. $\pi(\Sigma) = |\mathcal{P}_v|$
2. $\pi(\Sigma) \setminus \pi(\Sigma \setminus \Sigma^{-1}) = \mathcal{P}_v^{-1}$
3. $\mathcal{R}/\mathcal{J}[\Sigma, \Sigma^{-1}] \cong \mathcal{R}/\mathcal{J}[\mathcal{P}_v, \mathcal{P}_v^{-1}](-1) \otimes_R S$,

where the -1 in parentheses records a homological shift in dimension. In particular, if $\Sigma^{-1} = \partial\Sigma$, then $\mathcal{R}/\mathcal{J}[\Sigma, \partial\Sigma] \cong \mathcal{R}/\mathcal{J}[\mathcal{P}_v, \partial\mathcal{P}_v](-1) \otimes_R S$.

Proof. To show that $\pi(\Sigma) = |\mathcal{P}_v|$, let us first show that if $\gamma \subset \mathcal{P}$ is a face of $\mathcal{P}$ containing v, then $\pi(\widehat{\gamma}) = \text{cone}(\gamma - v)$. Let $\gamma = \text{conv}(v, v + q_1, \ldots, v + q_k)$ with coordinates $v = (v_1, \ldots, v_n)$ and $q_i = (q_i^1, \ldots, q_i^n)$ for $i = 1, \ldots, k$. Set $q_i' = (0, q_i^1, \ldots, q_i^n) \in V$ and $v' = (1, v_1, \ldots, v_n)$. We have $W = \text{aff}(\widehat{v}) = \text{aff}(v')$, $\widehat{\sigma} = \text{cone}(v', v' + q_1', \ldots, v' + q_k')$. Then $\pi(v' + q_i') = q_i'$ and $\pi(\widehat{\gamma}) = \text{cone}(\mathbf{0}, q_1', \ldots, q_k') = \text{cone}(\gamma - v)$. Now, suppose that $\gamma \in \mathcal{P}$ does not contain v. By definition of the star of a vertex, $\gamma \subset \psi$, where $\psi \in \mathcal{P}$ contains v. Since $\widehat{\gamma} \subset \widehat{\psi}$, $\pi(\widehat{\gamma}) \subset \pi(\widehat{\psi}) \subset |\mathcal{P}_v|$.

To show that $\pi(\Sigma) \setminus \pi(\Sigma \setminus \Sigma^{-1}) = \mathcal{P}_v^{-1}$, we claim that

$$\pi(\Sigma) \setminus \pi(\Sigma \setminus \Sigma^{-1}) = \bigcup_{\gamma \in \Sigma^{-1}, \widehat{v} \subset \gamma} \pi(\gamma).$$

To prove this, suppose $x \in \pi(\Sigma) \setminus \pi(\Sigma \setminus \Sigma^{-1})$. First we show that $x \in \pi(\gamma)$ for some $\gamma \in \Sigma^{-1}$ such that $\widehat{v} \subseteq \gamma$. Suppose not, and let $x' \in \gamma$ so that $\pi(x') = x$. Then $x' + w \notin \gamma$ for some positive multiple w of v'. But $x' + w \in \sigma$ for any facet σ containing x', since $\Sigma = \text{st}(\widehat{v})$. Hence it follows that $x' + w \in \Sigma \setminus \Sigma^{-1}$. Since

$\pi(x'+w) = \pi(x') = x$, this is a contradiction. So $x \in \pi(\gamma)$ for some $\gamma \in \Sigma^{-1}$ such that $\widehat{v} \subseteq \gamma$. We now claim that $\pi(\gamma) \cap \pi(\Sigma \setminus \Sigma^{-1}) = \emptyset$. To see this suppose that $y \in \gamma$ and $\pi(y) = \pi(y')$ for some $y' \notin \Sigma^{-1}$. Then $y' = y + w$ for some $w \in \text{aff}(v') = W$. Since only nonnegative muliplies of v' intersect nontrivially with Σ, we have either that $y' = y + w$ for some $w \in \widehat{v}$ or $y = y' + w$ for some $w \in \widehat{v}$. But $\widehat{v} \subset \gamma$, so we see that in either case, $y' \in \gamma$. This contradicts the choice of y', since we assumed $\gamma \in \Sigma^{-1}$.

We have

$$\begin{aligned}\mathcal{R}/\mathcal{J}[\Sigma,\Sigma^{-1}]_{i+1} &= \bigoplus_{\gamma\in\Sigma^{\geq 0}_{i+1}} \frac{S}{J(\gamma)} \\ &= \bigoplus_{\gamma\in\pi(\Sigma^{\geq 0}_{i+1})} \frac{R}{J(\pi(\gamma))} \otimes_R S \\ &= \mathcal{R}/\mathcal{J}[\mathcal{P}_v,\mathcal{P}_v^{-1}]_i \otimes_R S.\end{aligned}$$

The differential is in degree 0 and so commutes with tensoring, hence

$$\mathcal{R}/\mathcal{J}[\Sigma,\Sigma^{-1}] \cong \mathcal{R}/\mathcal{J}[\mathcal{P}_v,\mathcal{P}_v^{-1}](-1) \otimes_R S.$$

Finally, we must show that $\partial\pi(\Sigma) = \pi(\Sigma) \setminus \pi(\Sigma \setminus \partial\Sigma)$. Since interior faces of $\pi(\Sigma)$ are projections of interior faces of Σ, this is clear. □

The following theorem generalizes [47, Lemma 3.1] and [48, Lemma 2.4], precisely describing the form which associated primes of $\mathcal{R}/\mathcal{J}[\Sigma,\Sigma^{-1}]$ must take.

Theorem 7.2.4. *Let $\Sigma \subset \mathbb{R}^{n+1}$ be a pure, hereditary, $(n+1)$-dimensional fan with smoothness parameters α. For $1 \leq i \leq n$ we have*

1. $Ass_S(H_i(\mathcal{R}/\mathcal{J}[\Sigma,\Sigma^{-1}])) \subset \{I(W) | W \in L_{\Sigma,\Sigma^{-1}}, \dim(W) \leq i-1\}$
2. *If* $H_i(\mathcal{R}[\Sigma_{W,\sigma},\Sigma^{-1}_{W,\sigma}]) = 0$ *for every* $W \in L_{\Sigma,\Sigma^{-1}}$ *with* $\dim W = i-1$*, then*

$$Ass_S(H_i(\mathcal{R}/\mathcal{J}[\Sigma,\Sigma^{-1}])) \subset \{I(W) | W \in L_{\Sigma,\Sigma^{-1}}, \dim(W) \leq i-2\}$$

3. *If Σ is simplicial, then*

$$Ass_S(H_i(\mathcal{R}/\mathcal{J}[\Sigma,\Sigma^{-1}])) \subset \{I(\gamma) | \gamma \in \Sigma, \textit{aff}(\gamma) \in L_{\Sigma,\Sigma^{-1}}, \dim(\gamma) \leq i-2\}$$

Proof. Assume $H_i(\mathcal{R}/\mathcal{J}[\Sigma,\Sigma^{-1}]) \neq 0$, so $\text{Ass}_S(H_i(\mathcal{R}/\mathcal{J}[\Sigma,\Sigma^{-1}])) \neq \emptyset$. Let $P \in \text{Ass}_S(H_i(\mathcal{R}/\mathcal{J}[\Sigma,\Sigma^{-1}]))$ and set $W = \max_{V\in L_{\Sigma,\Sigma^{-1}}}\{I(V) | I(V) \subset P\}$. If $W = \mathbb{R}^{n+1}$, so that $I(W) = 0$, then $H_i(\mathcal{R}/\mathcal{J}[\Sigma,\Sigma^{-1}])_P = 0$ for $1 \leq i \leq n$. So if $P \in \text{Ass}_S(H_i(\mathcal{R}/\mathcal{J}[\Sigma,\Sigma^{-1}]))$, it must contain at least one ideal of the form $I(\text{aff}(\tau))$, $\tau \in \Sigma^{\geq 0}_n$, and W must be a proper subspace of $\mathbb{R}^{n+1}$. Then

$P \in \mathrm{Ass}_S(H_i(\mathcal{R}/\mathcal{J}[\Sigma, \Sigma^{-1}]))$

$$\begin{aligned} &\iff PS_P \in \mathrm{Ass}_{S_P}(H_i(\mathcal{R}/\mathcal{J}[\Sigma, \Sigma^{-1}])_P) \\ &\iff PS_P \in \mathrm{Ass}_{S_P}(H_i(\mathcal{R}/\mathcal{J}[\Sigma_{W,\sigma}, \Sigma^{-1}_{W,\sigma}])_P) \text{ for some } \sigma \in \Sigma_{n+1} \\ &\iff P \in \mathrm{Ass}_S(H_i(\mathcal{R}/\mathcal{J}[\Sigma_{W,\sigma}, \Sigma^{-1}_{W,\sigma}])) \text{ for some } \sigma \in \Sigma_{n+1}. \end{aligned}$$

The first and last equivalences follow from Proposition 2.2.2 part (1). The second equivalence follows from Lemma 7.1.10 and Lemma 2.2.2 part (2).

Now pick σ so that $P \in \mathrm{Ass}_S(H_i(\mathcal{R}/\mathcal{J}[\Sigma_{W,\sigma}, \Sigma^{-1}_{W,\sigma}]))$. It is clear that $W \subset \bigcap_{\tau \in (\Sigma^{\geq 0}_{W,\sigma})_n} \mathrm{aff}(\tau)$. Hence by Lemma 7.2.1, $P \subseteq I(W)$. But $I(W) \subseteq P$ by construction, so $P = I(W)$.

By Proposition 7.1.14, $H_k(\mathcal{R}[\Sigma_{W,\sigma}, \Sigma^{-1}_{W,\sigma}]) = 0$ for $k \leq \dim(W)$. By the long exact sequence in homology corresponding to the short exact sequence

$$0 \to \mathcal{J}[\Sigma_{W,\sigma}, \Sigma^{-1}_{W,\sigma}] \to \mathcal{R}[\Sigma_{W,\sigma}, \Sigma^{-1}_{W,\sigma}] \to \mathcal{R}/\mathcal{J}[\Sigma_{W,\sigma}, \Sigma^{-1}_{W,\sigma}] \to 0,$$

we obtain that $H_{\dim(W)}(\mathcal{R}/\mathcal{J}[\Sigma_{W,\sigma}, \Sigma^{-1}_{W,\sigma}]) = 0$ as well. Hence $i \geq \dim(W) + 1$ if $I(W) \in \mathrm{Ass}(H_i(\mathcal{R}/\mathcal{J}[\Sigma, \Sigma^{-1}]))$, which gives the condition on dimension. This concludes the proof of (1).

If in addition $H_{\dim(W)+1}(\mathcal{R}[\Sigma_{W,\sigma}), \Sigma^{-1}_{W,\sigma}]) = 0$, then the long exact sequence in homology yields

$$H_{\dim(W)+1}(\mathcal{R}/\mathcal{J}[\Sigma_{W,\sigma}, \Sigma^{-1}_{W,\sigma}]) \cong H_{\dim(W)}(\mathcal{J}[\Sigma_{W,\sigma}, \Sigma^{-1}_{W,\sigma}]).$$

If $\left(\Sigma^{\geq 0}_{W,\sigma}\right)_{\dim W} = \emptyset$ then $\mathcal{J}[\Sigma_{W,\sigma}, \Sigma^{-1}_{W,\sigma}]_{\dim(W)} = 0$ automatically. If $\left(\Sigma^{\geq 0}_{W,\sigma}\right)_{\dim W} \neq \emptyset$ then we saw in the proof of Proposition 7.1.14 that $\Sigma_{W,\sigma}$ must be the star of a face γ with $\mathrm{aff}(\gamma) = W$. In this case $H_{\dim(W)}(\mathcal{J}[\Sigma_{W,\sigma}, \Sigma^{-1}_{W,\sigma}])$ is the cokernel of the map

$$\bigoplus_{\psi \in \left(\Sigma^{\geq 0}_{W,\sigma}\right)_{\dim(W)+1}} J(\psi) \to J(\gamma).$$

Since γ is an interior face, the sum on the left hand side runs over all ideals of faces $\psi \in (\Sigma_{W,\sigma})_{\dim(W)+1}$ so that $\gamma \in \psi$. By definition,

$$\sum_{\substack{\gamma \in \psi \\ \psi \in \left(\Sigma^{\geq 0}_{W,\sigma}\right)_{\dim(W)+1}}} J(\psi) = \sum_{\substack{\gamma \in \tau \\ \tau \in \left(\Sigma^{\geq 0}_{W,\sigma}\right)_n}} J(\tau) = J(\gamma),$$

so the map above is surjective and $H_{\dim(W)}(\mathcal{J}[\Sigma_{W,\sigma}, \Sigma^{-1}_{W,\sigma}]) = 0$, hence $H_{\dim(W)+1}(\mathcal{R}/\mathcal{J}[\Sigma_{W,\sigma}, \Sigma^{-1}_{W,\sigma}]) = 0$ as well. This completes the proof of (2).

