ON QUADRATIC CORE PROJECTION PAYMENT RULES FOR COMBINATORIAL AUCTIONS

BY

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THESIS

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ABSTRACT

Auctions of licenses for electromagnetic spectrum conducted by the Federal Communications Commission (FCC) often involve the simultaneous sale of hundreds of licenses for wireless bandwidth in different geographic regions and in different spectral bands. The auctions can involve hundreds of bidding rounds over several weeks. A nontrivial open problem is to design an auction format that allows bidder flexibility, maximizes social welfare, and withstands legal scrutiny. We consider a recently introduced promising auction format called core projection auctions. It is based on a projection of a Vickrey price vector onto the core. The auction consists of two processes: winner determination process and payment determination process. The auction aims to make it easy for bidders to determine their bids by giving them little strategic advantage by having their bids deviate from their true valuation of the spectrum. This thesis explores properties of such a core projection mechanism with an emphasis on numerically analyzing the marginal incentive for bidders to bid untruthfully. By implementing solvers and running simulations, we conjecture that in general, the payment for a winner increases no faster than the corresponding bidding price.

To my parents, for their love and support,

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This thesis would not have been possible without the help and supervision of Professor Bruce Hajek. I have been very fortunate to have this opportunity of working with Professor Hajek for the past year. From a naive student transferring from Journalism to Electrical Engineering and claiming an interest in information science and systems, to a senior getting ready to continue research work in graduate school, I have learned so many things under the guidance of Professor Hajek. Professor Hajek has taught me not only how to do research work using various engineering tools, but also how to live a productive and healthy life. Though exploring the truly unknown problem in the world could be sometimes frustrating, it has also offered me many exciting and inspiring moments. The thesis work comes to an end for now, but I will not stop running on the road of science.

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CHAPTER 1

INTRODUCTION

1.1 Motivation for Combinatorial Auction

Bidders in an auction might have a desire for a bundle of items instead of a single item. For instance, customers usually buy complementary goods at the same time, such as toothbrush and toothpaste. In the case of a government radio spectral auction, companies tend to buy the licenses for the same bandwidth in neighboring areas to make the most use of the licenses. A combinatorial auction gives bidders freedom to bid for bundles of items. Motivated by the auctions of licenses for electromagnetic spectrum conducted by the Federal Communications Commission (FCC), we consider a combinatorial auction (also called 'package auction') mechanism: the core projection auction.

In a general combinatorial auction, bidders submit bidding prices for arbitrary bundles of items and then a core projection auction mechanism is applied to determine a winner set by maximizing the sum of bidding prices of winners subject to avoiding any overlapping of the bundles of the winners. The winner determination process is as complex as the weighted set-packing problem and thus is NP-hard [1]. However, dynamic programming is enough to solve such a problem of small scale. Once we determine the winner sets, the question of payment determination naturally rises. Combining core projection auction as an appropriate generalization of a second-price auction and a Vickrey auction mechanism, we will consider quadratic core projection payment rules for the payment determination process.

This thesis explores key features of this specific mechanism and focuses on a numerical analysis of the marginal incentive for bidders to bid untruthfully.

The principle application we have in mind is government sale of the radio frequency spectrum. Recently, core projection auctions have been proposed for dynamically allocating heterogeneous virtual machines in cloud computing [2].

1.2 Why Core-selecting Auctions?

Though a first-price auction is more intuitive and familiar, a second-price auction has its unique advantages compared with a first-price auction. The well-known facts about a second-price auction of a single item are as follows [3]:

1. Individual rationality: each bidder has a nonnegative payoff. In other words, losers do not pay and winners will not pay more than their bidding prices.

2. Efficiency: the highest-valued coalition wins.

3. Dominant strategy incentive compatibility: the best strategy for each bidder is to give a truthful bid.

4. The core property: there is no blocking feasible coalition. Namely, there is no feasible set of the bidders that can beat the winning set.

For combinatorial auctions, it is well known that a Vickrey auction satisfies the first three properties but does not necessarily satisfy the core property. The core projection mechanism treats properties 1, 2 and 4 as constraints and aims to minimize the deviation from property 3. This mechanism generates prices as a natural generalization of the second price for a single-item auction. The payment rules minimize the incentives for bidders as a whole to bid untruthfully. It is shown in [4] that for a bidder interacting with disjoint groups of other bidders in a star network setting, the price paid by a winner increases no faster than the bidding price. In the mathematical world, it simply means that the derivative of a winner's payment with respect to the winner's bidding price is not greater than 1. Inspired by the result in a star network setting and numerical simulation tests, we conjecture that the conclusion holds in general.

1.3 Overview of Thesis Organization

In Chapter 2, we define the core projection auction based on projection of a Vickrey price vector onto the core. Initially, we assume that bidders have only one single bid competing for one bundle (i.e. they are single-minded) in the default setting, but we will also consider the case in which bidders can have multiple bids. We also present modifications required for the case of multiple bids per bidder instead of one single bid per bidder. In Chapter 3, simulation results and interesting examples are given to help readers better understand the core projection mechanism. In Chapter 4, we will give general propositions about the core projection mechanism and corresponding mathematical proofs. Finally, a summary of the thesis and possible future work are given in Chapter 5.

CHAPTER 2

THE CORE PROJECTION AUCTION

We will first describe the core projection auction mechanism for the case of *single-minded bidders* - for which each bidder submits only one bid. Bids are submitted through some process. One option is sealed bids, in which all bids are submitted simultaneously in sealed envelopes. However, in a government spectral auction, bids are typically submitted in a process entailing multiple rounds of bidding. Our work does not focus on how the bids are collected. Rather, we assume that the bids have somehow been collected.

2.1 Winner Determination for Single-minded Bidder

Consider a combinatorial auction with n bidders indexed by N and m single items indexed by M. Bidder i will give the bid (i, S_i, b_i), such that b_i is the bidding price for a bundle of items S_i, with $S_i \subseteq M$. In this mechanism, we assume that every bidder is single-minded and thus only has one single bid. When all the bids are submitted, the first question would be how to determine the winner set.

A coalition C is simply a subset of N. A coalition is said to be feasible if $S_i \cap S_j = \emptyset$ for all $i, j \in C$ and $i \neq j$. Let $\mathcal{C}$ denote the set of all the feasible coalitions. Let W denote the winner set, and thus W is a feasible coalition. To determine the winner set, we just maximize the sum of bidding prices of the winners subject to avoiding selling any single item to different winners:

$$W = \underset{C \in \mathcal{C}}{\operatorname{argmax}} \sum_{i \in C} b_i \tag{2.1}$$

I implemented a solver using a version of dynamic programming for the winner determination process in *Python* and pseudo code is presented in

Algorithm 1.

