ENSEMBLE CONTROL OF HAMILTONIAN SYSTEMS

BY

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THESIS

Submitted in partial fulfillment of the requirements
for the degree of Master of Science in Electrical and Computer Engineering
in the Graduate College of the
University of Illinois at Urbana-Champaign, 2015

Urbana, Illinois

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Abstract

This thesis aims to demonstrate the conditions under which a particular kind of ensemble control system is controllable. Previous studies on this topic have tended to make simplifying assumptions on the systems in view, and this paper attempts to remove some of those assumptions. Even in a non-linear ensemble system, it can be demonstrated that controllability is achievable under some circumstances. This controllability is limited, however, by the Hamiltonian preservation of area in the position-velocity plane. At the end of this thesis, some example systems will be simulated, demonstrating the concepts of the paper and showing that the controllability result is sound.

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1 Introduction: Ensemble Control

Ensemble control is a term used to describe a particular class of controllable systems. In ensemble control, the same input is used to control multiple elements simultaneously. Often these have similar system dynamics. The group of these elements is called the ensemble.

A formal definition of ensemble controllability as given in [1] is as follows.

Definition 1

Consider a family of control systems

$$\dot{X}(t,s) = F(X(t,s), u(t), t, s)$$

$$X \in M \subset \mathbb{R}^n, s \in D \subset \mathbb{R}^d, u \in U \subset \mathbb{R}^m$$

where F is a smooth function of its arguments and D is a compact subset of $\mathbb{R}^d$. This family is called *ensemble controllable* in the function space $L_\infty(D, M)$ if and only if for all $\epsilon > 0$ and for all $g, X_0 \in L_\infty(D, M)$ there exists $T > 0$ and an open-loop piecewise-continuous control $u : [0, T] \to U$ such that starting from any initial state $\begin{cases} X_0 = X(0,s) \\ s \in D \end{cases}$, the final state $\begin{cases} X_T(s) = X(T,s) \in L_\infty(D,M) \\ s \in D \end{cases}$ satisfies $\|X_T - g\|_\infty \leq \epsilon$.

The following example is given to demonstrate this definition.

Simple Example

Consider a collection of points on the real line distributed uniformly from 0 to 1. The goal of this example is to move all such points to 2 with the same input signal. The system dynamics are as follows:

$$x(0, s) = s$$

$$\dot{x}(t, s) = u$$

Choosing $u = (-x + 2)$ would clearly give a velocity towards the desired destination, which grows smaller as the points get closer to the destination. Figure 1 is a graph of 10 such points over time with such an input selection.

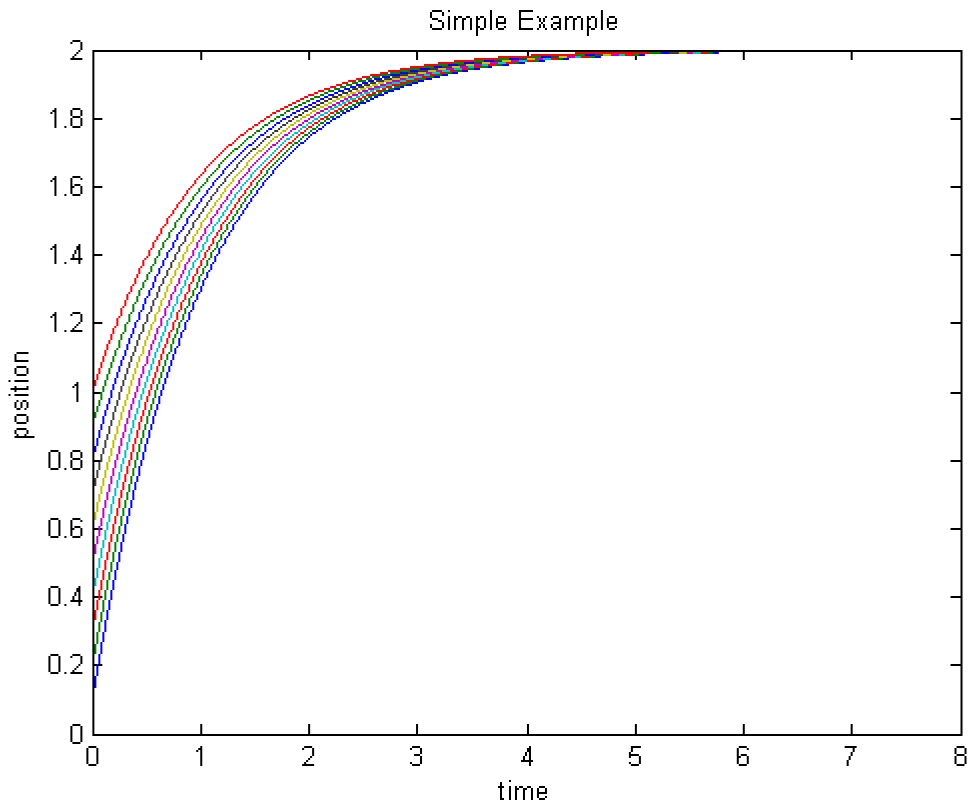

Figure 1: Simple Example Position Over Time

Convergence to a very small neighborhood of 2 happens for all points within about 6 seconds. This convergence to an arbitrary point demonstrates that this system is ensemble controllable.

1.1 Previous Work: Bilinear Systems

Bilinear ensemble control deals with the system with the following form:

$$\dot{x}(t,s) = (A(s) + \sum_i u_i(t)B_i(s))x(t,s) \quad (1)$$

where A is the $n \times n$ matrix of the internal system characteristics, and B_i is the $n \times n$ matrix multiplying the ith input. Here, s is the ensemble parameter, which represents slight variations in the system dynamics for different elements of the ensemble. For example, these system variations correspond to slight variations in initial position within a non-ensemble system. Previous papers such as [1] have dealt with the controllability of such systems. This typically means asking whether a system can be forced to move from one state to another by choosing the correct input.

One specific example of a bilinear system is the set of rotations around the origin, known as SO(n), where n is the dimension of x. Theorem 1 in [1] shows that such an example is ensemble controllable.

