# SYMPLECTIC CIRCLE ACTIONS WITH ISOLATED FIXED POINTS

BY

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DISSERTATION

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# Abstract

Consider a symplectic circle action on a closed symplectic manifold $M$ with non-empty isolated fixed points. Associated to each fixed point, there are well-defined non-zero integers, called *weights*. We prove that the action is Hamiltonian if the sum of an odd number of weights is never equal to the sum of an even number of weights (the weights may be taken at different fixed points). Moreover, we show that if $\dim M = 6$, or if $\dim M = 2n \leq 10$ and each fixed point has weights $\{\pm a_1, \cdots, \pm a_n\}$ for some positive integers $a_i$, the action is Hamiltonian if the sum of three weights is never equal to zero. As applications, we recover the results for semi-free actions, and for certain circle actions on six-dimensional manifolds. Finally, we prove that if there are exactly three fixed points, $M$ is equivariantly symplectomorphic to $\mathbb{CP}^2$.

*To my mother Keumja Jung, father Myungeok Jang, and sister Mini Jang.*

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# Chapter 1

# Introduction

The study of fixed points of a flow or a map is a classical and important topic in geometry and dynamical systems. In this paper, we focus on the case where a manifold admits a symplectic structure and a circle action on the manifold preserves the symplectic structure. A circle action in symplectic geometry corresponds to a periodic flow in mechanical systems. Fixed points by the action corresponds to equilibrium points by the flow. If a circle action has fixed points, a lot of information is encoded by the fixed point data of the action.

Any Hamiltonian action is symplectic but a symplectic action needs not be Hamiltonian. Hence it is a natural question to ask if there is a non-Hamiltonian symplectic action on a compact symplectic manifold. It is a classical fact that a Hamiltonian circle action on a compact symplectic manifold $(M, \omega)$ has at least $\frac{1}{2}\dim M + 1$ fixed points.

T. Frankel proves that a symplectic $S^1$-action on a Kähler manifold is Hamiltonian if and only if is has a fixed point [F]. The property that a symplectic $S^1$-action is Hamiltonian if and only if it has a fixed point holds on 4-dimensional symplectic manifolds [MD], and it also holds for semi-free actions on symplectic manifolds with discrete fixed points [TW]. On the other hand, in the same paper D. McDuff constructs an example of a non-Hamiltonian symplectic circle action with fixed points. However, in this case the fixed point set are tori. Therefore, the question has still remained, if there is a non-Hamiltonian symplectic circle action with isolated fixed points. Recently, S. Tolman constructes a non-Hamiltonian symplectic circle action on a six-dimensional compact symplectic manifold with 32 fixed points [T2].

Let the circle act symplectically on a compact, connected symplectic manifold $M$. First of all, there cannot be exactly one fixed point, unless $M$ is a point. Also, due to C. Kosniowski, if there are exactly two fixed points, then either $M$ is the 2-sphere or $\dim M = 6$ [Ko]. This is reproved by A. Pelayo and S. Tolman using another method [PT]. This result itself does not rule out the possibility that there is a 6-dimensional compact symplectic manifold $M$ with exactly two fixed points by the symplectic circle action.

In this paper, we study symplectic circle actions on compact, connected symplectic manifolds with isolated fixed points. It consists of two parts.

The first part (Chapter 3) concerns under which conditions a symplectic circle action is Hamiltonian. Let the circle act symplectically on a $2n$-dimensional closed symplectic manifold and suppose that the fixed points are isolated. Associated to each fixed point $p$, there are well-defined non-zero integers $w_p^i$, called *weights*, $1 \leq i \leq n$. We prove that if the weights at the fixed points satisfy certain conditions, then the action is Hamiltonian. Consider a collection of weights among all the fixed points, counted with multiplicity. For each integer $a$, the multiplicity of $a$ in the collection is precisely $\max_{p \in M^{S^1}} |\{i | a = w_p^i\}|$. For instance, if there are fixed points whose weights are $\{-1,-1,1,1\}$ and $\{-1,-1,-1,2\}$, then the multiplicity of $-1$ and 1 in the collection is at least 3 and 2, respectively. First, we show that the symplectic action is Hamiltonian if the sum of an odd number of weights in the collection is never equal to the sum of an even number of weights in the collection.

**Theorem 1.0.1.** *Consider a symplectic circle action on a closed symplectic manifold with non-empty isolated fixed points. The action is Hamiltonian if the sum of an odd number of weights among all fixed points is never equal to the sum of an even number of weights.*

For instance, if the action is semi-free, all the weights are either $+1$ or $-1$. Therefore, the sum of an odd number of weights cannot equal the sum of an even number of weights, and hence the action is Hamiltonian. In some cases, we only need to consider if the sum of three weights is never equal to zero.

**Theorem 1.0.2.** *Consider a symplectic circle action on a $2n$-dimensional closed symplectic manifold with non-empty fixed points, whose weights are $\{\pm a_1, \pm a_2, \cdots, \pm a_n\}$ for some positive integers $a_i$, $1 \leq i \leq n$. Assume that $n \leq 5$ and $\pm a_i \pm a_j \pm a_k \neq 0$ for all $i < j < k$. Then the action is Hamiltonian.*

**Theorem 1.0.3.** *Consider a symplectic circle action on a six-dimensional closed symplectic manifold with non-empty isolated fixed points. The action is Hamiltonian if the sum of three weights among all fixed points is never equal to zero.*

The condition that the sum of three weights among all fixed points is never equal to zero, seems to play a certain role for a symplectic circle action to be Hamiltonian. If a symplectic circle action on a closed symplectic manifold $M$ has two fixed points, then either $M$ is the 2-sphere, or $\dim M = 6$ and the weights at the two fixed points are $\{-a-b, a, b\}$ and $\{-a, -b, a+b\}$ for some positive integers $a$ and $b$ [Ka], [PT]. If $\dim M = 6$, then the action cannot be Hamiltonian, since a compact Hamiltonian $S^1$-manifold $M$ has at least $\frac{1}{2}\dim M + 1$ fixed points. Moreover, there is the sum of three weights that is equal to zero. In fact, the first Chern class at each fixed point, which is the sum of weights at the fixed point, is equal to zero. However, to the author's knowledge, we do not know, whether such a manifold exists or not. In S. Tolman's construction of a non-Hamiltonian symplectic $S^1$-action on a six-dimensional compact symplectic manifold with 32 fixed points, 16 fixed points have weights $\{1, 1, -2\}$ and the other 16 fixed points have weights $\{-1, -1, 2\}$.