Algorithm 1 Pseudo code for winner determination process using dynamic programming

```
INPUT: Bids in the format of (i, S_i, b_i) with 1 ≤ i ≤ n
OUTPUT: Winner set W
  function CONFLICT(i, j)
      if S_i ∩ S_j = ∅ then
          conflict(i, j) = 0
      else
          conflict(i, j) = 1
      end if
      return conflict
  end function
  Initialize b(C) = { −1  if C ≠ ∅
                    { 0   if C = ∅
  for i ∈ {1, 2, 3, ..., n} do
      for coalition C ⊂ {1, 2, ..., i − 1} do
          if ∀k ∈ C, conflict(k, i) = 0 then
              b(C ∪ {i}) = b(C) + b_i
          end if
      end for
  end for
  W = argmax_C b(C)
```

2.2 Payment Determination for Single-minded Bidder

After we determine the winner set, we need to figure out how much each winner should pay for items. As mentioned in Section 1.1, we will employ quadratic core projection payment rules that minimize the deviation from the Vickrey price vector. We will show readers how to define the minimization problem step by step.

2.2.1 Vickrey price

First, we will introduce the definition of *Vickrey price*. The Vickrey price for winner j is simply the least bidding price for winner j to still win the auction in the winner determination process for other bidders' bidding prices fixed.

Let $\mathbf{v}$ denote the Vickrey price vector for the winner set W, then the Vickrey price for Winner j is given by

$$v_j = \left\{ \max_{C \in \mathcal{C}: j \notin C} \sum_{i \in C} b_i \right\} - \sum_{i \in W/j} b_i \tag{2.2}$$

2.2.2 Generate the core

Let $\mathbf{p}$ denote a vector of payment prices for bidder set N and p_i be the payment for Bidder i. If Bidder i is not a winner, then $p_i = 0$. Otherwise, Bidder i should not pay more than his/her bidding price. Thus, the prices satisfy the following constraints:

$$p_i \begin{cases} \leq b_i & \text{if } i \in W \\ = 0 & \text{else} \end{cases} \tag{2.3}$$

In addition, payments for winners should beat any other feasible coalition C in terms of their bidding prices, indicated as follows:

$$\sum_{i \in W} p_i \geq \sum_{i \in C \setminus W} b_i + \sum_{i \in C \cap W} p_i \tag{2.4}$$

If we arrange the terms in Equation (2.4), we can cancel some terms and get a new equation:

$$\sum_{i \in W \setminus C} p_i \geq \sum_{i \in C \setminus W} b_i \tag{2.5}$$

If we put Equation (2.3) and Equation (2.5) in matrix form, we get the following complete core formulation.

$$\mathbf{p} \leq \mathbf{b}$$

$$A\mathbf{p} \geq \beta$$

Each row of A corresponds to a feasible coalition C. For each feasible coalition C, we have

$$A_{C,i} = \begin{cases} 1 & \text{if } i \in W \setminus C \\ 0 & \text{else} \end{cases}$$

$$\beta_C = \sum_{i \in C \setminus W} b_i$$

2.2.3 Projection of Vickrey price onto the core

Finally, quadratic payment rules require minimizing the Euclidean distance between the payment vector $\mathbf{p}$ and the Vickrey price vector $\mathbf{v}$. In other words, we will project the Vickrey price vector onto the core, and the projection will result in the final payment vector.

$$\mathbf{p} = \underset{\mathbf{x}}{\operatorname{argmin}} ||\mathbf{x} - \mathbf{v}||^2 \text{ subject to } \mathbf{x} \leq \mathbf{b}, A\mathbf{x} \geq \beta \tag{2.6}$$

I implemented a solver for payment determination process using the method of multipliers described in [5] in *Python* and pseudo code is provided in Algorithm 2.

Algorithm 2 Pseudo code for payment determination process using the method of multipliers

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INPUT: $\mathbf{b}, \mathbf{v}, A, \beta$ ▷ Check Equation (2.6)
OUTPUT: payment vector $\mathbf{p}$
  Initialize $c^0, \mu^0, \epsilon, \alpha$ and judge
  Define $f(\mathbf{x}) = ||\mathbf{x} - \mathbf{v}||^2, g(\mathbf{x}) = \begin{bmatrix} \mathbf{b} - \mathbf{x} & \beta - A\mathbf{x} \end{bmatrix}^\intercal$
  $\mathbf{x}^0 = \underset{\mathbf{x}}{\operatorname{argmin}}\ L_{c^0}(\mathbf{x}, \mu^0) = f(\mathbf{x}) + \frac{1}{2c^0} \sum_{j=1}^{r} \{(\max\{0, \mu_j^0 + c^0 g_j(\mathbf{x})\})^2 - (\mu_j^0)^2\}$
  while judge $> \epsilon$ **do**
    $c^{k+1} = c^k \alpha$
    $\mu^{k+1} = \max\{0, \mu_j^k + c^k g_j(\mathbf{x}^k)\}$
    $L_{c^{k+1}}(\mathbf{x}, \mu^{k+1}) = f(\mathbf{x}) + \frac{1}{2c^{k+1}} \sum_{j=1}^{r} \{(\max\{0, \mu_j^{k+1} + c^{k+1} g_j(\mathbf{x})\})^2 - (\mu_j^{k+1})^2\}$
    $\mathbf{x}^{k+1} = \underset{\mathbf{x}}{\operatorname{argmin}}\ L_{c^{k+1}}(\mathbf{x}, \mu^{k+1})$
    judge $= ||\mathbf{x}^{k+1} - \mathbf{x}^k||^2$
    $k \leftarrow k + 1$
  end while
  $\mathbf{p} = \mathbf{x}^{k+1}$
  return p

2.3 Modifications for Multiple Bids per Bidder

Multiple bids per bidder are typically allowed in an auction. Still, a bidder is allowed at most one winning bid by convention in order for bidders to communicate their preferences accurately. Specifically, every possible preference that bidders can declare when multiple winning bids are allowed can also be expressed if we only allow at most one winning bid per bidder. For instance, assume that there are three items indexed by $\{A, B, C\}$ in one auction. Suppose a bidder wants to get either a bundle $\{A, B\}$ or a single item $\{C\}$, but does not want to purchase all three items at the same time. In the context of one winning bid per bidder, the bidder just needs to give these two bids respectively in order to express his/her preference. However, it would be impossible to express the bidder's preference if multiple winning bids per bidder are allowed. If the bidder later changes his/her preference and is happy to buy all three items, all he/she needs to do is to add one more bid bidding for the bundle $\{A, B, C\}$.