Theorem 1 [1]: Let $s \in D = [a, b] \subset \mathbb{R}^+$. Let S(D) denote the set of all SO(3) valued continuous functions on D, where SO(3) is the set of all rotations around the origin of $\mathbb{R}^3$. Then an ensemble of bilinear control systems as in (2) is ensemble controllable on S(D).

$$\dot{X}(t,s) = s[u\Omega_y + v\Omega_x]X(t,s), X(0,s) = I \quad (2)$$

In more general ensemble control, $A(s)$ and $B_i(s)$ from (1) become $f(x, s)$ and $g_i(x, s)$, where f and g are vector functions (not restricted to SO(3)). The topic of this thesis is whether such a non-linear system is ensemble controllable.

1.2 Previous Work: Steering Robots

Further previous work has shown [2] that many ensemble systems are possible to analyze thoroughly and can produce practical results. In [2], however, some assumptions are made to simplify the situation and to make the system an accurate description of a real-life situation (specifically steering robots). Following is the system being used in this paper:

$$\dot{x_i} = \epsilon_i u_1(t) cos(\theta_i(t))$$
$$\dot{y_i} = \epsilon_i u_1(t) sin(\theta_i(t))$$
$$\dot{\theta}_i = \epsilon_i u_2(t)$$
$$u_1(t) = -\frac{1}{n}\sum_{i=1}^{n}(x_i cos(\theta_i(t)) + y_i sin(\theta_i(t)))$$
$$u_2(t) = 1$$

Note that the situation is almost linear, since it turns out that the one time-dependent variable on the right side of the differential equation (θ_i) is exactly $\epsilon_i t$, and is thus known. This allows for a Lyapunov function to be chosen for x_i and y_i that shows the system to be stable under the inputs shown above.

Using such an input allows all the robots, despite the variation in their parameters, to be steered to the origin simultaneously. An interesting note on the implementation of this system is that it was simulated in Matlab using a discrete-time model. This model alternates between two sets of inputs that reflect the idea that the robots are both turning and being moved forward by the control policy. This ″back-and-forth″ method of control is very useful for the purposes of this paper as well. Alternating between using one type of control and another allows for a simple policy that is capable of steering ensembles to a desired point.

In this thesis, the controls used in this ″back-and-forth″ fashion will be somewhat different. Rather than alternating between rotation and translation, we will alternate between an acceleration towards the desired point (which naturally separates the elements of the ensemble) and an acceleration which brings the elements back together near the desired point.

Other similarly real-world-inspired linear systems have been researched with just as robust and effective solutions. Two examples are a unicycle under bounded perturbation [3] and a plate-ball system under bounded perturbation [4]. These examples are interesting because they demonstrate that ensemble control is also capable of modeling the uncertainty in parameters. From this perspective, an ensemble controllable system is one that is controllable under a certain amount of perturbation of the system.

2 Nonlinear Ensemble Controllability

As stated in the proof of Theorem 1 from [1], if we can synthesize generators of orders of s, then we can produce an evolution of the form

$$R_x(s) = exp(c_0 sA)exp(c_1 s^2 A)...exp(c_n s^n A)$$

$$= exp(\sum_{k=0}^{n} c_k s^k A)$$

For instance,

$$ad^{2k}_{s\Omega_y}(s\Omega_x) = (-1)^k s^{2k+1}\Omega_x$$

$$R_x(s) = exp(\sum_{k=0}^{n} c_k s^{2k+1}\Omega_x)$$

Using the Stone-Weierstrass Theorem, we can show that this evolution can be approximately generated with a polynomial function of s. In the case above, $P(s) = \sum_{k=0}^{n} c_k s^{2k+1}$ is such a polynomial function.

Note: In the case of the examples in this research, A is a non-linear function of the system variables, and the approximation is not with a polynomial function of s, but rather with a polynomial in s and x (two variables). However, the result in [1] can be extended to this case using the Stone-Weierstrass theorem.

Stone-Weierstrass Theorem

If X is any compact space, let A be a subalgebra of the algebra C(X) over the reals $\mathbb{R}$ with binary operations + and $\times$. Then, if A contains the constant functions and separates the points of X (i.e., for any two distinct points x and y of X, there is some function f in A such that f(x)≠f(y)), A is dense in C(X) equipped with the uniform norm. [5]

This Theorem is a generalization of the Weierstrass approximation theorem:

Weierstrass Approximation Theorem

If f is a continuous, real-valued function defined on some real interval, for every $\epsilon > 0$ there exists a polynomial p(x) such that for all x in the real interval, $|f(x) - p(x)| < \epsilon$. [6]

The Stone-Weierstrass theorem can be used to prove that any continuous real-valued function in two variables can be approximated by a polynomial in two variables, since this is also a subalgebra of the algebra in question. In other words, the Weierstrass approximation theorem holds for two dimensions, x and v.

Theorem

The system:

$$\begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix} = f(x, v, s) + u_1 g(x, s) + u_2 h(x, s)$$

is ensemble controllable if:

1) Lie brackets of combinations of f, g, and h can generate two linearly

independent vectors of polynomials in s and x.

2) These polynomials (given enough brackets) can be generated for powers of s and x up to infinity.

Proof

By successive Lie brackets of f, g, and h, we obtain generators of the type $s^k x^k A(x,s)$, which leads us to the evolution

$$R_x(s) = exp(c_0 sxA)exp(c_1 s^2 x^2 A)...exp(c_n s^n x^n A)$$

$$= exp(\sum_{k=0}^{n} c_k s^k x^k A)$$

From the Stone-Weierstrass theorem, there exists a $P(x,s) = \sum_{k=0}^{n} c_k s^k x^k A$ such that $|P - R_x| < \eta, \quad \forall \eta > 0$. Thus the vector polynomial that we generate with successive Lie brackets of f,g, and h is sufficient to approximate the evolution of the system arbitrarily well.

It should be noted that these conditions are sufficient, but not necessary, thus the lack of "only if" in the theorem statement. Necessary conditions for the controllability of non-linear systems are subjects of current research. [7-8]

Since the above system is a two-dimensional system of position and velocity, it has certain properties that will now be discussed.