**Question 1.0.4.** *Let the circle act symplectically on a closed symplectic manifold with non-empty isolated fixed points. Suppose that the sum of three weights among all fixed points is never equal to zero. Then is the action Hamiltonian?*

In the second part (Chapter 4), we classify a symplectic circle action with exactly three fixed points; we prove that any symplectic circle action on a compact connected symplectic manifold with exactly three fixed points is equivariantly symplectomorphic to $\mathbb{CP}^2$ with some standard action on it. In particular, it follows that in this case the manifold must be 4-dimensional. Moreover, the action must be Hamiltonian.

**Theorem 1.0.5.** *Let the circle act symplectically on a compact, connected symplectic manifold $M$. If there are exactly three fixed points, $M$ is equivariantly symplectomorphic to $\mathbb{CP}^2$.*

# Chapter 2

# Background and Notation

A differential form $\alpha$ is called **closed** if $d\alpha = 0$. A two-form $\omega$ on a manifold $M$ is called **nondegenerate** if for each $v_p \in T_pM$ such that $v_p \neq 0$, there exists $w_p \in T_pM$ such that $\omega_p(v_p, w_p) \neq 0$. A **symplectic manifold** $(M, \omega)$ is a manifold with a closed, non-degenerate two-form $\omega$ on it.

**Example 2.0.6.** *Examples of symplectic manifolds.*

*(1)* $(\mathbb{R}^2, dx \wedge dy)$ *is a symplectic manifold.*

*(2) More generally,* $(\mathbb{R}^{2n}, \sum_{i=1}^{n} dx_i \wedge dy_i)$ *are symplectic manifolds.*

*(3) The two torus* $(\mathbb{T}^2 = (S^1)^2, d\theta_1 \wedge d\theta_2)$ *where we regard* $S^1$ *as* $\mathbb{R}/\mathbb{Z}$ *and* $\theta_i \in \mathbb{R}/\mathbb{Z}$ *is an example of compact symplectic manifolds.*

*(4) A complex projective spaces* $(\mathbb{CP}^n, \omega_{FS})$ *with the Fubini-Study form is another example of compact symplectic manifolds.*

Let $(M, \omega)$ be a $2n$-dimensional symplectic manifold. Then the wedge product of the symplectic form $\omega$ by $n$-times, $\omega^n \neq 0$. Therefore $\omega^n$ is a top-degree form and gives an orientation for $M$. It follows that every symplectic manifold is orientable. If $(M, \omega)$ is compact, it follows that $\omega^i$ are closed but not exact for $0 \leq i \leq n$. Therefore, it follows that $\dim H^{2i}(M) \geq 1$ for all $0 \leq i \leq n$.

**Example 2.0.7.** *Examples of even-dimensional manifolds that are not symplectic.*

*(1) The Möbius strip is not a symplectic manifold since it is not orientable.*

*(2) An even dimensional sphere* $S^{2n}$ *is not symplectic if* $n > 1$*, since it is compact but* $H^{2i}(M) = 0$ *for* $1 < i < n$.

Let the circle act on a symplectic manifold $(M, \omega)$. If the circle action on $M$ preserves $\omega$, the action is called **symplectic**, i.e $g^*\omega = \omega$ for each $g \in S^1$. Let $X_M$ be the vector field on $M$ generated by the circle action. The action is called **Hamiltonian**, if there exists a map $\mu : M \to \mathbb{R}$ such that

$$\iota_{X_M}\omega = -d\mu.$$

This implies that every symplectic action is Hamiltonian if $H_1(M; \mathbb{R}) = 0$, since $\iota_{X_M}\omega$ is closed. It is the classical fact due to Morse that any Hamiltonian $S^1$-action on a compact symplectic manifold has at least $\frac{1}{2}\dim M + 1$ fixed points, since the number of fixed points is equal to the number of critical points of $\mu$, which is equal to $\sum_i \dim H^{2i}(M)$. However, this is at least $\frac{1}{2}\dim M + 1$ since $\dim H^{2i}(M) \geq 1$ for all $0 \leq i \leq n$ as mentioned above.

**Example 2.0.8.** *Examples of symplectic $S^1$-actions*

*(1) Consider the unit 2-sphere $(S^2, dh \wedge d\theta)$ inside $\mathbb{R}^3$, where $\theta$ is the angle about the $z$-axis and $h$ is the height. Let the circle act on $M$ by rotation about the $z$-axis, i.e. it acts by $g \cdot (\theta, h) = (\theta + g, h)$, where $g \in S^1 = \mathbb{R}/\mathbb{Z}$. The north pole and the south pole are the fixed points of the action. The vector field $X_M$ generated by the action is $\frac{d}{d\theta}$ and $\iota_{X_M}\omega = \iota_{\frac{d}{d\theta}} dh \wedge d\theta = -dh$. Therefore the action is Hamiltonian.*

*(2) Let the circle act on the two torus $(\mathbb{T}^2 = (S^1)^2, d\theta_1 \wedge d\theta_2)$ by rotation on one factor of $S^1$, i.e. $g \cdot (\theta_1, \theta_2) = (\theta_1 + g, \theta_2)$, where $\theta_i, g \in S^1 = \mathbb{R}/\mathbb{Z}$. This action has no fixed points, and so it follows that the action cannot be Hamiltonian.*

Consider a circle action on a manifold $M$. **The equivariant cohomology of** $M$ is defined by $H^*_{S^1}(M) = H^*(M \times_{S^1} S^\infty)$. For instance, the equivariant cohomology of a point is $H^*_{S^1}(\{p\}) = H^*(\{p\} \times_{S^1} S^\infty) = H^*(\mathbb{CP}^\infty) = \mathbb{Z}[t]$, where $t$ is of degree 2. If $M$ is oriented and compact, then from the projection map $\pi : M \times_{S^1} S^\infty \to \mathbb{CP}^\infty$ we obtain a natural push-forward map

$$\pi_* : H^i_{S^1}(M; \mathbb{Z}) \to H^{i-\dim M}_{S^1}(\mathbb{CP}^\infty; \mathbb{Z})$$

for $i \in \mathbb{Z}$. This map is given by "integration over the fiber" and denoted by $\int_M$.

