EXTREMAL PROBLEMS IN DISJOINT CYCLES AND GRAPH SATURATION

BY

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DISSERTATION

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Abstract

In this thesis, we tackle two main themes: sufficient conditions for the existence of particular subgraphs in a graph, and variations on graph saturation.

Determining whether a graph contains a certain subgraph is a computationally difficult problem; as such, sufficient conditions for the existence of a given subgraph are prized. In Chapter 2, we offer a significant refinement of the Corrádi-Hajnal Theorem, which gives sufficient conditions for the existence of a given number of disjoint cycles in a graph. Further, our refined theorem leads to an answer for a question posed by G. Dirac in 1963 regarding the existence of disjoint cycles in graphs with a certain connectivity. This answer comprises Chapter 3.

In Chapter 4 we prove a result about equitable coloring: that is, a proper coloring whose color classes all have the same size. Our equitable-coloring result confirms a partial case of a generalized version of the much-studied Chen-Lih-Wu conjecture on equitable coloring. In addition, the equitable-coloring result is equivalent to a statement about the existence of disjoint cycles, contributing to our refinement of the Corrádi-Hajnal Theorem.

In Chapters 5 and 6, we move to the topic of graph saturation, which is related to the Turán problem. One imagines a set of n vertices, to which edges are added one-by-one so that a forbidden subgraph never appears. At some point, no more edges can be added. The Turán problem asks the maximum number of edges in such a graph; the saturation number, on the other hand, asks the minimum number of edges. Two variations of this parameter are studied.

In Chapter 5, we study the saturation of Ramsey-minimal families. Ramsey theory deals with partitioning the edges of graphs so that each partition avoids the particular forbidden subgraph assigned to it. Our motivation for studying these families is that they provide a convincing edge-colored (Ramsey) version of graph saturation. We develop a method, called iterated recoloring, for using results from graph saturation to understand this Ramsey version of saturation. As a proof of concept, we use iterated recoloring to determine the saturation number of the Ramsey-minimal families of matchings and describe the assiociated extremal graphs.

An induced version of graph saturation was suggested by Martin and Smith. [36] In order to offer a parameter that is defined for all forbidden graphs, Martin and Smith consider generalized graphs, called trigraphs. Of particular interest is the case when the induced-saturated trigraphs in question are equivalent to graphs. In Chapter 6, we show that a surprisingly large number of families fall into this case. Further, we define and investigate another parameter that is a version of induced saturation that is closer in spirit to the original version of graph saturation, but that is not defined for all forbidden subgraphs.

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List of Symbols

In the list below, n is a natural number, as are $k_1, \ldots, k_k$. G and H are graphs, S, $S_1, \ldots, S_k$, and T are sets of vertices, and u and v are vertices.

$\mathbb{Z}^+$	the positive integers
$\mathbb{N}$	the natural numbers (including zero)
$[n]$	the set $\{1, \ldots, n\}$
$V(G)$	the set of vertices of G
$E(G)$	the set of edges of G
$\|G\|$	number of vertices in G
$\\|G\\|$	number of edges in G
$\overline{G}$	complement of G
$\alpha(G)$	independence number of G
$\alpha'(G)$	size of maximum matching in G
$\omega(G)$	clique number of G
$d_G(v)$	degree of v in G
$d(v)$	degree of v, when the containing graph is clear
$\delta(G)$	minimum degree of G
$\Delta(G)$	maximum degree of G
$\sigma_2(G)$	$\min\{d(u) + d(v) : uv \in E(\overline{G})\}$
$\theta(G)$	$\max\{d(u) + d(v) : uv \in E(G)\}$
$G \to (H_1, \ldots, H_k)$	G forces $(H_1, \ldots, H_k)$
$\mathrm{sat}(n; H)$	saturation number of H over graphs on n vertices
$\mathrm{indsat}(n, H)$	induced saturation number of H over graphs on n vertices
$\mathcal{R}_{\min}(H_1, \ldots, H_k)$	Ramsey minimal family of forbidden subgraphs $(H_1, \ldots, H_k)$
$\\|S, T\\|$	number of edges with one endpoint in S and one endpoint in T
$N_G(v)$, $N(v)$	the set of neighbors of v in G; we do not specify G if it is clear
$N_G[v]$, N(v)	$N_G(v) \cup \{v\}$; we do not specify G if it is clear

$G[S]$ .... the subgraph of G induced by the vertices in S

K_n .... the complete graph on n vertices

$K(S)$ .... the complete graph on vertex set S

$K_{k_1,\dots,k_k}$ .... the complete k-partite graph with parts of size $k_1,\dots,k_k$

$K(S_1,\dots,S_k)$ .... the complete k-partite graph with parts $S_1,\dots,S_k$

P_k .... the path on k vertices

C_k .... the cycle on k vertices

$K_{1,3}^{+}$ .... the paw, $K_4 - P_3$

$\hat{C}_{2k}$ .... the cycle C_{2k} plus a triangle chord

C'_{2k} .... the cycle C_{2k} plus a pendant edge

$G + H$ .... the disjoint union of G and H

kH .... the disjoint union of k copies of H

$G \vee H$ .... the join of G and H

$G \,\square\, H$ .... the Cartesian product of G and H

Chapter 1

Overview

1.1 Definitions

Most of the notation and vocabulary in this thesis is in standard use. However, here we present a few definitions and notations that might not be familiar to the casual graph theorist.

1.1.1 Types of Graphs

Definition 1.1.1. By *multigraph*, we denote a graph that allows multiple edges and loops.

Definition 1.1.2. We denote the complement of G by $\overline{G}$; that is, for a graph $G = (V, E)$, $\overline{G} = \left(V, \binom{V}{2} - E\right)$.

Definition 1.1.3. The star with three leaves, $K_{1,3}$, is called the claw. The paw is the 4-vertex graph obtained by adding an edge to a claw, which we will denote $K_{1,3}^+$. (See Figure 1.1.)

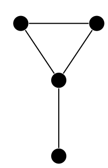


Figure 1.1: $K_{1,3}^+$

Definition 1.1.4. The graph formed by adding a chord in C_{2k} between two vertices of distance two is written $\hat{C}_{2k}$. We denote by C'_{2k} the graph obtained by adding a pendant edge to C_{2k}.

1.1.2 Graph Parameters

Definition 1.1.5. The number of vertices in a graph G is denoted $|G|$; the number of edges is $\|E\|$.

The number of edges with one endpoint in vertex set S and one endpoint in vertex set T is given by $\|S, T\|$, where perhaps $S \cap T \neq \emptyset$. Edges with both endpoints in $S \cap T$ are each counted only once.

Definition 1.1.6. The minimum degree of a graph G is denoted $\delta(G)$. The minimum degree sum of nonadjacent vertices, also called the Ore condition, is given by $\sigma_2(G) := \min\{d(x)+d(y) : xy \in E(\overline{G})\}$. The maximum degree sum of adjacent vertices in G is $\theta(G)$.

The independence number of G is denoted $\alpha(G)$. The largest size of a matching in G is $\alpha'(G)$. The chromatic number of G is $\chi(G)$.

Definition 1.1.7. Given a family of graphs $\mathcal{F}$, a graph G is *$\mathcal{F}$-saturated* if no element of $\mathcal{F}$ is a subgraph of G, but for any edge e in $\overline{G}$, some element of $\mathcal{F}$ is a subgraph of $G+e$. If $\mathcal{F} = \{F\}$, then we say that G is *F-saturated.*

The minimum number of edges over all n-verted graphs that are F-saturated is the *saturation number of* F, written $\mathrm{sat}(n, F)$.

1.1.3 Other Definitions

Definition 1.1.8. A vertex of degree 0 or 1 is called a *bud*.

We say a set S of vertices *dominates* a graph G if every vertex of $G - S$ is adjacent to some vertex in S, and we call S a *dominating set*; if $S = \{v\}$, we say v is a *dominating vertex*. We say a vertex u *dominates* S if u is adjacent to every vertex in S.

Definition 1.1.9. An *equitable k-coloring* of a graph G is a proper coloring of G with at most k colors in which any two color classes differ in size by at most one.

Definition 1.1.10. When we call cycles *disjoint*, we mean they share no vertices.

Definition 1.1.11. Given graphs G and H, we denote the *join* by $G \vee H$; that is, we obtain $G \vee H$ from the disjoint union of G and H by adding an edge between every vertex in G and every vertex in H.

Definition 1.1.12. Given graphs G and H, we denote the *Cartesian product* by $G \,\square\, H$. The graph $G \,\square\, H$ has vertex set $V(G) \times V(H) := \{(g,h) : g \in V(G), h \in V(H)\}$ and edge set $E = \{(g_1,h_1)(g_2,h_1) : g_1g_2 \in E(G)\} \cup \{(g_1,h_1)(g_1,h_2) : h_1h_2 \in E(H)\}$.

Definition 1.1.13. Given a graph G and forbidden subgraphs $H_1, \ldots, H_k$, we say G *forces* $(H_1, \ldots, H_k)$, written $G \to (H_1, \ldots, H_k)$, if given every k-edge-coloring of G there exists some $i \in [k]$ such that a copy of H_i appears as a subgraph of G with all its edge assigned color i.

Definition 1.1.14. For a graph G and a set S of vertices in G, $G[S]$ is the sugraph of G induced by the vertices in S. For $S = \{v_1, \ldots, v_k\}$, we will sometimes write $G[v_1, \ldots, v_k]$.

1.2 Disjoint Cycles, Chapter 2

In general, problems in extremal combinatorics seek to maximize or minimize a given graph parameter over a particular class of graphs. One celebrated theorem in extremal combinatorics is the Corrádi-Hajnal Theorem, [10] which gives the maximal allowable minimum degree over graphs with no k vertex-disjoint cycles.

Theorem 1.2.1 (Corrádi-Hajnal Theorem [10])**.** *Let $k \in \mathbb{Z}^+$. Every graph G with (i) $|G| \geq 3k$ and (ii) $\delta(G) \geq 2k$ contains k disjoint cycles.*

The Corrádi-Hajnal Theorem is a tidy generalization of the fact that forests have minimum degree at most one. Notice the condition $n \geq 3k$ is clearly necessary. The Corrádi-Hajnal Theorem is also a convenient theorem computationally. While checking for the existence of k disjoint cycles is computationally quite difficult, the minimum degree of a graph can be determined quickly.

Although the Corrádi-Hajnal theorem is sharp, it was refined by Enomoto and Wang ([13], [39]). They considered as a sufficient condition the minimum degree sum of nonadjacent vertices, rather than the minimum degree of the graph. The effect of this modified condition is that small-degree vertices are allowed, provided they form a clique and their nonneighbors compensate by having higher degree.

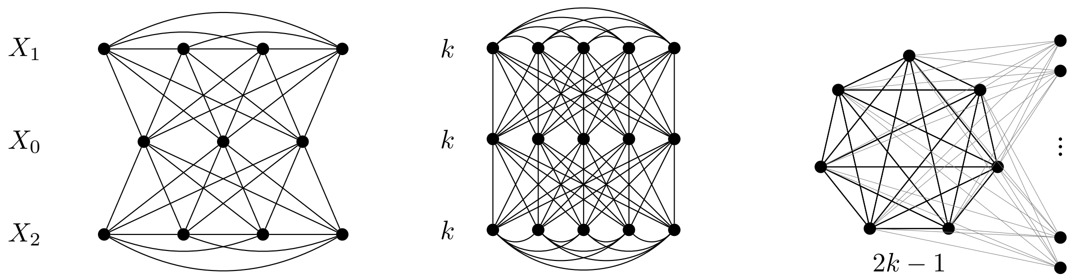
Theorem 1.2.2 (Enomoto [13], Wang [39])**.** *Let $k \in \mathbb{Z}^+$. Every graph G with $|G| \geq 3k$ and $\sigma_2(G) \geq 4k-1$ contains k disjoint cycles.*

Both the Corrádi-Hajnal Theorem and Enomoto and Wang's theorem are sharp, as shown by Examples 1.2.4 and 1.2.5.

Definition 1.2.3. Let $Y_{h,t} = \overline{K}_h \vee (K_t \cup K_t)$ (Figure 1.2(a)), where $V(\overline{K}_h) = X_0$ and the cliques have vertex sets X_1 and X_2. In other words, $V(Y_{h,t}) = X_0 \cup X_1 \cup X_2$ with $|X_0| = h$ and $|X_1| = |X_2| = t$, and a pair xy is an edge in $Y_{h,t}$ precisely when $\{x, y\} \subseteq X_1$, or $\{x, y\} \subseteq X_2$, or $|\{x, y\} \cap X_0| = 1$.

Example 1.2.4. For odd k, let $G_1 = Y_{k,k}$ (see Figure 1.2(b)). Then $|G_1| = 3k$, $\delta(G_1) = 2k-1$, and $\sigma_2(G_1) = 4k-2$. However, G_1 has no k disjoint cycles: any collection of k disjoint cycles would be a partition of $V(G)$ into triangles. In order to accomplish this, every vertex from the independent set $\overline{K_k}$ shares a triangle with two vertices from either copy of K_k; when k is odd, this is impossible.

Example 1.2.5. For any $n \geq 3k$, let $G_2 = K_{2k-1} \vee \overline{K_{n-2k+1}}$ (see Figure 1.2(c)). In other words, G_2 contains an independent set A of size $n-2k+1$, and $E(G_2) = \{uv : \{u, v\} \not\subseteq A\}$. Then $|G_2| \geq 3k$, $\delta(G_2) = 2k-1$, and $\sigma_2(G) = 4k-2$, but G_2 has no k disjoint cycles: any cycle contains at least two vertices of $V(G_2) - A$, but $|V(G_2) - A| = 2k-1$.



(a) $\mathbf{Y_{h,t}}$, with $h = 3$ and $t = 4$. (b) $G_1 = \mathbf{Y_{k,k}}$, with $k = 5$ (c) G_2, with $k = 4$

Figure 1.2

Chapter 2 consists of joint work with Henry Kierstead and Alexandr Kostochka, based on [29]. Our main result is that, for sufficiently large k, G_1 and G_2 are the only types of sharpness examples of Enomoto and Wang's theorem. More precisely:

Theorem 1.2.6. *Let $k \in \mathbb{Z}^+$ with $k \geq 4$. Every graph G with*

(H1) $|G| \geq 3k + 1$,

(H2) $\sigma_2(G) \geq 4k - 3$, *and*

(H3) $\alpha(G) \leq |G| - 2k$

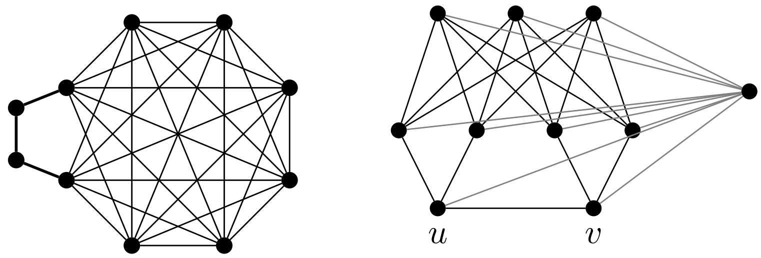
contains k disjoint cycles. Furthermore, for fixed k there is a polynomial time algorithm that either produces k disjoint cycles or demonstrates that one of the hypotheses fails.

Each condition (H1)–(H3) is sharp. Note *every* graph G with $\alpha(G) \geq |G| - 2k + 1$ contains at most $k - 1$ disjoint cycles, because every cycle uses at least two vertices outside of an independent set.

For $k \in [3]$ we characterize those graphs G that satistfy (H1)–(H3) but do not contain k disjoint cycles. We use a theorem of Lovász, and develop several other examples.

Theorem 1.2.7 (Lovász [33])**.** *Let G be a multigraph with $\delta(G) \geq 3$ and no two disjoint cycles. Then G is one of the following:* (1) K_5, (2) W_s^*, (3) $K_{3,|G|-3}$ *together with a multigraph on the vertices of the (first) 3-class, and* (4) *a forest F and a vertex x with possibly some loops at x and some edges linking x to F.*

Example 1.2.8. Let $k = 3$ and $\mathbf{Y}_1$ be the graph obtained by twice subdividing one of the edges wz of K_8, i.e., replacing wz by the path $wxyz$. Then $|\mathbf{Y}_1| = 10 = 3k+1$, $\sigma_2(\mathbf{Y}_1) = 9 = 4k-3$, and $\alpha(\mathbf{Y}_1) = 2 \leq |\mathbf{Y}_1| - 2k$. However, $\mathbf{Y}_1$ does not contain $k = 3$ disjoint cycles, since each cycle would need to contain three vertices of the original K_8 (see Figure 1.3(a)).



(a) $\mathbf{Y_1}$ (b) $\mathbf{Y_2}$

Figure 1.3

Example 1.2.9. Let $k = 3$. Let Q be obtained from $K_{4,4}$ by replacing a vertex v and its incident edges vw, vx, vy, vz by new vertices u, u' and edges $uu', uw, ux, u'y, u'z$; so $d(u) = 3 = d(u')$ and contracting uu' in Q yields $K_{4,4}$. Now set $\mathbf{Y}_2 := K_1 \vee Q$. Then $|\mathbf{Y}_2| = 10 = 3k + 1$, $\sigma_2(\mathbf{Y}_2) = 9 = 4k - 3$, and $\alpha(\mathbf{Y}_2) = 4 \leq |\mathbf{Y}_2| - 2k$. However, $\mathbf{Y}_2$ does not contain $k = 3$ disjoint cycles, since each 3-cycle contains the only vertex of K_1 (see Figure 1.3(b)).

Theorem 1.2.10. *Let $k \in \mathbb{Z}^+$. Let G be a graph with*

(H1) $|G| \geq 3k + 1$,

(H2) $\sigma_2(G) \geq 4k - 3$, *and*

(H3) $\alpha(G) \leq |G| - 2k$.

If $k = 1$, G contains k disjoint cycles unless G is a forest with at most one isolate.
If $k = 3$, G contains k disjoint cycles unless $G \in \{\mathbf{Y}_1, \mathbf{Y}_2\}$.
If $k = 2$, G contains k disjoint cycles unless G is one of the following (see Figure 1.4):

(a) $K_5 + K_2$;

(b) K_5 *with a pendant edge, possibly subdivided;*

(c) K_5 *with one edge subdivided and then a leaf added adjacent to the degree-2 vertex;*

(d) a graph of type (1–3) from Theorem 1.2.7 with no multiple edge, and possibly one edge subdivided once or twice, and if $|G| = 6 - i$ with $i \geq 1$ then some edge is subdivided at least i times;

(e) a graph G of type (2) or (3) from Theorem 1.2.7 with one edge of multiplicity two, and one of its parallel parts is subdivided once or twice—twice if $|G| = 4$.

Our proof of Theorem 1.2.6 is inductive. We suppose by way of contradiction that k is the smallest integer greater than 3 so that the theorem fails, and for this k we choose an edge-maximal counterexample

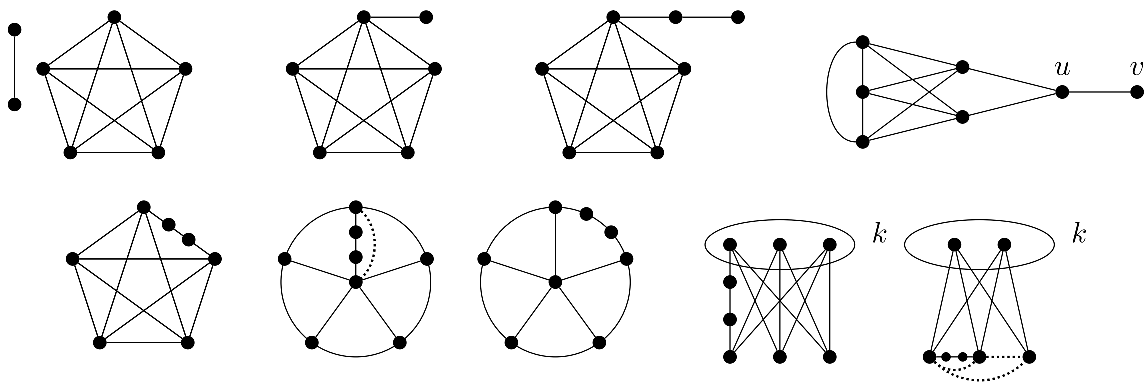


Figure 1.4

G. By edge maximality, G contains a collection of $(k-1)$ disjoint cycles. We choose a particular set $\mathcal{C}$ of $(k-1)$ cycles in G with extremal properties. For example, we choose $\mathcal{C}$ to have the minimum total number of vertices over all collections of $(k-1)$ disjoint cycles. These extremal properties force G to have a very particular structure, and eventually the requirements on the structure of G lead to a contradiction.

Following the proof of the Corrádi-Hajnal Theorem, Dirac [11] asked:

Question 1.2.11 (Dirac's Question)**.** Which $(2k-1)$-connected multigraphs do not have k disjoint cycles?

Notice that any $2k$-connected simple graph has minimum degree at least $2k$; so by the Corrádi-Hajnal Theorem, a $2k$-connected simple graph has k disjoint cycles if and only if it has at least $3k$ vertices.

In [11], Dirac answered his own question when $k = 2$ by describing all 3-connected multigraphs on at least 4 vertices in which every two cycles intersect. Indeed, the only simple 3-connected graphs with no two disjoint cycles are wheels. In Theorem 1.2.7, we saw that Lovász [33] fully described all multigraphs with minimum degree at least 3 in which every two cycles intersect. An easy corollary of this theorem describes all multigraphs (regardless of mnimum degree) in which every two cycles intersect.

In Chapter 2, we prove a result that yields a full answer to Dirac's question in the case of simple graphs. Indeed, we prove a more general result: we consider graphs with minimum degree at least $2k-1$. Our result is here:

Theorem 1.2.12. *Let $k \geq 2$. Every graph G with (i) $|G| \geq 3k$ and (ii) $\delta(G) \geq 2k-1$ contains k disjoint cycles if and only if*

(H3) $\alpha(G) \leq |G| - 2k$, *and*

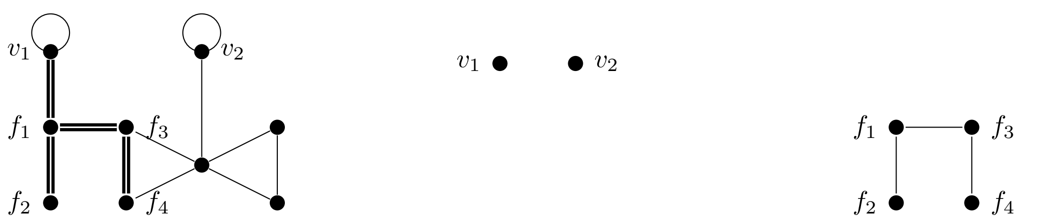
(H4) if k is odd and $|G| = 3k$, then $G \neq \mathbf{Y_{k,k}}$ and if $k = 2$ then G is not a wheel.

For fixed k, the conditions of Theorem 2.1.3 can be tested in polynomial time.

1.2.1 Disjoint Cycles in Multigraphs, Chapter 3

Chapter 3 is joint work with Henry Kierstead and Alexandr Kostochka, and is based on [30]. We heavily use the above theorem to obtain a characterization of $(2k-1)$-connected multigraphs that contain k disjoint cycles, answering Question 1.2.11 in full. Before we state this result, we need some specialized notation.

For every multigraph G, let $V_1 = V_1(G)$ be the set of vertices in G incident to loops (as in Figure 1.5(b)). Let $\widetilde{G}$ denote the *underlying simple graph of* G, i.e. the simple graph on $V(G)$ such that two vertices are adjacent in G if and only if they are adjacent in $\widetilde{G}$. Let $F = F(G)$ be the simple graph formed by the multiple edges in $G - V_1$; that is, if G' is the subgraph of $G - V_1$ induced by its multiple edges, then $G = \widetilde{G'}$ (as in Figure 1.5(c)). We will call the edges of $F(G)$ *the strong edges of* G, and define $\alpha' = \alpha'(F)$ to be the size of a maximum matching in F. A set $S = \{v_0, \ldots, v_s\}$ of vertices in a graph H is a *superstar with center* v_0 *in* H if $N_H(v_i) = \{v_0\}$ for each $1 \le i \le s$ and $H - S$ has a perfect matching.



(a) A multigraph G (b) $V_1(G) = \{v_1, v_2\}$ (c) $F(G)$, with $\alpha'(F) = 2$

Figure 1.5: Example to Illustrate Notation

For $v \in V$, we define $s(v) = |N(v)|$ to be the *simple degree* of v, and we say that $\mathcal{S}(G) = \min\{s(v) : v \in V\}$ is the *minimum simple degree* of G. We define $\mathcal{D}_k$ to be the family of multigraphs G with $\mathcal{S}(G) \ge 2k-1$. By the definition of $\mathcal{D}_k$, $\alpha(G) \le n - 2k + 1$ for every n-vertex $G \in \mathcal{D}_k$; so we call $G \in \mathcal{D}_k$ *extremal* if $\alpha(G) = n - 2k + 1$. A *big set* in an extremal $G \in \mathcal{D}_k$ is an independent set of size $\alpha(G)$.

The following is an easy extension of Theorem 2.1.1 to multigraphs.

Theorem 1.2.13. *For $k \in \mathbb{Z}^+$, let G be a multigraph with $\mathcal{S}(G) \ge 2k$, and set $F = F(G)$ and $\alpha' = \alpha'(F)$. Then G has no k disjoint cycles if and only if*

$$|V(G)| - |V_1(G)| - 2\alpha' < 3(k - |V_1| - \alpha'), \tag{1.1}$$

i.e., $|V(G)| + 2|V_1| + \alpha' < 3k$.

Theorem 1.2.13 yields the following.

Corollary 1.2.14. *Let G be a multigraph with $\mathcal{S}(G) \ge 2k-1$ for some integer $k \ge 2$, and set $F = F(G)$ and $\alpha' = \alpha'(F)$. Suppose G contains at least one loop. Then G has no k disjoint cycles if and only if*

$|V(G)| + 2|V_1| + \alpha' < 3k$.

So, in the case of a multigraph G with at least one loop, the answer to 1.2.11 is very similar to the Corrádi-Hajnal Theorem: G contains the desired number of disjoint cycles as long as it has the number of vertices that are trivially necessary.

If a multigraph G has no loop, there are more varieties of graphs G that have $\mathcal{S}(G) \geq 2k-1$ but no k disjoint cycles. A characterization of these graphs is the main result of Chapter 3 (given below as Theorem 1.2.15) and a complete answer to Question 1.2.11. It is worth noting that every graph in the Theorem 1.2.15 contains an element of Theorem 1.2.12: a subgraph $Y_{h,t}$, a large independent set, or a wheel.

Theorem 1.2.15. *Let $k \geq 2$ and $n \geq k$ be integers. Let G be an n-vertex multigraph in $\mathcal{D}_k$ with no loops. Set $F = F(G)$, $\alpha' = \alpha'(F)$, and $k' = k - \alpha'$. Then G does not contain k disjoint cycles if and only if one of the following holds: (see Figure 3.2)*

(a) $n + \alpha' < 3k$;

(b) $|F| = 2\alpha'$ *(i.e., F has a perfect matching) and either*

(i) k' is odd and $G - F = Y_{k',k'}$, or

(ii) $k' = 2 < k$ and $G - F$ is a wheel with 5 spokes;

(c) G is extremal and either

(i) some big set is not incident to any strong edge, or

(ii) for some two distinct big sets I_j and $I_{j'}$, all strong edges intersecting $I_j \cup I_{j'}$ have a common vertex outside of $I_j \cup I_{j'}$;

(d) $n = 2\alpha' + 3k'$, k' is odd, and F has a superstar $S = \{v_0, \ldots, v_s\}$ with center v_0 such that either

(i) $G - (F - S + v_0) = Y_{k'+1,k'}$, or

(ii) $s = 2$, $v_1v_2 \in E(G)$, $G - F = Y_{k'-1,k'}$ and G has no edges between $\{v_1, v_2\}$ and the set X_0 in $G - F$;

(e) $k = 2$ and G is a wheel, where some spokes could be strong edges;

(f) $k' = 2$, $|F| = 2\alpha' + 1 = n - 5$, and $G - F = C_5$.

If a multigraph G has at least one loop, Corollary 3.2.2 tells us precisely when G has k disjoint cyces. To prove Theorem 1.2.15, we may therefore assume that G has no loops. A multigraph G with no loops has at most $\alpha'(F)$ "short" cycles–that is, cycles with fewer than 3 vertices. If we know some cycles that are contained in a collection $\mathcal{C}$ of disjoint cycles, then we can investigate which other cycles might be in $\mathcal{C}$ by deleting the edges of the known cycles. If we delete the edges of all the short cycles used, the remainder of the cycles are simple cycles, so we can look at a simple subgraph of G and apply the Corrádi Hajnal theorem. These ideas form the backbone of the proof of Theorem 1.2.15.

1.2.2 Equitable Coloring, Chapter 4

Theorems 1.2.6 and 1.2.10 characterize graphs G with at least $3k+1$ vertices and $\sigma_2(G) \geq 4k-3$ that do not contain k disjoint cycles. However, missing from Chapter 2 is a characterization of graphs G with precisely $3k$ vertices and $\sigma_2(G) \geq 4k-3$ that do not contain k disjoint cycles. To achieve this characterization, we looked to a dual problem, equitable coloring, in Chapter 4. Chapter 4 is joint work with Henry Kierstead, Alexandr Kostochka, and Theodore Molla, and is based on [27].

If $|G| = 3k$, then G has an equitable k-coloring if and only if $\overline{G}$ contains k disjoint cycles (all triangles), because each color class has size three. In Chapter 4, we characterize graphs on $3k$ vertices with $d(x)+d(y) \geq 2k+1$ for every $xy \in E(G)$ that do not have an equitable k-coloring. This is equivalent to characterizing graphs G on $3k$ vertices with $\sigma_2(G) \geq 4k-3$ and no k disjoint cycles.

Example 1.2.16. We define a graph G_0 with vertex set $X \cup Y \cup Z$ with $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2, y_3\}$, and $Z = \{z_1, z_2, z_3\}$. We let $G[Z] = K_3$ and $G[X \cup Y] = K_{3,3} - x_3y_3$, and add edges x_3z_1, x_3z_2, and y_3z_3. (See Figure 1.6.)

G_0 is 3-colorable, $d(x_3) = 4$, and $d(v) = 3$ for every $v \in V(G) - x_3$. However, G_0 has no equitable 3-coloring. Any proper coloring of Z uses all three colors, and any equitable coloring of $X \cup Y$ puts the same color on x_3 and y_3, so this color cannot be used on Z. Therefore any proper coloring of G_0 is not equitable.

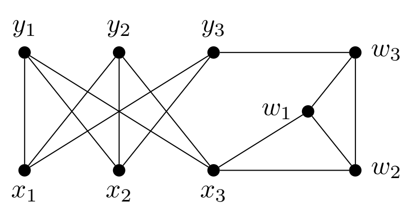


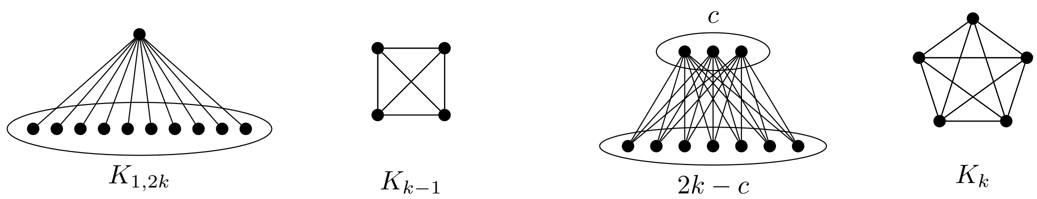
Figure 1.6: $\mathbf{G_0}$

Theorem 1.2.17 (Main Result of Chapter 4)**.** *Let G be a k-colorable graph on $3k$ vertices with $d(x)+d(y) \geq 2k+1$ for every $xy \in E(G)$. Then one of the following holds:*

(i) $G = K_{1,2k} + K_{k-1}$ (see Figure 1.7(a));

(ii) $G \supseteq K_{c,2k-c} + K_k$ for some odd c (see Figure 1.7(b));

(iii) $G = G_0$ and $k = 3$ (see Figure 1.6).



(a) $\mathbf{Y_k}$, $k = 5$ (b) $\mathbf{Z_{c,k}}$, $k = 5$, $c = 3$

Figure 1.7: Theorem 1.2.17

We note that if G is not k-colorable, then it is not equitably k-colorable. Therefore Theorem 1.2.17 completely characterizes graphs G on $3k$ vertices with $d(x) + d(y) \geq 2k + 1$ for every $xy \in E(G)$ that have no equitable k-coloring. In particular, translating Theorem 1.2.17 the language of disjoint cycles results in the following.

Theorem 1.2.18. *Let G be a graph on $3k$ vertices with $\sigma_2(G) \geq 4k - 3$. If G has no k disjoint cycles, then one of the following holds:*

(i) $G = K_{2k} \vee K_{1,k-1}$,

(ii) $G \subseteq (K_c + K_{2k-c}) \vee \overline{K_k}$ *for some odd c,*

(iii) $\overline{G} = G_0$ *and $k = 3$, or*

(iv) $\overline{G}$ *is not k-colorable.*

Together, Theorems 1.2.6, 1.2.10, and 1.2.18 completely characterize all graphs G on at least $3k$ vertices with $\sigma_2(G) \geq 4k - 3$ that have no k disjoint cycles. Interestingly, the case $|G| = 3k$ includes more types of graphs without k disjoint cycles than the case $|G| > 3k$.

Theorem 1.2.17 also proves a special case of an Ore-type version of the Chen-Lih-Wu Conjecture, which is an equitable-coloring version of Brooks's Theorem, discussed below.

The Hajnal-Szemerédi Theorem [18] tells us that, as in proper coloring, any graph G can be equitably colored using $\Delta(G) + 1$ colors. Brooks's Theorem states that every graph G can be properly colored using $\Delta(G)$ colors *unless* $\omega(G) = \Delta(G)+1$ or $\Delta(G) = 2$ and G contains an odd cycle. The Chen-Lih-Wu conjecture [6] attempts to likewise characterize when a graph G cannot be equitably colored using only $\Delta(G)$ colors.

Conjecture 1.2.19 (Chen-Lih-Wu [6])**.** *Let G be a connected graph with $\chi(G), \Delta(G) \leq k$. Then G is equitably k-colorable unless k is odd and $G = K_{k,k}$.*

The Chen-Lih-Wu conjecture was generalized using an Ore-type condition by Kierstead and Kostochka in [22]:

Conjecture 1.2.20 (Ore-type version of Chen-Lih-Wu Conjecture [22])**.** *Let G be a connected graph with $\chi(G) \leq k$ and $d(x) + d(y) \leq 2k + 1$ for every $xy \in E(G)$. Then G is equitably k-colorable unless k is odd and $K_{k,k} \subseteq G$.*

Our Theorem 1.2.17 settles Conjecture 1.2.20 in the case $|G| = 3k$. The conjecture is true in this case, *except* when $k = 3$: the conjecture must be expanded to include the graph G_0. Thorem 1.2.17 also strengthens a result of Kierstead and Kostochka [22] in the case $|G| = 3k$.

Theorem 1.2.21 ([22])**.** *Let G be a graph with $\theta(G) \leq 2k - 1$. Then G has an equitable k-coloring.*

Our Theorem 1.2.17 characterizes the sharpness examples for Theorem 1.2.21 when $|G| = 3k$.

Our proof of Theorem 1.2.17 proceeds by contradiction. Suppose G is a counterexample to the theorem, with k minimal, and further let G be edge-minimal. That is, if we delete any edge of G, it admits an equitable k-coloring. We show that G can be colored so that all but two classes have size 3, one "small" class has size 2, and one "large" class has size 4. Note that, if any vertex in the large class has no neighbors in the small class, we can simply move that vertex to the small class, creating an equitable coloring. We expand this idea: suppose there exists a vertex v_0 in the large class with no neighbors in a class V_1; there exists a vertex $v_1 \in V_1$ with no neighbors in a class V_2; and there exists a vertex $v_2 \in V_2$ with no neighbors in the small class. Then we move v_0 to V_1, move v_1 to V_2, and move v_2 to the small class, obtaining an equitable k-coloring. (See Figure 1.8.) We choose a coloring with certain extremal properties, and make use of this daisy-chaining of movable vertices to show that G must have a particular structure. These structural results eventually lead to a contradiction.

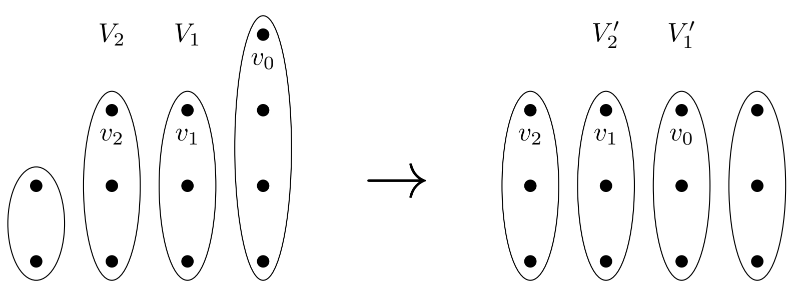


Figure 1.8: Moving Vertices to Obtain an Equitable Coloring

1.2.3 Graph Saturation: Background[1]

The field of extremal combinatorics is considered to have begun in earnest with Turán's Theorem [38], which answers the following question: given a graph G on n vertices that contains no complete k-vertex subgraph,

[1]Much of this discussion of important results in graph saturation is due to [15], Section 1.

what is the maxium attainable number of edges of G? Erdős, Hajnal, and Moon [14] elaborated on this idea. If G is a graph with no K_k subgraph attaining the maximum number of edges, then adding any edge to G creates a K_k subgraph. They then asked, over all n-vertex graphs G that avoid a forbidden subgraph, but have the property that adding any edge creates that forbidden subgraphs, what is the minimum nuber of edges in G? This number is the *saturation number* of the forbidden graph.

Definition 1.2.22. Given any forbidden graph H, and any natural number n, the *saturation number* $\mathrm{sat}(n, H)$ is defined as:

$$\mathrm{sat}(n, H) = \min\{||G|| : H \not\subseteq G \text{ and } \forall e \in \overline{G}, H \subseteq G + e\}$$

In [14], Erdős, Hajnal, and Moon determine $\mathrm{sat}(n, K_t)$ for all $n \geq t \geq 2$, and describe the graphs achieving the minimum number of edges.

Theorem 1.2.23 ([14])**.** *For every pair of integers n and k, with $2 \leq t \leq n$,*

$$\mathrm{sat}(n, K_t) = \binom{n}{2} - \binom{n-(t-2)}{2}$$

and the only n-vertex, K_t-saturated graph achieving this number of vertices is $K_{t-2} \vee \overline{K_{n-(t-2)}}$.

Bollobás used set pairs to generalize Theorem 1.2.23 to hypergraphs in [3]. In [4] (pp. 1269-1270), he simplified this idea to give a compact proof of the numerical portion of Theorem 1.2.23 using his well-known inequality, below.

Lemma 1.2.24 ([4])**.** *Given a finite index set I, let $\{\{A_i, B_i\} : i \in I\}$ be a collection of finite sets such that $A_i \cap B_j = \emptyset$ if and only if $i = j$. For $i \in I$, set $a_i = |A_i|$ and $b_i = |B_i|$. Then*

$$\sum_{i \in I} \binom{a_i + b_i}{a_i}^{-1} \leq 1.$$

Proof of the numerical portion of Theorem 1.2.23, [4]. Suppose a graph G is K_t-saturated. Let $\{A_i : i \in I\} = E(\overline{G})$. By the definition of saturation, for every $i \in I$, there exists a set C_i of t vertices such that $A_i \subseteq C_i$ and A_i is the only nonedge in $G[C_i]$. Define $B_i = V(G) - C_i$. If $A_i \cap B_j = \emptyset$ for some $i, j \in I$, then $A_i, A_j \subseteq C_j$, so $i = j$. It is clear that $A_i \cap B_i = \emptyset$ for all $i \in I$. So, by Lemma 1.2.24,

$$\sum_{i \in I} \binom{2 + (n-t)}{2}^{-1} \leq 1$$

and so $\|G\| = \binom{n}{2} - |I| \geq \binom{n}{2} - \binom{n-(t-2)}{2}$. $\square$

The above proof only uses the special case of Lemma 1.2.24 where all A_i resp. B_i have the same size. Lovász ([34], p. 83) used tensor calculus to prove generalizations to matroids and projective spaces of this version of Bollobás's inequality, demonstrating that linear algebraic methods are useful for studying saturation number. We present here a simplified version of Lovász's proof of Lemma 1.2.24 in the case that, for all $i \in I$, $a_i = a$ and $b_i = b$. This proof is taken from [1], pp. 94-95.

Proof of Lemma 1.2.24, in the case $a_i = a$ and $b_i = b$ for all $i \in I$. Let $V = (\cup_{i\in I} A_i) \cup (\cup_{i\in I} B_i)$. We will assign a vector to each $u \in V$, and use these vectors to define a polynomial f_{B_i} for every $i \in I$. The dimension of the space of homogeneous polynomials of degree b in $a+1$ variables is $\binom{b+(a+1)-1}{b} = \binom{a+b}{a}$. (This is easily seen by noting that a basis of such a space is the collection of polynomials $\{x_0^{e_0} x_1^{e_1} \cdots x_a^{e_a} : e_0 + \cdots + e_a = b\}$.) So, if our vectors and polynomials are defined in such a way that for every $i \in I$, f_{B_i} is homogeneous of degree b with $a+1$ variables, and the set $\{f_{B_i} : i \in I\}$ is linearly independent, then $|I| \leq \binom{a+b}{a}$, as desired.

To each $u \in V$, assign a vector $\mathbf{v}(u) = (v_0(u), \ldots, v_a(u)) \in \mathbb{R}^{a+1}$ so that the vectors of elements of V are in general position; that is, every collection of $a+1$ vectors is linearly independent. For $i \in I$, define a polynomial

$$f_{B_i}(\mathbf{x}) = f_{B_i}(x_0, \ldots, x_a) = \prod_{u \in B_i} (v_0(u)x_0 + \cdots + v_a(u)x_a) = \prod_{u \in B_i} \mathbf{v}(u) \cdot \mathbf{x}.$$

It is clear that f_{B_i} is a homogeneous polynomial of degree b in $a+1$ variables. So, it remains only to show that $\{f_{B_i} : i \in I\}$ is linearly independent. For every $j \in I$, the vectors of A_j form a space of dimension a; choose a nonzero vector $\mathbf{a_j} \in \mathbb{R}^{a+1}$ orthogonal to this space. Note $f_{B_i}(\mathbf{a_j}) = 0$ if and only if $\mathbf{v}(u) \cdot \mathbf{a_j} = 0$ for some $u \in B_i$. Since the vectors of V are in general position, $\mathbf{v}(u) \cdot \mathbf{a_j} = 0$ if and only if $u \in A_j$; so, $f_{B_i}(\mathbf{a_j}) = 0$ if and only if $i \neq j$. Now, if there exists constants $\alpha_1, \ldots, \alpha_{|I|}$ so that $\sum_{i\in I} \alpha_i f_{B_i}(\mathbf{x}) \equiv 0$, then for every $j \in I$, $0 = \sum_{i \in I} \alpha_i f_{B_i}(\mathbf{a_j}) = \alpha_j f_{B_j}(\mathbf{a}_j)$, and $f_{B_j}(\mathbf{a}_j) \neq 0$, so $\alpha_j = 0$. Thus $\{f_{B_i} : i \in I\}$ is linearly independent, as desired. $\square$

1.2.4 Saturation of Ramsey-Minimal Families, Chapter 5

Many variations of graph saturation have been studied. Chapter 5 is joint work with Michael Ferrara and Jaehoon Kim, based on [16]. In Chapter 5, we study a parameter that generalizes saturation to include multiple forbidden graphs, tied to different colors. We do this by using the idea of "forcing" from Ramsey theory, as in Definition 1.1.13.

Definition 1.2.25. Given forbidden subgraphs $(H_1, \ldots, H_k)$, the *Ramsey-minimal family* of $(H_1 \ldots, H_k)$ is defined as the family of graphs G with the properties $G \to (H_1, \ldots, H_k)$, and for any $e \in E(G)$, $G - e \not\to (H_1, \ldots, H_k)$. We denote the Ramsey minimal family by $\mathcal{R}_{\min}(H_1, \ldots, H_k)$.

It is readily shown that a graph saturated with respect to $\mathcal{R}_{\min}(H_1, \ldots, H_k)$ can be described by the equivalent definition below.

Definition 1.2.26. Given forbidden graphs $H_1, \ldots, H_k$, and any natural number n, an n-vertex graph G is saturated with respect to $\mathcal{R}_{\min}(H_1, \ldots, H_k)$ if $G \not\rightarrow (H_1, \ldots, H_k)$, but for any $e \in \overline{G}$, $G + e \rightarrow (H_1, \ldots, H_k)$.

That is, for the saturation number of a Ramsey minimal family, we want the minimum number of edges in an n-vertex graph G such that (i) there exists a k-edge-coloring of G such that no forbidden subgraph appears monochromatically in its assigned color, and (ii) G is edge-maximal with this property. This parameter is variously referred to in literature as the saturation number of Ramsey-minimal families [16], [7]; edge-colored saturation number [19]; and co-criticality [17], [37]. Definition 1.2.26 makes the motivation for studying this parameter clear: it is a Ramsey-type variation of graph saturation.

In Chapter 5, we introduce a technique we developed called *iterated recoloring*, which works as follows. Suppose a graph G is saturated with respect to forbidden subgraphs $\mathcal{R}_{\min}(H_1, \ldots, H_k)$. By definition, there exists a coloring ϕ of G such that no H_i appears with all edges assigned color i. We choose a color, say red. One by one, we examine each edge of ϕ that is not colored red, and consider changing it to red. If changing the edge to red does not result in a monochromatic red copy of the corresponding H_i, then we change that edge to red. At the end of this process, we have a coloring ϕ_i that we call *red-heavy*. If we change any non-red edge in ϕ_i to red, we create a monochromatic red H_i. Now, create an uncolored graph $G[i]$ by deleting every edge from G that is not colored red in ϕ_i. Our key observation is that $G[i]$ is H_i-saturated.

By manipulating ϕ in this way for each color, we are able to conclude that the subgraphs $G[1], G[2], \ldots, G[k]$ are saturated with respect to $H_1, \ldots, H_k$, respectively. This allows us to use results from saturation to gain information about various subgraphs of G. As a proof of concept, we use iterated recoloring, and a result of Mader [35] about graphs that are matching-saturated, to determine the saturation number of the Ramsey minimal family of any collection of matchings, for all n sufficiently large. We also characterize those graphs that are saturated with respect to a family of matchings and achieve the minimum number of edges.

Theorem 1.2.27 (Main Result of Chapter 5). *If $m_1, \ldots, m_k \geq 1$ and $n > 3(m_1 + \ldots + m_k - k)$, then*

$$\operatorname{sat}(n, \mathcal{R}_{\min}(m_1K_2, \ldots, m_kK_2)) = 3(m_1 + \ldots + m_k - k).$$

If $m_i \geq 3$ for some i, then the unique saturated graphs of minimum size consist solely of vertex-disjoint triangles and independent vertices. If $m_i \leq 2$ for every i, then the graphs achieving equality are unions of edge-disjoint triangles and independent vertices.

There is an important relationship between the saturation number of a Ramsey-minimal family and its Ramsey number. Suppose a collection of forbidden subgraphs $H_1, \dots, H_k$ has Ramsey number R. By definition of R, for any $n < R$ the edges of K_n can be colored with k colors avoiding all forbidden subgraphs. It is vacuously true that, for any edge $e \in E(\overline{K_n})$, $K_n + e \to (H_1, \dots, H_k)$. Therefore, whenever $n < R$, $\mathrm{sat}(n, \mathcal{R}_{\min}(H_1, \dots, H_k)) = \binom{n}{2}$. The Ramsey number for matchings is given by Cockayne and Lorimer in [9]:

Theorem 1.2.28 (Ramsey Number of Matchings [9])**.** *Given $m_1 \geq \cdots \geq m_k \geq 1$, the Ramsey number $r(m_1K_2, \dots, m_kK_2)$ is given by*

$$m_1 + 1 + \sum_{i=1}^{k}(m_i - 1).$$

Our Theorem 1.2.27 determines the saturation number of $\mathcal{R}_{\min}(m_1K_1, \dots, m_kK_2)$ when $n > 3\sum_{i=1}^{k} m_i$. So, if $m_1 = \max\{m_i : i \in [k]\}$, then the only values of n for which the saturation number of $\mathcal{R}_{\min}(m_1K_1, \dots, m_kK_2)$ is *not* known are $\left(m_1 + 1 + \sum_{i=1}^{k}(m_i - 1)\right) \leq n \leq 3\sum_{i=1}^{k}(m_i - 1)$.

1.2.5 Induced Saturation, Chapter 6

Chapter 6 is the result of joint work with Sarah Behrens, Catherine Erbes, Michael Santana, and Derrek Yager, and is largely based on [2]. In Chapter 6, we consider an induced version of graph saturation. Recall that, if a graph G is H-saturated for some forbidden subgraph H, then G contains no H-subgraph, but adding any edge creates an H-subgraph. However, this H-subgraph may not be induced. It is possible that a graph G contains no induced H-subgraph, but adding any edge creates an induced H-subgraph; it is likewise possible that G contains no induced H-subgraph, but *deleting* any edge creates an induced H-subgraph. One possible way to create an induced version of graph saturation is to say that G is H-induced-saturated if G does not contain any induced copy of H, but adding or deleting any edge from G creates an induced copy of H. This is a special case of the definition given by Martin and Smith [36] for induced saturation; however, the actual definition is more broad, because our suggested definition is undefined for many values of H.

Definition 1.2.29. A *trigraph* T is a quadruple $(V(T), E_B(T), E_W(T), E_G(T))$, where $V(T)$ is the vertex set and the other three elements partition $\binom{V(T)}{2}$ into a set $E_B(T)$ of black edges, a set $E_W(T)$ of white edges, and a set $E_G(T)$ of gray edges. These can be thought of as edges, non-edges, and potential edges, respectively.

A *realization* of T is a graph $G = (V(G), E(G))$ with $V(G) = V(T)$ and $E(G) = E_B(T) \cup S$ for some subset S of $E_G(T)$.

Definition 1.2.30 (Martin-Smith [36])**.** A trigraph T is *H-induced-saturated* if no realization of T contains H as an induced subgraph, but H occurs as an induced subgraph of some realization whenever any black or white edge of T is changed to gray.

The *induced saturation number* $\text{indsat}(n, H)$ of a forbidden graph H is the minimum size of $E_G(T)$ over all n-vertex trigraphs that are H induced saturated.

In the special case that an H-induced-saturated trigraph T exists with no gray edges, the unique representation G of T has the properties that G does not contain H as an induced subgraph, but adding or deleting any edge of G creates an induced copy of H. In this case, we say the graph G is H-induced saturated.

If $H \in \{K_k - e, \overline{K_k - e}\}$, then an H-induced-saturated n-vertex graph G exists for all sufficiently large n. Trivially, this G is a complete graph, or a graph with no edges. However, it is not immediately obvious that any non-trivial examples exist where a forbidden subgraph H has an H-induced-saturated graph. Indeed, before our work, no other such forbidden graphs were known. In [2], we show that a number of graphs have this property: induced-saturation number zero for all sufficiently large n. This motivated the study of a new parameter, $\text{indsat}^*(n, H)$, that minimizes the number of edges in an n-vertex graph that is H induced saturated.

Although the motivation for studying $\text{indsat}^*(n, H)$ was born of the families of graphs with induced-saturation number 0, to formally define $\text{indsat}^*(n, H)$ there is no need to restrict ourselves to these families.

Definition 1.2.31. Suppose H is a forbidden induced subgraph and n is an integer. Then

$$\text{indsat}^*(n, H) := \min\{|E_B(T)| : T \text{ is an } n\text{-vertex, } H\text{-induced-saturated trigraph with } |E_G(T)| = \text{indsat}(n, H)\}.$$

By simply constructing a graph (not trigraph) that is H-induced-saturated for a given H, we can show $\text{indsat}(n, H) = 0$. In Chapter 6, we provide constructions that show the paw, any matching, and a variety of cycles have induced saturation number 0 for all sufficiently large n. Also by construction, we provide upper bounds on $\text{indsat}^*(n, H)$ for the graphs mentioned. A variety of parameters, for example minimum degree, provide lower bounds for $\text{indsat}^*(n, H)$.

We completely characterize all paw-induced-saturated graphs for all $n \geq 7$. This, in turn, gives us an exact value of $\text{indsat}^*(n, K_{1,3}^+)$ for all $n \geq 7$. Interestingly, $\text{indsat}^*(n, K_{1,3}^+)$ is not monotone in n. This is reminiscent of graph saturation, which is also not necessarily monotone in n for a given forbidden subgraph.

We show that $\text{indsat}(n, C_4) = 0$ for all sufficiently large n by using graphs that generalize the icosahedron. Another construction involving an icosahedron shows that $\text{indsat}(n, kK_2) = 0$ for any $k \geq 2$ and for all n sufficiently large. Further, for a fixed k, $\text{indsat}^*(n, kK_2)$ is bounded above by a constant.

A construction involving the dodecahedron shows, for a restricted range of n, $\text{indsat}(n, C_8) = 0$. It remains an open quesiton whether there exists $k \geq 2$ such that $\text{indsat}(n, C_{2k}) = 0$ for all sufficiently large

n. This question is made even more compelling by our results regarding odd cycles, and two variations on even cycles. We show $\text{indsat}(n, C_{2k-1}) = 0$ for all $k \geq 3$ and for all n sufficiently large. The construction used–the product of cliques–gives a graph that is also induced saturated for $\hat{C}_{2k}$ and C'_{2k}. Our constructions for odd cycles are $n$ot induced-saturated for even cycles, and vice-versa.

Chapter 2

Disjoint Cycles

The following results are joint work with Henry Kierstead and Alexandr Kostochka; this chapter is based on [29].

2.1 Introduction

In 1963, Corrádi and Hajnal proved a conjecture of Erdős by showing the following:

Theorem 2.1.1 (Corrádi-Hajnal Theorem [10])**.** *Let $k \in \mathbb{Z}^+$. Every graph G with (i) $|G| \geq 3k$ and (ii) $\delta(G) \geq 2k$ contains k disjoint cycles.*

Clearly, hypothesis (i) in the theorem is sharp. Hypothesis (ii) also is sharp. Indeed, if a graph G has k disjoint cycles, then $\alpha(G) \leq |G| - 2k$, since every cycle contains at least two vertices of $G - I$ for any independent set I. Thus $H := \overline{K_{k+1}} \vee K_{2k-1}$ satisfies (i) and has $\delta(H) = 2k - 1$, but does not have k disjoint cycles, because $\alpha(H) = k + 1 > |H| - 2k$. There are several works refining Theorem 2.1.1. Dirac and Erdős [12] showed that if a graph G has many more vertices of degree at least $2k$ than vertices of degree at most $2k - 2$, then G has k disjoint cycles. Dirac [11] asked:

Question 2.1.2. Which $(2k - 1)$-connected graphs[1] do not have k disjoint cycles?

He also resolved his question for $k = 2$ by describing all 3-connected multigraphs on at least 4 vertices in which every two cycles intersect. It turns out that the only simple 3-connected graphs with this property are wheels. Lovász [33] fully described all multigraphs in which every two cycles intersect.

The following result in this chapter yields a full answer to Dirac's question for simple graphs.

Theorem 2.1.3. *Let $k \geq 2$. Every graph G with (i) $|G| \geq 3k$ and (ii) $\delta(G) \geq 2k - 1$ contains k disjoint cycles if and only if*

(H3) $\alpha(G) \leq |G| - 2k$, *and*

(H4) *if k is odd and $|G| = 3k$, then $G \neq 2K_k \vee \overline{K_k}$ and if $k = 2$ then G is not a wheel.*

[1] Dirac used the word *graphs*, but in [11] this appears to mean *multigraphs*.

For fixed k, the conditions of Theorem 2.1.3 can be tested in polynomial time.

It is likely that Dirac intended his question to refer to multigraphs; indeed, his result for $k = 2$ is for multigraphs. On the other hand, the above-mentioned paper [12] by Dirac and Erdős is about simple graphs. In Chapter 3, we will heavily use the results of this chapter to obtain a characterization of $(2k-1)$-connected multigraphs that contain k disjoint cycles, answering Question 2.1.2 in full.

Enomoto [13] and Wang [39] generalized the Corrádi-Hajnal Theorem in terms of the minimum Ore-degree $\sigma_2(G) := \min\{d(x) + d(y) : xy \notin E(G)\}$:

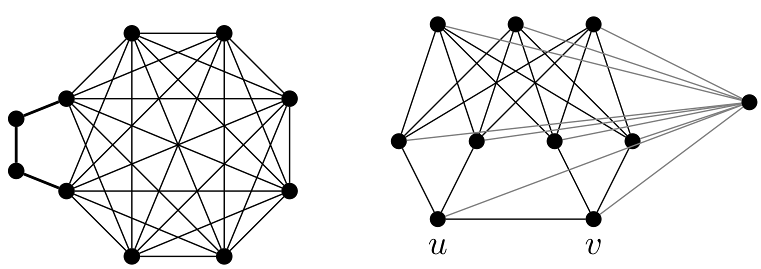
Theorem 2.1.4 ([13],[39]). *Let $k \in \mathbb{Z}^+$. Every graph G with (i) $|G| \geq 3k$ and*

(E2) $\sigma_2(G) \geq 4k - 1$

contains k disjoint cycles.

Again $H := \overline{K_{k+1}} \vee K_{2k-1}$ shows that hypothesis (E2) of Theorem 2.1.4 is sharp. What happens if we relax (E2) to (H2): $\sigma_2(G) \geq 4k - 3$, but again add hypothesis (H3)? Here are two interesting examples.

Example 2.1.5. Let $k = 3$ and $\mathbf{Y}_1$ be the graph obtained by twice subdividing one of the edges wz of K_8, i.e., replacing wz by the path $wxyz$. Then $|\mathbf{Y}_1| = 10 = 3k+1$, $\sigma_2(\mathbf{Y}_1) = 9 = 4k-3$, and $\alpha(\mathbf{Y}_1) = 2 \leq |\mathbf{Y}_1| - 2k$. However, $\mathbf{Y}_1$ does not contain $k = 3$ disjoint cycles, since each cycle would need to contain three vertices of the original K_8 (see Figure 2.1(a)).



(a) $\mathbf{Y_1}$ (b) $\mathbf{Y_2}$

Figure 2.1

Example 2.1.6. Let $k = 3$. Let Q be obtained from $K_{4,4}$ by replacing a vertex v and its incident edges vw, vx, vy, vz by new vertices u, u' and edges $uu', uw, ux, u'y, u'z$; so $d(u) = 3 = d(u')$ and contracting uu' in Q yields $K_{4,4}$. Now set $\mathbf{Y}_2 := K_1 \vee Q$. Then $|\mathbf{Y}_2| = 10 = 3k + 1$, $\sigma_2(\mathbf{Y}_2) = 9 = 4k - 3$, and $\alpha(\mathbf{Y}_2) = 4 \leq |\mathbf{Y}_2| - 2k$. However, $\mathbf{Y}_2$ does not contain $k = 3$ disjoint cycles, since each 3-cycle contains the only vertex of K_1 (see Figure 2.1(b)).

Our main result is:

Theorem 2.1.7. *Let $k \in \mathbb{Z}^+$ with $k \geq 3$. Every graph G with*

(H1) $|G| \geq 3k+1$,

(H2) $\sigma_2(G_k) \geq 4k-3$, *and*

(H3) $\alpha(G) \leq |G| - 2k$

contains k disjoint cycles, unless $k=3$ and $G \in \{\mathbf{Y}_1, \mathbf{Y}_2\}$. Furthermore, for fixed k there is a polynomial time algorithm that either produces k disjoint cycles or demonstrates that one of the hypotheses fails.

Theorem 2.1.7 is proved in Section 2. In Section 3 we discuss the case $k=2$. In Section 4 we discuss connections to equitable colorings and derive Theorem 2.1.3 from Theorem 2.1.7 and known results.

Now we discuss examples demonstrating the sharpness of hypothesis (H2) that $\sigma(G) \geq 4k-3$, and finally we review our notation.

Example 2.1.8. Let $k \geq 3$, $Q = K_3$ and $G_k := \overline{K_{2k-2}} \vee (\overline{K_{2k-3}} + Q)$. Then $|G_k| = 4k-2 \geq 3k+1$, $\delta(G_k) = 2k-2$ and $\alpha(G_k) = |G_k| - 2k$. If G_k contained k disjoint cycles, then at least $4k - |G_k| = 2$ would be 3-cycles; this is impossible, since any 3-cycle in G_k contains an edge of Q. This construction can be extended. Let $k = r + t$, where $k+3 \leq 2r \leq 2k$, $Q' = K_{2t}$, and put $H = G_r \vee Q'$. Then $|H| = 4r - 2 + 2t = 2k + 2r - 2 \geq 3k+1$, $\delta(H) = 2r - 2 + 2t = 2k-2$ and $\alpha(H) = 2r - 2 = |H| - 2k$. If H contained k disjoint cycles, then at least $4k - |H| = 2t + 2$ would be 3-cycles; this is impossible, since any 3-cycle in H contains an edge of Q or a vertex of Q'.

There are several special examples for small k. The constructions of $\mathbf{Y}_1$ and $\mathbf{Y}_2$ can be extended to $k=4$ at the cost of lowering σ_2 to $4k-4$. Below is another small family of special examples. The blow-up of G by H is denoted by $G[H]$; that is, $V(G[H]) = V(G) \times V(H)$ and $(x,y)(x',y') \in E(G[H])$ if and only if $xx' \in E(G)$, or $x = x'$ and $yy' \in E(H)$.

Example 2.1.9. For $k=4$, $G := C_5[\overline{K_3}]$ satisfies $|G| = 15 \geq 3k+1$, $\delta(G) = 2k-2$ and $\alpha(G) = 6 < |G| - 2k$. Since $\text{girth}(G) = 4$, G has at most $\frac{|G|}{4} < k$ disjoint cycles. This example can be extended to $k = 5, 6$ as follows. Let $I = \overline{K_{2k-8}}$ and $H = G \vee I$. Then $|G| = 2k+7 \geq 3k+1$, $\delta = 2k-2$ and $\alpha(G) = 6 < |G| - 2k = 7$. If H has k disjoint cycles then each of the at least $k - (2k-8) = 8-k$ cycles that do not meet I use 4 vertices of G, and the other cycles use at least 2 vertices of G. So $15 = |G| \geq 2k + 2(8-k) = 16$, a contradiction.

Notation. A *bud* is a vertex with degree 0 or 1. A vertex is *high* if it has degree at least $2k-1$, and *low* otherwise. For vertex subsets A, B of a graph $G = (V, E)$, let

$$\|A, B\| := \sum_{u \in A} |\{uv \in E(G) : v \in B\}|.$$

Note A and B need not be disjoint. For example, $\|V,V\| = 2\|G\| = 2|E|$. We will abuse this notation to a certain extent. If A is a subgraph of G, we write $\|A,B\|$ for $\|V(A),B\|$, and if $\mathcal{A}$ is a set of disjoint subgraphs, we write $\|\mathcal{A},B\|$ for $\left\|\bigcup_{H\in\mathcal{A}} V(H),B\right\|$. Similarly, for $u \in V(G)$, we write $\|u,B\|$ for $\|\{u\},B\|$. Formally, an edge $e = uv$ is the set $\{u,v\}$; we often write $\|e,A\|$ for $\|\{u,v\},A\|$.

If T is a tree or a directed cycle and $u,v \in V(T)$ we write uTv for the unique subpath of T with endpoints u and v. We also extend this: if $w \notin T$, but has exactly one neighbor $u \in T$, we write wTv for $w(T+w+wu)v$. Finally, if w has exactly two neighbors $u,v \in T$, we may write wTw for the cycle $wuTvw$.