If one bidder can have multiple bids, then we have to make some modifications to the case of single-minded bidders discussed before in order for the core projection mechanism to work.

2.3.1 Winner Determination

For the winner determination process, we add one bidder-specific dummy good to all the bids of each bidder to guarantee that a bidder will not have more than one winning bid. Other than this small modification, the winner determination process is the same as in the previous setting of bidders being single-minded.

2.3.2 Vickrey Price

While determining the Vickrey price of a given bidder, we should exclude other bids from that bidder. Otherwise, bidders would need to coordinate their bids to affect their possible payments, which requires strategy and is not desired in the auction. To be more specific, they would have an incentive to make their bids less than true values because they would basically be bidding against themselves.

Let us declare notations before we go any further. Let C_m denote a set of bids and $\mathcal{C}_m$ be the set of all sets of bids satisfying the following constraints: each set C_m in $\mathcal{C}_m$ has at most one bid from each bidder and is feasible, meaning that no two bids in C_m compete for any same item. Let B_i denote the set of bids of Bidder i and S_i be the winning bid's bundle for Bidder i. If Bidder i is not a winner, then $S_i = \emptyset$. Let $b_i(S)$ represent the bidding price for bundle S. Then

$$v_i = \left\{ \max_{C_m \in \mathcal{C}_m, C_m \cap B_i = \emptyset} \sum_{(j,S_j,b_j) \in C_m} b_j(S_j) \right\} - \sum_{(j,S_j,b_j) \in W : j \neq i} b_j(S_j)$$

2.3.3 Generate the Core

In order to determine payments for winners, we project the Vickrey price vector onto the core. To generate the core, we specify two types of constraints:

1. All payments should be less than or equal to corresponding bids.

2. There should be no blocking coalition with proposed alternate bids and payments. This is the same as Equation (2.4), but for each bidder in C_m, a choice of bid must be specified and losing bids of a winner should be modified. A winner would not sacrifice his/her own profit in any other coalition, which means the winner's payment in any other coalition should, at least, earn the winner a profit as much as the winner is awarded in the winning set. Otherwise, the new set that involves any losing bid of winners will not be taken as a reasonable blocking coalition. Let $\overline{S_i}$ be a losing bid's bundle for Bidder i, then for each Bidder i who has a winning bid's bundle S_i, and for each non-winning bid's bundle $\overline{S_i}$ of Bidder i, we should reduce the non-winning bid's bidding price from $b_i(\overline{S_i})$ to $b_i(\overline{S_i}) - (b_i(S_i) - p_i)$.

If we express the two constraints above in mathematical expressions, the first constraint should be as follows:

$$p_i \leq b_i(S_i) \text{ for } (i, S_i, b_i) \in W$$

Then we define $b'_i(S)$ for Bidder i as follows:

$$b'_i(S) = \begin{cases} b_i(S) - (b_i(S_i) - p_i) & \text{if } S_i \neq \emptyset \\ b_i(S) & \text{else} \end{cases} \tag{2.7}$$

Then we just follow the same rules as we do for the case of single-minded bidders. The mathematical expression satisfying the second constraint is in the following form:

$$\sum_{(i,S_i,b_i)\in W} p_i \geq \sum_{(i,S_i,b_i)\in C_m} b'_i(S_i) \; \forall \, C_m \in \mathcal{C}_m$$

Then we can express these constraints in matrix form and apply the quadratic payment rules again. In such a way we determine the winner set and corresponding payments by applying appropriate modifications.

CHAPTER 3

EXAMPLES AND SIMULATION RESULTS

3.1 An Example with Single-minded Bidders

In order to help readers better understand how the core projection mechanism works, we walk through a simple example by determining the winners and their corresponding payments. Consider an auction with five bidders indexed by $N = \{1, 2, 3, 4, 5\}$ bidding for items indexed by $M = \{A, B, C\}$. Bidder 1, Bidder 2 and Bidder 3 respectively submit the bids $(\{A\}, \$8)$, $(\{B\}, \$10)$, $(\{C\}, \$15)$. Bidder 4 is willing to pay \$15 for bundle $\{A, B\}$ and Bidder 5 gives a bidding price \$21 for bundle $\{B, C\}$. Table 3.1 and Figure 3.1 are an illustration of the bids for the auction.

Table 3.1: A simple example with single-minded bidders

Bidder	1	2	3	4	5
Bundle	A	B	C	A, B	B, C
Bidding price	\$8	\$10	\$15	\$15	\$21

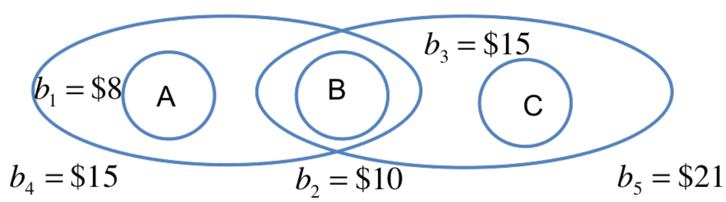

Figure 3.1: A simple example with single-minded bidders

3.1.1 Winner Determination

We can easily see that Bidder 1 and Bidder 2 beat Bidder 4 and Bidder 2 and Bidder 3 together beat Bidder 5. As a result, the feasible coalition $\{1, 2, 3\}$

beats any other feasible coalition. So winners are Bidder 1, Bidder 2 and Bidder 3, and thus $W = \{1, 2, 3\}$.

3.1.2 Payment Determination

Following the procedures, we will first determine the Vickrey prices for Bidder 1, Bidder 2 and Bidder 3. To determine the Vickrey price v_2 for Bidder 2, we just need to make sure that v_2 would satisfy the following conditions:

$$v_2 + b_1 \geq b_4 \tag{3.1}$$

$$v_2 + b_3 \geq b_5 \tag{3.2}$$

Then

$$v_2 = \max\{b_4 - b_1, b_5 - b_3\} \tag{3.3}$$

Finally, we get $v_1 = 5, v_2 = 7, v_3 = 11$ and thus $\mathbf{v} = \begin{bmatrix} 5 & 7 & 11 \end{bmatrix}^\intercal$.

Second, we generate the core in order to do the projection. Basically, we have two constraints.