**Theorem 2.0.9.** *(ABBV localization) [AB] Let the circle act on a compact oriented manifold $M$. Fix $\alpha \in H^*_{S^1}(M; \mathbb{Q})$. As elements of $\mathbb{Q}(t)$,*

$$\int_M \alpha = \sum_{F \subset M^{S^1}} \int_F \frac{\alpha|_F}{e_{S^1(N_F)}},$$

*where the sum is over all fixed components, and $e_{S^1}(N_F)$ denotes the equivariant Euler class of the normal bundle to $F$.*

Consider a circle action on an almost complex manifold $(M, J)$. Suppose that the action preserves the almost complex structure $J$. Let $p$ be an isolated fixed point. Then we can identify $T_pM$ with $\mathbb{C}^n$ and the action of $S^1$ at $p$ with $g \cdot (z_1, \cdots, z_n) = (g^{\xi_p^1} z_1, \cdots, g^{\xi_p^n} z_n)$, where $\xi_p^i$ are non-zero integers, regarding here $S^1$ as a subset of $\mathbb{C}$. These non-zero integers are called **weights** at the fixed point $p$. Any symplectic manifold $(M, \omega)$ admits an almost complex structure and hence is an almost complex manifold. Moreover, the set of almost complex structures on $M$ that are compatible with $\omega$ is contractible. Therefore, at each fixed point of a symplectic manifold $(M, \omega)$ by a symplectic circle action, the weights are well defined.

Denote $\sigma_i$ by the $i^{th}$-elementary symmetric polynomial in $n$ variables. Then the $i^{th}$-equivariant Chern class at the fixed point $p$ is given by

$$c_i(M)|_p = \sigma_i(\xi_p^1, ..., \xi_p^n)t^i,$$

where $t$ is the generator of $H^2_{S^1}(p; \mathbb{Z})$. For instance, the equivariant first Chern class at $p$ is $c_1(M)|_p = \Sigma \xi_p^i t$, and the equivariant Euler class of the normal bundle to $p$ is $e_{S^1}(N_p) = c_n(M)|_p = (\prod_{j=1}^n \xi_p^j)t^n$. Hence,

$$\int_p \frac{c_i(M)|_p}{e_{S^1}(N_p)} = \frac{\sigma_i(\xi_p^1, ..., \xi_p^n)}{\prod_{j=1}^n \xi_p^j} t^{i-n}.$$

Denote $\lambda_p$ by twice of the number of negative weights at $p$ for all $p \in M^{S^1}$. This is called **the index** at $p$ and this notion agrees with the index of a fixed point of a Hamiltonian circle action. Weights in the isotropy representation $T_pM$ satisfy the following:

**Lemma 2.0.10.** *[PT] Let the circle act symplectically on a $2n$-dimensional compact symplectic manifold $M$ with isolated fixed points. Then*

$$|\{p \in M^{S^1} \mid \lambda_p = 2i\}| = |\{p \in M^{S^1} \mid \lambda_p = 2n - 2i\}|, \textit{ for all } i \in \mathbb{Z}.$$

**Corollary 2.0.11.** *[PT] Let the circle act symplectically on a $2n$-dimensional compact symplectic manifold with $k$ isolated fixed points. If $k$ is odd, then $n$ is even.*

**Lemma 2.0.12.** *[PT] Let the circle act symplectically on a compact symplectic manifold $M$ with isolated fixed points. Then*

$$\sum_{p \in M^{S^1}} N_p(l) = \sum_{p \in M^{S^1}} N_p(-l), \text{ for all } l \in \mathbb{Z}.$$

*Here, $N_p(l)$ is the multiplicity of $l$ in the isotropy representation $T_pM$ for all weights $l \in \mathbb{Z}$, for $p \in M^{S^1}$.*

Consider a symplectic circle action on a compact, connected symplectic manifold. If there are exactly two fixed points, then A. Pelayo and S. Tolman give the classification of such a manifold:

**Theorem 2.0.13.** *[PT] Let the circle act symplectically on a compact, connected symplectic manifold $M$ with exactly two fixed points. Then either $M$ is the 2-sphere, or $\dim M = 6$ and there exist positive integers $a$ and $b$ so that the weights at the two fixed points are $\{a, b, -a-b\}$ and $\{a+b, -a, -b\}$.*

**Corollary 2.0.14.** *[PT] Let the circle act symplectically on a compact symplectic manifold $M$ with non-empty fixed point set. Then there are at least two fixed points, and if $\dim M \geq 8$, then there are at least three fixed points. Moreover, if the Chern class map is not identically zero and $\dim M \geq 6$, then there are at least four fixed points.*

Now we consider an elliptic differential operator on a $2n$-dimensional compact almost complex manifold $(M, J)$, where $J$ is an almost complex structure on $M$. By choosing an almost Hermitian metric on $M$, we can define the Hodge star operator $*$ and the formal adjoint operator $\overline{\partial}^*$ of the $\overline{\partial}$-operator. For each $i$ such that $0 \leq i \leq n$, we define an elliptic differential operator by

$$\overline{\partial} + \overline{\partial}^* : \bigoplus_{j:even} \Omega^{i,j}(M) \longrightarrow \bigoplus_{j:odd} \Omega^{i,j}(M),$$

where $\Omega^{i,j}(M) = \Gamma(\bigwedge^i T^*M \bigotimes \bigwedge^j \overline{T^*M})$. The index of the operator is defined to be $\chi^i(M) = \dim_{\mathbb{C}} ker(\overline{\partial} + \overline{\partial}^*) - \dim_{\mathbb{C}} coker(\overline{\partial} + \overline{\partial}^*)$. For more details, the readers are referred to [L] and the references therein.