Now suppose Σ is simplicial. Then $\Sigma_{W,\sigma} = \mathrm{st}_\Sigma(\gamma)$ for some γ with $W \subset \mathrm{aff}(\gamma)$, by Proposition 7.1.9. Replacing W with $\mathrm{aff}(\gamma)$ if necessary, we may assume that $W = \mathrm{aff}(\gamma)$, hence $P = I(W) = I(\gamma)$. We also have that

$H_{\dim(\gamma)+1}(\mathcal{R}[\Sigma_{W,\sigma}, \Sigma^{-1}_{W,\sigma}]) = 0$ by part (2) of Proposition 7.1.14. As in the proof of (2), this yields

$$H_{\dim(W)+1}(\mathcal{R}/\mathcal{J}[\Sigma_{W,\sigma}, \Sigma^{-1}_{W,\sigma}]) = 0,$$

so the condition on dimension follows. This concludes the proof of (3). □

Remark 7.2.5. Originally Billera defined the complex $\mathcal{R}/\mathcal{J}'[\Sigma, \partial\Sigma]$ with uniform smoothness r using the ideals $J'(\gamma) = I(\gamma)^{r+1}$ [9]. The proof of Theorem 7.2.4 shows precisely where using these ideals leads to associated primes of higher dimension: namely, the map

$$\bigoplus_{\substack{\psi \in \Sigma^{\geq 0}_{\dim(W)} \\ \gamma \in \psi}} J'(\psi) \to J'(\gamma)$$

is not necessarily surjective, so while (1) would hold for this complex, (2) and (3) would not. The price for using the ideals $J'(\gamma)$, which are simpler to understand, is more complicated homology modules.

In Example 7.6.5 we will see how to couple Theorem 7.2.4 with Lemma 7.2.3 to obtain some interesting relationships between associated primes and the existence of 'unexpected' splines of certain degrees.

We can be even more precise about associated primes $I(W)$ of $H_i(\mathcal{R}/\mathcal{J}[\Sigma])$ with $\dim(W) = i - 1$. This is a slight generalization of [48, Theorem 2.6].

Theorem 7.2.6. *Let $\Sigma \subset \mathbb{R}^{n+1}$ be a pure, hereditary, $(n+1)$-dimensional fan with smoothness parameters α. Let $W \in L_{\Sigma,\Sigma^{-1}}$ be a flat of dimension $d-1$, where $2 \leq d \leq n+1$. Then $I(W)$ is associated to $H_d(\mathcal{R}/\mathcal{J}[\Sigma_{W,\sigma}, \Sigma^{-1}_{W,\sigma}])$ iff one of the following equivalent conditions hold.*

1. $H_d(\mathcal{R}[\Sigma_{W,\sigma}, \Sigma^{-1}_{W,\sigma}]) \neq 0$
2. $H_d(\mathcal{R}[\Sigma_{W,\sigma}, \Sigma^{-1}_{W,\sigma}]) = S$
3. *$G_W(\Sigma_{W,\sigma}, \Sigma^{-1}_{W,\sigma})$ has no v_b vertex and $\Sigma_{W,\sigma}$ is not the star of a face.*

Moreover, we have

$$H_d(\mathcal{R}/\mathcal{J}[\Sigma_W, \Sigma^{-1}_W]) = \bigoplus_{\substack{\sigma \in \Upsilon_W \\ H_d(\mathcal{R}[\Sigma_{W,\sigma}, \Sigma^{-1}_{W,\sigma}]) \neq 0}} \left(\frac{S}{\sum_{\tau \in (\Sigma^{\geq 0}_{W,\sigma})_n} J(\tau)} \right),$$

where Υ_W runs across a set of representatives for the equivalence classes $[\sigma]_W$.

Proof. The equivalence of the three conditions is a consequence of Proposition 7.1.14. Assuming any one of these, the long exact sequence coming from

$$0 \to \mathcal{J}[\Sigma_{W,\sigma}, \Sigma^{-1}_{W,\sigma}] \to \mathcal{R}[\Sigma_{W,\sigma}, \Sigma^{-1}_{W,\sigma}] \to \mathcal{R}/\mathcal{J}[\Sigma_{W,\sigma}, \Sigma^{-1}_{W,\sigma}] \to 0$$

yields that

$$H_d(\mathcal{R}/\mathcal{J}[\Sigma_{W,\sigma},\Sigma^{-1}_{W,\sigma}]) = \frac{S}{\sum_{\tau\in(\Sigma^{\geq 0}_{W,\sigma})_n} J(\tau)}.$$

Now the result follows from

$$H_d(\mathcal{R}/\mathcal{J}[\Sigma_W,\Sigma^{-1}_W]) = \bigoplus_{\sigma\in\Upsilon_W} H_d(\mathcal{R}/\mathcal{J}[\Sigma_{W,\sigma},\Sigma^{-1}_{W,\sigma}]).$$

□

Corollary 7.2.7. *Let $\Sigma\subset\mathbb{R}^{n+1}$ be a pure, hereditary, $(n+1)$-dimensional fan with smoothness parameters α. Then for $2\leq i\leq n$, $\dim H_i(\mathcal{R}/\mathcal{J}[\Sigma,\Sigma^{-1}])\leq i-1$ with equality iff $H_i(\mathcal{R}[\Sigma_{W,\sigma},\Sigma^{-1}_{W,\sigma}])\neq 0$ for some $i-1$ dimensional flat $W\in L_{\Sigma,\Sigma^{-1}}$ and some $\sigma\in\Sigma_{n+1}$.*

Proof. The statement $\dim H_i(\mathcal{R}/\mathcal{J}[\Sigma,\Sigma^{-1}])\leq i-1$ is a consequence of Theorem 7.2.4 part (1). The backward implication for equality is Theorem 7.2.4 part (2), while the forward implication is provided by Theorem 7.2.6. □

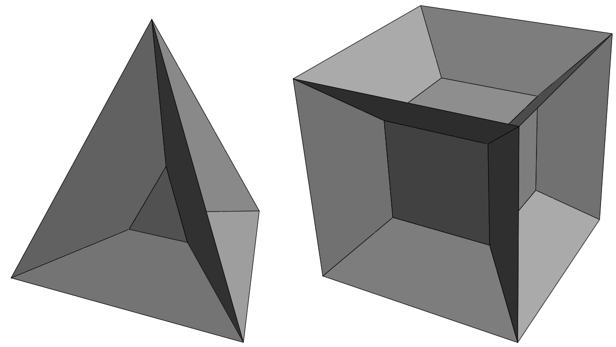
7.3 Examples

From Corollary 7.2.7 we see that if $\dim H_i(\mathcal{R}/\mathcal{J}[\Sigma,\Sigma^{-1}]) = i-1$ then there is some nontrivial topology of a lattice fan $\Sigma_{W,\sigma}$ relative to $\Sigma^{-1}_{W,\sigma}$ for a flat $W\in L_{\Sigma,\Sigma^{-1}}$ with $\dim W = i-1$. This behavior is far from generic, but it is not so difficult to construct examples manifesting such nontrivial topology. In the following example we provide two fans which illustrate such nongeneric behavior.

Example 7.3.1. The two polytopal complexes $\mathcal{P}_1$, $\mathcal{P}_2$ in Figure 7.6 (shown without boundary faces to clarify the inner structure) are both formed by placing a polytope inside of a scaled version of itself and connecting vertices as shown. In Figure 7.6a, we start with a tetrahedron which is the convex hull of $(0,0,8),(-4,-6,-3)$, $(-4,6,-3),(6,0,-3)$, then scale it up by a factor of 4 and place the smaller one inside. In Figure 7.6b we do the same procedure starting with the cube with vertices $(\pm1,\pm1,\pm1)$. Let $\Sigma_1=\widehat{\mathcal{P}}_1,\Sigma_2=\widehat{\mathcal{P}}_2$. Take $S=\mathbb{R}[w,x,y,z]$, where w is the cone variable. If we do not impose any vanishing along the boundaries of $\mathcal{P}_1,\mathcal{P}_2$, computations in Macaulay2 [30] yield the following information about associated primes. By dimension -1 we mean the module vanishes.

The only information in Table 7.1 that we cannot deduce from Theorem 7.2.6 is the fact that $H_3(\mathcal{R}/\mathcal{J}[\Sigma_1,\partial\Sigma_1])=0$. If we impose vanishing along all 6 codimension one boundary faces of Σ_2, then we obtain three additional codimension two associated primes of $H_3(\mathcal{R}/\mathcal{J}[\Sigma_2])$.

The associated primes $(x,w),(y,w),(z,w)$ correspond to intersections at infinity of the affine spans of the four codimension one faces parallel to the yz,



(a) $\mathcal{P}_1$ (b) $\mathcal{P}_2$

Figure 7.6

Module	Dimension	Minimal Associated Primes
$H_3(\mathcal{R}/\mathcal{J}[\Sigma_1, \partial\Sigma_1])$	-1	None
$H_2(\mathcal{R}/\mathcal{J}[\Sigma_1, \partial\Sigma_1])$	1	(x, y, z)
$H_3(\mathcal{R}/\mathcal{J}[\Sigma_2, \partial\Sigma_2])$	2	$(x, y), (y, z), (x, z)$
$H_2(\mathcal{R}/\mathcal{J}[\Sigma_2, \partial\Sigma_2])$	1	(x, y, z)

Table 7.1

Module	Dimension	Minimal Associated Primes
$H_3(\mathcal{R}/\mathcal{J}[\Sigma_2])$	2	$(x, y), (y, z), (x, z), (x, w), (y, w), (z, w)$

xz, and xy planes, respectively. Imposing vanishing on only three of four parallel affine spans will result in losing the corresponding associated prime. This is easily seen using the graph $G_W((\Sigma_2)_{W,\sigma}, (\Sigma_2^{-1})_{W,\sigma})$, where W is the line at infinity along which these affine spans intersect.

It is much more difficult to describe associated primes which do not arise from mere topological considerations. The following example, which we will continue in Section 7.6, is one such.

Example 7.3.2. Consider the fan $\Sigma = \widehat{\Delta}$, where Δ is the simplicial complex formed by placing an inverted tetrahedron symmetrically within a larger tetrahedron and connecting vertices as in Figure 7.7. The chosen coordinates for the inner tetrahedron in Figure 7.7 are $(0, 0, 8), (-4, -6, -3), (-4, 6, -3), (6, 0, -3)$ for the vertices labelled $0, 1, 2, 3$, respectively. The vertices of the outer tetrahedron are obtained by multiplying the coordinates of the inner tetrahedron by -5. In this simplicial complex there are 15 tetrahedra (listed by their vertices): 1234, 1678, 2578, 3568, 4567, 1278, 1368, 1467, 2358, 2457, 3456, 1238, 1346, 1247, 2345.

The important geometric consideration here is that the lines between vertices 0 and 4, 1 and 5, 2 and 6, 3 and 7 all intersect at the origin. This is the three dimensional analogue of an example due to Morgan-Scott [41] considered by Schenck [47, Example 5.3].