1. Bidders should not pay more than their bidding prices.

$$\mathbf{p} \leq \mathbf{b} \text{ where } \mathbf{b} = \begin{bmatrix} 8 & 10 & 15 \end{bmatrix}^\intercal$$

2. There is no blocking coalition.

$$p_1 + p_2 + p_3 \geq p_3 + b_4 = p_3 + 15$$

$$p_1 + p_2 + p_3 \geq p_1 + b_5 = p_1 + 21$$

If we write these two equations in matrix form, we get

$$A\mathbf{p} \geq \beta \text{ where } A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \text{ and } \beta = \begin{bmatrix} 15 \\ 21 \end{bmatrix}$$

Finally, we project the Vickrey price vector $\mathbf{v}$ onto the core above by minimizing the Euclidean distance between the payment vector $\mathbf{p}$ and the Vickrey price vector $\mathbf{v}$ as follows:

$$\mathbf{p} = \underset{\mathbf{x}}{\text{argmin}} ||\mathbf{x} - \mathbf{v}||^2 \text{ subject to } \mathbf{x} \leq \mathbf{b}, A\mathbf{x} \geq \beta \quad (3.4)$$

Solving the minimization problem, we get $\mathbf{p} = \begin{bmatrix} 6 & 9 & 12 \end{bmatrix}^\intercal$.

3.2 An Example in the Case of Multiple Bids per Bidder

The example in Section 3.1 is modified to illustrate how core projection works when multiple bids per bidder are allowed. Consider an auction with five bidders indexed by $N = \{1, 2, 3, 4, 5\}$ bidding for items indexed by $M = \{A, B, C\}$. Bidder 1 submits two bids $(\{A\}, \$8)$ and $(\{B\}, \$9)$. Bidder 2 submits two bids $(\{A\}, \$7)$ and $(\{B\}, \$10)$. Bidder 3 submits two bids $(\{C\}, \$15)$ and $(\{B, C\}, \$24)$. Bidder 4 submits one single bid $(\{A, B\}, \$15)$ and bidder 5 submits only one bid $(\{B, C\}, \$21)$. These bids are displayed in Table 3.2.

Table 3.2: An example for multiple bids per bidder

Bidder	1	1	2	2	3	3	4	5
Bundle	A	B	A	B	C	B, C	A, B	B, C
Bidding price	$8	$9	$7	$10	$15	$24	$15	$21

3.2.1 Winner Determination

We add one bidder-specific dummy good to each bidder's bids in order to avoid more than one winning bid per bidder. After the addition, Bidder 1 has two bids $(\{1, A\}, \$8)$ and $(\{1, B\}, \$9)$. Bidder 2 has two bids $(\{2, A\}, \$7)$ and $(\{2, B\}, \$10)$. Bidder 3 has two bids $(\{3, C\}, \$15)$ and $(\{3, B, C\}, \$24)$. Bidder 4 has one single bid $(\{4, A, B\}, \$15)$ and Bidder 5 has only one bid $(\{5, B, C\}, \$21)$. Then we can just proceed as we do in the case of single-minder bidders and winner sets are $\{1, 2, 3\}$ and their corresponding bids are $(\{1, A\}, \$8)$, $(\{2, B\}, \$10)$ and $(\{3, C\}, \$15)$.

3.2.2 Payment Determination

First, let us figure out the Vickrey prices for the winners. To determine Bidder 1's Vickrey price, we should exclude Bidder 1's losing bids. Thus

$$v_1 + 10 + 15 \geq 24 + 7$$

$$v_1 + 10 + 15 \geq 15 + 15$$

$$v_1 + 10 + 15 \geq 21 + 7$$

Similarly, we get $v_1 = 6, v_2 = 9, v_3 = 11$ and thus $\mathbf{v} = \begin{bmatrix} 6 & 9 & 11 \end{bmatrix}^\intercal$.

Second, we generate the core by listing two constraints.

1. Bidders should not pay more than their bidding prices.

$$\mathbf{p} \leq \mathbf{b} \text{ where } \mathbf{b} = \begin{bmatrix} 8 & 10 & 15 \end{bmatrix}^\intercal$$

2. There is no blocking coalition. However, we will update all the bids by Equation (2.7) first and then use the new bids while considering potential blocking coalitions. Finally, we get

$$\begin{aligned}
p_1 + p_2 + p_3 &\geq p_1 + (24 - (15 - p_3)) \\
p_1 + p_2 + p_3 &\geq p_1 + 21 \\
p_1 + p_2 + p_3 &\geq (7 - (10 - p_2)) + (9 - (8 - p_1)) + p_3 \\
p_1 + p_2 + p_3 &\geq (7 - (10 - p_2)) + (24 - (15 - p_3)) \\
p_1 + p_2 + p_3 &\geq (7 - (10 - p_2)) + 21 \\
p_1 + p_2 + p_3 &\geq p_3 + 15
\end{aligned}$$

If we write these two equations in matrix form, we get

$$A\mathbf{p} \geq \beta \text{ where } A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \text{ and } \beta = \begin{bmatrix} 9 \\ 21 \\ -2 \\ 6 \\ 18 \\ 15 \end{bmatrix}$$

Finally, we project the Vickrey price vector $\mathbf{v}$ onto the core above by minimizing the Euclidean distance between the payment vector $\mathbf{p}$ and the Vickrey price vector $\mathbf{v}$ as follows:

$$\mathbf{p} = \underset{\mathbf{x}}{\operatorname{argmin}} ||\mathbf{x} - \mathbf{v}||^2 \text{ subject to } \mathbf{x} \leq \mathbf{b}, A\mathbf{x} \geq \beta$$

Solving the minimization problem, we get $\mathbf{p} = \begin{bmatrix} \frac{19}{3} & \frac{28}{3} & \frac{35}{3} \end{bmatrix}^\intercal$.

Compared with the example for single-minded bidders, the modified example only adds three losing bids respectively submitted by Bidder1, Bidder 2 and Bidder 3. However, it turns out that both the Vickrey price vector and the payment vector change.

3.3 Simulations for Auctions with Single-minded Bidders

We first give definitions for three types of network topologies: *chain networks*, *star networks*, and *circle networks*. Assume we have n bidders indexed by N and m single items indexed by M. Bidder i will give the bid (i, S_i, b_i) such that b_i is the bidding price for a bundle of items S_i, with $S_i \subseteq M$.

3.3.1 Chain Network

Single-item bidders are those who only bid for one single item rather than a bundle of items and those bids are called single-item bids. In a chain network, imagine the items lined up in alphabetic order. Apart from single-item bids, each bundle is a consecutive set of two or more items and each item is in either one or two bundles and no bundle contains another one. Figure 3.2 presents a typical structure of a chain network.

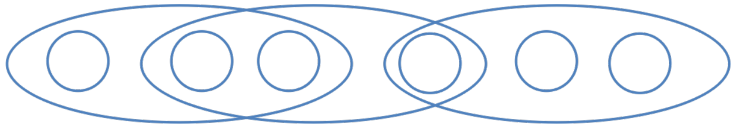

Figure 3.2: Chain network

3.3.2 Star Network

For a star network, all the bids include one specific item except for some single-item bids. Figure 3.3 gives a typical example of star network.