Let the circle act on a compact almost manifold $(M, J)$. Assume that the action preserves the almost complex structure $J$ and the fixed points are isolated. P. Li proves that the Dolbeault-type operator on an almost complex manifold is rigid under the circle action.

**Theorem 2.0.15.** *[L] Consider a circle action on a $2n$-dimensional compact almost complex manifold $M$. Suppose that the action preserves the action and the fixed points are isolated. Then*

$$\chi^i(M) = \sum_{p \in M^{S^1}} \frac{\sigma_i(t^{\xi_p^1}, \cdots, t^{\xi_p^n})}{\prod_{j=1}^n (1 - t^{\xi_p^j})} = (-1)^i N^i = (-1)^{n-i} N^{n-i},$$

*where $N^i$ is the number of fixed points of index $2i$ and $t$ is an indeterminate. In addition, assume that $M$ is a symplectic manifold and the action is symplectic. Then $\chi^0(M) = 1$ if the action is Hamiltonian, and $\chi^0(M) = 0$ if it is not Hamiltonian.*

**Example 2.0.16.** *Examples of symplectic $S^1$-actions with isolated fixed points.*

(1) *Consider an action of $S^1$ on the 2-sphere $S^2$ by rotating it a-times, where we regard $S^2$ as a subset of $\mathbb{R}^3$ and the action rotates with speed $a$ about the $z$-axis. The north pole $N$ and the south pole $S$ are the fixed points of the action. At $N$ and $S$, the action can be identified with $g \cdot z = g^{-a} z$ and $g \cdot z = g^a z$, for $g \in S^1 \subset \mathbb{C}$. Therefore, the weights at $N$ and $S$ are $\{-a\}$ and $\{a\}$. One can check Theorems and Lemmas above for this example.*

(2) *Let the circle act on $\mathbb{CP}^2$ by $g \cdot [z_0 : z_1 : z_2] = [g^{a+b} z_0 : g^a z_1 : z_2]$ for some positive integers $a$ and $b$. This action has three fixed points $[1 : 0 : 0]$, $[0 : 1 : 0]$, and $[0 : 0 : 1]$. Let $U_i = \{[z_0 : z_1 : z_2] \in \mathbb{CP}^2 | z_i \neq 0\}$. On $U_0$, the action is $g \cdot (\frac{z_1}{z_0}, \frac{z_2}{z_0}) = (\frac{g^a z_1}{g^{a+b} z_0}, \frac{z_2}{g^{a+b} z_0} = (g^{-b} \frac{z_1}{z_0}, g^{-a-b} \frac{z_2}{z_0})$. Therefore, the weights at $[1 : 0 : 0]$ are $\{-b, -a-b\}$. Similarly, one can show that the weights at $[0 : 1 : 0]$ and $[0 : 0 : 1]$ are $\{b, -a\}$ and $\{a, a+b\}$.*

**Remark 2.0.17.** *Consider a circle action on a compact almost complex (symplectic) manifold $(M, J)$ $((M, \omega))$. Assume that the action is effective and preserves the almost complex structure $J$ (the symplectic structure $\omega$). As a subgroup of $S^1$, $\mathbb{Z}_k$ also acts on $M$, for $k \in \mathbb{Z} \setminus \{-1, 0, 1\}$. The set $M^{\mathbb{Z}_k}$ of points fixed by the $\mathbb{Z}_k$-action is a union of smaller dimensional almost complex submanifolds (symplectic submanifolds). Moreover, the isotropy weights in $M^{\mathbb{Z}_k}$ are multiples of $k$. Suppose that $Z$ is a connected component of $M^{\mathbb{Z}_k}$ and $\dim Z = 2m$. If $p \in M^{S^1}$ is a point fixed by the $S^1$-action that lies in the connected component $Z$, then $p$ has exactly $m$-weights that are multiples of $k$.*

**Example 2.0.18.** *Let the circle act on* $M = S^2 \times S^2 = M_1 \times M_2$ *by rotating the first 2-sphere 3-times and the second 2-sphere 4-times, i.e., the action is given by* $g \cdot (\theta_1, h_1, \theta_2, h_2) = (\theta_1 + 6\pi g, h_1, \theta_2 + 8\pi g, h_2)$, *where* $\theta_i$ *are angles and* $h_i$ *are heights,* $i = 1, 2$, $g \in S^1 = \mathbb{R}/\mathbb{Z}$, $0 \leq g \leq 1$. *This action has four fixed points,* $n_1 \times n_2$, $n_1 \times s_2$, $s_1 \times n_2$, *and* $s_1 \times s_2$, *where* $n_i$ *and* $s_i$ *are the north pole and the south pole for each sphere,* $i = 1, 2$. *The weights at the fixed points are* $\Sigma_{n_1 \times n_2} = \{-3, -4\}$, $\Sigma_{n_1 \times s_2} = \{-3, 4\}$, $\Sigma_{s_1 \times n_2} = \{3, -4\}$, *and* $\Sigma_{s_1 \times s_2} = \{3, 4\}$. *Then as a subgroup of* $S^1$, $\mathbb{Z}_2$ *acts on* $M$ *by* $x \cdot (\theta_1, h_1, \theta_2, h_2) = (\theta_1 + 3\pi x, h_1, \theta_2 + 4\pi x, h_2)$, *for* $x = 0, 1$. *Therefore, the set of points fixed by the* $\mathbb{Z}_2$*-action is* $Z_1 = n_1 \times M_2$ *and* $Z_2 = s_1 \times M_2$. *In particular,* $\dim Z_i = 2 < 4 = \dim M$, *for* $i = 1, 2$. *Inside* $n_1 \times M_2$, $n_2$ *has the weight* $-4$ *and* $s_2$ *has the weight* $4$. *Inside* $s_1 \times M_2$, $n_2$ *has the weight* $-4$ *and* $s_2$ *has the weight* $4$.