Let us consider the algebra $C^1(\Sigma)$ - recall this means that we assign smooth-

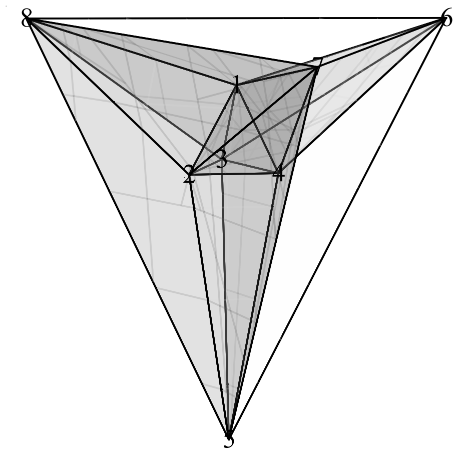


Figure 7.7: Three dimensional Morgan-Scott analogue

ness parameters $\alpha(\tau) = 1$ to every interior codimension one face and impose no vanishing conditions along the boundary, so $\Sigma^{-1} = \partial\Sigma = \widehat{\partial\Delta}$ and $C^1(\Sigma) = H_4(\mathcal{R}/\mathcal{J}[\Sigma, \partial\Sigma])$. Schenck computes that $H_2(\mathcal{R}/\mathcal{J}[\Sigma, \partial\Sigma]) = 0$, a computation readily verified in Macaulay2 (in his paper the homological degree is shifted down one from ours). He also finds that $H_3(\mathcal{R}/\mathcal{J}[\Sigma, \partial\Sigma])$ has associated primes in codimensions three and four. The associated prime of codimension four is the homogeneous maximal ideal of $S = \mathbb{R}[x_0, x_1, x_2, x_3]$. By Theorem 7.2.4 part (3), the associated primes of codimension three have the form $I(v)$, where v is a ray of Σ, corresponding to a vertex in Δ. Indeed, computations in Macaulay2 indicate that there are 8 associated primes of codimension 3 and these are precisely the homogeneous ideals of the vertices of Δ. We will return to this example in Section 7.6 to understand how these associated primes contribute to the fourth coefficient of the Hilbert polynomial of $C^1(\Sigma)$.

Remark 7.3.3. In general it is quite difficult to analyze $H_3(\mathcal{R}/\mathcal{J}[\Sigma, \Sigma^{-1}])$ for a fan $\Sigma \subset \mathbb{R}^4$. We will see in Section 7.6 that if Σ is simplicial then we can deduce the dimension of this module in large degrees if we are able to compute $H_2(\mathcal{R}/\mathcal{J}[\Sigma, \Sigma^{-1}])$ for simplicial $\Sigma \subset \mathbb{R}^3$. This latter module, while simpler than $H_3(\mathcal{R}/\mathcal{J}[\Sigma, \Sigma^{-1}])$, is still largely not understood.

The three dimensional analogue of the Morgan-Scott configuration in Example 7.3.2 gives rise to interesting associated primes in the case of uniform smoothness $r = 1$. One way to mimic a Morgan and Scott example with polytopal complexes is to start with a polytope $\mathcal{P}$ and fit it symmetrically within the polar polytope $\mathcal{P}^o$, and connect up vertices belonging to dual faces.

Example 7.3.4. Let $\mathcal{P}$ be the pure 3-dimensional polytopal complex constructed by starting with an octahedron having vertices $(\pm1, 0, 0), (0, \pm1, 0)$ $, (0, 0, \pm1)$. Fit this inside a cube with vertices $(\pm2, \pm2, \pm2)$. Then let the facets be the inner octahedron, the convex hull of each edge of the octahedron

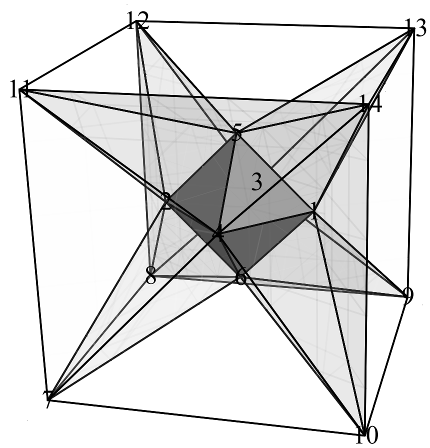


Figure 7.8: $\mathcal{P}$

with its dual edge on the cube, and the convex hull of each vertex of the octahedron with its dual cube face, yielding $f_3(\mathcal{P}) = 27$. Labelling each facet by its vertices, the 20 tetrahedra are: (1,3,9,13), (1,4,10,14), (2,4,7,11), (2,3,8,12), (1,5,13,14), (4,5,11,14), (2,5,11,12),(3,5,12,13), (1,6,9,10), (4,6,7,10), (2,6,7,8), (3,6,8,9), (1,4,5,14), (2,4,5,11), (2,3,5,12), (1,3,5,13), (1,4,6,10), (2,4,6,7), (2,3,6,8), (1,3,6,9). There are also 6 pyramids with square bases: (1,9,10,13,14), (2,7,8,11,12), (3,8,9,12,13), (4,7,10,11,14), (5, 11,12,13,14), (6,7,8,9,10)

Let $\Sigma = \widehat{\mathcal{P}}$, and consider uniform smoothness with $r = 1$. Set $S = \mathbb{R}[w, x, y, z]$, where w is the cone variable. Computations in Macaulay2 yield the results of Table 7.2.

Module	Dimension	Minimal Associated Primes
$H_3(\mathcal{R}/\mathcal{J}[\Sigma, \partial\Sigma])$	1	Ideals of vertices, $(x, y, w), (x, z, w), (y, z, w)$
$H_2(\mathcal{R}/\mathcal{J}[\Sigma, \partial\Sigma])$	-1	None

Table 7.2

Like the simplicial three dimensional analogue of the Morgan-Scott configuration, the ideals of the vertices are not something we can see arising in a topological manner. However, the three additional ideals are of interest and *are* in fact topological. Since they are all symmetric, consider the ideal (x, y, w). This is the ideal of the point $w \in L_{\Sigma,\Sigma^{-1}}$ at which the z-axis meets the hyperplane at infinity. The lattice fan $\Sigma_{w,\sigma}$ for any facet $\sigma \in \Sigma_4$ having a codimension one face τ with $w \in \operatorname{supp}(\tau)$, consists of 8 facets which surround the z-axis, namely the cones over the facets with labels (1,9,10,13,14), (1,4,10,14), (4,7,10,11,14), (2,4,7,11), (2,7,8,11,12), (2,3,8,12), (3,8,9,12,13), (1,3,9,13). Topologically, the pair $(\Sigma_{w,\sigma}, \partial\Sigma_{w,\sigma})$ is the cone over a torus $\mathbb{T}^2$ and its boundary. Via excision this yields $H_3(\mathcal{R}[\Sigma_{w,\sigma}, \partial\Sigma_{w,\sigma}]) \cong H_2(\mathbb{T}^2, \partial\mathbb{T}^2; S) = S$, hence via long exact sequences $H_3(\mathcal{R}/\mathcal{J}[\Sigma_{w,\sigma}, \partial\Sigma_{w,\sigma}]) \cong \dfrac{S}{\sum_{\tau\in\Sigma^0_{w,\sigma}} J(\tau)}$. This gives us the associated prime (x, y, w). The others follow analogously.

Remark 7.3.5. The facets of the lattice fan $\Sigma_{w,\sigma}$ form an equivalence class $[\sigma]_w$ which is not present in the equivalence relation defined by Yuzvinsky [63, § 2]. The reason for this is that the flat $w \in L_{\Sigma,\partial\Sigma}$ cannot be obtained by intersecting affine spans of codimension one faces of any single facet $\sigma \in \Sigma$.

7.4 Hilbert Polynomials

In this section we prove Proposition 7.4.1, which is the primary tool for translating our observations on associated primes into computations of Hilbert polynomials. The following two sections address computations of the third and fourth coefficients of the Hilbert polynomial of $C^\alpha(\Sigma)$. We begin by summarizing the commutative algebra which we will need.

If M is any $S = \mathbb{R}[x_0, \ldots, x_n]$-module, a *finite free resolution* of M of length r is an exact sequence of free S-modules

$$F_\bullet : \ 0 \to F_r \xrightarrow{\phi_r} F_{r-1} \xrightarrow{\phi_{r-1}} \cdots \xrightarrow{\phi_1} F_0$$

such that $\operatorname{coker} \phi_1 = M$. The Hilbert syzygy theorem guarantees that M has a finite free resolution. The *projective dimension* of M, denoted $\operatorname{pd}(M)$, is the minimum length of a finite free resolution. If M is a graded S-module with $\operatorname{pd}(M) = \delta$ then M has a minimal free resolution $F_\bullet \to M$ of length δ, unique up to graded isomorphism. This resolution is characterized by the property that the entries of any matrix representing the differentials $\phi.$ in $F_\bullet$ are contained in the homogeneous maximal ideal $(x_0, \ldots, x_n)$.

Recall that if M is a finitely generated nonnegatively graded S-module, we may write $M = \bigoplus_{i \geq 0} M_i$, where each M_i is an $\mathbb{R}$-vector space. The *Hilbert function* of M in degree d is $HF(M, d) = \dim M_d$. For $d \gg 0$ this agrees with a polynomial called the *Hilbert polynomial* of M, denoted $HP(M, d)$. If $HP(M, d) = 0$, then $M_d = 0$ for $d \gg 0$. Such modules are said to have finite length. If M is a module of finite length, then its *socle degree* is the largest degree k so that $M_k \neq 0$.

The standard use of the complex $\mathcal{R}/\mathcal{J}[\Sigma, \Sigma^{-1}]$ is to compute the dimensions of the vector spaces $C^\alpha(\Sigma)$ via an Euler characteristic computation, which we can state in terms of Hilbert functions as

$$\sum_{i=0}^{n+1} (-1)^i HF(\mathcal{R}/\mathcal{J}[\Sigma, \Sigma^{-1}]_{n+1-i}, d) = \sum_{i=0}^{n+1} (-1)^i HF(H_{n+1-i}(\mathcal{R}/\mathcal{J}[\Sigma, \Sigma^{-1}]), d).$$

Set

$$\begin{aligned} \chi(\mathcal{R}/\mathcal{J}[\Sigma, \Sigma^{-1}], d) &= \textstyle\sum_{i=0}^{n+1} (-1)^i HF(\mathcal{R}/\mathcal{J}[\Sigma, \Sigma^{-1}]_{n+1-i}, d) \\ &= \sum_{i=0}^{n+1} (-1)^i \left(\sum_{\gamma \in \Sigma^{\geq 0}_{n+1-i}} HF\left(\frac{S}{J(\gamma)}, d \right) \right). \end{aligned}$$

Recall from Lemma 3.3.15 that $H_{n+1}(\mathcal{R}/\mathcal{J}[\Sigma,\Sigma^{-1}]) = C^\alpha(\Sigma)$. This yields

$$
\begin{aligned}
HF(C^\alpha(\Sigma),d) = \chi(\mathcal{R}/\mathcal{J}[\Sigma,\Sigma^{-1}],d)& \\
-\sum_{i=1}^{n+1}(-1)^i HF(H_{n+1-i}(\mathcal{R}/\mathcal{J}[\Sigma,\Sigma^{-1}]),d).& \quad (7.1)
\end{aligned}
$$

Determining $HF(C^\alpha(\Sigma),d)$ from Equation 7.1 requires two tasks, both of which are unsolved in general. The first task is to determine the dimensions of the vector spaces $\left(\frac{S}{J(\gamma)}\right)_d$, which are quotients of the polynomial ring by an ideal generated by powers of linear forms. This is itself a rich field of research with connections to Waring's problem and fat point ideals [13, 27]. In [56], Shan exploits these connections (particularly an algorithm due to Geramita-Harbourne-Migliore [26] for computing Hilbert functions of certain fat point ideals) to obtain bounds on $\dim C^2(\Sigma)_d$ for $\Sigma \subset \mathbb{R}^3$.

The second task is to compute $\dim H_i(\mathcal{R}/\mathcal{J}[\Sigma,\Sigma^{-1}])_d$. There are few tools for dealing with these homology modules. Mourrain and Villamizar give bounds on the dimension of these homology modules when $\Sigma = \widehat{\Delta}$, the cone over a simplicial complex Δ, when $\Delta \subset \mathbb{R}^2$ and $\Delta \subset \mathbb{R}^3$ [42, 43]. Armed with these bounds and the current knowledge of fat point ideals, they obtain bounds on the dimension of the spline space $C^r(\widehat{\Delta})_d$ using Equation (7.1).