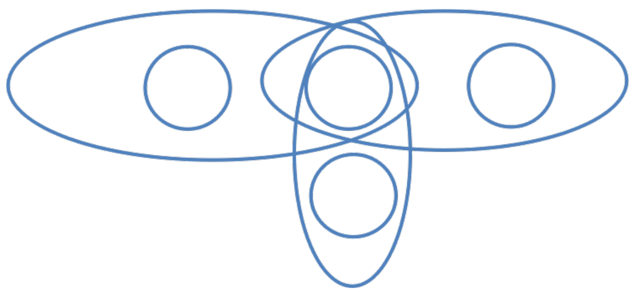

Figure 3.3: Star network

3.3.3 Circle Network

A circle network is similar to a chain network, but its items are arranged around a circle with the first item neighboring the last one. Again, each bundle is a consecutive set of items and each item is in two bundles except for those single-item bids that are subsets of some multiple-item bundles. No multiple-item bundle contains another multiple-item bundle. A typical example is as in Figure 3.4.

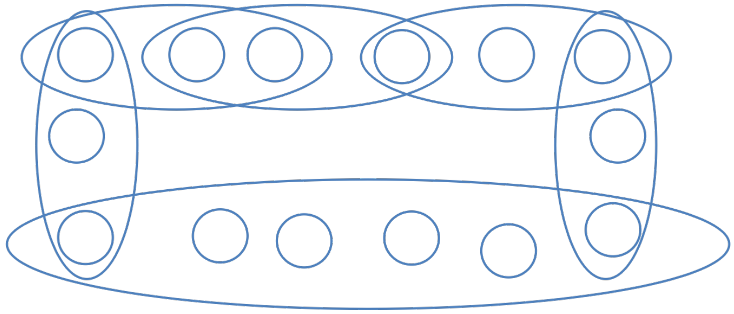

Figure 3.4: Circle network

For the simulations' setting, we intend to investigate how winners' payments change if one of the winners changes his/her bidding price. Importantly, we focus on the incentive to deviate from truthful bidding for winners. More specifically, we use the *marginal incentive to deviate* (MID) inspired by [6] to measure how a winner's payment changes per unit increase of the bidder's own bid, for the bids of other bidders fixed. The MID for a winning Bidder i for a particular bid vector is the rate of increase in payment of Bidder i as the bidding price of Bidder i increases. In these different networks, we take different examples and the settings are indicated in the following tables. Accordingly, results are shown in the form of figures as below. For each plot, the x-axis corresponds to one winner's bidding price while the y-axis corresponds to winners' payments.

Table 3.3: Chain network

Bidder	1	2	3	4	5	6	7
Bundle	A	B	C	D	A, B	B, C	C, D
Bidding price	$5	$7-$12	$6	$7	$12	$13	$12

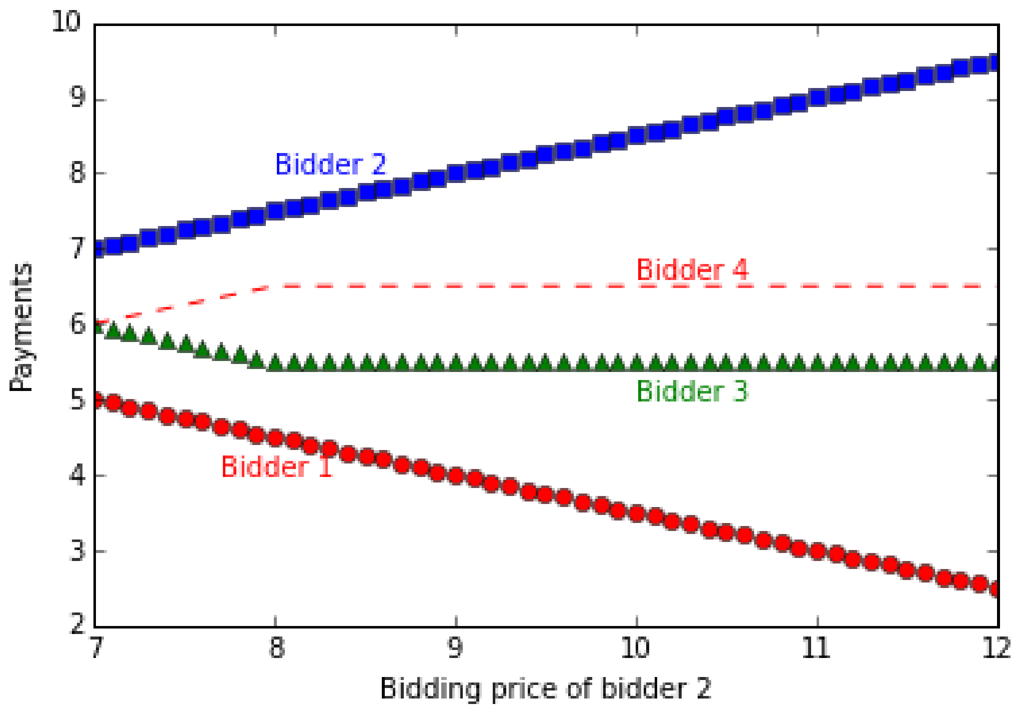

Figure 3.5: Chain network

Table 3.4: Star network

Bidder	1	2	3	4	5	6	7
Bundle	A	B	C	D	E	A, B	A, C
Bidding price	\$5-\$10	\$8	\$4	\$9	\$7	\$13	\$9
Bidder	8	9	10	11	12	13	14
Bundle	A, D	A, E	A	B	C	D	E
Bidding price	\$14	\$10	\$3	\$5	\$3	\$6	\$5