**Lemma 2.0.19.** *[T1] Let the circle act on a compact symplectic manifold* $(M, \omega)$. *Let* $p$ *and* $p'$ *be fixed points which lie in the same component* $N$ *of* $M^{\mathbb{Z}_k}$, *for some* $k > 1$. *Then the* $S^1$*-weights at* $p$ *and at* $p'$ *are equal modulo* $k$.

Let the circle act on a compact symplectic manifold $M$. Let $p$ and $p'$ be fixed points which lie in the same component $N$ of $M^{\mathbb{Z}_k}$, for some $k > 1$. Denote $\Sigma_p$ and $\Sigma_{p'}$ by the multisets of weights at $p$ and $p'$, respectively. Lemma 2.0.19 states that there exists a bijection between $\Sigma_p$ and $\Sigma_{p'}$ that takes each weight $\alpha$ at $p$ to a weight $\beta$ at $p'$ such that $\alpha \equiv \beta \mod k$.

# Chapter 3

# On symplectic $S^1$-actions with isolated fixed points

We begin with the proof of Theorem 1.0.1. Recall that for a symplectic circle action on a closed symplectic manifold $M$ with non-empty isolated fixed points, we consider a collection of weights among all the fixed points, counted with multiplicity, and for each integer $a$, the multiplicity of $a$ in the collection is precisely $\max_{p \in M^{S^1}} |\{i | a = w_p^i\}|$.

**Theorem 3.0.20.** *Consider a symplectic circle action on a closed symplectic manifold with non-empty isolated fixed points. The action is Hamiltonian if the sum of an odd number of weights among all fixed points is never equal to the sum of an even number of weights.*

*Proof.* The main idea of the proof is to manipulate the formula in Theorem 2.0.15; we consider $\chi^0(M)$, make each exponent in the denominator positive, and clear up the denominators by multiplying the least common multiple of the denominators. In such a way each term has the exponent that is the sum of the absolute values of the weights and we derive the conclusion.

Assume, on the contrary, that the action is not Hamiltonian. By Theorem 2.0.15, $\chi^0(M) = \chi^n(M) = 0$ and there are no fixed points of index 0 and $2n$. Moreover,

$$\begin{aligned}
\chi^0(M) = \sum_{p \in M^{S^1}} \frac{1}{\prod_{m=1}^n (1 - t^{\xi_p^m})} &= \sum_{p \in M^{S^1}} \frac{1}{\prod_{m=1}^n (1 - t^{\xi_p^m})} \frac{\prod_{\xi_p^m < 0} (-t^{-\xi_p^m})}{\prod_{\xi_p^m < 0} (-t^{-\xi_p^m})} \\
&= \sum_{p \in M^{S^1}} \frac{\prod_{\xi_p^m < 0} (-t^{-\xi_p^m})}{\prod_{\xi_p^m > 0} (1 - t^{\xi_p^m}) \prod_{\xi_p^m < 0} \{(1 - t^{\xi_p^m})(-t^{-\xi_p^m})\}} \\
&= \sum_{p \in M^{S^1}} \frac{(-1)^{\frac{\lambda_p}{2}} \prod_{\xi_p^m < 0} t^{-\xi_p^m}}{\prod_{\xi_p^m > 0} (1 - t^{\xi_p^m}) \prod_{\xi_p^m < 0} (1 - t^{-\xi_p^m})} \\
&= \sum_{p \in M^{S^1}} (-1)^{\frac{\lambda_p}{2}} \frac{t^{\sum_{\xi_p^m < 0} (-\xi_p^m)}}{\prod_{m=1}^n (1 - t^{|\xi_p^m|})} = 0.
\end{aligned}$$

Denote by $A = \{a_1, a_2, \cdots, a_l\}$ the collection of all the absolute values of weights among all the fixed points, counted with multiplicity, where for each positive integer $a$ the multiplicity of $a$ in $A$ is precisely $\max_{p \in M^{S^1}} |\{i | a = |w_p^i|\}|$. Therefore, the least common multiple of the denominators is $\prod_{i=1}^{l}(1-t^a)$. Denote by $B_p = A \setminus \{|w_p^1|, \cdots, |w_p^n|\} = \{b_p^1, b_p^2, \cdots, b_p^{l-n}\}$ the set of elements in $A$ minus the absolute values of the weights at $p$. We multiply the equation above by $\prod_{i=1}^{l}(1-t^{a_i})$ to get

$$\begin{aligned} 0 &= \sum_{p \in M^{S^1}} (-1)^{\frac{\lambda_p}{2}} t^{\sum_{\xi_p^m<0}(-\xi_p^m)} \prod_{a \in B_p} (1-t^a) \\ &= \sum_{p \in M^{S^1}} (-1)^{\frac{\lambda_p}{2}} t^{\sum_{\xi_p^m<0}(-\xi_p^m)} (1 - \sum_x t^{b_p^x} + \sum_{x_1<x_2} t^{b_p^{x_1}+b_p^{x_2}} + \cdots) \\ &= \sum_{p \in M^{S^1}} [(-1)^{\frac{\lambda_p}{2}} t^{\sum_{\xi_p^m<0}(-\xi_p^m)} - (-1)^{\frac{\lambda_p}{2}} \sum_x t^{b_p^x + \sum_{\xi_p^m<0}(-\xi_p^m)} \\ &\quad + (-1)^{\frac{\lambda_p}{2}} \textstyle\sum_{x_1<x_2} t^{\{\sum_{\xi_p^m<0}(-\xi_p^m)\}+(b_p^{x_1}+b_p^{x_2})} + \cdots]. \end{aligned}$$

In the last equation, each summand of the exponent of a term, $-\xi_p^m$ or $b_p^x$, is a positive integer and is an element of $A$. Each term whose exponent is the sum of an odd number of elements in $A$ has the coefficient $-1$ and each term whose exponent is the sum of an even number of elements in $A$ has the coefficient 1.