A slightly different approach, taken primarily by Schenck with various co-authors, is to compute the Hilbert polynomial of $C^\alpha(\Sigma)$ using Equation (7.1) [28, 38, 48]. This approach, which we also take, ignores information that 'eventually vanishes.' Our first step is to pull out the leading term of the Hilbert polynomial of the homology module $H_i(\mathcal{R}/\mathcal{J}[\Sigma,\Sigma^{-1}])$. This should be seen as a generalization of [38, Theorem 3.10] and [48, Corollary 2.7]. An important difference is that the aforementioned results only apply when $\dim H_i(\Sigma,\Sigma^{-1}) = i-1$, the maximal possible dimension. In particular, these formulas do not apply to simplicial complexes, where $\dim H_i(\Sigma,\Sigma^{-1}) < i-1$ by Theorem 7.2.4.

Recall that $H_i(\mathcal{R}/\mathcal{J}[\Sigma_W,\Sigma_W^{-1}]) = \bigoplus_{\sigma\in\Upsilon_W} H_i(\mathcal{R}/\mathcal{J}[\Sigma_{W,\sigma},\Sigma_{W,\sigma}^{-1}])$, where Υ_W is a set of representatives for the equivalence class of facets modulo the equivalence relation $\sim_W$ of Definition 7.1.2.

Proposition 7.4.1. *Let $\Sigma \subset \mathbb{R}^{n+1}$ be a pure,hereditary, $(n+1)$-dimensional fan with smoothness parameters α and set $k = \dim H_i(\mathcal{R}/\mathcal{J}[\Sigma,\Sigma^{-1}])$. Then*

$$
HP(H_i(\mathcal{R}/\mathcal{J}[\Sigma,\Sigma^{-1}]),d) = \sum_{\substack{W\in L_{\Sigma,\Sigma^{-1}},\\ \dim(W)=k}} HP(H_i(\mathcal{R}/\mathcal{J}[\Sigma_W,\Sigma_W^{-1}]),d) + O(d^{k-2}).
$$

If $k=1$, $O(d^{k-2})$ is understood to be 0.

Proof. For any $W \in L_{\Sigma,\Sigma^{-1}}$ there is a map of complexes

$$\mathcal{R}/\mathcal{J}[\Sigma,\Sigma^{-1}] \xrightarrow{q_W} \mathcal{R}/\mathcal{J}[\Sigma_W,\Sigma_W^{-1}],$$

The right hand side is the quotient of $\mathcal{R}/\mathcal{J}[\Sigma,\Sigma^{-1}]$ by the sub-chain complex $\mathcal{R}/\mathcal{J}[\Sigma_W^c,\Sigma_W^c \cup \Sigma^{-1}]$, where Σ_W^c is the subfan of faces whose affine span does not contain W. This descends to a map $\bar{q}_{W,i}$ in homology,

$$H_i(\mathcal{R}/\mathcal{J}[\Sigma,\Sigma^{-1}]) \xrightarrow{\bar{q}_{W,i}} H_i(\mathcal{R}/\mathcal{J}[\Sigma_W,\Sigma_W^{-1}]).$$

Summing over all $W \in L_{\Sigma,\Sigma^{-1}}$ with $\dim W = k$ and setting $\bar{q}_i = \sum_W \bar{q}_{W,i}$ we obtain:

$$H_i(\mathcal{R}/\mathcal{J}[\Sigma,\Sigma^{-1}]) \xrightarrow{\bar{q}_i} \bigoplus_W H_i(\mathcal{R}/\mathcal{J}[\Sigma_W,\Sigma_W^{-1}]). \tag{7.2}$$

If M is a graded $S = \mathbb{R}[x_0,\dots,x_n]$ module with $\dim M = k$, then $HP(M,d)$ has degree $k-1$. By the additivity of the Hilbert polynomial across exact sequences, we will be done if we can show that the kernel and cokernel of $\bar{q}_i$ both have dimension $\leq k-1$. This in turn will follow if we show

(A) The target of $\bar{q}_i$ in (7.2) has dimension k

(B) $\bar{q}_i$ becomes an isomorphism under localization at primes of codimension exactly $n+1-k$

We refer to the source of $\bar{q}_i$ in (7.2) as LHS and the target of $\bar{q}_i$ as RHS. Suppose that P is an associated prime of RHS. Then by Proposition 2.2.2 part (2), P is an associated prime of $H_i(\mathcal{R}/\mathcal{J}[\Sigma_{W,\sigma},\Sigma_{W,\sigma}^{-1}])$ for some W,σ, with $\dim W = k$.

Now set $\mathit{\Gamma} = \Sigma_{W,\sigma}$ and $\mathit{\Gamma}^{-1} = \Sigma_{W,\sigma}^{-1}$. Then $L_{\mathit{\Gamma},\mathit{\Gamma}^{-1}}$ is the sublattice of $L_{\Sigma,\Sigma^{-1}}$ consisting of the flats

$$\{V = \operatorname{aff}(\gamma) \in L_{\Sigma,\Sigma^{-1}} | W \in \operatorname{supp}(\gamma), \gamma \in \sigma' \text{ for some } \sigma' \sim_W \sigma\}.$$

Furthermore, for any $V \in L_{\mathit{\Gamma},\mathit{\Gamma}^{-1}}$ and $\sigma \in \mathit{\Gamma}_{n+1}$, $\mathit{\Gamma}_{V,\sigma} = \Sigma_{V,\sigma}$. By Theorem 7.2.4, $P = I(V)$ for some $V \in L_{\mathit{\Gamma},\mathit{\Gamma}^{-1}}$. Lemma 7.1.10 yields

$$\begin{aligned}(H_i(\mathcal{R}/\mathcal{J}[\mathit{\Gamma},\mathit{\Gamma}^{-1}]))_{I(V)} &= \bigoplus_{\sigma\in\Upsilon'_V} (H_i(\mathcal{R}/\mathcal{J}[\mathit{\Gamma}_{V,\sigma},\mathit{\Gamma}_{V,\sigma}^{-1}]))_{I(V)} \\ &= \bigoplus_{\sigma\in\Upsilon'_V} (H_i(\mathcal{R}/\mathcal{J}[\Sigma_{V,\sigma},\Sigma_{V,\sigma}^{-1}]))_{I(V)},\end{aligned}$$

where Υ'_V runs across representatives of the equivalence classes $[\sigma]_V$ for $\sigma \in \mathit{\Gamma}_{n+1}$. The final direct sum above appears as a summand of $H_i(\mathcal{R}/\mathcal{J}[\Sigma,\Sigma^{-1}])_{I(V)}$, according to Lemma 7.1.10. It follows from Proposition 2.2.2 parts (1) and (2) that $I(V)$ is an associated prime of $H_i(\mathcal{R}/\mathcal{J}[\Sigma,\Sigma^{-1}])$. Making use of the formula

$$\dim M = \max\{\dim R/P | P \in \operatorname{Ass}(M)\},$$

we find that $k = \dim H_i(\mathcal{R}/\mathcal{J}[\Sigma,\Sigma^{-1}]) \geq \dim V \geq \dim W = k \implies V = W$ and $P = I(W)$. By dimension considerations, $I(W)$ must be a minimal associated prime of $H_i(\mathcal{R}/\mathcal{J}[\Sigma,\Sigma^{-1}])$.

Thus the associated primes of RHS are precisely the minimal associated primes of LHS, and these are contained in the set of primes

$$\{I(W)|W \in L_{\Sigma,\Sigma^{-1}}, \dim(W) = k\}.$$

It follows immediately that dim RHS= k, proving (A). To prove (B) we need only show that $\bar{q}_i$ becomes an isomorphism under localization at primes of the form $I(V)$, $\dim V = k$. By Lemma 7.1.10,

$$H_i(\mathcal{R}/\mathcal{J}[\Sigma,\Sigma^{-1}])_{I(V)} = H_i(\mathcal{R}/\mathcal{J}[\Sigma_V,\Sigma_V^{-1}])_{I(V)}$$

The summands of RHS in (2) have the form $H_i(\mathcal{R}/\mathcal{J}[\Sigma_W,\Sigma_W^{-1}])$, where $\dim W = k$. As we have seen, each of these summands either has dimension less than k or has the unique minimal associated prime $I(W)$. It follows that a summand of RHS vanishes under localization at $I(V)$ unless it is precisely the summand $H_i(\mathcal{R}/\mathcal{J}[\Sigma_V,\Sigma_V^{-1}])$. This completes the proof of (2). □

7.5 Third Coefficient of Hilbert Polynomial

In this section we apply Proposition 7.4.1 and Theorem 7.2.4 to yield a formula for the third coefficient of the Hilbert polynomial $HP(C^\alpha(\Sigma), d)$ for any assignment of smoothness parameters α. Our approach synthesizes computations from two papers: Geramita and Schenck's computation for planar simplicial complexes with mixed smoothness in [28] and McDonald and Schenck's computation of the third coefficient of $HP(C^r(\widehat{\mathcal{P}}), d)$ for arbitrary polytopal complexes and uniform smoothness in [38]. Our main contributions to this story are twofold: we allow arbitrary vanishing conditions to be imposed along codimension one boundary faces, and we connect the third coefficient (in the polytopal case) to the topology of the lattice fans $\Sigma_{W,\sigma}$.

Looking back to Equation 7.1, we see that by dimension considerations there are 4 terms that will contribute to the first three coefficients of $HP(C^\alpha(\Sigma))$, for $\Sigma \subset \mathbb{R}^{n+1}$ a pure $(n+1)$-dimensional hereditary polyhedral fan. These are recorded in Table 7.3.

The Hilbert polynomials of the first two entries on the table are simple to derive. The first is well known and the second follows from the fact that $J(\tau) = \langle l_\tau^{\alpha(\tau)+1}\rangle$ and the one-step resolution

$$S(-\alpha(\tau)-1) \to S$$

for $J(\tau)$. The question marks in the table are resolved by understanding Hilbert functions of ideals of powers of linear forms in two variables, which is the heart

Dimension	Module	Hilbert Polynomial
$n+1$	$S^{f_{n+1}(\Sigma)}$	$f_{n+1}(\Sigma)\binom{d+n}{n}$
n	$\bigoplus_{\tau\in\Sigma_n^{\geq 0}} \dfrac{S}{J(\tau)}$	$\sum_{\tau\in\Sigma_n^{\geq 0}} \binom{d+n}{n} - \binom{d+n-\alpha(\tau)-1}{n}$
$n-1$	$\bigoplus_{\gamma\in\Sigma_{n-1}^{\geq 0}} \dfrac{S}{J(\gamma)}$	?
$n-1$	$H_n(\mathcal{R}/\mathcal{J}[\Sigma,\Sigma^{-1}])$	?

Table 7.3

of the paper by Geramita and Schenck [28]. We summarize this result, which is obtained using inverse systems to translate the problem into calculating dimensions of ideals of *fat points* in $\mathbb{P}^1$.

Theorem 7.5.1. *[28, Theorem 2.7] Suppose $\alpha_1,\ldots,\alpha_\mu$ are positive integers, $L_1,\ldots,L_\mu \in S=\mathbb{R}[x,y]$ are linear forms (not all multiples of the same linear form) and let J be the (x,y)-primary ideal minimally generated by $(L_1^{\alpha_1},\ldots,L_\mu^{\alpha_\mu})$. Let*

$$\Omega = \left\lfloor \frac{\sum_{i=1}^{\mu}\alpha_i - \mu}{\mu-1} \right\rfloor + 1.$$

Then $\Omega-1$ is the socle degree of S/J and the graded minimal free resolution of J has the form

$$0 \to S(-\Omega-1)^a \oplus S(-\Omega)^{t-1-a} \to \oplus_{i=1}^{\mu} S(-\alpha_i) \to J \to 0,$$

where $a=\sum_{i=1}^{\mu}\alpha_i + (1-\mu)\cdot\Omega$.