2.2 Proof of Theorem 2.1.7

Suppose $G = (V,E)$ is an edge-maximal counterexample to Theorem 2.1.7. That is, for some $k \geq 3$, (H1)–(H3) hold, and G does not contain k disjoint cycles, but adding any edge $e \in E(\overline{G})$ to G results in a graph with k disjoint cycles. The edge e will be in precisely one of these cycles, so G contains $k-1$ disjoint cycles, and at least three additional vertices. Choose a set $\mathcal{C}$ of disjoint cycles in G so that:

(O1) $|\mathcal{C}|$ is maximized;

(O2) subject to (O1), $\sum_{C\in\mathcal{C}} |C|$ is minimized;

(O3) subject to (O1) and (O2), the length of a longest path P in $R := G - \bigcup\mathcal{C}$ is maximized;

(O4) subject to (O1), (O2), and (O3), $\|R\|$ is maximized.

Call such a $\mathcal{C}$ an *optimal set*. We prove in Subsection 2.2.1 that R is a path, and in Subsection 2.2.2 that $|R| = 3$. We develop the structure of $\mathcal{C}$ in Subsection 2.2.3. Finally, in Subsection 2.2.4, these results are used to prove Theorem 2.1.7.

Our arguments will have the following form. We will make a series of claims about our optimal set $\mathcal{C}$, and then show that if any part of a claim fails, then we could have improved $\mathcal{C}$ by replacing a sequence $C_1,\dots,C_t \in \mathcal{C}$ of at most three cycles by another sequence of cycles $C'_1,\dots,C'_{t'}$. Naturally, this modification may also change R or P. We will express the contradiction by writing "$C'_1,\dots,C'_t,R',P'$ beats $C_1,\dots,C_t,R,P$," and may drop R' and R or P' and P if they are not involved in the optimality criteria.

This proof implies a polynomial time algorithm. We start by adding enough extra edges—at most $3k$—to obtain from G a graph with a set $\mathcal{C}$ of k disjoint cycles. Then we remove the extra edges in $\mathcal{C}$ one at a time. After removing an extra edge, we calculate a new collection $\mathcal{C}'$. This is accomplished by checking the series of claims, each in polynomial time. If a claim fails, we calculate a better collection (again in polynomial time) and restart the check, or discover an independent set of size greater than $|G| - 2k$. As there can be at most n^4 improvements, corresponding to adjusting the four parameters (O1)–(O4), this process ends in polynomial time.

We now make some simple observations. Recall that $|\mathcal{C}| = k-1$ and R is acyclic. By (O2) and our initial remarks, $|R| \geq 3$. Let a_1 and a_2 be the endpoints of P. (Possibly, R is an independent set, and $a_1 = a_2$.)

Claim 2.2.1. For all $w, w' \in V(R)$ and $C \in \mathcal{C}$, if $\|w, C\| \geq 2$ then $3 \leq |C| \leq 6 - \|w, C\|$. In particular, (a) $\|w, C\| \leq 3$, (b) if $\|w, C\| = 3$ then $|C| = 3$, and (c) if $|C| = 4$ then the two neighbors of w in C are nonadjacent.

Proof. Let $\overrightarrow{C}$ be a cyclic orientation of C. For distinct $u, v \in N(w) \cap C$, the cycles $wu\overrightarrow{C}vw$ and $wu\overleftarrow{C}vw$ have length at least $|C|$ by (O2). Thus $2\,\|C\| \leq \|wu\overrightarrow{C}vw\| + \|wu\overleftarrow{C}vw\| = \|C\| + 4$. So $|C| \leq 4$. Similarly, if $\|w, C\| \geq 3$ then $3\|C\| \leq \|C\| + 6$, and so $|C| = 3$. $\square$

Claim 2.2.2. If $xy \in E(R)$ and $C \in \mathcal{C}$ with $|C| \geq 4$ then $N(x) \cap N(y) \cap C = \emptyset$.

2.2.1 R is a path

Suppose R is not a path. Let L be the set of buds in R; then $|L| \geq 3$.

Claim 2.2.3. For all $C \in \mathcal{C}$, distinct $x, y, z \in V(C)$, $i \in [2]$, and $u \in V(R - P)$:

(a) $\{ux, uy, a_iz\} \nsubseteq E$;

(b) $\|\{u, a_i\}, C\| \leq 4$;

(c) $\{a_ix, a_iy, a_{3-i}z, zu\} \nsubseteq E$;

(d) if $\|\{a_1, a_2\}, C\| \geq 5$ then $\|u, C\| = 0$;

(e) $\|\{u, a_i\}, R\| \geq 1$; in particular $\|a_i, R\| = 1$ and $|P| \geq 2$;

(f) $4 - \|u, R\| \leq \|\{u, a_i\}, C\|$ and $\|\{u, a_i\}, D\| = 4$ for at least $|\mathcal{C}| - \|u, R\|$ cycles $D \in \mathcal{C}$.

Proof. (a) Else $ux(C - z)yu, Pa_iz$ beats C, P by (O3) (see Figure 2.2(a)).
(b) Else $|C| = 3$ by Claim 2.2.1. So there are distinct $p, q, r \in V(C)$ with $up, uq, a_ir \in E$, contradicting (a).
(c) Else $a_ix(C - z)ya_i, (P - a_i)a_{3-i}zu$ beats C, P by (O3) (see Figure 2.2(b)).
(d) Suppose $\|\{a_1, a_2\}, C\| \geq 5$ and $p \in N(u) \cap C$. By Claim 2.2.1, $|C| = 3$. Pick $j \in [2]$ with $pa_j \in E$, preferring $\|a_j, C\| = 2$. Then $V(C) - p \subseteq N(a_{3-j})$, contradicting (c).
(e) Since a_i is an end of the maximal path P, $N(a_i) \cap R \subseteq P$; so $a_iu \notin E$. By (b)

$$4(k-1) \geq \|\{u, a_i\}, V \smallsetminus R\| \geq 4k - 3 - \|\{u, a_i\}, R\|\,. \tag{2.1}$$

Thus $\|\{u, a_i\}, R\| \geq 1$. Hence $G[R]$ has an edge, $|P| \geq 2$, and $\|a_i, P\| = \|a_i, R\| = 1$.

(f) By (2.1) and (e), $\|\{u, a_i\}, V \smallsetminus R\| \geq 4|\mathcal{C}| - \|u, R\|$. Using (b), this implies the second assertion, and $\|\{u, a_i\}, C\| + 4(|\mathcal{C}| - 1) \geq 4|\mathcal{C}| - \|u, R\|$ implies the first assertion. ☐

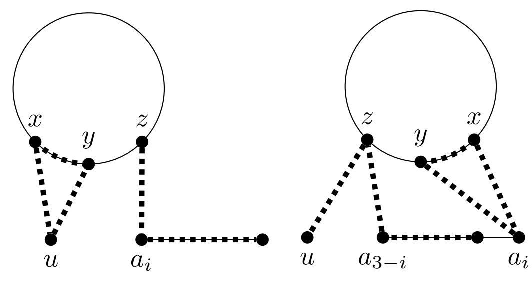


Figure 2.2: Claim 2.2.3

Claim 2.2.4. $|P| \geq 3$. In particular, $a_1a_2 \notin E(G)$.

Proof. Suppose $|P| \leq 2$. Then $\|u, R\| \leq 1$. As $|L| \geq 3$, there is a bud $c \in L \smallsetminus \{a_1, a_2\}$. By Claim 2.2.3(f), there exists $C = z_1 \ldots z_t z_1 \in \mathcal{C}$ such that $\|\{c, a_1\}, C\| = 4$ and $\|\{c, a_2\}, C\| \geq 3$.

If $\|c, C\| = 3$ then a_1c contradicts Claim 2.2.3(a). If $\|c, C\| = 1$ then $\|\{a_1, a_2\}, C\| = 5$, contradicting Claim 2.2.3(d). Therefore, we assume $\|c, C\| = 2 = \|a_1, C\|$ and $\|a_2, C\| \geq 1$. By Claim 2.2.3(a), $N(a_1) \cup N(a_2) = N(c)$. So there exists $z_i \in N(a_1) \cap N(a_2)$ and $z_j \in N(c) - z_i$. Then $a_1a_2z_ia_1, cz_jz_{j\pm1}$ beats C, P by (O3). ☐

Claim 2.2.5. Let $c \in L - a_1 - a_2$, $C \in \mathcal{C}$, and $i \in [2]$.

(a) $\|a_1, C\| = 3$ if and only if $\|c, C\| = 0$, and if and only if $\|a_2, C\| = 3$.

(b) There is at most one cycle $D \in \mathcal{C}$ with $\|a_i, D\| = 3$.

(c) For every $C \in \mathcal{C}$, $\|a_i, C\| \geq 1$ and $\|c, C\| \leq 2$.

(d) If $\|\{a_i, c\}, C\| = 4$ then $\|a_i, C\| = 2 = \|c, C\|$.

Proof. (a) If $\|c, C\| = 0$ then by Claims 2.2.1 and 2.2.3(f), $\|a_i, C\| = 3$. If $\|a_i, C\| \geq 3$ then by Claim 2.2.3(b), $\|c, C\| \leq 1$. By Claim 2.2.3(f), $\|a_{3-i}, C\| \geq 2$, and by Claim 2.2.3(d), $\|c, C\| = 0$.

(b) As $c \in L$, $\|c, R\| \leq 1$. Thus Claim 2.2.3(f) implies $\|c, D\| = 0$ for at most one cycle $D \in \mathcal{C}$.

(c) Suppose $\|c, C\| = 3$. By Claim 2.2.3(a), $\|\{a_1, a_2\}, C\| = 0$. By Claims 2.2.4 and 2.2.3(d):

$$4k - 3 \leq \|\{a_1, a_2\}, R \cup C \cup (V - R - C)\| \leq 2 + 0 + 4(k-2) = 4k - 6,$$

a contradiction. So $\|c, C\| \leq 2$. Thus by Claim 2.2.3(f), $\|a_i, C\| \geq 1$.

(d) Now (d) follows from (a). $\square$

Claim 2.2.6. R has no isolated vertices.

Proof. Suppose $c \in L$ is isolated. Fix $C \in \mathcal{C}$. By Claim 2.2.3(f), $\|\{c, a_1\}, C\| = 4$. By Claim 2.2.5(d), $\|a_1, C\| = 2 = \|c, C\|$; so $d(c) = 2(k-1)$. By Claim 2.2.3(a), $N(a_1) \cap C = N(c) \cap C$. Let $w \in V(C) \smallsetminus N(c)$. Then $d(w) \geq 4k - 3 - d(c) = 2k - 1 = 2|\mathcal{C}| + 1$. So, either $\|w, R\| \geq 1$ or $|N(w) \cap D| = 3$ for some $D \in \mathcal{C}$. In the first case, $c(C - w)c$ beats C by (O4). In the second case, by 2.2.5(c) there exists some $x \in N(a_1) \cap D$. So $c(C - w)c, w(D - x)w$ beats C, D by (O3). $\square$

Claim 2.2.7. L is an independent set.

Proof. Suppose $c_1c_2 \in E(L)$. By Claim 2.2.4, $c_1, c_2 \notin P$. By Claim 2.2.3(f) and using $k \geq 3$, there is $C \in \mathcal{C}$ with $\|\{a_1, c_1\}, C\| = 4$ and $\|\{a_1, c_2\}, C\|$, $\|\{a_2, c_1\}, C\| \geq 3$. By Claim 2.2.5(d), $\|a_1, C\| = 2 = \|c_1, C\|$; so $\|a_2, C\|$, $\|c_2, C\| \geq 1$. By Claim 2.2.3(a), $N(a_1) \cap C, N(a_2) \cap C \subseteq N(c_1) \cap C$. So there are distinct $x, y \in N(c_1) \cap C$ with $xa_1, xa_2, ya_1 \in E$. If $xc_2 \in E$ then $c_1c_2xc_1$, ya_1Pa_2 beats C, P by (O3). Else $a_1Pa_2xa_1$, $c_1(C - x)c_2c_1$ beats C, P by (O1). $\square$

Claim 2.2.8. If $|L| \geq 3$ then for some $D \in \mathcal{C}$, $\|l, C\| = 2$ for every $C \in \mathcal{C} - D$ and every $l \in L$.

Proof. Suppose some $D_1, D_2 \in \mathcal{C}$ and $l_1, l_2 \in L$ satisfy $D_1 \neq D_2$ and $\|l_1, D_1\| \neq 2 \neq \|l_2, D_2\|$.

CASE 1: $l_j \notin \{a_1, a_2\}$ for some $j \in [2]$. Say $j = 1$. For $i \in [2]$: $\|\{a_i, l_1\}, D_1\| \neq 4$ by Claim 2.2.5(d); $\|\{a_i, l_1\}, D_2\| = 4$ by Claim 2.2.3(f); $\|a_i, D_2\| = 2$ by Claim 2.2.5(d). So $l_2 \notin \{a_1, a_2\}$. By Claim 2.2.7, $l_1l_2 \notin E(G)$. So Claim 2.2.5(c) yields the contradiction:

$$4k - 3 \leq \|\{l_1, l_2\}, R \cup D_1 \cup D_2 \cup (V - R - D_1 - D_2)\| \leq 2 + 3 + 3 + 4(k-3) = 4k - 4.$$

CASE 2: $\{l_1, l_2\} \subseteq \{a_1, a_2\}$. Let $c \in L - l_1 - l_2$. As above, $\|\{l_1, c\}, D_1\| \neq 4$, and so $\|c, D_2\| = 2 = \|l_1, D_2\|$. This implies $l_1 \neq l_2$. By Claim 2.2.5(a,c), $\|l_2, D_2\| = 1$. Thus $\|\{l_2, c\}, D_1\| = 4$; so $\|c, D_1\| = 2$, and $\|l_1, D_1\| = 1$. With Claim 2.2.4, this yields the contradiction:

$$4k - 3 \leq \|\{l_1, l_2\}, R \cup D_1 \cup D_2 \cup (V - R - D_1 - D_2)\| \leq 2 + 3 + 3 + 4(k-3) = 4k - 4.$$

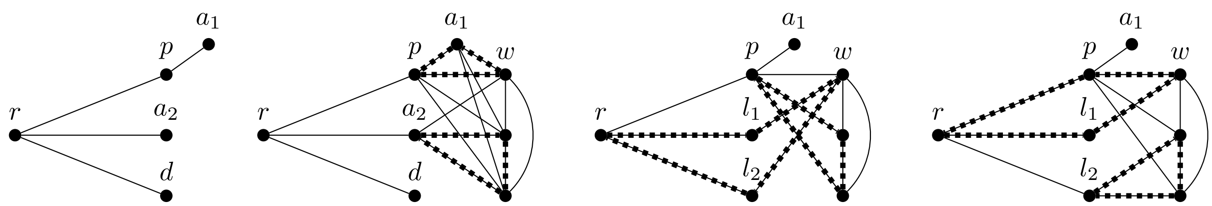
$\square$

Claim 2.2.9. R is a subdivided star (possibly a path).

Proof. Suppose not. Then we claim R has distinct leaves $c_1, d_1, c_2, d_2 \in L$ such that c_1Rd_1 and c_2Rd_2 are disjoint paths. Indeed, if R is disconnected then each component has two distinct leaves by Claim 2.2.6. Else R is a tree. As R is not a subdivided star, it has distinct vertices s_1 and s_2 with degree at least three. Deleting the edges and interior vertices of s_1Rs_2 yields disjoint trees containing all leaves of R. Let T_i be the tree containing s_i, and pick $c_i, d_i \in T_i$.

By Claim 2.2.8, using $k \geq 3$, there is a cycle $C \in \mathcal{C}$ such that $\|l, C\| = 2$ for all $l \in L$. By Claim 2.2.3(a), $N(a_1) \cap C = N(l) \cap C = N(a_2) \cap C =: \{w_1, w_3\}$ for $l \in L - a_1 - a_2$. Then replacing C in $\mathcal{C}$ with $w_1c_1Rd_1w_1$ and $w_3c_2Rd_2w_3$ yields k disjoint cycles. $\square$

Claim 2.2.10. R is a path or a star.



(a) (b) (c) (d)

Figure 2.3: Claim 2.2.10

Proof. By Claim 2.2.9, R is a subdivided star. If R is neither a path nor a star then there are vertices r, p, d with $\|r, R\| \geq 3$, $\|p, R\| = 2$, $d \in L - a_1 - a_2$ and (say) $pa_1 \in E$. Then a_2Rd is disjoint from pa_1 (see Figure 2.3(a)). By Claim 2.2.5(c), $d(d) \leq 1 + 2(k-1) = 2k - 1$. So

$$\|p, V - R\| \geq 4k - 3 - \|p, R\| - d(d) \geq 4k - 5 - (2k - 1) = 2k - 4 \geq 2. \tag{2.2}$$

In each of the following cases, $R \cup C$ has two disjoint cycles, contradicting (O1).

CASE 1: $\|p, C\| = 3$ for some $C \in \mathcal{C}$. Then $|C| = 3$. By Claim 2.2.5(a), if $\|d, C\| = 0$ then $\|a_1, C\| = 3 = \|a_2, C\|$. Then for $w \in C$, wa_1pw and $a_2(C - w)a_2$ are disjoint cycles (see Figure 2.3(b)). Else by Claim 2.2.5(c), $\|d, C\|$, $\|a_2, C\| \in \{1, 2\}$. By Claim 2.2.3(f), $\|\{d, a_2\}, C\| \geq 3$, so there are $l_1, l_2 \in \{a_2, d\}$ with $\|l_1, C\| \geq 1$ and $\|l_2, C\| = 2$; say $w \in N(l_1) \cap C$. If $l_2w \in E$ then wl_1Rl_2w and $p(C - w)p$ are disjoint cycles (see Figure 2.3(c)); else l_1wpRl_1 and $l_2(C - w)l_2$ are disjoint cycles (see Figure 2.3(d)).

CASE 2: There are distinct $C_1, C_2 \in \mathcal{C}$ with $\|p, C_1\|, \|p, C_2\| \geq 1$. By Claim 2.2.8, for some $i \in [2]$ and all $c \in L$, $\|c, C_i\| = 2$. Let $w \in N(p) \cap C_i$. If $wa_1 \in E$ then $D := wpa_1w$ is a cycle and $G[(C_i - w) \cup a_2Rd]$ contains cycle disjoint from D. Else, if $w \in N(a_2) \cup N(d)$, say $w \in N(c)$, then $a_1(C_i - w)a_1$ and $cwpRc$ are disjoint cycles. Else, by Claim 2.2.1 there exist vertices $u \in N(a_2) \cap N(d) \cap C_i$ and $v \in N(a_1) \cap C_i - u$. Then ua_2Rdu and $a_1v(C_i - u)wpa_1$ are disjoint cycles.

CASE 3: Otherwise. Then using (2.2), $\|p, V - R\| = 2 = \|p, C\|$ for some $C \in \mathcal{C}$. In this case, $k = 3$ and $d(p) = 4$. By (H2), $d(a_2), d(d) \geq 5$. Say $\mathcal{C} = \{C, D\}$. By Claim 2.2.3(b), $\|\{a_2, d\}, D\| \leq 4$. So

$$\|\{a_2, d\}, C\| = \|\{a_2, d\}, (V - R - D)\| \geq 10 - 2 - 4 = 4.$$

By Claim 2.2.5(c, d), $\|a_2, C\| = \|d, C\| = 2$ and $\|a_1, C\| \geq 1$. Say $w \in N(a_1) \cap C$. If $wp \in E$ then $dRa_2(C-w)d$ contains a cycle disjoint from wa_1pw. Else, by Claim 2.2.3(a) there exists $x \in N(a_2) \cap N(d) \cap C$. If $x \neq w$ then xa_2Rdx and $wa_1p(C - x)w$ are disjoint cycles. Else $x = w$, and xa_2Rdx and $p(C - w)p$ are disjoint cycles. □

Lemma 2.2.11. *R is a path.*

Proof. Suppose R is not a path. Then it is a star with root r and at least three leaves, any of which can play the role of a_i or a leaf in $L - a_1 - a_2$. Thus Claim 2.2.5(c) implies $\|l, C\| \in \{1, 2\}$ for all $l \in L$ and $C \in \mathcal{C}$. By Claim 2.2.8 there is $D \in \mathcal{C}$ such that for all $l \in L$ and $C \in \mathcal{C} - D$, $\|l, C\| = 2$. By Claim 2.2.3(f) there is $l \in L$ such that for all $c \in L - l$, $\|c, D\| = 2$. Fix distinct leaves $l', l'' \in L - l$.

Let $Z = N(l') - R$ and $A = V \smallsetminus (Z \cup \{r\})$. By the first paragraph, every $C \in \mathcal{C}$ satisfies $|Z \cap C| = 2$. So $|A| = |G| - 2k + 1$. For a contradiction, we show that A is independent.

Note $A \cap R = L$, so by Claim 2.2.7, $A \cap R$ is independent. By Claim 2.2.3(a),

$$\text{for all } c \in L \text{ and for all } C \in \mathcal{C},\ N(c) \cap C \subseteq Z. \tag{2.3}$$

So $\|L, A\| = 0$. By Claim 2.2.1(c), for all $C \in \mathcal{C}$, $C \cap A$ is independent. Suppose, for a contradiction, A is not independent. Then there exist distinct $C_1, C_2 \in \mathcal{C}$, $v_1 \in A \cap C_1$, and $v_2 \in A \cap C_2$ with $v_1v_2 \in E$. Subject to this choose C_2 with $\|v_1, C_2\|$ maximum. Let $Z \cap C_1 = \{x_1, x_2\}$ and $Z \cap C_2 = \{y_1, y_2\}$.

CASE 1: $\|v_1, C_2\| \geq 2$. Choose $i \in [2]$ so that $\|v_1, C_2 - y_i\| \geq 2$. Then define $C_1^* := v_1(C_2 - y_i)v_1$, $C_2^* := l'x_1(C_1 - v_1)x_2l'$, and $P^* := y_il''rl$ (see Figure 2.4(a)). By (3.4), P^* is a path and C_2^* is a cycle. So C_1^*, C_2^*, P^* beats C_1, C_2, P by (O3).

CASE 2: $\|v_1, C_2\| \leq 1$. Then for all $C \in \mathcal{C}$, $\|v_1, C\| \leq 2$ and $\|v_1, C_2\| = 1$; so $\|v_1, \mathcal{C}\| = \|v_1, C_2 \cup (\mathcal{C} - C_2)\| \leq 1 + 2(k - 2) = 2k - 3$. By (3.4) $\|v_1, L\| = 0$ and $d(l) \leq 2k - 1$. So by (H2), $\|v_1, r\| = \|v_1, R\| = (4k - 3) - \|v_1, \mathcal{C}\| - d(l) \leq (4k - 3) - (2k - 3) - (2k - 1) = 1$, and $v_1r \in E$. Let $C_1^* := l'x_1(C_1 - v_1)x_2l'$, $C_2^* := l''y_1(C_2 - v_2)y_2l''$, and $P^* := v_2v_1rl$ (see Figure 2.4(b)). Then C_1^*, C_2^*, P^* beats C_1, C_2, P by (O3). □

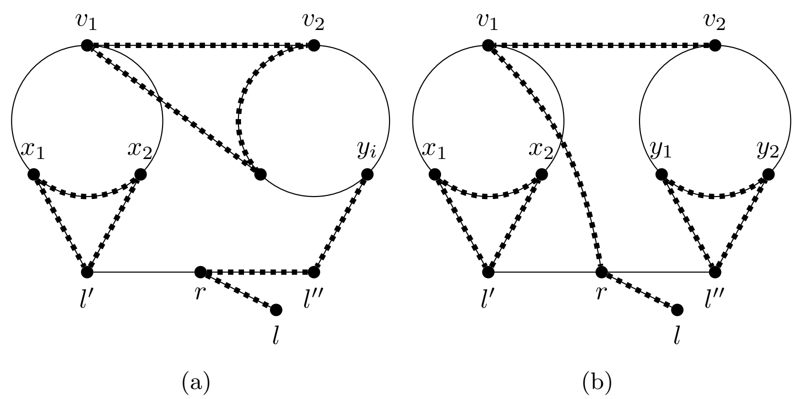


Figure 2.4: Claim 2.2.10

2.2.2 $|\mathbf{R}| = 3$

By Lemma 2.2.11, R is a path, and by Claim 2.2.4, $|R| \geq 3$. Next we prove $|R| = 3$. First, we prove a claim that will also be useful in later sections.

Claim 2.2.12. Let C be a cycle, $P = v_1v_2 \dots v_s$ be a path, and $1 < i < s$. At most one of the following two statements holds.

(1) (a) $\|x, v_1Pv_{i-1}\| \geq 1$ for all $x \in C$ or (b) $\|x, v_1Pv_{i-1}\| \geq 2$ for two $x \in C$;

(2) (c) $\|y, v_iPv_s\| \geq 2$ for some $y \in C$ or (d) $N(v_i) \cap C \neq \emptyset$ and $\|v_{i+1}Pv_s, C\| \geq 2$.

Proof. Suppose (1) and (2) hold. If (c) holds then the disjoint graphs $G[v_iPv_s + y]$ and $G[v_1Pv_{i-1} \cup C - y]$ contain cycles. Else (d) holds, but (c) fails; say $z \in N(v_i) \cap C$ and $z \notin N(v_{i+1}Pv_s)$. If (a) holds then $G[v_1Pv_i + z]$ and $G[v_{i+1}Pv_s \cup C - z]$ contain cycles. If (b) holds then $G[v_1Pv_{i-1} + w]$ and $G[v_iPv_s \cup C - w]$ contain cycles, where $\|w, v_1Pv_{i-1}\| \geq 2$. □

Suppose, for a contradiction, $|R| \geq 4$. Say $R = a_1a_1'a_1'' \dots a_2''a_2'a_2$. It is possible that $a_1'' \in \{a_2'', a_2'\}$, etc. Set $e_i := a_ia_i' = \{a_i, a_i'\}$ and $F := e_1 \cup e_2$.

Claim 2.2.13. If $C \in \mathcal{C}$, $h \in [2]$ and $\|e_h, C\| \geq \|e_{3-h}, C\|$ then $\|C, F\| \leq 7$; if $\|C, F\| = 7$ then

$$|C| = 3,\ \|a_h, C\| = 2,\ \ \|a_h', C\| = 3,\ \ \|a_h''Ra_{3-h}, C\| = 2,\ \text{and } N(a_h) \cap C = N(e_{3-h}) \cap C.$$

Proof. We will repeatedly use Claim 2.2.12 to obtain a contradiction to (O1) by showing that $G[C \cup R]$ contains two disjoint cycles. Suppose $\|C, F\| \geq 7$ and say $h = 1$. Then $\|e_1, C\| \geq 4$. So there is $x \in e_1$ with $\|x, C\| \geq 2$. Thus $|C| \leq 4$ by Claim 2.2.1, and if $|C| = 4$ then no vertex in C has two adjacent neighbors in F. So (1) holds with $v_1 = a_1$ and $v_i = a_2'$, even when $|C| = 4$.

If $\|e_1, C\| = 4$, as is the case when $|C| = 4$, then $\|e_2, C\| \geq 3$. If $|C| = 4$ there is a cycle $D := yza_2'a_2y$ for some $y, z \in C$. As (a) holds, $G[a_1Ra_2'' \cup C - y - z]$ contains another disjoint cycle. So $|C| = 3$. As (c) must fail with $v_i = a_2'$, (a) and (c) hold for $v_i = a_1'$ and $v_1 = a_2$, a contradiction. So $\|e_1, C\| \geq 5$. If $\|a_1, C\| = 3$ then (a) and (c) hold with $v_1 = a_1$ and $v_i = a_1'$. So $\|a_1, C\| = 2$, $\|a_1', C\| = 3$ and $\|a_1''Ra_2, C\| \geq 2$. If there is $b \in P - e_1$ and $c \in N(b) \cap V(C) \smallsetminus N(a_1)$ then $G[a_1'Ra_2 + c]$ and $G[a_1(C - c)a_1]$ both contain cycles. So for every $b \in R - e_1$, $N(b) \cap C \subseteq N(a_1)$. Then if $\|a_1''Ra_2, C\| \geq 3$, (c) holds for $v_1 = a_1$ and $v_1 = a_1''$, contradicting that (1) holds. So $\|a_1''Ra_2, C\| = \|e_1, C\| = 2$ and $N(a_1) = N(e_2)$. $\square$

Lemma 2.2.14. $|R| = 3$ *and* $m := \max\{|C| : C \in \mathcal{C}\} = 4$.

Proof. Let $t = |\{C \in \mathcal{C} : \|F, C\| \leq 6\}|$ and $r = |\{C \in \mathcal{C} : |C| \geq 5\}|$. It suffices to show $r = 0$ and $|R| = 3$: then $m \leq 4$, and $|V(\mathcal{C})| = |G| - |R| \geq 3(k-1) + 1$ implies some $C \in \mathcal{C}$ has length 4. Choose R so that:

(P1) R has as few low vertices as possible, and subject to this

(P2) R has a low end if possible.

Let $C \in \mathcal{C}$. By Claim 2.2.13, $\|F, C\| \leq 7$. By Claim 2.2.1, if $|C| \geq 5$ then $\|a, C\| \leq 1$ for all $a \in F$; so $\|F, C\| \leq 4$. Thus $r \leq t$. Hence

$$2(4k - 3) \leq \|F, (V \smallsetminus R) \cup R\| \leq 7(k-1) - t - 2r + 6 \leq 7k - t - 2r - 1. \tag{2.4}$$

So $5 - k \geq t + 2r \geq 3r \geq 0$. Since $k \geq 3$, this yields $3r \leq t + 2r \leq 2$, so $r = 0$ and $t \leq 2$, with $t = 2$ only if $k = 3$.

CASE 1: $k - t \geq 3$. That is, there exist distinct cycles $C_1, C_2 \in \mathcal{C}$ with $\|F, C_i\| \geq 7$. In this case, $t \leq 1$: if $k = 3$ then $\mathcal{C} = \{C_1, C_2\}$ and $t = 0$; if $k > 3$ then $t < 2$. For both $i \in [2]$, Claim 2.2.13 yields $\|F, C_i\| = 7$, $|C_i| = 3$, and there is $x_i \in V(C_i)$ with $\|x_i, R\| = 1$ and $\|y, R\| = 3$ for both $y \in V(C_i - x_i)$. Moreover, there is a unique index $j = \beta(i) \in [2]$ with $\left\|a_j', C_i\right\| = 3$. For $j \in [2]$, put $I_j := \{i \in [2] : \beta(i) = j\}$; that is, $I_j = \{i \in [2] : \|a_j', C_i\| = 3\}$. Then $V(C_i) - x_i = N(a_{\beta(i)}) \cap C_i = N(e_{3-\beta(i)}) \cap C_i$. As $x_i a_{\beta(i)} \notin E$, one of $x_i, a_{\beta(i)}$ is high. As we can switch x_i and $a_{\beta(i)}$ (by replacing C_i with $a_{\beta(i)}(C_i - x_i)a_{\beta(i)}$ and R with $R - a_{\beta(i)} + x_i$), we may assume $a_{\beta(i)}$ is high.

Suppose $I_j \neq \emptyset$ for both $j \in [2]$; say $\|a_1', C_1\| = \|a_2', C_2\| = 3$. Then for all $B \in \mathcal{C}$ and $j \in [2]$, a_j is high, and either $\|a_j, B\| \leq 2$ or $\|F, B\| \leq 6$. So since $t \leq 1$,

$$2k - 1 \leq d(a_j) = \|a_j, B \cup F\| + \|a_j, \mathcal{C} - B\| \leq \|a_j, B\| + 1 + 2(k-2) + t \leq 2k - 2 + \|a_j, B\|.$$

Thus $N(a_j) \cap B \neq \emptyset$ for all $B \in \mathcal{C}$. Let $y_j \in N(a_{3-j}) \cap C_j$. Then using Claim 2.2.13, $y_j \in N(a_j)$, and $a_1'(C_1 - y_1)a_1'$, $a_2'(C_2 - y_2)a_2'$, $a_1y_1a_2y_2a_1$ beats C_1, C_2 by (O1).

Otherwise, say $I_1 = \emptyset$. If $B \in \mathcal{C}$ with $\|F, B\| \le 6$ then $\|e_1, B\| + 2\|a_2, B\| \le \|F, B\| + \|a_2, B\| \le 9$. Thus, using Claim 2.2.13,

$$2(4k-3) \le d(a_1) + d(a_1') + 2d(a_2) = 5 + \|e_1, \mathcal{C}\| + 2\|a_2, \mathcal{C}\| \le 5 + 6(k-1-t) + 9t$$
$$\Rightarrow 2k \le 5 + 3t.$$

Since $k - t \ge 3$ (by the case), we see $3(k-t) + (5+3t) \ge 3(3) + 2k$ and so $k \ge 4$. Since $t \le 1$, in fact $k = 4$ and $t = 1$, and equality holds throughout: say B is the unique cycle in $\mathcal{C}$ with $\|F, B\| \le 6$. Then $\|a_2, B\| = \|e_1, B\| = 3$. Using Claim 2.2.13, $d(a_1) + d(a_1') = \|e_1, R\| + \|e_1, \mathcal{C} - B\| + \|e_1, B\| = 3 + 4 + 3 = 10$, and $d(a_1), d(a_2) \ge (4k-3) - d(a_2) = 13 - (1+4+3) = 5$, so $d(a_1) = d(a_2) = 5$. Note a_1 and a_2 share no neighbors: they share none in R because R is a path, they share none in $\mathcal{C} - B$ by Claim 2.2.13, and they share no neighbor $b \in B$ lest $a_1a_1'ba_1$ and $a_2(B-b)a_2$ beat B by (O1). Thus every vertex in $V - e_1$ is high.

Since $\|e_1, B\| = 3$, first suppose $\|a_1, B\| \ge 2$, say $B - b \subseteq N(a_1)$. Then $a_1(B-b)a_1$, $a_1'a_2'a_1b$ beat B, R by (P1) (see Figure 2.5(a)). Now suppose $\|a_1', B\| \ge 2$, this time with $B - b \subseteq N(a_1')$. Since $d(a_1) = 5$ and $\|a_1, R \cup B\| \le 2$, there exists $c \in C \in \mathcal{C} - B$ with $a_1c \in E(G)$. Now $c \in N(a_2)$ by Claim 2.2.13, so $a_1'(B-b)a_1'$, $a_2'(C-c)a_2'$, and a_1ca_2b beat B, C, and R by (P1) (see Figure 2.5(b)).

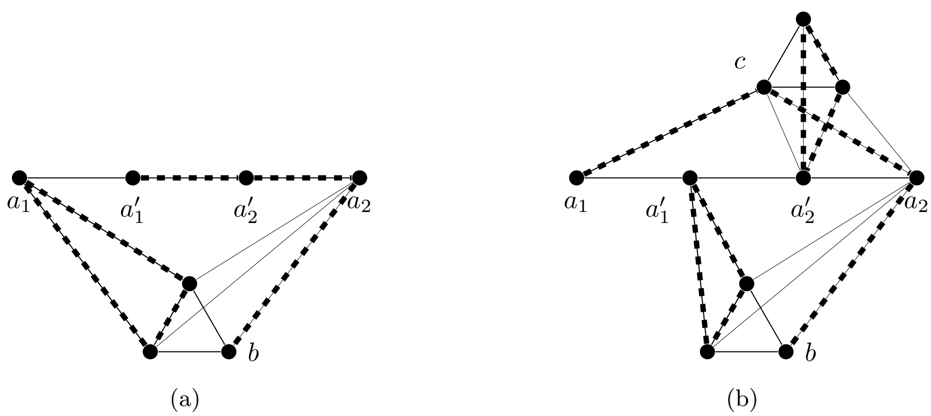


Figure 2.5: Lemma 2.2.14, Case 1

CASE 2: $k - t \le 2$. That is, $\|F, C\| \le 6$ for all but at most one $C \in \mathcal{C}$. Then, since $5 - k \ge t$, $k = 3$ and $\|F, V\| \le 19$. Say $\mathcal{C} = \{C, D\}$, so $\|F, C \cup D\| \ge 2(4k-3) - \|F, R\| = 2(4 \cdot 3 - 3) - 6 = 12$. By Claim 2.2.13, $\|F, C\|$, $\|F, D\| \le 7$. So $\|F, C\|$, $\|F, D\| \ge 5$. If $|R| \ge 5$, then for the (at most two) low vertices in R, we can choose distinct vertices in R not adjacent to them. So $\|R, V - R\| \ge 5|R| - 2 - \|R, R\| = 3|R|$. Thus we may assume $\|R, C\| \ge \lceil 3|R|/2 \rceil \ge |R| + 3 \ge 8$. Let $w' \in C$ be such that $q = \|w', R\| = \max\{\|w, R\| : w \in C\}$. Let $N(w') \cap R = \{v_{i_1}, \ldots, v_{i_q}\}$ with $i_1 < \ldots < i_q$. Suppose $q \ge 4$. If $\|v_1Rv_{i_2}, C - w'\| \ge 2$ or $\|v_{i_2+1}Rv_s, C - w'\| \ge 2$, then $G[C \cup R]$ has two disjoint cycles. Otherwise, $\|R, C - w'\| \le 2$, contradicting

$\|R, C\| \geq |R| + 3$. Similarly, if $q = 3$, then $\|v_1 R v_{i_2-1}, C - w'\| \leq 1$ and $\|v_{i_2+1} R v_s, C - w'\| \leq 1$ yielding $\|v_{i_2}, C\| = \|R, C\| - \|(R - v_{i_2}), C - w'\| - \|R - v_{i_2}, w'\| \geq (|R| + 3) - 2 - (3 - 1) \geq 4$, a contradiction to Claim 2.2.1(a). So $q \leq 2$ and hence $|R| + 3 \leq \|R, C\| \leq 2|C|$. It follows that $|R| = 5$, $|C| = 4$ and $\|w, R\| = 2$ for each $w \in C$. This in turn yields that $G[C \cup R]$ has no triangles and $\|v_i, C\| \leq 2$ for each $i \in [5]$. By Claim 2.2.13, $\|F, C\| \leq 6$, so $\|v_3, C\| = 2$. Thus we may assume that for some $w \in C$, $N(w) \cap R = \{v_1, v_3\}$. Then $\|e_2, C\| = \|e_2, C - w\| \leq 1$, lest there exist a cycle disjoint from $wv_1v_2v_3w$ in $G[C \cup R]$. So, $\|e_1, C\| \geq 8 - 1 - 2 = 5$, a contradiction to Claim 2.2.1(b). This yields $|R| \leq 4$.

Claim 2.2.15. Either a_1 or a_2 is low.

Proof. Suppose a_1 and a_2 are high. Then since $\|R, V\| \leq 19$, we may assume a_1' is low. Suppose there is $c \in C$ with $ca_2 \in E$ and $\|a_1, C - c\| \geq 2$. If $a_1'c \in E$, then $R \cup C$ contains two disjoint cycles; so $a_1'c \notin E$ and hence c is high. Thus either $a_1(C - c)a_1$ is shorter than C or the pair $a_1(C - c)a_1$, $ca_2a_2'a_1'$ beats C, R by $(P2)$. Thus if $ca_2 \in E$ then $\|a_1, C - c\| \leq 1$. As a_2 is high, $\|a_2, C\| \geq 1$ and hence $\|a_1, C\| = \|a_1, C \setminus N(a_2)\| + \|a_1, N(a_2)\| \leq 2$. Similarly, $\|a_1, D\| \leq 2$. Since a_1 is high, $\|a_1, C\| = \|a_1, D\| = 2$, and $d(a_1) = 5$. Hence

$$N(a_2) \cap C \subseteq N(a_1) \cap C \quad \text{and} \quad N(a_2) \cap D \subseteq N(a_1) \cap D. \tag{2.5}$$

As a_2 is high, $d(a_2) = 5$ and in (2.5) equalities hold. Also $d(a_1') = 4 \leq d(a_2')$.

If there are $c \in C$ and $i \in [2]$ with $ca_i, ca_i' \in E$ then by (O2), $|C| = 3$. Also $ca_i'a_ic$, $a_{3-i}'a_{3-i}(C - c)$ beats C, R by either (P1) or (P2). (Recall $N(a_1) \cap C = N(a_2) \cap C$ and neighbors of a_2 in C are high.) So $N(a_i) \cap N(a_i') = \emptyset$. Thus the set $N(a_1) - R = N(a_2) - R$ contains no low vertices. Also, if $\|a_1', C\| \geq 1$ then $|C| = 3$: else C has the form $c_1c_2c_3c_4c_1$, where $a_1c_1, a_1c_3 \in E$, and so $a_1a_1'c_1c_2a_1$, $c_3c_4a_2a_2'$ beats C, R by either (P1) or (P2). Thus $|C| = 3$ and $a_1'c \in E$ for some $c \in V(C) - N(a_1)$. If $\|a_2', C\| \geq 1$, we have disjoint cycles $ca_1'a_2'c$, $a_1(C - c)a_1$ and D. Then $\|a_1', C\| = 0$, so $d(a_1') \leq 2 + |D \setminus N(a_1)| \leq 4$. Now a_1' and a_2' are symmetric, and we have proved that $\|a_1', C\| + \|a_2', C\| \leq 1$. Similarly, $\|a_1', D\| + \|a_2', D\| \leq 1$, a contradiction to $d(a_1'), d(a_2') \geq 4$. □

By Claim 2.2.15, we can choose notation so that a_1 is low.

Claim 2.2.16. If a_1' is low then each $v \in V \setminus e_1$ is high.

Proof. Suppose $v \in V - e_1$ is low. Since a_1 is low, all vertices in $R - e_1$ are high, so $v \in C$ for some $C \in \mathcal{C}$. Then $C' := ve_1v$ is a cycle and so by (O2), $|C| = 3$. Since a_2 is high, $\|a_2, C\| \geq 1$. As v is low, $va_2 \notin E$. Since a_1' is low, it is adjacent to the low vertex v, and $\|a_1', C - v\| \leq 1$. So $C', a_2'a_2(C - v)$ beats C, R by (P1). □

Claim 2.2.17. If $|C| = 3$ and $\|e_1, C\|$, $\|e_2, C\| \geq 3$, then either

(a) $\|c, e_1\| = 1 = \|c, e_2\|$ for all $c \in V(C)$ or

(b) a'_1 is high and there is $c \in V(C)$ with $\|c, R\| = 4$ and $C - c$ has a low vertex.

Proof. If (a) fails then $\|c, e_i\| = 2$ for some $i \in [2]$ and $c \in C$. If $\|e_{3-i}, C - c\| \geq 2$ then there is a cycle $C' \subseteq C \cup e_{3-i} - c$, and $R \cup C$ contains disjoint cycles ce_ic and C'. Else,

$$\|c, R\| = \|c, e_i\| + (\|C, e_{3-i}\| - \|C - c, e_{3-i}\|) \geq 2 + (3 - 1) = 4 = |R|.$$

If $C - c$ has no low vertices then ce_1c, $e_2(C - c)$ beats C, R by (P1). So $C - c$ contains a low c'. If a'_1 is low then $c'a'_1a_1c'$ and $ca_2a'_2c$ are disjoint cycles. So (b) holds. □

CASE 2.1: $|D| = 4$. By (O2), $G[R \cup D]$ does not contain a 3-cycle. So $5 \leq d(a_2) \leq 3 + \|a_2, C\| \leq 6$. Thus $d(a_1), d(a'_1) \geq 3$.

Suppose $\|e_1, D\| \geq 3$. Pick $v \in N(a_1) \cap D$ with minimum degree, and $v' \in N(a'_1) \cap D$. Since $N(a_1) \cap D$ and $N(a'_1) \cap D$ are nonempty, disjoint and independent, $vv' \in E$. Say $D = vv'ww'v$. As $D = K_{2,2}$ and low vertices are adjacent, $D' := a_1a'_1v'va_1$ is a 4-cycle and v is the only possible low vertex in D. Note $a_1w \notin E$: else $a_1ww'va_1$, $v'a'_1a'_2a_2$ beats D, R by (P1). As $\|e_1, D\| \geq 3$, $a'_1w' \in E$. Also note $\|e_2, ww'\| = 0$: else $G[a_2, a'_2, w, w']$ contains a 4-path R', and D', R' beats D, R by (P1). Similarly, replacing D' by $D'' := a_1a'_1w'va_1$ yields $\|e_2, v'\| = 0$. So $\|e_1 \cup e_2, D\| \leq 3 + 1 = 4$, a contradiction. Thus

$$\|e_1, D\| \leq 2 \quad \text{and so} \quad \|R, D\| \leq 6. \tag{2.6}$$

Suppose $d(a'_1) = 3$. Then $\|a'_1, D\| \leq 1$. So there is $uv \in E(D)$ with $\|a'_1, uv\| = 0$. Thus $d(u), d(v), d(a_2) \geq 6$, and $\|a_2, C\| = 3$. So $|C| = 3$, $|G| = 11$, and there is $w \in N(u) \cap N(v)$. If $w \in C$ put $C' = a_2(C - w)a_2$; else $C' = C$. In both cases, $|C'| = |C|$ and $|wuvw| = 3 < |D|$, so C', $wuvw$ beats C, D by (O2). Thus $d(a'_1) \geq 4$. If $d(a_1) = 3$ then $d(a_2), d(a'_2) \geq 9 - 3 = 6$, and $\|a_2, C\| \geq 3$. By (2.6),

$$\|R, C\| \geq 3 + 4 + 6 + 6 - \|R, R\| - \|R, D\| \geq 19 - 6 - 6 = 7,$$

contradicting Claim 2.2.13. So $d(a_1) = 4 \leq d(a'_1)$ and by (2.6), $\|e_1, C\| \geq 3$. Thus (2.6) fails for C in place of D; so $|C| = 3$. As $\|a_2, C\| \geq 2$ and $\|a'_2, C\| \geq 1$, Claim 2.2.17 implies either (a) or (b) of Claim 2.2.17 holds. If (a) holds then (a) and (d) of Claim 2.2.12 both hold, and so $G[C \cup R]$ has two disjoint cycles. Else, Claim 2.2.17 gives a'_1 is high and there is $c \in \mathcal{C}$ with $\|c, R\| = 4$. As a'_1 is high, $\|R, C\| \geq 7$. So $\|c, R\| = 4$ contradicts Lemma 2.2.13.

CASE 2.2: $|C| = |D| = 3$ and $\|R, V\| = 18$. Then $d(a_1) + d(a_2') = 9 = d(a_1') + d(a_2)$, a_1 and a_1' are low, and by Claim 2.2.16 all other vertices are high. Moreover, $d(a_1') \le d(a_1)$, since

$$18 = \|R, V\| = d(a_1') - d(a_1) + 2d(a_1) + d(a_2') + d(a_2) \ge d(a_1') - d(a_1) + 9 + 9.$$

Suppose $d(a_1') = 2$. Then $d(v) \ge 7$ for all $v \in V - a_1a_1'a_2'$. In particular, $C \cup D \subseteq N(a_2)$. If $d(a_1) = 2$ then $d(a_2') \ge 7$, and $G = \mathbf{Y_1}$. Else $\|a_1, C \cup D\| \ge 2$. If there is $c \in C$ with $V(C) - c \subseteq N(a_1)$, then $a_1(C - c)a_1$, $a_1'a_2'a_2c$ beats C, R by (P1). Else $d(a_1) = 3$, $d(a_2') = 6$, and there are $c \in C$ and $d \in D$ with $c, d \in N(a_1)$. If $ca_2' \in E$ then $C \cup R$ contains disjoint cycles $a_1ca_2'a_1'a_1$ and $a_2(C - c)a_2$, so assume not. Similarly, assume $da_2' \notin E$. Since $d(d) \ge 7$ and $a_1', a_2' \notin N(d)$, $cd \in E(G)$. Then there are three disjoint cycles $a_2'(C - c)a_2'$, $a_2(D - d)a_2$, and a_1cda_1. So $d(a_1') \ge 3$.

Suppose $d(a_1') = 3$. Say $a_1'v \in E$ for some $v \in D$. As $d(a_2) \ge 6$, $\|a_2, D\| \ge 2$. So $e_2 + D - v$ contains a 4-path R'. Thus $a_1v \notin E$: else ve_1v, R' beats D, R by (P1). Also $\|a_1, D - v\| \le 1$: else $a_1(D - v)a_1, va_1'a_2'a_2$ beats D, R by (P1). So $\|a_1, D\| \le 1$.

Suppose $\|a_1, C\| \ge 2$. Pick $c \in C$ with $C - c \subseteq N(a_1)$. Then (*) $a_2c \notin E$: else $a_1(C - c)a_1$, $a_1'a_2'a_2c$ beats C, R by (P1). So $\|a_2, C\| = 2$ and $\|a_2, D\| = 3$. Also $a_1c \notin E$: else picking a different c violates (*). As $a_1'c \notin E$, $\|c, D\| = 3$ and $a_2'c \in E(G)$. So $a_1(C - c)a_1$, $a_2(D - v)a_2$ and $cva_1'a_2'c$ are disjoint cycles. Otherwise, $\|a_1, C\| \le 1$ and $d(a_1) \le 3$. Then $d(a_1) = 3$ since $d(a_1) \ge d(a_1')$.

Now $d(a_2') = 6$. Say $D = vbb'v$ and $a_1b \in E$. As $b'a_1' \notin E$, $d(b') \ge 9 - 3 = 6$. Since $\|e_2, V\| = 12$, a_2 and a_2' have three common neighbors. If one is b' then $D' := a_1a_1'vba_1$, $b'e_2b'$, and C are disjoint cycles; else $\|b', C\| = 3$ and there is $c' \in C$ with $\|c', e_2\| = 2$. Then D', $c'e_2c'$ and $b'(C - c')b'$ are disjoint cycles. So $d(a_1') = 4$.

Since a_1 is low and $d(a_1) \ge d(a_1')$, $d(a_1) = d(a_1') = 4$ and $\|\{a_1, a_1'\}, C \cup D\| = 5$, so we may assume $\|e_1, C\| \ge 3$. If $\|e_2, C\| \ge 3$, then because a_1' is low, Claim 2.2.17(a) holds. So $V(C) \subseteq N(e_1)$ and there is $x \in e_1 = xy$ with $\|x, C\| \ge 2$. First suppose $\|x, C\| = 3$. As x is low, $x = a_1$. Pick $c \in N(a_2) \cap C$, which exists because $\|a_2, C \cup D\| \ge 4$. Then $a_1(C - c)a_1$, $a_1'a_2'a_2c$ beats C, R by (P1). Now suppose $\|x, C\| = 2$. Let $c \in C \smallsetminus N(x)$. Then $x(C - c)x$, yce_2 beats C, R by (P1).

CASE 2.3: $|C| = |D| = 3$ and $\|R, V\| = 19$. Say $\|C, R\| = 7$ and $\|D, R\| = 6$.

CASE 2.3.1: a_1' is low. Then $\|a_1', C \cup D\| \le 4 - \|a_1', R\| = 2$, so by Claim 2.2.13 $\|e_2, C\| = 5$ with $\|a_2, C\| = 2$. Then $5 \le d(a_2) \le 6$.

If $d(a_2) = 5$ then $d(a_1) = d(a_1') = 4$ and $d(a_2') = 6$. So $\|a_2, D\| = 2$ and $\|a_2', D\| = 1$. Say $D = b_1b_2b_3b_1$, where $a_2b_2, a_2b_3 \in E$. As a_1' is low, (a) of Claim 2.2.17 holds. So $\|b_1, a_1a_1'a_2'\| = 2$, and there is a cycle $D' \subseteq G[b_1a_1a_1'a_2']$. Then $a_2(D - b_1)a_2$ and D' are disjoint.

If $d(a_2) = 6$ then $\|a_2, D\| = 3$. Let $c_1 \in C - N(a_2)$. By Claim 2.2.13, $\|c_1, R\| = 1$, so c_1 is high, and $\|c_1, D\| \geq 2$. If $\|a_2', D\| \geq 1$, then (a) and (d) hold in Claim 2.2.12 for $v_1 = a_2$ and $v_i = a_2'$, so $G[D \cup c_1 a_2' a_2]$ has two disjoint cycles, and $c_2 e_1 c_3 c_2$ contains a third. So assume $\|a_2', D\| = 0$, and so $d(a_2') = 5$. Thus $d(a_1) = d(a_1') = 4$. Again, $\|e_1, D\| = 3 = \|a_2, D\|$. So there are $x \in e_1$ and $b \in V(D)$ with $D - b \subseteq N(x)$. As a_1' is low and has two neighbors in R, if $\|x, D\| = 3$ then $x = a_1$. Anyway, using Claim 2.2.17, $G[R + b - x]$ contains a 4-path R', and $x(D-b)x$, R' beats D, R by (P1).

CASE 2.3.2: a_1' is high. Since $19 = \|R, V\| \geq d(a_1) + d(a_1') + 2(9 - d(a_1)) \geq 23 - d(a_1)$, we get $d(a_1) = 4$ and $d(a_1') = d(a_2') = d(a_2) = 5$. Choose notation so that $C = c_1 c_2 c_3 c_1$, $D = b_1 b_2 b_3 b_1$, and $\|c_1, R\| = 1$. By Claim 2.2.13, there is $i \in [2]$ with $\|a_i, C\| = 2$, $\|a_i', C\| = 3$, and $a_i c_1 \notin E$. If $i = 1$ then every low vertex is in $N(a_1) - a_1' \subseteq D \cup C'$, where $C' = a_1 c_2 c_3 a_1$. So C', $c_1 a_1' a_2' a_2$ beats C, R by (P1). Thus let $i = 2$. Now $\|a_2, C\| = 2 = \|a_2, D\|$.

Say $a_2 b_2, a_2 b_3 \in E$. Also $\|a_2', D\| = 0$ and $\|e_1, D\| = 4$. So $\|b_j, e_1\| = 2$ for some $j \in [3]$. If $j = 1$ then $b_1 e_1 b_1$ and $a_2 b_2 b_3 a_2$ are disjoint cycles. Else, say $j = 2$. By inspection, all low vertices are contained in $\{a_1, b_1, b_3\}$. If b_1 and b_3 are high then $b_2 e_1 b_2$, $b_1 b_3 e_2$ beats D, R by (P1). Else there is a 3-cycle $D' \subseteq G[D + a_1]$ that contains every low vertex of G. Pick D' with $b_1 \in D'$ if possible. If $b_2 \notin D'$ then D' and $b_2 a_1' a_2' a_2 b_2$ are disjoint cycles. If $b_3 \notin D'$ then D', $b_3 a_2 a_2' a_1'$ beats D, R by (P1). Else $b_1 \notin D'$, $a_1 b_1 \notin E$, and b_1 is high. If $b_1 a_1' \in E$ then D', $b_1 a_1' a_2' a_2$ beats D, R by (P1). Else, $\|b_1, C\| = 3$. So D', $b_1 c_1 c_2 b_1$, and $c_3 e_2 c_3$ are disjoint cycles. □

2.2.3 Key Lemma

Now $|R| = 3$; say $R = a_1 a' a_2$. By Lemma 2.2.14 the maximum length of a cycle in $\mathcal{C}$ is 4. Fix $C = w_1 \dots w_4 w_1 \in \mathcal{C}$.

Lemma 2.2.18. *If $D \in \mathcal{C}$ with $\|R, D\| \geq 7$ then $|D| = 3$, $\|R, D\| = 7$ and $G[R \cup D] = K_6 - K_3$.*

Proof. Since $\|R, D\| \geq 7$, there exists $a \in R$ with $\|a, D\| \geq 3$. So $|D| = 3$ by Claim 2.2.1. If $\|a_i, D\| = 3$ for any $i \in [2]$, then (a) and (c) in Claim 2.2.12 hold, violating (O1). Then $\|a_1, D\| = \|a_2, D\| = 2$ and $\|a', D\| = 3$. If $G[R \cup D] \neq K_6 - K_3$ then $N(a_1) \cap D \neq N(a_2) \cap D$. Then there is $w \in N(a_1) \cap D$ with $\|a_2, D - w\| = 2$. Then $w a_1 a' w$ and $a_2 (D - w) a_2$ are disjoint cycles. □

Lemma 2.2.19 (Key Lemma)**.** *Let $D \in \mathcal{C}$ with $D = z_1 \dots z_t z_1$. If $\|C, D\| \geq 8$ then $\|C, D\| = 8$ and*

$$W := G[C \cup D] \in \{K_{4,4}, \quad K_1 \vee K_{3,3}, \quad \overline{K}_3 \vee (K_1 + K_3)\}.$$

Proof. First suppose $|D| = 4$. Suppose

(*) W contains two disjoint cycles T and C' with $|T| = 3$.

Then $\mathcal{C}' := \mathcal{C} - C - D + T + C'$ is at least as good as $\mathcal{C}$. So by Lemma 2.2.14, $|C'| \leq 4$. Thus $\mathcal{C}'$ beats $\mathcal{C}$ by (O2).

CASE 1: $\Delta(W) = 6$. By symmetry, assume $d_W(w_4) = 6$. Then $\|\{z_i, z_{i+1}\}, C - w_4\| \geq 2$ for some $i \in \{1, 3\}$. So (*) holds with $T = w_4 z_{4-i} z_{5-i} w_4$.

CASE 2: $\Delta(W) = 5$. Say $z_1, z_2, z_3 \in N(w_1)$. Then $\|\{z_i, z_4\}, C - w_1\| \geq 2$ for some $i \in \{1, 3\}$. So (*) holds with $T = w_1 z_{4-i} z_2 w_1$.

CASE 3: $\Delta(W) = 4$. Then W is regular. If W has a triangle then (*) holds. Else, say $w_1 z_1, w_1 z_3 \in E$. Then $z_1, z_3 \notin N(w_2) \cup N(w_4)$, so $z_2, z_4 \in N(w_2) \cup N(w_4)$, and $z_1, z_3 \in N(w_3)$.

Now, suppose $|D| = 3$.

CASE 1: $d_W(z_h) = 6$ for some $h \in [3]$. Say $h = 3$. If $w_i, w_{i+1} \in N(z_j)$ for some $i \in [4]$ and $j \in [2]$, then $z_3 w_{i+2} w_{i+3} z_3$, $z_j w_i w_{i+1} z_j$ beats C, D by (O2). Else for all $j \in [2]$, $\|z_j, C\| = 2$, and the neighbors of z_j in C are nonadjacent. If $w_i \in N(z_1) \cap N(z_2) \cap C$, then $z_3 w_{i+1} w_{i+2} z_3$, $z_1 z_2 w_i z_1$ are preferable to C, D by (O2). Wence $W = K_1 \vee K_{3,3}$.

CASE 2: $d_W(z_h) \leq 5$ for every $h \in [3]$. Say $d(z_1) = 5 = d(z_2)$, $d(z_3) = 4$, and $w_1, w_2, w_3 \in N(z_1)$. If $N(z_1) \cap C \neq N(z_2) \cap C$ then $W - z_3$ contains two disjoint cycles, preferable to C, D by (O2); if $w_i \in N(z_3)$ for some $i \in \{1, 3\}$ then $W - w_4$ contains two disjoint cycles. So $N(z_3) = \{w_2, w_4\}$, and so $W = \overline{K}_3 \vee (K_1 + K_3)$, where $V(K_1) = \{w_4\}$, $w_2 z_1 z_2 w_2 = K_3$, and $V(K_3) = \{w_1, w_3, z_3\}$. $\square$

Claim 2.2.20. For $D \in \mathcal{C}$, if $\|\{w_1, w_3\}, D\| \geq 5$ then $\|C, D\| \leq 6$. If also $|D| = 3$ then $\|\{w_2, w_4\}, D\| = 0$.

Proof. Assume not. Let $D = z_1 \dots z_t z_1$. Then $\|\{w_1, w_3\}, D\| \geq 5$ and $\|C, D\| \geq 7$. Say $\|w_1, D\| \geq \|w_3, D\|$, $\{z_1, z_2, z_3\} \subseteq N(w_1)$, and $z_l \in N(w_3)$.

Suppose $\|w_1, D\| = 4$. Then $|D| = 4$. If $\|z_h, C\| \geq 3$ for some $h \in [4]$ then there is a cycle $B \subseteq G[w_2, w_3, w_4, z_h]$; so B, $w_1 z_{h+1} z_{h+2} w_1$ beats C, D by (O2). Else there are $j \in \{l-1, l+1\}$ and $i \in \{2, 3, 4\}$ with $z_i w_j \in E$. Then $z_l z_j [w_i w_3] z_l$, $w_1 (D - z_l - z_j) w_1$ beats C, D by (O2), where $[w_i w_3] = w_3$ if $i = 3$.

Else, $\|w_1, D\| = 3$. By assumption, there is $i \in \{2, 4\}$ with $\|w_i, D\| \geq 1$. If $|D| = 3$, applying Claim 2.2.12 with $P := w_1 w_i w_3$ and cycle D yields two disjoint cycles in $(D \cup C) - w_{6-i}$, contradicting (O2). So suppose $|D| = 4$. Because $w_1 z_1 z_2 w_1$ and $w_1 z_2 z_3 w_1$ are triangles, there do not exist cycles in $G[\{w_i, w_3, z_3, z_4\}]$ or $G[\{w_i, w_3, z_1, z_4\}]$ by (O2). Then $\|\{w_i, w_3\}, \{z_3, z_4\}\|$, $\|\{w_i, w_3\}, \{z_1, z_4\}\| \leq 1$. Since $\|\{w_i, w_3\}, D\| \geq 3$, one has a neighbor in z_2. If both are adjacent to z_2, then $w_i w_3 z_2 w_i$, $w_1 z_1 z_4 z_3 w_1$ beat C, D by (O2). Then $\|\{w_i, w_3\}, z_2\| = 1 = \|\{w_i, w_3\}, z_1\| = \|\{w_i, w_3\}, z_3\|$. Let z_m be the neighbor of w_i. Then $w_i w_1 z_m w_i$, $w_3 (D - z_m) w_3$ beat C, D by (O2).

Suppose $|D| = 3$ and $\|\{w_1, w_3\}, D\| \geq 5$. If $\|\{w_2, w_4\}, D\| \geq 1$, then $C \cup D$ contains two triangles, and these are preferable to C, D by (O2). $\square$

For $v \in N(C)$, set $\text{type}(v) = i \in [2]$ if $N(v) \cap C \subseteq \{w_i, w_{i+2}\}$. Call v *light* if $\|v, C\| = 1$; else v is *heavy.* For $D = z_1 \ldots z_t z_1 \in \mathcal{C}$, put $H := H(D) := G[R \cup D]$.

Claim 2.2.21. If $\|\{a_1, a_2\}, D\| \geq 5$ then there exists $i \in [2]$ such that

(a) $\|C, H\| \leq 12$ and $\|\{w_i, w_{i+2}\}, H\| \leq 4$;

(b) $\|C, H\| = 12$;

(c) $N(w_i) \cap H = N(w_{i+2}) \cap H = \{a_1, a_2\}$ and $N(w_{3-i}) \cap H = N(w_{5-i}) \cap H = V(D) \cup \{a'\}$.

Proof. By Claim 2.2.1, $|D| = 3$. Choose notation so that $\|a_1, D\| = 3$ and $z_2, z_3 \in N(a_2)$.

(a) Using that $\{w_1, w_3\}$ and $\{w_2, w_4\}$ are independent and Lemma 2.2.19:

$$\|C, H\| = \|C, V - (V - H)\| \geq 2(4k - 3) - 8(k - 2) = 10. \tag{2.7}$$

Let $v \in V(H)$. As $K_4 \subseteq H$, $H - v$ contains a 3-cycle. If $C + v$ contains another 3-cycle then these 3-cycles beat C, D by (O2). So type(v) is defined for all $v \in N(C) \cap H$, and $\|C, H\| \leq 12$. If only five vertices of H have neighbors in C then there is $i \in [2]$ such that at most two vertices in H have type i. So $\|\{w_i, w_{i+2}\}, H\| \leq 4$. Else every vertex in H has a neighbor in C. By (2.7), H has at least four heavy vertices.

Let H' be the spanning subgraph of H with $xy \in E(H')$ iff $xy \in E(H)$ and $H - \{x, y\}$ contains a 3-cycle. If $xy \in E(H')$ then $N(x) \cap N(y) \cap C = \emptyset$ by (O2). So if x and y have the same type they are both light. By inspection, $H' \supseteq z_1 a_1 a' a_2 z_2 + a_2 z_3$.