3 Hamiltonian Systems

An important notion in control is that of a Hamiltonian system. This is a system in which the state is described by two vectors p and q, often representing momentum and position, respectively. Of course, if unit mass is assumed, momentum and velocity are the same. Such a system can be completely described by the Hamiltonian scalar function H(q,p,t). In physics, this Hamiltonian often represents energy, since energy is a function of position and velocity, and energy is conserved. In general, H(q,p,t) is defined by the equations:

$$\dot{q} = \partial H / \partial p, \quad \dot{p} = -\partial H / \partial q$$

Hamiltonian systems have many interesting properties, but the most important one to the topics of this paper is the symplectic structure. This refers to the fact that the evolution of p over time depends only on the change of the Hamiltonian with respect to q, and vice versa.

Consequently, the phase-space volume of the system is preserved [9]. This result is known as Liouville's theorem [10]. For instance, bringing a number of particles which are spread out in position and close together in velocity towards the same position will cause their velocities to spread out, assuming said particles do not form a single line through phase space.

The notion of a Hamiltonian system drives the example below, which is a particular non-linear ensemble control system that happens to be Hamiltonian. In the simulations of this system, it ought to be clear that it is Hamiltonian.

Example 1

We will investigate the ensemble controllability of the following example system:

$$\begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix} = 1/s \begin{pmatrix} v \\ 0 \end{pmatrix} + u_1 \begin{pmatrix} 0 \\ e^{sx} \end{pmatrix} + u_2 \begin{pmatrix} 0 \\ sx \end{pmatrix}$$

$s \in (0, 2)$

Let the three vectors on the right side of the equation be known as h_1 , h_2 and h_3 respectively. Note that h_1 carries with it a coefficient of $1/s$.

Taking the Lie bracket of the first vector with the second and third vectors, we get:

$$1/s[h_1, h_2] = 1/s \begin{pmatrix} -e^{sx} \\ sve^{sx} \end{pmatrix}$$

$$1/s[h_1, h_3] = 1/s \begin{pmatrix} -sx \\ sv \end{pmatrix}$$

Let the vectors on the right side of these equations be known as h_4 and h_5 respectively. Note that both carry with them a coefficient of $1/s$.

In order to show that this system can generate the entire 2-dimensional space for an evolution in the x variable, and that it is thus ensemble controllable for x, it suffices to show that two linearly independent vectors of polynomials in s can be generated for any power of s up to infinity. This conclusion can be drawn from the Stone-Weierstrass theorem.

To this end, we now take successive Lie brackets of h_4 and h_5.

$$1/s^2[h_4, h_5] = 1/s^2 \begin{pmatrix} sE - s^2xE \\ s^2xvE \end{pmatrix}$$

where E is e^{sx}.

It is apparent that the power of s in the x-coordinate of the vector has increased by 1 from h_4 to $[h_4, h_5]$. Thus we could guess that the highest order term of the x-coordinate polynomial will be $-s^{n-1}x^nE$.

Lemma 1: The highest order term in the x-coordinate of $ad^n_{1/sh_4}(1/sh_5)$ is $-s^{n-1}x^nE$.

Basis Step:

$$h_4 = 1/s \begin{pmatrix} -e^{sx} \\ sve^{sx} \end{pmatrix}$$

This checks out with the formula above for n=0.

Induction Step:

$$[-s^{n-1}x^nEdx, -xdx]$$

$$= (s^{n-1}x^nE - s^nx^{n+1}E - s^{n-1}x^nE)dx$$

$$= -s^n x^{n+1} E dx$$

Note that the first item in the bracket is simply our induction assumption that the formula holds for n. The second item in the bracket is the x-coordinate of h_5. The result is that the formula holds for n+1. Thus the formula for the highest-order term is proven by induction.

From Lemma 1, we have one of the two polynomial vectors in s that we need in order to show ensemble controllability in x. The other can be obtained by starting with h_4 and alternating between h_3 and $1/sh_1$.

$$1/s[h_4, h_3] = 1/s \begin{pmatrix} 0 \\ -sE - s^2 xE \end{pmatrix}$$

$$1/s^2[[h_4, h_3], h_1] = 1/s^2 \begin{pmatrix} sE + s^2 xE \\ -2s^2 vE - s^3 xE \end{pmatrix}$$

$$1/s^2[[[h_4, h_3], h_1], h_3] = 1/s^2 \begin{pmatrix} 0 \\ s^4 x^2 E + 3s^3 xE + s^2 E \end{pmatrix}$$

The power of epsilon in the v-coordinate increases by one each time h_3

enters into the bracket. Thus we could guess that the highest order term of the v-coordinate polynomial is $(-1)^n s^n x^n E$, where n is the number of times h_3 appears in the bracket.

Lemma 2: The highest order term in the v-coordinate of the given alternation is $(-1)^n s^n x^n E$, where n is the number of times h_3 appears in the bracket.

Basis Step (n=1):

$$1/s[h_4, h_3] = 1/s \begin{pmatrix} 0 \\ -sE - s^2 xE \end{pmatrix}$$

This clearly fits the formula, since we derived the formula from it.

Induction Step:

$$[(-1)^n s^n x^n E dv, 1/s h_1] = 1/s h_6$$

where

$$h_6 = \begin{pmatrix} (-1)^n s^n x^n E \\ n(-1)^{n+1} s^n x^{n-1} vE + (-1)^{n+1} s^{n+1} x^n vE \end{pmatrix}$$

$$[h_6, h_3] = 1/s \begin{pmatrix} 0 \\ Ax^n E + (-1)^{n+1} s^{n+2} x^{n+1} E \end{pmatrix}$$

where

$$A = (-1)^n s^{n+1} + n(-1)^{n+1} s^{n+1}$$

The last term of the v-coordinate polynomial is exactly the formula of the lemma for n+1. Thus the formula for the highest-order term is proven by induction.

From Lemma 1 and Lemma 2, we have two independent vectors of polynomials in s. Further, these vectors can have arbitrarily high orders of s and x. By the Stone-Weierstrass theorem and the argument above, we can generate the entire two-dimensional space for an evolution, and thus the system is ensemble controllable.