Corollary 7.5.2. *Suppose $L_1,\ldots,L_\mu \in S=\mathbb{R}[x_0,\ldots,x_n]$ are linear forms which vanish on a common codimension 2 linear subspace W. Let $\alpha_1,\ldots,\alpha_\mu$ and Ω be as defined in Theorem 7.5.1, so that $L_1^{\alpha_1},\ldots,L_\mu^{\alpha_\mu}$ are minimal generators for the ideal the generate. Then J has minimal free resolution*

$$0 \to S(-\Omega-1)^a \oplus S(-\Omega)^{\mu-1-a} \to \oplus_{i=1}^{t} S(-\alpha_i) \to J \to 0,$$

where $a=\sum_{i=1}^{\mu}\alpha_i + (1-\mu)\cdot\Omega$.

Proof. Choose linear forms $l_1,\ldots,l_{n-1}$ which do not vanish on W. These form a regular sequence on S/J. Cutting down by these to the case of Theorem 7.5.1 yields the result. The exact statement we need is the following: let $f\in S_1$ be a linear form which is a nonzerodivisor on S/J. Then $F_\bullet \to S/J$ is exact if and only if $F_\bullet/fF_\bullet \to (S/J+(f))$ is exact. This is an easy consequence of the long exact sequence in homology induced by the short exact sequence $0\to S(-1)/J \xrightarrow{\cdot f} S/J \to S/(J+(f)) \to 0$. Now induct. □

Both question marks in Table 7.3 are resolved by applying Corollary 7.5.2. We mainly need some notation to state the results.

Set $\alpha'(\tau) = \alpha(\tau) + 1$. For each $W \in L_{\Sigma,\Sigma^{-1}}$, let $\mu(W,\sigma)$ be the number of minimal generators of the ideal $\langle l_\tau^{\alpha'(\tau)} | \tau \in \left(\Sigma^{\geq 0}_{W,\sigma}\right)_n\rangle$ and $\beta(W,\sigma) = (\alpha'(\tau_1), \ldots, \alpha'(\tau_{\mu(W,\sigma)}))$ the exponent vector for a set of minimal generators. Set

$$\Omega(W,\sigma) = \left\lfloor \frac{\sum_{i=1}^{\mu(W,\sigma)} \alpha'(\tau_i) - \mu(W,\sigma)}{\mu(W,\sigma) - 1} \right\rfloor + 1.$$

Also let $a(W,\sigma) = \sum_{\alpha'(\tau)\in\beta(W,\sigma)} \alpha'(\tau) + (1 - \mu(W,\sigma)) \cdot \Omega(W,\sigma)$ and $b(W,\sigma) = \mu(W,\sigma) - 1 - a(W,\sigma)$. If $\Sigma_{W,\sigma} = \mathrm{st}(\gamma)$, then replace $\mu(W,\sigma), \beta(W,\sigma), \Omega(W,\sigma), a(W,\sigma), b(W,\sigma)$ by $\mu(\gamma), \beta(\gamma), \Omega(\gamma), a(\gamma), b(\gamma)$, respectively. Now we can finish off Table 7.3.

Corollary 7.5.3. *Let $\Sigma \subset \mathbb{R}^{n+1}$ be a pure $(n+1)$-dimensional fan with smoothness parameters α. Using the notation above, Table 7.3 may be completed as in Table 7.4.*

Module	Hilbert Polynomial
$S^{f_{n+1}(\Sigma)}$	$f_{n+1}(\Sigma)\binom{d+n}{n}$
$\bigoplus_{\tau\in\Sigma^{\geq 0}_n} \frac{S}{J(\tau)}$	$\sum_{\tau\in\Sigma^{\geq 0}_n} \binom{d+n}{n} - \binom{d+n-\alpha(\tau)-1}{n}$
$\bigoplus_{\gamma\in\Sigma^{\geq 0}_{n-1}} \frac{S}{J(\gamma)}$	$\sum_{\gamma\in\Sigma^{\geq 0}_{n-1}} \Big(\binom{d+n}{n} - \sum_{\alpha'(\tau)\in\beta(\gamma)} \binom{d+n-\alpha'(\tau)}{n} + a(\gamma)\binom{d+n-\Omega(\gamma)-1}{n} + b(\gamma)\binom{d+n-\Omega(\gamma)}{n} \Big)$
$H_n(\mathcal{R}/\mathcal{J}[\Sigma,\Sigma^{-1}])$	$\sum_{\substack{W\in L_{\Sigma,\Sigma^{-1}} \\ \dim W = n-1}} \sum_{\substack{\sigma\in\Upsilon_W \\ H_{n-1}(\mathcal{R}[\Sigma_{W,\sigma},\Sigma^{-1}_{W,\sigma}])\neq 0}} \Big(\binom{d+n}{n} - \sum_{\alpha'(\tau)\in\beta(W,\sigma)} \binom{d+n-\alpha'(\tau)}{n} + a(W,\sigma)\binom{d+n-\Omega(W,\sigma)-1}{n} + b(W,\sigma)\binom{d+n-\Omega(W,\sigma)}{n} \Big) + O(d^{n-3})$

Table 7.4

Proof. The third entry is a direct application of Corollary 7.5.2 to the quotients $S/J(\gamma)$. By Theorem 7.2.6 and Proposition 7.4.1,

$$HP(H_{n-1}(\mathcal{R}/\mathcal{J}), d) = \sum_{\substack{W\in L_{\Sigma,\Sigma^{-1}}, \dim W = n-1, \sigma\in\Upsilon_W \\ H_{n-1}(\mathcal{R}[\Sigma_{W,\sigma},\Sigma^{-1}_{W,\sigma}])\neq 0}} HP\left(\frac{S}{\sum_{\tau\in(\Sigma^{\geq 0}_{W,\sigma})_n} J(\tau)}, d \right) + O(d^{n-3}),$$

where $\mathcal{R}/\mathcal{J}$ is short for $\mathcal{R}/\mathcal{J}[\Sigma,\Sigma^{-1}]$. Recognizing $\sum_{\tau\in(\Sigma^{\geq 0}_{W,\sigma})_n} J(\tau)$ as the ideal $\langle l_\tau^{\alpha'(\tau)} | \tau \in \left(\Sigma^{\geq 0}_{W,\sigma}\right)_n \rangle$, we are done. □

At this point we can extract the first three coefficients of $HP(C^\alpha(\Sigma), d)$ by taking the appropriate alternating sums of the expressions in Corollary 7.5.3. Instead of doing this in full generality, we restrict to the case where $\Sigma \subset \mathbb{R}^3$, in which case we recover the full Hilbert polynomial. The second and third entries in the table above give the constant term.

Corollary 7.5.4. *Let $\Sigma \subset \mathbb{R}^3$ be a pure, hereditary, 3-dimensional fan with smoothness parameters α. Then the Hilbert polynomial $HP(C^\alpha(\Sigma), d)$ is given by*

$$f_{n+1}(\Sigma)\binom{d+2}{2} - \left(\sum_{\tau\in\Sigma_2^{\geq 0}} \binom{d+2}{2} - \binom{d+2-\alpha(\tau)-1}{2}\right) + \sum_{\substack{W\in L_{\Sigma,\Sigma^{-1}} \\ \dim W=1}} c_W,$$

where

$$c_W = \sum_{\sigma\in\Upsilon_W} c_{W,\sigma},$$

and

$$c_{W,\sigma} = \begin{cases} 1 - \sum_{\alpha'(\tau)\in\beta(\gamma)} \binom{\alpha'(\tau)-1}{2} + a(\gamma)\binom{\Omega(\gamma)}{2} + b(\gamma)\binom{\Omega(\gamma)-1}{2} & \text{if } \Sigma_{W,\sigma} = st(\gamma), \gamma\in\Sigma_1^{\geq 0} \\ 1 - \sum_{\alpha'(\tau)\in\beta(W,\sigma)} \binom{\alpha'(\tau)-1}{2} + a(W,\sigma)\binom{\Omega(W,\sigma)}{2} + b(W,\sigma)\binom{\Omega(W,\sigma)-1}{2} & \text{if } H_1(\mathcal{R}[\Sigma_{W,\sigma},\Sigma^{-1}_{W,\sigma}]) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

Example 7.5.5. Let Σ be the cone over the Schlegel diagram of a cube (Example 3.2.7). Impose vanishing of order r along interior codimension one faces and vanishing of order s along boundary codimension one faces. If $s = -1$, i.e. no vanishing is imposed along codimension one boundary faces, then the only lattice fan with nontrivial H_1 is the lattice fan $\Sigma_{W,\sigma}$ where W is the z-axis, shown in Figure 7.2b. The ideal generated by the forms l_τ, $\tau\in\Sigma_1^{\geq 0}$, vanishing on W is a complete intersection with two generators in degree $r+1$. We have $\Omega(W,\sigma) = 2r+1$, $a(W,\sigma) = 1$, $b(W,\sigma) = 0$, hence $c_{W,\sigma} = 1 - 2\binom{r}{2} + \binom{2r+1}{2}$. The ideal of every interior ray γ of Σ is generated by three forms of degree $r+1$, so $\Omega(\gamma) = \lfloor 3r/2\rfloor + 1$, $a(\gamma) = 3r+3-2\lfloor(3r+2)/2\rfloor$, $b(\gamma) = 2\lfloor(3r+2)/2\rfloor - 3r - 1$. Using Corollary 7.5.4 and simplifying, we have

$$\begin{aligned} HP(C^\alpha(\Sigma), d) = \; & \tfrac{5}{2}d^2 + \left(-8r - \tfrac{1}{2}\right)d \\ & -4\left\lfloor \tfrac{3r}{2}\right\rfloor^2 + 12r\left\lfloor\tfrac{3r}{2}\right\rfloor - r^2 + 4r + 2 \end{aligned}$$

Now suppose $s \geq 0$, so vanishing of degree s is imposed along the boundary of Σ. In addition to the lattice fan around the z-axis, there are two others corresponding to the x and y-axes (see Figure 7.4a) for which $H_1(\mathcal{R}[\Sigma, \Sigma^{-1}])$ is nontrivial. The corresponding ideals are generated by two forms of degree $s+1$ and two forms of degree $r+1$. For each of these we have $\Omega = \lfloor 2(r+s)/3 \rfloor + 1$, $a = 2(r+s) + 1 - 3\lfloor 2(r+s)/3 \rfloor$, $b = 3\lfloor 2(r+s)/3 \rfloor - 2(r+s) + 2$. In addition to the interior rays, we also must incorporate the four boundary rays of Σ_1. The corresponding ideals of these rays are generated by two forms of degree $s+1$ and a form of degree $r+1$. For each of these we have $\Omega = \lfloor (2s+r)/2 \rfloor$, $a = 2s+r+1-2\lfloor (2s+r)/2 \rfloor$, $b = 1-r-2s+2\lfloor (2s+r)/2 \rfloor$. Using Corollary 7.5.4 and simplifying, we have

$$\begin{aligned} HP(C^\alpha(\Sigma), d) = \quad & \tfrac{5}{2}d^2 + \left(-8r - 4s - \tfrac{9}{2}\right) d \\ & -3\left\lfloor \tfrac{2(r+s)}{3} \right\rfloor^2 + 4r\left\lfloor \tfrac{2(r+s)}{3} \right\rfloor + 4s\left\lfloor \tfrac{2(r+s)}{3} \right\rfloor - \left\lfloor \tfrac{2(r+s)}{3} \right\rfloor \\ & -4\left\lfloor \tfrac{r}{2} \right\rfloor^2 - 4\left\lfloor \tfrac{3r}{2} \right\rfloor^2 + 4r\left\lfloor \tfrac{r}{2} \right\rfloor + 12r\left\lfloor \tfrac{3r}{2} \right\rfloor \\ & -5r^2 + 4rs + 8r + 4s + 4 \end{aligned}$$

This formula is correct when the number of generators of the ideals above are as indicated in the preceding paragraph. When one of r, s is small compared to the other, then the number of minimal generators of the above ideals may drop, which will change the formula. For instance, when $r = 3$ and $s = 0$, the above formula gives a constant term of 81, while the actual constant is 87. This is because the minimal number of generators of several of the ideals drops for these values.

Remark 7.5.6. In [42], Mourrain and Villamizar bound the dimension of the homology module $H_2(\mathcal{R}/\mathcal{J}[\Sigma, \partial\Sigma])$ in the case of uniform smoothness, where $\Sigma = \widehat{\Delta}$ for $\Delta \subset \mathbb{R}^2$ a simplicial complex. In the simplicial case this module has finite length and so vanishes in high degree. Their are no known formulas for $\dim H_2(\mathcal{R}/\mathcal{J}[\Sigma, \partial\Sigma])_d$ in small degrees d. We revisit this module in the next section.