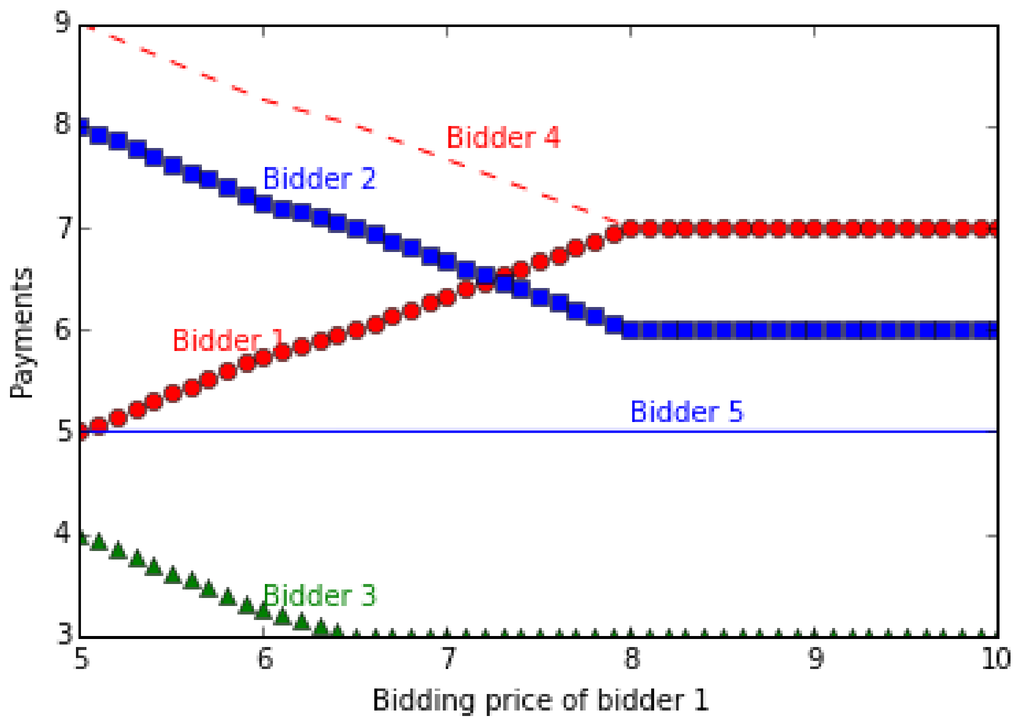

Figure 3.6: Star network

Table 3.5: Circle network

Bidder	1	2	3	4	5	6	7	8	9	10
Bundle	A	B	C	D	E	F	G	H	A,B,C	C,D,E
Bidding price	$3-$6	$7	$6	$8	$9	$4	$10	$6	$16	$22
Bidder	11	12	13	14	15	16	17	18	19	20
Bundle	E,F,G	A,G,H	A	B	C	D	E	F	G	H
Bidding price	$20	$15	$3	$6	$5	$4	$5	$2	$6	$5

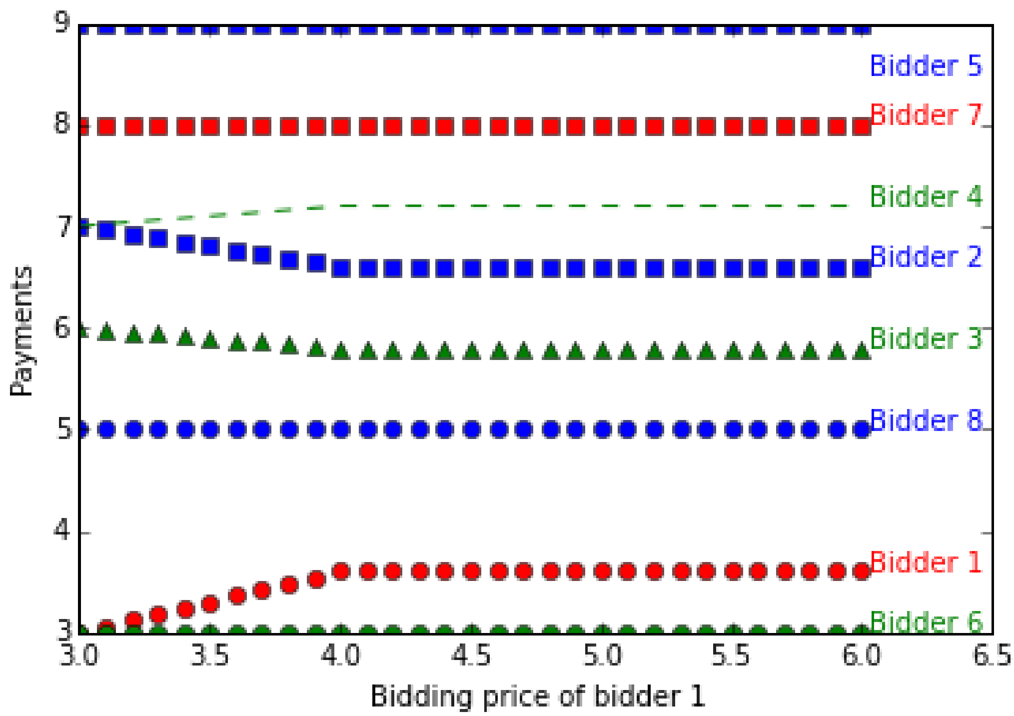

Figure 3.7: Circle network

The simulation results match our expectations and coincide with the conjecture about MID. One interesting fact that we notice in the simulation experiments is non-monotonicity of MID. MID of bidder 1 in the star network simulation is an example but not obvious from the figure. The following example is more obvious and direct. In Figure 3.8, it is clear to see that the slope of the payment of Bidder 1 (red line) is not monotone.

Table 3.6: Non-monotonicity of MID

Bidder	1	2	3	4	5	6	7
Bundle	A	B	C	A	B	C	A, B, C
Bidding price	\$5 - \$11	\$9	\$4	\$4	\$3	\$2	\$17