4 Simulation 1: Fixing Position

The first simulation of the system will use the equation from Example 1. This equation is reproduced below:

$$\begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix} = 1/s \begin{pmatrix} v \\ 0 \end{pmatrix} + u_1 \begin{pmatrix} 0 \\ e^{sx} \end{pmatrix} + u_2 \begin{pmatrix} 0 \\ sx \end{pmatrix}$$

The initial position is close to zero and the initial velocity is zero for all particles. This is simpler than having exactly zero initial position, since one of the vectors multiplying the controls is sx, which would be zero if x was zero. Moving the particles slightly away from zero could be accomplished with the $exp(sx)$ vector.

The goal of the first simulation is to bring all the particles to a non-zero position simultaneously, regardless of velocity.

This simulation went back and forth between u_1 being zero and u_2 being zero at different time intervals, to demonstrate the difference between how the two inputs modify the system.

We begin with a positive control for u_1, which separates the particles slightly. Then we use a positive control for u_2, which moves all the particles forward without causing them to separate very much. Finally, we use a negative control for u_2, which moves the particles back towards the origin. The particles all move simultaneously to a position of -1, but have different velocities. This happens to occur at time 56.46. Equations for u_1 and u_2 are shown below:

$$u_1 = \begin{cases} 1/t : t \in [0, 25] \\ 0 : t \in [25, 50] \\ -50/t : t \in [50, 56.46] \end{cases}$$

$$u_2 = \begin{cases} 0 : t \in [0, 25] \\ 1/45 : t \in [25, 50] \\ 0 : t \in [50, 56.46] \end{cases}$$

One way of understanding this method is that u_2 shifts the equilibrium point of the particles (the point where they can all be at the same position), while u_1 brings the particles closer together or farther apart.

Eight snapshots of the particles at different times are shown in Figure 2. These snapshots are at times 0, 15, 25, 40, 45, 51, 55, and 56.46 respectively. In addition, a trace of the position (Figure 3) and velocity (Figure 4) over time of a few of the particles is given. An animation that shows many more time instants in a row is available at [11].