Let type$(a_2) = i$. If a_2 is heavy then its neighbors a', z_2, z_3 have type $3 - i$. Either z_1, a_1 are both light or they have different types. Anyway, $\|\{w_i, w_{i+2}\}, H\| \leq 4$. Else a_2 is light. Then because there are at least four heavy vertices in H, at least one of z_1, a_1 is heavy and so they have different types. Also any type-i vertex in a', z_2, z_3 is light, but at most one vertex of a, z_2, z_3 is light because there are at most two light vertices in H. So $\|\{w_i, w_{i+2}\}, H\| \leq 4$.

(b) By (a), there is i with $\|\{w_i, w_{i+2}\}, H\| \leq 4$; thus

$$\|\{w_i, w_{i+2}\}, V - H\| \geq (4k - 3) - 4 = 4(k - 2) + 1.$$

So $\|\{w_i, w_{i+2}\}, D'\| \geq 5$ for some $D' \in \mathcal{C} - C - D$. By (a), Claim 2.2.20, and Lemma 2.2.19,

$$12 \geq \|C, H\| = \|C, V - D' - (V - H - D')\| \geq 2(4k - 3) - 6 - 8(k - 3) = 12.$$

(c) By (b), $\|C, H\| = 12$, so each vertex in H is heavy. Thus type(v) is the unique proper 2-coloring of H', and (c) follows. $\square$

Lemma 2.2.22. *There exists $C^* \in \mathcal{C}$ such that $3 \leq \|\{a_1, a_2\}, C^*\| \leq 4$ and $\|\{a_1, a_2\}, D\| = 4$ for all $D \in \mathcal{C} - C^*$. If $\|\{a_1, a_2\}, C^*\| = 3$ then one of a_1, a_2 is low.*

Proof. Suppose $\|\{a_1, a_2\}, D\| \geq 5$ for some $D \in \mathcal{C}$; set $H := H(D)$. Using Claim 2.2.21, choose notation so that $\|\{w_1, w_3\}, H\| \leq 4$. Now

$$\|\{w_1, w_3\}, V - H\| \geq 4k - 3 - 4 = 4(k-2) + 1.$$

Thus there is a cycle $B \in \mathcal{C} - D$ with $\|\{w_1, w_3\}, B\| \geq 5$; say $\|\{w_1, B\}\| = 3$. By Claim 2.2.20, $\|C, B\| \leq 6$. Note by Claim 2.2.21, if $|B| = 4$ then for an edge $z_1z_2 \in N(w_1)$, $w_1z_1z_2w_1$ and $w_2w_3a_2a'w_2$ beat B, C by (O2). So $|B| = 3$. Using Claim 2.2.21(b) and Lemma 2.2.19,

$$2(4k-3) \leq \|C, V\| = \|C, H \cup B \cup (V - H - B)\| \leq 12 + 6 + 8(k-3) = 2(4k-3).$$

So $\|C, D'\| = 8$ for all $D' \in \mathcal{C} - C - D$. By Lemma 2.2.19, $\|\{w_1, w_3\}, D'\| = \|\{w_2, w_4\}, D'\| = 4$. By Claim 2.2.21(c) and Claim 2.2.20,

$$4k - 3 \leq \|\{w_2, w_4\}, H \cup B \cup (V - H - B)\| \leq 8 + 1 + 4(k-3) = 4k - 3,$$

and so $\|\{w_2, w_4\}, B\| = 1$. Say $\|w_2, B\| = 1$. Since $|B| = 3$, by Claim 2.2.12, $G[B \cup C - w_4]$ has two disjoint cycles that are preferable to C, B by (O2). This contradiction implies $\|\{a_1, a_2\}, D\| \leq 4$ for all $D \in \mathcal{C}$. Since $\|\{a_1, a_2\}, V\| \geq 4k - 3$ and $\|\{a_1, a_2\}, R\| = 2$, $\|\{a_1, a_2\}, D\| \geq 3$, and equality holds for at most one $D \in \mathcal{C}$, and only if one of a_1 and a_2 is low. $\square$

2.2.4 Completion of the proof of Theorem 2.1.7.

For an optimal $\mathcal{C}$, let $\mathcal{C}_i := \{D \in \mathcal{C} : |D| = i\}$ and $t_i := |\mathcal{C}_i|$. For $C \in \mathcal{C}_4$, let $Q_C := Q_C(\mathcal{C}) := G[R(\mathcal{C}) \cup C]$. A 3-path R' is *$\mathcal{D}$-useful* if $R' = R(\mathcal{C}')$ for an optimal set $\mathcal{C}'$ with $\mathcal{D} \subseteq \mathcal{C}'$; we write *$D$-useful* for $\{D\}$-useful.

Lemma 2.2.23. Let $\mathcal{C}$ be an optimal set and $C \in \mathcal{C}_4$. Then $Q = Q_C \in \{K_{3,4}, K_{3,4} - e\}$.

Proof. Since $\mathcal{C}$ is optimal, Q does not contain a 3-cycle. So for all $v \in V(C)$, $N(v) \cap R$ is independent and $\|a_1, C\|, \|a_2, C\| \leq 2$. By Lemma 2.2.22, $\|\{a_1, a_2\}, C\| \geq 3$. Say $a_1w_1, a_1w_3 \in E$ and $\|a_2, C\| \geq 1$. So type(a_1) and type(a_2) are defined.

Claim 2.2.24. $\text{type}(a_1) = \text{type}(a_2)$.

Proof. Suppose not. Then $\|w_i, R\| \leq 1$ for all $i \in [4]$. Say $a_2w_2 \in E$. If $w_ia_j \in E$ and $\|a_{3-j}, C\| = 2$, let $R_i = w_ia_ja'$ and $C_i = a_{3-j}(C - w_i)a_{3-j}$ (see Figure 2.6). Then R_i is $(\mathcal{C} - C + C_i)$-useful. Let $\lambda(X)$ be the number of low vertices in $X \subseteq V$. As Q does not contain a 3-cycle, $\lambda(R) + \lambda(C) \leq 2$. We claim:

$$\forall D \in \mathcal{C} - C, \quad \|a', D\| \leq 2. \tag{2.8}$$

Fix $D \in \mathcal{C} - C$, and suppose $\|a', D\| \geq 3$. By Claim 2.2.1, $|D| = 3$. Since

$$\begin{aligned} \|C, D\| &= \|C, \mathcal{C}\| - \|C, \mathcal{C} - D\| \\ &\geq 4(2k-1) - \lambda(C) - \|C, R\| - 8(k-2) \\ &= 12 - \|C, R\| - \lambda(C) \geq 6 + \lambda(R), \end{aligned} \tag{2.9}$$

$\|w_i, D\| \geq 2$ for some $i \in [4]$. If R_i is defined, R_i is $\{C_i, D\}$-useful. By Lemma 2.2.22, $\|\{w_i, a'\}, D\| \leq 4$. As $\|w_i, D\| \geq 2$, $\|a', D\| \leq 2$, proving (2.8). Then R_i is not defined, so a_2 is low with $N(a_2) \cap C = \{w_2\}$ and $\|w_2, D\| \leq 1$. Then by (2.9), $\|C - w_2, D\| \geq 6$. Note $G[a' + D] = K_4$, so for any $z \in D$, $D - z + a'$ is a triangle, so by (O2) the neighbors of z in C are independent. Then $\|C - w_2, D\| = 6$ with $N(z) \cap C = \{w_1, w_3\}$ for every $z \in D$. Then $\|w_2, D\| = 1$, say $zw_2 \in E(G)$, and now $w_2w_3zw_2$, $w_1(D - z)w_1$ beat C, D by (O2).

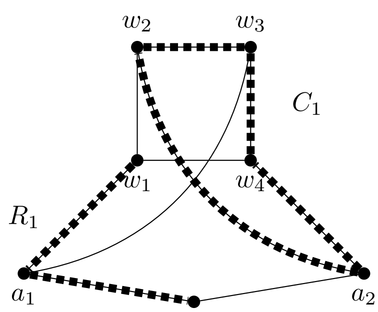


Figure 2.6: Claim 2.2.24

If $\|a', C\| \geq 1$ then $a'w_4 \in E$ and $N(a_2) \cap C = \{w_2\}$. So R_2 is C_2-useful, $\text{type}(a') \neq \text{type}(w_2)$ with respect to C_2, and the middle vertex a_2 of R_2 has no neighbors in C_2. So we may assume $\|a', C\| = 0$. Then a' is low:

$$d(a') = \|a', C \cup R\| + \|a', \mathcal{C} - C\| \leq 0 + 2 + 2(k-2) = 2k - 2. \tag{2.10}$$

Thus all vertices of C are high. Using Lemma 2.2.19, this yields:

$$4 \geq \|C, R\| = \|C, V - (V - R)\| \geq 4(2k-1) - 8(k-1) = 4. \tag{2.11}$$

As this calculation is tight, $d(w) = 2k-1$ for every $w \in C$. Thus $d(a') \geq 2k-2$. So (2.10) is tight. Hence $\|a', D\| = 2$ for all $D \in \mathcal{C} - C$.

Pick $D = z_1 \dots z_t z_1 \in \mathcal{C} - C$ with $\|\{a_1, a_2\}, D\|$ maximum. By Lemma 2.2.22, $3 \leq \|\{a_1, a_2\}, D\| \leq 4$. Say $\|a_i, D\| \geq 2$. By (2.11), $\|C, D\| = 8$. By Lemma 2.2.19,

$$W := G[C \cup D] \in \{K_{4,4},\ \overline{K}_3 \vee (K_3 + K_1),\ K_1 \vee K_{3,3}\}.$$

CASE 1: $W = K_{4,4}$. Then $\|D, R\| \geq 5 > |D| = 4$, so $\|z, R\| \geq 2$ for some $z \in V(D)$. Let $w \in N(z) \cap C$. Either w and z have a comon neighbor in $\{a_1, a_2\}$ or z has two consecutive neighbors in R. Regardless, $G[R + w + z]$ contains a 3-cycle D' and $G[W - w - z]$ contains a 4-cycle C'. Thus C', D' beats C, D by (O2).
CASE 2: $W = \overline{K}_3 \vee (K_3 + K_1)$. As $\|\{a', a_i\}, D\| \geq 4 > |D|$, there is $z \in V(D)$ with $D' := za'a_iz \subseteq G$. Also $W - z$ contains a 3-cycle C'. So C', D' beats C, D by (O2).
CASE 3: $W = K_1 \vee K_{3,3}$. Some $v \in V(D)$ satisfies $\|v, W\| = 6$. There is no $w \in W - v$ such that w has two adjacent neighbors in R: else a and v would be contained in disjoint 3-cycles, contradicting the choice of C, D. So $\|w, R\| \leq 1$ for all $w \in W - v$, because $\text{type}(a_1) \neq \text{type}(a_2)$. Similarly, no $z \in D - v$ has two adjacent neighbors in R. Thus

$$2 + 3 \leq \|a', D\| + \|\{a_1, a_2\}, D\| = \|R, D\| = \|R, D - v\| + \|R, v\| \leq 2 + 3.$$

So $\|\{a_1, a_2\}, D\| = 3$, $R \subseteq N(v)$, and $N(a_i) \cap K_{3,3}$ is independent. By Lemma 2.2.22 and the maximality of $\|\{a_1, a_2\}, D\| = 3$, $k = 3$. Thus $G = \mathbf{Y}_2$, a contradiction. $\square$

Returning to the proof of Lemma 2.2.23, we have $\text{type}(a_1) = \text{type}(a_2)$. Using Lemma 2.2.22, choose notation so that $a_1w_1, a_1w_3, a_2w_1 \in E$. Then Q has bipartition $\{X, Y\}$ with $X := \{a', w_1, w_3\}$ and $Y := \{a_1, a_2, w_2, w_4\}$. The only possible nonedges between X and Y are $a'w_2$, $a'w_4$ and a_2w_3. Let $C' := w_1Rw_1$. Then $R' := w_2w_3w_4$ is C'-useful. By Lemma 2.2.22, $\|\{w_2, w_4\}, C'\| \geq 3$. Already $w_2, w_4 \in N(w_1)$; so because Q has no C_3, (say) $a'w_2 \in E$. Now, let $C'' := a_1a'w_2w_3a_1$. Then $R'' := a_2w_1w_4$ is C''-useful; so $\|\{a_2, w_4\}, C''\| \geq 3$. Again, Q contains no C_3, so $a'w_4$ or a_2w_3 is an edge of G. Thus $Q \in \{K_{3,4}, K_{3,4} - e\}$. $\square$

Proof of Theorem 2.1.7. Using Lemma 2.2.23, one of two cases holds:

(C1) For some optimal set $\mathcal{C}$ and $C' \in \mathcal{C}_4$, $Q_{C'} = K_{3,4} - x_0y_0$;

(C2) for all optimal sets $\mathcal{C}$ and $C \in \mathcal{C}_4$, $G[R \cup C] = K_{3,4}$.

Fix an optimal set $\mathcal{C}$ and $C' \in \mathcal{C}_4$, where $R = y_0x'y$ with $d(y_0) \leq d(y)$, such that in (C1), $Q_{C'} = K_{3,4} - x_0y_0$. By Lemmas 2.2.22 and 2.2.23, for all $C \in \mathcal{C}_4$, $1 \leq \|y_0, C\| \leq \|y, C\| \leq 2$ and $\|y_0, C\| = 1$ only in Case (C1) when $C = C'$. Put $H := R \cup \bigcup \mathcal{C}_4$, $S = S(\mathcal{C}) := N(y) \cap H$, and $T = T(\mathcal{C}) := V(H) \smallsetminus S$. As $\|y, R\| = 1$ and $\|y, C\| = 2$ for each $C \in \mathcal{C}_4$, $|S| = 1 + 2t_4 = |T| - 1$.

Claim 2.2.25. H is an S,T-bigraph. In case (C1), $H = K_{2t_4+1,2t_4+2} - x_0y_0$; else $H = K_{2t_4+1,2t_4+2}$.

Proof. By Lemma 2.2.23, $\|x', S\| = \|y, T\| = \|y_0, T\| = 0$.

By Lemmas 2.2.22 and 2.2.23, $\|y_0, S\| = |S| - 1$ in (C1) and $\|y_0, S\| = |S|$ otherwise. We claim that for every $t \in T - y_0$, $\|t, S\| = |S|$. This clearly holds for y, so take $t \in H - \{y, y_0\}$. Then $t \in C$ for some $C \in \mathcal{C}_4$. Let $\mathcal{R}^* := tx'y_0$ and $\mathcal{C}^* := y(C - t)y$. (Note R^* is a path and C^* is a cycle by Lemma 2.2.23 and the choice of y_0.) Since R^* is C^*-useful, by Lemmas 2.2.22 and 2.2.23, and by choice of y_0, $\|t, S\| = \|y, S\| = |S|$. Then in (C1), $H \supseteq K_{2t_4+1,2t_4+2} - x_0y_0$ and $x_0y_0 \notin E(H)$; else $H \supseteq K_{2t_4+1,2t_4+2}$.

Now we easily see that if any edge exists inside S or T, then $C_3 + (t_4 - 1)C_4 \subseteq H$, and these cycles beat $\mathcal{C}_4$ by (O2). $\square$

By Claim 2.2.25 all pairs of vertices of T are the ends of a $\mathcal{C}_3$-useful path. Now we use Lemma 2.2.22 to show that they have essentially the same degree to each cycle in $\mathcal{C}_3$.

Claim 2.2.26. If $v \in T$ and $D \in \mathcal{C}_3$ then $1 \leq \|v, D\| \leq 2$; if $\|v, D\| = 1$ then v is low and for all $C \in \mathcal{C}_3 - D$, $\|v, C\| = 2$.

Proof. By Claim 2.2.25, $H + x_0y_0$ is a complete bipartite graph. Let $y_1, y_2 \in T - v$ and $u \in S - x_0$. Then $R' = y_1uv$, $R'' = y_2uv$, and $R''' = y_1uy_2$ are $\mathcal{C}_3$-useful. By Lemma 2.2.22,

$$3 \leq \|\{v, y_1\}, D\|, \|\{v, y_2\}, D\|, \|\{y_1, y_2\}, D\| \leq 4.$$

Say $\|y_1, D\| \leq 2 \leq \|y_2, D\|$. Thus

$$1 \leq \|\{v, y_1\}, D\| - \|y_1, D\| = \|v, D\| = \|\{v, y_2\}, D\| - \|y_2, D\| \leq 2.$$

Suppose $\|v, D\| = 1$. By Claim 2.2.25 and Lemma 2.2.22, for any $v' \in T - v$,

$$4k - 3 \leq \|\{v, v'\}, H \cup (\mathcal{C}_3 - D) \cup D\| \leq 2(2t_4 + 1) + 4(t_3 - 1) + 3 = 4k - 3.$$

Thus for all $C \in \mathcal{C}_3 - D_0$, $\|\{v, v'\}, C\| = 4$, and so $\|v, C\| = 2$. Hence v is low. $\square$

Next we show that all vertices in T have essentially the same neighborhood in each $C \in \mathcal{C}_3$.

Claim 2.2.27. Let $z \in D \in \mathcal{C}_3$ and $v, w \in T$ with w high.

1. If $zv \in E$ and $zw \notin E$ then $T - w \subseteq N(z)$.

2. $N(v) \cap D \subseteq N(w) \cap D$.

Proof. (1) Since w is high, Claim 2.2.26 implies $\|w, D\| = 2$. Since $zw \notin E$, $D' := w(D-z)w$ is a 3-cycle. Let $u \in S - x_0$. Then $zvu = R(\mathcal{C}')$ for some optimal set $\mathcal{C}'$ with $\mathcal{C}_3 - D + D' \subseteq \mathcal{C}'$. By Claim 2.2.25, $T(\mathcal{C}') = S + z$ and $S(\mathcal{C}') = T - w$. If (C2) holds, then $T - w = S(\mathcal{C}') \subseteq N(z)$, as desired. Suppose (C1) holds, so there are $x_0 \in S$ and $y_0 \in T$ with $x_0y_0 \notin E$. By Claims 2.2.25 and 2.2.26, $d(y_0) \leq (|S|-1)+2(t_3) = 2k-2$, so y_0 is low. Since w is high, $y_0 \in T-w$. But now apply Claims 2.2.25 and 2.2.26 to $T(\mathcal{C}')$: $d(x_0) \leq |S(\mathcal{C}')|-1+2t_3 = 2k-2$, and x_0 is low. As $x_0y_0 \notin E$, this is a contradiction. So $T - w = S(\mathcal{C}') \subseteq N(z)$.

(2) Suppose there exists $z \in N(v) \cap D \setminus N(w)$. By (1), $T - w \subseteq N(z)$. Let $w' \in T - w$ be high. By Claim 2.2.26, $\|w', D\| = 2$. So there exists $z' \in N(w) \cap D \setminus N(w')$ and $z \neq z'$. By (1), $T - w' \subseteq N(z')$. As $|T| \geq 4$ and at least three of its vertices are high, there exists a high $w'' \in T - w - w'$. Since $w''z, w''z' \in E$, there exists $z'' \in N(w) \cap D \setminus N(w'')$ with $\{z, z', z''\} = V(D)$. By (1), $T - w'' \subseteq N(z'')$. Since $|T| \geq 4$ there exists $x \in T \setminus \{w, w', w''\}$. So $\|x, D\| = 3$, contradicting Claim 2.2.26. $\square$

Let $y_1, y_2 \in T - y_0$ and let $x \in S$ with $x = x_0$ if $x_0y_0 \notin E$. By Claim 2.2.25, y_1xy_2 is a path, and $G - \{y_1, y_2, x\}$ contains an optimal set $\mathcal{C}'$. Recall y_0 was chosen in T with minimum degree, so y_1 and y_2 are high and by Claim 2.2.26 $\|y_i, D\| = 2$ for each $i \in [2]$ and each $D \in \mathcal{C}_3$. Let $N = N(y_1) \cap \bigcup \mathcal{C}_3$ and $M = \bigcup \mathcal{C}_3 \setminus N$ (see Figure 2.7). By Claim 2.2.25, T is independent. By Claim 2.2.27, for every $y \in T$, $N(y) \cap \bigcup \mathcal{C}_3 \subseteq N$, so $E(M, T) = \emptyset$. Since $y_2 \neq y_0$, also $N(y_2) \cap \bigcup \mathcal{C}_3 = N$.

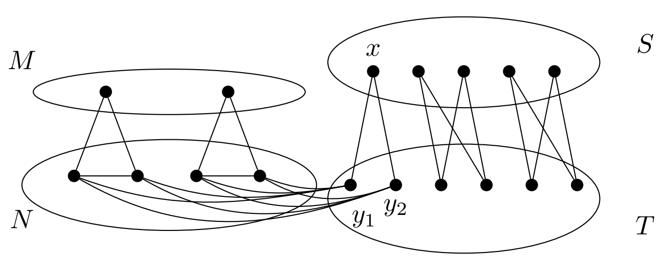


Figure 2.7

Claim 2.2.28. M is independent.

Proof. First, we show (*) $\|z,S\| > t_4$ for all $z \in M$. If not then there exists $z \in D \in \mathcal{C}_3$ with $\|z,S\| \leq t_4$. Since $\|M,T\| = \|T,T\| = 0$,

$$\|\{y_1,z\},\mathcal{C}_3\| \geq 4k-3-\|\{z,y_1\},S\| \geq 4(t_4+t_3+1)-3-(2t_4+1+t_4) = t_4+4t_3 > 4t_3.$$

So there is $D' = z'z'_1z'_2z' \in \mathcal{C}_3$ with $\|\{z,y_1\},D'\| \geq 5$ and $z' \in M$. As $\|y_1,D\| = 2$, $\|z,D'\| = 3$. Since $D^* := zz'z'_2z$ is a cycle, $xy_2z'_1$ is D^*-useful. As $\|z'_1,D^*\| = 3$, this contradicts Claim 2.2.26, proving (*).

Suppose $zz' \in E(M)$; say $z \in D \in \mathcal{C}_3$ and $z' \in D' \in \mathcal{C}_3$. By (*) there is $u \in N(z) \cap N(z') \cap S$. So $zz'uz$, $y_1(D-z)y_1$ and $y_2(D'-z')y_2$ are disjoint cycles, contrary to (O1). $\square$

By Claims 2.2.25 and 2.2.28, M and T are independent; as remarked above $E(M,T) = \emptyset$. So $M \cup T$ is independent. This contradicts (H3), since

$$|G|-2k+1 = 3t_3+4t_4+3-2(t_3+t_4+1)+1 = t_3+2t_4+2 = |M \cup T| \leq \alpha(G).$$

The proof of Theorem 2.1.7 is now complete. $\square$

2.3 The case $k = 2$

Lovász [33] observed that any (simple or multi-) graph can be transformed into a multigraph with minimum degree at least 3, without affecting the maximum number of disjoint cycles in the graph, by using a sequence of operations of the following three types: (i) deleting a bud; (ii) suppressing a vertex v of degree 2 that has two neighbors x and y, i.e., deleting v and adding a new (possibly parallel) edge between x and y; and (iii) increasing the multiplicity of a loop or edge with multiplicity 2. Here loops and two parallel edges are considered cycles, so forests have neither. Also K_s and $K_{s,t}$ denote simple graphs. Let W_s^* denote a wheel on s vertices whose spokes, but not outer cycle edges, may be multiple. The following theorem characterizes those multigraphs that do not have two disjoint cycles.

Theorem 2.3.1 (Lovász [33])**.** *Let G be a multigraph with $\delta(G) \geq 3$ and no two disjoint cycles. Then G is one of the following:* (1) K_5, (2) W_s^*, (3) $K_{3,|G|-3}$ *together with a multigraph on the vertices of the (first) 3-class, and* (4) *a forest F and a vertex x with possibly some loops at x and some edges linking x to F.*

Let $\mathcal{G}$ be the class of simple graphs G with $|G| \geq 6$ and $\sigma_2(G) \geq 5$ that do not have two disjoint cycles. Fix $G \in \mathcal{G}$. A vertex in G is low if its degree is at most 2. The low vertices form a clique Q of size at most

2—if $|Q| = 3$, then Q is a component-cycle, and $G - Q$ has another cycle. By Lovász's observation, G can be reduced to a graph H of type (1–4). Reversing this reduction, G can be obtained from H by adding buds and subdividing edges. Let $Q' := V(G) \smallsetminus V(H)$. It follows that $Q \subseteq Q'$. If $Q' \neq Q$, then Q consists of a single leaf in G with a neighbor of degree 3, so G is obtained from H by subdividing an edge and adding a leaf to the degree-2 vertex. If $Q' = Q$, then Q is a component of G, or $G = H + Q + e$ for some edge $e \in E(H, Q)$, or at least one vertex of Q subdivides an edge $e \in E(H)$. In the last case, when $|Q| = 2$, e is subdivided twice by Q. As G is simple, H has at most one multiple edge, and its multiplicity is at most 2.

In case (4), because $\delta(H) \geq 3$, either F has at least two buds, each linked to x by multiple edges, or F has one bud linked to x by an edge of multiplicity at least 3. So this case cannot arise from G. Also, $\delta(H) = 3$, unless $H = K_5$, in which case $\delta(H) = 4$. So Q is not an isolated vertex, lest deleting Q leave H with $\delta(H) \geq 5 > 4$; and if Q has a vertex of degree 1 then $H = K_5$. Else all vertices of Q have degree 2, and Q consists of the subdivision vertices of one edge of H. So we have the following lemma.

Lemma 2.3.2. *Let G be a graph with $|G| \geq 6$ and $\sigma_2(G) \geq 5$ that does not have two disjoint cycles. Then G is one of the following (see Figure 2.8):*

(a) $K_5 + K_2$*;*

(b) K_5 *with a pendant edge, possibly subdivided;*

(c) K_5 *with one edge subdivided and then a leaf added adjacent to the degree-2 vertex;*

(d) a graph H of type (1–3) with no multiple edge, and possibly one edge subdivided once or twice, and if $|H| = 6 - i$ with $i \geq 1$ then some edge is subdivided at least i times;

(e) a graph H of type (2) or (3) with one edge of multiplicity two, and one of its parallel parts is subdivided once or twice—twice if $|H| = 4$.

2.4 Connections to Equitable Coloring

A proper vertex coloring of a graph G is *equitable* if any two color classes differ in size by at most one. In 1970 Hajnal and Szemerédi proved:

Theorem 2.4.1 ([18]). *Every graph G with $\Delta(G) + 1 \leq k$ has an equitable k-coloring.*

For a shorter proof of Theorem 2.4.1, see [31]; for an $O(k|G|^2)$-time algorithm see [28].

Motivated by Brooks' Theorem, it is natural to ask which graphs G with $\Delta(G) = k$ have equitable k-colorings. Certainly such graphs are k-colorable. Also, if k is odd then $K_{k,k}$ has no equitable k-coloring.

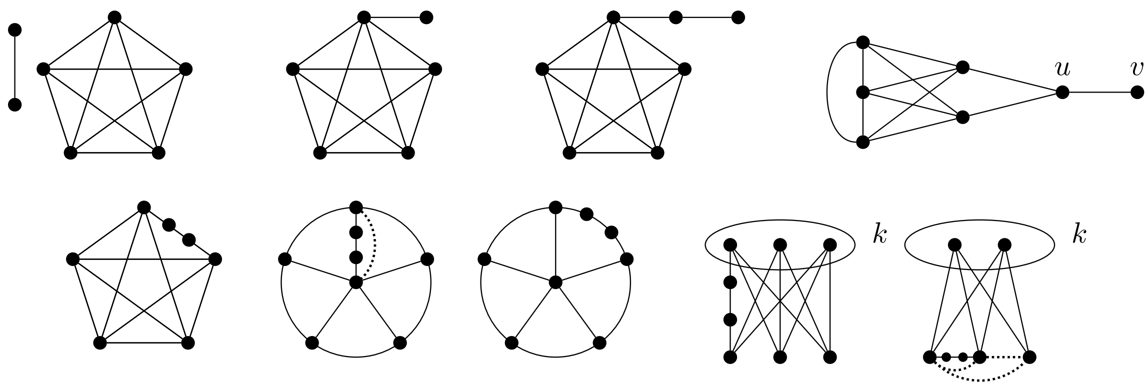


Figure 2.8: Theorem 2.3.2

Chen, Lih, and Wu [6] conjectured (in a different form) that these are the only obstructions to an equitable version of Brooks' Theorem:

Conjecture 2.4.2 ([6])**.** *If G is a graph with $\chi(G), \Delta(G) \leq k$ and no equitable k-coloring then k is odd and $K_{k,k} \subseteq G$.*

In [6], Chen, Lih, and Wu proved Conjecture 2.4.2 holds for $k = 3$. By a simple trick, it suffices to prove the conjecture for graphs G with $|G| = ks$. Combining the results of the two papers [25] and [26], we have:

Theorem 2.4.3. *Suppose G is a graph with $|G| = ks$. If $\chi(G), \Delta(G) \leq k$ and G has no equitable k-coloring, then k is odd and $K_{k,k} \subseteq G$ or both $k \geq 5$ [25] and $s \geq 5$ [26].*

A graph G is *k-equitable* if $|G| = ks$, $\chi(G) \leq k$ and every proper k-coloring of G has s vertices in each color class. The following strengthening of Conjecture 2.4.2, if true, provides a characterization of graphs G with $\chi(G), \Delta(G) \leq k$ that have an equitable k-coloring.

Conjecture 2.4.4 ([24])**.** *Every graph G with $\chi(G), \Delta(G) \leq k$ has an no equitable k-coloring if and only if k is odd and $G = H + K_{k,k}$ for some k-equitable graph H.*

The next theorem collects results from [24]. Together with Theorem 2.4.3 it yields Corollary 2.4.6.

Theorem 2.4.5 ([24])**.** *Conjecture 2.4.2 is equivalent to Conjecture 2.4.4. Indeed, for any k_0 and n_0, Conjecture 2.4.2 holds for $k \leq k_0$ and $|G| \leq n_0$ if and only if Conjecture 2.4.4 holds for $k \leq k_0$ and $|G| \leq n_0$.*

Corollary 2.4.6. *A graph G with $|G| = 3k$ and $\chi(G), \Delta(G) \leq k$ has no equitable k-coloring if and only if k is odd and $G = K_{k,k} + K_k$.*

We are now ready to complete our answer to Dirac's question for simple graphs.

Proof of Theorem 2.1.3. Assume $k \geq 2$ and $\delta(G) \geq 2k-1$. It is apparent that if any of (i), (H3), or (H4) in Theorem 2.1.3 fail, then G does not have k disjoint cycles. Now suppose G satisfies (i), (H3), and (H4). If $k = 2$ then $|G| \geq 6$ and $\delta(G) \geq 3$. Thus G has no subdivided edge, and only (d) of Lemma 2.3.2 is possible. By (i), $G \neq K_5$; by (H4), G is not a wheel; and by (H3), G is not type (3) of Theorem 3.2.5. So G has 2 disjoint cycles. Finally, suppose $k \geq 3$. Since G satisfies (ii), $G \notin \{\mathbf{Y}_1, \mathbf{Y}_2\}$ and G satisfies (H2). So, if $|G| \geq 3k+1$ then G has k disjoint cycles by Theorem 2.1.7. Otherwise, $|G| = 3k$ and G has k disjoint cycles if and only if its vertices can be partitioned into disjoint K_3's. This is equivalent to $\overline{G}$ having an equitable k-coloring. By (ii), $\Delta(\overline{G}) \leq k$, and by (H3), $\omega(\overline{G}) \leq k$. So by Brooks' Theorem, $\chi(\overline{G}) \leq k$. By (H4) and Corollary 2.4.6, $\overline{G}$ has an equitable k-coloring. □

Next we turn to Ore-type results on equitable coloring. To complement Theorem 2.1.7, we need a theorem that characterizes when a graph G with $|G| = 3k$ that satisfies (H2) and (H3) has k disjoint cycles, or equivalently, when its complement $\overline{G}$ has an equitable coloring. The complementary version of $\sigma_2(G)$ is the *maximum Ore-degree* $\theta(H) := \max_{xy \in E(H)}(d(x) + d(y))$. So $\theta(\overline{G}) = 2|G| - \sigma_2(G) - 2$, and if $|G| = 3k$ and $\sigma_2(G) \geq 4k-3$ then $\theta(\overline{G}) \leq 2k+1$. Also, if G satisfies (H3) then $\omega(\overline{G}) \leq k$. This would correspond to an Ore-Brooks-type theorem on equitable coloring.

Several papers, including [22, 23, 32], address equitable colorings of graphs G with $\theta(G)$ bounded from above. For instance, the following is a natural Ore-type version of Theorem 2.4.1.

Theorem 2.4.7 ([22])**.** *Every graph G with $\theta(G) \leq 2k-1$ has an equitable k-coloring.*

Even for ordinary coloring, an Ore-Brooks-type theorem requires forbidding some extra subgraphs when θ is 3 or 4. It was observed in [23] that for $k = 3, 4$ there are graphs for which $\theta(G) \leq 2k+1$ and $\omega(G) \leq k$, but $\chi(G) \geq k+1$. The following theorem was proved for $k \geq 6$ in [23] and then for $k \geq 5$ in [32].

Theorem 2.4.8. *Let $k \geq 5$. If $\omega(G) \leq k$ and $\theta(G) \leq 2k+1$, then $\chi(G) \leq k$.*

Chapter 3

Disjoint Cycles in Multigraphs

The following results are joint work with Henry Kierstead and Alexandr Kostochka; this chapter is based on [30].

3.1 Introduction

As mentioned in Chapter 2, after the proof of the Corrádi-Hajnal Theorem, Dirac [11] described the 3-connected multigraphs not containing two disjoint cycles and asked the more general question:

Question 2.1.2. Which $(2k-1)$-connected graphs[1] do not have k disjoint cycles?

In Chapter 2, we characterized simple graphs G with minimum degree $\delta(G) \geq 2k-1$ that do not contain k disjoint cycles. We use this result to answer Dirac's question in full.

Below is the definition of the graph $Y_{h,t}$, used heavily in this chapter. $Y_{h,t}$ is a generalized version of $2K_k \vee \overline{K_k}$, used in Theorem 2.1.3. For easier reference, we repeat below Theorem 2.1.3, our result from Chapter 2 characterizing graphs with minimum degree $2k-1$ and no k disjoint cycles.

Example 3.1.1. Let $Y_{h,t} = \overline{K}_h \vee (K_t \cup K_t)$ (Figure 3.1), where $V(\overline{K}_h) = X_0$ and the cliques have vertex sets X_1 and X_2. In other words, $V(Y_{h,t}) = X_0 \cup X_1 \cup X_2$ with $|X_0| = h$ and $|X_1| = |X_2| = t$, and a pair xy is an edge in $Y_{h,t}$ iff $\{x, y\} \subseteq X_1$, or $\{x, y\} \subseteq X_2$, or $|\{x, y\} \cap X_0| = 1$.

Theorem 2.1.3. *Let $k \geq 2$. Every graph G with (i) $|G| \geq 3k$ and (ii) $\delta(G) \geq 2k-1$ contains k disjoint cycles if and only if*

(H3) $\alpha(G) \leq |G| - 2k$, and

(H4) if k is odd and $|G| = 3k$, then $G \neq 2K_k \vee \overline{K_k}$ and if $k = 2$ then G is not a wheel.

Question 2.1.2 asks about graph that are $(2k-1)$-connected. We consider the broader class $\mathcal{D}_k$ of multigraphs in which each vertex has at least $2k-1$ distinct neighbors. We describe several classes of

[1] Dirac used the word *graphs*, but in [11] this appears to mean *multigraphs*.

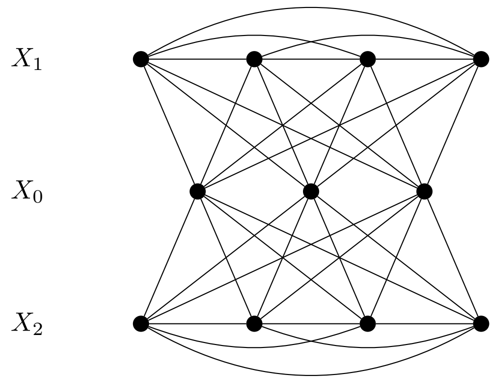


Figure 3.1: $Y_{h,t}$, shown with $h = 3$ and $t = 4$.

multigraphs that do not have k disjoint cycles for simple reasons, and prove that if a multigraph in $\mathcal{D}_k$ has no k disjoint cycles, then it belongs to one of these classes. This characterization is our main result, Theorem 3.2.6.

Every $(2k-1)$-connected multigraph is in D_k, so this provides a complete answer to Question 2.1.2. Determining whether a multigraph is in $\mathcal{D}_k$, and determining whether a multigraph is $(2k-1)$-connected, can be accomplished in polynomial time.

In the next section, we introduce notation, discuss existing results to be used later on, and state our main result, Theorem 3.2.6. In the last two sections, we prove Theorem 3.2.6.

3.2 Preliminaries and statement of the main result

3.2.1 Notation

For every multigraph G, let $V_1 = V_1(G)$ be the set of vertices in G incident to loops. Let $\widetilde{G}$ denote the *underlying simple graph of* G, i.e. the simple graph on $V(G)$ such that two vertices are adjacent in G if and only if they are adjacent in $\widetilde{G}$. Let $F = F(G)$ be the simple graph formed by the multiple edges in $G - V_1$; that is, if G' is the subgraph of $G - V_1$ induced by its multiple edges, then $G = \widetilde{G'}$. We will call the edges of $F(G)$ *the strong edges of* G, and define $\alpha' = \alpha'(F)$ to be the size of a maximum matching in F. A set $S = \{v_0, \ldots, v_s\}$ of vertices in a graph H is a *superstar with center* v_0 *in* H if $N_H(v_i) = \{v_0\}$ for each $1 \le i \le s$ and $H - S$ has a perfect matching.

For $v \in V$, we define $s(v) = |N(v)|$ to be the *simple degree* of v, and we say that $\mathcal{S}(G) = \min\{s(v) : v \in V\}$ is the *minimum simple degree* of G. We define $\mathcal{D}_k$ to be the family of multigraphs G with $\mathcal{S}(G) \ge 2k-1$. By the definition of $\mathcal{D}_k$, $\alpha(G) \le n - 2k + 1$ for every n-vertex $G \in \mathcal{D}_k$; so we call $G \in \mathcal{D}_k$ *extremal* if

$\alpha(G) = n - 2k + 1$. A *big set* in an extremal $G \in \mathcal{D}_k$ is an independent set of size $\alpha(G)$. If I is a big set in an extremal $G \in \mathcal{D}_k$, then since $s(v) \geq 2k - 1$, each $v \in I$ is adjacent to each $w \in V(G) - I$. Thus

$$\text{every two big sets in any extremal } G \text{ are disjoint.} \tag{3.1}$$

3.2.2 Preliminaries and main result

Since every cycle in a simple graph has at least 3 vertices, the condition $|G| \geq 3k$ is necessary in the Corrádi-Hajnal Theorem, Theorem 2.1.1. However, it is not necessary for multigraphs, since loops and multiple edges form cycles with fewer than three vertices. Theorem 2.1.1 can easily be extended to multigraphs, although the statement is no longer as simple:

Theorem 3.2.1 (Multigraph Corrádi-Hajnal)**.** *For $k \in \mathbb{Z}^+$, let G be a multigraph with $\mathcal{S}(G) \geq 2k$, and set $F = F(G)$ and $\alpha' = \alpha'(F)$. Then G has no k disjoint cycles if and only if*

$$|V(G)| - |V_1(G)| - 2\alpha' < 3(k - |V_1| - \alpha'), \tag{3.2}$$

i.e., $|V(G)| + 2|V_1| + \alpha' < 3k$.

Proof. If (3.2) holds, then G does not have enough vertices to contain k disjoint cycles. If (3.2) fails, then we choose $|V_1|$ cycles of length one and α' cycles of length two from $V_1 \cup V(F)$. By Theorem 2.1.1, the remaining (simple) graph contains $k - |V_1| - \alpha'$ disjoint cycles. □

Theorem 3.2.1 yields the following.

Corollary 3.2.2. *Let G be a multigraph with $\mathcal{S}(G) \geq 2k - 1$ for some integer $k \geq 2$, and set $F = F(G)$ and $\alpha' = \alpha'(F)$. Suppose G contains at least one loop. Then G has no k disjoint cycles if and only if $|V(G)| + 2|V_1| + \alpha' < 3k$.*

Instead of the $(2k-1)$-connected multigraphs of Question 2.1.2, we consider the wider family $\mathcal{D}_k$. Since acyclic graphs are exactly forests, Theorem 2.1.3 can be restated as follows:

Theorem 3.2.3. *For $k \in \mathbb{Z}^+$, let G be a simple graph in $\mathcal{D}_k$. Then G has no k disjoint cycles if and only if one of the following holds:*

(α) *$|G| \leq 3k - 1$;*

(β) *$k = 1$ and G is a forest with no isolated vertices;*

(γ) *$k = 2$ and G is a wheel;*

(δ) $\alpha(G) = n - 2k + 1$; *or*

(ϵ) $k > 1$ *is odd and* $G = Y_{k,k}$.

Dirac [11] described all multigraphs in $\mathcal{D}_2$ that do not have two disjoint cycles:

Theorem 3.2.4 ([11]). *Let G be a 3-connected multigraph. Then G has no two disjoint cycles if and only if one of the following holds:*

(A) $\widetilde{G} = K_4$ *and the strong edges in G form either a star (possibly empty) or a 3-cycle;*

(B) $G = K_5$;

(C) $\widetilde{G} = K_5 - e$ *and the strong edges in G are not incident to the ends of e;*

(D) $\widetilde{G}$ *is a wheel, where some spokes could be strong edges; or*

(E) G is obtained from $K_{3,|G|-3}$ by adding non-loop edges between the vertices of the (first) 3-class.

Going further, Lovász [33] described *all* multigraphs with no two disjoint cycles. He observed that it suffices to describe such multigraphs with minimum (ordinary) degree at least 3, and proved the following:

Theorem 3.2.5 ([33]). *Let G be a multigraph with $\delta(G) \geq 3$. Then G has no two disjoint cycles if and only if G is one of the following:*

(1) K_5;

(2) *A wheel, where some spokes could be strong edges;*

(3) $K_{3,|G|-3}$ *together with a loopless multigraph on the vertices of the (first) 3-class; or*

(4) *a forest F and vertex x with possibly some loops at x and some edges linking x to F.*

By Corollary 3.2.2, in order to describe the multigraphs in $\mathcal{D}_k$ not containing k disjoint cycles, it is enough to describe such multigraphs with no loops. Our main result is the following:

Theorem 3.2.6. *Let $k \geq 2$ and $n \geq k$ be integers. Let G be an n-vertex multigraph in $\mathcal{D}_k$ with no loops. Set $F = F(G)$, $\alpha' = \alpha'(F)$, and $k' = k - \alpha'$. Then G does not contain k disjoint cycles if and only if one of the following holds: (see Figure 3.2)*

(a) $n + \alpha' < 3k$;

(b) $|F| = 2\alpha'$ *(i.e., F has a perfect matching) and either*

(i) k' *is odd and* $G - F = Y_{k',k'}$, *or*

(ii) $k' = 2 < k$ *and $G - F$ is a wheel with 5 spokes;*

(c) G is extremal and either

(i) some big set is not incident to any strong edge, or

(ii) for some two distinct big sets I_j and $I_{j'}$, all strong edges intersecting $I_j \cup I_{j'}$ have a common vertex outside of $I_j \cup I_{j'}$;

(d) $n = 2\alpha' + 3k'$, k' is odd, and F has a superstar $S = \{v_0, \ldots, v_s\}$ with center v_0 such that either

(i) $G - (F - S + v_0) = Y_{k'+1,k'}$, or

(ii) $s = 2$, $v_1v_2 \in E(G)$, $G - F = Y_{k'-1,k'}$ and G has no edges between $\{v_1, v_2\}$ and the set X_0 in $G - F$;

(e) $k = 2$ and G is a wheel, where some spokes could be strong edges;

(f) $k' = 2$, $|F| = 2\alpha' + 1 = n - 5$, and $G - F = C_5$.

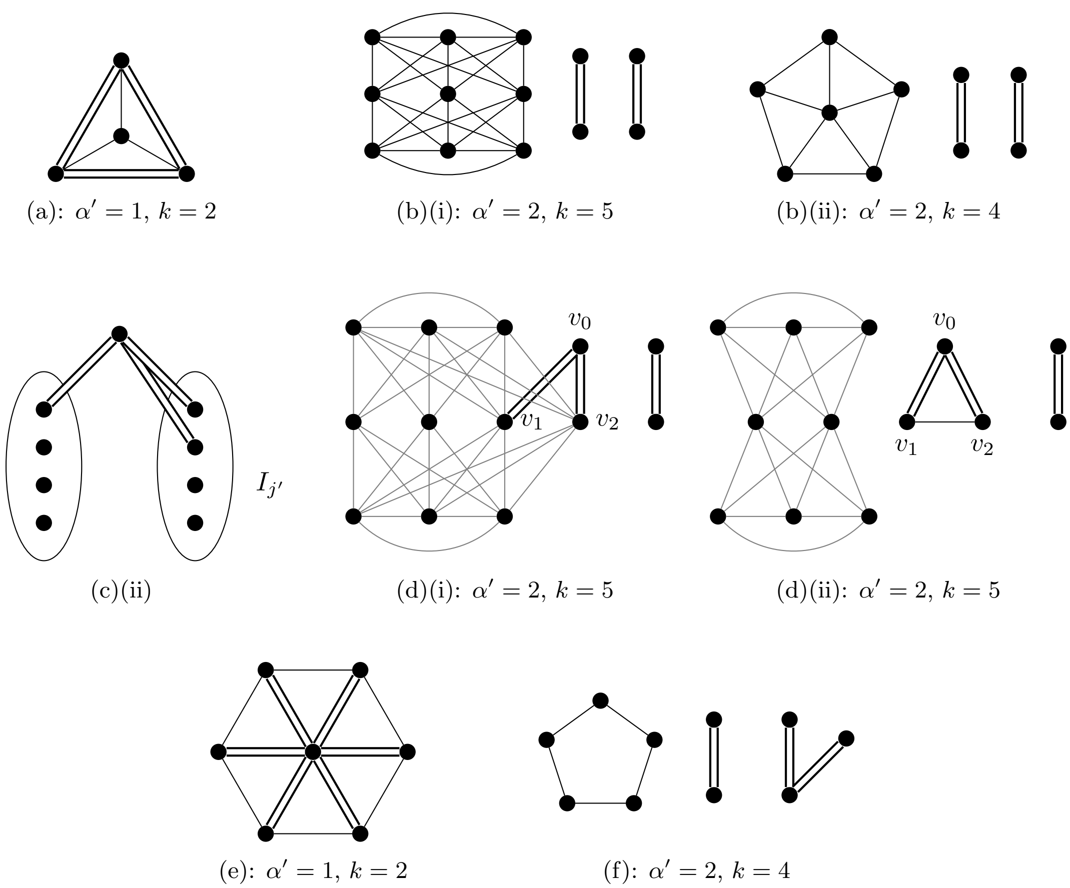


Figure 3.2: Examples of Subgraphs of Multigraphs Listed in Theorem 3.2.6

The six infinite classes of multigraphs described in Theorem 3.2.6 are exactly the family of multigraphs in $\mathcal{D}_k$ with no k disjoint cycles. So, the $(2k-1)$-connected multigraphs with no k disjoint cycles are exactly the $(2k-1)$-connected multigraphs that are in one of these classes. For any multigraph G, we can check in polynomial time whether $G \in \mathcal{D}_k$ and whether G is $(2k-1)$-connected. If $G \in \mathcal{D}_k$, we can check in polynomial time whether any of the conditions (a)–(f) hold for G. Note that to determine the extremality of G we need only check whether G has an independent set of size $n-2k+1$. Such a set will be the complement of $N(v)$ for some vertex v with $s(v) = 2k-1$; so all big sets can be found in polynomial time.

Note if G is $(2k-1)$-connected, and (b)(i), (d)(i), or d(ii) holds, then $k' \leq 1$.

3.3 Proof of sufficiency in Theorem 3.2.6

Suppose G has a set $\mathcal{C}$ of k disjoint cycles. Our task is to show that each of (a)–(f) fails. Theorem 3.2.5, case (2) implies (e) fails. Let $M \subseteq \mathcal{C}$ be the set of strong edges (2-cycles) in $\mathcal{C}$, $h = |M|$, and $W = V(M)$. Now $h \leq \alpha'$; so $n \geq 2h + 3(k-h) \geq 3k - \alpha'$. Thus (a) fails. If $n = 3k - \alpha'$ as in cases (b), (d) and (f), then $h = \alpha'$ and $G' = G - W$ is a simple graph of minimum degree at least $2k'-1$ with $3k'$ vertices and k' cycles. By Theorem 2.1.3 all of (i)–(iii) hold for G'. In case (b), $G' = G - F$; so (ii) and (iii) imply (b)(i) and (b)(ii) fail. In case (f), $G' = G - (F - v) = v \vee C_5$ for some vertex $v \in F$. So (iii) implies (f) fails. In case (d), M consists of a strong perfect matching in $F - S$ together with a strong edge $v_0 v \in S$. If $G - (F - S + v_0) = Y_{k'+1,k'}$ then either $\alpha(G') = k'+1$ or $G' = Y_{k',k'}$, contradicting (i) or (ii). So(d)(i) fails. Similarly, in case (d)(ii), $G' \subseteq Y_{k',k'}$, another contradiction.

In case (c), G is extremal. Every big set I satisfies $|V(G) - I| < 2k$. So some cycle $C_I \in \mathcal{C}$ has at most one vertex in $V(G) - I$. Since I is independent, C_I has at most one vertex in I. Thus C_I is a strong edge and $(c)(i)$ fails. Let J be another big set; then $I \cap J = \emptyset$. As cycles in $\mathcal{C}$ are disjoint, $C_I = C_J$ or $C_I \cap C_J = \emptyset$. Regardless, $C_I \cap C_J \subseteq I \cup J$. So (c)(ii) fails.

3.4 Proof of necessity in Theorem 3.2.6

Suppose G does not have k disjoint cycles. Our goal is to show that one of (a)–(f) holds. If $k = 2$ then one of the cases (1)–(4) of Theorem 3.2.5 holds. If (1) holds then $\alpha' = 0$, and so (a) holds. Case (2) is (e). Case (3) yields (c)(i), where the partite set of size $n-3$ is the big set. As $G \in \mathcal{D}_k$, it has no vertex l with $s(l) < 3$. So (4) fails, because each leaf l of the forest satisfies $s(l) \leq 2$. Thus below we assume

$$k \geq 3. \tag{3.3}$$

Choose a maximum strong matching $M \subseteq F$ with $\alpha(G - W)$ minimum, where $W = V(M)$. Then

$|M| = \alpha'$, $G' := G - W$ is simple, and $\delta(G') \geq 2k-1-2\alpha' = 2k'-1$. So $G' \in \mathcal{D}_{k'}$. Let $n' := |V(G')| = n-2\alpha'$. Since G' has no k' disjoint cycles, Theorem 3.2.3 implies one of the following: (α) $|G'| \leq 3k'-1$; (β) $k' = 1$ and G' is a forest with no isolated vertices; (γ) $k' = 2$ and G' is a wheel; (δ) $\alpha(G') = n'-2k'+1 = n-2k+1$; or ($\epsilon$) $k' > 1$ is odd and $G' = Y_{k',k'}$. If (α) holds then so does (a). So suppose $n' \geq 3k'$. In the following we may obtain a contradiction by showing G has k disjoint cycles.

Case 1: (β) holds. By (3.3), there are strong edges $yz, y'z' \in M$. As $\mathcal{S}(G) \geq 2k-1$, each vertex $v \in V(G')$ is adjacent to all but $d_{G'}(v) - 1$ vertices of W.

Case 1.1: G' contains a path on four vertices, or G' contains at least two components. Let $P = x_1 \dots x_t$ be a maximum path in G'. Then x_1 is a leaf in G', and either $d_{G'}(x_2) = 2$ or x_2 is adjacent to a leaf $l \neq x_1$. So vx_1x_2v or vx_1x_2lv is a cycle for all but at most one vertex $v \in W$. If $t \geq 4$, let $s_1 = x_t$ and $s_2 = x_{t-1}$. Otherwise, G' is disconnected and every component is a star; in a component not containing P, let s_1 be a leaf and let s_2 be its neighbor. As before, for all but at most one vertex $v' \in W$, either $v's_1s_2v'$ is a cycle or $v's_1s_2l'v'$ is a cycle for some leaf l'. Thus $G[(V \smallsetminus W) \cup \{u,v\}]$ contains two disjoint cycles for some $uv \in \{yz, y'z'\}$. These cycles and the $\alpha'-1$ strong edges of $M - uv$ yield k disjoint cycles in G, a contradiction.

Case 1.2: G' is a star with center x_0 and leaf set $X = \{x_1, x_2, \dots, x_t\}$. Since $n' \geq 3k'$, $t \geq 2$ and X is a big set in G. If (c)(i) fails then some vertex in X, say x_1, is incident to a strong edge, say x_1y. If $t \geq 3$, then G has k disjoint cycles: $|M - yz + yx_1|$ strong edges and $zx_2x_0x_3z$. Else $t = 2$. Then $n = 3\alpha' + 3k' = 2k+1$, as in (d); and each vertex of G is adjacent to all but at most one other vertices. If $x_0z \in E(G)$ then again G has k disjoint cycles: $|M - yz + yx_1|$ strong edges and zx_0x_2z, a contradiction. So $N(x_0) = V(G) - z - x_0$, and $G[\{x_0, x_1, x_2, z\}] = C_4 = Y_{2,1}$. Also y is the only possible strong neighbor of x_1 or x_2: if $u \in \{x_1, x_2\}$, $y'z' \in M$ with $y' \neq y$ (maybe $y' = z$) and $uy' \in E(F)$, using the same argument as above, if $z'x_0 \in E(G)$ then G has k disjoint cycles consisting of $|M - y'z' + y'u|$ strong edges and $G[G' - u + z']$, a contradiction. Then $x_0z' \notin E(G)$, so $z' = z$, and $y' = y$. Thus $S = N_F(y) \cap \{z, x_0, x_1, x_2\} + y$ is a superstar. So(d)(i) holds.

Case 2: (γ) holds. Then $k' = 2$ and G' is a wheel with center x_0 and rim $x_1x_2 \dots x_tx_1$. By (3.3), there exists $yz \in M$. Since (a) fails, $t \geq 5$. For $i \in [t]$,

$$s(x_i) \geq 2k-1 = 2\alpha' + 3 = 2\alpha' + |N(x_i) \cap G'|,$$

so x_i is adjacent to every vertex in W. If $t \geq 6$, then G' has k disjoint cycles: $|M - yz|$ strong edges, yx_1x_2y, zx_3x_4z and $x_0x_5x_6x_0$. Thus $t = 5$. If no vertex of G' is incident to a strong edge, then (b)(ii) holds. Therefore, we assume y has a strong edge to G'. The other endpoint of the strong edge could be in the outer cycle, or could be x_0. If some vertex in the outer cycle, say x_1, has a strong edge to y, then we have

k disjoint cycles: $|M - yz + yx_1|$ strong edges, zx_2x_3z and $x_0x_4x_5x_0$. The last possibility is that x_0 has a strong edge to y, and (f) holds.

Case 3: (ϵ) holds. Then $k' > 1$ is odd, $G' = Y_{k',k'}$ and $n = 2\alpha' + 3k'$. Let $X_0 = \{x_1, \dots, x_{k'}\}$, $X_1 = \{x'_1, \dots, x'_{k'}\}$, and $X_2 = \{x''_1, \dots, x''_{k'}\}$ be the sets from the definition of $Y_{k',k'}$. Observe

$$\overline{K}_{s+t} \vee (K_{2s} \cup K_{2t}) \text{ contains } s+t \text{ disjoint triangles.} \tag{3.4}$$

By degree conditions, each $x' \in X_1 \cup X_2$ is adjacent to each $v \in W$ and each $x \in X_0$ is adjacent to all but at most one $y \in W$. If (b)(i) fails then some strong edge uy is incident with a vertex $u \in V(G')$. If possible, pick $u \in X_1 \cup X_2$. By symmetry we may assume $u \notin X_2$. Let yz be the edge of M incident to y. Set $v_0 = y$ and $\{v_1, \dots, v_s\} = V(F \cap G') + z$. We will prove that $\{v_0, \dots, v_s\}$ is a superstar, and use this to show that (d)(i) or (d)(ii) holds. Let $G^* = G - (W - z)$, and observe that $Y_{k'+1,k'}$ is a spanning subgraph of G^* with equality if $X_0 + z$ is independent.

Suppose $xz \in E(G)$ for some $x \in X_0 - u$. Then G has k disjoint cycles: $|M - yz + yu|$ strong edges, zxx''_1z, and $k' - 1$ disjoint cycles in $G^* - \{x, x''_1, u\}$, obtained by applying (3.4) directly if $u \in X_1$, or by using $T := x'_1x'_2x'_3x'_1$ and applying (3.4) to $G^* - \{x, x''_1, u\} - T$ if $u \in X_0$. This contradiction implies zu is the only possible edge in $G[X_0 + z]$. Thus if y has two strong neighbors in X_0 then $X_0 + z$ is independent, and $G^* = K_{k'+1,k'}$. Also by degree conditions, every $x \in X_0 - u$ is adjacent to every $w \in W - z$. So if $y'z' \in M$ with $y' \neq y$ and $u' \in V(G')$, then $u'y' \notin E(F)$: else $x \in X_0 - u - u'$ satisfies $xz' \in E(G)$ and $xz' \notin E(G)$. So $\{v_0, \dots, v_s\}$ is a superstar. If $X_0 + z$ is independent then (d)(i) holds; else (d)(ii) holds.

Case 4: (δ) holds. Then $\alpha(G') = n' - 2k' + 1 > n'/3$, since $n' \geq 3k'$. So G' is extremal. Let J be a big set in G'. Then $|J| = n' - 2k' + 1 = n - 2k + 1$. So G is extremal and J is a big set in G. Also each $x \in J$ is adjacent to every $y \in V(G) - J$. If (c)(i) fails then some $x \in J$ has a strong neighbor y. Let yz be the edge in M containing y. In F, consider the maximum matching $M' = M - yz + xy$, and set $G'' = G - V(M')$. By the choice of M, G'' contains a big set J', and J' is big in G. Since $x \notin J'$, (3.1) implies $J' \cap J = \emptyset$ (possibly, $z \in J'$). If (c)(ii) fails then there is a strong edge vw such that $v \in J \cup J'$ and $w \neq y$. Moreover, by the symmetry between J and J', we may assume $v \in J'$. Let uw be the edge in M containing w. Since M is maximum, $u \neq z$. Let $M'' = M' - uw + vw$. Again by the case, $G - V(M'')$ contains a big set J''. Since $x, v \notin J''$, J'' is disjoint from $J \cup J'$. So $n' \geq 3|J| > n'$, a contradiction. $\square$

Chapter 4

Equitable Coloring

The following results are joint work with Henry Kierstead, Alexandr Kostochka, and Theodore Molla; this chapter is based on [27].

In this chapter, we prove that under certain conditions a graph is guaranteed to have an equitable coloring. This result confirms a partial case of a generalized version of the Chen-Lih-Wu conjecture on equitable coloring. In addition, our result is equivalent to a statement about disjoint cycles, and so completes the work of Theorem 2.1.7 of characterizing graphs G with $\sigma_2(G) \geq 4k-3$ that have no k disjoint cycles.

4.1 Introduction

It is a trivial result that a graph with maximum degree Δ can be properly colored using at most $\Delta+1$ colors. Brooks' Theorem [5] famously characterizes those graphs with maximum degree Δ that can be colored using only Δ colors. The Chen-Lih-Wi Conjecture [6] attempts to extend Brooks' Theorem to equitable colorings.

Conjecture 4.1.1 (Chen-Lih-Wu Conjecture). *Every k-colorable graph G with $\Delta(G) \leq k$ is k-equitably-colorable unless k is odd and G contains $K_{k,k}$.*

Kierstead and Kostochka proposed an Ore-type version of the Chen-Lih-Wu conjecture in [22]:

Conjecture 4.1.2 ([22]). *Let $k \geq 3$. If $\theta(G) \leq 2k+1$, then G is equitably k-colorable unless G contains K_{k+1} or $K_{m,2k-m}$ for some odd m.*

In the same paper, Kierstead and Kostochka proved the following result, which will be of use to us:

Theorem 4.1.3 ([22]). *Every graph G with $\theta(G) < 2k$ has an equitable k-coloring.*

Kierstead and Kostochka have also proved results on equitable coloring in [25] and [21] which are equivalent to the following theorem:

Theorem 4.1.4 ([25], [21]). *Let G be a graph with $|G| = ks$ and $\chi(G), \Delta(G) \leq k$ that has no equitable k-coloring. If either $s \leq 4$ or $k \leq 4$ then k is odd, $K_{k,k} \subseteq G$, and $G - K_{k,k}$ is k-equitable. In particular, if $s = 3$ then $G = K_{k,k} + K_k$.*

In this chapter, we prove an Ore-type version of Theorem 4.1.4 for the case $s \leq 3$. This settles the partial case of Conjecture 4.1.2 when $|G| \leq 3k$. (Indeed, in the case $k = 3$ and $|G| = 9$, in order for the conjecture to be true it must be modified to include one more graph.)

First we dispense with the easy cases $s \leq 2$. If $s = 1$ then G has k vertices and trivially has an equitable k-coloring. The next theorem completes the case $s = 2$. Notice that if $c \in [k]$ is odd, then $K_{c,2k-c}$ has no equitable k-coloring.

Theorem 4.1.5. *Let G be a graph satisfying $|G| = 2k$, (H1) $\chi(G) \leq k$ and (H2) $\theta(G) \leq 2k+1$. If G has no equitable k-coloring then $G = K_{c,2k-c}$ for some odd $c \in [k]$.*

Proof. By (H1), G has a k-coloring. If $k = 1$ it is equitable, and if $k = 2$ it can be made equitable unless $G = K_{1,3}$. So suppose $k \geq 3$, and G has no equitable 2-coloring. Then $\overline{G}$ has no 2-factor. By Tutte's Theorem, there is a set $T \subseteq V(G)$ with $|T| = t$ such that $\overline{G} - T$ has $t + 2i$ odd components, where $i \in \mathbb{Z}^+$. If $t = 0$, then $K_{2i} \subseteq G$, so $i = 1$ and $K_{c,2k-c} \subseteq G$ for some odd $c \in [k]$.

For a contradiction, it suffices to prove that if $t > 0$ then (H1) or (H2) fails. If $t \geq k-1$ then $\chi(G) \geq \omega(G) \geq t + 2 \geq k + 1$. Otherwise $t \in [k-2]$. Let X and Y be the two smallest components of $\overline{G} - T$, $x \in X$ and $y \in Y$. Then $|X \cup Y| \leq \lfloor 2(2k-t)/(t+2) \rfloor$. So

$$\theta(G) \geq d(x) + d(y) \geq 2(2k-t) - |X \cup Y| \geq 4k - 2t - \left\lfloor \frac{4k-2t}{t+2} \right\rfloor \tag{4.1}$$

$$\geq f(t) := 4k - 2t + 2 - \frac{4k+4}{t+2} \tag{4.2}$$

If $k = 3$, then $t = 1$ and $|X| = 1 = |Y|$ by parity; then $\theta(G) \geq 10 - 2 = 8 > 2 \cdot 3 + 1$. So assume $k \geq 4$. Now $f(1) = 8k/3 - 4/3 > 2k + 1$, so assume $t > 1$. If $k = 4$, then $t = 2$ and $|X| = 1 = |Y|$ by parity; so $\theta(G) \geq 10 > 2 \cdot 4 + 1$. Then $k > 4$, so $f(k-2) = 2k + 2 - 4/k > 2k + 1$. Finally, for fixed k, $f(t)$ is concave, since

$$\frac{d^2}{dt^2} f(t) = -8 \frac{k+1}{(t+2)^3} < 0$$

So $\theta(G) \geq f(t) \geq \min\{f(1), f(k-2)\} > 2k + 1$. □

Next we consider some examples for the case $s = 3$. Let $K(X)$ denote the complete graph with vertex set X, and $K(X, Y)$ denote the X, Y-partite graph.