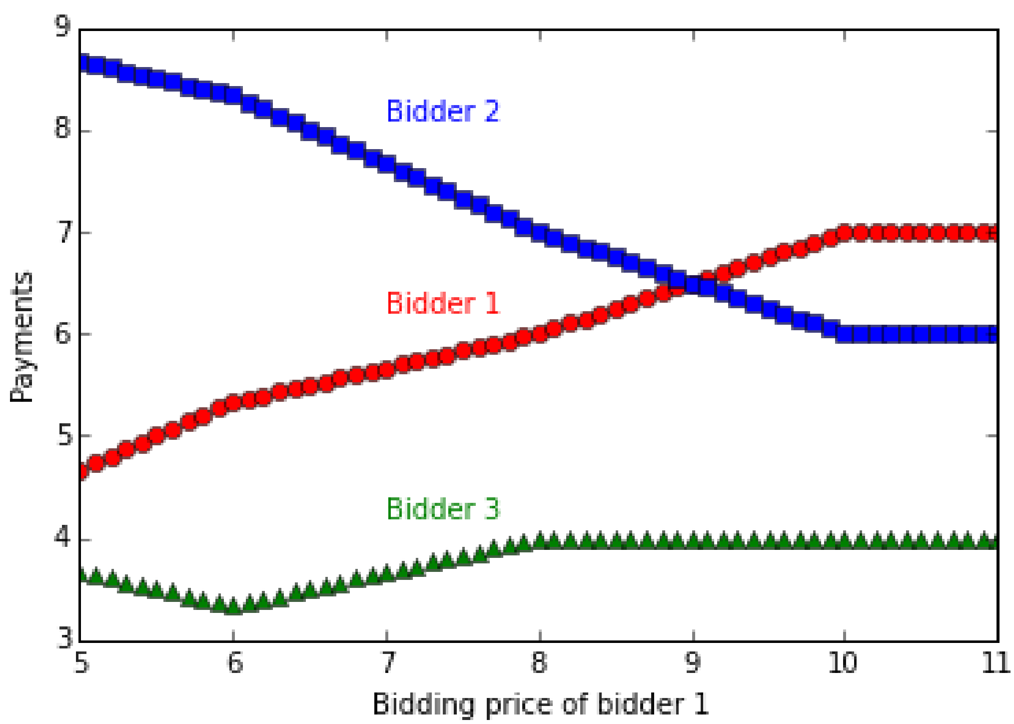

Figure 3.8: Non-monotonicity of MID

CHAPTER 4

GENERAL PROPOSITIONS

This chapter shows some properties of the core projection auction mechanism and presents some mathematical proofs under the assumption that bidders are single-minded.

Proposition 4.0.1. *If one winner bids more, some other winner can either pay more or less.*

Proof. It suffices to give an example to prove the statement as follows. Bidder 1 bid only for item A, Bidder 2 bids for item B, and Bidder 3 bids for item C. Bidder 4 bids for a combination of items A and B while bidder 5 bids for item B and C. In this setting, winners are Bidder 1, 2 and 3. Figure 4.1 gives a Venn digram for the specific bids. Assume Bidder 1 bids more and thus winners set does not change. However, the payments for Bidder 2 and Bidder 3 would change.

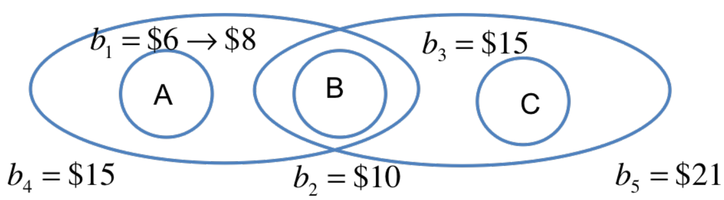

Figure 4.1: Venn diagram for the example

The original bidding price vector for bidders is $\mathbf{b} = \begin{bmatrix} 6 & 10 & 15 & 15 & 21 \end{bmatrix}^\intercal$. Thus the corresponding payment vector is $\mathbf{p} = \begin{bmatrix} \frac{16}{3} & \frac{29}{3} & \frac{34}{3} \end{bmatrix}^\intercal$. (Note: only winners have to pay.) After bidder 1 increases its bidding price from \$6 to \$8, the new bidding price vector is $\mathbf{b}' = \begin{bmatrix} 8 & 10 & 15 & 15 & 21 \end{bmatrix}^\intercal$. Thus the corresponding payment vector changes into $\mathbf{p}' = \begin{bmatrix} 6 & 9 & 12 \end{bmatrix}^\intercal$. In this case, the payment for Bidder 2 decreases from $\frac{29}{3}$ to 9 and the payment for Bidder

3 increases from $\frac{34}{3}$ to 12. In conclusion, Bidder 2 pays less and Bidder 3 pays more if Bidder 1 bids more in this case. □

Proposition 4.0.2. *The payment price for any winner is never less than the Vickrey price for that winner.*

Proof. According to the definition of the Vickrey price of a given winner, it is the minimum price of the bidder that still wins the auction in the winner determination process for other bidders' bidding prices fixed. For winner j,

$$v_j = \left\{ \max_{C \in \mathcal{C}: j \notin C} \sum_{i \in C} b_i \right\} - \sum_{i \in W/j} b_i$$

To determine the payment, we solve a quadratic minimization problem:

$$\mathbf{p} = \underset{\mathbf{x}}{\text{argmin}} ||\mathbf{x} - \mathbf{v}||^2 \text{ subject to } \mathbf{x} \leq \mathbf{b}, A\mathbf{x} \geq \beta$$

Case 1: If $\mathbf{v} \in$ core, then $\mathbf{p}^* = v$;
Case 2: If $\mathbf{v} \notin$ core, let $\mathbf{p}^* = \begin{bmatrix} p_1 & p_2 & \cdots & p_n \end{bmatrix}^\intercal$ and assume that $p_1 < v_1$ without loss of generality. Then we can show that $\exists\ \mathbf{p}' \in$ core such that $||\mathbf{p}' - \mathbf{v}||^2 < ||\mathbf{p}^* - \mathbf{v}||^2$ as follows. Let $p_1' = \max\{p_1, v_1\}$ and $p_j' = p_j$ for $j \neq 1$, then $||\mathbf{p}' - \mathbf{v}||^2 \leq ||\mathbf{p}^* - \mathbf{v}||^2$ with equality only when $p_1 = v_1$. Since $\mathbf{p}^* \in$ core by definition and core is upward closed, we know that $\mathbf{p}' \in$ core as well. Then it illustrates $\mathbf{p}^*$ is not the optimal solution in this case and it is impossible to have $p_i < v_i$.
All in all, the payments of winners can never be smaller than the corresponding Vickrey prices in any case. □

Proposition 4.0.3. *If one winner bids more, other winners' Vickrey prices can only decrease or stay the same.*

Proof. Assume winner k raises the bidding price from b_k to $b_k' = b_k + \delta$. First of all, winner set W does not change at all. Before winner k increases his/her bidding price, by the definition of the Vickrey price for winner j, we have

$$v_j = \left\{ \max_{C \in \mathcal{C}: j \notin C} \sum_{i \in C} b_i \right\} - \sum_{i \in W/j} b_i$$

After winner k raises the bidding price, we have

$$v_j = \max\left\{\left\{\max_{C\in\mathcal{C}:j\notin C,k\in C}\sum_{i\in C} b_i\right\} + \delta - \left(\sum_{i\in W/j} b_i + \delta\right), \left\{\max_{C\in\mathcal{C}:j,k\notin C}\sum_{i\in C} b_i\right\} - \left(\sum_{i\in W/j} b_i + \delta\right)\right\}$$