Remark 7.5.7. In Chapter 6 we address the question of how large d must be in order for the formula in Corollary 7.5.4 to hold. We obtain a combinatorial bound for such d which is valid for any fan $\Sigma \subset \mathbb{R}^3$. This is the first such bound obtained for arbitrary polyhedral fans. In the simplicial case we can tighten this bound and, reducing to the case of uniform smoothness, recover the $3r + 2$ bound of Ibrahim and Schumaker [33]. This is the best known bound for arbitrary planar simplicial complexes, although Alfeld and Schumaker have reduced this bound to $3r + 1$ in the case of *generic* simplicial complexes [3].

7.6 Simplicial Fourth Coefficient and the Generic Dimension of C^1 Tetrahedral Splines

In this section we consider the computation of the Hilbert polynomial of $C^r(\Sigma)$, where $\Sigma = \widehat{\Delta}$ and $\Delta \subset \mathbb{R}^3$. We then revisit the computation by Alfeld, Schumaker, and Whiteley of $\dim C^1(\widehat{\Delta})_d$ for $d \geq 8$ [7].

As in the previous section, we start by describing the modules of relevant dimension for the computation of $HP(C^r(\Sigma), d)$, referring back to Equation 7.1. We leave out $H_2(\mathcal{R}/\mathcal{J}[\Sigma, \Sigma^{-1}])$ since by Theorem 7.2.4 it has finite length and will not contribute to the Hilbert polynomial.

Dimension	Module	Hilbert Polynomial
4	$S^{f_4(\Sigma)}$	$f_4(\Sigma)\binom{d+3}{3}$
3	$\bigoplus_{\tau \in \Sigma_3^{\geq 0}} \frac{S}{J(\tau)}$	$\sum_{\tau \in \Sigma_3^{\geq 0}} \binom{d+3}{3} - \binom{d+3-\alpha(\tau)-1}{3}$
2	$\bigoplus_{e \in \Sigma_2^{\geq 0}} \frac{S}{J(e)}$	see Table 7.4
1	$\bigoplus_{v \in \Sigma_1^{\geq 0}} \frac{S}{J(v)}$	?
1	$H_3(\mathcal{R}/\mathcal{J}[\Sigma, \Sigma^{-1}])$	?

Table 7.5

If we approach the first question mark appearing in Table 7.5 as we did ideals of codimension 2 faces in the previous section, we would transfer the problem to one of computing Hilbert functions of ideals of fat points in $\mathbb{P}^2$. We refer the reader to [43], where Mourrain and Villamizar bound the dimension of this piece using the Fröberg sequence in the case of uniform smoothness. Our main contribution to this story is to elucidate the term $H_3(\mathcal{R}/\mathcal{J}[\Sigma, \Sigma^{-1}])$, the second question mark in Table 7.5.

Proposition 7.6.1. *Let $\Delta \subset \mathbb{R}^3$ be a simplicial complex with smoothness parameters α,$\Sigma = \widehat{\Delta}$, and for $v \in \Delta_0$ let $\Delta_v \subset \mathbb{R}^3$ be the fan with smoothness parameters as in Lemma 7.2.3. Let $S = \mathbb{R}[x_0, x_1, x_2, x_3]$ and $R = \mathbb{R}[x_1, x_2, x_3]$. Then $HP(H_3(\mathcal{R}/\mathcal{J}[\Sigma, \Sigma^{-1}]), d)$ is the constant given by*

$$HP(H_3(\mathcal{R}/\mathcal{J}[\Sigma, \Sigma^{-1}]), d) = \sum_{\substack{v \in \Delta_0 \\ \widehat{v} \in L_{\Sigma, \Sigma^{-1}}}} \sum_{i \geq 0} \dim H_2(\mathcal{R}/\mathcal{J}[\Delta_v, \Delta_v^{-1}])_i.$$

Note that the sum $\sum_{i \geq 0} \dim H_2(\mathcal{R}/\mathcal{J}[\Delta_v, \Delta_v^{-1}])_i$ is finite since $H_2(\mathcal{R}/\mathcal{J}[\Delta_v, \Delta_v^{-1}])$ is a module of finite length by Theorem 7.2.4.

Proof. Applying Proposition 7.4.1 yields

$$HP(H_3(\mathcal{R}/\mathcal{J}[\Sigma, \Sigma^{-1}]), d) = \sum_{\substack{v \in L_{\Sigma,\Sigma^{-1}} \\ \dim(v)=1}} HP(H_3(\mathcal{R}/\mathcal{J}[\Sigma_v, \Sigma_v^{-1}]), d),$$

where $\Sigma_v = \bigsqcup_{\sigma \in \Upsilon_v} \Sigma_{v,\sigma}$. By Lemma 7.1.9, $\Sigma_{v,\sigma} = \mathrm{st}_\Sigma(\gamma)$ for some face $\gamma \in \Sigma$. If $\dim(\gamma) \geq 2$, then $\Sigma_{v,\sigma} = \Sigma_{W,\sigma}$, where $W = \mathrm{aff}(\gamma)$. Then $H_3(\mathcal{R}/\mathcal{J}[\Sigma_{v,\sigma}, \Sigma_{v,\sigma}^{-1}]) = H_3(\mathcal{R}/\mathcal{J}[\Sigma_{W,\sigma}, \Sigma_{W,\sigma}^{-1}) = 0$ by the same as in the proof of Theorem 7.2.4. So we have

$$HP(H_3(\mathcal{R}/\mathcal{J}[\Sigma, \Sigma^{-1}])) = \sum_{\substack{v \in \Delta_0 \\ \widehat{v} \in L_{\Sigma,\Sigma^{-1}}}} HP(H_3(\mathcal{R}/\mathcal{J}[\mathrm{st}_\Sigma(\widehat{v}), \mathrm{st}_\Sigma(\widehat{v})^{-1}]), d).$$

Since $\mathrm{st}_\Sigma(\widehat{v}) = \widehat{\mathrm{st}_\Delta(v)}$, we conclude by Lemma 7.2.3 that

$$\mathcal{R}/\mathcal{J}[\mathrm{st}_\Sigma(\widehat{v}), \mathrm{st}_\Sigma(\widehat{v})^{-1}] = \mathcal{R}/\mathcal{J}[\Delta_v, \Delta_v^{-1}](-1) \otimes_R S.$$

In particular,

$$H_3(\mathcal{R}/\mathcal{J}[\mathrm{st}_\Sigma(\widehat{v}), \mathrm{st}_\Sigma(\widehat{v})^{-1}]) = H_2(\mathcal{R}/\mathcal{J}[\Delta_v, \Delta_v^{-1}]) \otimes_R S.$$

Now the result follows, since, if M is a finitely generated graded R-module, there is an isomorphism between the vector spaces $\bigoplus_{i \leq d} M_i$ and $(M \otimes S)_d$. $\square$

Corollary 7.6.2. *Let $\Delta \subset \mathbb{R}^3$ be a pure, hereditary simplicial complex, with smoothness parameters α, and let $\Sigma = \widehat{\Delta}$. Set*

$$\begin{aligned} \chi(\mathcal{R}/\mathcal{J}[\Sigma, \Sigma^{-1}], d) &= \sum_{i=0}^{4} (-1)^i \left(\sum_{\gamma \in \Sigma_{4-i}^{\geq 0}} HF\left(\frac{S}{J(\gamma)}, d\right) \right) \\ \chi_H(\mathcal{R}/\mathcal{J}[\Sigma, \Sigma^{-1}], d) &= \sum_{i=0}^{3} (-1)^i \left(\sum_{\gamma \in \Sigma_{4-i}^{\geq 0}} HP\left(\frac{S}{J(\gamma)}, d\right) \right) \end{aligned}$$

Then

$$HP(C^\alpha(\Sigma), d) = \chi_H(\mathcal{R}/\mathcal{J}[\Sigma, \Sigma^{-1}], d) + C,$$

where

$$\begin{aligned} C &= \sum_{\substack{v \in \Delta_0 \\ \widehat{v} \in L_{\Sigma,\Sigma^{-1}}}} \textstyle\sum_{i \geq 0} \dim H_2(\mathcal{R}/\mathcal{J}[\Delta_v, \Delta_v^{-1}])_i \\ &= \sum_{\substack{v \in \Delta_0 \\ \widehat{v} \in L_{\Sigma,\Sigma^{-1}}}} \textstyle\sum_{i \geq 0} (\dim C^\alpha(\Delta_v)_i - \chi(\mathcal{R}/\mathcal{J}[\Delta_v, \Delta_v^{-1}], i)) \end{aligned}$$

Proof. Write out Equation 7.1 for Σ, applying Proposition 7.6.1 to the term

$H_2(\mathcal{R}/\mathcal{J}[\Sigma, \Sigma^{-1}])$. To get the final equality, apply Equation 7.1 to Δ_v, yielding

$$\dim H_2(\mathcal{R}/\mathcal{J}[\Delta_v, \Delta_v^{-1}])_i = \dim C^\alpha(\Delta_v)_i - \chi(\mathcal{R}/\mathcal{J}[\Delta_v, \Delta_v^{-1}], i).$$

□

Remark 7.6.3. Corollary 7.6.2 makes precise the well known fact that, in order to compute $\dim C^\alpha(\widehat{\Delta})_d$ for $\Delta \subset \mathbb{R}^3$, even for $d \gg 0$, one must know $\dim C^\alpha(\Sigma)_d$ for $\Sigma \subset \mathbb{R}^3$ arbitrary simplicial fans and all d. See for instance [7, Remark 65].

Remark 7.6.4. In the case of uniform parameters and $\Sigma \subset \mathbb{R}^3$ a simplicial fan, $\dim C^r(\Sigma)_d - \chi(\mathcal{R}[\Sigma, \partial\Sigma], d) = 0$ for all d iff $C^r(\Sigma)$ is a free module over the polynomial ring in 3 variables [50, Corollary 4.2]. Hence we see that $C = 0$ in Corollary 7.6.2 characterizes when the sheaf associated to $C^r(\widehat{\Delta})$ is a vector bundle over $\mathbb{P}^3$. This is the first nontrivial instance of a general characterization due to Schenck and Stiller [51, Theorem 3.1].

To understand the significance of the constant appearing in Corollary 7.6.1, we return to $C^1(\Sigma)$ for the fan $\Sigma = \widehat{\Delta}$ from Example 7.3.2.

Example 7.6.5. Referring back to Example 7.3.2, let w denote the vertex $(0, 0, 8)$, labelled with a 0 in Figure 7.7, so $\widehat{w}$ denotes the cone in $\mathbb{R}^4$ over this vertex. Let $X \subset \mathbb{R}^3$ denote the cone over $\mathrm{st}_\Delta(w)$, shown in Figure 7.9. Note that $X^{-1} = \partial X$ since we are considering uniform smoothness.

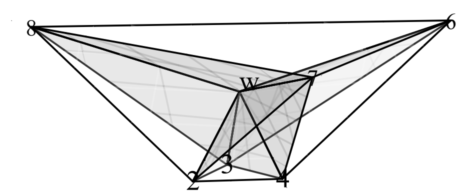


Figure 7.9: Star of the vertex w

Localizing $\mathcal{R}/\mathcal{J}[\Sigma, \partial\Sigma]$ at $I(\widehat{w})$ yields

$$\mathcal{R}/\mathcal{J}[\Sigma, \partial\Sigma]_{I(\widehat{w})} = \mathcal{R}/\mathcal{J}[X, \partial X]_{I(\widehat{w})}$$

by Lemma 7.1.10. By Lemma 7.2.3,

$$\mathcal{R}/\mathcal{J}[X, \partial X] = \mathcal{R}/\mathcal{J}[\Delta_w, \partial\Delta_w](-1) \otimes_R S,$$

where $R = \mathbb{R}[x_1, x_2, x_3]$ is the polynomial ring in 3 variables corresponding to the inclusion of $\mathbb{R}^3$ into $\mathbb{R}^4$ as the hyperplane $x_0 = 0$, and Δ_w is obtained by translating $\mathrm{st}_\Delta(w)$ to the origin and taking the positive hull of each facet containing w. Since $\partial\Delta_w = \emptyset$ we have

$$H_3(\mathcal{R}/\mathcal{J}[X, \partial X]) = H_2(\mathcal{R}/\mathcal{J}[\Delta_w]) \otimes_R S.$$

By Proposition 2.2.2, $I(\widehat{w})$ is associated to $H_3(\mathcal{R}/\mathcal{J}[\Sigma,\partial\Sigma])$ if and only if the homogeneous maximal ideal is associated to $H_2(\mathcal{R}/\mathcal{J}[\Delta_w])$. This is true iff $H_2(\mathcal{R}/\mathcal{J}[\Delta_w]) \neq 0$, since by Theorem 7.2.4 the maximal ideal of R is the only ideal that can be associated to $H_2(\mathcal{R}/\mathcal{J}[\Delta_w,\partial\Delta_w])$.