Example 4.1.6. Let $Q := K(\{x_1, x_2, x_3\}, \{y_1, y_2, y_3\})$, $K = K(\{w_1, w_2, w_3\})$, and

$$\mathbf{X} \quad = \quad Q - x_3y_3 + K + x_3w_1 + x_3w_2 + y_3w_3. \tag{4.3}$$

(See Figure 4.1.) Then $|\mathbf{X}| = 9 = 3 \cdot 3$, $\chi(\mathbf{X}) = 3$, and $\theta(\mathbf{X}) = 2 \cdot 3 + 1$, but $\mathbf{X}$ has no equitable 3-coloring: Any 3-coloring f gives distinct colors to K and satisfies $f(x_3) = f(w_3) \neq f(y_3)$. So if f is an equitable 3-coloring of $\mathbf{X}$ then it is also an equitable 2-coloring of Q, a contradiction. Also

$$\mathbf{X} \simeq Q - x_3y_3 - x_3y_2 + K + x_3w_1 + x_3w_2 + y_3w_3 + y_2w_3. \tag{4.4}$$

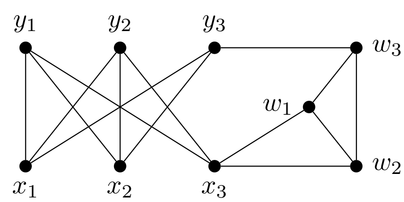
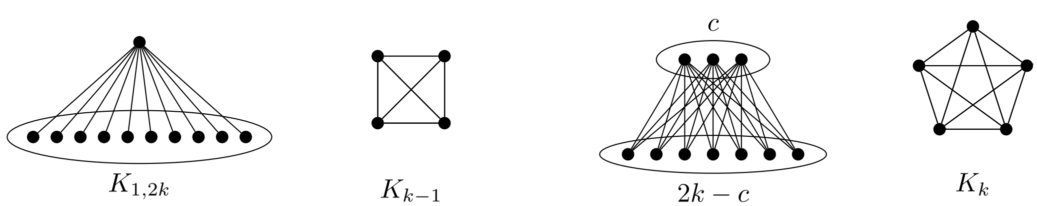


Figure 4.1: $\mathbf{X}$, Example 4.1.6

Example 4.1.7. Let $k \geq 2$, and $\mathbf{Y} = \mathbf{Y_k} = K_{1,2k} + K_{k-1}$. (See Figure 4.2(a).) Then $|\mathbf{Y}| = 3k$, $\chi(\mathbf{Y}) \leq k$, and $\theta(\mathbf{Y}) = 2k + 1$, but $\mathbf{Y}$ has no equitable k-coloring: for any k-coloring the class of the vertex r with $d(r) = 2k$ contains at most r and one vertex from K_{k-1}.

Example 4.1.8. For $k \geq 2$ and odd $c \leq k$, let $V = B_1 \cup B_2 = C_1 \cup C_2 \cup B_2$, where C_1, C_2, B_2 are disjoint, $|C_1| = c$, $|C_2| = 2k - c$, and $|B_2| = k$. Set $\mathbf{Z_c} = \mathbf{Z_{c,k}} = Q + K$, where $Q = K(C_1, C_2)$ and $K = K(B_2)$. (See Figure 4.2(b).) Then $|\mathbf{Z_c}| = 3k$, $\chi(\mathbf{Z_c}) = k$, and $\theta(\mathbf{Z_c}) = 2k$, but $\mathbf{Z_c}$ has no equitable k-coloring. Indeed, each class of an equitable coloring of $\mathbf{Z_c}$ must contain one vertex of K and two vertices from the same part of Q. As c is odd, this is impossible.



(a) $\mathbf{Y_k}$, $k = 5$ (b) $\mathbf{Z_{c,k}}$, $k = 5$, $c = 3$

Figure 4.2: Examples 4.1.7 and 4.1.8.

Theorem 4.1.9. *Let G be a graph with (H1) $\chi(G) \leq k$, (H2) $\theta(G) \leq 2k + 1$, and (H3) $|G| = 3k$. If G has no equitable k-coloring then $G \in \{\mathbf{X}, \mathbf{Y_k}\}$ or $\mathbf{Z_{c,k}} \subseteq G$ for some odd c.*

Notation. For a graph $G = (V, E)$ and sets $X, Y \subseteq V$, let $E(X) := E_G(X) = E(G[X])$ and let $E(X, Y) := E_G(X, Y)$ be the set of edges with one end in X and one end in Y. A k-coloring of G is a partition $\mathcal{V}$ of V into k independent sets. We may express this partition as a function $f : V \to [k]$, where $f^{-1}(i) \in \mathcal{V}$ for each $i \in [k]$.

4.2 Setup and preliminaries

Suppose $G = (V, E)$ is a counterexample to Theorem 4.1.9 with k minimum, and subject to this $\|G\|$ minimum. So G satisfies (H1–H3), $G \notin \{\mathbf{X}, \mathbf{Y}\}$, $\mathbf{Z}_c \nsubseteq G$ for any odd c, and

$$G \text{ has no equitable } k\text{-coloring, but } G - e \text{ has an equitable } k\text{-coloring for all } e \in E. \tag{4.5}$$

By minimality of k,

$$\text{Theorem 4.1.9 holds for all } k' \in [1, \ldots, k-1]. \tag{4.6}$$

Call a vertex v *high* if $d(v) \geq k+1$, and *low* otherwise. For a subset W of $V(G)$, let $H(W)$ denote the set of high vertices in W and $L(W) = W \smallsetminus H(W)$ denote the set of low vertices. An edge is *high* if it has a high end. By (H2), $H(V)$ is independent; so a high edge also has a low vertex.

Lemma 4.2.1. *$k < \Delta(G) \leq 2k-2$. In particular, $k \geq 3$.*

Proof. By Theorem 4.1.4, if $\Delta(G) \leq k$ then k is odd and $\mathbf{Z_k} \subseteq G$, a contradiction. Suppose $d(v) = d := \Delta(G) \geq 2k-1$ for some $v \in V$. As every neighbor of v has positive degree, (H2) implies $d \leq 2k$. Let $X = N(v)$ and $Y = V(G) \smallsetminus N[v]$. If Y is a clique then G contains $\mathbf{Y}$ or $\mathbf{Z_1}$; else choose distinct nonadjacent vertices $y_1, y_2 \in Y$ with $\|\{y_1, y_2\}, X\|$ maximum. Let $V_1 = \{v, y_1, y_2\}$ be one color class.

If $d = 2k$ then X is independent and $\|X, Y\| = 0$. Since $G - \{v, y_1, y_2\} \subseteq K_{k-3} + \overline{K}_{2k}$, it has an equitable $(k-1)$-coloring. Thus G has an equitable k-coloring, contradicting (4.5). So $d = 2k-1$. If $k = 2$ then X is independent by (H1), contradicting (4.5). Thus $k \geq 3$.

By (H2), each $x \in X$ has at most one neighbor in $V - v$. So $M := E(X)$ is a matching, the vertices of Y are not adjacent to vertices saturated by M, and $\|X, Y\| \leq d - 2t$, where $t = |M|$. Say $M = \{e_i : i \in [t]\}$. Order the vertices in $Y - y_1 - y_2$ so that $\|y_3, X\| \geq \ldots \geq \|y_k, X\|$.

Note that $\|y_3, X\| \leq k$, and if equality holds then $d(y_3) = d$: If not then $\|y_3, Y\| \leq d - (k+1) = k-2$; so there is $y \in Y - y_3$ with $yy_3 \notin E$. Thus $\|\{y_1, y_2\}, X\| \geq \|\{y_3, y\}, X\| \geq k$, so $\|X, Y\| \geq 2k > d$, a contradiction. Thus $|X \smallsetminus N(y_3)| \geq k-1 \geq 2$. Then there exist distinct nonadjacent vertices $x_1, x_2 \in X \smallsetminus N(y_3)$: if not, $X \smallsetminus N(y_3) = K_2$, $\|y_3, X\| = k$, $d(y_3) = d$, and $V \smallsetminus N[y_3] = K_3 = K_k$, so $\mathbf{Z_1} \subseteq G$.

Using that M is a matching, choose x_1 and x_2 to be in distinct edges of M if possible; that is, label X and M so that for each $j \leq \max\{2, t\}$, $x_j \in e_j$.

Let $V_2 = \{x_1, x_2, y_3\}$ be the second color class. Put $X_3 = X \smallsetminus \{x_1, x_2\}$. If $k = 3$ then X_3 is independent, and we are done. So assume $k \geq 4$.

We recursively construct color classes $V_i = \{y_{i+1}, x_{2i-3}, x_{2i-2}\}$ for $i \in \{3, \dots, k-1\}$. Suppose we have chosen $V_1, \dots, V_{i-1}$, and set $X_i := N(v) \smallsetminus \{x_1, \dots, x_{2i-4}\}$. By our choice of labels in $Y \smallsetminus \{y_1, y_2\}$, $\|y_{i+1}, X\| \leq \left\lfloor \frac{\|Y,X\|}{i-1} \right\rfloor \leq \left\lfloor \frac{2k-2t-1}{i-1} \right\rfloor$. Also $|X_i| = 2(k-i)+3$, so

$$\begin{aligned}|X_i - N(y_{i+1})| &\geq |X_i| - \|y_{i+1}, X\| \geq 2(k-i) + 3 - \left\lfloor \frac{2k-2t-1}{i-1} \right\rfloor \\ &= \left\lceil 3 + 2(k-i)\left(1 - \frac{1}{i-1}\right) - \frac{2i-2t-1}{i-1} \right\rceil \qquad (*) \\ &\geq \left\lceil 3 + (k-i) - \frac{2i-1}{i-1} \right\rceil \geq \left\lceil 3 + 1 - \frac{5}{2} \right\rceil = 2.\end{aligned}$$

Note that if $|X_i - N(y_{i+1})| = 2$, the starred line shows $i > t$. Now we select distinct, nonadacent x_{2i-3}, x_{2i-2} in $X_i \smallsetminus N(y_{i+1})$. If we can choose $x_{2i-3} \in e_i$, we do so. More precisely: using that $V(M) \subseteq X \smallsetminus N(y_i)$, if $i \leq t$ and $e_i \cap X_i \neq \emptyset$, we choose $x_{2i-3} \in e_i$; then, since $|X_i - N(y_{i+1})| \geq 3$, we select $x_{2i-2} \in X_i \smallsetminus (e_i \cup N(y_{i+1}))$. Suppose $i > t$, or $e_i \cap X_i = \emptyset$. If $|X_i \smallsetminus N(y_{i+1})| = 2$, since $i > t$ and by our choice of $V_1, \dots, V_{i-1}$, the two vertices of $X_i \smallsetminus N(y_{i+1})$ are nonadjacent. Otherwise, since M is a matching, we let x_{2i-3}, x_{2i-2} be any two distinct, nonadjacent vertices in $X_i \smallsetminus N(y_{i+1})$. Finally, let $V_k := X_{k-1}$ be the last color class. Since $|M| \leq k-1$, V_k is independent. □

Lemma 4.2.2. $\omega(G) \leq k - 1$.

Proof. Suppose K is a k-clique in G, and set $H = G - K$. As $\mathbf{Z_c} \nsubseteq G$ for any odd c, $K_{c,2k-c} \nsubseteq H$ for any odd c. By (H2),

$$\|xy, H\| \leq 3 \text{ for all } x, y \in K. \tag{4.7}$$

By Theorem 4.1.5, H has an equitable k-coloring f.

First suppose (i) $K \nsubseteq N(U)$ for all classes U of f and (ii) no vertex $x \in K$ has neighbors in all classes of f. Extend f to an equitable k-coloring f' of G by first greedily adding vertices of K into distinct classes of f starting with the vertex x with $\|x, H\|$ maximum. By (ii) and (4.7) the process will not get stuck before the last vertex $z \in K$. If z cannot be greedily added to the last remaining class W, (4.7) implies W is the only class z is adjacent to. By (i) there is $y \in K \smallsetminus N(W)$. Move y to W and z to the former class of y to finish. As this contradicts (4.5), (i) or (ii) fails.

If $k \geq 4$ then (4.7) implies (ii). Suppose $k = 3$ and (ii) fails because $x \in K$ has a neighbor z_i in each class Z_i of f. As $G[N[x]] = \mathbf{Z_5} \nsubseteq G$, there is $y \in Y := V - N[x]$ with $Y' := Y - y + x$ independent. Set $G' = G - Y'$. By (H2), $d_{G'}(w) \leq 1$ for all $w \in N(x)$. Also $d_{G'}(y) \leq 2$. Thus by Theorem 4.1.3, G' has an equitable 2-coloring, contradicting 4.5. So (ii) holds and (i) fails.

Say $K \subseteq N(Z)$ for some class $Z = \{z, z'\}$ of f. Put $H^+ = H + zz'$. Then $d_{H^+}(z) \leq d_G(z)$ and $d_{H^+}(z') \leq d_G(z')$. So $\theta(H^+) \leq 2k+1$. Suppose H^+ has no equitable k-coloring. By Theorem 4.1.5, $Q := K_{c,2k-c} \subseteq H^+$ for some odd $c \leq k$, and $zz' \in E(Q)$. Say $d_Q(z') = c$. Note each vertex of $\{z, z'\}$ has a neighbor in K because $\chi(G) \leq k$. Then there exist $x \in K$ and $y \in V(H)$ with $xz, yz' \in E$. As $\Delta(G) \leq 2k-2$ and $G \neq \mathbf{X}$, $k \geq 4$. By (H2)

$$4k+2 \geq \theta(xz) + \theta(yz') \geq \|Z, K\| + k + (2k-c-1) + (2k-1) \geq 6k-2-c.$$

So $2k-4 \leq c \leq k$. As c is odd and $k \geq 4$, this is a contradiction. Thus H^+ has an equitable k-coloring f'.

Since (i) fails, there is a class Y of f' such that $K \subseteq N(Y)$. As $zz' \in E(H^+)$, $Y \neq Z$. As $\|K, H^+\| \leq k+1$, and $\chi(G) \leq k$, there are vertices $u \in K$ and $z'' \in V(H)$ with (say) $Y = \{z, z''\}$, $N(z) \cap K = K - u$, $uz', uz'' \in E$, and $N(K) = \{z, z', z''\}$. If $H^* := H^+ + zz''$ has an equitable coloring then it satisfies (i), and we are done. Otherwise, $Q := K_{c,2k-c} \subseteq H^*$ for some odd $c \leq k$, with $zz'' \in E(Q)$. By Lemma 4.2.1, $3 \leq c$. If $k = 3$ then $G = \mathbf{X}$ by (4.4). Else, for $w \in N_Q(z) \smallsetminus \{z', z''\}$,

$$2k+1 \geq \theta(zw) \geq \|z, K\| + \theta_{H^*}(wz) - 2 \geq k-1+2k-2 = (2k+1)+(k-4),$$

so $k = 4$ and z', z'' are in one part Q' of Q. Since $d(u) = k+1$, $d(z'), d(z'') \leq k$, so $|Q'| = 5$. But now for $x \in V(K) - u$, $d(z) + d(u) \geq 6 + 4 = 2k+2$, a contradiction. $\square$

Lemma 4.2.3. $k \geq 4$.

Proof. For a contradiction, suppose $k \leq 3$. By Lemma 4.2.1, $k = 3$ and $\Delta(G) = 4$. Let $d(v) = 4$, $N = N(v)$, $G' = G - N[v]$, and $V(G') = N'$. By Lemma 4.2.2,

$$\omega(G) \leq 2. \tag{i}$$

So N is independent and, since $|G'| = 4$, G' is bipartite. Thus (H2) implies

$$\|x, N'\| \leq 2 \text{ for all } x \in N \tag{ii}$$

and $\|N, N'\| \leq 8$.

Suppose $d_{G'}(w) = 3$ for some $w \in N'$. Then $\|w, N\| \leq 1$ because $\Delta(G) = 4$, and $N(w) \cap N(w') = \emptyset$ for all $w' \in N' - w$ by (i) Because $\|N, N'\| \leq 8$, $\|w', N\| \leq 2$ for some $w' \in N' - w$. Choose $x_1, x_2 \in N \smallsetminus N(w')$, including the neighbor of w if it exists. Then $\{\{w', x_1, x_2\}, N - x_1 - x_2 + w, N' - w - w' + v\}$ is an equitable 3-coloring of G.

Otherwise $\Delta(G') \leq 2$, so N' has an equitable 2-coloring.

$$\text{If } Y \text{ is a class of an equitable 2-coloring of } N' \text{ then } N(x) \cap Y \neq \emptyset \text{ for all } x \in N: \tag{iii}$$

else $\{(N' \smallsetminus Y) + v, Y + x, N - x\}$ is an equitable 3-coloring of G. Let $N' = \{y_1, y_2, y_3, y_4\}$ and $x \in N$. As N' has an equitable 2-coloring g, (ii) and (iii) imply $\|x, N'\| = 2$. Say $N(x) = \{y_1, y_2\}$. By (iii), $y_3y_4 \in E$. By (i), $y_1y_2 \notin E$ and $N(y_3) \cap N(y_4) = \emptyset$. If $\|y_3, N\|, \|y_4, N\| \leq 2$, then because they share no neighbrs there exist disjoint 2-sets $X_1, X_2 \subseteq N$ with $N(y_3) \cap N \subseteq X_1$ and $N(y_4) \cap N \subseteq X_2$. So $\{\{v, y_1, y_2\}, X_1 + y_4, X_2 + y_3\}$ is an equitable 3-coloring of G. Thus (say) $N(y_3) \cap N = N - x$. Say $g(y_1) = g(y_3)$. By (i) and (iii), $N(y_2) = N$. By (H2), $y_2y_3 \notin E$. By (ii) $\|y_1, N\| = 1$ and $\|y_4, N\| = 0$. Let $x' \in N - x$. Then $\{\{v, y_2, y_3\}, \{x, x', y_4\}, N - x - x' + y_1\}\}$ is an equitable 3-coloring of G. ☐

4.3 Nearly equitable colorings

A coloring of G is nearly equitable if one color class has size 2, one color class has size 4, and all other color classes have size 3.

Proposition 4.3.1. *G admits a nearly-equitable k-coloring.*

Proof. Suppose not. By Lemma 4.2.1, $\Delta(G) \geq k+1$. Let $xy \in E$ with $d(x) \geq d(y)$. By (4.5), $G - xy$ has an equitable k-coloring f with $f(x) = f(y)$. Let $\mathcal{C}$ be the set of color classes of f, and $X = \{x, y, z\} \in \mathcal{C}$. Choose xy and f so that $d(z)$ is minimum. If x (or y) has no neighbor in some class $W \in \mathcal{C} - X$ then moving it to W yields a nearly equitable k-coloring; so assume not. As y is low, $d(y) = k$, and $\Delta(G) = d(x) = k+1$. Furthermore,

$$y \text{ has exactly one neighbor in every class,} \tag{i}$$

and

$$x \text{ has exactly two neighbors in one class, and exactly one neighbor in every other class.} \tag{ii}$$

For $W \in \mathcal{C} - X$ let $G_W := G[W \cup X]$. If G_W is bipartite, then its parts form an equitable or nearly equitable 2-coloring unless $G_W = K_{1,4}$. However, $\Delta(G_W) \leq 3$, so $G_W \neq K_{1,4}$; thus if G_W is bipartite, it has an equitable or nearly equitable coloring. If G_W has an equitable or nearly equitable coloring, then G has an equitable or nearly equitable k-coloring. Thus G_W contains an odd cycle C_W with $xy \in C$. Let $\mathcal{C}_1 = \{W \in \mathcal{C} - X : |C_W| = 3\}$ and $\mathcal{C}_2 = \mathcal{C} - X \smallsetminus \mathcal{C}_2$. For $W \in \mathcal{C}_1$ let $C_W = xv_Wyx$. If v_W is movable to some class U then moving y to W and v_W to U yields a nearly equitable k-coloring. As $v_w \in N(x)$, it is low. Thus v_w has two neighbors in X and one neighbor in each class of $\mathcal{C} - X - W$. In particular,

$$v_w z \notin E. \tag{iii}$$

For $W \in \mathcal{C}_2$ let $C_W = xx_W zy_W yx$, where $x_W, y_W \in W$. Then $G_W - z$ is bipartite. So z is not movable. Thus,

$$\text{if } |\mathcal{C}_2| \neq 0, \text{ then } |\mathcal{C}_1| + 2|\mathcal{C}_2| \leq d(z) \leq k+1. \tag{iv}$$

So $|C_1| \leq 1$.

If there are distinct $W, W' \in \mathcal{C}_1$ with $v_W v_{W'} \notin E$ then, using (ii), choose notation so that $\|x, W\| = 1$. By (i) and (iii), moving x to W, y to W', and both v_W and $v_{W'}$ to X yields an equitable k-coloring. So $Q := \{v_W : W \in \mathcal{C}_1\} \cup \{x, y\}$ is a clique. By Lemma 4.2.2, $|Q| \leq k-1$. So $|\mathcal{C}_2| \geq 2$; by (iv) $d(z) = k+1$. Consider distinct $W, W' \in \mathcal{C}_2$. Using (ii) choose notation so that $\|x, W\| = 1$. Switching x and x_W yields an equitable k-coloring of $G - zx_W$, with color class $\{z, x_W, y\}$. As $d(y) < d(z)$, this contradicts the choice of f. □

Fix a nearly equitable k-coloring $f := \{V_1, \ldots, V_k\}$, where $V^- = V_1$ and $V^+ = V_k$. As our proof progresses we will put more and more stringent conditions on f.

Construct an auxiliary digraph $\mathcal{H} := \mathcal{H}(G, f)$ as follows. The vertices of $\mathcal{H}$ are the color classes $V_1, \ldots, V_k$. A directed edge $V'V''$ belongs to $E(\mathcal{H})$ if some vertex $x \in V'$ has no neighbors in V''. In this case we say that *x is movable to V''* and that *x witnesses* the edge $V'V''$. A vertex $v \in V_i$ is *movable* if it is movable to some accessible class; otherwise it is unmovable. Let $M = M(f)$ be the set of movable vertices and $\overline{M} = \overline{M(f)}$ be the set of unmovable vertices. Call a color class V_i of f *accessible* if V^- is reachable from V_i in the digraph $\mathcal{H}$. By definition, V^- is accessible. Let $\mathcal{A} := \mathcal{A}(f)$ denote the family of accessible classes, $\mathcal{B}$ denote the family of inaccessible classes, $A := \bigcup \mathcal{A}$, and $B := \bigcup \mathcal{B} = V - A$. If $V_k \in \mathcal{A}$ then switching witnesses along a path from V^+ to V^- yields an equitable r-coloring; so $V^+ \in \mathcal{B}$. Let $a := |\mathcal{A}|$ and $b := |\mathcal{B}| = ks - a$. Then $|A| = as - 1$ and $|B| = bs + 1$.

An *in-tree* is a digraph T with a *root* $r \in V(T)$ such that every $v \in V(T)$ has a unique vr-walk. So the undirected graph underlying T is acyclic. A vertex $v \in T$ is a *leaf* if $d^-(v) = 0$. Fix a spanning in-tree $\mathcal{F} \subseteq \mathcal{H}[\mathcal{A}]$ with the most leaves possible. Write $W\mathcal{F}$ for the unique W, V^--path in $\mathcal{F}$,and let w_x be the witness for its first edge. Let $\mathcal{D} \subseteq \mathcal{H}[\mathcal{A}]$ be the spanning graph with $UW \in E(\mathcal{D})$ if and only if $UW \in E(\mathcal{H})$ and $U \notin W\mathcal{F}$.

A class $Z \in \mathcal{A}$ is *terminal* if there is a UV^--path in $\mathcal{H} - Z$ for every $U \in \mathcal{A} - Z$. For example, any leaf of $\mathcal{F}$ is terminal. Class V^- is terminal if and only if $a = 1$. Let $\mathcal{A}' = \mathcal{A}'(f)$ be the set of terminal classes, $A' := \bigcup \mathcal{A}'$ and $a' := |\mathcal{A}'|$.

4.4 Normal colorings

A *nearly equitable* k-coloring is *normal* if

$$\text{among nearly equitable } k\text{-colorings } a \text{ is maximum,} \tag{C1}$$

and

$$\text{there are at least two in-leaves whenever } a \geq 3. \tag{C2}$$

Lemma 4.4.1. *There exists a normal coloring.*

Proof. Suppose f is a nearly equitable k-coloring with a maximum. If $a \leq 2$, (C2) is vacuously true, so we may suppose $a \geq 3$. If $\mathcal{F}$ has at least two leaves then we are done; else $\mathcal{F}$ is a dipath with leaf Z and last edge UV^- witnessed by w. As $a \geq 3$, $U \neq Z$. Shifting w to V^- yields a normal k-coloring with in-leaves $V^- + w$ and Z. □

Fix a normal coloring f. A vertex $y \in B$ is *good* if $G[B - y]$ has an equitable b-coloring; else y is *bad*. A major goal of this section is to show that every vertex in B is good.

Lemma 4.4.2. $a = a(f) \geq 2$.

Proof. Assume $a = 1$ for all nearly equitable k-colorings of G, and choose one with

$$d(x_1) + d(x_2) \text{ minimal,} \tag{*}$$

where $V^- = \{x_1, x_2\}$. Say $d(x_1) \leq d(x_2)$. By Lemma 4.2.1, $d(x_2) \leq 2k - 2$. As $N(V^-) = V - V^-$, $d(x_1) + d(x_2) \geq 3k - 2 + |N(x_1) \cap N(x_2)|$.

Case 1: $N(x_1) \cap N(x_2) = \emptyset$. If $\|x_1, V^-\| = \|x_2, V^-\| = 2$, then coloring x_1 resp. x_2 with its non-neighbors in V^+ yields an equitable k-coloring. Therefore we suppose $\|x, V^+\| \geq 3$ for some $x \in V^-$. Pick $Y \in \mathcal{B}$ with $\|x, Y\|$ minimum. If $\|x, Y\| = 0$ then moving x to Y and $v \in N(x) \cap V^+$ to V^- yields a nearly equitable k-coloring with $a \geq 2$: any vertex $N(x) \cap V^+ - v$ is movable to the new small class $V^- - x + v$. Else, since $d(x) \leq 2k - 2 = 2b$, $\|x, Y\| = 1$ and $d(x) \geq k + 1$. Switching x with $y \in N(x) \cap Y$ yields a nearly equitable coloring, contradicting (*) since $d(y) \leq (2k + 1) - d(x) \leq k$.

Case 2: $N(x_1) \cap N(x_2) \neq \emptyset$. Then $d(x_1) \geq k+1$ and $d(x_2) \geq k+2$. Put $G' = G[B]$. Then $\chi(G') \leq b$. By (H2), $\Delta(G') \leq 2k+1-d(x_1)-1 \leq b$. If $S \subseteq V$ with $|S| = 2k$ then there is $v \in N(x_2) \cap S$, and $d_{G'}(v) \leq b-1$. So $K_{b,b} \nsubseteq G'$. Pick $w \in N(x_2) \smallsetminus N(x_1)$. Theorem 4.1.4 implies $G' - w$ has an equitable b-coloring $\mathcal{Y}$. As $\|x_2, B - w\| < 2b$, some class $Y \in \mathcal{Y}$ satisfies $\|x_2, Y\| \leq 1$. Move w to $V^- - x_2$ and x_2 to Y; if x_2 has a neighbor $v \in Y$ then move v to a class X in which it has no neighbors; X exists as $d(v) \leq k-1$. This yields an equitable k-coloring, or a nearly equitable k-coloring, contradicting (4.5) or (*) since $d(w) < d(x_2)$. □

An edge xy with $x \in X \in \mathcal{A}$ and $y \in B$ is *solo* if $\|y, X\| = 1$; else it is nonsolo. If xy is solo then x *and* y *are solo neighbors* of each other of x. For $x \in A$ and $y \in B$ let S_x denote the set of solo neighbors of x in B and S^y denote the set of solo neighbors of y in A.

Lemma 4.4.3. *Let $z \in Z \in \mathcal{A}$, $y \in S_z$, and g be an equitable b-coloring $G[B-y]$. Then*

0. if $\mathcal{P}$ is a W, V^--path in $\mathcal{H} - Z$ and w witnesses $WW' \in E(\mathcal{P})$ then $\|z, W - w\| \geq 1$.

If (a) the nonsolo neighbors of y are unmovable (as when $\|y, A\| = a$) or (b) $Z \in \mathcal{A}'$ then

1. z is unmovable;

2. If (c) $\|z, A\| \leq a-1$, then z has no movable neighbor $w \in W \in \mathcal{A}$.

Proof. (0) If not, shift witnesses along $\mathcal{P}$, move z to W, and move y to Z to obtain an equitable a-coloring h of $A + y$. Then $g \cup h$ contradicts (4.5).

(1) Suppose (a) or (b) holds and z is movable to $U \in \mathcal{A}$. Pick U and a U, V^--path $\mathcal{P}$ in $\mathcal{H}$. By (0), $Z \in \mathcal{P}$; in particular, there is no Z, V^- path in $\mathcal{H}$ where z is the witness to the first edge. Then (b) fails, so (a) holds; say x witnesses $XZ \in \mathcal{P}$. By (0) applied to x, x is not a solo neighbor of y; by (a) applied to x, x is not a neighbor of y at all. We move z to U, then shift witnesses along $\mathcal{P}$, noting that the witness from Z is not z; then we move y to $Z - z + x$. The coloring obtained in this way is proper, contradicting (4.5).

(2) Suppose (a) or (b) holds; further suppose (c) holds and $wz \in E$ with w movable to $U \in A$. Note by (1) and (c), z has precisely one neighbor in every class of $\mathcal{A} - Z$. Pick a U, V^--path $\mathcal{P}$ in $\mathcal{H}$ so that $Z \notin \mathcal{P}$ if (b) holds. Choose $w, W, U, \mathcal{P}$ so that $|\mathcal{P}|$ minimum. For every vertex w' that is the witness of an edge of $\mathcal{P}$, $zw' \notin E(G)$, because otherwise w' is preferable to w by minimality. As above, if x witnesses $XZ \in \mathcal{P}$, then (0) implies x is not a solo neighbor of y; since $Z \in \mathcal{P}$, (b) fails for Z, so (a) holds, and $xy \notin E$. Since $\|z, W\| = 1$ and z is unmovable (hence not a witness to any edge of $\mathcal{P}$), switching witnesses on $\mathcal{P}$, and moving w to U, z to W and y to Z yields an equitable a-coloring h of $A + y$. Then $g \cup h$ contradicts (4.5). □

Lemma 4.4.4. *Every color class in $\mathcal{A}$ contains at most one unmovable vertex.*

Proof. Suppose $Z \in \mathcal{A}$ has two unmovable vertices z_1 and z_2. If $Z \neq V^-$ then let $Z = \{z_1, z_2, z_3\}$. Let $B_0 = B + z_1 + z_2$ and $A_0 = A - z_1 - z_2$. Since z_3 (if it exists) is the witness for the first edge ZZ' of $\mathcal{P}_0 := Z\mathcal{F}$, shifting witnesses on $\mathcal{P}_0$ yields an equitable $(a-1)$-coloring f_0 of $G[A_0]$. Thus $G' := G[B_0]$ has no equitable $(b+1)$-coloring, but $g := f|B_0$ is a nearly equitable $(b+1)$-coloring. As each $v \in B_0$ is unmovable,

$$\text{(a) } d(v) \geq a - 1 + d_{G'}(v) + \|v, z_3\|\,, \text{ and (b) } \theta(G') \leq 2b + 3. \tag{4.8}$$

By Lemma 4.4.2, $b + 1 < a + b = k$. As G' has no equitable $(b+1)$-coloring, our choice of k minimum in the setup implies $G' \in \{\mathbf{X}, \mathbf{Y_{b+1}}\}$ or $G' \supseteq \mathbf{Z_{b+1,c}}$ for some odd c. Now consider several cases, always assuming all previous cases fail for all choices of Z.

Case 0: $G' = \mathbf{X}$. Use the notation of (4.3). By (4.8), $\theta(w_1w_2) \geq 2k + \|\{w_1, w_2\}, z_3\|$. By (H2), (say) $w_1z_3 \notin E$. Also, by (4.8)(a), $\theta(x_3y_i) \geq 2k + 1 + \|w_i, z_3\|$ for $i \in [2]$. By (H2), $y_1z_3, y_2z_3 \notin E$. So

$$f' := f|(A - Z) \cup \{\{w_1, y_1, z_3\}, \{w_2, y_2, y_3\}, \{x_1, x_2, x_3, w_3\}\}$$

is a nearly equitable k-coloring with y_2 movable to $\{w_1, y_1, z_3\} \in \mathcal{A}(f')$, contradicting (C1).

Case 1: $G' = K_{1,2b+2} + K_b$. Let $K = K_b$ and $r \in B_0$ with $d_{G'}(r) = 2b + 2$. Then $d_{G'}(w) = b - 1$ for all $w \in K$. As r is not contained in an independent 3-set, $r \in Z - z_3$. By (4.8), $d(r) \geq a + 2b + 1$ and $d(v) \geq a$ for every $v \in N_{G'}(r)$. By (H2), these bounds are sharp. Let $y \in N(r) \cap B$. Then $\|y, A\| = a$, and so $\|y, B_0 - r\| = 0$. Thus ry is solo. Also r is good. Let $u \in N(r) \cap A$. Lemma 4.4.3(2) implies all neighbors of r are unmovable. So $\|u, B_0\| \leq 2$, and witnesses of edges of $\mathcal{P}_0$ are not adjacent to r. Switch u and r in f_0 to obtain a new $(a-1)$-equitable coloring of $G[A_0]$. Finally, as $\|u, B_0 - r\| \leq 1$, $\Delta(G' - r + u) \leq b$. By Theorem 4.1.3, $G' - r + u$ has an equitable $(b+1)$-coloring, contradicting (4.5).

Case 2: $G' \supseteq K_{c,2b+2-c} + K_{b+1}$ for some odd $c \in [b+1]$. Use the notation of Example 4.1.8, but with $V = B_0 = B_1 \cup B_2$, and $c \in [2b+1]$. As the clique B_2 has one vertex in every class of g, assume $z_2 \in B_2$. Then $z_1 \in B_1$. Say $z_1 \in C_1$. Since c is odd, $V^+ \smallsetminus B_2 \subseteq C_2$.

Case 2.1: $c \geq 3$. Then $C_1 - z_1 \neq \emptyset$. Let $y_1 \in C_1 - z_1$ and $y_2, y_2' \in V^+ \smallsetminus B_2 \subseteq C_2$

$$d(y_1) \geq \|y_1, A \cup (B_1 - z_1) \cup (B_2 - z_2)\| \geq a + |C_2| + \|y_1, B_2 - z_2\|\,;$$

$$d(y_2), d(y_2') \geq \|y_2, (A \smallsetminus Z) \cup B_1 \cup (B_2 + z_3)\| \geq a - 1 + |C_1| + \|y_2, B_2 + z_3\|\,; \text{ and}$$

$$d(z_1) \geq \|z_1, (A \smallsetminus Z) \cup B_1 \cup B_2\| \geq a - 1 + |C_2| + \|z_1, B_2 \cup C_1\|\,.$$

So $\theta(y_1y_2) = 2k + 1$, $\|y_1, C_1 + B_2 - z_2\| = \|\{y_2, y_2'\}, B_2 + z_3\| = 0$ and $\|y_2, A\| = a$. Also $\theta(z_1y_2) \geq 2k$ and

$\|z_1, B_2 \cup C_1\| \leq 1$. Let $Y = \{y_1, y_1', w\}$ be the class in $\mathcal{B}$ containing y_1, with $y_1' \in C_1$ and $w \in B_2$. Note $\|y_1', C_1 + B_2 - z_2\| = \|y_1, C_1 + B_2 - z_2\| = 0$. Let $w' \in V^+ \cap B_2$. Move y_2 to $Z - z_1$, z_1 to Y, and if $z_1 w \in E$ then switch w and w'. This yields a new nearly equitable k-coloring f_1. Since $\mathcal{A}(f_0) - Z = \mathcal{A}(f_1) - (Z - z_1 + y_2)$, and z_3 witnesses $Z \in \mathcal{A}(f_0)$, still $Z - z_1 + y_2 \in \mathcal{A}(f_1)$. Since y_2' is movable to $A - a_1 + y_2$, it follows $a(f_0) < a(f_1)$, contradicting (C1).

Case 2.2: $c = 1$. Then $C_1 = \{z_1\}$ and $|C_2| = 2b+1$. So (i) $d(z_1) \geq a + 2b$, (ii) $d(y) \leq a+1$ for all $y \in N(z_1)$. For any $y \in B_2$, $d(y) \geq k-1$, so by (H2): (iii) $d(y) \leq k+2$ for all $y \in B_2$. Because Case 1 does not hold, $\|z_1, B\| = 2b+1$. We now prove the following:

Subclaim 4.4.4.1. If some $y \in Y \in \mathcal{B}$ is bad then $b = 2$, $d(z_1) = a + 2b$, $Y \neq V^+$, and the unique $u \in B_2 \cap Y$ is high and satisfies $\|u, B\| \geq 3$. In particular, there are at most two bad vertices.

Proof of Claim 4.4.4.1. Suppose $G_y := G[B-y] = G' - \{z_1, z_2, y\}$ has no equitable b-coloring. Then $y \notin V^+$; so $Y \neq V^+$ and $b \geq 2$. By (ii, iii), $\Delta(G_y) \leq \Delta(G[B]) \leq b+2$, and $d_{G_y}(y') \leq 1$ for all $y' \in C_2$. Recall $\theta(G[B]) \leq 2b+1$, so $\theta(G_y) \leq 2b+1$. By choice of k minimum in the setup, $G_y \in \{\mathbf{X}, \mathbf{Y_k}\}$, or $\mathbf{Z_{c,k}} \subseteq G_y$ for some odd c. Since $d_{G_y}(y') \leq 1$ for all $y' \in C_2$, this implies $\Delta(G_y) \geq 2b$ or there are at least $b+1$ vertices $v \in B - y$ with $d_{G_y}(v) \geq b-1$. So $b = 2$, $d_{Gy}(y') = 1$ for some $y' \in C_2$, and there is $u \in B_2 - y$ such that $\|u, G_y\| \geq 3$. As $\theta(y' z_1) \leq 2k+1$, (i) implies $d(z_1) = a + 2b$. As $|Y - y| = 2$, $u \in Y \cap B_2$, so both vertices of $Y - u$ are in C_2. Since $b = 2$, $\mathcal{B} = \{Y, V^+\}$. Then u is not bad, since $\Delta(G[B-u]) \leq 2$. So if any vertex v is bad, $v \in Y - u$. □

Case 2.2.0: Every $X \in \mathcal{A}$ has a unmovable vertex v_X with $\|v_X, B\| \geq 2b+1$. By Lemma 4.4.2, $a \geq 2$. For all $T \in \mathcal{A} - V^-$ let $T = \{u_T, v_T, w_T\}$, where w_T witnesses the edge of $\mathcal{F}$ leaving T. Since $d(v_T) \geq (a-1) + 2b + 1 = k + b$, the set $D = \{v_T : T \in \mathcal{A}\}$ is independent. Let $v = v_{V^-}$ and $V^- = \{v, v'\}$. Since v_T is unmovable and D is independent, $v_T v' \in E$. Hence $D - v \subseteq N(v')$; so v' is unmovable. Use V^- for Z, so $v = z_1$ and $v' = z_2$. Then

$$k - 1 \leq \|v', A\| + b \leq d(v') \leq 2k + 1 - d(v) \leq k - b + 1, \tag{4.9}$$

so $b \in \{1, 2\}$. It follows that we can choose a leaf X of $\mathcal{F}$ so that $\|v', X\| = 1$: If $\mathcal{F}$ has only one leaf X then by (C2) $a = 2$, by Lemma 4.2.3 $b = 2$, and $\|v', X\| = 1$ because equality holds in (4.9). Otherwise, $\mathcal{F}$ has two leaves T and X and (say) $\|v', X\| = 1$. Switch v' and v_X to obtain $Z' = \{v, v_X\}$, $X' = \{v', u_X, w_X\}$, and a new nearly equitable k-coloring f'. For all $T \in \mathcal{A} - X - Z$, v_T witnesses that $TZ' \in \mathcal{H}(f')$, and w_X witnesses an edge of $\mathcal{H}(f')$. So f is normal. Since both vertices in Z' are high, all vertices in B are low, so Claim 4.4.4.1 implies every vertex in B is good.

If $a = 2$ then by Lemma 4.2.3, $b = 2$. Also $\|v', B\| = 2$ and $E(A) = \{v'v_X, vu_X\}$. Moving w_X to Z' in f' shows that $B \subseteq N(v') \cup N(u_X)$: otherwise, we move a vertex $y \in B$ to $\{v', u_x\}$, and equitably color $B - y$, since y is good. Then $d(u_x) + d(v) \geq 2(1 + |B \smallsetminus N(v')|) = 12$, contradicting $\theta(vu_X) \leq 9$. So $a \geq 3$ and by (C2) there is a leaf $T \neq X$. As v_T is movable to Z', $\|B, T\| \geq 3b + 1 + \|v_T, B\| \geq 5b + 2$. If $\|v', T\| = 1$ then by symmetry $\|B, X\| \geq 5b + 2$. Else $\|v', T\| = 2$ because w_X is moveable to V^-. Then $\|v', B\| = d(v') - a \leq (k - b + 1) - a = 1$. Considering the coloring f', and using (4.9), $\|B, X\| \geq 3b + 1 + \|v_X, B\| - \|v', B\| \geq (3b + 1) + (2b + 1) - 1 \geq 4b + 2$. Regardless, $\|B, T \cup X\| > 9b + 3$. So there exists $y \in B$ with $\|y, A\| \geq 4 + a - 2 = a + 2$. As f' is a nearly equitable coloring of A, and y is good, $yz \in E$ for some $z \in Z'$, and this gives the contradiction $\theta(yz) \geq k + b + a + 2 = 2k + 2$.

Case 2.2.1: $\|y, A\| = a$ for all $y \in C_2$. First suppose (*) for every $X \in \mathcal{A}$ and $y \in C_2$ the unique $x \in S^y \cap X$ is unmovable. If $X \in \mathcal{A}$ has a unique unmovable vertex v_X then $\|v_X, B\| \geq 2b+1$. Else X has two unmovable vertices. Using X for Z, yields some unmovable v_X with $\|v_X, B\| \geq 2b+1$. Regardless, Case 2.2.0 holds. So (*) fails.

Pick $X \in \mathcal{A}$ and $y \in C_2$ with $x_3 \in S^y \cap X$ movable, and $|X\mathcal{F}|$ maximum. By Lemma 4.4.3(1), y is bad. By Claim 4.4.4.1, $\mathcal{B}$ has the form $\{U, V^+\}$, where $U = \{u, y, y'\}$, $w, w' \in V^+ \cap C_2$, $u \in B_2$, $\|u, V^+\| \geq 3$, u high, and all vertices in V^+ are good. Since $\|y', B\| \leq 1$ we can label so $w'y' \notin E$. By Lemma 4.4.3(1), each $v \in C_2 \cap V^+$ is adjacent to an unmovable $x_v \in X$. If $x_w \neq x_{w'}$ then X is a candidate for Z, and so y has an unmovable neighbor, a contradiction. So, since $\|C_2 \cap V^+\| = 3$, $d(x_w) \geq (a-1) + 3 + \|x_w, u\| = k + \|x_w, u\|$. By (H2), $ux_w \notin E$. If $x_w y' \in E$, switch x_w and y'. Since the sole neighbor of y in X is x_3, and the sole neighbor of y' and w' in X is x_w, this yields a nearly equitable k-coloring f' with w' movable to $X - x_w + y'$. By maximality of $|X\mathcal{F}|$, y' is not adjacent to any witness of an edge $TX \in \mathcal{F}$. So $a(f') > a(f)$, contradicting (C1). If $x_w y' \notin E$, then move x_w to U and w to $X - x_w$. This yields a nearly equitable k-coloring f'' with w' movable to $X - x_w + w$. Again, by maximality of $|X\mathcal{F}|$, w is not adjacent to any witness of an edge $TX \in \mathcal{F}$, so $a(f'') > a(f)$, contradicting (C1).

Case 2.2.2: $\|w, A\| = a$ for some $w \in C_2$. If possible, choose w to be good. By $\theta(z_1 w) \leq 2k + 1$ and not Case 2.2.1, there exists a vertex in C_2 with degree at least $a + 1$, so $\|z_1, A\| = a - 1$. If w is bad, then by Claim 4.4.4.1, $b = 2$ and there exists a good $y \in C_2 \cap V^+$ with $\|y, B\| \geq 1$. As $\theta(z_1 y) \leq 2k + 1$, $\|y, A\| \leq a$. But then we would have chosen y instead of w, so w is good. As $z_1 \in S^w$, $wz_2 \notin E$.

By Lemma 4.4.3, the unique $w_X \in N(w) \cap X$ is unmovable for every $X \in \mathcal{A}$, and z_1 has an unmovable neighbor z_X in every $X \in \mathcal{A} - Z$. If $X \in \mathcal{A}$ has two unmovable vertices, then by Case 2.2, one of them has $2b + 1$ neighbors in B. By not Case 2.2.0, there is $X \in \mathcal{A}$ with a unique unmovable vertex $v_X = z_X = w_X$. By (H2), $d(v_X), d(w) \leq a + 1$. If $y \in N(z_2) \cap C_2$ is good then $\|y, A\| = a + 1$ by (H2) and $yv_X \in E$ by Lemma 4.4.3(1).

Consider f_0, the equitable k-coloring of $G[A_0]$ defined in the beginning of this proof, obtained by shifting witnesses along $Z\mathcal{F}$ starting with z_3. As unmovable vertices remained in their color classes, v_X still is the unique neighbor of z_1 and w in the new X. Replacing v_X with z_1 in f_0 yields an equitable $(a-1)$-coloring f_1 of $G[A_0 + z_1 - v_X]$. Suppose $v_X z_2 \notin E$. Since $d(v_X) = a + 1$ and v_X is unmovable, $\|v_X, B\| \leq 2$. Since $|V^+ \cap C_2| = 3$, we can choose $y \in (V^+ \cap C_2) \smallsetminus N(v_X)$. As y is good, $yz_2 \notin E$, and there is an equitable b-coloring g of $B - y$. Then $f_1 \cup g + \{v_X, z_2, y\}$ is an equitable k-coloring, contradicting (4.5). Otherwise, $v_X z_2 \in E$. Then $\|v_X, B - w\| = 0$. As w is good there is an equitable b-coloring g of $B - w$. Let $y \in V^+ \smallsetminus N[w]$, and g' be the result of replacing y with v_X in g. As $v_X y \notin E$, $yz_2 \notin E$. So $f_1 \cup g' + \{z_2, w, y\}$ contradicts (4.5).

Case 2.2.3: There does not exist $y \in C_2$ such that $\|y, A\| = a$. That is, $\|y, A\| = a + 1$ for all $y \in C_2$.

$$\text{For each } y \in C_2 \text{ there is } T \in \mathcal{A} \text{ with } N(y) \cap (A - T) \subseteq S^y. \tag{i}$$

Also

$$\|z_1, A\| = a - 1, \tag{ii}$$

$$\|z_1, B\| = 2b + 1, \tag{iii}$$

$$\|C_2, B\| = 0, \tag{iv}$$

and

$$\text{every vertex in } B \text{ is good.} \tag{v}$$

Let $X \in \mathcal{A}' - Z$. As z_1 is unmovable, (ii) implies it has a unique neighbor $v_X \in X$, and

$$d(v_X) \leq a + 1. \tag{vi}$$

Suppose $x \in X$ and $y, y' \in S_x \cap C_2$ are distinct, and note $yy' \notin E$. By Lemma 4.4.3(1), x is unmovable. By symmetry in B, we may assume $y, y' \in V^+$. If x is low then $\|x, B\| \leq b + 1$, and so switching x with y and y', and switching witnesses on a X, V^--path in $\mathcal{F}$ contradicts (4.5). So

$$\text{if } x \in X \text{ is low it has at most one solo neighbor in } C_2. \tag{vii}$$

Suppose $\mathcal{A} = \{V^-, X\}$. By Lemma 4.2.3, $b \geq 2$. Assume $V^- = \{z_1, z_2\}$, as otherwise moving z_3 to V^- yields this. By (vi), $\|v_X, B_0 - z_1\| \leq 2 \leq b$. Using this and (iv), $G[B_0 - z_1 + v_X]$ has an equitable $(b+1)$-coloring, and by (ii), $X - v_X + z_1$ is independent, contracting (4.5). So $a \geq 3$, and $\mathcal{F}$ has two leaves.

An unmovable vertex $x \in A$ is *big* if $\|x, B\| \geq 2b+1$, and *small* if $\|x, B\| \leq 2b$. By Case 2.2,

$$\text{no class has two small vertices.} \tag{viii}$$

Suppose z_1 and z_2 are big. Then $|N(z_1) \cap N(z_2) \cap C_2| \geq b+1$. Let $y_1, y_2 \in N(z_1) \cap N(z_2) \cap C_2$. Each $x \in X \cap N(\{y, y'\})$ is solo by (i). By Lemma 4.4.3 each $v \in N_A[x]$ is unmovable; so $x \in N(\{z_1, z_2\})$. As z_1 and z_2 are high, x is low. By (vii) $|S_x \cap C_2| \leq 1 < b+1$. So X has a low solo vertex $x' \neq x$. Lemma 4.4.3(1) implies x and x' are unmovable. So $\|x, B\|, \|x', B\| \leq b+1$. Thus x and x' are small, contradicting (viii). So

$$\text{no class has two big vertices.} \tag{ix}$$

For a class $U \in \mathcal{A}$ let $S(U) := \{v \in C_2 : \|v, U\| = 1\}$. Over all color classes in $\mathcal{A}$ with two unmovable vertices, pick Z, with $S(Z) \neq \emptyset$ if possible; subject to this, choose Z to be a leaf if possible; and subject to these, choose $|S(Z)|$ maximum. Suppose $S(Z) = \emptyset$ or Z is not a leaf. By (i) there is a leaf X with $S(X) \geq \frac{1}{2}|C_2| \geq b+1$. By (vi) and (vii), $|S_{v_X} \cap C_2| \leq 1$. So there is a solo vertex $x \in X - v_X$. By Lemma 4.4.3(1), the solo vertices in X are unmovable. Because we did not choose X for Z, both $v \in X - x$ are movable. So $S_x = S(X)$. Say v_X is movable to $W \in \mathcal{A}$.

As X is a leaf, $X \notin \mathcal{P} := W\mathcal{F}$. If $Z \in \mathcal{P}$, let u witness $UZ \in \mathcal{P}$. Consider any $y \in C_2$. By (i), $y \in S_x \cup S_{z_1}$. Suppose $y \in S_{z_1}$. If $uy \notin E$ or u is undefined then moving y to $Z - z_1$, z_1 to $X - v_X$, v_X to W, and shifting witnesses along $\mathcal{P}$ contradicts (4.5). So $uy \in E$. By Lemma 4.4.3(1), uy is not solo. By (i), $y \in S_x$. Thus $C_2 \subseteq S_x$. So x is big. By (H2), $xz_1 \notin E$. Now $X \in \mathcal{A}'$, $y \in S_x$ for some $y \in C_2$, and $\|x, A\| \leq a-1$, so Lemma 4.4.3(2) implies $xz_2 \in E$. By (H2), $d(z_2) \leq a+1$, and so $\|z_2, C_2\| \leq 2-b \leq 1$. Let $V^+ = \{y_0, y_1, y_2, y^*\}$, where $y^* \in B_2$ and $N(z_2) \cap V^+ \subseteq \{y_0, y^*\}$. Shifting vertices starting with z_3 on $Z\mathcal{F}$, and recoloring $X, Z - z_3, V^+$ as $X - x + y_0$, $\{z_2, y_1, y_2\}$, $\{z_1, x, y^*\}$ contradicts (4.5). So $S(Z) \neq \emptyset$ and Z is a leaf.

Let $X = \{v_X, x_2, x_3\} \neq Z$ be a leaf, where x_3 witnesses an edge of $\mathcal{F}$. Put $H = G[X \cup Z \cup V^+]$. By (4.5),

$$\text{if some } v \in V(H) \text{ is movable to } \mathcal{A} - X - Z \text{ then } H - v \text{ has no equitable 3-coloring.} \tag{x}$$

By (ix), z_2 is small, so $|C_2 \smallsetminus N(z_2)| \geq b+1 \geq 2$. Using (iv), choose $V^+ = \{y_1, y_2, y_3, y^*\}$ so that $y^* \in B_2$ and $y_1, y_2 \in C_2 \smallsetminus N(z_2)$. By (ii) and Lemma 4.4.3(2), v_X is unmovable. Now $\|v_X, B \cup \{z_2, z_3\}\| \leq d(v_X) - (a-1) \leq 2$. As z_3 witnesses an edge of $\mathcal{F}$, (x) implies $\{\{x_2, x_3, z_1\}, \{z_2, y_1, y_2\}, \{y_3, y^*, v_X\}\}$ is not a coloring of $H - z_3$. So $\|v_X, \{y_3, y^*\}\| \geq 1$ and $v_X y_i \notin E$ for some $i \in [2]$. Also $\{\{x_2, x_3, z_1\}, \{z_2, v_X, y_i\}, V^+ - y_i\}$ is not a coloring. So $v_X z_2 \in E$, $v_X z_3 \notin E$ and $\|v_X, B\| = 1$. In particular, $v_X y_1, v_X y_2 \notin E$.

Suppose x_2 is unmovable. By (viii), x_2 is big. So $C_2 \subseteq S_{x_2}$, $\|x_2, A\| = a-1$, and by Lemma 4.4.3(2) x_2 has an unmovable neighbor in Z. By (H2), $x_2z_1 \notin E$ and so $x_2z_2 \in E$. For each color class $T \notin \{V^+, Z\}$, $\|y^*z_2, T\| \geq 2$ and each $y \in V^+$ satisfies $\|yz_1, T\| \geq 2$. Let $Q = z_1v_Xz_2x_2$. Note Q induces P_4. By inspection, $d_H(z_1) = 4 = d_H(x_2)$, $d_H(z_2) = 3 = d_H(v_X)$, and $\|V^+, \{x_3, z_3\}\| \leq 5$. Say $d_H(z_3) \leq d_H(x_3)$. Let $H' = H - x_3$. Then $\Delta(H') \leq 4$, $\theta(H') \leq 7$, $\chi(H') \leq 3$, and $d_{H'}(z_3) \leq 2$. Since H' contains an induced P_4, and $d_{H'}(z_3) \leq 2$, by (4.6), H' has a nearly equitable 3-coloring. An analogous argument works if $d_{H'}(x_3) \leq d_{H'}(z_3)$. So x_2 is movable. By Lemma 4.4.3(1), for $j \in \{1,2\}$, $\|y_j, X\| = 2$, so $\{x_2, x_3\} \subseteq N(y_j)$. Also $y_jz_3 \notin E$ by Case 2.2.3. Let $i \in \{2,3\}$. By (x), $\{\{v_X, z_3, y_1\}, \{z_1, z_2, x_i\}, V^+ - y_1\}$ is not a coloring of $H - x_{5-i}$. So $x_iz_2 \in E$.

Now suppose $v_Xy^* \in E$. Then by (vi), $v_Xy_3 \notin E$. Because v_X is the only unmovable vertex in X, then $y_3x_2, y_3x_3 \in E$ by Lemma 4.4.3(1). By Case 2.2.3, $\{z_2, z_3, y_3\}$ is an independent set. For $i \in \{2,3\}$, consider $\{\{z_2, z_3, y_3\}, \{x_i, z_1, y^*\}, \{v_X, y_1, y_2\}\}$. Since x_{5-i} is moveable, (x) implies this is not a proper coloring, so by (ii) and (iii), $y^*x_i \in E$. But now $d(y^*) + d(z_2) \geq (a+2+b-1) + (a+1+b) = 2k+2$, contradicting (H2). Therefore $v_Xy^* \notin E$, and so $v_Xy_3 \in E$. Now by Lemma 4.4.3(1), $y^*x_2, y^*x_3 \in E$. Then $d(y^*) + d(z_2) \geq (a+1+b-1) + (a+1+b) = 2k+1$; so equality holds, and in particular $z_2y_3 \notin E$. Now $\{\{z_2, y_2, y_3\}, \{v_X, y_1, y^*\}, \{z_1, x_2, x_3\}\}$ is a proper coloring of $H - z_3$, contradicting (x). $\square$

If $T \in \mathcal{A}$ and $T \cap \overline{M} \neq \emptyset$, let $T = \{u_T, m_T, w_T\}$, where $u_T \in \overline{M}$.

Lemma 4.4.5. *Every $y \in B$ is good.*

Proof. Suppose not. Say $G_0 := G[B - y_0]$ has no equitable b-coloring. Then $b \geq 2$. Also $|B - y_0| = 3b$, $\chi(G[B]) \leq b$, and, as every $y \in B$ is unmovable, $\theta(G[B]) \leq 2b+1$. So (4.6) implies $G_0 \in \{\mathbf{X}, \mathbf{Y_b}\}$ or $\mathbf{Z_{c,b}} \subseteq G_0$ for some odd c. If $y, y' \in V(G_0)$ with $\|yy', B\| = 2b+1$ then define yy', y and y' to be *B-heavy*. If $\|y, B\| > b$ then y is *B-high*. If y is B-heavy then $\|y, A\| = a$, and so y has a solo neighbor v in every class $X \in \mathcal{A}$. If y is good then Lemmas 4.4.3(1) and 4.4.4 imply v is the unique unmovable vertex $u_X \in X$. Observe that

$$\text{if } b+2 \text{ vertices are good and } B\text{-heavy then none of them is } B\text{-high,} \tag{4.10}$$

since if y is a counterexample then $\theta(u_Xy) \geq 2a - 1 + 2b + 3 = 2k+2$, contradicting (H2).

Consider several cases, always assuming previous cases fail for all bad $y_0 \in B$.

Case 1: $G_0 = \mathbf{X}$. Then $\Delta(G[B]) = 4$. Using the notation (4.3), x_3 is B-high and all five $v \in N[x_3]$ are B-heavy. By (4.10), there is a bad $v \in N[x_3]$. Since y_0 is not adjacent to any B-heavy vertex, $\|y_0, B\| \leq 4$; however, neighbors of y_0 in B are high, so $\|y_0, B\| \leq 3$.

Suppose $\|y_0, B\| = 3$. Since the neighbors of y_0 are high, $N(y_0)$ is independent; thus $N(y_0) = \{x_1, x_2, w_3\}$.

The B-high vertices x_1, x_2, x_3, w_3 are good and B-heavy; by inspection, w_1 is B-heavy and good. This contradicts (4.10). Then $\|y_0, B\| \le 2$. Since $\Delta(G[B-v]) = 3$, using (4.6), $\mathbf{Z}_{3,3} = G[B-v]$ and $\|y_0, B\| = 2$. By considering degrees, $N_B(y_0) \subseteq N_B(v)$; since the B-neghbors of y_0 are adjacent, $y_3 \in N(y_0)$. But this contradicts $v \in N[x_3]$.

Case 2: $G_0 = \mathbf{Y_b}$. Let y be the vertex with degree $2b$. Then the color class of y is $\{y, y_0, w\}$, where $w \in K_{b-1}$. So $V^+ \subseteq N(y)$. Since $\|N[y], B - y\| = 0$, the vertices of $N(y)$ all good; by inspection, also y is good. But the vertices of $N[y]$ are B-heavy and y is B-high, contradicting (4.10).

Case 3: $G_0 \supseteq \mathbf{Z_{c,b}}$, for some odd $c \le b$. Recall $M = \{v \in A : v \text{ is movable}\}$ and $\overline{M} = A \smallsetminus M$, and use the notation of Example 4.1.8 with $V = B - y_0$.

Case 3.1: $a = 2$. Then $x \in A$ is movable if and only if it has no neighbors in A. Thus an unmovable vertex has an unmovable neighbor. By Lemma 4.4.4, $|M| \ge 3$. So $\|A\| \le 1$, and $\{S, A \smallsetminus S\}$ is an equitable coloring for any 2-set $S \subseteq A$ with $|S \cap \overline{M}|, |(A - S) \cap \overline{M}| \le 1$. Thus (C1) implies every $w \in B$ satisfies $\|w, M\| \ge 3$ or $\left\|w, \overline{M}\right\| \ge 2$. Let $e \in E(Q)$. Then $\theta(e) \ge 2b + \|e, A\|$. By (H2), e has an end w_0 with $\|w_0, A\| = 2$; say $N(w_0) \cap A = \{u_1, u_2\}$. So $u_1u_2 \in E$ and $u_1, u_2 \in \overline{M}$. Set $R = \{w \in B : \|w, M\| \ge 3\}$ and $P = \{w \in B : \left\|w, \overline{M}\right\| \ge 2\}$. As $\theta(u_1u_2) \le 2k+1$, $|P| \le b+1$. Let $v \in M$. Then $2b \le |R| \le d(v)$. Thus there is $y_2 \in R \cap B_1$. Then $d(y_2) \ge 3 + c$. Since $2b + 3 + c \le \theta(vy_2) \le 2k+1$ and c is odd, $c = 1$, and $y_2 \in C_2$. Let $C_1 = \{y_1\}$. Then $y_1 \in P$, and $d(y_1) \ge 2b+1$. By Lemma 4.2.2, there is $w^* \in R \cap B_2$. As $d(w^*) \ge b+2$, (H2) implies $|R| \le d(v) \le b+3$. So $|P| \ge 2b-2$ and $d(u_1) \ge 2b-1$. By (H2), $4b \le \theta(u_1y_1) \le 2k+1$. Thus $b = 2$, and by Lemma 4.2.2 P is independent. By (H2), $N(C_2) = M + y_1$ and $d(v) \le 5$. If there is $y \in P \cap R$ then $|R| \ge 5$ and $d(y) \ge 5$, contradicting $\theta(vy) \le 9$. Else $y^*u_i \notin E$ for some $i \in [2]$. If $|P| = 3$ then $\{\{u_i, y^*, y_2\}, C_2 - y_2 + u_{3-i}, M, P\}$ contradicts (4.5). Else $|R| = 5$, and $\{\{u_i, y^*, y_2\}, M - v + u_{3-i}, P + v, R - y^* - y_2\}$ contradicts (4.5).

Case 3.2: There is a bad $y_1 \in B_1$. Say $G[B - y_1] \supseteq Q' + K' := K(C_1', C_2') + K(B_2')$. Set $B_0 = B_1 + y_0$. Then each $v \in V^+$ is good, $V^+ \smallsetminus B_2 \subseteq C_i$ and $V^+ \smallsetminus B_2' \subseteq C_{i'}'$ for some $i, i' \in [2]$. By (C2) and $a \ge 3$, there are distinct $Z_1, Z_2 \in \mathcal{A}'$. For distinct $v_1, v_2 \in B_2$,

$$2k + 1 \ge \theta(v_1v_2) \ge 2(a-2) + \|v_1v_2, Z_1 \cup Z_2\| + 2(b-1) \ge 2k - 6 + \|v_1v_2, Z_1 \cup Z_2\|.$$

So there exists $Z^* = \{z, z^*, z'\} \in \{Z_1, Z_2\}$ and $v^* \in \{v_1, v_2\}$ such that $z^*, z' \in M$ and $z^*v^* \notin E$. Shifting witnesses on $Z^*\mathcal{F}$, starting with z', yields an equitable $(a-1)$-coloring $\mathcal{A}^*$ of $A - z - z^*$.