Since $\chi^0(M) = 0$, this implies that each term whose exponent is the sum of an odd number of elements must cancel out with another term whose exponent is the sum of an even number of elements.

Suppose that there is a fixed point $p_0$ of index $2i_0$, for some $i_0$ such that $0 < i_0 < n$. Then $p_0$ contributes a summand $(-1)^{i_0} t^{\sum_{\xi_{p_0}^m<0}(-\xi_{p_0}^m)}$, where $\xi_{p_0}^m < 0$ are the negative weights at $p_0$. Since $\chi^0(M) = 0$, this term must be cancelled out. The coefficient of the term is $(-1)^{i_0}$. Therefore, if the term is cancelled out by another term, then its exponent must be the sum of $j_0$-elements in $A$, where $j_0$ and $i_0$ have different parities. Suppose that the summand $(-1)^{i_0} t^{-\sum_{\xi_{p_0}^m<0} \xi_{p_0}^m}$ is cancelled out by another term, say $(-1)^{j_0} t^{d_1+d_2+\cdots+d_{j_0}}$, $d_i > 0$. These $d_i$'s form a subset of $\{-\xi_q^1, \cdots, -\xi_q^n\} \cup B_q$ for some fixed point $q$, i.e., $\{d_1, \cdots, d_{j_0}\} \subset \{-\xi_q^1, \cdots, -\xi_q^n\} \cup B_q$. Let us rewrite $\sum_{\xi_{p_0}^m<0}(-\xi_{p_0}^m) = c_1 + c_2 + \cdots + c_{i_0}$, i.e., $c_1 + c_2 + \cdots + c_{i_0} = d_1 + d_2 + \cdots + d_{j_0}$. For each positive integer $a$, the multiplicity of $a$ on each side does not exceed $\max_{p \in M^{S^1}} |\{i : a = |w_p^i|\}|$.

For the equation $c_1 + c_2 + \cdots + c_{i_0} = d_1 + d_2 + \cdots + d_{j_0}$, we do the following: if $c_k = d_{k'}$ for some $k$ and $k'$, then we cancel these terms out on the equation.

By performing these steps as many as possible and by permuting $c_i$'s and $d_i$'s if necessary, assume that we have $c_1 + c_2 + \cdots + c_{i'_0} = d_1 + d_2 + \cdots + d_{j'_0}$ $(*)$, $c_i, d_i > 0$. For any positive integer $a$ that appears in $(*)$, either it appears only on the left hand side or only on the right hand side. Moreover, the multiplicity of $a$ in the equation does not exceed $\max_{p \in M^{S^1}} |\{i : a = |w_p^i|\}|$.

Denote by $C$ the collection of the equations, each of which is the sum of an odd number of weights being equal to the sum of an even number of weights, where weights are taken among all the fixed points, counted with multiplicity. Consider an element of $C$. It is an equation of the form $w_1 + w_2 + \cdots + w_{i'} = w_{i'+1} + \cdots + w_{2j'+1}$ for some $i', j'$, where each $w_k$ is a weight at some fixed point. For each integer $a$, the multiplicity of $a$ on each side is at most $\max_{p \in M^{S^1}} |\{i | a = w_p^i\}|$. For the equation we do the following: if $w_k = -w_{k'}$ and they appear on the same side of the equation, we cancel out these terms. If $w_k = w_{k'}$ and they appear on the opposite side of the equation, we also cancel out these terms. If $w_k$ is a negative weight, we move the term to the opposite side as $-w_k$. By performing these steps as many as possible, assume that we have $e_1 + e_2 + \cdots + e_{i''} = f_1 + f_2 + \cdots + f_{2j''+1}$, $e_i, f_i > 0$. For any positive integer $a$ that appears in the last equation, either it appears only on the left hand side or only on the right hand side. Moreover, the multiplicity of $a$ in the equation does not exceed $\max_{p \in M^{S^1}} |\{i | -a = w_p^i\}| + \max_{p \in M^{S^1}} |\{i | a = w_p^i\}|$.

Note that for each positive integer $a$, we have that $\max_{p \in M^{S^1}} |\{i : a = |w_p^i|\}| \leq \max_{p \in M^{S^1}} |\{i | -a = w_p^i\}| + \max_{p \in M^{S^1}} |\{i | a = w_p^i\}|$. Therefore, the equation $(*)$ is an element of $C$. However, by the assumption that the sum of an odd number of weights is never equal to the sum of an even number of weights, $C$ is an emptyset. Therefore, there are no fixed points of index $2i$ for all $0 < i < n$, which is a contradiction. $\square$

We can generalize Theorem 1.0.1 further. Let the circle act symplectically on a closed symplectic manifold $M$ with isolated fixed points. As in the proof of Theorem 3.0.20, denote by $A = \{a_1, a_2, \cdots, a_l\}$ the collection of all the absolute values of weights among all the fixed points counted with multiplicity, and $A_i = \{a_{j_1} + a_{j_2} + \cdots + a_{j_i}\}_{a_{j_1} < a_{j_2} < \cdots < a_{j_i}}$ the collection of sums of $i$ elements of $A$, for $1 \leq i \leq l$. For each positive integer $a$, the multiplicity of $a$ in $A$ is precisely $\max_{p \in M^{S^1}} |\{i | a = |w_p^i|\}|$. Note that here we consider the collection of the absolute values of the weights, and hence it is different from the one in the introduction. For instance, with the fixed points of the same weights $\{-1,-1,1,1\}$ and $\{-1,-1,-1,2\}$ as before, the multiplicity of 1 in the collection is 4.