Since $\mathcal{P}(\Delta_w)$ relative to $\mathrm{lk}(\Delta_w)$ has the topology of a 3-sphere, $H_2(\mathcal{R}[\Delta_w]) = H_1(\mathcal{R}[\Delta_w]) = 0$. From the long exact sequence in homology corresponding to

$$0 \to \mathcal{J}[\Delta_w] \to \mathcal{R}[\Delta_w] \to \mathcal{R}/\mathcal{J}[\Delta_w] \to 0$$

we see that $H_2(\mathcal{R}/\mathcal{J}[\Delta_w]) \cong H_1(\mathcal{J}[\Delta_w])$. Since $f_2(\Delta_w) = 12$ and $f_1(\Delta_w) = 6$, $\mathcal{J}[\Delta_w]$ has the form (nonzero in homological degrees $0,1,2$):

$$0 \to \bigoplus_{i=1}^{12} J(\tau_i) \to \bigoplus_{j=1}^{6} J(e_j) \to J(v) \to 0.$$

Each of the ideals in this sequence are generated in degree two. In particular, we have isomorphisms (for all i,j)

$$\begin{aligned} J(\tau_i) &\cong \langle x^2\rangle \\ J(e_j) &\cong \langle x,y\rangle^2 \\ J(v) &\cong \langle x,y,z\rangle^2. \end{aligned}$$

In fact, using the arguments in the proof Lemma 7.6.7 we can also show that $H_2(\mathcal{J}[\Delta_w])$ is generated in degree two, so it is of particular interest to examine $\mathcal{J}[\Delta_w]$ in degree two.

From the isomorphisms above, $\dim J(\tau_i)_2 = 2, \dim J(e_j)_2 = 3$, and $\dim J(v)_2 = 6$. Hence $\mathcal{J}[\Delta_w]_2$ has the form

$$0 \to \mathbb{R}^{12} \xrightarrow{\delta_2} (\mathbb{R}^3)^6 \xrightarrow{\delta_1} \mathbb{R}^6 \to 0.$$

Taking the Euler characteristic yields that the alternating sums of homologies is 0, so

$$\dim H_2(\mathcal{J}[\Delta_w])_2 = \dim H_1(\mathcal{J}[\Delta_w])_2.$$

But the homology $H_2(\mathcal{J}[\Delta_w])_2$ can be identified with nontrivial splines on Δ_w of degree 2. It follows that $H_1(\mathcal{J}[\Delta_w]) \neq 0$ precisely when there is an 'unexpected' nontrivial spline on Δ_w of degree 2, which is indeed the case. We check in Macaulay2 that

$$\dim C^1(\Delta_w)_d - \chi(\mathcal{R}/\mathcal{J}[\Delta_w,\partial\Delta_w],d) = \begin{cases} 1 & d=2 \\ 0 & d\neq 2 \end{cases}$$

This is the same at each of the eight vertices of Δ. Hence, by Corollary 7.6.2

we arrive at the conclusion

$$\begin{aligned} HP(C^1(\Sigma), d) &= \chi(\mathcal{R}/\mathcal{J}[\Sigma, \partial\Sigma], d) + 8 \\ &= \frac{5}{2}d^3 - 13d^2 + \frac{51}{2}d - 3 \end{aligned}$$

We emphasize that, unlike the two dimensional case, $HP(C^r(\widehat{\Delta}), d)$ may not be either an upper or a lower bound for $HF(C^r(\widehat{\Delta}), d) = \dim C^r_d(\Delta)$ in low degree. However, for $d \gg 0$, $HF(C^r(\widehat{\Delta}), d) = HP(C^r(\widehat{\Delta}), d)$. For our current example, here is a table of values computed in Macaulay2.

d	$HF(C^1(\widehat{\Delta}), d)$	$HP(\widehat{\Delta}, d) = \frac{5}{2}d^3 - 13d^2 + \frac{51}{2}d - 3$
1	4	12
2	11	16
3	25	24
4	54	51
5	113	112
6	222	222
7	396	396

We conclude, as promised, by computing $\dim C^1(\widehat{\Delta})_d$ for $\Delta \subset \mathbb{R}^3$ generic and $d \gg 0$. This formula was shown by Alfeld, Schumaker, and Whiteley to hold for $d \geq 8$ [7]. For simplicity, set $f_i(\Delta) = f_i, f^0_i(\Delta) = f^0_i$.

Theorem 7.6.6. *Suppose $\Delta \subset \mathbb{R}^3$ is a generic triangulation of a 3-ball. Then, for $d \gg 0$,*

$$\dim C^1(\widehat{\Delta})_d = \chi(\mathcal{R}/\mathcal{J}[\widehat{\Delta}, \partial\widehat{\Delta}], d) = f_3\binom{d+3}{3} - f^0_2(d+1)^2 + f^0_1(3d+1) - 4f^0_0$$

In [7, Remark 5], the authors note that Theorem 7.6.6 can be derived by a simple heuristic, which is in fact the computation of $\chi_H(\mathcal{R}/\mathcal{J}[\widehat{\Delta}, \partial\widehat{\Delta}], d)$. We make this heuristic argument rigorous by showing that the one relevant homology module vanishes in large degree. There are two key steps. First, we use Corollary 7.6.2 to reduce the argument to three dimensional simplicial fans. Second, for a generic three dimensional simplicial fan Σ, we must show that $H_2(\mathcal{R}/\mathcal{J}[\Sigma, \partial\Sigma]) = 0$, or equivalently show that $\dim C^1(\Sigma)_d = \chi(\mathcal{R}/\mathcal{J}[\Sigma, \partial\Sigma], d)$. This is accomplished in [7, Corollaries 40,41] using projections of generalized triangulations. We accomplish this by using some homological algebra to reduce to the case of noncomplete fans, where the methods of Whiteley [61] can be applied directly. We postpone the proof of Theorem 7.6.6 until we have accomplished this second step.

If $\Sigma \subset \mathbb{R}^3$ has the form Δ_w for $w \in \Delta$, where Δ triangulates a three-ball, then $\mathrm{lk}(\Sigma)$. In this case,

$$H_2(\mathcal{R}/\mathcal{J}[\Sigma, \partial\Sigma]) \cong H_1(\mathcal{J}[\Sigma, \partial\Sigma]),$$

so we will work with $H_1(\mathcal{J}[\Sigma, \partial\Sigma])$.

If the union of the cones of Σ is $\mathbb{R}^3$, Σ is called *complete*. We have the following fact for generic C^1 splines as a result of the presentation for $H_1(\mathcal{J}[\Sigma])$ given in Lemma 3.4.3.

Lemma 7.6.7. *Let $\Sigma \subset \mathbb{R}^3$ be a complete, generic, simplicial fan. For uniform smoothness $r = 1$, $H_1(\mathcal{J}[\Sigma])$ is generated in degree two.*

Proof. Since Σ is generic, we may assume there are at least 6 codimension one faces, so $J(\nu) \cong \langle x, y, z\rangle^2$. There is an eight dimensional space of linear syzygies on $J(\nu)$ and no syzygies of higher degree (the free resolution is in fact *linear*). It follows that V^1 consists of an eight dimensional space of linear syzygies (of degree three), and perhaps many more of degree two. We need only check that these eight linear syzygies live in the submodule K^1, i.e. that these eight linear syzygies may be obtained as syzygies around rays. Since Σ is generic, we may assume that Σ_1 consists of at least 4 rays v_1, v_2, v_3, v_4 whose linear spans are linearly independent. Hence $J(v_i) \cong \langle x, y\rangle^2$. An easy check yields that there are two linear syzygies on this ideal. We claim that the two linear syzygies contributed from each of these four rays form a vector space of linear syzygies of dimension eight, which spans the entire space of linear syzygies on $J(\nu)$. We can do this explicitly by making a projective change of coordinates so that v_1 points along the positive x-axis, v_2 along the positive y-axis, v_3 along the positive z-axis, and v_4 points in the direction of the vector $\langle -1, -1, -1\rangle$. Then we have

$$\begin{array}{ll} J(v_1) = \langle y^2, yz, z^2\rangle & J(v_2) = \langle x^2, xz, z^2\rangle \\ J(v_3) = \langle x^2, xy, y^2\rangle & J(v_4) = \langle (x-z)^2, (x-z)(y-z), (y-z)^2\rangle. \end{array}$$

The claim is that the linear syzygies on $J(v_1), \dots, J(v_4)$ generate the linear syzygies on $J(v_1) + \cdots + J(v_4) = J(\nu)$. This is readily checked by hand or in Macaulay2. □

Theorem 7.6.8. *Let $\Sigma \subset \mathbb{R}^3$ be a generic hereditary simplicial fan with lk(Σ) simply connected. Then*

$$\dim C^1(\Sigma)_d = \chi(\mathcal{R}/\mathcal{J}[\Sigma, \partial\Sigma], d).$$

Equivalently,

$$H_2(\mathcal{R}/\mathcal{J}[\Sigma, \partial\Sigma]) = 0.$$

Proof. It is equivalent to prove that $H_1(\mathcal{J}[\Sigma, \partial\Sigma]) = 0$. We argue by induction on the number of interior vertices. First assume Σ is complete and let Σ' be the simplicial fan obtained by removing any three dimensional cone. Let τ_1, τ_2, τ_3 be the codimension one faces and v_1, v_2, v_3 the rays of the cone removed. We have the following commutative diagram with exact columns.

$$
\begin{array}{cccccc}
 & 0 & & 0 & & 0 \\
 & \downarrow & & \downarrow & & \downarrow \\
 & \bigoplus_{i=1}^{3} J(\tau_i) & \longrightarrow & \bigoplus_{i=1}^{3} J(v_i) & \longrightarrow & J(\nu) \\
 & \downarrow & & \downarrow & & \downarrow \\
J[\Sigma] & \bigoplus_{\tau\in\Sigma_1} J(\tau) & \longrightarrow & \bigoplus_{v\in\Sigma_1} J(v) & \longrightarrow & J(\nu) \\
 & \downarrow & & \downarrow & & \downarrow \\
J[\Sigma', \partial\Sigma'] & \bigoplus_{\tau\in(\Sigma')_1^0} J(\tau) & \longrightarrow & \bigoplus_{v\in(\Sigma')_1^0} J(v) & \longrightarrow & 0 \\
 & \downarrow & & \downarrow & & \\
 & 0 & & 0 & &
\end{array}
$$

By induction, $H_1(\mathcal{J}[\Sigma', \partial\Sigma']) = 0$. By Lemma 7.6.7, to prove that $H_1(\mathcal{J}[\Sigma]) = 0$ it suffices to prove that $H_1(\mathcal{J}[\Sigma])_2 = 0$. The ideals across the top row have the form

$$
\begin{aligned}
J(\tau_i) &\cong \quad \langle x^2 \rangle \\
J(v_i) &\cong \quad \langle x, y\rangle^2 \\
J(\nu) &\cong \quad \langle x, y, z\rangle^2
\end{aligned}
$$

Hence in degree two we have

$$
\begin{aligned}
\bigoplus_{i=1}^{3} J(\tau_i)_2 &\cong \quad \mathbb{R}^3 \\
\bigoplus_{i=1}^{3} J(v_i)_2 &\cong \quad \mathbb{R}^9 \\
J(\nu)_2 &\cong \quad \mathbb{R}^3.
\end{aligned}
$$

The leftmost map of the top row is injective in degree two and the rightmost is surjective. Since the Euler characteristic is zero, we have that the top row is exact in degree two. The long exact sequence in homology then yields $H_1(\mathcal{J}[\Sigma])_2 = 0$.