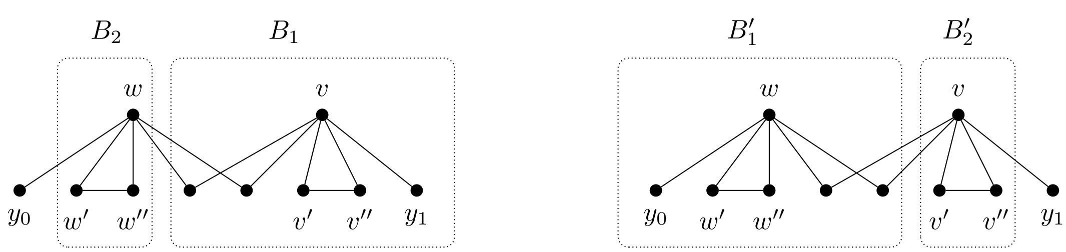
Case 3.2.1: $b = 2$. Say $\mathcal{B} = \{Y, V^+\}$. Then $Q = K_{1,3}$, $C_2 = V^+ \smallsetminus B_2$, and $C_1 = \{y_1\}$. So $Y = \{y_0, y_1, y_2\}$, where $y_2 \in B_2$. Since Case 2 fails, $\Delta(G[B]) \le 3$. We note here by inspection, using Lemma 4.2.2,

(*) if a graph H with $\alpha(H) \geq 4$, $|H| = 6$, and $\Delta(H) \leq 3$ is not equitably 2-colorable, then $K_{1,3} + K_1 \subseteq H$.

Since y_1 is bad, it follows that every vertex of V^+ has a neighbor in $\{y_0, y_2\}$, and $\|y^*, V^+\| = 3$ for some $y^* \in \{y_0, y_2\}$. So y_1 and y^* are high. Thus each $v \in V^+ \cap N(\{y, y^*\})$ satisfies $\|v, B\| \leq 2$. If there exists $v \in B - N(\{y, y^*\})$, then $\|v, B\| = 1$ because $\|v, B\| = \|v, Y - \{y_1, y^*\}\| \leq 1$. So V^+ has the form $\{v_1, v_2, v_3, v'\}$, where $\{v_1, v_2, v_3\} = C_2$, $1 \leq \|v', B\| \leq 2$, and $\|v_i, B\| = 2$ for $i \in [3]$. Thus $\|v_i, A\| = a$. As v_i is good, $z \in \bigcap_{i \in [3]} N_A(v_i) = \overline{M}$. So $d(z) \geq a + 2 + \|z, \{y_1, y^*\}\|$, and $\{z, y_1, y^*\}$ is independent because y_1 and y^* are high. Also $U := V^+ + z^* - v'$ is independent, $\|v', U\| \leq 1$, and $\|y', U\| \leq 2$. So using (*), $U + v' + y'$ has an equitable 2-coloring $\mathcal{B}^*$. So $\mathcal{A}^* \cup \mathcal{B}^* + \{z, y_1, y_2\}$ contradicts (4.5).

Case 3.2.2: $B_2 = B_2'$ and $b \geq 3$. Then $V(Q \cap Q') = B_0 - y_0 - y_1$. As Q and Q' are connected, so is $Q \cup Q'$. If $O \subseteq Q \cup Q'$ is an odd cycle then $y_0 \in O$, and $O - y_0 := v_1 \dots y_{2r} \subseteq Q$. So $v_1 v_{2r} \in E$ and $\theta(v_1 v_{2r}) = 2a + 2b + 2$, contradicting (H2). Thus $Q \cup Q'$ is bipartite. Since it has bad vertices, it is complete. So $\theta_{Q \cup Q'}(e) = 2b + 1$ for every $e \in E(B_0)$, and every $w \in B_0$ satisfies $\|w, A\| = a$ and $\|w, B_2\| = 0$. Let $\{D_1, D_2\}$ be the unique 2-coloring of $Q \cup Q'$, where $|D_1|$ is odd. Consider any $w_1 \in D_1$. Then w_1 is good, so $y_0, y_1 \in D_2$. By Lemmas 4.4.3 and 4.4.4, $N(w_1) \cap A = \overline{M}$. Let $z \in Z^* \cap \overline{M}$. Then $D_1 \subseteq N(z)$, and $\theta(zw_1) \geq 2a - 1 + 2b + 1 + \|z, D_2\|$. Thus $\|z, D_2\| \leq 1$. If $\|z, D_2\| = 0$ then (*) $|D_2 \smallsetminus N(z)| \geq 2$. Else there is $w_2 \in N(z) \cap D_2$. Then $\theta(zw_2) \geq 2a - 1 + 2|D_1|$. So $|D_1| \leq b + 1$, $|D_2| \geq b \geq 3$, and again (*) holds. So there are distinct $y', y'' \in D_2 \smallsetminus N(z)$. Let $B^* = B_0 + z^* - y' - y''$. Then $D_1 + z^*$ and $D_2 - y' - y''$ are even independent sets, and $N(B^*) \cap B_2 = N(z^*) \cap B_2 \neq B_2$. So B^* has an equitable b-coloring $\mathcal{B}^*$. Thus $\mathcal{A}^* \cup \mathcal{B}^* + \{z, y', y''\}$ contradicts (4.5).

Case 3.2.3: $B_2 \neq B_2'$ and $b \geq 3$. Let $w \in B_2 \cap B_1'$. As $|B_2| \geq 3$ and $\|w, B_2'\| \leq 1$, there is $w' \in B_2 \cap B_1' - w$. As $\|ww', B_2'\| \leq 1$, $B_2 \subseteq B_1'$. Thus $b = 3$. Now there are $i \in [2]$ and distinct $w', w'' \in C_i' \cap B_2$. Then $\theta(w'w'') \geq 2(a+1+|C_{3-i}'|)$, and $|C_{3-i}'| = 1$. Say $C_1' = \{w\}$. Similarly, $C_1 = \{v\}$, where $B_2' = \{v, v', v''\} \subseteq B_1$. (See Figure 4.3.) So all vertices of $B - \{y_0, y_1\}$ are B-heavy and good, and w is B-high, contradicting (4.10).



(a) $B - y_0 = B_1 \cup B_2$

(b) $B - y_1 = B_1' \cup B_2'$

Figure 4.3: $G[B]$ in Case 3.2.2, perhaps missing the edge $y_0 y_1$

Case 3.3: Every $y \in B_1$ is good. There is $i \in [2]$ with $\|w, A\| = a$ for all $w \in C_i$ and $\|w, A\| \leq a + 1$

for all $w \in C_{3-i}$. We set $|C_i| = c$, for some odd $c \in [2b-1]$. By Lemma 4.4.3(1) and Lemma 4.4.4, $C_i \subseteq N(x)$ for all $x \in \overline{M}$ and $|S_z| \geq |C_{3-i}|/2$ for some $z \in \overline{M}$ with $z \in Z \in \mathcal{A}'$. Suppose $|C_i| \geq |C_{3-i}|$. Let $z' \in \overline{M} - z$ with $z' \in Z' \in \mathcal{A}'$ and $w \in C_i$. As $b \geq 2$ and c is odd, $2b - c \geq 3$. If $C_{3-i} \subseteq N(z')$ then $\theta(z'w) \geq a - 1 + 2b + a + 2b - c \geq 2k + 2$, contradicting (H2). So there is $y' \in C_{3-i} \smallsetminus N(z')$. By Lemma 4.4.3(1), $\|y', Z'\| = 2$ and $y'z \in E$. Now

$$\theta(zy') \geq a - 1 + |C_i| + |C_{3-i}|/2 + a + 1 + |C_i| \geq 2k + |C_i| - |C_{3-i}|/2 > 2k + 1,$$

another contradiction. So $|C_i| < |C_{3-i}|$. Say $i = 1$. For $y \in C_1$,

$$2k + 1 \geq \theta(zy) \geq a - 1 + |C_1| + |C_2|/2 + a + |C_2| \geq 2k - 1 + |C_2|/2.$$

So $|C_2| = 3$, $|C_1| = 1$, and $b = 2$. Let $\mathcal{B} = \{W, V^+\}$ and $C_1 = \{w\}$. Then $C_2 = V^+ \smallsetminus B_2$ and $d(w) \geq a + 3$. Also $d(z) \geq a - 1 + |C_1| + |C_2|/2$. As $wz \in E$, $d(z) = a + 2$ and $d(w) = a + 3$. So z has exactly two neighbors $v_1, v_2 \in V^+$, and $v_1, v_2 \in S_z$ by the choice of z. Switching witnesses on $Z\mathcal{F}$, and switching z with v_1 and v_2 yields an equitable k-coloring. □

Lemma 4.4.6. *Every solo $x \in X \in \mathcal{A}'$ satisfies $\|x, B\| \leq 2b$.*

Proof. Suppose $\|x, B\| \geq 2b + 1$, and let $y \in S_x$. Since $\theta(xy) \leq 2k + 1$, Lemmas 4.4.3 and 4.4.5 imply $a + 2b \leq d(x) \leq a + 2b + 1$. First suppose $d(x) = a + 2b + 1$. Consider any $w \in N(x) \cap B$. Then $\theta(xw) \leq 2k + 1$ implies $\|w, A\| = a$. Thus $S^w = N(w) \cap A = \overline{M}$. So for unmovable$u_Z \in Z \in \mathcal{A}$, $d(u_Z) \geq a - 1 + \|x, B\| \geq k + 1$. Thus the set $\{u_Z : U \in \mathcal{A}\}$ is independent. By Lemma 4.4.4, the unique vertex $v \in V^- - u_{V^-}$ is movable; say v is movable to $U \in \mathcal{A}$. Since u_U is not movable to V^-, it is adjacent to u_{V^-}, a contradiction.

So $d(x) = a + 2b$, $\|x, A\| = a - 1$ and $\|w, A\| \leq a + 1$ for every $w \in N(x) \cap B$. As $X \in \mathcal{A}'$, Lemmas 4.4.3 and 4.4.5 imply $N[x] \cap A = \overline{M}$. Some $W \in \mathcal{B}$ satisfies $\|x, W\| \geq 3$; set $W' = N(x) \cap W$. Each $w \in W'$ has at most one neighbor in $\{x_1, x_2\} := X - x$. Thus $\|x_i, W' - w'\| = 0$, for some $i \in [2]$ and $w' \in W'$. Say x_i is movable to $U \in \mathcal{A}$, and $x_U \in N(x) \cap U$. Then

$$\|x_U, B \cup \{x_1, x_2\}\| \leq 2k + 1 - d(x) - \|x_U, A - X + x\| \leq a + 1 - a - 1 \leq 2. \tag{4.11}$$

If $x_U x_{3-i} \notin E$ then switch x and x_U. As $N[x] \cap A = \overline{M}$, this yields a new normal k-coloring f' with $X' := X - x + x_U \in \mathcal{A}'(f')$. By (4.11), some $w \in W'$ is not adjacent to x_U. By Lemmas 4.4.3 and 4.4.5, $\|w, X'\| \geq 2$, a contradiction.

Else $x_U x_{3-i} \in E$. By (4.11), $\|x_U, W\| \leq 1$. So there is $w \in W$ with $\{w, x_U, x_i\}$ independent. Shift witnesses, starting with x_{3-i}, on an X, V^--path in $\mathcal{H}$. This does not affect neighbors of x since they are unmovable. Now switch x with x_U, move w to $X - x - x_{3-i} + x_U$, and equitably b-color $B - w$. This yields an equitable k-coloring of G. $\square$

Theorem 4.4.7. *If $x \in X \in A'$, $y_1 \in S_x$, $y_2 \in N(x) \cap B - y_1$ and $\|y_2, X\| \leq 2$ then $y_1 y_2 \in E$.*

Proof. If not, pick a counterexample $y \in S_x$, $y' \in N(x) \cap B - N[y]$ with $\|y', X\| \leq 2$ and $\|y, B\|$ maximum. By Lemmas 4.4.3 and 4.4.5, x is unmovable; so $\|x, A - X\| \geq a - 1$. Put $A^* = A - x + y$, $X^* = X - x + y$ and $B^* = B - y$. By Lemma 4.4.5, $G[B^*]$ has an equitable b-coloring $\mathcal{B}^*$; say $y' \in Y \in \mathcal{B}^*$. Then $\mathcal{A}^* := \mathcal{A} - X + X^*$ is an equitable a-coloring of A^*. By Lemma 4.4.6, $\|x, B\| \leq 2b$. So $\|x, W\| \leq 1$ for some $W \in \mathcal{B}^*$; consider any such W.

Since x is unmovable and $X \in \mathcal{A}'$, if $\mathcal{B}^+$ is a b-equitable coloring of $B^* + x$ then $f^+ := \mathcal{A}^* \cup \mathcal{B}^+$ is a normal k-coloring with $X^* \in \mathcal{A}(f^+)$. As y is unmovable in f and $yy' \notin E$, $\|y', X^*\| \geq 2$, a contradiction. So $B^* + x$ has no equitable b-coloring. Thus x has a neighbor in every class of $\mathcal{B}^* - W$. In particular, $N(x) \cap W = \{w\}$. Then $\|w, A - X + x\| \geq a$, and w (like x) has a neighbor in every class of $\mathcal{B}^* - W$.

For $x_0 \in X - x$, $G[A - X + x_0]$ has an equitable $(a-1)$-coloring obtained by shifting witnesses, starting with x_0, on an X, V^--path in $\mathcal{H}$. If $G[B^* + x - u]$ has an equitable b-coloring, where $u \in B^*$, then (4.5) implies $X^* + u - x_0$ is not independent. Thus w is not movable to X^*, and $\|w, Y - y'\|, \|x, Y - y'\| \geq 1$, where $y' \in Y \in \mathcal{B}^*$. So $d(w) \geq \|w, A - X + x\| + \|w, B^*\| + \|w, X^*\| \geq k$ and $d(x) \geq k + 1$. By (H2), $d(x) = k + 1$, $d(w) = k$, $\|x, B\| = b + 2$, $\|x, A\| = a - 1$, and $\|w, X^*\| = 1$. So $wy \in E$, $w \in S_x$, $\|w, A\| = a$, $\|w, B\| = b$, and $\|w, Y\| = 1$. Thus $wy' \notin E$.

As $\theta(xy) \leq 2k + 1$, $\|y, B\| \leq b$. So any $w' \in S := N(x) \cap B \smallsetminus Y$ can play the role of y. By maximality, $\|w', B\| = b$ and $\|w', A\| = a$ for all $w' \in S$. By Lemmas 4.4.3, 4.4.4 and 4.4.5, $N(w') \cap A = \overline{M}$ for each $w' \in S$, and $N(x) \cap A = \overline{M} - x$. Let $u_Z \in Z \cap \overline{M}$ for $Z \in \mathcal{A}$. By Lemma 4.2.2, $\omega(G) < k$. Since S is a clique, there are distinct $Z, Z' \in \mathcal{A} - X$ with $u_Z u_{Z'} \notin E$. First, we note $\|u_Z, Z\| \geq 2$ by Lemmas 4.4.3(0), 4.4.4, and 4.4.5. Since $u_Z x \in E$, by (H2) $d(u_Z) \leq k$, so $\|u_Z, A\| = a$ and $\|u_Z, B\| = b = |S|$. In particular, $u_Z y' \notin E$. Then switching x and u_Z yields a normal k-coloring in which y' has a movable, solo, terminal neighbor, a contradiction. $\square$

4.5 Optimal colorings

A normal k-coloring f of G is *optimal* if

(C3) among normal k-colorings, $|H(B)|$ is minimum, and

(C4) subject to (C3), a' is maximum.

Let f be optimal.

Lemma 4.5.1. *If $y \in H(B)$ then $S^y \cap A' = \emptyset$.*

Proof. Suppose $y \in H(B)$, $X \in \mathcal{A}'$ and $x \in S^y \cap X$. By Lemmas 4.4.3 and 4.4.5, x is unmovable and $G[B-y]$ has an equitable b-coloring $\mathcal{B}^*$. Thus if $G[B+x-y]$ has an equitable b-coloring then putting y in $X-x$ yields a normal k-coloring with fewer high vertices in B, contradicting (C3). Thus $\|x, Y\| \geq 1$ for all $Y \in \mathcal{B}^*$. Because $xy \in E$ and y is high, $k \leq d(x)$; but by the above, $d(x) \geq (a-1)+b+1$, so indeed x has precisely one neighbor in every class of $\mathcal{B}^*$. Further, $N[x] \cap A = \overline{M}$ and $d(y) = k+1$. Suppose there exists $y' \in N(x) \cap B - y$ in class $Y' \in \mathcal{B}^*$ that is moveable to class $Y'' \in \mathcal{B}^*$. Then we move y' to Y'' and move x to $Y' - y'$; this is an equitable b-coloring of $G[B+x-y]$, a contradiction. Therefore each $y' \in N(x) \cap B - y$ satisfies $\|y', B-y\| \geq b-1$.

Let $W = B \cap N(x) \cap N(y)$ and $W' = B \cap N(x) \smallsetminus N[y]$. Let $w \in W$; then w is low. So $\|w, A\| = a$ and $\|w, B\| = b$. Thus $W + y \subseteq S_x$, and $S^w = N[w] \cap A = \overline{M}$. By Lemma 4.4.7, S_x is a clique. As $G[B]$ is b-colorable, $|W| \leq b-1$, and so $|W'| \geq 1$. Consider any $w' \in W'$. As $w'y \notin E$, Lemma 4.4.7 implies $X \subseteq N(w')$. So $d(w') \geq (b-1)+3+(a-1) = k+1$. Let $X = \{x, x', x''\}$. Every $u \in B \smallsetminus N(x) + w'$ is adjacent to x' by Lemmas 4.4.3(1) and 4.4.4. Thus $2k+1 \geq \theta(x'w') \geq 2b+1+k+1$. So $a > b$; as $k \geq 4$, $a \geq 3$. Thus there is $Z \in \mathcal{A}' - X$. Then $u_Z \in S^{w'}$. So $W \cup W' \subseteq S_{u_Z}$ is a b-clique. As w' is high, $|W'| = 1$. Also Z, u_Z, w' can play the role of X, x, y. Thus there is a high w'' with $\|w'', W\| = b-1$ and $\|w'', Z\| = 3$. Indeed: we can choose $w'' = y$. So $N[u_Z] \cap A = \overline{M}$.

Choose $T \in \mathcal{A} \smallsetminus \{X, Z\}$. By Lemma 4.4.3(1), $W \subseteq N(u_T)$. As $u_T x \in E$,

$$k+1 \geq d(u_T) \geq a-3+|W|+\|u_T, X+y\|+\|u_T, Z+w'\|.$$

So $\|u_T, X+y\|+\|u_T, Z+w'\| \leq 5$. Say $\|u_T, X+y\| \leq 2$. Then there is $x' \in X-x$ with $\|u_T, X-x-x'\| = 0$. Suppose $u_T y \notin E$. Let x' be moveable to $U \in \mathcal{A}' - X$; move x' to U, and switch witnesses along a UV^- path in $\mathcal{A} - X$; moving u_T and y to $X - x - x'$, and moving x to $T - u_T$ contradicts (4.5). So $u_T y \in E$ and $\|u_T, X+y\| \geq 2$. As y is high, $d(u_T) \leq k$. So $\|u_T, Z+w'\| \leq 2$. By an analogous argument $u_T w' \in E$. Now $w', y \in S_{u_T}$, but $w'y \notin E(G)$, contradicting Lemma 4.4.7. $\square$

For $X \in \mathcal{A}$, let $\mathcal{T}(X)$ be set of $U \in \mathcal{A} - X$ such that every U, V^--path in $\mathcal{H}$ contains X. Then $\mathcal{T}(X) = \emptyset$ if and only if $X \in \mathcal{A}'$, and if $X' \in \mathcal{T}(X)$ then $\mathcal{T}(X') \subsetneq \mathcal{T}(X)$. So $\mathcal{T}(X)$ contains a terminal class for every nonterminal class X. Choose $X_0 \in \mathcal{A} \smallsetminus \mathcal{A}'$ such that $|\mathcal{T}(X_0)|$ is minimum, and set $\mathcal{A}'' = \mathcal{T}(X_0)$. As usual, set $A'' := \bigcup \mathcal{A}''$, and $a'' := |\mathcal{A}''|$. Note if $a' = a-1$, then $X_0 = V^-$ and $\mathcal{A}'' = \mathcal{A}'$. Note further $(X_0) \subseteq \mathcal{A}'$:

otherwise, there is some $X \in \mathcal{T}(X_0) \smallsetminus \mathcal{A}'$ that is preferable to X_0. So $\emptyset \subsetneq \mathcal{A}'' \subseteq \mathcal{A}'$ and $1 \leq a'' \leq a'$. Also

$$\forall w \in A'', \ \ \|w, A\| \geq a - a'' - 1. \tag{4.12}$$

Proposition 4.5.2. *If $a'' = a'$, then $a = a' + 1$.*

Proof. Argue by contraposition. If $a' \leq a - 2$ then $X_0 \neq V^-$ and $\mathcal{A}'' = \mathcal{T}(X_0) \subseteq \mathcal{A}'$. Let $\mathcal{P}$ be a minimum X_0, V^--path in $\mathcal{H}$, and let its last edge be UV^-. If there exists $W \neq U$ such that $WV^- \in E(\mathcal{H})$, then $W \notin V(\mathcal{P})$ by minimality. So $\mathcal{T}(W) \cap \mathcal{T}(X_0) = \emptyset$ and $\mathcal{T}(W)$ contains a terminal class. So $a'' < a'$. Else $\mathcal{A}' \subseteq \mathcal{T}(U) = \mathcal{A} - V^- - U$. Shifting a witness w of UV^- to V^- yields a normal k-coloring f' with small class $U - w$, $A(f) = A(f')$ and $\mathcal{A}'(f') = \mathcal{A}'(f) + (V^- + w)$, preserving (C3) and contradicting (C4). $\square$

4.6 Almost all color classes in $\mathcal{A}$ are terminal

A vertex $y \in B$ is *petite* if $d(y) \leq a + a' - 1$ or if $d(y) = a + a'$ and either y has 3 neighbors in a terminal class or at least two neighbors in a nonterminal class of $\mathcal{A}$. For a subset C of B, let $L'(C)$ denote the set of the petite vertices in C and $H'(C) = C - L'(C)$. By Lemma 4.5.1,

$$L'(B) \subseteq L(B). \tag{4.13}$$

Lemma 4.6.1. *If $b \leq a' - 1$ then $|L(B)| \leq b + 1$. Moreover, if $|L(B)| = b + 1$, then $b = a' - 1$, $G[L(B)]$ is the disjoint union of cliques, and $d(y) = k$ for every $y \in L(B)$. Even moreover, if $|L(B)| = b + 1$ and $a' = a - 1$, then $b \leq 2$.*

Proof. Suppose $L = L(B)$ and $|L| \geq b + 1$. Let I be an inclusion maximal independent subset of L of size at least 2. Since $G[B]$ is b-colorable, such I exists. The total number of solo neighbors in A' of vertices in I is at least

$$\sum_{y \in I}(a' - b + \|y, B\|) \geq |I|(a' - b) + |L| - |I| = |I|(a' - b - 1) + |L| \geq (|I| - 1)(a' - b - 1) + (a' - b - 1 + |L|).$$

But A' has at most a' unmovable vertices. Since no vertex in A' is a solo neighbor of two non-adjacent vertices, we conclude

$$(|I| - 1)(a' - b - 1) - b - 1 + |L| \leq 0.$$

It follows that $|L| = b + 1$ and $a' = b + 1$. Moreover in order to have the total number of solo neighbors in A' of vertices in L exactly $\sum_{y \in I}(a' - b + \|y, B\|)$, we need that for every $y \in I$, $d(y) = k$, $N(y) \cap B \subseteq L$

and that for all distinct $y, y' \in I$, $N(y) \cap N(y') \cap B = \emptyset$. If some $y \in L$ is adjacent to all other vertices in L, then two its non-adjacent neighbors y' and y'' both have y in their neighborhoods, a contradiction to the previous sentence. So each $y \in L$ is in an inclusion maximal independent subset I_y of L of size at least 2. Thus $\|L, B - L\| = 0$ and each component of $G[L]$ is a complete graph. This proves the first two statements of the lemma.

Suppose $a' = a - 1$. Let C_1 and C_2 be the vertex sets of a smallest and a second smallest components of $G[L]$, respectively. Let $x \in X \in \mathcal{A}'$ be a solo neighbor of some $y_1 \in C_1$ and $X = \{x, x', x''\}$. By Lemma 4.4.7, each $y \in B - C_1$ is adjacent to both, x' and x''. So if $|C_1| \le b - 2$, then $d(x') \ge 2b + 3$. On the other hand, since for every $y \in H(B)$, $2a - 1 \le d(y)$, we have $d(x') \le 2k + 1 - d(y) \le 2b + 2$, a contradiction. But for $b \ge 4$, we have $\lfloor \frac{b+1}{2} \rfloor \le b - 2$. So, $b \le 3$. If $b = 3$ and $|C_1| \ge b - 1$, then $|C_1| = |C_2| = 2$. Let $z \in Z \in \mathcal{A}'$ be a solo neighbor of some $y_2 \in C_2$ and $Z = \{z, z', z''\}$. Since $y_1 y_2 \notin E(G)$, $Z \ne X$. Repeating the argument in this paragraph we get $d(x') = d(z') = 2b + 2$ and $\|\{x', x''\}, A\| = \|\{z', z''\}, A\| = 0$. Then switching x with z, we increase $\mathcal{A}$, since the class of y_1 is in the new $\mathcal{A}$. □

Lemma 4.6.2. *$|L'(B)| \le a'$.*

Proof. Suppose $L' = L'(B)$ and $|L'| \ge a' + 1$. Similarly to the proof of Lemma 4.6.1, let I be an inclusion maximal independent subset of L'. We claim that

$$\text{each } y \in L'(B) \text{ has at least } 1 + \|y, B\| \text{ solo neighbors in } A'. \tag{4.14}$$

Indeed, if $d(y) \le a + a' - 1$, then $\|y, A\| = d(y) - \|y, B\|$ and the number of classes in $\mathcal{A}'$ with at least two neighbors of y is at most

$$\|y, A\| - a \le (a + a' - 1) - \|y, B\| - a = a' - 1 - \|y, B\|.$$

So the remaining $a' - (a' - 1 - \|y, B\|)$ classes in $\mathcal{A}'$ have solo neighbors of y. If $d(y) = a + a'$ and a class $X \in \mathcal{A}'$ has 3 neighbors of y, then $\|y, A' - X\| \le (a + a') - \|y, B\| - (a - a') - 3 = 2(a' - 1) - 1 - \|y, B\|$. So again at least $1 + \|y, B\|$ classes in $\mathcal{A}' - X$ have solo neighbors of y. Finally, if $d(y) = a + a'$ and a class $X' \in \mathcal{A} - \mathcal{A}'$ has at least 2 neighbors of y, then $\|y, A'\| \le (a + a') - \|y, B\| - (a - a' + 1) - 3 = 2(a' - 1) - 1 - \|y, B\|$. This proves (4.14).

By (4.14), the total number of solo neighbors in A' of vertices in I is at least

$$\sum_{y \in I} (1 + \|y, B\|) \ge |L'| \ge a' + 1.$$

But A' has at most a' unmovable vertices. $\square$

Recall that for a class $X \in \mathcal{A}$, $\mathcal{T}(X)$ is the set of classes in $\mathcal{A} - X$ from which there are no paths to V^- in digraph $\mathcal{H} - X$. If $\mathcal{T}(X) \neq \emptyset$ (i.e., X is not terminal), let $\mathcal{T}'(X)$ be a smallest nonempty subset $\mathcal{D}$ of $\mathcal{T}(X)$ with no outneighbors in $\mathcal{A} - \mathcal{D} - X$. By definition, if $\mathcal{T}(X) \neq \emptyset$, then $\mathcal{T}'(X) \neq \emptyset$.

Suppose $a' < a - 1$. Choose $X_0' \in \mathcal{A} \smallsetminus \mathcal{A}'$ such that $|\mathcal{T}'(X_0)|$ is minimum, and set $\mathcal{A}''' = \mathcal{T}'(X_0')$. As usual, set $A''' := \bigcup \mathcal{A}'''$, and $a''' := |\mathcal{A}'''|$. Since X_0' is nonterminal, $a''' > 0$. Also, for all $w \in W \in \mathcal{A}'''$, $\|w, A\| \geq a - a''' - 1$.

Lemma 4.6.3. *For every $z \in A'''$,*

(a) $\|z, B\| \leq \max\{b, 2b + 2 + a''' - a'\}$; and

(b) if $||z, A|| \geq a - a'''$, then $||z, B|| \leq \max\{b, 2b + 1 + a''' - a'\}$; and

(c) if every vertex in $N(z) \cap B$ is petite, then $||z, B|| \leq \max\{b, 2b + 1 + a''' - a'\}$.

Proof. Let $z \in Z \in \mathcal{A}'''$ and $B_1 = N(z) \cap B$. Suppose the lemma does not hold for z. Then $\|z, B\| \geq b + 1$, in particular, $B_1 \neq \emptyset$. Also, either:

$$\|z, B\| \geq 2b + 3 + a''' - a' \tag{4.15}$$

(in which case, $d(z) \geq (2b + 3 + a''' - a') + (a - a''' - 1) = 2k - a - a' + 2$); or

$$\|z, A\| \geq a - a''' \text{ and } \|z, B\| \geq 2b + 2 + a''' - a' \tag{4.16}$$

(in which case, $d(z) \geq (2b + 2 + a''' - a') + (a - a''') = 2k - a - a' + 2$, again); or

$$\text{every vertex in } B \text{ is petite and } d(z) \geq (2b + 2 + a''' - a') + (a - a''' - 1) = 2k - a - a' + 1. \tag{4.17}$$

If there exists any $y_0 \in B_1$ that is not petite, then (4.15) or (4.16) holds, so for every $y \in B_1$ $d(y) \leq 2k + 1 - d(z) \leq a + a' - 1$. Then y_0 is petite, a contradiction. So every $y \in B_1$ is petite. For every $y \in B_1$, $d(y) \leq 2k + 1 - d(z) \leq a + a'$. Let I be a largest independent subset of B_1.

Case 1: $a' \leq b+1$. If (4.15) or (4.16) holds, each $y \in I \subseteq B_1$ has at least $\|y, B\| + a + a' - d(y) \geq 1 + \|y, B\|$ solo neighbors in $\mathcal{A}'$. If (4.17) holds, then (again) each $y \in I \subseteq B_1$ has at least $\|y, B\| + a + a' - d(y) + 1 \geq 1 + \|y, B\|$ solo neighbors in $\mathcal{A}'$.

Then the total number of solo neighbors of vertices in I is at least

$$\sum_{y \in I} (1 + \|y, B\|) \geq |I| + (|B_1| - |I|) = |B_1| \geq 2b + 2 + a''' - a' \geq 2(a' - 1) + 2 + a''' - a' > a'.$$

Since A' has at most a' solo vertices, some distinct vertices in I share solo neighbors, contradicting Lemma 4.4.7.

Case 2: $a' \geq b+2$. Since each petite vertex has a solo neighbor in A', and every vertex in B_1 is petite, by Lemma 4.5.1, all vertices in B_1 are low. So by Lemma 4.6.1, $|B_1| \leq b$. $\square$

Lemma 4.6.4. $a' \leq a''' + 1$.

Proof. Suppose $a' \geq a''' + 2$ and let $Z = \{z, z', z''\} \in \mathcal{A}'''$. Consider the discharging from B to Z such that each $y \in B$ gives to each neighbor in Z the value $1/\|y, Z\|$. If $z \in Z$ has no solo neighbors in B, then by Lemma 4.6.3(a),

$$ch(z) \leq \frac{\|z, B\|}{2} \leq \max\left\{\frac{b}{2}, \frac{2b+2+a'''-a'}{2}\right\} \leq b.$$

So, since the total charge on Z is $3b+1$, Z contains a solo vertex, say z'', and $ch(z'') \geq b+1$. For $i = 1, 2, 3$, let z'' have exactly c_i $1/i$-neighbors in B. We have $b+1 \leq ch(z'') = c_1 + c_2/2 + c_3/3$.

Case 1: $c_2 = 0$. Then each $w \in B - S_{z''}$ is adjacent to both z and z'. So by Lemma 4.6.3(a), $|S_{z''}| \geq |B| - \|z, B\| \geq 3b + 1 - 2b = b + 1$, a contradiction to Lemma 4.4.7.

Case 2: $c_2 \geq 1$. Then either z or z' has at least $3b+1-c_1-c_2+1$ neighbors in B. By Lemma 4.6.3(a), this is at most $2b$. So, $c_1 + c_2 \geq b+2$. On the other hand, by Lemma 4.4.7, each $y \in S_{z''}$ has at least $c_1 + c_2 - 1$ neighbors in B. Then for such y we have $d(y) \geq a + (b+2) - 1 = k+1$, a contradiction to Lemma 4.5.1. $\square$

Lemma 4.6.5. *If $a' = a''' + 1$, then G has an optimal coloring f' (possibly, $f' = f$) such that*

(i) $a'(f') = a(f') - 1$, or

(ii) $a''(f') = 1$ and $a'(f') = 2$.

Proof. Let $X \in \mathcal{A} - \mathcal{A}'$ have the minimum nonempty $\mathcal{T}(X)$. By the minimality of $\mathcal{T}(X)$, $\mathcal{T}(X) \subseteq \mathcal{A}'$ and for each vertex $U \in \mathcal{T}(X)$ there is a U, X-path in $\mathcal{H}[\mathcal{T}(X) + X]$. If $a = a' - 1$, then (i) holds. Let $a' \neq a - 1$. Then $X \neq V^-$. By Proposition 4.5.2, since $a' \neq a - 1$, $\mathcal{T}(X) \neq \mathcal{A}'$. Since $a' - 1 = a''' \leq a''$, there is exactly one $Z \in \mathcal{A}' - \mathcal{T}(X)$.

Let $\mathcal{H}' = \mathcal{H} - \mathcal{B} - \mathcal{A}' - X$. We first prove that

$$\text{for every } W \in V(\mathcal{H}'),\ V^- \text{ is reachable from } W \text{ in } \mathcal{H}'. \tag{4.18}$$

Indeed, suppose V^- is not reachable in $\mathcal{H}'$ from $W = \{w, w', w''\} \in V(\mathcal{H}')$, and let W have the smallest $\mathcal{T}(W)$ among the vertices with this property. Since $W \notin \mathcal{A}'$, $\mathcal{T}(W) \neq \emptyset$. By the minimality of $\mathcal{T}(W)$, it is contained in $\mathcal{A}' = \mathcal{T}(X) + Z$. If $Z \in \mathcal{T}(W)$, then by the definition of W, $W \in \mathcal{T}(X)$, a contradiction to the minimality of $\mathcal{T}(X)$. So $Z \notin \mathcal{T}(W)$, and thus there is $U \in \mathcal{T}(X) \cap \mathcal{T}(W)$. Since $W \notin \mathcal{T}(X)$ and $\mathcal{T}(Z) = \emptyset$, $\mathcal{H}[\mathcal{A}]$ contains a W, Z-path avoiding X. So if there would be a U, W-path avoiding X, then $U \notin \mathcal{T}(X)$. Therefore, each U, W-path goes through X and so there is a U, X-path avoiding W. Then, since $U \in \mathcal{T}(W)$, also $X \in \mathcal{T}(W)$. Then by the definition of W, $W \in \mathcal{T}(Z)$, contradicting $Z \in \mathcal{A}'$. This proves (4.18).

Let $\mathcal{F}'$ be a rooted-at-V^- spanning in-tree of $\mathcal{H}'$ with most leaves. Let $\mathcal{L}$ denote the set of leaves in $\mathcal{F}'$. Since $X \notin \mathcal{T}(X)$ and $\mathcal{T}(Z) = \emptyset$, each of X and Z has an outneighbor, $O(X)$ and $O(Z)$, respectively, in $V(\mathcal{F}')$. Since $V(\mathcal{F}') \subset \mathcal{A} - \mathcal{A}'$, $\mathcal{L} \subseteq \{O(X), O(Z)\}$.

Case 1: $|\mathcal{L}| = 2$. Then $O(Z) \in \mathcal{L}$ and is the only outneighbor of Z in $\mathcal{H}[\mathcal{A}]$, since otherwise $O(Z) \in \mathcal{A}'$. Thus we may assume that $\mathcal{T}'(O(Z)) = \{Z\}$. So $a''' = 1$ and hence $a' = 2$ and $|\mathcal{T}(X)| = 1$. In particular, $a'' = 1$, i.e. (ii) holds. If the first common vertex on an X, V^--path and a Z, V^--path is $U \neq V^-$, then let U' be the penultimate vertex on a U, V^--path in $\mathcal{H}$. In this case, we move a witness u of $U'V^- \in E(\mathcal{H})$ to V^-. This way, we obtain a new coloring with more terminal classes in $\mathcal{H}[\mathcal{A}]$.

Case 2: $|\mathcal{L}| = 1$. Let $\mathcal{L} = \{W\}$. Then $\mathcal{F}'$ is a W, V^--path. If $W = V^-$, then $\mathcal{A} = \{V^-, Z, X\} \cup \mathcal{T}(X)$. In this case, if Z has no outneighbors apart from V^-, then $\mathcal{T}'(V^-) = \{Z\}$ and so $a''' = 1$. This implies $|\mathcal{T}(X)| = 1$, and (ii) holds. If Z has an outneighbor $Z' \in \mathcal{A} - V^-$, then we move a witness x of $XV^- \in E(\mathcal{H})$ to V^- and get a new coloring f'. In f', the class $V^- + x$ is terminal, because of Z'. If $Z' \notin \mathcal{T}(X)$ in f or is terminal in f', then $a'(f') > a'(f)$, a contradiction. Suppose $Z' \in \mathcal{T}(X)$ and is not terminal in f'. Then the only class blocked by Z' is Z and so $a''(f') = 1$. Thus (ii) holds.

Now suppose $W \neq V^-$. Let W' be the penultimate vertex on a W, V^--path in $\mathcal{H} - Z - X$. If each of X and Z has an outneighbor in $\mathcal{A} - V^-$, then moving a witness of $W'V^- \in E(\mathcal{H})$ to V^- yields a new coloring with more terminal classes in $\mathcal{H}[\mathcal{A}]$. So exactly one of Z and X has V^- as the unique outneighbor in $\mathcal{A} - \mathcal{T}(X)$, and the other has W as the unique outneighbor in $\mathcal{A} - \mathcal{T}(X)$.

Case 2.1: $O(Z) = W$. Then Z has no outneighbors in $\mathcal{A} - W$, since otherwise W would be terminal. So $\mathcal{T}(W) = \{Z\}$; thus $a'' = 1$ and (ii) holds.

Case 2.2: $O(Z) = V^-$. Then we practically repeat the argument of the first paragraph of Case 2. □

Lemma 4.6.6. *If $a'' = 1$, $a' = 2$, and $\mathcal{A}' = \{W, Z\}$ then $\mathcal{H}[\mathcal{A}]$ has a W, V^--path $\mathcal{P} = W, X_1, \ldots, X_s, V^-$ and a Z, V^--path $\mathcal{P}' = Z, U_1, \ldots, U_t, V^-$ such that $V(\mathcal{P}) \cup V(\mathcal{P}') = \mathcal{A}$ and $V(\mathcal{P}) \cap V(\mathcal{P}') = \{V^-\}$. Moreover, each of W, Z has exactly one outneighbor in $\mathcal{H}[\mathcal{A}]$.*

Proof. Since $a'' = 1$, there is $W \in \mathcal{A}' = \{W, Z\}$ and $X_1 \in \mathcal{A}$ such that $\mathcal{T}(X_1) \cap \mathcal{A}' = \{W\}$. We may choose X_1 with this property and the smallest $|\mathcal{T}(X_1)|$. Then simply $\mathcal{T}(X_1) = \{W\}$. Since $Z \notin \mathcal{T}(X_1)$, $X_1 \neq V^-$. Then X_1 is the only outneighbor of W in $\mathcal{A}$. Since $Z \in \mathcal{A}'$, $\mathcal{H}$ has a shortest W, V^--path $\mathcal{P} = W, X_1, \ldots, X_s = V^-$ avoiding Z. Since $Z \notin \mathcal{T}(X_1)$, $\mathcal{H}$ has a shortest Z, V^--path $\mathcal{P}' = Z, U_1, \ldots, U_t = V^-$ avoiding X_1. We can choose such a shortest path with the most common edges with $\mathcal{P}$. If $\mathcal{C} = \mathcal{A} - (V(\mathcal{P}) \cup V(\mathcal{P}')) \neq \emptyset$, then $\mathcal{H}[\mathcal{A}]$ has a spanning in-tree with root V^- with a leaf in $\mathcal{C}$. But any such leaf is in $\mathcal{A}'$, a contradiction. Thus $V(\mathcal{P}) \cup V(\mathcal{P}') = \mathcal{A}$.

Suppose that for some i and j, $X_i = U_j \neq V^-$. Then by the choice of $\mathcal{P}'$, $X_{i+1} = U_{j+1}$ and so on. Then moving a witness from X_{s-1} to $X_s = V^-$, we obtain a coloring with more terminal classes. Thus $V(\mathcal{P}) \cap V(\mathcal{P}') = \{V^-\}$.

To prove the "Moreover" part, observe that if $U_1 \neq V^-$ and Z has an outneighbor $Z' \in \mathcal{A} - U_1$, then $U_1 \in \mathcal{A}'$, a contradiction. □

Lemma 4.6.7. *$a' = a - 1$.*

Proof. By Lemmas 4.6.4, and 4.6.5, if $a' < a - 1$, then we may assume that $a'' = 1$ and $a' = 2$. Then by Lemma 4.6.6, there are $X_1 \in \mathcal{A} - \mathcal{A}' - V^-$, $U_1 \in \mathcal{A} - \mathcal{A}' - X$ and $W, Z \in \mathcal{A}'$ such that $\mathcal{T}(X_1) = \{W\}$ and U_1 is the only outneighbor of Z in $\mathcal{H}[\mathcal{A}]$. In particular, if $U_1 \neq V^-$, then $\mathcal{T}(U_1) = \{Z\}$. Also, there are chordless paths $\mathcal{P} = W, X_1, \ldots, X_s, V^-$ and a $\mathcal{P}' = Z, U_1, \ldots, U_t, V^-$ such that $V(\mathcal{P}) \cup V(\mathcal{P}') = \mathcal{A}$ and $V(\mathcal{P}) \cap V(\mathcal{P}') = \{V^-\}$. Observe that we can choose $\mathcal{A}''' = \{W\}$ or $\mathcal{A}''' = \{Z\}$, so Lemma 4.6.3 applies to both W and Z. Let $W = \{w, w', w''\}$, $Z = \{z, z', z''\}$, $U_1 \subseteq \{u, u', u''\}$ and $X_1 = \{x, x', x''\}$ with x'' being a witness of $X_1X_2 \in E(\mathcal{F})$. Also if $U_1 = V^-$, then u'' does not exist, otherwise, let u'' be a witness of $U_1U_2 \in E(\mathcal{F})$.

Suppose first that $X_1 \cup W - x''$ is independent. Then each $y \in B$ has at least four neighbors in $X_1 \cup W - x''$, since otherwise we can color equitably $X_1 \cup W - x'' + y$ with two colors, $B - y$ with b colors, and $A - X_1 - W + x''$ with $a - 2$ colors. So $\|B, X_1 \cup W - x''\| \geq 4(3b + 1) > 5(2b + 1)$ and there is $s \in W \cup X_1 - x''$ with $\|s, B\| \geq 2b + 2$. If $s \in W$ or could be swapped with a vertex in W, then we get a contradiction with Lemma 4.6.3(a). Otherwise, the only reason that we cannot swap it with a vertex in W is that each vertex in W is adjacent to x''. But each vertex in W adjacent to a vertex in X_1 is unmovable by the definition of $\mathcal{T}(X_1)$, and W cannot have 3 unmovable vertices. If $U_1 \neq V^-$, then the same argument shows that $U_1 \cup Z - u''$ is not independent. Suppose now that $U_1 = V^-$ and $V^- \cup Z$ is independent. Then as above, each $y \in B$ has at least four neighbors in $V^- \cup Z$ and $\|B, V^- \cup Z\| \geq 4(3b + 1)$. Since $\|V^-, B\| \leq |V^-| \cdot |B| = 6b + 2$, $\|B, Z\| \geq 6b + 2$, so there exists $z \in Z$ with $\|z, B\| \geq 2b + 1 = 2b + 2 + a''' - a'$. Then by Lemma 4.6.3(c), there exists some non-petite neighbor of z in B. Since every vertex in B has two neighbors in V^- or three in Z, then the non-petite neighbor y of z in B has $d(y) > a + a' = a + 2$. But now $d(z) + d(y) > 2b + 1 + a - 2 + a + 2 = 2k + 1$, contradicting the degree conditions of G. Thus

$$\text{neither of } X_1 \cup W - x'' \text{ and } U_1 \cup Z - u'' \text{ is independent.} \tag{4.19}$$

Since each vertex in W (respectively, Z) with a neighbor in X_1 (respectively, U_1) is unmovable, by (4.19) we may assume using Lemma 4.4.4 that the unique such vertex in W is w and in Z is z. Also by (4.19) we may assume that $wx, zu \in E(G)$. Then by Lemma 4.6.3(b),

$$\text{each of } w \text{ and } z \text{ has at most } 2b \text{ neighbors in } B. \tag{4.20}$$

Since $WZ, ZW \notin E(\mathcal{H})$, if $\|W, Z\| \leq 3$, then $\|W, Z\| = 3$ and these edges form a matching. Then by symmetry, we may assume $N(z') \cap W = \{w'\}$ and $N(w') \cap Z = \{z'\}$. In this case, we switch w' with z'. Since Z and W are terminal, we can still reach V^- from every class in $\mathcal{A} - Z - W$ in the new coloring f^*. Moreover, X_1 and U_1 are outneighbors of $W^* = W - w' + z'$ and so X_1 is a new terminal class in f^*, a contradiction to the maximality of $\mathcal{A}'$. Thus,

$$\|W, Z\| \geq 4. \tag{4.21}$$

Case 1: Vertex w is not solo. By Lemma 4.6.3(a) and (4.20), $\|w, B\| = 2b$ and $\|w', B\| = \|w'', B\| = 2b + 1$. Then by Lemma 4.6.3(b), each of w', w'' has exactly one neighbor in each class in $\mathcal{A} - W - X_1$. By Lemma 4.6.3(c), there exists $y \in B \cap N(w')$ that is not petite. Since $\|y, W\| = 2$ and $W \in \mathcal{A}'$, $d(y) \geq a + a' + 1 = a + 3$. Now $d(w') + d(y) \geq (2b + 1 + a - 2) + (a + 3) = 2k + 2$, contradicting the degree conditions of G.

The proof of the case when z is not solo is exactly the same (with the switched roles of W and Z).

Case 2: Both w and z are solo. By the case, $B_1(w) \neq \emptyset$ and $B_1(z) \neq \emptyset$. Since each $y' \in B_0(w) \cup B_3(w)$ is adjacent to both w' and w'', using Lemma 4.6.3(a), $b_0(w) + b_3(w) \leq \|B, w'\| \leq 2b + 1$. So, $b_1(w) + b_2(w) \geq |B| - (2b + 1) = b$. Similarly, $b_1(z) + b_2(z) \geq b$.

Case 2.1: $b_1(w) + b_2(w) \geq b + 1$. Let $y \in b_1(w)$. Since each $y' \in b_1(w) \cup b_2(w) - y$ is adjacent to y (by Lemma 4.4.7), $d(y) + d(w) \geq (b_1(w) + b_2(w) - 1 + a) + (b_1(w) + b_2(w) + b_3(w) + a - 1) \geq 2k + 2(b_1(w) + b_2(w) - 1 - b)$. So, $b_1(w) + b_2(w) \leq b + 1$, and by the case, $b_1(w) + b_2(w) = b + 1$. Since $G[B]$ is b-colorable, there are $y_1, y_2 \in b_1(w) \cup b_2(w)$ with $y_1y_2 \notin E(G)$. Then by Lemma 4.4.7, $y_1, y_2 \in b_2(w)$ and $b + 1 \geq 3$. In particular, each of y_1, y_2 has a neighbor in $W - w$ and $\|\{w', w''\}, B\| \geq 2(b_0(w) + b_3(w)) + |b_2(w)| \geq 2(2b) + 2$. Then by Lemma 4.6.3(a), $b_2(w) = \{y_1, y_2\}$, the neighbors of y_1 and y_2 in W are distinct, and each of w', w'' has exactly $a - 2$ neighbors in A. So by (4.21), $\|w, Z\| \geq 2$ and $d(w) \geq (a - 1 + 1) + b + 1 = k + 1$. Hence $\|y_1, A\| \leq 2k + 1 - d(w) - \|y_1, B\| \leq k - \|y_1, B\|$. Since $b_2(w) = \{y_1, y_2\}$, $\|y_1, b_1(w) \cup b_2(w)\| = b - 1$. Thus $\|y_1, A\| \leq k - (b - 1) = a + 1$. Since $\|y_1, W\| = 2$, y_1 has a solo neighbor in Z. Similarly, y_2 has a solo neighbor in Z, a contradiction to Lemma 4.4.7.

The proof of the case $b_1(z) + b_2(z) \geq b + 1$ is exactly the same. So, since $b_1(w) + b_2(w) \geq b$ and $b_1(z) + b_2(z) \geq b$, the last subcase is:

Case 2.2: $b_1(w) + b_2(w) = b$ and $b_1(z) + b_2(z) = b$. Then $b_0(w) + b_3(w) = b_0(z) + b_3(z) = 2b + 1$; so Lemma 4.6.3(a) and (b) applied to the vertices of $W - w'$ and $Z - z$ yields $b_2(w) = b_2(z) = 0$ and $\|s, A\| = a - 2$ for all $s \in \{w', w'', z', z''\}$. In particular, $\|s, W \cup Z\| = 1$ for all $s \in \{w', w'', z', z''\}$. If say, $z'w' \in E(G)$, then as in the proof of (4.21), switching w' with z' leads to a coloring with more terminal classes, a contradiction. Thus, $\{wz', wz'', zw', zw''\} \subset E(G)$. Now, if $wz \in E(G)$, then since both are immovable, and by the case: $d(w) + d(z) \geq 2(a + 1 + b) = 2k + 2$, contradicting the degree conditions of G. So $wz \notin E(G)$. If there is $y \in B - N(w) - N(z)$, then we transform f into an equitable k-coloring as follows: take an equitable b-coloring of $B - y$, add classes $\{y, z, w\}$, $\{w', z', z''\}$, recolor the witnesses along $\mathcal{P}$ (starting from w'') and keep the obtained classes in $\mathcal{A} - W - Z$. Thus $B \subset N(w) \cup N(z)$. In particular, since $b_1(w) + b_2(w) = b_1(z) + b_2(z) = b < |B|/2$, there exists $y \in B_3(w) \cup B_3(z)$; say $y \in B_3(z)$ (the other case is the same). Now $\|y, A\| \geq a+2$, so $d(y)+d(w') \geq (a+2)+(a-2+2b+1) = 2k+1$. Since $yw' \in E(G)$, by the degree conditions of G, y has precisely one neighbor in every class of $A - W$ and y is isolated in B. Since y has only one neighor in Z, y is a solo neighbor of z, but since $b_1(z) = b > 0$ the isolation of y in B violates Lemma 4.4.7. $\square$

4.7 $\mathcal{F}$ is a star

In this section we will prove the following lemma:

Lemma 4.7.1. *If there exist an optimal coloring f such that $a(f) = a'(f) + 1$, then there exists an optimal coloring f such that $\mathcal{F}(f')$ is a star.*

We begin with a simple lemma.

Lemma 4.7.2. *If $\mathcal{F}$ is not a star and $a' = a - 1$, then $a \geq 4$ and there exist two classes $Z \in \mathcal{A}'$ and $W \in \mathcal{A}'$ such that ZV^- and WV^- are both edges in $\mathcal{F}$.*

Proof. If $\mathcal{F}$ is not star, there exists $X \in \mathcal{A}'$ such that XV^- is not an edge in $\mathcal{F}$. Since $a' = a - 1$, there exists $Z \in \mathcal{A}'$ such that ZV^- is an edge in $\mathcal{F}$. Because Z is in $\mathcal{A}'$, there exists an X, V^--path $X \dots WV^-$ in $\mathcal{F}$ that avoids Z. Since $a' = a - 1$, $W \in \mathcal{A}' - Z - Z$ and $a' \geq 3$. $\square$

The following lemma is crucial to the proof of Lemma 4.7.1. We make the following definitions which are used throughout the rest of the paper. For every $x \in A$, let $B_0(x)$ denote the set of nonneighbors of x in B, and for $i = 1, 2, 3$, let $B_i(x)$ denote the set of neighbors of x in B that have exactly i neighbors in the class of X. For $i = 0, 1, 2, 3$, let $b_i(x) = |B_i(x)|$.

Lemma 4.7.3. *Assume that $a'(f) = a(f) - 1$ for some optimal coloring f and there does not exists an optimal coloring f' for which $\mathcal{F}(f')$ is a star. For every $X \in \mathcal{A}'$, if $\|u, A\| \geq 1$ for every $u \in X$, then X has a solo vertex. Furthermore, if x is the solo vertex in X, then $b_1(x) + b_2(x) \in \{b, b+1\}$ and $\|x', A\| = 1$ for every $x' \in X - x$.*

Proof. First assume that X does not have a solo vertex. We then have that $\|X, B\| \geq 6b + 2$. If we let $\|x'', B\| \geq \|x', B\| \geq \|x, B\|$, then $\|x'', B\| \geq 2b + 1$ and $d(x'') \geq 2b + 2$. Since $|L'(B)| \leq b + 1 < 2b + 1$, we also have that x'' is adjacent to a vertex $y \in H'(B)$. So,

$$\|x'', B\| \leq 2a + 2b + 1 - d(y) - \|x'', A\| \leq 2b + 1$$

and $\|x'', B\| = 2b + 1$ and $\|x'', A\| = 1$. Similar logic gives that $\|x', B\| = 2b + 1$ and $\|x', A\| = 1$.

We now have that $\|x, B\| \geq 2b$. Suppose x is movable. Then moving x to a class $U \in \mathcal{A}$ and switching witnesses along a U, V^- in $\mathcal{F}$ that avoids X gives a nearly equitable coloring with small class $\{x', x''\}$ and $\|\{x', x''\}, A\| = 2 < 3$. This contradicts the fact that there are no optimal coloring in which $\mathcal{F}$ is a star. Therefore, x is not movable. If x is adjacent to a vertex $y \in H'(B)$, then $d(x) + d(y) \geq a - 1 + 2b + 2a - 1 \geq a + 2a + 2b - 2$. Since, by Lemma 4.7.1, $a \geq 4$, this is a contradiction. So $2b \leq \|x, B\| \leq |L'(B)| \leq b + 1$, which implies $b = 1$ and $N(x) \cap B = L'(B) \subseteq L(B)$ which further implies $|L(B)| \geq 2$. Let y and y' be distinct vertices in $L(B)$. Since they are low, $d(y), d(y') \leq a + b \leq a + 1$. Since they both have two neighbors in X, they both have a solo neighbor in every class of $\mathcal{A}' - X$. Since y and y' are not adjacent (they are in the same color class) and $a' \geq 3$, this is a contradiction.

So we can assume there exists a solo vertex $x \in X$. Let $\{x', x''\} = X - x$ and $u \in \{x', x''\}$. If $\|u, B\| \geq 2b + 1$, then u is adjacent to a vertex in $y \in H'(B)$. So, since $\|u, A\| \geq 1$ and $d(y) \geq 2a - 1$, we have that $d(y) = 2a - 1$, $\|u, B\| = 2b + 1$ and $\|u, A\| = 1$. This implies that if $\|\{x', x''\}, B\| \geq 4b + 2$, then $\|x', A\| = \|x'', A\| = 1$ and $\|\{x', x''\}, B\| \leq 4b + 2$. Therefore, we know that $b_1(x) + b_2(x) \geq b$, since $B_0(x) \cup B_3(x) \subseteq N(u) \cap B$. Furthermore, if $b_1(x) + b_2(x) = b$, then we have the desired conclusion. Hence, we can assume $b_1(x) + b_2(x) \geq b + 1$.

There exists $y \in B_1(x)$, because x is solo. By Lemma 4.5.1, y is low, so it has at most b neighbors in B. Since, by Lemmas 4.4.7 and 2.2.22, y is adjacent to every vertex in $B_1(x) \cup B_2(x)$, $b_1(x) + b_2(x) = b + 1$. We have that $\|\{x', x''\}, B_0(x) \cup B_3(x)\| = 2(3b + 1 - (b + 1)) = 4b$. Note that $B_1(x) \cup B_2(x)$ is not a $(b+1)$-clique since $G[B]$ is b-colorable, so there exist distinct $y', y'' \in B_1(x) \cup B_2(x)$ that are not adjacent. By Lemmas 4.4.7 and 2.2.22, this implies $\{y', y''\} \subseteq B_2(x)$. By the definition of $B_2(x)$, y' and y'' have a neighbor in $\{x', x''\}$, so $\|\{x', x''\}, B\| \geq 4b + 2$ which, from a previous argument, gives us the desired conclusion. $\square$

Let $\{x, x', x''\} \in X \in \mathcal{A}'$ such that XV^- is not in $\mathcal{F}$ and there does not exist an optimal coloring f' for which $\mathcal{F}(f')$ is a star. Since every vertex in X is adjacent to a vertex in V^-, we can apply Lemma 4.7.2, so we can label such that x is a solo vertex, $\|x, B\| \geq b$ and x' and x'' both have exactly one neighbor in V^- and no neighbors in A'.

We make the following two claims.

Claim 1. For every $Z \in \mathcal{A}'$, $\|x, Z\| \leq 2$.

Proof. Suppose $\|x, Z\| = 3$ for some $Z \in \mathcal{A}'$. By Lemma 4.7.2, we can assume that there exists $z \in Z$ such that z is solo, $\|z, B\| \geq b$ and that for any $u \in Z - z$, $N(u) \cap A = \{x\}$. Since $\|x, B\| \geq b$, $\|x, A\| \geq a + 1$ and x is adjacent to z we have that $d(z) \leq a + b$. Since $\|z, B\| \geq b$, we have that $\|z, A\| \leq a$. This implies that $\|z, U\| \leq 2$ for every $U \in \mathcal{A}'$. If we let $\{z', z''\} = Z - z$, then we can move z'' to V^-. Now $\{z, z'\}$ is the small class of a nearly equitable coloring. In this new coloring, the classes $V^- + z''$ and $\{x, x', x''\}$ are clearly movable to $\{z, z'\}$. Furthermore, any class $U \in \mathcal{A}' - Z - X$ is still a class of the new coloring, and it is movable to $\{z, z'\}$ since the only neighbor of z' in A is x and z has at most two neighbors in U. $\square$

Claim 2. For every $u \in A' - X$, if x is not adjacent to u, then u is not movable to V^-.

Proof. Suppose there exists a vertex $z' \in A' - X$ such that $z' \in Z \in \mathcal{A}'$ is not adjacent to x and z' is movable to V^-. Form a new nearly equitable coloring by moving z' to V^- and x'' to $Z - z'$. Note that $\{x, x'\}$ is the small class in this coloring and that z', and hence $V^- + z'$, is movable to $\{x, x'\}$. Clearly $Z - z' + x''$ is movable to $\{x, x'\}$. Every $U \in \mathcal{A}' - Z - X$ is a color class of the new coloring and, since $\|x', U\| = 0$ and $\|x, U\| \leq 2$, U is movable to $\{x, x'\}$. This implies that the new coloring is a star. $\square$

By Lemma 4.7.1, there exist distinct $Z, W \in \mathcal{A}' - X$ such that ZV^- and WV^- are both edges in $E(\mathcal{F})$. By Claim 2, every vertex in both $Z \cup W$ has a neighbor in A: either x or a vertex in V^-. Therefore, by Lemma 4.7.2, there exists $z \in Z$ and $w \in W$ that are both solo and such that $\|z, B\|, \|w, B\| \geq b$. Furthermore, the vertices in $Z \cup W - z - w$ have exactly one neighbor in $V^- + x$ and no neighbors in $A' - x$.

Note that since both z and w are solo, and hence unmovable, they both have neighbors in X. The only neighbors of $\{x', x''\}$ in A are in V^-, so x is adjacent to both w and z. Furthermore, there exists $w' \in W - w$ and $z' \in Z - z$ that witness the WV^- and ZV^- edges, respectively. Claim 2 then implies x is adjacent to both w' and z'. This with the fact that x is solo and unmovable, implies that $\|x, A\| \geq a + 1$. Recall that $\|x, B\|, \|w, B\|, \|z, B\| \geq b$, so $\|w, A\|, \|z, A\| \leq 2a + 2b + 1 - (a + 1) + 2b = a$. Therefore, both w and z have at most 2 neighbors in any class of $\mathcal{A}$. Let $\{z''\} = Z - z - z'$. Note that the only neighbor of z'' in A is either x or a vertex in V^-. Moving z' to V^-, then creates a coloring f' with small class $\{z, z''\}$. We have that z', x' and x'' are movable to to $\{z, z''\}$. This implies that the classes $V^- + z'$ and X are both movable to $\{z, z''\}$. We also have that for any class $U \in \mathcal{A}' - X - Z$, z' has no neighbors in $A' - X \supseteq U$ and z has at most 2 neighbors in U. This implies that U is movable to $\{z, z''\}$ and $\mathcal{F}(f')$ is a star.

Because $\mathcal{F}$ attains the maximum number of leaves over all spanning rooted in-trees of $\mathcal{H}$ rooted at V^-, $\mathcal{F}$ is a star.

4.8 $a' \geq b + 1$

Lemma 4.8.1. *If $b \leq a' - 1 = a - 2$, then each class $X \in \mathcal{A}'$ has a neighbor in V^-.*

Proof: Suppose $V^- = \{v, v'\}$, $X = \{x, x', x''\} \in \mathcal{A}'$ and $V^- \cup X$ is independent. Then each $y \in B$ has at least 4 neighbors in $V^- \cup X$, so some $w \in V^- \cup X$ has at least $\left\lceil \frac{4(3b+1)}{5} \right\rceil \geq 2b + 2$ neighbors in B. By Lemma 4.6.1, at least one, say y_0, of these neighbors is in $H(B)$. Then y_0 has at least $2(a-2)$ neighbors in $A - X - V^-$. Thus $d(v) + d(y_0) \geq (2b + 2) + 2(a - 2) + 4 = 2k + 2$, a contradiction. □

Lemma 4.8.2. *If $b \leq a' - 1 = a - 2$, then for each class $X \in \mathcal{A}'$, $||X, A|| \geq a - 1 = a'$.*

Proof: Suppose $||X, A|| \leq a - 2$. Since we know that $||X, V^-|| \geq 1$, there is $Z \in \mathcal{A}' - X$ s.t. $X \cup Z$ is independent. If each $y \in B$ has at least 5 neighbors in $X \cup Z$, then there is $x \in X \cup Z$ with

$$||x, B|| \geq \left\lceil \frac{5(3b+1)}{6} \right\rceil = 2b + \left\lceil \frac{3b+5}{6} \right\rceil \geq 2b + 2.$$

Then $d(y) \leq 2a - 1$ for each $y \in B$ and so each high vertex has exactly two neighbors in each class of $\mathcal{A}'$; thus it cannot have more than 4 neighbors in $X \cup Z$.