**Theorem 3.0.21.** *Let the circle act symplectically on a closed symplectic manifold $M$ with non-empty isolated fixed points. Let $A=\{a_1,a_2,\cdots,a_l\}$ be the collection of all the absolute values of weights among all the fixed points, counted with multiplicity, and $A_i=\{a_{j_1}+a_{j_2}+\cdots+a_{j_i}\}_{a_{j_1}<a_{j_2}<\cdots<a_{j_i}}$ the collection of sums of $i$-elements of $A$, for $1\leq i\leq l$. If there exists $0<i<n$ such that $A_i\cap A_j=\emptyset$ for all $j$ such that $j\neq i \mod 2$, then the action is Hamiltonian.*

*Proof.* The idea of the proof is similar to that of Theorem 3.0.20. However, we consider $\chi^i(M)$ for many $i$'s.

Assume, on the contrary, that the action is not Hamiltonian. By Theorem 2.0.15, $\chi^0(M)=\chi^n(M)=0$ and there are no fixed points of index 0 and $2n$. As in the proof of Theorem 3.0.20, we have

$$\begin{aligned}\chi^0(M)&=\sum_{p\in M^{S^1}}\frac{1}{\prod_{m=1}^n(1-t^{\xi_p^m})}\\&=\sum_{p\in M^{S^1}}(-1)^{\frac{\lambda_p}{2}}\frac{t^{\sum_{\xi_p^m<0}(-\xi_p^m)}}{\prod_{m=1}^n(1-t^{|\xi_p^m|})}=0.\end{aligned}$$

Denote by $B_p=A\setminus\{|w_p^1|,\cdots,|w_p^n|\}=\{b_p^1,b_p^2,\cdots,b_p^{l-n}\}$ for each fixed point $p$. We multiply the equation above by $\prod_{i=1}^l(1-t^{a_i})$ to get

$$\begin{aligned}0&=\sum_{p\in M^{S^1}}(-1)^{\frac{\lambda_p}{2}}t^{\sum_{\xi_p^m<0}(-\xi_p^m)}\prod_{a\in B_p}(1-t^a)\\&=\sum_{p\in M^{S^1}}(-1)^{\frac{\lambda_p}{2}}t^{\sum_{\xi_p^m<0}(-\xi_p^m)}(1-\sum_x t^{b_p^x}+\sum_{x<y}t^{b_p^x+b_p^y}+\cdots)\\&=\sum_{p\in M^{S^1}}[(-1)^{\frac{\lambda_p}{2}}t^{\sum_{\xi_p^m<0}(-\xi_p^m)}-(-1)^{\frac{\lambda_p}{2}}\sum_x t^{b_p^x+\sum_{\xi_p^m<0}(-\xi_p^m)}\\&\quad+(-1)^{\frac{\lambda_p}{2}}\textstyle\sum_{x_1<x_2}t^{\{\sum_{\xi_p^m<0}(-\xi_p^m)\}+(b_p^{x_1}+b_p^{x_2})}+\cdots].\end{aligned}$$

In the equation, $-\xi_p^m\in A$ and $b_p^x\in A$ for all $-\xi_p^m,b_p^x$. Therefore, if a term has the exponent that is the sum of $i$-elements in $A$, then the exponent is an element of $A_i$. Each term whose exponent is the sum of an odd number of elements in $A$ has the coefficient $-1$ and each term whose exponent is the sum of an even number of elements in $A$ has the coefficient 1.

Since $\chi^0(M) = 0$, this implies that each term whose exponent is the sum of an odd number of elements must cancel out with another term whose exponent is the sum of an even number of elements.

Suppose that there is a fixed point $p$ of index $2i$, $0 < i < n$. Then $p$ contributes a summand $(-1)^i t^{\sum_{\xi_p^m<0}(-\xi_p^m)}$, where $\xi_p^m < 0$ are the negative weights at $p$. Since $\chi^0(M) = 0$, this term must be cancelled out. The coefficient of the term is $(-1)^i$. Therefore, if the term is cancelled out by another term, then its exponent must be the sum of $j$-elements, where $j$ and $i$ have different parities. By the assumption that $A_i \cap A_j = \emptyset$ for $j \neq i$ mod 2, the summand $(-1)^i t^{\sum_{\xi_p^m<0}(-\xi_p^m)}$ cannot be cancelled out, which is a contradiction. Therefore, there are no fixed points of index $2i$.

From now on we seperate into several cases, depending on if $i \leq \frac{n}{2}$ or $i > \frac{n}{2}$, if $i$ is odd or even, and if $n$ is odd or even. Each case is a slight variation of the other cases. If $i > \frac{n}{2}$, we use the symmetry that $N^j = N^{n-j}$ for all $j$, where $N_j$ is the number of fixed points of index $2j$. With the symmetry, the case where $i > \frac{n}{2}$ is a slight variation of the case where $i \leq \frac{n}{2}$.

First, suppose that $i \leq \frac{n}{2}$ and $i$ is odd. By Theorem 2.0.15,

$$\chi^i(M) = \sum_{p \in M^{S^1}} \frac{\sigma_i(t^{\xi_p^1}, \cdots, t^{\xi_p^n})}{\prod_{m=1}^n (1 - t^{\xi_p^m})} = 0.$$

As in the proof of Theorem 3.0.20, we make the exponent of each term in the denominators positive to get

$$\chi^i(M) = \sum_{p \in M^{S^1}} (-1)^{\frac{\lambda_p}{2}} \frac{t^{\sum_{\xi_p^m<0}(-\xi_p^m)} \sigma_i(t^{\xi_p^1}, \cdots, t^{\xi_p^n})}{\prod_{m=1}^n (1 - t^{|\xi_p^m|})} = 0.$$

Multiplying the equation above by $\prod_{i=1}^l (1 - t^{a_i})$, we have

$$\sum_{p \in M^{S^1}} (-1)^{\frac{\lambda_p}{2}} t^{\sum_{\xi_p^m<0}(-\xi_p^m)} \sigma_i(t^{\xi_p^1}, \cdots, t^{\xi_p^n}) \prod_{a \in B_p} (1 - t^a) = 0.$$

Let us consider $t^{\sum_{\xi_p^m<0}(-\xi_p^m)} \sigma_i(t^{\xi_p^1}, \cdots, t^{\xi_p^n})$. When we expand the terms, the exponent of any term is the sum of positive integers that are in $A$. No negative integer appears in the exponent of any term when expanded. Therefore, when we multiply $t^{\sum_{\xi_p^m<0}(-\xi_p^m)} \sigma_i(t^{\xi_p^1}, \cdots, t^{\xi_p^n})$ by $\prod_{a \in B_p}(1 - t^a)$ and expand the terms, each summand in the equation has the exponent that belongs to $A_i$ for some $i$. If it belongs to $A_i$, then it has the sign $(-1)^i$.