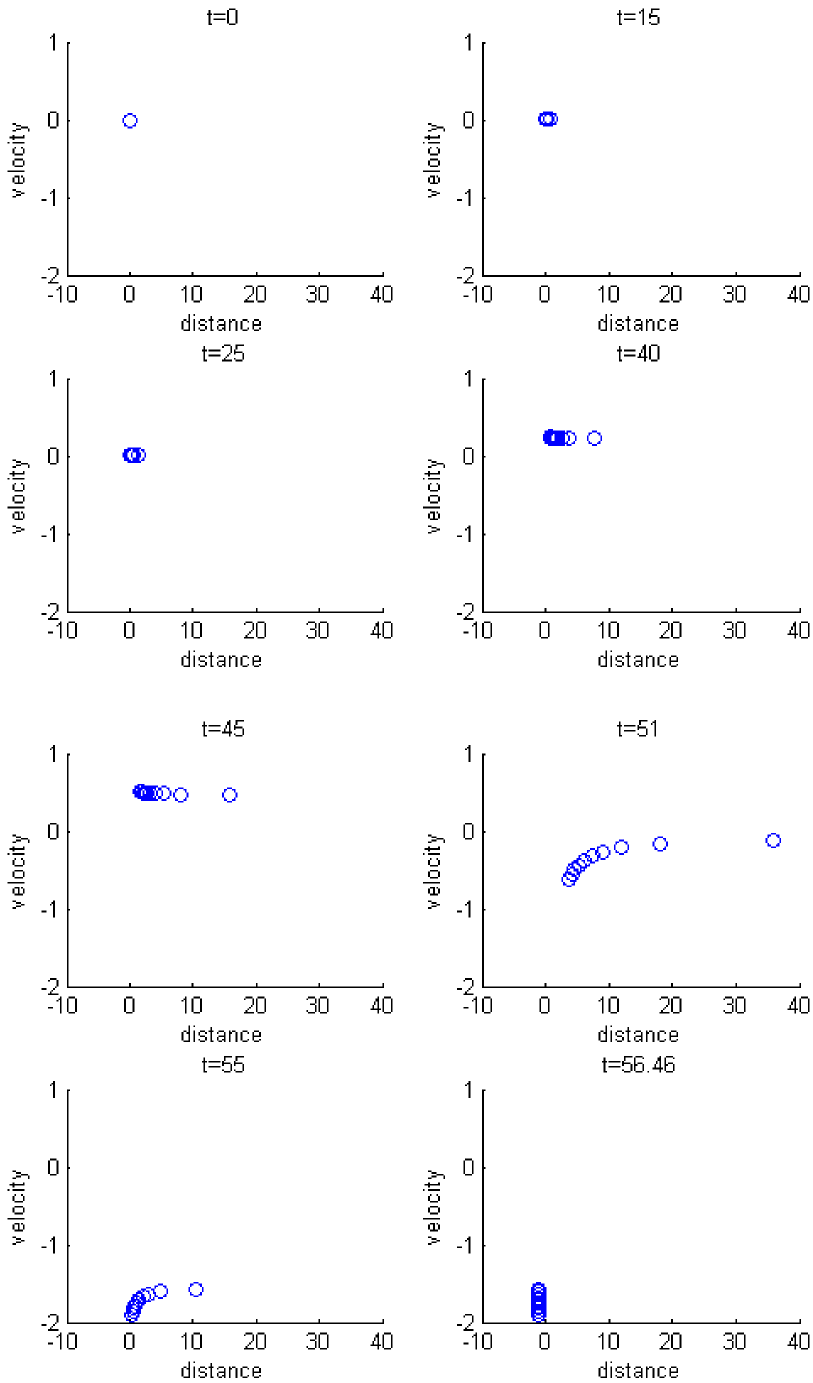

Figure 2: Example 1 Simulation Snapshots

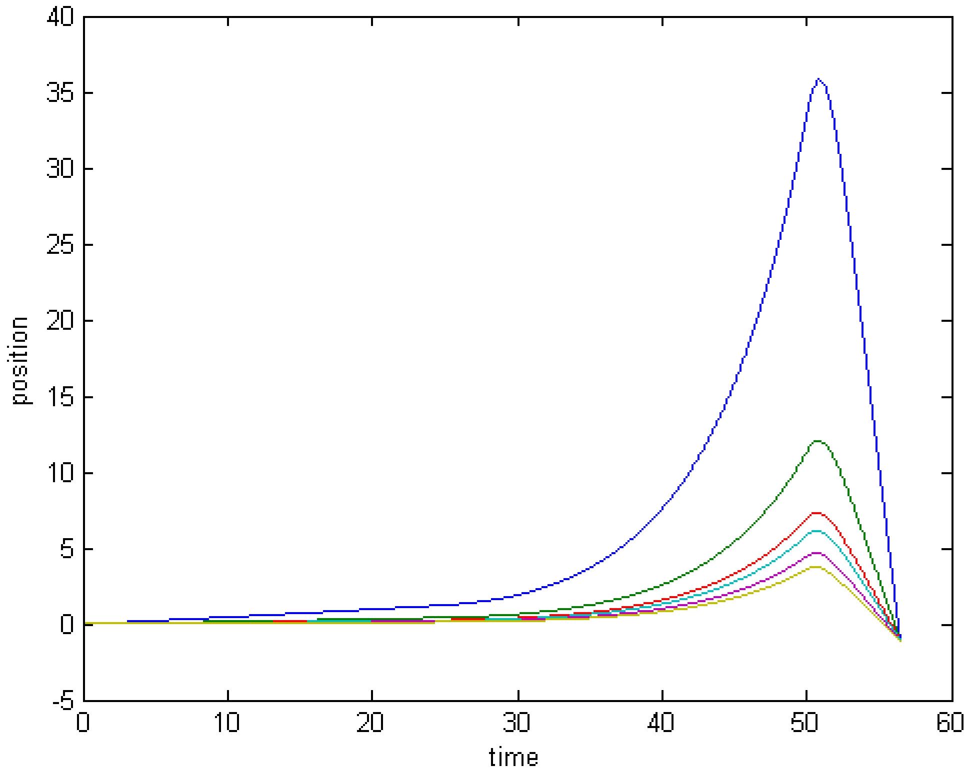

Figure 3: Position Trace for Example 1

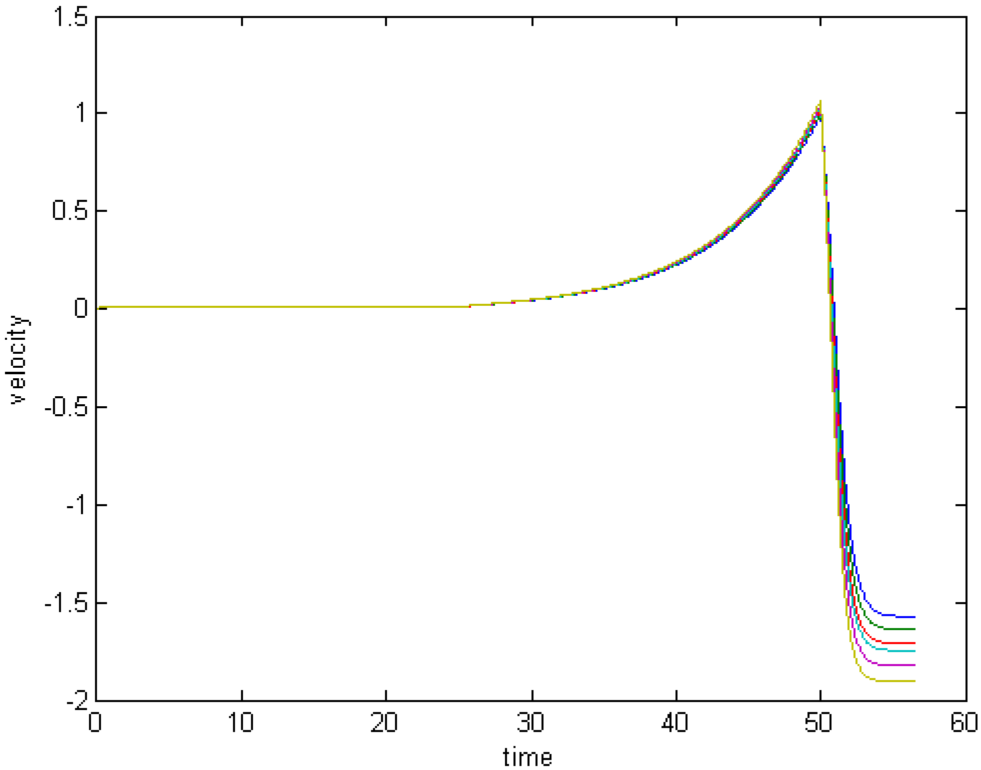

Figure 4: Velocity Trace for Example 1

5 Hamiltonian Area Conservation

As stated in chapter 3, Hamiltonian systems (e.g. Example 1) have the property of conservation of area in the position-velocity plane. If we were to introduce to a system such as Example 1 a second variation parameter which would affect the velocity (let us call it "r"), the system would have a non-zero phase-space volume (area). Thus, from the fact that this volume is conserved, we know that the system is no longer ensemble controllable, since there are limitations to the position and velocity of the ensemble.

Example 2

$$\begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix} = 1/s \begin{pmatrix} rv \\ 0 \end{pmatrix} + u_1/r \begin{pmatrix} 0 \\ e^{sx} \end{pmatrix} + u_2/r \begin{pmatrix} 0 \\ sx \end{pmatrix}$$

Here, r is defined exactly how s was defined previously. Such a system is not ensemble controllable, because the area created by having two-dimensional variation cannot be reduced or increased. Thus determining the position and velocity of all ensemble particles to an arbitrarily small degree of error is impossible.