Thus there is $y \in B$ with $||y, X \cup Z|| \leq 4$. Let $x, z \in X \cup Z - N(y)$. If there is $v \in X \cup Z - z - x$ that is movable to a class U outside of $X \cup Z$, then we move it to U, then (if $U \neq V^-$) move a witness from U to V^-, and create color classes $\{y, x, z\}$ and $X \cup Z - x - z - v$. Thus, there are no such v, and each $u \in X \cup Z - z - x$ has neighbors in each of $a - 2$ classes of $\mathcal{A}' - X - Z$. But each of X and Z has a vertex movable to V^-. So, x and z are in distinct classes and so $||X, A|| \geq 2(a - 2) \geq a - 1$, as claimed. □

Lemma 4.8.3. *If $b \leq a' - 1 = a - 2$, then $||y, V^-|| = 1$ for each $y \in H(B)$.*

Proof: Suppose $V^- = \{v, v'\}$ and some $y \in H(B)$ is adjacent to both, v and v'. Then $||V^-, B|| \geq 3b+2$. So by Lemma 4.8.1, $d(v) + d(v') \geq (3b + 2) + (a - 1) \geq 4b + 3$. So we may assume $d(v) \geq 2b + 2$. But $d(y) \geq 2a$ and so $d(y) + d(v) \geq 2k + 2$. □

Lemma 4.8.4. *If $b \leq a' - 1 = a - 2$, then $||y, X|| = 2$ for each $X \in \mathcal{A}'$ and $y \in H(B)$.*

Proof: Let $X = \{x, x', x''\} \in \mathcal{A}'$ and $y \in H(B)$. By Lemma 4.5.1, $||y, X|| \geq 2$. Suppose $||y, X|| = 3$. Then $d(y) \geq 2a$ and so

$$d(x), d(x'), d(x'') \leq 2b + 1. \tag{4.22}$$

If X does not have a solo vertex, then, since $||y, X|| = 3$, $||X, B|| \geq 2|B| + 1 = 6b + 3$. So by (4.22), $||X, A|| = d(x) + d(x') + d(x'') - ||X, B|| \leq 3(2b + 1) - (6b + 3) = 0$, a contradiction to Lemma 4.8.2. Thus we may assume that x is solo (and so unmovable).

Since $||x, A|| \geq a - 1||$, by (4.22), $||x, B|| \leq (2b + 1) - (a - 1)$. So $|B - N(x)| \geq a - 1 + b$. Since each of x', x'' is adjacent to each vertex in $(B - N(x)) + 1$, this and (4.22) yield

$$2b + 1 \geq d(x') \geq (a - 1 + b) + 1 = a' + b + 1$$

and thus $a' \leq b$, a contradiction. □

Lemma 4.8.5. *If $b \leq a' - 1 = a - 2$, then each class $X \in \mathcal{A}'$ contains an unmovable vertex w_X.*

Proof: Suppose that $X = \{x, x', x''\} \in \mathcal{A}'$ has no unmovable vertices. Then it has no solo vertices and $||X, B|| \geq 6b+2$. Rename the vertices in X so that $||x, B|| \leq ||x', B|| \leq ||x'', B||$. Then $||\{x', x''\}, B|| \geq 4b+2$ and $||x'', B|| \geq 2b + 1$.

CASE 1: $||\{x', x''\}, A|| \leq 2$. Since x is movable, move it to a class U with no conflict, and if $U \neq V^-$, then move a witness from U to V^-. By the case, every new class has a vertex movable to $X' = X - x$. By Lemma 4.8.1 for the new coloring and again by the case, $a \leq 3$. Since $1 \leq b \leq a - 2$, we conclude that $b = 1$, $a = 3$ and $||\{x', x''\}, A|| = 2$. In particular, $|B| = 4$. Since $||\{x', x''\}, B|| \geq 4b + 2 = 6$, there are $y, y' \in B$ adjacent to both x', x''. By Lemma 4.8.3, $y, y' \in L(B)$. Then each of y and y' has a solo neighbor in the other class of $\mathcal{A}'$, a contradiction to Lemma 4.4.7.

CASE 2: $||\{x', x''\}, A|| \geq 3$. Then $d(x') + d(x'') \geq 4b + 2 + 3$ and so $d(w) \geq 2b + 3$ for some $w \in \{x', x''\}$. If w has a neighbor $y \in H(B)$, then $d(w) + d(y) \geq 2b + 3 + 2a - 1 = 2k + 2$, a contradiction. Thus by Lemma 4.6.1, $||w, B|| \leq |L(B)| \leq b + 1$. Since $||x'', B|| \geq 2b + 1$, $w = x'$ and $||x'', B|| \leq 2b + 2$. So $||x', B|| \geq 4b+2-||x'', B|| \geq 2b$. Again by Lemma 4.6.1, $2b \leq b+1$ and thus $b = 1$, $a' = 2$, $||x'', B|| = 2b+2 = 4$ and $N(x') \cap B = L(B)$. As at the end of Case 1, then each of the two vertices in $L(B)$ has a solo neighbor in the other class of $\mathcal{A}'$, a contradiction. □

Lemma 4.8.6. *If $b \leq a' - 1 = a - 2$, then for each unmovable $x \in X \in \mathcal{A}'$,*
$||x, B|| \geq b - 1$. If in addition, x is a high vertex, then $||x, B|| \leq b$.

Proof: Let x be unmovable in $X = \{x, x', x''\} \in \mathcal{A}'$ and $Y = B \cap N(x)$. If $|Y| \leq b - 2$, then $||B - Y|| \geq$

$2b+3$ and each $y \in B-Y$ is adjacent to both x' and x''. By Lemma 4.6.1, there is $y' \in (B-Y) \cap H(B)$. Then $d(y')+d(x') \geq (2a-1)+(2b+3) = 2k+2$, a contradiction. So $|Y| \geq b-1$.

Suppose now that $|Y| \geq b+1$ and $d(x) \geq k+1$. For every $y \in Y$, $d(y) \leq 2k+1-d(x) \leq k$. So $Y \subseteq L(B)$. By Lemma 4.6.1, $|L(B)| \leq b+1$. Thus $|Y| = b+1$, and again by Lemma 4.6.1, $a' = b+1$ and $G[L(B)]$ is the disjoint union of some cliques $C_1, \dots, C_s$. Since $G[B]$ is b-colorable, $s \geq 2$. If x is not a solo vertex, then G has at most $a'-1$ solo vertices in A'. In this case, repeating the proof of Lemma 4.6.1, instead of (4.6) we get $(|I|-1)(a'-b-1)-b-1+|L(B)| \leq -1$ and thus $|L(B)| \leq b$, a contradiction to above.

So suppose x is a solo neighbor of some $y_1 \in C_1$. Then by Lemmas 4.4.7 and 2.2.22 each $y_2 \in C_2$ is adjacent to both x' and x''. It follows that y_2 has at least $a'-b+||y,B||+1$ solo neighbors in A', and repeating the proof of Lemma 4.6.1, we derive that there are more than a' solo vertices in A'. □

Lemma 4.8.7. *If $b \leq a'-1 = a-2$, then each unmovable high vertex $x \in X \in \mathcal{A}'$ is adjacent to all other unmovable vertices in A'. In particular, A' contains at most one high unmovable vertex.*

Proof: Suppose x is the unmovable vertex in $X = \{x, x', x''\} \in \mathcal{A}'$, z is the unmovable vertex in $Z = \{z, z', z''\} \in \mathcal{A}'$, $Z \neq X$, $xz \notin E(G)$ and $d(x) \geq k+1$. Let $Y = B \cap N(x)$ and $Y' = B - Y$. Then each $y \in Y'$ is adjacent to both, x' and x''. Since z is unmovable and $zx \notin E(G)$, we may assume that $x'z \in E(G)$. By Lemma 4.8.6, $b-1 \leq |Y| \leq b$.

CASE 1: $|Y| = b-1$. By Lemma 4.6.1, Y' contains a high vertex y'. Then $d(y')+d(x') \geq (2a-1)+|Y'|+1 = 2a-1+2b+3 = 2k+2$, a contradiction.

CASE 2: $|Y| = b$. If z is low, then $||z,B|| \leq k-(a-1) = b+1$. Otherwise, by Lemma 4.8.6, $||z,B|| \leq b$. In any case, there is $y'' \in Y' - N(z)$. Suppose first that x'' is movable to V^-. Then we try to move x'' to V^-, color equitably $B - y''$, create color classes $\{y'', x, z\}$ and $\{x', z', z''\}$. The problem occurs only if x' is adjacent to $\{z', z''\}$, but in this case $d(x') \geq |Y'|+2 = 2b+3$, and again for any $y' \in Y' \cap H(B)$, $d(x')+d(y') \geq 2k+2$. So we may assume $||x'', V^-|| \geq 1$. But then x' is the witness of $XV^- \in E(\mathcal{H})$ and we can repeat our attempt of recoloring with the switched roles of x' and x''. This fails only if x'' is adjacent to $\{z', z''\}$, and in this case $d(x'') \geq |Y'|+2 = 2b+3$. So we again get a contradiction. □

4.9 $b = 1$

This section is devoted to the case $b = 1$. A helpful situation in this case is that

$$\forall y \in L(B), \text{ at most one } X \in \mathcal{A}' \text{ has no solo neighbors of } y\text{; if such } X \text{ exists, } ||y,X|| = 2. \tag{4.23}$$

We handle the case in three steps.

Lemma 4.9.1. *If $b = 1$ then for each $y \in L(B)$, $||y, V^-|| = 1$.*

Proof: Suppose $y \in L(B)$ and $V^- \subset N(y)$. Then by (4.13), each $X \in \mathcal{A}'$ contains a solo (unmovable) neighbor w_X of y. This implies that y is the only low vertex in B, since otherwise the other low vertex in B would share a solo neighbor in A' with y. This also yields that for each $y' \in H(B) = B - y$ and each $X \in \mathcal{A}'$, $N(y') \cap X = X - w_X$.

CASE 1: Some $v \in V^-$ is adjacent to all vertices in B. Since $d(v) \leq 2k + 1 - (2a - 1) = 4$, $N(v) = B$. Let $\{v'\} = V^- - v$. Then v' must be adjacent to each unmovable vertex in A'. Let $W = \{w_X : X \in mathcalA'\} \cup \{v', y\}$. Then by above, $G[W] = K_k$ and $G - W$ contains $K_{3,2k-3}$ with partite sets $B - y$ and $A - W$ (the latter contains v and two vertices in each $X \in mathcalA'$). This contradicts the choice of G.

CASE 2: Each of $v, v' \in V^-$ has a neighbor in $B - y$. Then $d(v) + d(v') \leq 2(2k + 1 - (2a - 1)) = 8$. Since $||V^-, B|| = 5$, $||V^-, A'|| \leq 3$. So there is $X \in \mathcal{A}'$ with $||X, V^-|| \leq 1$. Since w_X has a neighbor in V^-, we may assume that $E(G[X \cup V^-]) = \{w_X v'\}$. By the case, there is $y' \in B - y$ not adjacent to v'. Let the color classes of the new coloring be $\{y', w_X, v'\}$, $X - w_X + v$, $B - y'$, and the classes in $\mathcal{A}' - X$. This is an equitable coloring, a contradiction. □

Lemma 4.9.2. *If $b = 1$ then each class $X \in \mathcal{A}'$ has a solo vertex.*

Proof: Suppose x is the unmovable vertex in $X = \{x, x', x''\} \in \mathcal{A}'$ and x is not solo. Then by (4.23) and Lemma 4.8.4, $||X, B|| = 8$.

CASE 1: $||x, B|| = 0$. Then $||x', B|| = ||x'', B|| = 4$ and so $||x', A|| = ||x'', A|| = 0$. Let $V^- = \{v.v'\}$. By Lemmas 4.9.1 and 4.8.3, each $y \in B$ has exactly one neighbor in V^-. So we may assume that $|N(v) \cap B| \leq 2$ and $y, y' \in B - N(v)$. Let the color classes of the new coloring be $\{y, y', v\}$, $X - x + v'$, $B - y' - y + x$, and the classes in $\mathcal{A}' - X$. This is an equitable coloring, a contradiction.

CASE 2: $||x, B|| \geq 1$. We may assume that $||x', B|| \leq ||x'', B||$, so that x and x'' have a common neighbor in B. Since x' is movable, we move it into a class $Z \in \mathcal{A} - X$ and if $Z \neq V^-$, then we move a witness u_Z of ZV^- into V^-. If in the new coloring every color class in the new $\mathcal{A}$ has a vertex movable to $X - x'$ then we get a contradiction either with Lemma 4.9.1 or with Lemma 4.8.3. So there exists a class W of the new coloring f' in which every vertex has a neighbor in $\{x, x''\}$. Moreover, W is not the new class of x'.

CASE 2.1: $||x'', B|| \leq 3$. This yields $||x, B|| \geq 8 - 2||x'', B|| \geq 2$. If B contained two low vertices y and y', then by (4.23), y and y' would have a common solo neighbor in each other class in $\mathcal{A}'$, a contradiction. So $|L(B)| \leq 1$. Thus x has a neighbor in $H(B)$ and so $d(x) \leq 4$. But $d(x) \geq a - 1 + ||x, B|| \geq 2 + 2$. It follows that $a = 3$, $||x, B|| = 2$ and $||x'', B|| = 3$. Also then $||\{x, x''\}, A|| \leq d(x) + d(x'') - ||\{x, x''\}, B|| \geq 4 + 4 - 5 = 3$. But we already have 3 neighbors of $\{x, x''\}$ in W. So the third class in the new A has no neighbors of x. However, originally every class in $\mathcal{A}$ had a neighbor of x and every class in $\mathcal{A}'$ had an unmovable neighbor of x. This is a contradiction.

CASE 2.2: $||x'', B|| = 4$. Then every $w \in W$ is adjacent to x, so $||x, A|| \geq (a-1)+2 = k$. It follows that $d(x) \geq k+1$ and so $N(x) \cap H(B) = \emptyset$. By the case, this means that $|L(B)| = 1$. Let $L(B) = \{y\}$. Then $N(x') \cap B = H(B)$. Since $d(y) = a+1$, $d(x)$ is exactly $k+1$, and x has exactly one neighbor in each class in $\mathcal{A} - W$. By Lemma 4.9.1, W is not obtained from V^- by adding a vertex. So, W was already a color class in f, let $W = \{w, w', w''\}$ with unmovable w. By (4.23), w is the solo neighbor of y in W. Then $N(w') \cap B = N(w'') \cap B = B - y$ and $d(w') \leq 4$ and $d(w'') \leq 4$. Since $xw', xw'' \in E(G)$, x' is adjacent neither to w' nor to w''. Thus if $x'w \notin E(G)$, then we can choose W as the class Z at the beginning of the case and obtain another class with many neighbors of x, a contradiction. So, $x'w \in E(G)$. As a neighbor of x, w is a low vertex, and so $||w, V^-|| = 1$ (w has a neighbor in each class of $\mathcal{A} - W$ plus an extra neighbor in X plus y). Let $V^- = \{v, v'\}$ with $wv' \in E(G)$. If v has a nonneighbor $y' \in H(B)$, then we create new color classes $\{w, v, y'\}$, $B - y'$, $W - w + v'$ and use the old classes in $\mathcal{A}' - W$. So $H(B) \subseteq N(v)$. If $yv' \notin E(G)$, then we take some $y' \in H(B)$ and create new color classes $\{y, v', y'\}$, $B - y' - y + w$, $W - w + v$ and use the old classes in $\mathcal{A}' - W$. So $yv' \in E(G)$. If $xv' \notin E(G)$ then similarly we take some $y' \in H(B)$ and create new color classes $\{x, v', y'\}$, $B - y'$, $X - x + v$ and use the old classes in $\mathcal{A}' - W$. Thus $G[\{y, x, w, v'\}] = K_4$ and $G[(X \cup W \cup B \cup V^-) - \{y, x, w, v'\}]$ contains $K_{3,5}$ with one of partite sets $H(B)$. Therefore, if $a = 3$, then we have a contradiction to the choice of G. So, let $a \geq 4$ and $U = \{u, u', u''\}$ be the third class in $\mathcal{A}'$ with unmovable u. By the degree conditions on x, x' and x'', the only edge in $G[X \cup U]$ is xu. In particular, u is low. Then switching u with x we again get Case 2.2, but now the unmovable vertex is low, a contradiction to above. $\square$

Lemma 4.9.3. $b \neq 1$.

Proof: Suppose $b = 1$. Since $k \geq 4$, $a \geq 3$. If $a' = 1$, then $a'' = a'$ and by Proposition 4.5.2, $a' = a - 1 \geq 2$, a contradiction. So $a' \geq 2$ and by Lemma 4.7.1 $\mathcal{F}$ is a star. Then by Lemma 4.9.2, $L(B) \neq \emptyset$.

Let $y \in L(B)$. If $d(y) = k$, then by Lemma 4.9.1 there is a class $X = \{x, x'x''\} \in \mathcal{A}'$ with $||y, X|| = 2$. By Lemma 4.9.2, some $x \in X$ is the solo neighbor for some $y' \in B$. Then $y' \in L(B)$ and since B is independent, $N(x') \cap B = N(x'') \cap B = B - y'$. By (4.23), $a' = 2$, and we may assume that the other class in $\mathcal{A}'$ is $W = \{w, w', w''\}$ with $wy \in E(G)$ and $N(w') \cap B = N(w'') \cap B = B - y$. If x has no neighbors in $W - w$ and w has no neighbors in $X - x$, then swapping x with w yields a coloring in which y has no neighbors in $W - w + x$. By symmetry, we may assume that $wx' \in E(G)$. Let $V^- = \{v, v'\}$ and $B - y - y' = \{y'', y'''\}$. We construct an equitable 4-coloring as follows. Two classes will be $\{y, x, y''\}$ and $\{y', w, y'''\}$. In the remaining 6-vertex subgraph G' of G, w' is isolated, the degrees of x', x'' and w'' do not exceed 1, and the vertices v and v' are not adjacent to each other. This means that at least one component of G' is an isolated vertex and the other are stars with at most 3 rays. Each such 6 forest has an equitable 2-coloring.

Thus $d(y) = a$. Then each neighbor of y is solo and by (4.23), $L(B) = \{y\}$. By Lemma 4.4.3, all neighbors of y are unmovable. For every $X \in \mathcal{A}$, let u_X denote the unmovable vertex in X. Then by Lemmas 4.4.7 and 2.2.22, for every $X \in \mathcal{A}'$,

$$N(u_X) \cap B = \{y\}, \text{ and for each } x \in X - u_X,\ N(x) \cap B = B - y. \tag{4.24}$$

Let $V^- = \{v, v'\}$ with $v' = u_{V^-}$. If $X \in \mathcal{A}'$ and u_X is low, then switching y with u_X again creates an appropriate coloring. So,

$$\text{for every low unmovable } u \in A',\ d(u) = a \text{ and } u \text{ is adjacent to } v' \text{ in } V^-. \tag{4.25}$$

If there is $y' \in B - y = H(B)$ with $y'v \notin E(G)$, then $y'v' \in E(G)$ and so $d(v') \leq 4$. But $||v', A|| \geq a - 1$ and $N(v') \cap B \supseteq \{y, y'\}$. It follows that $a - 1 = 2$ and $N(v') \cap B = \{y, y'\}$. Then we can replace color classes B and V^- with $\{y, y', v\}$ and $B + v' - y - y'$. Thus $N(v) \supseteq B - y$, and

$$G[V(G) - N(y) - y] \text{ contains } K_{3,2k-3} \text{ with partite sets } B - y \text{ and } A - N(y). \tag{4.26}$$

By Lemma 4.8.7, at most one solo neighbor of y in A' is high. Suppose that if such neighbor exists, then it is $w \in W \in \mathcal{A}'$.

CASE 1: All solo neighbors of y in A' are low or $wy' \in E(G)$. In this case, by (4.24), (4.25), (4.26), and Lemma 4.8.7, $G[N(y) + y] = K_k$ and $G[V(G) - N(y) - y]$ contains $K_{3,2k-3}$, a contradiction to its choice.

CASE 2: There is a high neighbor $w \in W \in \mathcal{A}'$ of y, and $wv' \notin E(G)$. Then $wv \in E(G)$ and so $N(v) = B - y + w$. We choose any $y' \in B - y$ and replace the color classes B, W and V^- with the classes $B - y'$, $\{w, v', y'\}$ and $W - w + v$. $\square$

4.10 $a' > b > 1$

In this section, we assume that $a' > b$. By Lemma 4.9.3, $b > 1$.

Lemma 4.10.1. $|L(B)| \leq b$.

Proof: Suppose $|L(B)| \geq b + 1$. Then by Lemma 4.6.1, $b = 2$, $a = 4$, $|L(B)| = 3$, $G[L(B)]$ is the disjoint union of at least two cliques and each class in $\mathcal{A}'$ has a solo vertex. Then $L(B)$ contains a vertex y isolated in $G[L(B)]$. Let $L(B) = \{y, y', y''\}$. Let $X = \{x, x'x''\} \in \mathcal{A}'$ contain a solo neighbor x of y. Then each

$y' \in B - y$ is adjacent to x' and x'' and so $||x', B|| = ||x'', B|| = 6$. It follows from degree conditions that

$$N(x') = N(x'') = B - y \text{ and all vertices in } H(B) \text{ are isolated in } G[B]. \tag{4.27}$$

In particular, the only possible edge in $G[B]$ is $y'y''$. Let $W = \{w, w'w''\} \in \mathcal{A}'$ contain a solo neighbor w of y'. Then $N(w) \cap B \subseteq \{y', y''\}$ and each of w' and w'' is adjacent to all vertices in $B - y' - y''$ and thus has at most one neighbor in A. We construct an equitable k-coloring of G as follows. Let $H(B) = \{u_1, \ldots, u_4\}$. The three classes involving vertices of B are $\{x, y', u_1\}$, $\{w, y, u_2\}$ and $\{y'', u_3, u_4\}$. One more class is $Z \in \mathcal{A}' - X - W$. In the subgraph G' induced by the remaining 6 vertices $V^- \cup \{w', w'', x', x''\}$, vertices x' and x'' are isolated, vertices w' and w'' have degree at most 1, and the vertices v and v' of V^- are not adjacent to each other. Every such forest on six vertices is equitably 2-colorable. □

Lemma 4.10.2. *Each class $X \in \mathcal{A}'$ has a solo vertex.*

Proof: Suppose x is the unmovable vertex in $X = \{x, x', x''\} \in \mathcal{A}'$ and x is not solo. Then $||X, B|| \geq 6b + 2$. Let $Y = N(x) \cap B$.

CASE 1: $|Y| \leq b - 1$. Then some of x', x'', say x', is adjacent to at least $\lceil \frac{|B-Y|}{2} \rceil \geq 2b + 3$ vertices in B. By Lemma 4.6.1, x' has a high neighbor y in B, a contradiction to $d(x') + d(y) \geq (2b+3) + (2a-1) = 2k+2$.

CASE 2: $|Y| \geq b + 1$. Then by Lemma 4.10.1, x has a high neighbor y in B, and so $d(x) \leq 2b + 2$. It follows that

$$||x, A|| \leq 2b + 2 - |Y| \leq b + 1 \leq a - 1. \tag{4.28}$$

Since x is unmovable, $||x, A|| \geq a - 1 \geq b + 1$. So $|Y| = b + 1$ and $a = b + 2$. Since $||\{x', x''\}, B|| \geq 6b + 2 - |Y| = 5b + 1$, if say x' has no high neighbors in B, then by Lemma 4.10.1, x'' has at least $5b + 1 - b > b$ neighbors in B and thus by the same lemma, $d(x'') \leq 2b + 2$. But then $||x', B|| \geq 5b + 1 - (2b + 2) = 3b - 1 > b$, a contradiction. Thus each of x' and x'' has a high neighbor in B, hence $d(x') \leq 2b + 2$ and $d(x'') \leq 2b + 2$. It follows that $||\{x', x''\}, A|| \leq 2(2b + 2) - 5b - 1 = 3 - b \leq 1$.

Let x' be the other neighbor of y in X. If $||x'', V^-|| = 0$, then move x'' into V^-. If every class in $\mathcal{A}' - X$ has a vertex movable to $X - x''$, then we get a new optimal coloring in which y has two neighbors in the small class, a contradiction. Thus there is $W \in \mathcal{A}' - X$ in which every vertex has a neighbor in $X - x''$. But only one can be adjacent to x' and, since x is unmovable, by (4.28) only one can be adjacent to x. Hence x'' has a neighbor V^-. Then x' has no neighbors in A. In this case, move x'' to some class Z (since x'' is movable) and move a witness of $ZV^- \in E(\mathcal{H})$ into V^-. Then either in some $W \in \mathcal{A}' - X$ each vertex is adjacent to x or both vertices in V^- are adjacent to x. Each of the possibilities contradicts (4.28).

CASE 3: $|Y| = b$ and x has a high neighbor y in B. Then we essentially repeat the argument of Case 2 with two changes: On the one hand, instead of (4.28), x may have two neighbors in a class $W \in \mathcal{A}' - X$; on the other hand, since $||\{x', x''\}, B|| \geq 5b + 2 \geq 2(2b + 2)$, neither of x' and x'' has neighbors in A. So, we simply move x'' into V^- and get that in some $W \in \mathcal{A}' - X$ each vertex is adjacent to x.

CASE 4: $|Y| = b$ and $Y \subseteq L(B)$. Then by Lemma 4.10.1, $Y = L(B)$. As in Case 3, since $||\{x', x''\}, B|| \geq 2|B| - b \geq 5b + 2$, in order to have $d(x') \leq 2b + 2$ and $d(x'') \leq 2b + 2$, we need $b = 2$ and $||\{x', x''\}, A|| = 0$. Moreover, since X has no solo vertices, the number of solo vertices in A is at most $a' - 1$. So, repeating the argument of Lemma 4.6.1, we obtain that each vertex in $L(B)$ has no neighbors in $H(B)$ and exactly one neighbor in V^-. Also, since $d(x') = 2b + 2$ each $y \in H(B)$ has exactly $2a - 1$ neighbors, and so $H(B)$ is independent. Let $L(B) = \{y, y'\}$ and $H(B) = \{y_1, \ldots, y_5\}$. Then the only possible edge in $G[B]$ is yy'. Since each $w \in B$ has exactly one neighbor in V^-, we may assume that $v' \in V^-$ has at most 3 neighbors in B and $v \in V^-$ has at least 4 such neighbors. If $N(v') \cap B$ does not contain $L(B)$, then assuming $yv' \notin E(G)$, we create new color classes $\{y, v', y_1\}$, $\{y', y_2, y_3\}$, $\{x, y_4, y_5\}$, $\{v, x', x''\}$ and keep the color classes in $\mathcal{A}' - X$. If $N(v') \cap B$ does contain $L(B)$ but contains also some $y_1 \in H(B)$, then we create new color classes $\{y, v, y_1\}$, $\{y', y_2, y_3\}$, $\{x, y_4, y_5\}$, $\{v', x', x''\}$ and keep the color classes in $\mathcal{A}' - X$.

So let $N(v') \cap B = L(B)$. If $yy' \notin E(G)$, then we create new color classes $\{y, v, y'\}$, $\{y_1, y_2, y_3\}$, $\{x, y_4, y_5\}$, $\{v', x', x''\}$ and keep the color classes in $\mathcal{A}' - X$.

Suppose $yy' \in E(G)$ and let $U = \{u_Z : Z \in \mathcal{A}'\}$ be the set of unmovable vertices in A'. Then each neighbor in $\mathcal{A}' - X$ of y or y' is solo, and thus $N(y) \cap A - X = N(y') \cap A - X = U + v'$. Let $Z \in \mathcal{A}'$ and $z = u_Z \in U$. By Lemmas 4.4.7 and 2.2.22, vertices in $H(B)$ are not adjacent to z and thus $N(w) \cap Z = Z - z$ for each $w \in H(B)$. So, $N(w) = (A' - x - U) + v$ for all $w \in H(B)$ and $N(z) \cap B = Y$ for all $z \in U + x + v'$.

If $v'z \notin E(G)$ for some $z \in U + x$, then since z is unmovable, $zv \in E(G)$. Thus $N(v) = H(B) + z$ and we can replace the color classes in $B \cup V^- \cup Z$ (where Z is the class of z) with $\{v', z, y_1\}, \{y, y_2, y_3\}$, $\{y', y_4, y_5\}$ and $Z - z + v$, a contradiction. So $v'u \in E(G)$ for each $u \in U + x$. Thus if $G[U + x]$ is a complete graph then G contains disjoint subgraphs $K_k = G[U \cup Y + x + v']$ and $K_{5,2k-5}$ with $H(B)$ as one of the partite sets, a contradiction to the choice of G. So, there are $z_1, z_2 \in U + x$ with $z_1z_2 \notin E(G)$. Let Z_i be the class of z_i for $i = 1, 2$. Since $|N(v) - B| \leq 1$, we may assume that $||v, Z_1|| = 0$. Then we replace the color classes in $B \cup V^- \cup Z_1 \cup Z_2$ with $\{z_1, z_2, y_1\}, \{v', y_2, y_3\}$, $\{y', y_4, y_5\}$, $Z_1 - z_1 + v$ and $Z_2 - z_2 + y$, a contradiction. ☐

Lemma 4.10.3. *For every unmovable $x \in A'$, $N(x) \cap B \subseteq L(B)$.*

Proof: Suppose some unmovable $x \in A'$ has a neighbor $y \in H(B)$. Let $X = \{x, x', x''\} \in \mathcal{A}'$ be the class of x. By Lemma 4.10.2, x has a solo neighbor $y' \in Y = N(x) \cap B$. By Lemma 2.2.22 $yy' \in E(G)$. Then $d(y) \geq 2a$ and so $d(x) \leq 2b + 1$. Since x is unmovable, $|Y| \leq d(x) - (a - 1) \leq b$. Each $w \in B - Y$ is adjacent to x' and x''. By symmetry, we may assume $x'y \in E(G)$. Then $d(x') \geq 1 + |B - Y| \geq 2b + 2$ and so $d(x') + d(y) \geq 2k + 2$, a contradiction. $\square$

Lemma 4.10.4. $b \geq a'$.

Proof: Suppose $b \leq a' - 1$, and recall $\mathcal{H}$ is a star, $b \geq 2$ and every $X \in \mathcal{A}'$ has a solo vertex u_X by Lemmas 4.7.1, 4.9.3, and 4.10.2. Suppose $X = \{x, x', x''\} \in \mathcal{A}'$ and $x = u_X$. Let $Y = N(x) \cap B$. By Lemma 4.10.3, $Y \subseteq L(B)$. By Lemma 4.10.1, $|L(B)| \leq b$. Since each $w \in B - Y$ is adjacent to both x' and x'' and $B - Y$ contains a high vertex, $|B - Y| \leq d(x'), d(x'') \leq 2b + 2$, which yields $|Y| \geq b - 1$.

CASE 1: $|Y| = b - 1$. Then $N(x') = N(x'') = B - Y$ and so $N(y) \cap X = \{x\}$ for every $y \in Y$. By Lemma 4.4.7, $G[Y] = K_{|Y|}$. Also the vertices in $H(B)$ are isolated in $G[B]$ and x' and x'' are isolated in $G[A]$. Let $Z = \{z, z', z''\} \in \mathcal{A}' - X$. By Lemma 4.10.1, $|L(B) - Y| \leq 1$. If $L(B) - Y \neq \emptyset$, then let $y' \in L(B) - Y$, otherwise, let y' be any vertex in $B - Y$. Let $Y = \{y_1, \ldots, y_{b-1}\}$ and $B - Y - y' = \{w_1, \ldots, w_{2b+1}\}$. The color classes in our new coloring will be all classes in $\mathcal{A}' - X - Z$, $V^- + x'$, $Z - z + x''$, $\{x, y', w_{2b+1}\}$, $\{z, w_{2b-1}, w_{2b}\}$ and for every $1 \leq i \leq b - 1$, the class $\{y_i, w_{2i-1}, w_{2i}\}$.

CASE 2: $|Y| = b$ for every choice of a solo $x \in A'$. Then by Lemma 4.10.3, $Y = L(B)$ for each choice of x. Let U be the set of unmovable vertices in A' and $M = A' - U$. Then by the case, $N(w) \cap A' = M$ for every $w \in H(B)$. We claim that among the colorings of A there is such that the vertices v and v' in V^- satisfy

$$N(v) \supseteq H(B) \text{ and } v' \text{ is a low unmovable vertex with } N(v') \cap B = Y. \tag{4.29}$$

Indeed, by Lemma 4.8.7, there is $X = \{x, x', x''\} \in \mathcal{A}'$ with low $x = u_X$. By the first paragraph, $||x', A|| \leq 1$ and $||x'', A|| \leq 1$. We may assume that x' is a witness of $XV^- \in E(\mathcal{H})$. Move x' to V^-. If every class in $\mathcal{A}' - X$ has a vertex movable to $X - x'$, then we get a coloring satisfying (4.29), so suppose the contrary: that there is $Z \in \mathcal{A}'$ in which every vertex is adjacent to $\{x, x''\}$. At most one of them is adjacent to x'' and since x is low and unmovable,

$$||x, Z|| \leq k - ||x, B|| - ||x, A - Z|| \leq (a + b) - b - (a - 2) = 2.$$

So, the only possibility of failure is that $||x'', Z|| = 1$, $||x, Z|| = 2$ and $||x, W|| = 1$ for every $W \in \mathcal{A} - X - Z$. But then instead of x' we can move x'' to V^-, and by the previous sentence we do not fail this time.

By (4.29), $N(w) = M + v$ for every $w \in H(B)$; in particular, $G[H(B) \cup M + v] \supseteq K_{2b+1,2a-1}$. Since G is a counter-example, $G[U \cup Y + v']$ has two nonadjacent vertices u and u'. Let $Y = \{y_1, \ldots, y_b\}$ and $B - Y = \{w_1, \ldots, w_{2b+1}\}$. By the case and (4.29), either $\{u, u'\} \subseteq Y$ or $\{u, u'\} \cap Y = \emptyset$.

CASE 2.1: $u = y_b$ and $u' = y_{b-1}$. Since $||v, A|| \leq 1$, we may choose $X = \{x, x', x''\} \in \mathcal{A}'$ with $||v, X|| = 0$. The color classes of our new coloring will be all classes in $\mathcal{A}' - X$, $\{u, u', w_{2b+1}\}$, $\{v', w_{2b-1}, w_{2b}\}$,

$\{x, w_{2b-3}, w_{2b-2}\}$, $\{v, x', x''\}$ and for every $1 \le i \le b-2$, the class $\{y_i, w_{2i-1}, w_{2i}\}$.

CASE 2.2: $\{u, u'\} \subset A$ and $v' \notin \{u, u'\}$. Then there are $X = \{x, x', x''\} \in \mathcal{A}'$ and $Z = \{z, z', z''\} \in \mathcal{A}'$ such that $u = x = u_X$ and $u' = z = u_Z$. For a new coloring, we use an equitable b-coloring of $B - w_1$ (which exists by Lemma 4.4.5), the color classes of $\mathcal{A}' - X - Z$, the class $\{x, z, w_1\}$ and consider the remaining set $D = \{x', x'', z', z'', v', v\}$. By construction, each vertex in $D - v'$ has at most one neighbor in A, and by (4.29), $||v', D|| \le 3$. Such a forest is equitably 2-colorable, unless v' is adjacent to three of x', x'', z', z'' and v is adjacent to the fourth, say, z''. In the last case, v' has at most one neighbor in each class in $\mathcal{A}' - X - Z$, and the remaining vertices in D have no neighbors in $A - X - Z$. In this case, we switch v with a movable vertex r in a class $R \in \mathcal{A}' - X - Z$ and color $D - v + r$ equitably with two colors.

CASE 2.3: $u = v'$ and $u' \in X = \{x, x', x''\} \in \mathcal{A}'$. Let $u' = x$. Since x is unmovable, $xv \in E(G)$ and so $||v, A - x|| = 0$. Similarly to Case 2.3, we use an equitable b-coloring of $B - w_1$, the color classes of $\mathcal{A}' - X$, and the classes $\{x, v', w_1\}$ and $X - x + v$. □

4.11 Preliminaries and small cases for $b \ge a'$

By Lemma 4.6.7, $a' = a - 1$, and by Lemma 4.7.1 $\mathcal{F}$ is a star. We will use some analogues of lemmas in Sections 4.6 and 4.8, but proofs and some notions will somewhat differ.

Note that the definition of L' is changed in Section 4.6. An analog of Lemma 4.8.1 is:

Lemma 4.11.1. *If $b \ge a' = a - 1$, then each class $X \in \mathcal{A}'$ has a neighbor in V^-.*

Proof: Suppose $V^- = \{v, v'\}$, $X = \{x, x', x''\} \in \mathcal{A}'$ and $V^- \cup X$ is independent. Then each $y \in B$ has at least 4 neighbors in $V^- \cup X$, so some $w \in V^- \cup X$ has at least $\left\lceil \frac{4(3b+1)}{5} \right\rceil \ge 2b + 2$ neighbors in B. By Lemma 4.6.3, at least one, say y_0, of these neighbors is in $H'(B)$. Since $||y_0, V^- \cup X|| \ge 4$, either $||y_0, X|| = 3$ or $||y_0, V^-|| = 2$. Then by the definition of petite vertices, $d(y_0) \ge a + a' + 1 = 2a$. Thus $d(v) + d(y_0) \ge (2b + 2) + 2a = 2k + 2$, a contradiction. □

An analog of Lemma 4.8.5 is:

Lemma 4.11.2. *If $b \ge a' = a - 1$, then each class $X \in \mathcal{A}'$ contains an unmovable vertex w_X.*

Proof: Suppose that $X = \{x, x', x''\} \in \mathcal{A}'$ has no unmovable vertices. Then X has no solo vertices and $||X, B|| \ge 6b+2$. Rename the vertices in X so that $||x, B|| \le ||x', B|| \le ||x'', B||$. Then $||\{x', x''\}, B|| \ge 4b+2$ and $||x'', B|| \ge 2b + 1$. If there is $w \in \{x', x''\}$ with $d(w) \ge 2b + 3$, then $d(y) \le 2k + 1 - 2b - 3 = 2a - 2$ for every $y \in N(w)$. In particular, each $y \in B \cap N(w)$ is petite. So by Lemma 4.11.1, $||w, B|| \le a' \le b$. Then by the ordering of x, x' and x'', $||w, B|| \le a' \le b$. Therefore, $||x'', B|| \ge 6b + 2 - b - b > |B|$, a contradiction.

So,

$$d(x') \leq 2b+2 \text{ and } d(x'') \leq 2b+2. \tag{4.30}$$

If $||\{x', x''\}, A|| \geq 3$, then $d(x') + d(x'') \geq 4b+2+3$ and so $d(w) \geq 2b+3$ for some $w \in \{x', x''\}$, a contradiction to (4.30). Thus $||\{x', x''\}, A|| \leq 2$. Since x is movable, move it to a class U with no conflict, and if $U \neq V^-$, then move a witness from U to V^-. By the case, every class in the new coloring f' has a vertex movable to $X' = X - x$. So, f' is optimal. By Lemma 4.11.1 for f' and again by the case, $a \leq 3$. By Lemma 4.11.1 for the original coloring, $a \geq 3$. So $a = 3$ and $||\{x', x''\}, A|| = 2$. Then $d(x')+d(x'') \geq 4b+2+2$ and by (4.30), $d(x') = d(x'') = 2b+2$. Since $B \subset N(x') \cup N(x'')$, $d(y) \leq 2k+1-2b-2 = 2a-1 = 5$ for every $y \in B$. Since $||y, X|| \geq 2$, we have $||y, A|| \geq a+1 = 4$ and thus $||y, B|| \leq 1$ for every $y \in B$. Let $B' = \{y \in B : ||y, B|| = 1\}$ and $Z = \{z, z', z''\}$ be the other class in $\mathcal{A}'$. If $B' \neq \emptyset$, then for every $y \in B'$, $||y, A|| = 4$ and y has a solo neighbor z in Z. So $G[B']$ is a complete graph, and therefore $|B'| = 2$.

CASE 1: $B' = \{y, y'\}$. Then $N(z) \cap B = B'$, since each neighbor $y'' \in B - B'$ of z either has 3 neighbors in Z or is adjacent to y and y' (in both cases, contradicting $d(y'') \leq 5$). But each nonneighbor of z in B is adjacent to z' and z'', making $||z', B||, ||z'', B|| \geq 3b-1$. This is more than $a' = 2$, so each of them has a non-petite neighbor in B; thus $d(z'), d(z'') \leq 2b+2$. It follows that $b \leq 3$.

Suppose first that $b = 3$. Then $d(z') = d(z'') = ||z', B|| = ||z'', B|| = 2b+2 = 8$. It follows that z has a neighbor in $\{x', x''\}$, and the second neighbor of $\{x', x''\}$ in A is in V^-. Since all vertices of X are movable, we may assume that $zx' \in E(G)$ and $x''v \in E(G)$, where $v \in V^-$. If x is movable to Z, then instead of moving x there, we move x' to V^-. Then Z has no neighbors in $X - x'$, a contradiction to Lemma 4.11.1 for the new coloring. Otherwise, x is adjacent to z and by the case, is movable to V^-. Then move x'' to Z and z' to V^-. In the new coloring, $X - x''$ has no neighbors in $V^- + z'$, again a contradiction to Lemma 4.11.1. Thus $b = 2$.

Then each of z', z'' has at most one neighbor in A. Moreover, if say z' has a neighbor in A, then $d(z') \geq 5+1 > k = 5$, and so this neighbor is not in $\{x', x''\}$. So we may assume $x'z \in E(G)$. Since z is unmovable and in the coloring f' defined above both size 3 classes have had neighbors in $\{x', x''\}$, we may assume $x''v \in E(G)$. Since $d(w) \leq 5$ for every $w \in B$, each such w has exactly one neighbor in V^- and exactly two neighbors in X. So $||x, B|| = 4$ and there is $y_0 \in B - B'$ not adjacent to x. So if $xz \notin E(G)$, then we color $B - y_0$ with two colors and add the class $\{y_0, x, z\}$. In the subgraph G' of G induced by $\{x', x'', z', z'', v, v'\}$, x' is isolated, x'', z', z'' have degrees at most 1 and v is not adjacent to v'. Each such 6-vertex forest is equitably 2-colorable. Thus $xz \in E(G)$. Since we already know 5 neighbors of x, it is not adjacent to $\{z', z''\}$, because high vertices are not adjacent to each other. So if x'' has a nonneighbor $y_0 \in B - B'$, then again can do the recoloring with the roles of x and x'' switched. Therefore, $N(x'') = B - B' + v$. Since

v is adjacent to the high vertex x'', there is a nonneighbor y_0 of v in $B-B'$ and, as above, we can color $B-y_0$ with two colors, add the class $\{z,v,y_0\}$ and color $G[X\cup Z-z+v']$ with two colors. So $vz\in E(G)$. We now know 5 neighbors of z and x' is high. Then $v'z\notin E(G)$. Again if there is a nonneighbor y_0 of v' in $B-B'$, then we color $B-y_0$ with two colors and add the classes $\{z,v',y_0\}$, $\{x,x',v\}$ and $\{x'',z',z''\}$. So $N(v')\supseteq B-B'$. Then v' has no neighbors in A. Also, since each $y\in B$ has only one neighbor in V^-, $(B-B')\cap N(v)=\emptyset$. Since $N(x')\cup N(x'')\supset B$, there is $y_0\in N(x'')-N(x')=(B-B')-N(x')$. We color $B-y_0$ with two colors, and add classes $\{y_0,x',v\}$, Z, and $X-x'+v'$.

CASE 2: $B'=\emptyset$ and some $y_0\in B$ has a solo neighbor $z\in Z$. Since $B'=\emptyset$, $N(z)\cap B=\{y_0\}$. Thus $N(z')\cap B=N(z'')\cap B=B-z$. Since z' and z'' have non-petite neighbors in B, $d(z')\le 2b+2$ and $d(z'')\le 2b+2$. It follows that $b=2$ and z' and z'' have no neighbors in A. So, as in Case 1, the set $\{x',x''\}$ has neighbors in both, Z and V^-. Since each of x' and x'' is movable, we may assume $x'z,x''v\in E(G)$. If $xz\notin E(G)$, then choose a nonneighbor y of x in $B-y_0$, color $B-y$ with 2 colors and add color classes $\{x,z,y\}$, $X-x+v'$ and $Z-z+v$. So $xz\in E(G)$ and therefore x has no neighbors in V^-. Then choose a nonneighbor y' of x'' in $B-y_0$, color $B-y'$ with 2 colors and add color classes $\{x'',z,y'\}$, $X-x''+v'$ and $Z-z+v$.

CASE 3: $B'=\emptyset$ and no vertex in B has any solo neighbor in A'. Then by the degree restrictions, each $y\in B$ has a solo neighbor in V^-. Thus if each of v and v' has at most $|B|-2$ neighbors in $B=\{y_1,\dots,y_{3b+1}\}$, then we can choose y_1,y_2 not adjacent to v, and y_3,y_4 not adjacent to v and form an equitable coloring of G as follows: keep all color classes in $\mathcal{A}'$ and add classes $\{v,y_1,y_2\}$, $\{v',y_3,y_4\}$, $\{y_5,y_6,y_7\},\dots,\{y_{3b-1},y_{3b},y_{3b+1}\}$. So we may assume that $|B\cap N(v')|\ge 3b$. Since B has at least two non-petite vertices, $d(v')\le 2b+2$. It follows that $b=2$ and there is $y_0\in B$ such that $N(v')=B-y_0$. Since $\{x'x''\}$ has a neighbor in Z, we may assume that x' has a neighbor in Z. Then x' has no neighbor in V^- and at most 5 neighbors in B. In particular, there is $y\in B-y_0$ not adjacent to x'. Then we color $B-y$ with two colors and add classes $\{y,v,x'\}$, Z, and $X-x'+v'$. □

Lemma 4.11.3. $a\ge 3$.

Proof: Suppose $\mathcal{A}=\{X,V^-\}$, $V^-=\{v,v'\}$ and $X=\{x,x',x''\}$. By Lemma 4.11.1, we may assume that $xv\in E(G)$. Since each vertex in X having a neighbor in V^- is unmovable, $||\{x',x''\},V^-||=0$. If also $xv'\in E(G)$, then moving x' to V^- we obtain an optimal coloring in which the class V^-+x' has two unmovable vertices, a contradiction. So, the only edge in $G[A]$ is xv. By symmetry, we may assume $d(x)\le k$. If a vertex $y\in B$ has a nonneighbor $w\in\{x,v\}$ and a nonneighbor $w'\in A-\{x,v\}$, then we color $G[B-y]$ equitably with b colors and add classes $\{y,w,w'\}$ and $A-w-w'$, a contradiction. Thus $B=Y_1\cup Y_2$ where $Y_1=B\cap N(x)\cap N(v)$ and $Y_2=B\cap N(x')\cap N(x'')\cap N(v')$. Since x is low, $|Y_1|\le k-1=b+1$. If there is $y \in Y_1 \cap Y_2$ then $d(y) \geq 5$ and $|Y_2| \geq |B| - |Y_1| + 1 \geq 2b + 1$. So $d(y) + d(x') \geq 5 + 2b + 1 = 2k + 2$, a contradiction. Thus $|Y_2| = 3b + 1 - |Y_1|$.

CASE 1: $|Y_1| \leq b - 1$. Then $|Y_2| \geq 2b + 2$. Since $|Y_2| \leq d(v') \leq (2k + 1) - 3 = 2b + 2$, we conclude that $|Y_2| = 2b+2$ and $N(w) = Y_2$ for each $w \in \{x', x'', v'\}$ and $N(y) = \{x', x'', v'\}$ for each $y \in Y_2$. Then we color G as follows: one class is $\{x', x'', v'\}$ and every other class consists of one vertex in $B - Y_2 + x + v$ and two vertices in Y_2.

CASE 2: $|Y_1| = b$. Then $|Y_2| = 2b + 1$. If $G[Y_1 + x + v]$ is complete, then G is as the theorem claims. So suppose there are nonadjacent $y, y' \in Y_1$. Since they cannot be both solo neighbors of x and we can permute v' with x' or x'', some of y, y' has at least two neighbors in $\{x', x'', v'\}$. Then for such vertex , say y, and any its neighbor $w \in \{x', x'', v'\}$ we have $d(y) + d(w) \geq 4 + (|Y_2| + 1) = 2k + 2$, a contradiction.

CASE 3: $|Y_1| = b + 1$. Then $|Y_2| = 2b + 1$. Since $G[B]$ is b-colorable, there are nonadjacent $y, y' \in Y_1$. As in Case 2, we may assume that y is adjacent to x' and x''. Since Y_1 and Y_2 are disjoint, $yv' \notin E(G)$. If $y'v' \notin E(G)$ then after swapping v' with x', y' will be a solo neighbor of x and y a 1/2-neighbor not adjacent to y', a contradiction to Lemma 13 of the main text. Thus $y'v' \in E(G)$. Since $b + 1 \geq 3$, there is $y'' \in Y_1 - y - y'$. Since $d(y) \leq 2k+1-d(x') \leq 2k+1-2b-1 = 4$, $yy' \notin E(G)$ and $N(x') = N(x'') = Y_2 + y$. So y'' is a solo neighbor of x and, as above, $y''v' \in E(G)$. Also by Lemma 4.4.7, $y'y'' \in E(G)$ and so $d(y'') + d(v') \geq 4 + |Y_2| + 2 = 2k + 2$, a contradiction. □

Lemma 4.11.4. *If $b \geq a' = a - 1$, then for each unmovable $x \in X \in \mathcal{A}'$,*
$b_1(x) + b_2(x) \geq b - 1$.

Proof: Let x be unmovable in $X = \{x, x', x''\} \in \mathcal{A}'$. Since x' and x'' have no solo neighbors, each $y \in B_0(x) \cup B_3(x)$ is adjacent to x' and x''. If $b_1(x) + b_2(x) \leq b - 2$, then $b_0(x) + b_3(x) \geq 2b + 3$. Then by Lemma 4.6.2, there is $y' \in (B_0(x) \cup B_3(x)) \cap H'(B)$. So $d(y') + d(x') \geq (2a - 1) + (2b + 3) = 2k + 2$, a contradiction. □

Lemma 4.11.5. *If $b \geq a' = a - 1$, then no unmovable vertex in A' is adjacent to all 3 vertices in some color class in $\mathcal{A}'$.*

Proof: Suppose $X = \{x, x', x''\} \in \mathcal{A}'$, x is unmovable and $N(x) \supset Z = \{z, z', z''\} \in \mathcal{A}'$ with unmovable z.

CASE 1: z is not solo. Then $||B, Z|| \geq 2|B|$ and $||z', B||, ||z'', B|| \leq 2b + 1$. So, $d(z) \geq (a - 1) + 2b$ and by Lemma 4.11.4, $d(x) \geq (a + 1) + b - 1 = k$. Thus $2k + 1 \geq d(x) + d(z) \geq k + a - 1 + 2b = 2k + b - 1$. It follows that $b = 2, a = 3, k = 5$, $d(z) = (a - 1 + 2b) = 6$, $d(x) = 5$ and $b_1(x) + b_2(x) = b - 1 = 1$. Let $B_1(x) \cup B_2(x) = \{y_0\}$. Let $B' = B - y_0$. If $y_0 \in B_2(x)$, then each $y \in B'$ is adjacent to x' and x'', and also

y_0 is adjacent to one of them. Thus in this case $||x', B|| + ||x'', B|| \geq 6b + 1 = 13$, and some of x' and x'' has at least 7 neighbors in B, a contradiction to Lemma 4.6.3. Therefore, $y_0 \in B_1(x)$. Then again each $y \in B'$ is adjacent to x' and x''. Thus each $y \in B'$ has two neighbors in $X - x$, at least two neighbors in Z, and at least one neighbor in V^-. Since $d(y) \leq 2k + 1 - d(x') \leq 5$, we conclude that $d(y) = 5$ for every $y \in B'$, and $N(x') = N(x'') = B'$. In particular, $b_3(x) = 0$ and B is independent. Let v be the vertex in V^- adjacent to z. Then v has at most 5 neighbors in B, since otherwise $d(v) \geq 7$. Let y_1, y_2 be two nonneighbors of v in B and y_3, y_4 be two other nonneighbors of x in B (recall that $||x, B|| = 1$). The classes of our coloring will be $\{y_1, y_2, v\}$, $\{y_3, y_4, x\}$, $Y - y_1 - y_2 - y_3 - y_4$, Z, and $X - x + v'$.

CASE 2: z is solo and $b_1(z) + b_2(z) \leq b$. Since each $y \in B_0(z) \cup B_3(z)$ is adjacent to z' and z'', by Lemma 4.6.3(b), $b_0(z) + b_3(z) \leq 2b + 1$. So $b_1(z) + b_2(z) = b$ and $N(z') = N(z'') = B_0(z) \cup B_3(z) + x$. In particular, $B_2(z) = \emptyset$ and $|B_1(z)| = b$. Since x is adjacent to vertex z' of degree $2b + 2$, $||x, B|| \leq (2k + 1) - (2b + 2) - ||x, A|| \leq a - 2$. So by Lemma 4.11.4, $a = b + 1$, $b_1(x) + b_2(x) = b - 1$, $||x, A|| = a + 1$ and $b_3(x) = 0$. In particular, x has exactly one neighbor in each class of $\mathcal{A} - X - Z$. Since each of the $2b + 2$ vertices in $B_0(x)$ is adjacent to x' and x'', by Lemma 4.6.3, $N(x') = N(x'') = B_0(x)$. In particular, $b_2(x) = 0$. Let $y_1 \in B_1(z) - B_1(x)$. Then y_1 has two neighbors of degree $2b + 2$ in X. Hence, on the one hand, $d(y_1) \leq 2a - 1 = k$ and on the other hand,

$$d(y_1) \geq ||y_1, B|| + ||y_1, A|| \geq (b - 1) + (a + 1) = k.$$

So, $d(y_1) = k$, thus y_1 has exactly one neighbor in each class of $\mathcal{A} - X$ and no neighbors in $B - B_1(z)$. If x and y_1 have a common nonneighbor w in $A - X$, then we color w, y_1 and x with the same color, color $B - y_1$ with b colors, move x' to the class of w and x'' to V^-. So this is not the case. But by the above, each class in $\mathcal{A} - X - Z$ has at most one neighbor of x and at most one neighbor of y'. It follows that $a = 3$, $b = 2$, and we may assume that $vx, v'y_1 \in E(G)$. Let y_0 be the only neighbor of x in B. If v' has a nonneighbor $y \in B - y_0$, then as above, we color v', y_0 and x with the same color, color $B - y_0$ with 2 colors, keep Z and merge $X - x$ with $V^- - v'$. We conclude that $N(v') \supset B - y_0$. Since v' is adjacent to y_1 of degree 5, $N(v') = B - y_0$. Every $y \in B - B_1(z) - y_0$ is adjacent to two vertices in X (of degree 6), two vertices in Z and to v'. So each such y has no neighbors in $B + v$. Since for each $y_1 \in B_1(z) - y_0$, the only neighbor of y_1 in B is the other vertex in $B_1(z)$, $G[B]$ has only one edge, namely, between the two vertices in $B_1(z)$. So, we color y_1, x and a vertex $y' \in B - B_1(z) - y_0$ with one color, v and two vertices in $B - B_1(z) - y_0 - y'$ with the second color, the remaining 3 vertices in B with the third color, keep Z and use $X - x + v'$.

CASE 3: z is solo and $b_1(z)+b_2(z) \geq b+1$. Let y_0 be a solo neighbor of z. By Lemmas 4.4.7 and 2.2.22,

$$2k+1 \geq d(y)+d(z) \geq (a+b_1(z)+b_2(z)-1)+(a-1+b_1(z)+b_2(z)) = 2k+2(b_1(z)+b_2(z)-(1+b)).$$

So $b_1(z)+b_2(z) = b+1$, all neighbors of y_0 in B are in $B_1(z) \cup B_2(z)$ and $||y_0, A|| = a$. The last equality means that every neighbor of y_0 in A is its solo neighbor and is unmovable. In particular, x is a solo neighbor of y_0 and $B_1(x) \supseteq B_1(z)$.

Since $G[B]$ is b-colorable, $B_1(z) \cup B_2(z)$ contains two nonadjacent vertices, say y_1 and y_2. Since they are not adjacent, they both are in $B_2(z)$, and each of them has a neighbor in $Z-z$. Apart from this, z' and z'' are both adjacent to every vertex in $B' = B_0(z) \cup B_3(z)$. So $||z', B|| \geq 2b+1$ and $||z'', B|| \geq 2b+1$. By Lemma 4.6.3(b), this yields $||z', B|| = ||z'', B|| = 2b+1$ and $d(z') = d(z'') = 2b+2$. If there would be a third vertex y_3 in $B_2(z)$, then the degree of either z' or z'' would exceed $2b+2$. Thus $B_2(z) = \{y_1, y_2\}$ and $G[B_1(z) \cup B_2(z)] = K_{b+1} - y_1y_2$. In particular, $|B_1(z)| = b-1$.

As in Case 2, since x is adjacent to z' of degree $2b+2$, we get $||x, B|| \leq a-2$ and thus by Lemma 4.11.4, $a = b+1$, $b_1(x)+b_2(x) = b-1$, $||x, A|| = a+1$ and $b_3(x) = 0$. In particular, x has exactly one neighbor in each class of $\mathcal{A} - X - Z$. Since each of the $2b+2$ vertices in $B_0(x)$ is adjacent to x' and x'', by Lemma 4.6.3, $N(x') = N(x'') = B_0(x)$. In particular, $b_2(x) = 0$. So $B_1(x) = B_1(z)$ and $y_1x, y_2x \notin E(G)$. It follows that $||y_1, X|| = 2$ and

$$d(y_1) = ||y_1, B|| + ||y_1, Z \cup X|| + ||y_1, A - Z - X|| \geq (b-1)+4+(a-2) = k+1,$$

a contradiction to the fact that y_1 is adjacent to x' of degree at least $2b+2$. □

4.12 Super-optimal colorings

A vertex $x \in A$ is *free* if it has no neighbors in A.

An optimal coloring f is *super-optimal* if

(C5) it has the most free vertices in V^- among all optimal colorings, and

(C6) modulo (C5), as many as possible color classes of f contain free vertices.

Lemma 4.12.1. *Let f be a super-optimal coloring and $b \geq a' = a-1$. If $X = \{x, x', x''\} \in \mathcal{A}'$ has two free vertices, x' and x'', then*

(a) every class in $\mathcal{A}$ has a free vertex, and

(b) all unmovable vertices in A are adjacent to each other.

Proof: Let $X = \{x, x', x''\} \in \mathcal{A}'$ have free vertices x' and x''. Then x is unmovable. If V^- has no free vertices, then we move x' to V^-. By Lemma 4.11.5, every color class in $\mathcal{A}' - X$ has a vertex movable to $X - x'$. This would contradict the super-optimality of f. Similarly, if some $Z = \{z, z', z''\} \in \mathcal{A}' - X$ has no free vertices, then we choose any nonneighbor $z \in Z$ of x (which again exists by Lemma 4.11.5) and switch it with x'. By (C6), the new coloring will contradict the super-optimality of f. This proves (a).

Since x is the only non-free vertex in its class, it is adjacent to each unmovable vertex in A. The same holds for the unmovable vertex, say v, in V^-. By (a), if x is high, then we can switch it with v and have a super-optimal coloring in which x is low. Suppose there are $Z = \{z, z', z''\} \in \mathcal{A}' - X$ and $W = \{w, w', w''\} \in \mathcal{A}' - X - Z$ with unmovable z and w such that $zw \notin E(G)$. By (a), we may assume that z'' and w'' are free. Then, since z and w are unmovable, $zw', z'w \in E(G)$. If w is the only neighbor of z' in A, then switching z' with x' creates an optimal coloring with no unmovable vertices in W (which contradicts Lemma 4.11.2). So $||z', A|| \geq 2$. Similarly, $||w', A|| \geq 2$. Since $(d(w) + d(z')) + (d(w') + d(z)) \leq 2(2k + 1)$, by the symmetry between W and Z, we may assume that $d(w) + d(w') \leq 2k + 1$, and if equality holds, then $d(w) \leq k$. So

$$||\{w, w'\}, B|| \leq 2k + 1 - ||\{w, w'\}, A|| \leq 2k + 1 - (a - 1) - 2 = 2b + a.$$

If W has no solo neighbors, then $||w'', B|| \geq 2|B| - 2b - a = 2 + 4b - a \geq 3b + 1 \geq 2b + 3$, a contradiction. Suppose now that y is a solo neighbor of w in B. Let U a class in $\mathcal{A}$ to which we can move z'. Since $z'w \in E(G)$, $U \neq W$. We color $B - y$ with b colors, move y to W, w to Z, z' to U, and if $U \neq V^-$, then move a witness from U to V^-. $\square$

Lemma 4.12.2. *Let f be a super-optimal coloring and $b \geq a' = a - 1$. Each $X = \{x, x', x''\} \in \mathcal{A}'$ contains at most one free vertex.*

Proof: Let $X = \{x, x', x''\} \in \mathcal{A}'$ have free vertices x' and x''. Then x is unmovable. By Lemma 4.12.1, we may assume that V^- contains a free vertex v' and unmovable v adjacent to x. If x is high, then we switch x with v and further assume x is low. Let $Y = N(x) \cap B$ and $Y' = B - Y$. Since $Y' \subseteq N(x')$, $|Y'| \leq 2b + 2$. So, since x is low, $b - 1 \leq |Y| \leq b + 1$.

Since swapping two free vertices does not break super-optimality. So, if some $y \in Y'$ is not adjacent to some free w, then swapping w with x' creates a super-optimal coloring in which y has the movable solo neighbor x'' in $X - x' + w$, a contradiction. And every vertex in B adjacent to a movable vertex in some $W \in \mathcal{A}'$ has another neighbor in this class. Thus

$$\text{each } y \in Y' \text{ is adjacent to each free } w \in A \text{ and to at least two vertices in each } W \in \mathcal{A}'. \tag{4.31}$$

Let F be the set of free vertices in A. By Lemma 4.12.1, $|F| \geq a+1$.

CASE 1: $|Y| = b-1$. Then $N(x') = N(x'') = Y'$. By this and (4.31), all vertices of Y' are isolated in $G[B]$ and are adjacent in V^- only to the vertex v'. Let $Y' = \{y'_1, \dots, y'_{2b+2}\}$. Create a coloring of G as follows: (i) color $B - \{y'_1, y'_2, y'_3, y'_4\}$ with $b-1$ colors putting into each class one vertex from Y and two vertices from $Y' - \{y'_1, y'_2, y'_3, y'_4\}$, (ii) add classes $\{y'_1, y'_2, v\}$ and $\{x, y'_3, y'_4\}$, (iii) move v' to $X - x$, (iv) keep the classes in $\mathcal{A} - X - V^-$ as they are.

CASE 2: $|Y| = b$. By (4.31), for each $y' \in Y'$, $d(y') \geq 2a-1$.

CASE 2.1: For each $y' \in Y'$, $d(y') = 2a-1$. Then again by (4.31), each $y' \in Y'$ has exactly two neighbors in each class of $\mathcal{A}'$ and is adjacent to v'. In particular, y' is isolated in $G[B]$ and v has no neighbors in Y'. If some $y_1, y_2 \in Y$ are not adjacent, then we create a coloring of G as follows: (i) color $B - \{y'_1, y'_2, y'_3, y'_4, y'_5, y_1, y_2\}$ with $b-2$ colors putting into each class one vertex from $Y - y_1 - y_2$ and two vertices from $Y' - \{y'_1, y'_2, y'_3, y'_4, y'_5\}$, (ii) add classes $\{y'_1, y'_2, v\}$, $\{x, y'_3, y'_4\}$, and $\{y_1, y_2, y'_5\}$, (iii) move v' to $X - x$, (iv) keep the classes in $\mathcal{A} - X - V^-$ as they are.

So we may assume $G[Y] = K_b$. Suppose now that some $y \in Y$ is not adjacent to some unmovable $z \in A$ and to a free $w \in A$. Then we can switch w with a free z' in the class Z of z, and the remaining movable vertex $z'' \in Z - z - z'$ will be the solo neighbor of y in $Z - z' + w$, a contradiction. Thus either y is adjacent to all unmovable vertices in A or to all vertices in F. In the latter case, since $|F| \geq a+1$ and $xy \in E(G)$,

$$2k+1 \geq d(y) + d(x') \geq (||y, Y|| + ||y, F|| + 1) + (2b+2) \geq (b-1) + (a+2) + (2b+2) \geq 2k+2.$$

So, denoting by U the set of unmovable vertices, $G[Y \cup U] = K_k$.