So

$$\frac{dv_j}{db_i} = \{0, -1\}$$

Plots of v_j versus b_i look like either Figure 4.2 or Figure 4.3. □

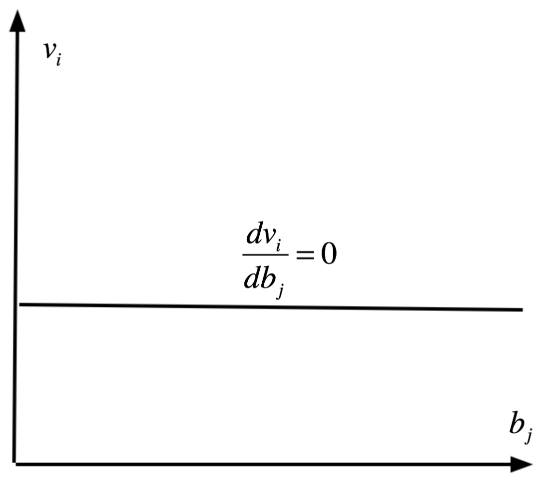

Figure 4.2: Case 1: $\frac{dv_j}{db_i}$

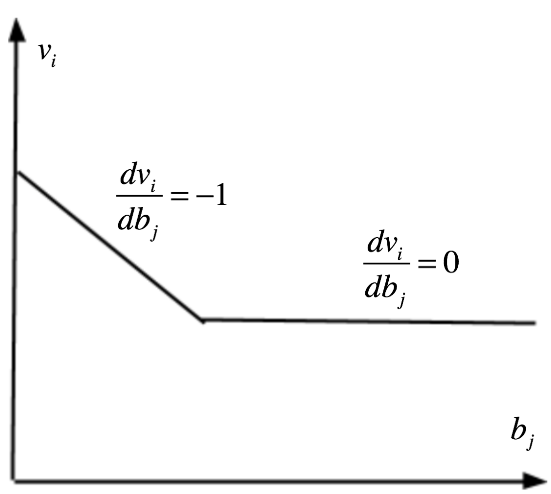

Figure 4.3: Case 2: $\frac{dv_j}{db_i}$

Theorem 4.0.4. *Let*

$$S = \{\mathbf{x} : A\mathbf{x} \leq \beta\} \subset \mathbb{R}^n$$

$$\mathbf{v}(t) = \mathbf{a} + \mathbf{b}t$$

$$\mathbf{x}(t) = \Pi_S(\mathbf{v}(t))$$

For any $\mathbf{x} \in S$*, let* $C(\mathbf{x}) =$ *set of constraints that* $\{i : \sum_j a_{ij}x_{ij} = \beta_j\} \subset [m]$*. Suppose* $\mathbf{x}_1 = \Pi_S(\mathbf{v}_1), \mathbf{x}_2 = \Pi_S(\mathbf{v}_2)$ *and* $C(\mathbf{x}_1) = C(\mathbf{x}_2)$*, then for* $0 \leq \lambda \leq 1$*,*

$$\Pi_S(\mathbf{v}_\lambda) = \mathbf{x}_\lambda \text{ where } \mathbf{v}_\lambda = (1-\lambda)\mathbf{v}_1 + \lambda\mathbf{v}_2 \text{ and } \mathbf{x}_\lambda = (1-\lambda)\mathbf{x}_1 + \lambda\mathbf{x}_2$$

Note that dimensions are as follows: $A : m \times n; \mathbf{x}, \mathbf{v}, \mathbf{a}, \mathbf{b} : n \times 1; \beta : m \times 1$.

Proof. According to KKT conditions for the optimization problem defining $\Pi_S(\mathbf{v}) = \mathbf{x}$, there exists a vector of Lagrange Multipliers μ such that

$$\mu \geq 0, \mu \in \mathbb{R}^m \tag{4.1}$$

$$\mu^T(A\mathbf{x} - \beta) = 0 \tag{4.2}$$

$$A\mathbf{x} - \beta \leq 0 \tag{4.3}$$

$$\mathbf{x} = \mathbf{v} - A^T\mu \tag{4.4}$$

Starting with the KKT conditions for $\mathbf{v}_1$ and $\mathbf{v}_2$, we show that the KKT conditions for $\mathbf{v}_\lambda$ are satisfied. It's easy to verify Equation (4.1), Equation (4.3) and Equation (4.4) by taking $\mu_\lambda = (1-\lambda)\mu_1 + \lambda\mu_2$. Since we assume that $C(\mathbf{x}_1) = C(\mathbf{x}_2)$ and $\mathbf{x}_\lambda = (1-\lambda)\mathbf{x}_1 + \lambda\mathbf{x}_2$, we have $C(\mathbf{x}_\lambda) = C(\mathbf{x}_1) = C(\mathbf{x}_2)$. As μ_λ is a linear combination of μ_1 and μ_2, we know that $\mu_\lambda \geq 0$ holds as well. For $i \notin C(\mathbf{x}_\lambda) = C(\mathbf{x}_1) = C(\mathbf{x}_2)$, we have $\mu_1(i) = \mu_2(i) = 0$. Thus for $i \notin C(\mathbf{x}_\lambda) = C(\mathbf{x}_1) = C(\mathbf{x}_2)$, it is true that $\mu_\lambda(i) = 0$. In such a way we veirfy the complementary slackness for μ_λ and thus Equation (4.2) is verified. In summary, the KKT conditions for $\mathbf{v}_\lambda$ are satisfied and thus $\Pi_S(\mathbf{v}_\lambda) = \mathbf{x}_\lambda$. □

Lemma 4.0.5. *In general, one winner's bidding price and the corresponding payment price are piecewise linear in change for other bidders' bidding prices fixed.*

Proof. Theorem 4.0.4 obviously implies that a single winner's payment is piecewise linear in his/her own bidding price for other bidders' bidding prices fixed. A geometric explanation is as follows. We can divide the constraint set into different parts where either the payment is linear to the bidding price or payments will be fixed however the bidding price changes between those linear regions. Thus in a nutshell, one winner's payment is in piecewise linear in the bidding price for other bids fixed. □

CHAPTER 5

CONCLUSION

By investigating the simulation results, we believe that the following conjectures are true in general.

1. MID ≤ 1
2. If one winner's Vickrey price decreases due to an increase in another bidder's bidding price, that winner's payment will decrease as well.
3. In general, all the winners' payments and one winner's bidding price are piecewise linear in change for other winners' bidding prices fixed.

For future work, one possible direction is to make the most of the connections between bidding prices, Vickrey prices, and payments. Otherwise, in the process of turning the payment determination problem into a quadratic minimization problem, we lose key information concerning the connection. Another possible way is to extend the star network to a general network by doing mathematical inductions. Since all of the propositions that we have in Chapter 4 are based on the assumption that bidders are single-minded, we can consider those propositions for the case where bidders are allowed to bid for a bundle of items. In addition, we can introduce the core projection mechanism into areas of other resource allocation problems.

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