6 Simulation 2: Fixing Position and Velocity

The second simulation of the system will use the dual identity of ensemble systems in which a variation in the system (given by parameter s) is equivalent to a variation in initial position or velocity (but not both). Thus, the system being simulated has no s parameter, but is otherwise the same as Example 1. The equation is shown below:

$$\begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} v \\ 0 \end{pmatrix} + u_1 \begin{pmatrix} 0 \\ e^x \end{pmatrix} + u_2 \begin{pmatrix} 0 \\ x \end{pmatrix}$$

We start the particles at 0 position and various velocities, and move them simultaneously arbitrarily close to 0 position and 0 velocity. In order to do this, all we must use for a control is some piecewise continuous u_2 which slows the particles down while moving them towards the origin. This can be accomplished by any negative control. With the particular controls chosen, the particles come close to the origin at time 46.5. Equations for u_1 and u_2 are shown below:

$$u_1 = 0$$

$$u_2 = \begin{cases} -10/sqrt(t) : t \in [0, 25] \\ -1/45 : t \in [25, 32] \\ -1 : t \in [32, 40.65] \\ 0 : t \in [40.65, 46.5] \end{cases}$$

Also in the simulation are six other groups of particles which are similarly spread out in velocity, but start at different initial positions. Each color of particles is a different group that starts at a separate initial position from the other groups. This difference in initial position represents a second variation parameter (index) which would cause the system to have two-dimensional variations. This makes the system no longer ensemble controllable. These new groups of particles also move to near-zero velocity, but their final positions are widely different from those of the central group of particles. This difference demonstrates graphically the Hamiltonian preservation of area in the position-velocity plane.

Four snapshots of the particles at different times are shown in Figure 5. These snapshots are at times 0, 25, 45, and 46.5 respectively. A trace of the position (Figure 6) and velocity (Figure 7) over time of a few of the particles from the center group is shown as well. A n animation that shows many more time instants in a row is available at [11].