If for each $y' \in Y'$, $N(y') = A - U$, then G contains disjoint K_k (induced by U) and $K_{2b+1,2a-1}$ (with partite sets Y' and $A - U$), a contradiction. So there is $y'_1 \in Y'$ and a class $Z = \{z, z', z''\}$ with unmovable z such that $N(y'_1) \cap Z = \{z, z'\}$. Then by (4.31), z' is free and z'' is not free. If there is $y_1 \in Y$ with $z''y_1 \notin E(G)$, then we color G as follows: (i) color $B - \{y'_1, y'_2, y'_3, y_1\}$ with $b-1$ colors putting into each class one vertex from $Y - y_1$ and two vertices from $Y' - \{y'_1, y'_2, y'_3\}$, (ii) add classes $\{y'_1, y_1, z''\}$ and $\{x, y'_2, y'_3\}$, (iii) move x' to $Z - z''$ and x'' to V^-, (iv) keep the classes in $\mathcal{A} - X - V^- - Z$ as they are.

So z'' is adjacent to all of Y. If $z''x \notin E(G)$ then we color $B - y'_1$ with b colors, add color class $\{y'_1, z'', x\}$, move x' to $Z - z''$ and x'' to V^- and keep the remaining classes as they are. Similarly, if $z''v \notin E(G)$ then we color $B - y'_1$ with b colors, add color class $\{y'_1, z'', v\}$, move v' to $Z - z''$ and keep the remaining classes in $\mathcal{A}$ as they are. Thus $N(z'') \supseteq Y + x + v$, and hence $||z'', Y'|| \leq 2b + 2 - b - 2 = b$. Also $d(y) = k$ for each $y \in Y$, and hence $d(z) \leq k+1$. So $||z, Y'|| \leq k + 1 - ||z, U \cup Y|| = 2$. Since $|Y'| = 2b+1 > b+2 \geq ||z'', Y'|| + ||z, Y'||$, there is y'_2 not adjacent to both, z and z''. Then we color $B - y'_2$ with b colors, add color class $\{y'_2, z'', z\}$, move z' to V^- and keep the remaining classes in $\mathcal{A}$ as they are.

CASE 2.2: There is $y' \in Y'$ with $d(y') \geq 2a$. Then by (4.31),

$$\text{each free vertex } w \text{ has degree at most } 2b+1 \text{ and } N(w) = Y'. \tag{4.32}$$

In particular, every vertex in Y is a solo neighbor of x and thus $G[Y] = K_b$. If some $y \in Y$ is not adjacent to some unmovable $z \in Z \in \mathcal{A}'$, then $||y, Z|| = 2$ and y is adjacent to a free vertex in Z, a contradiction. Similarly, the only possible neighbor of y in V^- is the unmovable v. Thus $G[Y \cup U] = K_k$. If $G - Y - U$ contains a $K_{2b+1,2a-1}$, then we are done. So there is $y_1' \in Y'$ and a class $Z = \{z, z', z''\}$ with unmovable z such that $z''y_1' \notin E(G)$. Then by (4.31), $N(y_1') \cap Z = \{z, z'\}$, z' is free and z'' is not free. If $z''x \notin E(G)$ then we color $B - y_1'$ with b colors, add color class $\{y_1', z'', x\}$, move x' to $Z - z''$ and x'' to V^- and keep the remaining classes as they are. So $z''x \in E(G)$. Similarly, if v is a common nonneighbor of z'' and y_1', then we color $B - y_1'$ with b colors, add color class $\{y_1', z'', v\}$, move v' to $Z - z''$ and keep the remaining classes in $\mathcal{A}$ as they are. If there is a common nonneighbor $y \in Y$ of y_1' and z'', then, by (4.32), y is a solo neighbor of z not adjacent to the 1/2-neighbor y_1' of z, a contradiction. Thus $Y + v + x \subset N(z'') \cup N(y')$ and y_1' has $2a - 1$ neighbors outside of $Y + v + x$. So

$$||z'', Y'|| \leq 2k + 1 - d(y') - ||z'', A \cup Y|| \leq 2k + 1 - (2a - 1) - (b + 2) = b.$$

Since $Y \subset N(z'') \cup N(y')$, $d(y_1) \geq k$ for each $y_1 \in Y$. So as in Case 2.1, $d(z) \leq k + 1$ and $||z, Y'|| \leq k + 1 - ||z, U \cup Y|| = 2$. Then there is a common nonneighbor $y_2' \in Y'$ of z and z''. Then we color $B - y_2'$ with b colors, add color class $\{y_2', z'', z\}$, move z' to V^- and keep the remaining classes in $\mathcal{A}$ as they are.

CASE 3: $|Y| = b + 1$. By (4.31), each free vertex w has $2b$ neighbors in Y' and so at most two neighbors in Y. Since $|F| \geq 4$ and $|Y| \geq 3$, we can choose $x', x'' \in F$ so that $Y \not\subset N(x') \cup N(x'')$. Then Y contains a solo neighbor y_1 of x. Since x is low, $||x, A|| = a - 1$ and thus $N(x) \cap A = U - x$. If a movable $z'' \in Z = \{z, z', z''\} \in \mathcal{A}'$ with unmovable z is not adjacent to some $y_1' \in Y'$, then we color $B - y_1'$ with b colors, add color class $\{y_1', z'', x\}$, move x' to $Z - z''$ and x'' to V^- and keep the remaining classes as they are. So

$$\text{each } y' \in Y' \text{ is adjacent to each } w \in A - U. \tag{4.33}$$

CASE 3.1: There is $y_2 \in Y$ adjacent to all free vertices. Then it has at least two neighbors in each class of $\mathcal{A}'$ and 3 neighbors in X (in particular, $y_2 \neq y_1$). So for each $w \in F$, $d(w) + d(y_2) \geq (2b+1) + 2a = 2k+1$. It follows that $N(w) = Y' + y_2$ and y_2 is isolated in $G[B]$. Then all vertices in $Y - y_2$ are solo neighbors of x and $G[Y - y_2] = K_b$. As in Case 2.2, if some $y \in Y - y_2$ is not adjacent to some unmovable $z \in Z \in \mathcal{A}'$, then

$||y, Z|| = 2$ and y is adjacent to a free vertex in Z, a contradiction. Similarly, the only possible neighbor of y in V^- is the unmovable v. Thus $G[Y \cup U - y_2] = K_k$. Now we practically repeat part of the argument of Case 2.2: in order not to have disjoint K_k and $K_{2b+1,2a-1}$, there should be $y_1' \in Y' + y_2$ and a class $Z = \{z, z', z''\}$ with unmovable z such that $z''y_1' \notin E(G)$. Then by (4.31), $N(y_1') \cap Z = \{z, z'\}$, z' is free and z'' is not free. Since $z''x \notin E(G)$, we color $B - y_1'$ with b colors, add color class $\{y_1', z'', x\}$, move x' to $Z - z''$ and x'' to V^- and keep the remaining classes as they are.

CASE 3.2: No $y \in Y$ is adjacent to all free vertices and there are $y_2 \in Y$ and $u \in U$ with $y_2 u \notin E(G)$. By the case, there is a free w not adjacent to y_2. Then we color $B - y_2$ with b colors, add color class $\{y_2, u, w\}$, move x to the class of u (since x is adjacent only to U), move x' to the class of w and x'' to V^- and keep the remaining classes as they are.

CASE 3.3: No $y \in Y$ is adjacent to all free vertices and for every $y \in Y$ and every $u \in U$, $yu \in E(G)$. Since $G[B]$ is b-colorable, there are $y_2, y_3 \in Y$ with $y_2 y_3 \notin E(G)$. Then by Lemmas 4.4.7 and 2.2.22, $||y_2, X|| + ||y_3, X|| \geq 4$. We claim that

$$\text{one can choose } y_2 \text{ and } y_3 \text{ above distinct from } y_1 \text{ (possibly shuffling free vertices).} \tag{4.34}$$

Indeed, if (4.34) fails and $y_3 = y_1$, then $||y_2, X|| = 3$, y_2 is adjacent to each $y \in Y - y_1 - y_2$ and each such y has a neighbor in $X - x$. Therefore, $b \leq 3$ and $\max\{d(x'), d(x'')\} = 2b+2$. Thus $d(y_2) + \max\{d(x'), d(x'')\} \geq (a + 2 + b - 1) + (2b + 2) = k + 2b + 3 \geq 2k + 2$, a contradiction. This proves (4.34).

By (4.34), $||y_2, X + y_1|| + ||y_3, X + y_1|| \geq 5$. Also $||y_2, Z|| + ||y_3, Z|| \geq 4$ for every $Z \in \mathcal{A}'$. So $||y_2, A + y_1|| + ||y_3, A + y_1|| \geq 4a - 1$. Assuming $||y_2, A + y_1|| \geq ||y_3, A + y_1||$, we have $||y_2, A + y_1|| \geq 2a$. In particular, y_2 is adjacent to some $w \in A - U$. By (4.33), $||w, B|| \geq |Y'| + 1 = 2b + 1$. Hence

$$2k + 1 \geq d(y_2) + d(w) \geq ||y_2, A + y_1|| + ||w, B|| \geq 2a + 2b + 1,$$

which yields $d(y_2) = 2a$, $N(y_2) \subset A + y_1$, $d(w) = 2b + 1$ and $N(w) = Y' + y_2$. In particular, w is free and y is adjacent only to free vertices in $A - U$. So, at least two free vertices are adjacent to y_2. Switching them with x' and x'' if needed, we may assume that $X \subset N(y_2)$ and all vertices in $Y - y_2$ are solo neighbors of x. Then $G[Y - y_2] = K_b$ and each vertex in $Y - y_2$ is low.

Since $d(y_2) = 2a$ and $U \subset N(y_2)$, there are at least $a - 1$ nonneighbors of y_2 in $A - U$, and by the case some free w is among them. If $A - U$ contains a not free $z'' \in Z = \{z, z', z''\} \in \mathcal{A}'$ with unmovable z, then z' is free and switching z' with w we obtain a super-optimal coloring in which y_2 is a solo neighbor of z. Since y_3 is low and has $k - 1$ neighbors in $Y \cup U$, it has at most one neighbor in $\{z, w, z''\}$, a contradiction to either Lemma 4.4.7 or Lemma 2.2.22. Otherwise, all vertices in $A-U$ are free, and we can rearrange them so that some $Z=\{z,z',z''\}\in\mathcal{A}'$ has a solo neighbor of y_2, again a contradiction with $y_2y_3\notin E(G)$. $\square$

Corollary 4.12.3. *Let f be a super-optimal coloring, $b\geq a'=a-1$ and $\mathcal{F}$ be a star. For each $X=\{x,x',x''\}\in\mathcal{A}'$ with unmovable x, $b_1(x)+b_2(x)\geq b$.*

Proof: Suppose there is $X=\{x,x',x''\}\in\mathcal{A}'$ with unmovable x and $b_1(x)+b_2(x)\leq b-1$. Then $b_0(x)+b_3(x)\geq 2b+2$ and each of x' and x'' is adjacent to every vertex in $B_0(x)\cup B_3(x)$. So by Lemma 4.6.3(b), x' and x'' are free, contradicting Lemma 4.12.2. $\square$

Corollary 4.12.4. *Let f be a super-optimal coloring, $b\geq a'=a-1$ and $\mathcal{F}$ be a star. Then each petite $y\in B$ cannot have at least two neighbors in any $X\in\mathcal{A}'$.*

Proof: Suppose $y_0\in B$ is petite and there is $X=\{x,x',x''\}\in\mathcal{A}'$ with unmovable x such that $||y_0,X||\geq 2$. Then also there is $Z=\{z,z',z''\}\in\mathcal{A}'$ with unmovable z such that z is the solo neighbor of y_0 in Z. Let $Y=B_1(z)\cup B_2(z)$ and $Y'=B-Y$. By Lemmas 4.4.7 and 2.2.22, $||y_0,B||\geq |Y|-1$. By Corollary 4.12.3, $|Y|\geq b$. So, since $||y_0,A||\geq a+1$ (because of X), $d(y_0)\geq b-1+a+1\geq 2a-1$. Then by the definition of petite vertices, $d(y_0)=2a-1$ and y_0 has either 3 neighbors in some class of $\mathcal{A}'$ or two neighbors in V^-. In both cases, $||y_0,A||\geq a+2$ and so $d(y_0)\geq b-1+a+2\geq 2a$, a contradiction. $\square$

Lemma 4.12.5. *Let f be a super-optimal coloring, $b\geq a'=a-1$ and $\mathcal{F}$ be a star. For each $X=\{x,x',x''\}\in\mathcal{A}'$ with unmovable x, $||x',B||\geq 2b+1$.*

Proof: Suppose $X=\{x,x',x''\}\in\mathcal{A}'$ with unmovable x and $||x',B||\leq 2b$. Let $Y=B_1(x)\cup B_2(x)$ and $Y'=B-Y$. Since each of x' and x'' is adjacent to every vertex in $B_0(x)\cup B_3(x)$, $|Y|\geq b+1$.

CASE 1: There is $y_1\in B_1(x)$. Then y_1 is low and is adjacent to all vertices in $Y-y_1$. Thus $|Y|=b+1$. Let y_2,y_3 be nonadjacent vertices in Y. By Lemmas 4.4.7 and 2.2.22, y_2 and y_3 are 1/2-neighbors of x. Since all $2b$ neighbors of x' in B are in Y', $y_2x'',y_3x''\in E(G)$. In particular, $d(x'')\geq 2b+2$. Then we know all neighbors in B of x' and x'', so all vertices in $Y-y_2-y_3$ are solo neighbors of x, and $G[Y-y_2-y_3]=K_{b-1}$. Also each of y_2 and y_3 is adjacent to all vertices in $Y-y_2-y_3$. Since for $y\in\{y_1,y_2\}$,

$$||y,A-X||\leq 2k+1-d(x'')-||y,B||-||y,X||\leq 2k+1-(2b+2)-(b-1)-2=2a-b-2\leq a-1,$$

all neighbors of y_2 and y_3 in $A-X$ are solo, a contradiction to Lemma 4.4.7.

CASE 2: $B_1(x)=\emptyset$. Then $||B,X||\geq 6b+2$, $||x',B||\leq 2b$ and $||x'',B||\leq 2b+2$. So $||x,B||\geq 2b$ and $d(x)\geq 2b+a-1$. Since x is adjacent to a non-petite vertex $y\in B$,

$$2b+a-1\leq d(x)\leq 2k+1-(2a-1)=2b+2.$$

In order this to be possible, we need $a = 3$ and all the following equalities: $||B, X|| = 6b + 2$, $||x', B|| = 2b$, $||x'', B|| = 2b + 2$, $||x, B|| = 2b$, $d(y) = 2a - 1$, and $d(x) = 2b + a - 1$. In particular, each $y \in B$ has exactly two neighbors in X and x'' is free. Also if $Z = \{z, z', z''\} \in \mathcal{A}'$ with unmovable z is the other class in $\mathcal{A}'$, then (since $d(y) \leq 2a-1 = 5$ for each $y \in B$), no vertex in B has more than two neighbors in Z. Thus if some $y \in B$ is a solo neighbor of z, then y is adjacent to every vertex in B and $d(y) \geq 3b+a+1$. On the other hand, since y is adjacent to at least one of the vertices x and x'' of degree $2b+2$, $d(y) \leq 2k+1-2b-2 = 2a-1 < 3b+a+1$, a contradiction. Thus B is an independent set of vertices of degree 5 in G. Since each $y \in B$ has exactly two neighbors in X, $|N(x) \cap N(x'') \cap B| = |N(x') \cap N(x'') \cap B| = b + 1$. Let $y_1, y_2 \in N(x) \cap N(x'') \cap B$ and $y_3, y_4 \in N(x') \cap N(x'') \cap B$. We color $B - \{y_1, y_2, y_3, y_4\}$ with $b - 1$ colors and add color classes $\{x', y_1, y_2\}$, $\{x, y_3, y_4\}$, Z, and $V^- + x''$. □

Lemma 4.12.6. *Let f be a super-optimal coloring, $b \geq a' = a - 1$ and $\mathcal{F}$ be a star. Then each $X = \{x, x', x''\} \in \mathcal{A}'$ with unmovable x contains a free vertex.*

Proof: Suppose $X = \{x, x', x''\} \in \mathcal{A}'$ with unmovable x, and each of x' and x'' has a neighbor in A. Let $Y = B_1(x) \cup B_2(x)$ and $Y' = B - Y$. Since each of x' and x'' is adjacent to every vertex in $B_0(x) \cup B_3(x)$, $|Y| \geq b$.

CASE 1: $B_1(x) = \emptyset$. (Repeats Case 2 in Lemma 4.12.5). Then $||B, X|| \geq 6b + 2$, $||x', B|| \leq 2b + 1$ and $||x'', B|| \leq 2b + 1$. So $||x, B|| \geq 2b$ and $d(x) \geq 2b + a - 1$. Since x is adjacent to a non-petite vertex $y \in B$,

$$2b + a - 1 \leq d(x) \leq 2k + 1 - (2a - 1) = 2b + 2.$$

In order this to be possible, we need $a = 3$ and all the following equalities: $||B, X|| = 6b + 2$, $||x', B|| = ||x'', B|| = 2b + 1$, $||x, B|| = 2b$, $d(y) = 2a - 1$, and $d(x) = 2b + a - 1$. In particular, each $y \in B$ has exactly two neighbors in X and $||x', A|| = ||x'', A|| = 1$. Also if $Z = \{z, z', z''\} \in \mathcal{A}'$ with unmovable z is the other class in $\mathcal{A}'$, then (since $d(y) \leq 2a - 1 = 5$ for each $y \in B$), no vertex in B has more than two neighbors in Z. Thus if some $y \in B$ is a solo neighbor of z, then y is adjacent to every vertex in B and $d(y) \geq 3b+a+1$. On the other hand, since y is adjacent to at least one of the vertices x' and x'' of degree $2b+2$, $d(y) \leq 2k+1-2b-2 = 2a-1 < 3b+a+1$, a contradiction. Thus B is an independent set of vertices of degree 5 in G. Since each $y \in B$ has exactly two neighbors in X, $|N(x) \cap N(x'') \cap B| = |N(x') \cap N(x'') \cap B| - 1 = b$. Let $y_1, y_2 \in N(x) \cap N(x'') \cap B$ and $y_3, y_4 \in N(x') \cap N(x'') \cap B$. We color $B - \{y_1, y_2, y_3, y_4\}$ with $b - 1$ colors and add color classes $\{x', y_1, y_2\}$, $\{x, y_3, y_4\}$, Z, and $V^- + x''$.

CASE 2: There is $y_1 \in B_1(x)$ and $|Y| \geq b + 1$. (Repeats Case 1 in Lemma 4.12.5) Then y_1 is low and is adjacent to all vertices in $Y - y_1$. Thus $|Y| = b+1$. Let y_2, y_3 be nonadjacent vertices in Y. By Lemmas 4.4.7

and 2.2.22, y_2 and y_3 are 1/2-neighbors of x. Since each of x' and x'' has at most one neighbor in Y, we may assume that $y_2x', y_3x'' \in E(G)$. Then we know all neighbors in B of x' and x'', so all vertices in $Y - y_2 - y_3$ are solo neighbors of x, and $G[Y - y_2 - y_3] = K_{b-1}$. Also each of y_2 and y_3 is adjacent to all vertices in $Y - y_2 - y_3$. In particular, $d(x'), d(x'') \geq 2b+2$. Then we know all neighbors in B of x' and x'', so all vertices in $Y - y_2 - y_3$ are solo neighbors of x, and $G[Y - y_2 - y_3] = K_{b-1}$. Also each of y_2 and y_3 is adjacent to all $b-1$ vertices in $Y - y_2 - y_3$. Since for $y \in \{y_1, y_2\}$,

$$||y, A - X|| \leq 2k + 1 - (2b + 2) - ||y, B \cup X|| \leq 2k + 1 - (2b + 2) - (b - 1) - 2 = 2a - b - 2 \leq a - 1,$$

all neighbors of y_2 and y_3 in $A - X$ are solo, a contradiction to Lemma 4.4.7.

CASE 3: There is $y_1 \in B_1(x)$ and $|Y| = b$. Since each of x' and x'' is adjacent to all vertices in Y', all vertices in Y are solo neighbors of x. In particular, $G[Y] = K_b$ and all vertices in Y are low. Also $d(y') \leq 2a - 1$ for each $y' \in Y'$.

CASE 3.1: $b = 2$. Let $Z = \{z, z', z''\}$ be the other class in $\mathcal{A}'$. Since by Lemma (4.12.5), each of x', x'', z', z'' has at least $2b + 1 = 5$ neighbors in B, the set $P = \{x', x'', z', z''\}$ is independent. Also, edges XV^- and ZV^- have witnesses, so we may assume

$$x' \text{ and } z' \text{ are movable to } V^-. \tag{4.35}$$

This means that $x'z \in E(G)$. Since x' is high, $d(z) \leq 5$ and $||z, B|| \leq d(z) - 2 \leq 3$. If some $y' \in Y'$ is adjacent to x, then $||y', X|| = 3$ and $||y', Z \cup V^-|| \leq 5 - 3 = 2$. In this case y' has a solo neighbor in Z, and so $zy' \in E(G)$. Thus $N(x) \cap Y' \subseteq N(z)$ and $|Y' - N(x) - N(z)| = |Y' - N(z)| \geq 5 - 3 > 0$. Therefore, if $xz \notin E(G)$, then we can choose $y' \in Y' - N(x) - N(z)$, color $B - y'$ with 2 colors, add class $\{x, z, y'\}$ and color $G[P \cup V^-]$ with 2 colors (we can do it by (4.35)). So let

$$xz \in E(G). \tag{4.36}$$

Then $||z, A|| \geq 3$ and so $||z, B|| \leq d(z) - 3 \leq 2$. Let $Q = N(z) \cap B$ and $Q' = B - Q$. Since each of z', z'' is adjacent to every vertex in Q', and some of z', z'' also has a neighbor in A, $|Q| \geq 2$. So $N(z) \cap B = Q$ and $|Q| = 2$. Then each of the vertices in $B - Y - Q = Y' \cap Q'$ has at least 5 neighbors in A (at least two in X, two in Z, one in V^-), and since $d(x') \geq 6$,

$$\text{each } y' \in Y_1' \cap Q' \text{ is isolated in } B,\ ||y', X|| = ||y', Z|| = 2, \text{ and } ||y', V^-|| = 1. \tag{4.37}$$

If Z had no solo neighbors, then $||B, Z-z|| \geq 2|B|-|N(z)\cap B| \geq 14-2 = 12$, and by (4.37), both z' and z'' are free, a contradiction. So $B_1(z) \neq \emptyset$ and thus $G[Q] = K_2$. Furthermore, each $q \in Q - Y$ has at least two neighbors in X, at least two neighbors in $Z \cup V^-$ and one neighbor in Q. This means that q has no other neighbors and is a solo neighbor of z. This and (4.37) also yield that $B_3(x) = \emptyset$ and thus $Y = N(x) \cap B$. Thus if $Y = \{y_1, y_2\}$ and $Q = \{q_1, q_2\}$ (Y and Q may intersect and even coincide), then

$$\text{the only edges in } G[B] \text{ are } y_1y_2 \text{ and } q_1q_2 \text{ (possibly, } q_1q_2 = y_1y_2). \tag{4.38}$$

Since z is not adjacent to all vertices in X, we may assume that $x''v \in E(G)$. Then v is low and there is $y' \in Y'$ with $vy' \notin E(G)$. So if $vx \notin E(G)$, then we can color $B - y'$ with two colors and add classes Z, $\{y', x, v\}$ and $X - x + v'$. Therefore, $xv \in E(G)$ and so $||v, Y'|| \leq d(v) - 2 \leq 3$. So if $vy_1 \notin E(G)$, then by (4.38) there is $y'' \in Y' - N(v) - N(y_1)$. In this case, let y_0 be a vertex in $Y' - y''$ of maximum degree in $G[B - y_1 - y'']$ and y_0' be a non-neighbor of y_0 in $Y' - y'' - y_0$. Then $Y_0 = B - \{y_1, y'', y_0, y_0'\}$ is independent, and we can color G using color classes Y_0, $\{x, y_0, y_0'\}$, $\{y'', y_1, v\}$, Z, and $X - x + v'$. Thus by the symmetry between y_1 and y_2, we have

$$N(v) \supseteq \{x, x'', y_1, y_2\}. \tag{4.39}$$

If Z contains no free vertices, then the above argument works with the roles of Z and X switched, and similarly to (4.39), we have that for some $w \in V^-$, $N(w) \supseteq \{z, z'', q_1, q_2\}$. Since v is low and already has 4 neighbors, $w = v'$, and hence v' is low. But $d(v) + d(v') \geq |B| + |\{x, x'', z, z''\}| = 11$, a contradiction. Thus we may assume that z' is free.

If v' also is free, then $zv \in E(G)$ and we know all neighbors of v. So the only possible neighbor for z'' is x and we know all neighbors of x. By switching z' with x'' we obtain our case for $Z - z' + x''$ in place of X and conclude that $Q = N(z) \cap B = N(v) \cap B = Y$ and that $N(z'') \cap B = N(x'') \cap B = Q'$. Since $N(v) \cap B = Y$, $N(v') \supseteq Y'$. Since we can switch free vertices v' and z', $N(z') \supseteq Y'$. Thus G contains $G[\{Y + x + z + v\}] = K_5$ and the complete bipartite graph with partite sets Y' and $A - x - z - v$, a contradiction.

Thus v' is not free. If z'' had no neighbors in V^-, then we can move it there and obtain another coloring with a free vertex z' in the small class $Z - z''$, a contradiction to the super-optimality of f. Thus $||\{z, z''\}, V^-|| \geq 2$ and $d(v) + d(v') \geq |B| + ||v, X|| + ||\{z, z''\}, V^-|| \geq 11$. Since $d(v) \leq 5$, this gives $d(v') \geq 6$ and so $v'z'' \notin E(G)$. It follows that $vz'' \in E(G)$ and we know all neighbors of v. Then $N(v') \supseteq Y' + z$, and $v'y_1 \notin E(G)$. Then we color $B - y_1$ with two colors and add classes $\{y_1, x'', v'\}$, $Z - z'' + v$ and $X - x'' + z''$. This proves Case 3.1.

CASE 3.2: $b \geq 3$. We claim that

$$\text{each } y' \in Y' \text{ has no solo neighbors in } A'. \tag{4.40}$$

Indeed, suppose $y'_1 \in Y'$ has solo neighbor z in $Z = \{z, z', z''\} \in \mathcal{A}'$. Since $b_1(z) + b_2(z) \geq b$, y'_1 has $b-1$ neighbors in B. Since $G[B]$ does not contain K_{b+1}, there is $y_1 \in Y$ not adjacent to y'_1. Similarly to Case 2, $||y_2, W|| + ||y_3, W|| \geq 4$ for every $W \in \mathcal{A}'$. But the low vertex y_1 has more than one neighbor in at most one class (namely, Z), so $d(y'_1) \geq b - 1 + 2a - 2$. It follows that $b - 1 + 2a - 2 \leq 2a - 1$, i.e., $b \leq 2$, which is not the case. This proves (4.40).

Thus each $y' \in Y'$ is isolated in B, has exactly one neighbor in V^- and exactly two neighbors in each $Z \in \mathcal{A}'$. Now we can strengthen (4.40):

$$\text{each } y' \in Y' \text{ is adjacent to all movable vertices in } A'. \tag{4.41}$$

Indeed, suppose $y'_1 \in Y'$ is not adjacent a movable vertex in $Z = \{z, z', z''\} \in \mathcal{A}'$ with unmovable z. Since y'_1 is isolated in B, z has no solo neighbors in B. By Case 1, Z has a free vertex, say z''. By Lemma 4.12.2, then z' is not free and thus has at most $2b+1$ neighbors in B. Thus $||z, B|| \geq 6b + 2 - (2b+2) - (2b+1) = 2b - 1$ and so $d(z) \geq (2b-1) + a - 1 = k + b - 2 > k$. If z' has no neighbors in V^-, then moving z' to V^- creates an optimal coloring with a free vertex in the small class. So by the definition of super-optimal colorings, V^- has a free vertex, say v'. Then $vz \in E(G)$ and so z is low. Thus switching z with v creates a new super-optimal coloring but the vertices in $B \cap (N(z) - N(v))$ are solo neighbors of movable vertices in $Z - z + v$. Thus z' has a neighbor in V^- and so no neighbors in X. Also high vertex z is not adjacent to high vertices x' and x''. So the only edge in $G[X \cup Z]$ is xz. Switching x with z, we get Case 2 of our lemma, which is proved. This proves (4.41).

This means that for each $y' \in Y'$, $N(y') - V^- = A' - U$. Since each of $w \in A' - U$ has $2b+1$ neighbors in B, no two of them are adjacent to each other, i.e.

$$A' - U \text{ is independent.} \tag{4.42}$$

So, if $Z = \{z, z', z''\} \in \mathcal{A}'$ and $W = \{w, w', w''\} \in \mathcal{A}'$ with unmovable z and w and $wz \notin E(G)$, then take any $y' \in Y'$, color $B - y'$ with b colors, add class $\{y', w, z\}$, move the witness of ZV^- to V^- and the last vertex of Z to $W - w$ and keep the remaining classes unchanged. If $Z = \{z, z', z''\} \in \mathcal{A}'$ with unmovable z and $y \in Y$ are such that $zy \notin E(G)$, then $yz', yz'' \in E(G)$ and thus z' and z'' are free, a contradiction to

Lemma 4.12.2. It follows that

$$G[Y \cup U - V^-] = K_{k-1}. \tag{4.43}$$

Suppose $N(v') \supseteq Y'$. Then $G[Y' \cup (A' - U) + v'] \supseteq K_{2b+1,2a-1}$. Also v' is not adjacent to any vertex in $A' - U$. Since each $y' \in Y'$ is adjacent to only one vertex in V^-, $N(v) \cap Y' = \emptyset$. In order G not to contain a disjoint union of a K_k and a $K_{2b+1,2a-1}$, by (4.43), v is not adjacent either to some $y \in Y$ or to some unmovable z is a $Z = \{z, z', z''\} \in \mathcal{A}'$. In the first case, choose any 3 vertices $y_1', y_2', y_3' \in Y'$, we color $B - \{y, y_1', y_2', y_3'\}$ with $b-1$ colors, add classes $\{y, y_1', v\}$ and $\{x, y_2', y_3'\}$, move v' into $X - x$ and keep the remaining classes in $\mathcal{A}'$. In the second case, choose any vertex $y' \in Y'$, color $B - y'$ with b colors, add class $\{y', z, v\}$, move v' into $Z - z$ and keep the remaining classes in $\mathcal{A}'$. So by the symmetry between v and v' we may assume

$$\text{there are } y_1', y_2' \in Y' \text{ with } y_1'v, y_2'v' \notin E(G). \tag{4.44}$$

Suppose now that there is $y_1 \in Y$ with $y_1 v \notin E(G)$. Since $||y', B|| \leq 2b + 2 \leq 3b - 1$ (AGAIN use $b \geq 3$), B contains a nonneighbor y_2 of y' distinct from y_2'. Then we color $B - \{y_1', y_2', y_1, y_2\}$ with $b-1$ colors, add classes $\{y_1, y_1', v\}$ and $\{y_2, y_2', v'\}$, and keep $\mathcal{A}'$. Thus each of v, v' is adjacent to each vertex in Y. So one of them, say v' has at least $\left\lceil \frac{|Y|+|B|}{2} \right\rceil = 2b + 1$ neighbors in B. Hence v' has no neighbors in $A' - U$. If $vz \notin E(G)$ for some unmovable $z \in Z \in \mathcal{A}'$, then we color $B - y_1'$ with b colors, add class $\{y_1', z, v\}$, move v' into $Z - z$ and keep the remaining classes in $\mathcal{A}'$. So $N(v) \supset U$. Moreover, if for some $Z = \{z, z', z''\} \in \mathcal{A}'$, the only edge in $Z \cup V^-$ is zv, then we color $B - y_2'$ with b colors, add class $\{y_2', z, v'\}$, move v into $Z - z$ and keep the remaining classes in $\mathcal{A}'$. So, $||V^-, Z|| \geq 2$ for each $Z \in \mathcal{A}'$ and

$$d(v) + d(v') \geq 2(a-1) + |Y| + |B| = 2a + 4b - 1 = 2k + 2b - 1 \geq 2k + 3.$$

On the other hand each of v and v' is adjacent to vertices in Y of degree k, and so $d(v) + d(v') \leq 2(k+1)$, a contradiction. $\square$

4.13 Finishing the proof

Let f be a super-optimal coloring of G. By Lemmas 4.11.2, 4.12.2 and 4.12.6, every $Z = \{z, z', z''\} \in \mathcal{A}'$ has one unmovable vertex z, one vertex z' with exactly one neighbor in A and one free vertex z''. We will always use this notation below.

Lemma 4.13.1. *Set V^- contains a free vertex.*

Proof: Suppose $V^- = \{v, v'\}$ and each of v and v' has a neighbor in A. If for some $Z = \{z, z', z''\} \in \mathcal{A}'$,

z' has no neighbors in V^-, then moving z' into V^- creates (by Lemma 4.11.5) an optimal coloring with a free vertex, a contradiction to super-optimality of f. Thus for every $Z = \{z, z', z''\} \in \mathcal{A}'$, z' has a neighbor in V^-. In particular,

$$\text{no vertex in } A' - U \text{ has a neighbor in } A' \text{ and hence } G[U - V^-] = K_{a-1}. \tag{4.45}$$

Let $Z = \{z, z', z''\} \in \mathcal{A}'$. By Lemma 4.11.5, moving z'' into V^- creates another super-optimal coloring f'. Then by Lemma 4.11.2, a vertex in V^-, say v, is unmovable and by (4.45) it is adjacent to all vertices in $U - v - z$. Furthermore, then v' has only one neighbor in A and this neighbor is in $Z - z$. Using another class $W \in \mathcal{A}'$ instead of Z, we obtain that also $zv \in E(G)$ and that v' has a neighbor in W, a contradiction. □

So below we assume that $V^- = \{v, v'\}$ and v' is free. Then v is unmovable and adjacent to all unmovable vertices in A'.

Lemma 4.13.2. $G[U] = K_a$.

Proof: Suppose $X = \{x, x', x''\} \in \mathcal{A}'$, $Z = \{z, z', z''\} \in \mathcal{A}'$, and $xz \notin E(G)$. Since x and z are unmovable, $xz', x'z \in E(G)$. Then swapping x with z' creates a super-optimal coloring with two unmovable vertices in $Z - z' + x$, a contradiction to the main text. □

By Lemma 4.13.2, we can choose $Z = \{z, z', z''\} \in \mathcal{A}'$ with low z.

Lemma 4.13.3. *One can construct a super-optimal coloring in which a low unmovable vertex in $\mathcal{A}'$ has a solo neighbor in B.*

Proof: Suppose z has no solo neighbors in B. Then $||B, Z|| \geq 6b+2$, $||z', B|| \leq 2b+1$ and $||z'', B|| \leq 2b+2$. So $||x, B|| \geq 2b - 1$. Since z is low, $||x, B|| \leq b + 1$. Thus $b = 2$, $a = 3$ and all inequalities used above are equalities: $||B, Z|| = 6b + 2 = 14$, $||z', B|| = 2b + 1 = 5$, $||z'', B|| = 2b + 2 = 6$, $d(z) = k = 5$ and $||z, A|| = 2$. In particular, each $y \in B$ has exactly two neighbors in Z. Then switching z'' with another free vertex should give us the same pattern. So $N(w) = N(z'')$ for every free w. Let $X = \{x, x', x''\}$ be the other class in $\mathcal{A}'$. Since z' is not free, the unique neighbor of z in A is v or x or x'. If $z'x' \in E(G)$ then we can switch x' with z' and get another super-optimal coloring. If z is not solo in it, then $N(x') \cap B = N(z') \cap B$, but two high vertices cannot be adjacent. So $z'x' \notin E(G)$. If $z'x \in E(G)$, then x is low but has two neighbors in Z; thus $||x, B|| \leq 2$ and it has a solo neighbor in B. Therefore, $z'v \in E(G)$. The only possible neighbor of x' in A also is v. Since switching x' with z' does not create a solo neighbor for z, $N(x') = N(z')$. So for every of the four $y \in B \cap N(z') \cap N(z'')$, $N(y) = A - U$ and y is isolated in $G[B]$. Then x has two neighbors in A and at most 3 neighbors in B. In particular, x is low. So $N(x) \cap B = N(z) \cap B$ and for each of the two vertices

$y' \in N(x) \cap B$, $N(y') = F + x + z$. In particular, B is independent. Let $B \cap N(z') \cap N(z'') = \{y_1, \ldots, y_4\}$. Then our color classes will be $N(z) \cap B$, $\{x, y_1, y_2\}$, $\{z, y_3, y_4\}$, $X - x + z'$ and $V^- + z''$. $\square$

Let f be a super-optimal coloring and z be a low unmovable vertex in $\mathcal{A}'$ with a solo neighbor in B guaranteed by Lemma 4.13.3. Let $Z = \{z, z', z''\} \in \mathcal{A}'$ be the class of z. Let $Y = B_1(z) \cup B_2(z)$ and $Y' = B - Y$. By the choice of f and z, there is $y_1 \in B_1(z)$. Since each of z' and z'' is adjacent to every vertex in $B_0(z) \cup B_3(z)$, $|Y| \geq b$.

CASE 1: $|Y| \geq b+1$. Since y_1 is low and is adjacent to all vertices in $Y - y_1$, $|Y| = b+1$. Since z is low, $B_3(z) = \emptyset$ and $||z, A|| = a - 1$. Let y_2, y_3 be nonadjacent vertices in Y. By Lemmas 4.4.7 and 2.2.22, y_2 and y_3 are 1/2-neighbors of z. Since z' has at most one neighbor in Y, we may assume that $y_2 z'' \in E(G)$.

CASE 1.1: $y_3 z'' \in E(G)$. If some $w \in F$ is not adjacent to y_3 then switching w with z'' creates a super-optimal coloring in which solo neighbor y_3 of z is not adjacent to y_2 not adjacent to z'. This contradicts Lemma 4.4.7 or 2.2.22. Thus each of y_2 and y_3 is adjacent to each $w \in F$ and thus to at least $2a-1$ vertices in A. Since also $y_1 y_2 \in E(G)$, $d(y_2) + d(z'') \geq (2a - 1 + 1) + (2b + 2) = 2k + 2$, a contradiction.

CASE 1.2: $y_3 z' \in E(G)$. If there is a nonneighbor y_4 of y_2 or y_3 in $Y - y_2 - y_3$, then it is a half-neighbor of z and thus must be adjacent to z''. Because of Case 1.1, $y_4 y_2 \in E(G)$, and every other vertex in Y is a solo neighbor of z. Then $||y_4, Y|| = ||y_2, Y|| = b - 1$. Furthermore, one of the adjacent vertices y_2, y_4, say y_2, is low. Then $||y_2, A - Z|| \leq k - ||y_2, Y \cup X|| = k - (b - 1 + 2) = a - 1$ and each neighbor of y_2 in $A' - Z$ is solo. Since $y_2 y_3 \notin E(G)$, this yields $||y_3, A|| \geq 2(a-1) + 1$ and $d(y_3) \geq b - 2 + 2a - 1 = k + a - 3$. Since $d(z') \geq 2b + 2$, we get $2b + 2 + k + a - 3 \leq 2k + 1$, i.e. $b \leq 2$. So $b = 2$, but then we have no room for y_4. Thus every $y \in Y - y_2 - y_3$ is adjacent to both, y_2 and y_3. Since $d(y_3) \leq 2k + 1 - d(z') \leq 2a - 1$, $||y_3, A - Z|| \leq (2a - 1) - (b - 1) - 2 = 2a - b - 2 \leq a - 1$. Hence $a = b + 1$ and each neighbor of y_3 in $A' - Z$ is solo. Then $||y_2, A|| \geq 2(a - 1) + 1$ and $d(y_2) \geq b - 1 + 2a - 1 = k + a - 2$. So $b = 2 = a - 1$ and $d(y_2) = 6$. Let $X = \{x, x', x''\}$ be the other class in $\mathcal{A}'$. Then $||y_2, X|| = 2$. Since y_2 is not adjacent to solo neighbor y_3 of x, $x' y_2 \in E(G)$ and so $d(x') \leq 5$. Then $||x', B|| \leq 4$, a contradiction to Lemma 4.12.5.

CASE 2: $|Y| \leq b$. By Corollary 4.12.3, $|Y| = b$. Since each of z' and z'' is adjacent to all vertices in Y', vertices in Y are not adjacent to z' and at most one of them is adjacent to z''. So at least $b - 1$ vertices in Y are solo neighbors of z and thus $G[Y] = K_b$.

CASE 2.1: There is $y_1' \in B_3(z)$. If y_1' has no solo neighbors in A' then $d(y_1') \geq 2a$, otherwise by Corollary 4.12.3, $||y_1', B|| \geq b - 1$ and $d(y_1') \geq b - 1 + a + 2 \geq 2a$, again. But it is adjacent to z' of degree $2b + 2$, a contradiction.

CASE 2.2: $B_3(z) = \emptyset$, i.e., $Y' = B_0(z)$. If any $y' \in Y'$ is not adjacent to any free w, then switching w with z'' we get a coloring in which the solo neighbor of y' in $Z - z'' + w$ is movable z'. Thus each $y' \in Y'$ is adjacent to each $w \in F$ and thus to at least two vertices in each $X \in \mathcal{A}'$. Since $y' z' \in E(G)$, we can write "exactly two" instead of "at least two",

$$\text{each } y' \in Y' \text{ is isolated in } G[B], \text{ and } N(y') \cap V^- = \{v'\}. \tag{4.46}$$

Let $Y' = \{y'_1, \dots, y'_{2b+1}\}$ and $Y = \{y_1, \dots, y_b\}$. Suppose an unmovable x in $X = \{x, x', x''\} \in \mathcal{A}'$ is not adjacent to some $y_1 \in Y$ and some $y'_1 \in Y'$. By (4.46), we can color $B - \{y_1, y'_1, y'_2, y'_3\}$ with $b-1$ colors including into each color class one vertex of $Y - y_1$ and two vertices in $Y' - \{y'_1, y'_2, y'_3\}$, then add classes $\{y_1, y'_1, x\}$ and $\{y'_2, y'_3, v\}$, move v' to $X - x$ and keep the remaining classes in $\mathcal{A}'$. So if an unmovable x in $X = \{x, x', x''\} \in \mathcal{A}'$ is not adjacent to some $y_1 \in Y$, then it is adjacent to every $y' \in Y'$ and y_1 is adjacent to x' and x''. Also, then y_1 is adjacent to every $w \in F$. So y_1 is adjacent to two vertices in each class of $\mathcal{A}'$ and $d(y_1) \geq (2a-1) + (b-1) = k + a - 2 \geq k+1$, implying that y_1 is the only nonsolo neighbor of z in Y. But then similarly x must be adjacent to each of the $b-1$ low vertices in Y, and so $d(x) \geq 3b + a - 1$ and $d(x) + d(z) \geq (3b + a - 1) + (k-1) = 2k + 2b - 2$, a contradiction. Therefore each $y \in Y$ is adjacent to each unmovable $x \in A'$. Similarly, if some $y_1 \in Y$ is not adjacent to v, then by (4.46), we can color $B - \{y_1, y'_1, y'_2, y'_3\}$ with $b-1$ colors including into each color class one vertex of $Y - y_1$ and two vertices in $Y' - \{y'_1, y'_2, y'_3\}$, then add classes $\{y_1, y'_1, v\}$ and $\{y'_2, y'_3, z\}$, move v' to $Z - z$ and keep the remaining classes in $\mathcal{A}'$. Hence

$$G[Y \cup U] = K_k. \tag{4.47}$$

Suppose that some $y'_1 \in Y'$ is adjacent to an unmovable x in $X = \{x, x', x''\} \in \mathcal{A}'$. If some low $y_1 \in Y$ is not adjacent to x', then we can shuffle free vertices so that x is the solo neighbor of y_1, a contradiction. If a high $y_2 \in Y$ is not adjacent to x', then it is adjacent to x'' with $d(x'') \geq |Y'| + 1 = 2b + 2$, a contradiction again. So $Y \subset N(x')$ and hence $|Y' - N(x')| \geq b - 1$. If $x'v \notin E(G)$, then choose $y'_1 \in Y' - N(x')$, color $B - y'_1$ with b colors, add class $\{y'_1, v, x'\}$, move v' to $X - x'$ and keep the remaining classes in $\mathcal{A}'$. So $x'v \in E(G)$ and $x'z \notin E(G)$. If also $xz' \notin E(G)$ then by switching z with x we get a coloring in which the vertices in $Y' - N(x')$ are solo neighbors of the movable vertex x'' in $X - x + z$. Thus $xz' \in E(G)$ and $z'v \notin E(G)$. Then choose $y'_1 \in Y' - N(x')$, color $B - y'_1$ with b colors, add class $\{y'_1, z, x'\}$, move v' to $X - x'$ and v to $Z - z$ and keep the remaining classes in $\mathcal{A}'$. Therefore, $N(y') = A - U$ for every $y' \in Y'$ and G contains the union of disjoint $K_k = G[Y \cup U]$ and $K_{2b+1, 2a-1}$ with partite sets Y' and $A - U$, as claimed. □

Chapter 5

Saturation of Ramsey-Minimal Families

The following results are joint work with Michael Ferrara and Jaehoon Kim; this chapter is based on [16].[1]

Ramsey theory deals with partitioning the edges of graphs so that each partition avoids the particular forbidden subgraph assigned to it. In this chapter, we study the saturation of Ramsey-minimal families. Our motivation for studying these families is that they provide a convincing edge-colored (Ramsey) version of graph saturation. We develop a method, called iterated recoloring, for using results from graph saturation to understand this new Ramsey version of saturation. As a proof of concept, we use iterated recoloring to determine the saturation number of the Ramsey-minimal families of matchings and describe the assiociated extremal graphs.

5.1 Introduction

Given an edge coloring ϕ of a graph G, let G_ϕ denote the edge-colored graph obtained by applying ϕ to G, and let $G_\phi[i]$ denote the spanning subgraph of G_ϕ induced by all edges of color i. When the context is clear, we will simply write G and $G[i]$ in place of the more cumbersome G_ϕ and $G_\phi[i]$.

In this paper, we are concerned with saturation number. This parameter was introduced by Erdős, Hajnal and Moon in [14], wherein they determined $sat(n, K_t)$ and characterized the unique saturated graphs of minimum size.

Theorem 5.1.1. *If n and t are positive integers such that $n \geq t$, then*

$$sat(n, K_t) = \binom{t-2}{2} + (t-2)(n-t+2).$$

Furthermore, $K_{t-2} \vee \overline{K}_{n-t+2}$ is the unique K_t-saturated graph of order n with minimum size.

Subsequently, $\mathrm{sat}(n, \mathcal{F})$ has been determined for a number of families of graphs and hypergraphs. For a thorough dynamic survey, see [15].

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The (classical) Ramsey number $r(H_1, \ldots, H_K)$ is the smallest positive integer n such that $K_n \rightarrow (H_1, \ldots, H_k)$. A graph G is $(H_1, \ldots, H_k)$-*Ramsey-minimal* if $G \rightarrow (H_1, \ldots, H_k)$ but for any $e \in G$, $(G - e) \not\rightarrow (H_1, \ldots, H_k)$. Let $\mathcal{R}_{\min}(H_1, \ldots, H_k)$ denote the family of $(H_1, \ldots, H_k)$-Ramsey-minimal graphs.

Here we are interested in the following general problem.

Problem 5.1.2. Let $H_1, \ldots, H_k$ be graphs, each with at least one edge. Determine

$$\mathrm{sat}(n, \mathcal{R}_{\min}(H_1, \ldots, H_k)).$$

It is straightforward to prove that $G \rightarrow (H_1, \ldots, H_k)$ if and only if G contains an $(H_1, \ldots, H_k)$-Ramsey-minimal subgraph. Hence Problem 5.1.2 is equivalent to finding the minimum size of a graph G of order n such that there is some k-edge-coloring of G that contains no copy of H_i in color i for any i, yet for any $e \in \overline{G}$ every k-edge-coloring of $G + e$ contains a monochromatic copy of H_i in color i for some i. We observe as well that

$$\mathrm{sat}(n, \mathcal{R}_{\min}(H, K_2, \ldots, K_2)) = \mathrm{sat}(n, H),$$

so that Problem 5.1.2 not only represents an interesting juxtaposition of classical Ramsey theory and graph saturation, but is also a direct extension of the problem of determining $\mathrm{sat}(n, H)$. Problem 5.1.2 is inspired by the following 1987 conjecture of Hanson and Toft [19].

Conjecture 5.1.3. *Let $r = r(K_{t_1}, K_{t_2}, \ldots, K_{t_k})$ be the standard Ramsey number for complete graphs. Then*

$$sat(n, \mathcal{R}_{\min}(K_{t_1}, \ldots, K_{t_k})) = \begin{cases} \binom{n}{2} & n < r \\ \\ \binom{r-2}{2} + (r-2)(n-r+2) & n \geq r. \end{cases}$$

In [7] it was shown that

$$\mathrm{sat}(n, \mathcal{R}_{\min}(K_3, K_3)) = 4n - 10$$

for $n \geq 54$, thereby verifying the first nontrivial case of Conjecture 5.1.3. At this time, however, it seems that a complete resolution of the Hanson-Toft conjecture remains elusive. As such, one goal of the study of Problem 5.1.2 is to develop a collection of techniques that might be useful in attacking Conjecture 5.1.3.

Here, we solve Problem 5.1.2 completely in the case where each H_i is a matching, and further completely characterize all saturated graphs of minimum size. Specifically, we prove the following.

Theorem 5.1.4. *If $m_1, \ldots, m_k \geq 1$ and $n > 3(m_1 + \ldots + m_k - k)$, then*

$$sat(n, \mathcal{R}_{\min}(m_1K_2, \ldots, m_kK_2)) = 3(m_1 + \ldots + m_k - k).$$

If $m_i \geq 3$ for some i, then the unique saturated graphs of minimum size consist solely of vertex-disjoint triangles and independent vertices. If $m_i \leq 2$ for every i, then the graphs achieving equality are unions of edge-disjoint triangles and independent vertices.

As noted in [20], a result of Mader [35] implies that the unique minimum saturated graph of order $n \geq 3m-3$ for $H = mK_2$ is $(m-1)K_3 \cup (n-3m+3)K_1$. Hence, the minimum saturated graphs in Theorem 5.1.4 are precisely a union of m_iK_2-saturated graphs of minimum size. This provides an interesting contrast to both Conjecture 5.1.3 and the main result in [7] which posit and demonstrate, respectively, a stronger relationship between $r(K_{t_1}, K_{t_2}, \ldots, K_{t_k})$ and $sat(n, \mathcal{R}_{\min}(K_{t_1}, \ldots, K_{t_k}))$.

The proof of Theorem 5.1.4 uses *iterated recoloring*, a new technique that utilizes the structure of H_i-saturated graphs to gain insight into $\mathcal{R}_{\min}(H_1, \ldots, H_k)$-saturated graphs. We describe this approach next.

5.2 Iterated Recoloring

Given graphs $G, H_1, \ldots, H_{k-1}$ and H_k, a k-edge coloring of G is an $(H_1, \ldots, H_k)$-*threshold-coloring* if under this coloring G contains no monochromatic copy of H_i in color i, but for any e in $\overline{G}$ and any $i \in [k]$, the addition of e to G in color i creates a monochromatic copy of H_i in color i. In the interest of concision, we will frequently refer to an $(H_1, \ldots, H_k)$-threshold-coloring of G as an $(H_1, \ldots, H_k)$-coloring. Central to our approach here is the following observation.

Observation 5.2.1. If G is an $\mathcal{R}_{\min}(H_1, \ldots, H_k)$-saturated graph, then every k-edge-coloring of G that contains no monochromatic copy of H_i in color i for any i is an $(H_1, \ldots, H_k)$-coloring. In particular, G has at least one $(H_1, \ldots, H_k)$-coloring.

An $(H_1, \ldots, H_k)$-coloring of a graph G is *i-heavy* if for any edge e in G with color not equal to i, recoloring e with color i creates a monochromatic copy of H_i in color i. The next proposition connects the structure of H_i-saturated graphs with the monochromatic subgraph $G[i]$ in an i-heavy $(H_1, \ldots, H_k)$-coloring of G.

Lemma 5.2.2. *If G is an $\mathcal{R}_{\min}(H_1, \ldots, H_k)$-saturated graph and ϕ is an i-heavy $(H_1, \ldots, H_k)$-coloring of G for some $i \in [k]$, then $G_\phi[i]$ is H_i-saturated.*

Proof. Throughout the proof, it suffices to treat $G[i]$ as an uncolored graph. As ϕ is an $(H_1, \ldots, H_k)$-coloring of G, it follows that $G[i]$ contains no subgraph isomorphic to H_i. It remains to prove that for any edge $e \in E(G[i])$, $G[i] + e$ has a subgraph isomorphic to H_i.

If $e \in E(G) - E(G[i])$, then $\phi(e) \neq i$. Because ϕ is i-heavy, changing e to color i in G_ϕ creates a copy of H_i in color i. Therefore, adding e to $G[i]$ creates a subgraph isomorphic to H_i. On the other hand, if $e \in E\left(\overline{G}\right)$, then the fact that ϕ is an $(H_1, \ldots, H_k)$-coloring of G implies that adding e to G_ϕ in color i creates a copy of H_i in color i. Consequently, $H_i \subseteq G[i] + e$. $\square$

The general technique is as follows. Starting with an $(H_1, \ldots, H_k)$-coloring ϕ of an $\mathcal{R}_{\min}(H_1, \ldots, H_k)$-saturated graph G, we iteratively recolor edges in G_ϕ to obtain a 1-heavy $(H_1, \ldots, H_k)$-coloring ϕ_1, and then recolor edges in G_{ϕ_1} to obtain a 2-heavy coloring ϕ_2, and so on until we have successively created i-heavy $(H_1, \ldots, H_k)$-colorings ϕ_i for every $i \in [k]$.

By Lemma 5.2.2, the monochromatic subgraph $G[i]$ corresponding to each ϕ_i is H_i-saturated. The goal is to then use any knowledge we may have about (uncolored) H-saturated graphs to force additional extra structure within G.

For instance, here we will use the following characterization of large enough mK_2-saturated graphs due to Mader [35]. A *dominating vertex* in a graph G of order n is a vertex of degree $n-1$.

Theorem 5.2.3. *If G is an mK_2-saturated graph of order $n \geq 2m-1$, then:*

1. *G is disconnected and every component is an odd clique, or*

2. *G has a dominating vertex v and $G - v$ is $(m-1)K_2$-saturated.*

5.3 Proof of Theorem 5.1.4

We begin by proving the upper bound in Theorem 5.1.4.

Proposition 5.3.1. *$sat(n, \mathcal{R}_{\min}(m_1K_2, \ldots, m_kK_2)) \leq 3(m_1 + \ldots + m_k - k)$ whenever $n > 3(m_1 + \ldots + m_k - k)$.*

Proof. Let G be the vertex-disjoint union of $(m_1 + \ldots + m_k - k)$ triangles and $n - 3(m_1 + \ldots + m_k - k)$ independent vertices. We can create an $(m_1K_2, \ldots, m_kK_2)$-coloring ϕ of G by coloring the edges of $m_i - 1$ triangles with color i, for each i. A monochromatic matching can use at most one edge from each triangle, so for any i, the size of the largest matching in color i is $m_i - 1$.

Note that in any coloring of G containing no monochromatic m_iK_2 in color i for any i, each triangle is monochromatic and each color i is used in $m_i - 1$ triangles. There are at most $m_i - 1$ triangles containing an edge of color i, lest there exist an i-colored m_iK_2. Therefore, by the pigeonhole principle, the only way to color G without creating a forbidden subgraph, up to isomorphism, is ϕ.

Consequently, for any $e = uv$ in $\overline{G}$, G_ϕ contains a copy of $(m_i - 1)K_2$ in color i that is disjoint from u and v. Given a k-edge coloring of $G + e$ in which G does not contain a copy of m_iK_2 in color i, it then follows that e lies in a monochromatic copy of $m_{\phi(e)}K_2$. Thus, G is $\mathcal{R}_{\min}(H_1, \ldots, H_k)$-saturated. $\square$

We note that if each $m_i = 2$, then there are minimum saturated graphs aside from kK_3. Indeed, let $n \geq 8$ and let G be the disjoint union of K_7 and $n-7$ isolated vertices. Note K_7 is the edge-disjoint union of seven triangles, so that any $(m_1K_2, \ldots, m_kK_2)$-coloring necessarily assigns a distinct color to each triangle. Then for any $e \in E(\overline{G})$, $G + e \to (H_1, \ldots, H_k)$, so G is $\mathcal{R}_{\min}(H_1, \ldots, H_k)$-saturated.

To prove the upper bound in Theorem 5.1.4, we will utilize the iterated recoloring technique described in Section 5.2. Assume that G is an $\mathcal{R}_{min}(m_1K_2, \ldots, m_kK_2)$-saturated graph of order $n > 3(m_1 + \cdots + m_k - k)$ with at most $3(m_1 + \cdots + m_k - k)$ edges. If G has a dominating vertex, then necessarily G is a star of order $3(m_1 + \cdots + m_k - k) + 1$, which is clearly not $\mathcal{R}_{min}(m_1K_2, \ldots, m_kK_2)$-saturated. Hence we may assume that G contains no dominating vertex.

The following claims establish several important properties of G. The first follows immediately from Lemma 5.2.2 and the fact that G has no dominating vertex.

Proposition 5.3.2. *If ϕ is an i-heavy $(m_1K_2, \ldots, m_kK_2)$-coloring of G, then $G[i]$ is the disjoint union of odd cliques.*

Next we show that no component of any $G[i]$ arising from an $(m_1K_2, \ldots, m_kK_2)$-coloring can have a cut edge.

Proposition 5.3.3. *If ϕ is an $(m_1K_2, \ldots, m_kK_2)$-coloring of G, then each component of $G[i]$ is 2-edge-connected. In particular, each component C of $G[i]$ has at least $|V(C)|$ edges.*

Proof. Suppose ϕ is an $(m_1K_2, \ldots, m_kK_2)$-coloring of G and that C is a component of $G[i]$ with cut-edge uv. As G_ϕ contains no m_i-matching in color i, every $(m_i - 1)$-matching assigned color i in G_ϕ necessarily uses either u or v. Let $C - uv = C_1 \cup C_2$ for disjoint subgraphs C_1 and C_2 of C with $u \in C_1$ and $v \in C_2$.

Because G has no dominating vertex, there exist (not necessarily distinct) vertices x and y such that $ux, vy \in E(\overline{G})$. By the saturation of G, if we extend ϕ to $G + ux$ or $G + vy$ by assigning $\phi(ux) = i$ or $\phi(vy) = i$, respectively, then we create an m_i-matching in color i. Let M_u be an m_i-matching in color i in $G + ux$ that uses n_1 edges from $C_1 - u$ and n_2 edges from C_2. Then M_u restricted to G gives an $(m_i - 1)$-matching that does not use u, and so uses v. Indeed, any matching on C_2 that has n_2 edges must use v.

Now let M_v be an m_i-matching in color i in $G + vy$. M_v restricted to G does not use v, so $C_2 - v$ contributes at most $n_2 - 1$ edges to M_v. Then C_1 contributes at least $n_1 + 1$ edges. Now, if we take the matching formed by restricting M_v to C_1 and M_u to C_2, then G has a matching in color i with at least $n_1 + 1 + n_2 = m_i$ edges, a contradiction.

The assertion that C has at least as many edges as vertices then follows from the fact that C has no leaves.

□

Let ϕ be an $(H_1, \dots, H_k)$-coloring of a graph G. An edge e in G is *inflexible* if changing the color of e to any $j \neq \phi(e)$ creates a monochromatic copy of H_j. The next proposition follows immediately from Proposition 5.3.3.

Proposition 5.3.4. *If ϕ is an $(m_1K_2, \dots, m_kK_2)$-coloring of G, and H is a component of some $G[i]$ that is isomorphic to a triangle, then every edge of H is inflexible.*

Let ϕ be an $(m_1K_2, \dots, m_kK_2)$-coloring of G, and let C be an i-component of G_ϕ. If ψ is a coloring of G obtained from ϕ by iteratively recoloring edges of G in a manner such that each successive coloring is an $(m_1K_2, \dots, m_kK_2)$-coloring, then we say that ψ is obtained from ϕ by *flexing*, or that we *flex* ϕ to ψ. In particular, it is always possible to flex to an i-heavy $(m_1K_2, \dots, m_kK_2)$-coloring of G from any other $(m_1K_2, \dots, m_kK_2)$-coloring of G.

Proposition 5.3.5. *Let ϕ be an $(m_1K_2, \dots, m_kK_2)$-coloring of G, and let C be a component of $G_\phi[i]$. If ψ is obtained from from ϕ by flexing, then $V(C)$ induces a component of $G_\psi[i]$.*

Proof. Suppose that there is some edge e such that recoloring e causes the order of C to increase or decrease in $G[i]$. If recoloring e to color i causes the order of C to increase, then e is necessarily a cut-edge in $G[i]$. On the other hand, if recoloring e causes the order of C to decrease, then prior to recoloring, e was a cut-edge in $G[i]$. In either case, we have contradicted Proposition 5.3.3, completing the proof. □

Let ϕ be an $(m_1K_2, \dots, m_kK_2)$-coloring of G and flex ϕ to a 1-heavy $(m_1K_2, \dots, m_kK_2)$-coloring ϕ_1. For $2 \leq i \leq k$, we flex ϕ_{i-1} to an i-heavy $(m_1K_2, \dots, m_kK_2)$-coloring ϕ_i. Consider then the nontrivial components of $G_{\phi_i}[i]$, all of which are odd cliques by Proposition 5.3.2. In particular, suppose that these components have order $2x_j + 1$ for $1 \leq j \leq \ell$. Then, as ϕ_i is an $(m_1K_2, \dots, m_kK_2)$-coloring, we have that $x_1 + \cdots + x_\ell = m_i - 1$. Further, since the components of G_i do not change order via flexing, a component C of order $2x + 1$ in $G_{\phi_j}[i]$ must have a maximum matching of size x.

Propositions 5.3.2 and 5.3.5 imply that a set X of vertices in G induces a component of $G_{\phi_i}[i]$ if and only if X induces a component of $G_{\phi_j}[i]$ for all $i, j \in [k]$. This, in turn, implies that if ϕ' and ϕ'' are i-heavy colorings obtained via flexing from ϕ, then $G_{\phi'}[i] = G_{\phi''}[i]$. This yields the following proposition.

Proposition 5.3.6. *Let C be a component of $G_{\phi_i}[i]$. Then there are at least $|V(C)|$ edges e in C such that $\phi_j(e) = i$ for all $1 \leq j \leq k$.*

Proof. Let $S \subset E(C)$ be those edges e in C such that $\{\phi_j(e) : 1 \leq j \leq k\} = \{i\}$ and suppose that $|S| < |V(C)|$. Every edge of C that is not in S lies in some component C' of $G_{\phi_j}[j]$ for some $j \neq i$. Iteratively recoloring each $e \notin S$ with any such j does not create a matching of size m_ℓ in color ℓ for any ℓ, as all edges colored ℓ lie within some component of $G_{\phi_\ell}[\ell]$. However, this means that at most $|S| < |V(C)|$ edges of C remain colored with color i, contradicting Proposition 5.3.5. □

Our final proposition shows that no edge in G receives more than two colors under $\phi_1, \ldots, \phi_k$.

Proposition 5.3.7. *If Q is a component of $G_{\phi_i}[i]$ on $2m+1$ vertices, with $m \geq 1$, then any edge of Q is assigned at most 2 colors under $\phi_1, \ldots, \phi_k$. Furthermore, if Q is a triangle, then every edge of Q is inflexible in every G_{ϕ_i}.*

Proof. Note first that if $m = 1$, so that Q is a triangle, then this is the result of Proposition 5.3.4. Hence we will assume that $m \geq 2$.

Suppose Q is a component of $G_{\phi_1}[1]$, and an edge $uv \in E(Q)$ appears in components Q_2 and Q_3 of $G_{\phi_2}[2]$ and $G_{\phi_3}[3]$, respectively. Recall that by Proposition 5.3.2, Q_2 and Q_3 are necessarily odd cliques.