For the remainder of the induction we assume Σ is not complete and follow the argument of Whiteley [61, § 4]. To prove that $H_1(\mathcal{J}[\Sigma, \partial\Sigma]) = 0$, it suffices to show that, in degree 2,

$$\dim H_2(\mathcal{J}[\Sigma, \partial\Sigma])_2 = \chi(\mathcal{J}[\Sigma, \partial\Sigma], 2).$$

$\mathcal{J}[\Sigma, \partial\Sigma]$ is concentrated in homological degrees one and two, and has the form

$$\bigoplus_{\tau\in\Sigma_2^0} J(\tau) \xrightarrow{\delta_2} \bigoplus_{v\in\Sigma_1^0} J(v).$$

The codomain of δ_2 is contained in the free module $\bigoplus_{v\in\Delta_1^0} S$, where $S = \mathbb{R}[x, y, z]$ is the polynomial ring in three variables. Since $J(\tau) \cong \langle l_\tau^2 \rangle \cong S(-2)$, we can identify δ_2 as a graded map between free S-modules of the form

$$\bigoplus_{\tau\in\Sigma_2^0} S(-2) \xrightarrow{\delta_2} \bigoplus_{v\in\Sigma_1^0} S.$$

Let $e_\tau, \tau \in \Sigma_2^0$ be the generators of $\bigoplus_{\tau\in\Sigma_2^0} S(-2)$, and $e_v, v \in \Sigma_1^0$ be generators of $\bigoplus_{v\in\Sigma_1^0} S$. Then $\delta_2(e_\tau) = l_\tau^2 e_{v_1} \pm l_\tau^2 e_{v_2}$, where e_{v_1}, e_{v_2} are the interior rays on the boundary of τ and the signs come from orientations of the codimension one and two faces of Σ. Hence δ_2 is given in the chosen free basis by the matrix $N(\Sigma)$ whose columns are labelled by interior codimension one faces τ, whose rows are labelled by interior rays, and whose entries are given by

$$N_{v,\tau} = \begin{cases} \pm l_\tau^2 & \text{if } v \in \tau \\ 0 & \text{otherwise.} \end{cases}$$

Let $N_2(\Sigma)$ be the restriction of $N(\Sigma)$ to degree 2, i.e. $N_2(\Sigma)$ represents the map δ_2 in degree 2. For generic Σ, $J(\tau) \cong \langle x^2\rangle$ and $J(v) \cong \langle x^2, xy, y^2\rangle$. So in degree 2,

$$\begin{aligned} \dim \textstyle\bigoplus_{\tau\in\Sigma_2^0} J(\tau)_2 &= f_2^0 \\ \dim \textstyle\bigoplus_{v\in\Sigma_1^0} J(v)_2 &= 3f_1^0. \end{aligned}$$

Hence, viewed as a map between $\mathbb{R}$-vector spaces, $N_2(\Sigma)$ has $3f_1^0$ rows (three rows corresponding to each interior ray) and f_2^0 columns. We obtain an explicit form for $N_2(\Sigma)$ by choosing a basis A_v, B_v, C_v for forms of degree 2 vanishing on any $v \in \Sigma_1^0$. Then replace the entry l_τ^2 in $N(\Sigma)$ by the 3×1 column vector of coefficients expressing l_τ^2 in terms of this basis. An important observation is that $N_2(\Sigma)$ has entries which depend continuously on the (homogeneous) coordinates of the rays v. Explicitly, if τ is a codimension one face joining two rays $v_1 = \mathbb{R}_+(x_1, y_1, z_1), v_2 = \mathbb{R}_+(x_2, y_2, z_2)$, then

$$l_\tau = \det \begin{bmatrix} x_1 & x_2 & x \\ y_1 & y_2 & y \\ z_1 & z_2 & z \end{bmatrix}.$$

Let f_i^b denote the number of boundary i-faces of Σ. We have the relations

$$\begin{aligned} 3f_3 = {} & 2f_2^0 + f_2^b \\ f_1^0 - f_2^0 + f_3 = {} & 1 \end{aligned}$$

Together these yield $3f_1^0 - f_2^0 = 3 - f_2^b$. Since $f_2^b \geq 3$, we have $3f_1^0 \leq f_2^0$, so to show that $H_1(\mathcal{J}[\Sigma, \partial\Sigma]) = 0$ generically, it suffices to show that $N_2(\Sigma)$ has full rank. We show there are no dependencies among the rows of N_2 for generic Σ.

First we reduce to the case where $\mathrm{lk}(\Sigma)$ has triangular boundary. Let Σ' be a subfan of any non-complete fan $\Sigma \subset \mathbb{R}^3$, and order the interior rays and codimension one faces of Σ so that those which are also interior faces of Σ'

appear first. Then $N_2(\Sigma)$ has block form

$$\begin{bmatrix} N_2(\Sigma') & 0 \\ A & B \end{bmatrix},$$

where the upper right block is a block of zeros. This block of zeros is present because any codimension one face $\tau \in \Sigma_2^0$ which contains a ray $v \in (\Sigma')_1^0$ is also an interior codimension one face of Σ'. It follows that any relation among the rows of $N_2(\Sigma')$ immediately gives a relation among the rows of $N_2(\Sigma)$. For an arbitrary non-complete fan Σ', it is simple to build a fan Σ having Σ' as a subfan so that $\mathrm{lk}(\Sigma)$ has triangular boundary. Since Σ' is not complete, take a simplicial cone σ so that $\sigma \cap \Sigma' = 0$. Σ' is contained in a component C of $\mathbb{R}^3 \setminus \partial\sigma$. Fill in the region $C \setminus \Sigma'$ with simplicial cones. Together with Σ', this creates a fan Σ whose boundary is $\partial\sigma$. Hence it suffices to prove that there are no relations among the rows of $N_2(\Sigma)$ when $\mathrm{lk}(\Sigma)$ has triangular boundary.

If $\mathrm{lk}(\Sigma)$ has triangular boundary, then $3f_1^0 = f_2^0$ and $N_2(\Sigma)$ has the same number of rows as columns. We now argue that the columns of $N_2(\Sigma)$ are independent. This is the main induction, and it is performed on the number of interior rays. As the base case, consider the fan Σ with a single interior ray, three codimension one interior faces, and three codimension one boundary faces. By changing coordinates we may assume the interior ray is the z-axis and the three interior codimension one faces are given by $x = 0, y = 0$, and $x - y = 0$. Then we have

$$N(\Sigma) = \begin{bmatrix} x^2 & y^2 & (x-y)^2 \end{bmatrix}.$$

These form a basis for forms of degree two in x and y, hence the columns of $N_2(\Sigma)$ are independent.

We will be terse in the remainder of the proof, since the argument for [61, Theorem 6] carries over almost verbatim (Whiteley uses the transpose of the matrix $N_2(\widehat{\Delta})$ we use here). For the inductive step, we apply *vertex splitting* and the inverse process of *edge contraction* (edge *shrinking* in [61]). Applied to the fan Σ this process is one of splitting the rays $v \in \Sigma_1$ and contracting codimension one faces $\tau \in \Sigma_2$. The split of an interior ray adds one interior ray, three interior codimension one faces, and two simplicial facets. This process does not affect $\partial\Sigma$, hence $\mathrm{lk}(\Sigma)$ remains triangular. Likewise, the reverse process of contracting an interior codimension one face joining two interior rays does not affect $\partial\Sigma$. Any noncomplete fan $\Sigma \subset \mathbb{R}^3$ with at least two interior rays has a contractible codimension one face joining two interior rays. This follows by taking a stereographic projection of $\mathrm{lk}(\Sigma)$ with center outside of Σ, and then applying [61, Lemma 5], which says that any triangulated disk with at least two interior vertices has a contractible edge joining two interior vertices. Such an edge has the property that it is not an edge of any non-facial three-cycle.

Now choose a contractible codimension one face τ of Σ joining two interior rays v_1, v_2. In the process of contracting τ, two codimension one faces, call them τ_1, τ_2, collapse to two corresponding codimension one faces e_1, e_2 when τ is fully contracted. Let Σ' denote the fan obtained by contracting τ, and let v be the vertex which splits to create v_1, v_2. $M_2(\Sigma)$ is obtained from $M_2(\Sigma')$ by replacing the row corresponding to v by two rows corresponding to v_1, v_2 adding the column corresponding to τ, and replacing the columns corresponding to e_1, e_2 each by two columns corresponding to e_1, τ_1, e_2, τ_2. Since $M_2(\Sigma)$ has entries which depend continuously on the homogeneous coordinates of the ray v_1, we consider $\lim_{v_1 \to v_0} M_2(\Sigma)$. Whiteley shows that a nontrivial relation among the columns of $\lim_{v_1 \to v_0} M_2(\Sigma)$ implies a relation among the columns of $M_2(\Sigma')$. By induction, we assume $M_2(\Sigma')$ has independent columns, so no such relation exists. Again, arguing by the continuous dependence of $\det M_2(\Sigma)$ on the homogeneous coordinates of v_1, most choices of coordinate v_1 with v_1 close to v_0 and $v_1 \neq v_0$ yield $\det M_2(\Sigma) \neq 0$. Hence the columns of $M_2(\Sigma)$ are linearly independent for generic choices of Σ. □

Proof of Theorem 7.6.6. Set $\Sigma = \widehat{\Delta}$, $S = \mathbb{R}[w, x, y, z]$. Since we are considering uniform smoothness, $\Sigma^{-1} = \partial\Sigma$. $\mathcal{R}/\mathcal{J}[\Sigma, \partial\Sigma]$ is concentrated in homological degrees $4, 3, 2$, and 1 and has the form

$$S^{f_4(\Sigma)} \to \bigoplus_{\tau \in \Sigma_3^0} \frac{S}{J(\tau)} \to \bigoplus_{\gamma \in \Sigma_2^0} \frac{S}{J(\gamma)} \to \bigoplus_{v \in \Sigma_1^0} \frac{S}{J(v)}. \tag{7.3}$$

We show first that $\chi_H(\mathcal{R}/\mathcal{J}[\Sigma, \partial\Sigma], d) = f_3\binom{d+3}{3} - f_2^0(d+1)^2 + f_1^0(3d+1) - 4f_0^0$. Since we assume Δ is generic, there are at least three interior codimension one faces meet along every edge $\gamma \in \Delta_1^0$ and at least 6 codimension one faces meet at every vertex $v \in \Delta_0^0$. Under these assumptions, for $\tau \in \Delta_2^0, \gamma \in \Delta_1^0, v \in \Delta_0^0$ we have

$$\begin{aligned} J(\tau) &\cong \langle x \rangle^2 \\ J(\gamma) &\cong \langle x, y \rangle^2 \\ J(v) &\cong \langle x, y, z \rangle^2. \end{aligned}$$

Given these equalities, it follows easily that $HP(S/J(\tau), d) = (d+1)^2$, $HP(S/J(\tau), d) = 3d + 1$, and $HP(S/J(v), d) = 4$. Also, $f_i(\Sigma) = f_{i-1}(\Delta)$ and $f_i^0(\Sigma) = f_{i-1}^0(\Delta)$. Taking an alternating sum and using the well-known fact that $HP(S, d) = \binom{d+3}{3}$ gives $\chi_H(\mathcal{R}/\mathcal{J}[\Sigma, \partial\Sigma], d) = f_3\binom{d+3}{3} - f_2^0(d+1)^2 + f_1^0(3d + 1) - 4f_0^0$.

To complete the proof, it suffices by Corollary 7.6.2 to show that

$$H_2(\mathcal{R}/\mathcal{J}[\Delta_v, \partial\Delta_v]) = 0$$

for all $v \in \Delta_0$. Equivalently we need to show that

$$\dim C^1(\Delta_v)_d = \chi(\mathcal{R}/\mathcal{J}[\Delta_v, \partial\Delta_v], d),$$

for all $d \geq 0$ and all $v \in \Delta_0^0$. This follows from Theorem 7.6.8. □

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