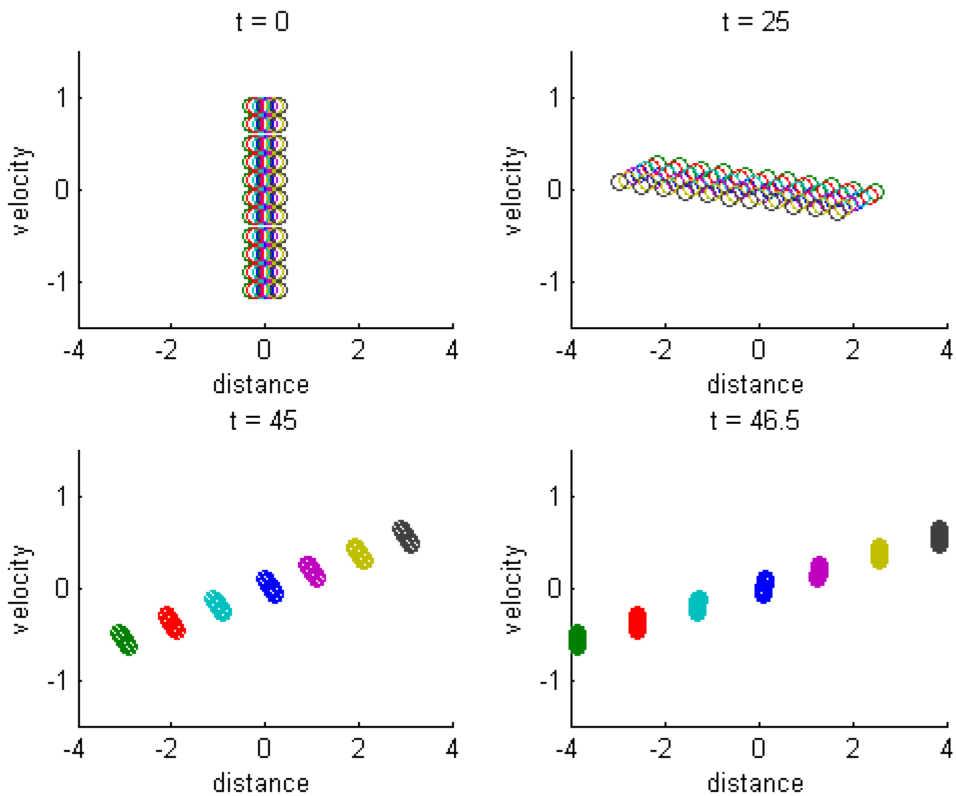

Figure 5: Snapshots of Simulation with Initial Velocity

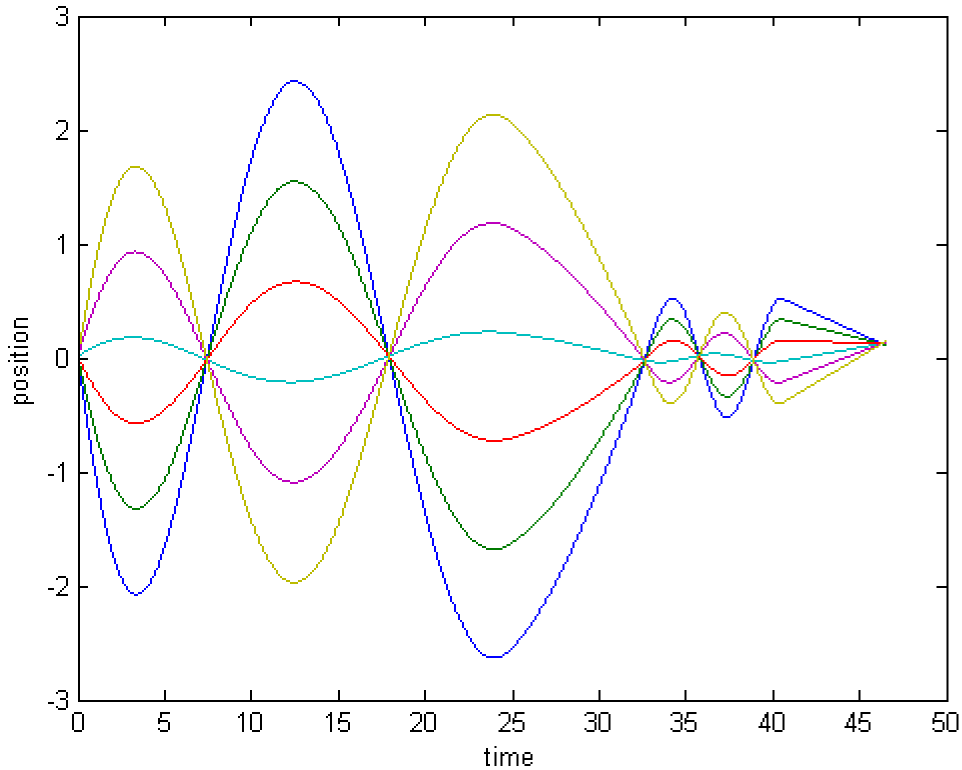

Figure 6: Position Trace with Initial Velocity

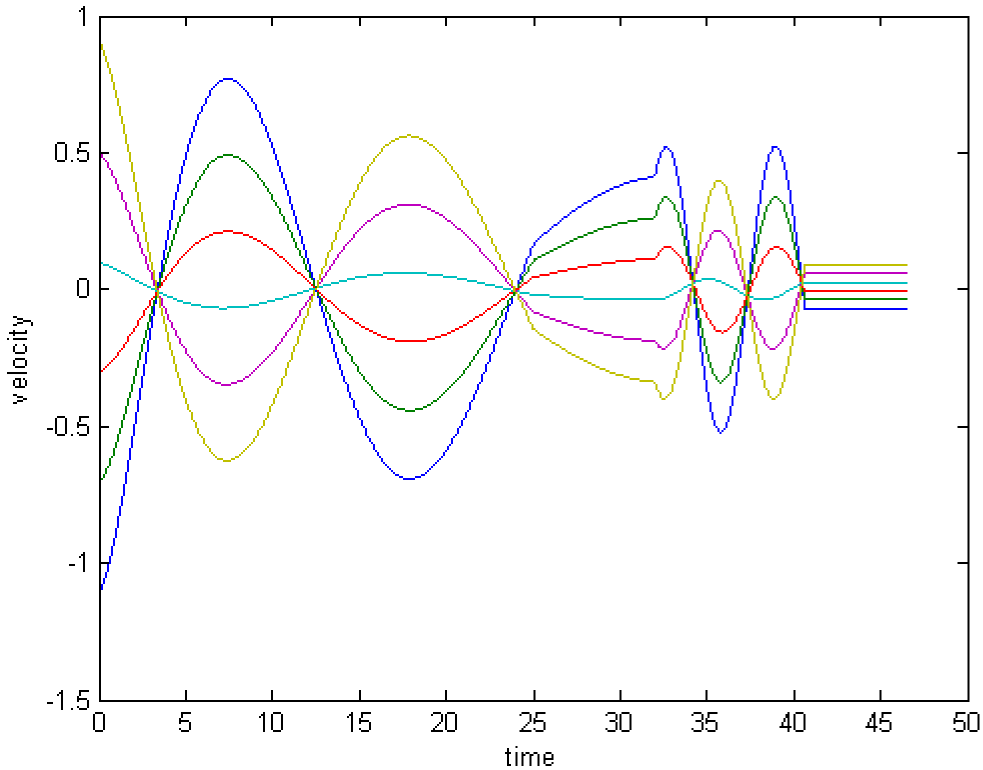

Figure 7: Velocity Trace with Initial Velocity

7 Simulation 3: A Simple Four-Dimensional System

It can be assumed that the same reasoning would apply in a situation with two dimensions of position and two dimensions of velocity. In fact, this is the case. For simplicity, let us assume that one can provide acceleration in these two dimensions separately and that the system is the same for both dimensions. Specifically, let us use a system with characteristics similar to, but more complex than, those of Example 1.

Example 3

$$\begin{pmatrix} \dot{x} \\ \dot{v_x} \end{pmatrix} = 1/s \begin{pmatrix} v_x \\ 0 \end{pmatrix} + u_{x1} \begin{pmatrix} 0 \\ sinh(sx) \end{pmatrix} + u_{x2} \begin{pmatrix} 0 \\ s^2x^2 \end{pmatrix}$$

$$\begin{pmatrix} \dot{y} \\ \dot{v_y} \end{pmatrix} = 1/s \begin{pmatrix} v_y \\ 0 \end{pmatrix} + u_{y1} \begin{pmatrix} 0 \\ sinh(sy) \end{pmatrix} + u_{y2} \begin{pmatrix} 0 \\ s^2y^2 \end{pmatrix}$$

Such a system can be controlled in a similar fashion for each dimension separately to create the desired effect in two dimensions at once.

For instance, we may go from the point (1,1) to the point (-2,2) by finding a set of controls that will go in one dimension from 1 to -2 and another set that will go from 1 to 2 in the same amount of time. The following controls accomplish this:

$$u_{x1} = \begin{cases} -100e^{-t} : t \in [0, 10] \\ 0 : t \in [10, 50] \end{cases}$$

$$u_{x2} = \begin{cases} 0 : t \in [0, 10] \\ -1/t : t \in [10, 40] \\ 100/t : t \in [40, 49.7] \\ -10000/t : t \in [49.7, 50] \end{cases}$$

$$u_{y1} = 0$$

$$u_{y2} = \begin{cases} 0 : t \in [0, 19] \\ -20/t : t \in [19, 23] \\ 0 : t \in [23, 42] \\ -60/t : t \in [42, 49] \\ 800/t : t \in [49, 50] \end{cases}$$

Four snapshots of the particles at different times are shown in Figure 8. These snapshots are at times 0, 10, 40, and 49.85 respectively. The same snapshots are given with the surrounding groups of particles included in Figure 9. An animation that shows many more time instants in a row is available at [11].