Let $V(Q) - \{u, v\} = \{x_1, x_2, \ldots, x_{2m-1}\}$. First, we define a coloring ψ' of Q.

$$\psi'(e) := \begin{cases} 2 & \text{if } e = x_2 x_j \\ 3 & \text{if } e = x_3 x_j \text{ with } j \neq 2 \\ 1 & \text{otherwise} \end{cases}$$

Now:

$$\psi(e) := \begin{cases} \phi(e) & \text{if } e \notin Q \cup Q_2 \cup Q_3 \\ 1 & \text{if } e \text{ is in } Q \cup Q_2 \cup Q_3 \text{ and incident to } u \text{ or } v. \\ \psi'(e) & \text{if } e \text{ is not incident to } u, v \text{ and } e \text{ is in } Q \\ 2 & e \text{ is not incident to } u \text{ or } v, \text{ and } e \in Q_2 \setminus Q_3 \\ 3 & e \text{ is not incident to } u \text{ or } v, \text{ and } e \in Q_3 \end{cases}$$

In this coloring, the $(2m-3)$ vertices $\{x_1, x_4, \ldots, x_{2m-1}\}$ form a clique of color 1, contributing at most $m-2$ edges to any matching in color 1. Further, edges incident to u or v also contribute at most two matching edges, so any matching in color 1 has at most m edges with an endpoint in Q. As Proposition 5.3.5 implies that the other ℓ nontrivial components of $G_\psi[1] - V(Q)$ are odd cliques with total order $2m_1 - 2m + \ell - 2$, the maximum size of a matching with color 1 in G_ψ is $m_1 - 1$.

Let Q_2 have $2n_2+1$ vertices, and let Q_3 have $2n_3+1$ vertices. Note that in G_{ϕ_1}, Q_2 contributes n_2 edges to any maximum monochromatic matching of color 2 and Q_3 contributes n_3 edges to any maximum monochromatic matching of color 3. As we have recolored all edges in $Q \cup Q_2 \cup Q_3$ that are incident to u or v with color 1, for color $i \in \{2,3\}$, $Q_i - u - v$ contains a matching of size $n_i - 1$. One more edge of color i incident with x_i completes a matching of size at most n_i in $Q \cup Q_2 \cup Q_3$. Outside $Q \cup Q_1 \cup Q_2$, $\psi = \phi$, so ψ is a $(H_1, \ldots, H_k)$-coloring.

If x is a vertex in G that is not adjacent to u, then adding the edge ux to G in color 1 does not increase the size of a maximum 1-colored matching. Thus G is not $\mathcal{R}_{\min}(m_1K_2, \ldots, m_kK_2)$-saturated, a contradiction. □

We are now ready to prove Theorem 5.1.4.

Proof. Let G and $\phi_1, \ldots, \phi_k$ be as given above, and further assume that

$$|E(G)| = \mathrm{sat}(n, \mathcal{R}_{min}(m_1K_2, \ldots, m_kK_2)) \leq 3(m_1 + \cdots + m_k - k).$$

For each i, we let $Q_{i,1}, \ldots, Q_{i,p_i}$ be the (clique) components of $G_{\phi_i}[i]$, and suppose that each $Q_{i,j}$ has $2t_{i,j}+1$ vertices. Recall that $\sum_{j=1}^{p_i} t_{i,j} = m_i - 1$.

For any $e \in E(G)$, we define $w(e) = |\{\phi_i(e) \ : \ 1 \leq i \leq k\}|$. That is, $w(e)$ is the number of colors assigned to e by the heavy colorings $\phi_1, \ldots, \phi_k$. Note

$$|E(G)| = \sum_{i=1}^{k} \sum_{e \in G_i[i]} \frac{1}{w(e)}.$$

By Proposition 5.3.7, $w(e) \leq 2$ for every edge of G. Further, by Proposition 5.3.6, $w(e) = 1$ for at least $|V(Q)|$ edges of Q. Therefore,

$$\begin{aligned} |E(G)| &= \sum_{i=1}^{k} \sum_{e \in G[i]} \frac{1}{w(e)} \\ &\geq \sum_{i=1}^{k} \sum_{j=1}^{p_i} \left((2t_{i,j}+1) + \frac{1}{2}\left[\binom{2t_{i,j}+1}{2} - (2t_{i,j}+1) \right] \right) \qquad (1) \\ &\geq \sum_{i=1}^{k} \sum_{j=1}^{p_i} 3t_{i,j} = \sum_{i=1}^{k} 3(m_i - 1) = 3(m_1 + \ldots + m_k - k). \end{aligned}$$

We therefore conclude that

$$sat(n, \mathcal{R}_{\min}(m_1K_2, \ldots, m_kK_2)) = 3(m_1K_2 + \ldots + m_kK_2).$$

Additionally, equality holds in all equations above, leading us to conclude that every component of every $G_{\phi_i}[i]$ is a triangle. By Proposition 5.3.4, also every component of every $G_{\phi_i}[j]$ is a triangle.

It remains only to show that if $m_i \geq 3$ for at least one i, then G consists of triangles that are vertex disjoint. Suppose not. Then there exists at least one "bow-tie" B: a subgraph of G consisting of two triangles that share one vertex. We can create an $(H_1, \ldots, H_k)$-coloring ϕ of G by assigning color i to $m_i - 1$ of the edge-disjoint triangles in a triangle decomposition of G. Let ϕ be a such a coloring, in which both triangles of B are assigned color i. If we flex ϕ to be i-heavy, then Proposition 5.3.2 implies that the vertices of B must lie in a clique on at least five vertices. However, as equality holds throughout (1) and ϕ was selected arbitrarily, each component of $G[i]$ under any valid coloring is a triangle, a contradiction. $\square$

Chapter 6

Induced Saturation

The results in this chapter are joint work with Sarah Behrens, Catherine Erbes, Michael Santana, and Derrek Yager; this chapter is based on [2].

6.1 Background and Introduction

Consider an n-vertex forbidden graph H. If a graph G with $|G| \geq |H|$ does not contain an induced copy of H, then every collection of n vertices in G either does not contain H as a subgraph, or contains H as a subgraph that is not induced. So, to extend the notion of graph saturation to induced subgraphs, it is natural to consider not only adding edges to create H, but also deleting them.

To this end, we offer the following definition: a graph is *H-induced-saturated* if H is not an induced subgraph of G, but it we add or delete any edge of G, H arises as an induced subgraph. Unfortunately, under this definition, there exist graphs H and positive integers n so that *no* graph G on n vertices is H-induced-saturated. (A simple example is $n = 4$ and $H = K_{1,3}$.)

In order to offer a definition that is well-defined, Martin and Smith [36] consider generalized graphs, called *trigraphs*, objects also used by Chudnovsky and Seymour in their structure theorems on claw-free graphs [8]. The definition given above is equivalent to the definition of Martin and Smith in the case that an n-vertex graph G exists that is H-induced-saturated. We concern ourselves almost entirely with this case.

It was not previously known that for any non-trivial graphs H there exists a graph that was H-induced-saturated. However, we found a surprising number of graphs H for which H-induced-saturated graphs exist. Motivated by this, we began examining the minimum number of edges over all n-vertex, H-induced-saturated graphs. This is a natural extension of saturation number to induced subgraphs, and leads to many unexpected and beautiful constructions. For example, several Platonic solids are H-induced-saturated for appropriate graphs H. (See Figures 6.1 and 6.2, as well as [2], for examples.)

6.1.1 Definitions

Definition 6.1.1. A *trigraph* T is a quadruple $(V(T), E_B(T), E_W(T), E_G(T))$, where $V(T)$ is the vertex set and the other three elements partition $\binom{V(T)}{2}$ into a set $E_B(T)$ of black edges, a set $E_W(T)$ of white edges, and a set $E_G(T)$ of gray edges. These can be thought of as edges, nonedges, and potential edges, respectively. For any $e \in E_B(T) \cup E_W(T)$, let T_e denote the trigraph where e is changed to a gray edge, i.e. $T' = (V(T), E_B(T) - e, E_W(T) - e, E_G(T) + e)$.

The *complement* of a trigraph T, denoted $\overline{T}$, is the trigraph with $V(\overline{T}) = V(T)$, $E_B(\overline{T}) = E_W(T)$, $E_W(\overline{T}) = E_B(T)$, and $E_G(\overline{T}) = E_G(T)$.

In a trigraph, the black (resp. gray) degree of a vertex is the number of black (resp. gray) edges incident to that vertex.

Definition 6.1.2. A *realization* of T is a graph $G = (V(G), E(G))$ with $V(G) = V(T)$ and $E(G) = E_B(T) \cup S$ for some $S \subseteq E_G(T)$. Let $\mathcal{R}(T)$ be the family of graphs that are a realization of T.

Definition 6.1.3. A trigraph T is *H-induced-saturated* if no realization of T contains H as an induced subgraph, but H occurs as an induced subgraph of some realization whenever any black or white edge of T is changed to gray. The *induced saturation number* $\mathrm{indsat}(n, H)$ of a forbidden H is the minimum number of gray edges over all n-vertex, H-induced-saturated trigraphs.

The *induced saturation number* of a graph H with respect to n, written $\mathrm{indsat}(n, H)$, is the minimum number of gray edges in an H-induced-saturated trigraph with n vertices.

Notice that a trigraph with $E_G(T) = \emptyset$ has a unique realization, so if $\mathrm{indsat}(n, H) = 0$, there is a graph G that has no induced copy of H yet adding or removing any edge creates an induced copy of H. We will call such a graph *H-induced-saturated*.

6.1.2 Observations and Previous Results

By definition, the only trigraphs on fewer than $|H|$ vertices that are H-induced-saturated are those in which all edges are gray. Thus we will usually assume that $n \geq |H|$ when we compute $\mathrm{indsat}(n, H)$.

The following theorem summarizes the results of Martin and Smith [36]:

Theorem 6.1.4. *Let H be a graph.*

- *For all $n \geq |H|$, $\mathrm{indsat}(n, H) \leq \mathrm{sat}(n; H)$. By [20], $\mathrm{sat}(n; H) \in O(n)$, so in particular $\mathrm{indsat}(n, H) \in O(n)$.*

- *For all $n \geq m \geq 3$, $\mathrm{indsat}(n, K_m) = \mathrm{sat}(n; K_m)$. (Note that $\mathrm{sat}(n; K_m)$ was determined by Erdős, Hajnal, and Moon in [14].)*

- *For all $n \geq m \geq 2$, and for $e \in E(K_m)$,* $\operatorname{indsat}(n, K_m - e) = 0$. *In particular, for all $n \geq 3$,* $\operatorname{indsat}(n, P_3) = 0$.
- *For all $n \geq 4$,* $\operatorname{indsat}(n, P_4) = \lceil \frac{n+1}{3} \rceil$.

Observation 6.1.5. A trigraph T is H-induced-saturated if and only if $\overline{T}$ is $\overline{H}$-induced-saturated. In particular, $\operatorname{indsat}(n, H) = \operatorname{indsat}(n, \overline{H})$.

Proof. Suppose a trigraph T has a realization G such that H is an induced subgraph of G. Then $\overline{H}$ is an induced subgraph of $\overline{G}$. Using the definition of $\overline{T}$, $\overline{G}$ is a representation of $\overline{T}$. It follows that a trigraph T is H-induced-saturated if and only if $\overline{T}$ is $\overline{H}$-induced-saturated. $\square$

6.1.3 Minimally H-induced-saturated Graphs

In this paper we show that for several graphs H, $\operatorname{indsat}(n, H) = 0$. That is, there exists a graph that is H-induced-saturated. This leads to a natural saturation question: What is the minimum number of edges in such a graph?

Definition 6.1.6. For a graph H and whole number n with $\operatorname{indsat}(n, H) = 0$, we define

$$\operatorname{indsat}^*(n, H) := \min\{\|G\| : |G| = n \text{ and } G \text{ is } H\text{-induced-saturated}\}.$$

We say a graph G on n vertices with $\operatorname{indsat}^*(n, H)$ edges is *minimally H-induced-saturated.*

By Observation 6.1.5, the *maximum* number of edges in an n-vertex H-induced-saturated graph is $\binom{n}{2} - \operatorname{indsat}^*(n, \overline{H})$.

In this chapter, we show that the following graphs (and their complements) have induced-saturation number zero for n sufficiently large: $K_{1,3}^+$, C_4, odd cycles of length at least 5, C'_{2k}, $\hat{C}_{2k}$, and matchings. Additionally, we provide bounds on $\operatorname{indsat}^*(n, H)$ for the graphs listed above. In particular, we characterize the $K_{1,3}^+$-induced-saturated graphs, which in turn completely determines $\operatorname{indsat}^*(n, K_{1,3}^+)$.

6.2 The Paw

In this section we provide a construction that shows $\operatorname{indsat}(n, K_{1,3}^+) = 0$ for $n \geq 7$. We then show that our construction characterizes all $K_{1,3}^+$-induced-saturated graphs, allowing us to completely determine $\operatorname{indsat}^*(n, K_{1,3}^+)$ for $n \geq 7$.

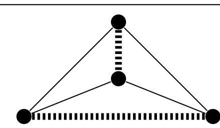
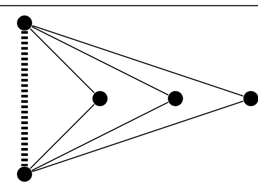
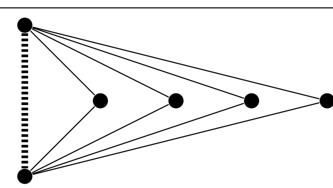
This construction, given in Construction 6.2.1 requires $n \geq 7$, and since Theorem 6.2.4 will show that these are the only $K_{1,3}^+$-induced-saturated graphs, we deduce that $\text{indsat}(n, K_{1,3}^+)$ is nonzero for $n \in \{4, 5, 6\}$. The exact values for such n are provided in Table 6.1.

Table 6.1 exhibits paw-induced-saturated trigraphs on n vertices with only one gray edge for $n \in \{5, 6\}$. Since $\text{indsat}(n, K_{1,3}^+) > 0$, this establishes $\text{indsat}(n, K_{1,3}^+) = 1$ for such n.

For $n = 4$, Table 6.1 gives a 4-vertex, paw-induced-saturated trigraph with two gray edges. To show that $\text{indsat}(4, K_{1,3}^+) = 2$, we argue that any 4-vertex trigraph T with only one gray edge is not $K_{1,3}^+$-induced-saturated.

T has at least two black edges, otherwise chaning a white edge to gray does not result in a realization with an induced $K_{1,3}^+$. Now suppose T has no white edges. Since it has precisely one gray edge, its black edges form $K_4 - e$, and changing the black edge whose endpoints have black degree three to a gray edge does not result in a realization with an induced $K_{1,3}^+$. Next, suppose T has at least two white edges. Since $K_{1,3}^+$ has precisely two nonedges, changing a black edge to gray does not result in a realization with an induced $K_{1,3}^+$., unless T already had such a realization. Therefore T has precisely one white edge. If the gray edge of T is incident to the white edge, then $K_{1,3}^+$ is a realization, so the black edges induce C_4. Since $C_4 \not\subseteq K_{1,3}^+$, changing the white edge to gray does not create an induced $K_{1,3}^+$.

Table 6.1: Values of $\text{indsat}(n, K_{1,3}^+)$ for $4 \leq n \leq 6$ and trigraphs realizing those values

	trigraph
$\text{indsat}(4, K_{1,3}^+) = 2$	
$\text{indsat}(5, K_{1,3}^+) = 1$	
$\text{indsat}(6, K_{1,3}^+) = 1$	

Having established $\text{indsat}(n, K_{1,3}^+)$ for small values of n, we now present our construction.

Construction 6.2.1. Let G be a graph with at most one trivial component, where each nontrivial component is complete multipartite, each with at least three parts, at most one of which contains only one vertex, and the remainder of which have order at least three.

Proposition 6.2.2. *The graph G in Construction 6.2.1 is $K_{1,3}^+$-induced-saturated.*

Proof. Since $K_{1,3}^+$ is not an induced subgraph of a complete multipartite graph, G contains no induced $K_{1,3}^+$. Suppose we add an edge xy such that x and y are in distinct components, say F_x and F_y, respectively. Since at least one of these components, say F_x, has at least three parts, x is in some triangle xab in F_x. Because y is in a different component, y is adjacent to x but not a or b. Thus $\{x, y, a, b\}$ induces a $K_{1,3}^+$.

Suppose we add an edge xy such that x and y are in the same component. Then in particular, they are in the same part. This part has at least two vertices, so by construction it has at least three vertices; choose z distinct from x and y from this part, and let a be in another part of the component. Then $\{x, y, z, a\}$ induces a $K_{1,3}^+$.

Suppose we delete an edge xy. Then x and y were in different parts of one component, say F. As F is complete multipartite with at least three parts, there exists a vertex z in a third part of that component. Since at most one part has only one vertex, there is a vertex a in the same part as either x or y; say x. Then $\{x, y, z, a\}$ induces a $K_{1,3}^+$. □

Corollary 6.2.3. *For $n \geq 7$,* $\mathrm{indsat}(n, K_{1,3}^+) = 0$.

We now show that Construction 6.2.1 describes all $K_{1,3}^+$-induced-saturated graphs.

Theorem 6.2.4. *A graph is $K_{1,3}^+$-induced-saturated if and only if it is as described in Construction 6.2.1.*

To prove this theorem, we begin by making several observations.

Lemma 6.2.5. *Let G be a $K_{1,3}^+$-induced-saturated graph. Then G has the following properties:*

(a) Every edge of G is in a triangle.

(b) The neighborhood of any vertex of G is a complete multipartite graph.

(c) Given any non-isolated vertex $v \in V(G)$, there exists a (possibly empty) independent set $S = S(v)$ such that for every $x \in N(v)$, $S = N(x) \setminus N[v]$.

Proof. Lemma 6.2.5(a) holds because deleting any edge in G creates an induced $K_{1,3}^+$. As a consequence, any vertex has degree either zero or at least two.

Since G does not contain an induced $K_{1,3}^+$, the neighborhood of any vertex cannot contain an induced copy of $K_2 \cup K_1$. This is equivalent to the neighborhood being a complete multipartite graph. This gives us Lemma 6.2.5(b).

To prove Lemma 6.2.5(c), suppose there exists $x \in N(v)$ that has a neighbor not in $N[v]$. (If no such x exists, the claim holds with $S = \emptyset$.) Let $S := N(x) \setminus N[v]$. If $G[S]$ has an edge ss', then $G[v, x, s, s']$ is a paw. Since G is $K_{1,3}^+$-induced-saturated, we conclude that S is independent.

By Lemma 6.2.5(a), there exists $y \in N(v) \cap N(x)$. If any element $s \in S$ is not adjacent to y, then $G[v, x, y, s]$ is a paw with s as the pendant vertex. Therefore, $S \subseteq N(y)$, but also $N(y) \backslash N[v] \subseteq S$ or else we would have a paw. Because $N(v)$ is complete multipartite by Lemma 6.2.5(b), every vertex in $N(v) \setminus \{x, y\}$ is adjacent to x or y. By symmetry, we conclude that for every $z \in N(v)$, $N(z) \setminus N[v] = S$. □

We proceed to the proof of Theorem 6.2.4.

Proof of Theorem 6.2.4. Let G be a $K_{1,3}^{+}$-induced-saturated graph. Then G has at most one nontrivial component, since adding an edge between two isolated vertices does not create an induced $K_{1,3}^{+}$. We now show that every nontrivial component of G is a complete multipartite graph. Let v be a non-isolated vertex in G and let S be the set given by Lemma 6.2.5(c). By Lemmas 6.2.5(b) and 6.2.5(c), $G[N[v] \cup S]$ is a complete multipartite graph, with v and S sharing a part. So, we need only show $N[v] \cup S$ is a component of G. If not, then there exists some vertex $s \in S$ with a neighbor $t \notin N[v] \cup S$, since we have included the neighborhood of every $x \in N[v]$ and S is an independent set. If there exists an edge xy in $G[N(v)]$, then $G[x, y, s, t]$ is a paw, so $N(v)$ is an independent set. This violates Lemma 6.2.5(a).

Now, by Lemma 6.2.5(a), every nontrivial component of G has at least three parts. Next, we show that no part in any component of G has order two, and any component has at most two parts of order one. Suppose x and y either make up a part of order two, or are each a part of order one in a component F. Then $\{x, y\}$ dominates $F \setminus \{x, y\}$, and so x and y do not appear together in an induced paw, so adding or deleting the edge xy does not create an induced paw. Hence, G being $K_{1,3}^{+}$-induced-saturated implies that it can be formed by Construction 6.2.1. □

Corollary 6.2.6. *For $n \geq 7$, let $n \equiv r \mod 7$, where $0 \leq r \leq 6$. Then*

$$\operatorname{indsat}^*(n, K_{1,3}^{+}) = \begin{cases} \dfrac{15}{7}n & \text{if } r = 0 \\ 15\lfloor n/7 \rfloor + 4(r-1) & \text{if } r \neq 0 \end{cases}.$$

Proof. Let G be a minimally $K_{1,3}^{+}$-induced-saturated graph on n vertices. From Theorem 6.2.4, each nontrivial component of G is a complete multipartite graph with at least three parts. If some nontrivial component F of G has at least three parts, then we form a $K_{1,3}^{+}$-induced-saturated graph with strictly fewer edges by dropping edges between two of the parts and forming a single larger part. Thus each nontrivial component of G is tripartite.

The number of edges of a complete tripartite graph on m vertices with parts of size s, t, and $m-(s+t)$ is given by $(m-[s+t])(s+t)+st$. Given the constraints $s \geq 1, t \geq 3$, and $m \geq t$, we see that $(m-[s+t])(s+t)$

is minimized when $s+t$ is minimized, i.e. $s+t=4$; also st is minimized when $s+t$ is minimized. Therefore, $K_{1,3,m-4}$ obtains the smallest number of edges among all complete tripartite graphs on m vertices.

Now, we may assume G has components $F_0, F_1, \ldots, F_i$ with $|F_0| \in \{0,1\}$ and for $i>0$, $F_i = K_{1,3,n_i-4}$, where $|F_0| + \sum_{i=1}^{k} n_i = n$. Then:

$$e(G) = \sum_{i=1}^{k} e(F_i) = \sum_{i=1}^{k} (4n_i - 13) = 4n - 13k - 4|F_0|.$$

This is minimized by taking k as big as possible and, subject to this, $|F_0| = 1$. That is, we take $k = \lfloor n/7 \rfloor$ and

$$|F_0| = \begin{cases} 0 & \text{if 7 divides } n \\ 1 & \text{else.} \end{cases}$$

□

Observation 6.2.7. Given H for which $\text{indsat}^*(n,H)$ is defined for all sufficiently large n, the function $\text{indsat}^*(n,H)$ is not necessarily monotone in n. In particular, from Corollary 6.2.6 we see for any integer $k \geq 2$, $\text{indsat}^*(7k, K_{1,3}^+) < \text{indsat}^*(7k+2, K_{1,3}^+) < \text{indsat}^*(7k-1, K_{1,3}^+)$. This is a similarity between minimal induced saturation and saturation: as noted in [15], the function $\text{sat}(n;H)$ is not necessarily monotone in n for fixed H.

6.3 Small Cycles

6.3.1 C_4 and its complement

In this section we show that the induced saturation number of C_4 is zero for sufficiently large n, and we compute some bounds on $\text{indsat}^*(n, C_4)$. Additionally, using Observation 6.1.5 and the fact that $\overline{C_4} = 2K_2$, we use C_4-induced-saturated graphs to obtain results for matchings.

Construction 6.3.1. For $j \geq 5$ and $k \geq 2$, let I_j^k be the graph that combines k copies of a wheel with j spokes. Label the copies $W^1, \ldots, W^k$, and label the vertices of W^i so that its center is w_0^i, and the outer cycle of W^i is $w_1^i, \ldots, w_j^i$. For $1 \leq i < i^*$, add the edges $w_\ell^i w_\ell^{i^*}$ and $w_\ell^i w_{\ell+1}^{i^*}$ for every $\ell \in [j]$, defining $j+1 := 1$.

I_5^2 is the icosahedron, shown in Figure 6.1. The icosahedron can be thought of as two wheels with 5 spokes whose outer-cycle vertices are joined by a zig-zag pattern (as described precisely in Construction 6.3.1). Construction 6.3.1 generalizes the icosahedron by allowing the number of wheels and the length of their outer cycles to vary.

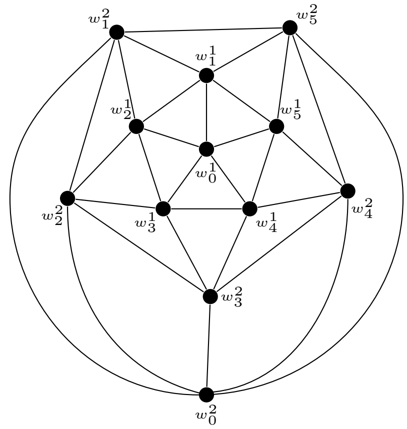


Figure 6.1: The icosahedron graph.

Proposition 6.3.2. *For $j \in \{5, 6, 7\}$, and $k \geq 2$, I_j^k is C_4-induced-saturated.*

Proof. We first show that I_j^k does not contain an induced C_4. Suppose to the contrary that it does. Since a single wheel does not contain an induced C_4, this C_4 must contain vertices from at least two different wheels. Suppose that w_0^p is in this C_4. Recall that w_0^p is the center of wheel W^p. Then, this C_4 must contain w_r^p and w_s^p such that $|s - r| \geq 2$. However, since $|s - r| \geq 2$, w_r^p and w_s^p contain no common neighbors outside of W^p. Thus, all four vertices of this induced C_4 must be inside of W^p, a contradiction. So our induced C_4 contains no centers of wheels.

If this C_4 contains exactly three vertices from a single W^p, then they must be consecutive along their cycle. That is, C_4 contains w_s^p, w_{s+1}^p, and w_{s+2}^p. However, as above, w_s^p and w_{s+2}^p have no common neighbors outside of W^p. Thus, our induced C_4 contains at most two vertices from each W^p.

If this C_4 contains exactly two vertices from a single W^p, then by the above arguments, they are adjacent in W^p, say w_s^p and w_{s+1}^p. No vertex of the form w_s^q, with $q < p$, or w_{s+1}^r, with $r > p$, can be in our C_4, as either produces a triangle with w_s^p and w_{s+1}^p.

Now, w_{s+1}^p must have another neighbor in our C_4. Suppose it is in W^t. If $t > p$, then it must be w_{s+2}^t by the above. However, the only common neighbors w_{s+2}^t and w_s^p have are of the form w_{s+1}^q where $q > p$, a contradiction. So $t < p$, and the other neighbor of w_{s+1}^p is w_{s+1}^t. Again though, the only common neighbors of w_s^p and w_{s+1}^t are either of the form w_{s+1}^q where $q > p$, or w_s^r where $r < p$. In either case, we have a contradiction to the above. Thus, our C_4 has exactly one vertex from each wheel.

Suppose our induced C_4 contains the vertices $w_{t_1}^p, w_{t_2}^q, w_{t_3}^r, w_{t_4}^s$. If $|\{t_1, t_2, t_3, t_4\}| \leq 2$, then we have a triangle, a contradiction. If $|\{t_1, t_2, t_3, t_4\}| = 4$, then some vertex is not adjacent to two of the others, a contradiction. So $|\{t_1, t_2, t_3, t_4\}| = 3$, and two vertices have the same subscript. We may assume that it is $w_{t_1}^p, w_{t_2}^q = w_{t_1}^q$, and that $p < q$. Then, $w_{t_1}^p$ must have a neighbor not adjacent to $w_{t_1}^q$ in this C_4, say it is $w_{t_3}^r$.

However, in order for this to be possible, we must have $t_3 = t_1 + 1$ and $p < r < q$. Thus, $w_{t_4}^s$ is adjacent to both $w_{t_1+1}^r$ and $w_{t_1}^q$. However, since t_4 must be distinct from both t_1 and $t_1 + 1$, this cannot happen, a contradiction. So I_j^k is C_4-free.

By inspection we see that I_j^k has the property that every edge is the lone diagonal of a C_4. Thus, removing any edge results in an induced C_4. So we only need to consider adding edges. Adding an edge within one wheel (say W^m) is simply adding a chord $w_i^m w_p^m$ to a 5-, 6-, or 7-cycle. If $p \neq i+2$ or $j = 5$, then this chord creates an induced 4-cycle. If $p = i+2$ and $j = 6$ or $j = 7$, then if $m \neq k$, $w_i^m w_{i+1}^\ell w_{i+2}^\ell w_{i+2}^m w_i^1$ is an induced 4-cycle, where $\ell > m$, and if $m = k$, then $w_i^m w_{i+1}^m w_{i+1}^\ell w_i^\ell$ is an induced C_4.

Now suppose we add an edge between wheels, say W^m and W^ℓ, where we may assume $m < \ell$. If the new edge is between the centers of these wheels, that is, $w_0^m w_0^\ell$, then $w_0^m w_0^\ell w_1^\ell w_1^m w_0^m$ is an induced C_4. If it is from the center of W_m to a vertex on the cycle of W^ℓ, say w_i^ℓ, then $w_0^m w_i^\ell w_{i+1}^\ell w_{i+1}^m w_0^m$ is an induced C_4; a similar cycle is also created if the new edge is $w_0^\ell w_i^m$. Finally, if we add an edge $w_i^m w_p^\ell$, note that w_i^m is not adjacent to at least one of w_p^m and w_{p-1}^m; label this vertex u. Since u is adjacent to w_p^ℓ, the vertices w_0^m, w_i^m, w_p^ℓ, and u induce a C_4. □

Proposition 6.3.2 implies that for many values of n, $\mathrm{indsat}(n, C_4) = 0$. In fact, this is the case for $n \geq 12$. To show this, we use the following proposition regarding kK_2. While we only employ the proposition in the case $k = 2$, the more general statement which we present is not difficult.

Proposition 6.3.3. *Let $s := (s_1, \ldots, s_n)$ be a sequence of positive integers. Let G be a graph with vertex set $\{v_1, \ldots v_n\}$, and let G_s be the graph obtained from G by replacing each vertex v_i with an independent set of order s_i and each edge with a complete bipartite graph between the corresponding independent sets. For $k \geq 2$, G is kK_2-induced-saturated if and only if G_s is kK_2-induced-saturated.*

Proof. For each vertex $v_i \in V(G)$, let V_i be the independent set in G_s that corresponds to it. We will call this collection of vertices in G_s that replaces a single vertex in G a *part*.

Note that no induced matching in G_s uses two vertices from the same part, and the same holds if we add or remove a single edge from G_s. We claim that if w_i and w_j are vertices from different parts V_i and V_j, respectively, of G_s, then G_s (or $G_s + w_i w_j$ or $G_s - w_i w_j$) contains an induced matching if and only if G (resp. $G + v_i v_j$, or $G - v_i v_j$) contains an induced matching M. Suppose M_s is such an induced matching in G_s (or $G_s + w_i w_j$ or $G_s - w_i w_j$). Then each vertex in M_s comes from a different part of G_s (resp. $G_s + w_i w_j$ or $G_s - w_i w_j$), and thus they correspond to distinct vertices in $V(G)$. This is an induced matching in G.

If G (or $G + v_i v_j$ or $G - v_i v_j$) has an induced matching M, then when the graph is expanded, no new adjacencies have been added between the parts corresponding to the endpoints of vertices in M (except for

w_iw_j in the case of $G+v_iv_j$). Thus, we can find an induced matching in G_s (resp. $G_s+w_iw_j$ or $G_s-w_iw_j$). This shows that if G_s is kK_2-induced-saturated, then so is G.

To show that if G is kK_2-induced-saturated, then so is G_s, it remains to consider adding edges between vertices in one part of G_s. First we note that G has no dominating vertex. Indeed, if u is a dominating vertex, then deleting an edge incident to u, say uw, does not create an induced $2K_2$, let alone an induced kK_2, as u dominates $N_G(w)$.

Now, suppose we add w_iw_i' to G_s, in the part V_i corresponding to v_i. Since v_i is not dominating, there exists w not adjacent to v_i. Since G is kK_2-induced-saturated, $G+v_iw$ contains an induced matching $M=\{v_iw, x_2y_2, \ldots, x_ky_k\}$. Then $M_s=\{w_iw_i', X_2Y_2, \ldots, X_kY_k\}$ is an induced matching in $G_s+w_iw_i'$, where X_j and Y_j are vertices in the parts corresponding to x_j and y_j, respectively. ☐

Corollary 6.3.4. *For $n \geq 12$,* $\text{indsat}(n, C_4)=0$*.*

Proof. Applying Observation 6.1.5 to case $k=2$ in Proposition 6.3.3, allows us to begin with a graph that is C_4-induced-saturated, replace a single vertex with a clique of any order, replace the affected edges with complete bipartite graphs, and produce another graph that is C_4-induced-saturated. Thus, beginning with I_5^2, applying these operations obtains C_4-induced-saturated graphs for all values of $n \geq 12$. ☐

Observation 6.3.5. Recall that Observation 6.1.5 states that a graph is H-induced-saturated if and only if its complement is $\overline{H}$-induced-saturated. Thus, beginning with a graph that is C_4-induced-saturated, the operations of replacing a single vertex with a clique of some order and replacing the affected edges with complete bipartite graphs produces another graph that is C_4-induced-saturated. This shows that we can create C_4-induced-saturated graphs for any $n \geq 12$ by applying these operations to the graphs in Construction 6.3.1. Thus, $\text{indsat}(n, C_4)=0$ for all $n \geq 12$.

For $4 \leq n \leq 10$, a computer search showed $\text{indsat}(n, C_4) > 0$. At this time, whether $\text{indsat}(11, C_4)$ is zero or not, is yet unknown. We now turn our attention to $\text{indsat}^*(n, C_4)$.

Theorem 6.3.6. *For sufficiently large n, $(5/2)n \leq \text{indsat}^*(n, C_4) \leq (7/64)n^2 + o(n)$.*

Proof. To prove the lower bound we show that $\delta(G) \geq 5$. Suppose G is a C_4-induced-saturated graph. Let $x \in V(G)$, and let $H := G[N(x)]$. Since deleting any edge produces an induced C_4, every edge is the diagonal of a C_4 and $d(x) \geq 3$. In particular, there exists $v_1, v_2, v_3 \in V(H)$ such that v_1v_3 is not an edge, but v_1v_2 and v_2v_3 are edges. Now, $G-xv_1$ contains an induced C_4 that contains both x and v_1, but not v_3. If v_2 is not in this C_4, then there exists two other vertices distinct from v_1, v_2, v_3 in H. Thus, $d(x) \geq 5$. If v_2 is in this C_4, then there exists $v_4 \in V(H)$ distinct from v_1, v_2, v_3 such that v_1v_4 is an edge, but v_2v_4 is not. By a similar argument, considering $G - xv_3$ gives at least one additional vertex in H distinct from v_1, v_2, v_3, v_4. So in any case, $d(x) \geq 5$, and as x was arbitrary, $\delta(G) \geq 5$. Thus, provided $n \geq 12$, $\mathrm{indsat}^*(n, C_4) \geq (5/2)n$.

To prove the upper bound, we choose $n \geq 56$ and create a graph G of order n. Let $r \equiv n \mod 8$, where $0 \leq r \leq 7$. Set $k = \lfloor n/8 \rfloor$ so that $k \geq r$ and $|I_7^k| = 8k$. If $r = 0$, choose $G = I_7^k$. If $r > 0$, we create G by adding r vertices to I_7^k. Recall, as discussed after Proposition 6.3.3, by replacing the vertices of I_7^k with cliques, and its edges with complete bipartite graphs, we preserve the property of being C_4-induced-saturated. Accordingly, using the notation of Construction 6.3.1, we replace $w_0^1, \ldots, w_0^r$ with copies of K_2 and make each new vertex adjacent to the neighborhood of the vertex it replaces.

Now we determine $e(G)$. The first r wheels have 22 edges, and the rest have 14. Between any two wheels there are 14 edges. So $e(G) = 14\left[\binom{k}{2} + k\right] + 8r$. Since $r \in [0, 7]$ and $k = \lfloor n/8 \rfloor$, $e(G) \leq \frac{7}{64}n^2 + \frac{7}{8}n + 56$.

□

6.3.2 Matchings

Another graph that is C_4-induced-saturated is the join $I_5^2 \vee K_{n-12}$. Observation 6.1.5 implies that the complement of this graph is $2K_2$-induced-saturated. We can further generalize this to get a kK_2-induced-saturated graph for any $k \geq 2$.

Proposition 6.3.7. *Let $\overline{I_5^2}$ be the complement of the icosahedron. For fixed k and $n \geq 12(k-1)$, the graph $(k-1)\overline{I_5^2} + (n - 12(k-1))K_1$ is kK_2-induced-saturated. Thus, for $n \geq 12(k-1)$, $\mathrm{indsat}(n, kK_2) = 0$.*

Proof. By Proposition 6.3.2 and Observation 6.1.5, the complement of an icosahedron is $2K_2$-induced-saturated. Let G denote $(k-1)\overline{I_5^2} + (n - 12(k-1))K_1$. Clearly, G contains $(k-1)K_2$ as an induced subgraph, but no induced kK_2. If we add or delete any edge inside a component, or add an edge among the isolates, we create an induced kK_2. Note that every vertex v in $\overline{I_5^2}$ is in an induced copy of $K_2 + K_1$ where v is the isolate. Thus, adding any edge with an endpoint in a copy of $\overline{I_5^2}$ creates an induced kK_2. □

Corollary 6.3.8. *For $n \geq 12(k-1)$, $\mathrm{indsat}^*(n, kK_2) \leq 36(k-1)$.*

In particular, for fixed k, $\mathrm{indsat}^*(n, kK_2)$ is bounded above by a constant.

6.3.3 C_8

Construction 6.3.9. The dodecahedron (Figure 6.2) is C_8-induced-saturated.

Proof. Clearly, the dodecahedron is C_8 free. Because every edge is on the boundary between two 5-faces, deleting any edge creates an induced C_8. To check what happens when we add edges, note that any pair of vertices at distance d from each other is symmetric with any other pair at distance d. Therefore, it suffices to check only a generic pair of vertices at distance d for $d \in \{2, 3, 4, 5\}$, Figure 6.3. $\square$

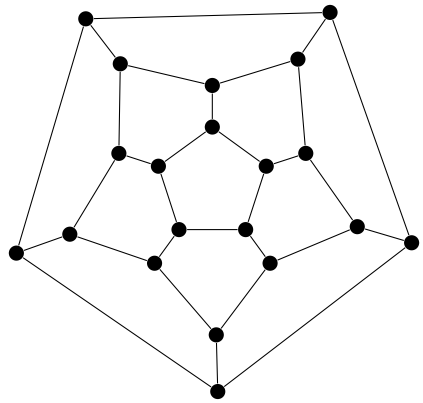


Figure 6.2: Dodecahedron

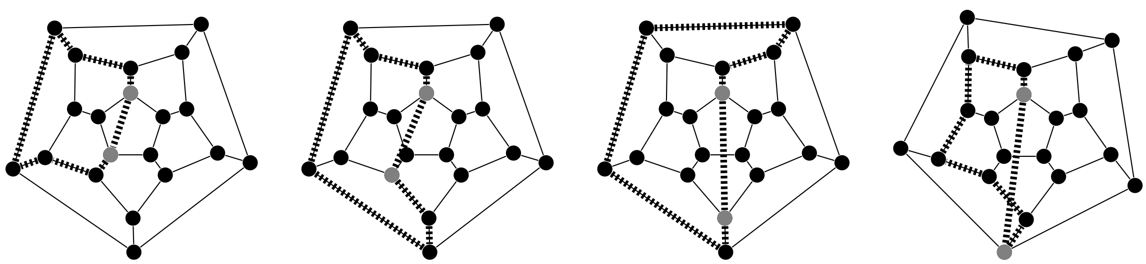


Figure 6.3: Adding Edges to Dodecahedron to obtain C_8

While we cannot add a dominating clique to the dodecahedron to obtain graphs that are C_8-induced-saturated, we can add a few vertices in a nice way to show that the induced saturation number of C_8 is zero for a small range of n.

Construction 6.3.10. Let D be a dodecahedron. Note that we can partition $V(D)$ into five sets S_1, S_2, S_3, S_4, S_5 such that each S_i has exactly four vertices in it that are pairwise distance 3 apart in D. An example of one such S_i is in Figure 6.4. It is easy to see from the figure how we can obtain the other sets in the partition by rotating the first five times along the dodecahedron.

Let G_k be the graph obtained from D by adding vertices $x_1, \ldots, x_k$ such that $N_G(x_i) = S_i$, $1 \leq i \leq k \leq 5$. Then, G_i is C_8-induced-saturated.

Proof. The proof that G_1 is C_8-induced-saturated is based on two observable facts. First, given any pair of vertices u, v in S_1, there exists an induced path between them of length six that contains exactly one other vertex in S_1 and does not contain x_1 (see Figure 6.4). Second, for any vertex $u \in S_1$ and $v \notin S_1$, $v \neq x_1$, there exists an induced path between them of length six that contains no internal vertex in $S_1 \cup \{x_1\}$.

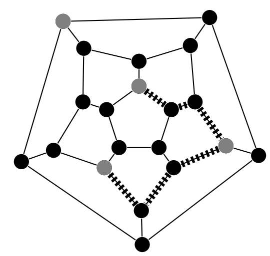


Figure 6.4: Set S in a dodecahedron, with induced P_7

These facts correspond to removing and adding an edge incident to x_1, respectively. As we know that the dodecahedron is C_8-induced-saturated, this suffices to show that G_1 is C_8-induced-saturated.

The only additional fact needed to show that G_k is C_8-induced-saturated, for $2 \leq k \leq 5$, is that for any i, j, where $1 \leq i < j \leq k$, and for any $u \in S_i, v \in S_j$, there exists an induced path between u and v of length five that contains no internal vertex in $S_i \cup S_j \cup \{x_1, \ldots, x_k\}$.

This additional fact corresponds to adding the edge $x_i x_j$. As the dodecahedron is C_8-induced-saturated, and adding or removing edge incident to a single x_i produces an induced C_8, we deduce that G_k is indeed C_8-induced-saturated. □

Corollary 6.3.11. $\text{indsat}(n, C_8) = 0$*, for* $20 \leq n \leq 25$.

While the icosahedron is C_4-induced-saturated and the dodecahedron is C_8-induced-saturated, the cube is not C_6-induced-saturated.

6.4 Odd Cycles and Modified Cycles

In this section we provide a construction proving that odd cycles also have induced saturation number zero for n sufficiently large. As it is already known that $\text{indsat}(n, C_3) = \text{sat}(n; C_3)$ [36], we only consider odd cycles of length at least five. Additionally, this construction is also H-induced-saturated when H is a modification of an even cycle as described below.

Let C'_{2k} denote a cycle of length $2k$ with a pendant vertex, and let $\hat{C}_{2k}$ denote an even cycle with a chord between two vertices at distance 2 from each other (sometimes called a triangle chord, or hop).

For a given k and $n \geq (k+1)^2 + 2$, we can write n as $(k+1)t - s$ where t and s are integers with $t \geq k+2$ and $0 \leq s \leq t - 3$. In particular, we choose $t = \lceil \frac{n}{k+1} \rceil$. Using this expression for n, we give the following construction.

Construction 6.4.1. For $k \geq 3$ and $n \geq (k+1)^2 + 2$, let $n = (k+1)t - s$, where $t = \lceil \frac{n}{k+1} \rceil \geq k + 2$ and $0 \leq s \leq t - 3$. Let $G_{n,k}$ be formed from the Cartesian product $K_{k+1} \square K_t$ by removing s vertices from one copy of K_t.

Proposition 6.4.2. *If $H \in \{C_{2k-1}, C'_{2k}, \hat{C}_{2k}\}$ for some $k \geq 3$, then the graph $G_{n,k}$ in Construction 6.4.1 is H-induced-saturated.*

Proof. Let $G_{n,k}$ be as described in Construction 6.4.1. We first show that $G_{n,k}$ is H-free for $H \in \{C_{2k-1}, C'_{2k}, \hat{C}_{2k}\}$. Any induced subgraph of $G_{n,k}$ that is triangle-free has at most two vertices from any copy of K_{k+1} or K_t. Since $2k-1$ is odd, an induced C_{2k-1} would contain precisely one vertex v from some copy of K_{k+1}. Then the neighbors of v must be in the same copy of K_t, which means they form a triangle. Thus, $G_{n,k}$ has no induced odd cycle larger than a triangle. Since $\hat{C}_{2k}$ contains C_{2k-1} as an induced subgraph, neither C_{2k-1} nor $\hat{C}_{2k}$ are induced subgraphs of $G_{n,k}$. Similarly, if $G_{n,k}$ contained an induced C'_{2k} subgraph, then because C'_{2k} is triangle-free with an odd number of vertices, there would be one copy of K_t containing precisely one vertex v of the subgraph. If v is on the cycle, it has at least two neighbors, but these can only be other copies of v, forming a triangle in some copy of K_{k+1}. If v is the pendant vertex, suppose it has neighbor u on the cycle. Then u has some neighbor u' in a different copy of K_t from itself, and u, u', and v are all in one copy of K_{k+1}, forming a triangle. Thus, $G_{n,k}$ has no induced C'_{2k}.

In the remainder of this proof we view $K_{k+1} \square K_t$ as a t-by-$(k+1)$ grid with vertices $v_{i,j}$ for $1 \leq i \leq t$ and $1 \leq j \leq k+1$, where two vertices are adjacent if and only if they share a row or column. Note that we can permute rows, or columns, by changing only the labeling of the vertices. We form $G_{n,k}$ by removing s vertices from a copy of K_t. Let j^* be the index of the column with the s removed vertices. Since $s \leq t-3$ and $k \geq 3$, $G_{n,k}$ has at least three vertices in each row and column.

To complete this proof, we show that adding or deleting any edge of $G_{n,k}$ creates an induced copy of H, for every $H \in \{C_{2k-1}, \hat{C}_{2k}, C'_{2k}\}$. In order to show this, we first add or delete the edge in $K_{k+1} \square K_t$, and find an induced copy of H in that graph. Since $G_{n,k}$ is an induced subgraph of $K_{k+1} \square K_t$, it remains only to show that by permuting rows and columns appropriately, $V(H) \subseteq V(G_{n,k})$.

Consider adding an edge to $K_{k+1} \square K_t$. Up to relabeling, we may assume that $v_{1,1}v_{k+1,k+1}$ is the added edge, and $j^* \neq 1$. Let $T' := \{v_{i,i+1}, v_{i+1,i+1} : 1 \leq i \leq k-2\}$, and let $T := T' \cup \{v_{1,1}, v_{k+1,k+1}\}$. Then $V_1 := T \cup \{v_{k-1,k+1}\}$ induces C_{2k-1}, $V_2 := T \cup \{v_{k-1,k}, v_{k-1,k+1}\}$ induces $\hat{C}_{2k}$, and $V_3 := T \cup \{v_{k-1,k}, v_{k,k}, v_{k,1}\}$ induces C'_{2k}. Below, we show how to permute rows and columns of $G_{n,k}$ so that $V_i \subseteq V(G_{n,k})$ for every $i \in [3]$. Note that, since we assume we are adding edge $v_{1,1}v_{k+1,k+1}$, we do not permute rows or columns containing the endpoints of this edge. That is, we leave fixed row 1, column 1, row $k+1$, and column $k+1$.

Case 6.4.2.1. $j^* = k+1$.

Since $G_{n,k}$ has at least three vertices in every column, there is at least one vertex of column $k+1$ that is not in row 1 or $k+1$; then we arrange rows so that $v_{k-1,k+1} \in V(G_{n,k})$. That is, the s deleted vertices of column j^* were deleted from rows other than $k-1$ and $k+1$, and we achieve this indexing by permuting only rows from $\{2, \ldots, k\}$, leaving the indexing of the added edge intact. With this new indexing, $V_i \subseteq V(G_{n.k})$ for every $i \in [3]$.

Case 6.4.2.2. $k+1 \neq j^*$, and $v_{1,j^*} \notin V(G_{n,k})$.

We permute columns so that $j^* = k$. By the case, there exist at least 2 vertices in column j^* that lie in rows other than 1 and $k+1$. We arrange rows so that $v_{k-1,k}$, $v_{k,k} \in V(G_{n,k})$. With this labeling, $V_i \in V(G_{n,k})$ for every $i \in [3]$.

Case 6.4.2.3. $k+1 \neq j^*$, and $v_{1,j^*} \in V(G_{n,k})$.

Permute columns so that $j^* = 2$. Then $v_{1,2} \in V(G_{n,k})$, and there is at least one vertex in column 2 in a row other than 1 or $k+1$. Permute rows so that $v_{2,2} \in V(G_{n,k})$. Now $V_i \in V(G_{n,k})$ for every $i \in [3]$.

Now consider deleting an edge of $K_{k+1} \,\square\, K_t$. Up to relabeling, we need only consider deleting $v_{1,1}v_{1,2}$ or $v_{1,2}v_{2,2}$. Suppose first we delete $v_{1,1}v_{1,2}$. Without loss of generality, we may assume $j^* \neq 2$. Now $U_1 := T' \cup \{v_{1,1}, v_{k-1,1}, v_{1,k}\}$ induces C_{2k-1}; $U_2 := T' \cup \{v_{1,1}, v_{k-1,1}, v_{k-1,k}, v_{1,k+1}\}$ induces $\hat{C}_{2k}$; and $U_3 := T' \cup \{v_{1,1}, v_{k-1,k}, v_{k,k}, v_{k,k+1}, v_{1,k+1}\}$ induces C'_{2k}. Below, we show how to permute rows and columns of $G_{n,k}$ so that $U_i \subseteq V(G_{n,k})$ for every $i \in [3]$. Note that, since we delete edge $v_{1,1}v_{1,2}$, we do not permute row 1, column 1, or column 2.

Case 6.4.2.4. $j^* = 1$.

There exists some vertex in column 1 other than $v_{1,1}$. Permute rows so that $v_{k-1,1} \in V(G_{n,k})$. Now $U_i \in V(G_{n,k})$ for every $i \in [3]$.

Case 6.4.2.5. $j^* \geq 3$ and $v_{1,j^*} \in V(G_{n,k})$.

Permute columns so $j^* = k+1$. There exists some vertex of $G_{n,k}$ in column j^* not from row 1; permute rows so that $v_{k,k+1} \in V(G_{n,k})$. Now $U_i \in V(G_{n,k})$ for every $i \in [3]$.

Case 6.4.2.6. $j^* \geq 3$ and $v_{1,j^*} \notin V(G_{n,k})$.

First, permute columns so that $j^* = k+1$. Then $U_1 \in V(G_{n,k})$. Second, permute columns so that $j^* = k$. There exist at least two vertices in column j^* not in row 1, so permute rows so that $v_{k-1,k}, v_{k,k} \in V(G_{n,k})$. Now $U_2, U_3 \in V(G_{n,k})$.

Finally, suppose we delete $v_{1,2}v_{2,2}$ from $K_{k+1} \,\square\, K_t$. Now $W_1 := T' \cup \{v_{1,k}, v_{k,2}, v_{k-1,k}\}$ induces C_{2k-1}; $W_2 := T' \cup \{v_{1,k+1}, v_{k,2}, v_{k-1,k}, v_{k-1,k+1}\}$ induces $\hat{C}_{2k}$; and
$W_3 := T' \cup \{v_{k,2}, v_{k,k}, v_{k+1,k}, v_{k+1,k+1}, v_{k-1,k+1}\}$ induces C_{2k-1}. Below, we show that by permuting rows and columns of $G_{n,k}$ (other than row 1, row 2, or column 2), we find $W_i \subseteq V(G_{n,k})$ for every $i \in [3]$.

Case 6.4.2.7. $j^* = 2$

Since we deleted edge $v_{1,2}v_{2,2}$, both its endpoints are in $G_{n,k}$. There exists some vertex in column 2 other than these; permute rows so that $v_{k,2} \in V(G_{n,k})$. Now $W_i \in V(G_{n,k})$ for every $i \in [3]$.

Case 6.4.2.8. $j^* \neq 2$

Permute columns so that $j^* = 1$. Now $W_i \in V(G_{n,k})$ for every $i \in [3]$.

We conclude the graph $G_{n,k}$ in Construction 6.4.1 is H-induced-saturated for every $H \in \{C_{2k-1}, C'_{2k}, \hat{C}_{2k}\}$ and$k \geq 3$. $\square$

Corollary 6.4.3. *For all $k \geq 3$, if $n \geq (k+1)^2 + 2$ and $H \in \{C_{2k-1}, C'_{2k}, \hat{C}_{2k}\}$, then* $\operatorname{indsat}(n, H) = 0$.

In the following discussion assume $H \in \{C_{2k-1}, C'_{2k}, \hat{C}_{2k}\}$. Using Construction 6.4.1 we obtain an upper bound on $\operatorname{indsat}^*(n, H)$ with order of magnitude n^2, which is trivial. We can improve this order of magnitude slightly in the case when $\lceil \sqrt{n}\, \rceil$ is not prime. To do so we note that if n can be written as a product of two integers s and t that are both at least k, then the graph $K_s \,\square\, K_t$ is H-induced-saturated.

Proposition 6.4.4. *Fix $k \geq 3$ and choose n such that $n^{1/4} \geq k+1$. For $H \in \{C_{2k-1}, C'_{2k}, \hat{C}_{2k}\}$, if $\lceil \sqrt{n}\, \rceil$ is divisible by some $t \geq 3$,* $\operatorname{indsat}^*(n, H) \leq cn^{7/4} + O(n^{3/2})$ *for some constant c.*

Proof. As noted above, the Cartesian product of two sufficiently large cliques is H-induced-saturated. So, consider $G := K_{\lceil \sqrt{n}\, \rceil/t} \,\square\, K_{t\lceil \sqrt{n}\, \rceil}$. Simple computation shows $n \leq |G| \leq n + 2\sqrt{n} + 1$. So, $|G|$ can be written as $n + s$, where $0 \leq s \leq 2\sqrt{n} + 1 \leq t\sqrt{n} - 3$, as $t \geq 3$. Let G' be obtained from G by removing s vertices from a single copy of $K_{3\lceil \sqrt{n}\, \rceil}$ as in Construction 6.4.1. An argument similar to that in Proposition 6.4.2 shows that G' is H-induced-saturated. Observe:

$$e(G') \leq t\lceil \sqrt{n}\, \rceil \binom{(1/t)\lceil \sqrt{n}\, \rceil}{2} + (1/t)\lceil \sqrt{n}\, \rceil \binom{t\lceil \sqrt{n}\, \rceil}{2} = \frac{\lceil \sqrt{n}\, \rceil^2}{2} \left(\left(t + \frac{1}{t} \right) \lceil \sqrt{n}\, \rceil) - 2 \right).$$

Since t divides $\lceil \sqrt{n}\, \rceil$, $t \leq \sqrt{\lceil \sqrt{n}\, \rceil} \leq c' n^{1/4}$ for some $c' > 1$. Using this and $\lceil \sqrt{n}\, \rceil \leq \sqrt{n} + 1$ gives $e(G') \leq \frac{c'}{2} n^{7/4} + O(n^{3/2})$. $\square$

Considering odd cycles points out another property of the induced saturation number. That is, if $\operatorname{indsat}(n, H) = 0$ for a particular n, it is not necessarily the case that $\operatorname{indsat}(k, H) = 0$ for all $k > n$. For example, Construction 6.4.1 shows $\operatorname{indsat}(n, C_5) = 0$ for $n = 9$ and $n \geq 12$. However, a computer search showed that for $n = 10$ and $n = 11$, we have $\operatorname{indsat}(n, C_5) > 0$. (A C_5-induced-saturated trigraph on 10 vertices with one gray edge is shown in Figure 6.5, so that $\operatorname{indsat}(10, C_5) = 1$.)

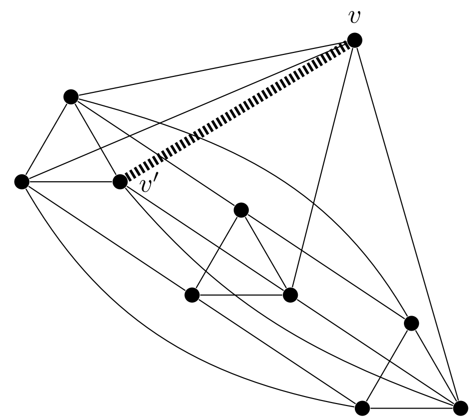


Figure 6.5: This trigraph, with the gray edge vv', is C_5-induced-saturated.

The fact that, for sufficiently large n, $\mathrm{indsat}(n, H) = 0$ when H is an odd cycle or a modification of an even cycle raises the question of whether a similar construction exists for even cycles. Note that $\mathrm{indsat}(n, C_4) = 0$ for sufficiently large n, and $\mathrm{indsat}(n, C_8) = 0$ for a narrow interval of values for n. If $\mathrm{indsat}(n, C_8) > 0$ for sufficiently large n, this would be the first known example of a graph H with induced saturation number zero for a number of non-trivial n, but not for large n. Conversely, if $\mathrm{indsat}(n, C_8) = 0$ for all sufficiently large n, this would make the case of even cycles all the more interesting. In particular, is $\mathrm{indsat}(n, C_6) = 0$ for sufficiently large n? Is there some condition on k that predicts whether $\mathrm{indsat}(n, C_{2k}) = 0$?

Further resuts regarding graphs with induced saturation number zero can be found in [2]. In particular, we prove $\mathrm{indsat}(n, K_{1,k}) = 0$ for all $k \geq 3$ and n sufficiently large. There is an in-depth discussion of $\mathrm{indsat}^*(n, K_{1,3})$, and an extension of the definiton of induced saturation to forbidden families of graphs.

References

[1] L. Babai, P. Frankl, Linear Algebra Methods in Combinatorics with Applications to Geometry and Computer Science. Preliminary Version 2 (September 1992). *Department of Computer Science, The University of Chicago, unpublished book manuscript.*

[2] S. Behrens, C. Erbes, M. Santana, D. Yager, E. Yeager, Graphs with induced-saturation number zero. *submitted*

[3] B. Bollobás, On generalized graphs. *Acta Math. Acad. Sci. Hungary*, 16:447-452, 1965.

[4] B. Bollobás, Extremal Graph Theory. In R. Graham, M. Grötschel, L. Lovász, editors, *Handbook of Combinatorics, Vol. 2*, 1231-1292. Elsevier, 1995.

[5] R. L. Brooks, On colouring the nodes of a network. *Proc. Cambridge Philos. Soc.* 37, (1941). 194-197.

[6] B.-L. Chen, K.-W. Lih, and P.-L. Wu, Equitable coloring and the maximum degree. *Europ. J. Combinatorics*, 15 (1994) 443–447.

[7] G. Chen, M. Ferrara, R. Gould, C. Magnant and J. Schmitt, Saturation Numbers for Families of Ramsey-minimal Graphs. *J. Combinatorics* 2 (2011), 435-455.

[8] Chudnovsky, M.; Seymour, P.; The structure of claw-free graphs. *Surveys in combinatorics 2005, 153 - 171.*

[9] Cockayne, E. J.; Lorimer, P. J., The Ramsey number for stripes. *J. Austral. Math. Soc.* 19 (1975), 252-256.

[10] K. Corrádi and A. Hajnal, On the maximal number of independent circuits in a graph. *Acta Math. Acad. Sci. Hungar.* 14 (1963) 423–439.

[11] G. Dirac, Some results concerning the structure of graphs. *Canad. Math. Bull.* 6 (1963) 183–210.

[12] G. Dirac and P. Erdős, On the maximal number of independent circuits in a graph. *Acta Math. Acad. Sci. Hungar.* 14 (1963) 79–94.

[13] H. Enomoto, On the existence of disjoint cycles in a graph. *Combinatorica* 18(4) (1998) 487-492.

[14] P. Erdős, A. Hajnal and J.W. Moon, A problem in graph theory. *Amer. Math. Monthly* 71 (1964), 1107–1110.

[15] J. Faudree, R. Faudree, and J. Schmitt, A Survey of Minimum Saturated Graphs. *Electron. J. Combin.* 18 (2011), Dynamic Survey 19, 36 pp.

[16] M. Ferrara, J. Kim, E. Yeager, Ramsey-minimal saturation numbers for matchings. *Discrete Mathematics* 322(2014), 26-30.

[17] A. Galluccio, M. Simonovits, G. Simonyi, On the structure of co-critical graphs. *Graph Theory, Combinatorics and Algorithms, Vol. 1, 2 (Kalamazoo, MI, 1992).* Wiley-Intersci. Publ., Wiley, New York, 1053-1071.

[18] A. Hajnal and E. Szemerédi, Proof of a conjecture of P. Erdős. *Combinatorial Theory and its Application*, pp. 601-623, North-Holland, London, 1970.

[19] D. Hanson and B. Toft, Edge-colored saturated graphs. *J. Graph Theory* 11 (1987), 191–196.

[20] L. Kászonyi and Z. Tuza. Saturated Graphs with Minimal Number of Edges. *J. Graph Theory*, 10 (1986) 203-210.

[21] H. A. Kierstead and A. V. Kostochka, A refinement of a result of Corrádi and Hajnal, *submitted*.

[22] H. A. Kierstead and A. V. Kostochka, An Ore-type theorem on equitable coloring. *J. Combinatorial Theory Series B*, 98 (2008) 226-234.

[23] H. A. Kierstead and A. V. Kostochka, Ore-type versions of Brooks' theorem. *J. Combin. Theory Ser. B*, 99 (2009) 298–305.

[24] H. A. Kierstead and A. V. Kostochka, Equitable versus nearly equitable coloring and the Chen-Lih-Wu Conjecture. *Combinatorica*, 30 (2010) 201–216.

[25] H. A. Kierstead and A. V. Kostochka, Every 4-colorable graph with maximum degree 4 has an equitable 4-coloring. *J. Graph Theory* 71 (2012) 31–48.

[26] H. A. Kierstead and A. V. Kostochka, A refinement of a result of Corrádi and Hajnal. *J. Graph Theory*, to appear.

[27] H. A. Kierstead, A.V. Kostochka, T.N. Molla, E.C. Yeager, Sharpening an Ore-type version of the Corrádi-Hajnal Theorem. *In preparation*

[28] H. A. Kierstead, A. V. Kostochka, M. Mydlarz and E. Szemerédi, A fast algorithm for equitable coloring. *Combinatorica*, 30 (2010) 217–224.

[29] H. A. Kierstead, A. V. Kostochka and E.C. Yeager, On the Corrádi-Hajnal Theorem and a question of Dirac. *submitted*

[30] H. A. Kierstead, A. V. Kostochka and E.C. Yeager, The $(2k-1)$-connected graphs with no k disjoint cycles. *Combinatorica*, to appear.

[31] H. A. Kierstead, A. V. Kostochka and Gexin Yu, Extremal graph packing problems: Ore-type versus Dirac-type. *Surveys in Combinatorics 2009* (eds. S. Huczynska, J. Mitchell and C. Roney-Dougal), London Mathematical Society Lecture Note Series 365, Cambridge University Press, Cambridge (2009) 113–136.

[32] A. V. Kostochka, L. Rabern, and M. Stiebitz, Graphs with chromatic number close to maximum degree. *Discrete Math.*, 312 (2012), 1273–1281.

[33] L. Lovász, On graphs not containing independent circuits, (Hungarian. English summary). *Mat. Lapok* 16 (1965), 289–299.

[34] L. Lovász, Flats in matroids and geometric graphs. in P. Cameron, editor, *Combinatorial surveys: Proc. Sixth British Combinatorial Conf., 45-86. Academic Press, 1977.*

[35] Mader, W. 1-Faktoren von Graphen. Math. Ann. 201 (1973), 269–282.

[36] R. Martin and J. Smith, Induced saturation number. Discrete Mathematics*, 312(21) (2012), pp. 3096–3106.*

[37] T. Szabo, On nearly regular co-critical graphs. Discrete Math. *160 (1996) 279-281.*

[38] P. Turán, Eine Extremalaufgabe aus der Graphentheorie. (Hungarian) Mat. Fiz. Lapok *48, (1941). 436452.*

[39] H. Wang, On the maximum number of disjoint cycles in a graph. Discrete Mathematics *205 (1999) 183-190.*