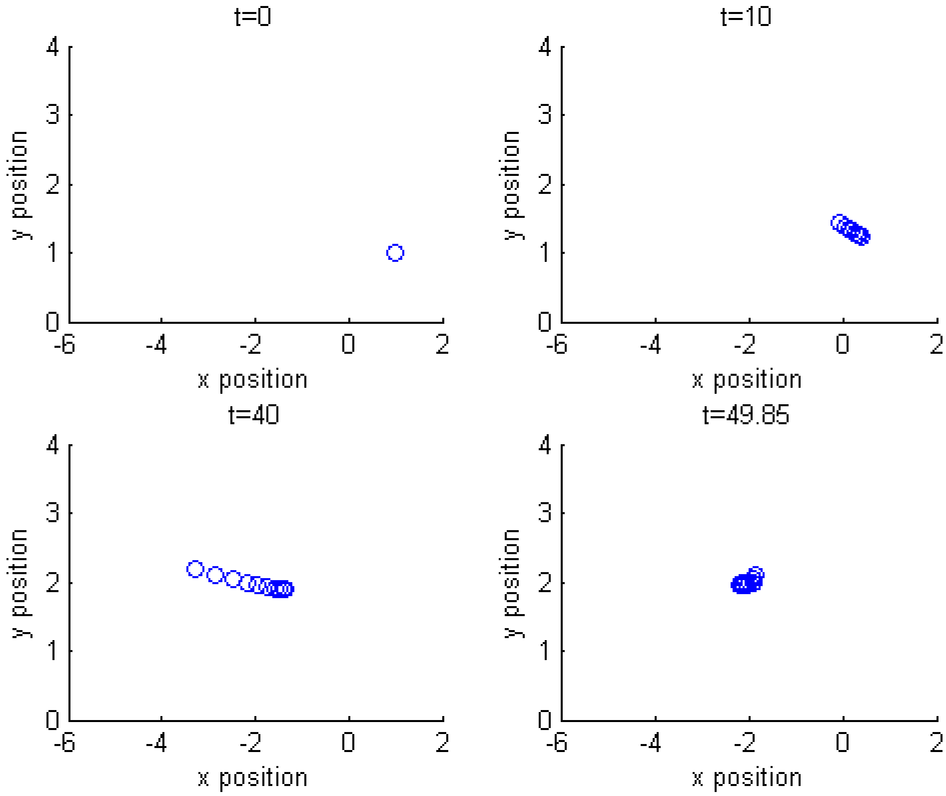

Figure 8: Snapshots of Simulation of X/Y System

As before, we may surround these particles with other groups of particles to observe the Hamiltonian preservation of area.

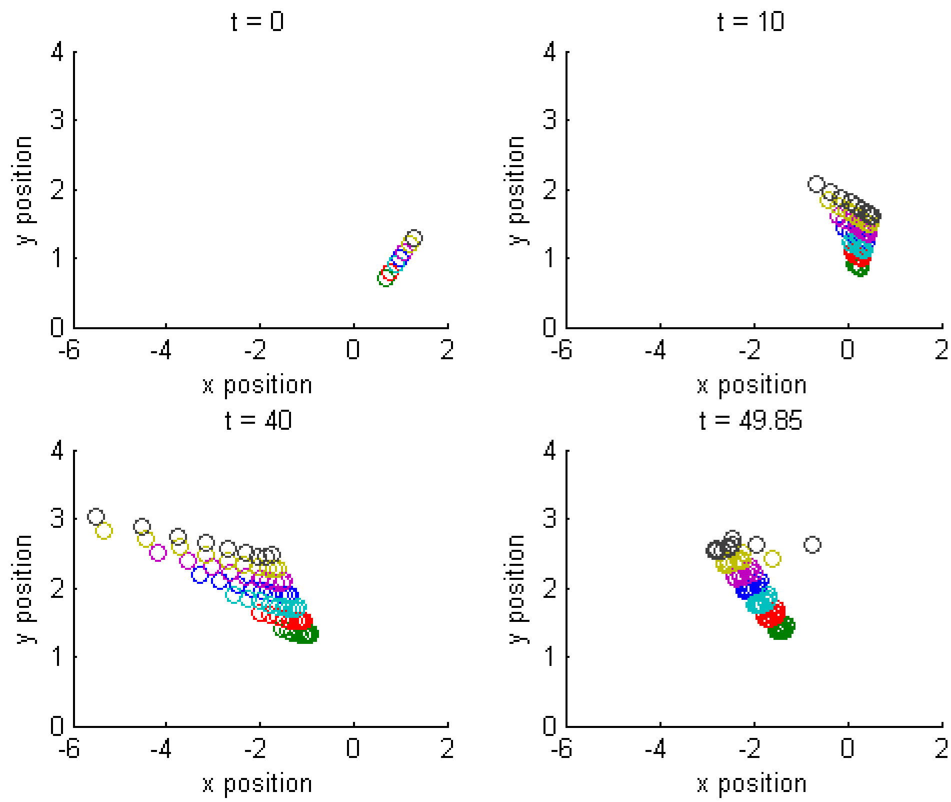

Figure 9: Snapshots of Simulation of Surrounded X/Y System

It is not as immediately clear that area has been preserved, since the variation parameter s has not been removed, but notice that in the final frame, the sets of particles have moved farther apart from each other than they started. The particles went from a line of length .85 to something resembling a line of length greater than 1.5. In addition, a couple of the groups of particles have had some of the particles break off from the group. Thus in this case as well, the inputs that cause one group of particles to behave as desired will not do so for small variations in initial position. In fact, these small variations will cause much larger variations in outcome.

8 Conclusions

In this paper, it has been shown that there is a method of analysis by which we are able to show that certain Hamiltonian ensemble control systems given by (3) are completely controllable.

$$dx(t,s)/dt = (A(s) + \sum_i u_i(t)B_i(s))x(t,s) \quad (3)$$

Thus in some cases, we are able to take a group of individual objects governed by similar, but varied, differential equations, and with a uniform input, cause them all to behave exactly how we wish with a piecewise continuous control. Likewise, we are able to take a group of individual objects with precisely the same differential equations, but varied initial positions (or velocities, but not both), and produce the desired behavior with a piecewise continuous control.

On the other hand, in a two-dimensional case, a system varied with two parameters cannot be controlled the same way. It is impossible for such a system to be completely controllable, since it will have some inherent area in the position-velocity plane which cannot be modified.

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