Suppose that a fixed point $p$ has the index $2k$, where $k$ is even and $0 \leq k \leq 2i$. In the last equation, such a point contributes a summand whose exponent is the sum of $i$ elements. By the assumption, such a term cannot be cancelled out. Hence there are no fixed points of index $0, 4, \cdots, 4i$, i.e. $N^0 = N^4 = \cdots = N^{4i} = 0$ and thus $N^{2n} = N^{2n-4} = \cdots = N^{2n-4i} = 0$, by Theorem 2.0.15. In particular, $\chi^{i+1}(M) = (-1)^{i+1} N^{i+1} = 0$.

Next, we consider $\chi^{i+1}(M) = 0$. Using the same argument, one can show that there are no fixed points of index $2k$ where $k$ is odd and $0 \leq k \leq i+2$. And then we consider $\chi^{i+2}(M) = 0$ to conclude that there are no fixed points of index $2k$ where $k$ is even and $0 \leq k \leq i+3$. We continue this to conclude that there are no fixed points of any index, which is a contradiction.

Second, suppose that $i \leq \frac{n}{2}$ and $i$ is even. Using the same argument as in the first case, by considering $\chi^i(M) = 0$, one can show that there are no fixed points of index $2k$, where $k$ is even and $0 \leq k \leq 2i$. Next, consider $\chi^{i+2}(M) = 0$ and conclude that there are no fixed points of index $2k$ where $k$ even and $k \leq i+4$. And then we consider $\chi^{i+4}(M) = 0$ to conclude that there are no fixed points of index $2k$ where $k$ is even and $k \leq i+6$. We continue this until $\chi^{n-i}(M)$ if $n$ is even and $\chi^{n-i-1}(M)$ if $n$ is odd, to conclude that there are no fixed points of index that is a multiple of 4, which contradicts Lemma 3.0.28 below that there must be fixed points whose indices differ by 2.

Third, suppose that $n$ is odd, $i > \frac{n}{2}$, and $i$ is odd. Considering $\chi^i(M) = 0$, it follows that there are no fixed points of index $2k$ such that $k$ is even and $0 \leq k \leq 2(n-i)$. By Theorem 2.0.15, since $N^j = N^{n-j}$ for all $j$, there are no fixed points of index $2k$, where $k$ is odd and $n - (2n-2i) = 2i - n \leq k \leq n$. In particular, there are no fixed points of index $i-2$. Next, considering $\chi^{i-2}(M) = 0$, we have that there are no fixed points of index $2k$, where $k$ is even and $k \leq 2n - 2i + 2$. By the symmetry that $N^j = N^{n-j}$ for all $j$, there are no fixed points of index $2k$ such that $k$ is odd and $2i - n - 2 \leq k \leq n$. We continue this to have that there are no fixed points of any index, which is a contradiction.

As a slight variation of the arguments above, the other cases, (4) $n$ is odd, $i > \frac{n}{2}$, and $i$ is even, (5) $n$ is even, $i > \frac{n}{2}$, and $i$ is odd, and (6) $n$ is even, $i > \frac{n}{2}$, and $i$ is even, are proved. $\square$

**Corollary 3.0.22.** *[TW], [L] A semi-free, symplectic circle action on a closed symplectic manifold $M$ with isolated fixed points is Hamiltonian if* *and only if it has a fixed point.*

*Proof.* All the weights are either 1 or $-1$. Therefore, $A_i = \{i\}$ for each $i$ and so the corollary follows. $\square$

We show that, in certain cases, we can only look at the sum of three weights. The first instance is the following (Theorem 1.0.2):

**Theorem 3.0.23.** *Consider a symplectic circle action on a $2n$-dimensional closed symplectic manifold $M$ with non-empty fixed points, whose weights are $\{\pm a_1, \pm a_2, \cdots, \pm a_n\}$ for some positive integers $a_i$, where $1 \leq i \leq n$. Assume that $n \leq 5$ and $\pm a_i \pm a_j \pm a_k \neq 0$ for all $i < j < k$. Then the action is Hamiltonian.*

*Proof.* Assume, on the contrary, that the action in not Hamiltonian. By Theorem 2.0.15, $\chi^0(M) = \chi^n(M) = 0$ and there are no fixed points of index 0 and $2n$. Denote by $A = \{a_1, a_2, \cdots, a_n\}$ and $A_i = \{a_{j_1} + a_{j_2} + \cdots + a_{j_i}\}_{a_{j_1} < a_{j_2} < \cdots < a_{j_i}}$ the collection of sums of $i$ elements of $A$, where $1 \leq i \leq n$. Then the problem is equivalent to showing that if $A_1 \cap A_2 = \emptyset$, the action is Hamiltonian. We consider $A_i \cap A_j$ for all $i, j$ such that $i \neq j \mod 2$, $1 \leq i \leq n-1$, $1 \leq j \leq n$.

First, assume that $n \leq 3$. Then $A_1 \cap A_2$ is the only intersection that we consider, so the result follows from Theorem 3.0.21.

Second, assume that $n = 4$. Then $A_1 \cap A_2$ and $A_2 \cap A_3$ are the only ones that we consider. However, $A_1 \cap A_2 = \emptyset$ if and only if $A_2 \cap A_3 = \emptyset$. Therefore the result follows from Theorem 3.0.21.

Finally, assume that $n = 5$. The only possible non-empty intersections that we consider are $A_1 \cap A_4 \neq \emptyset$, $A_2 \cap A_3 \neq \emptyset$, and $A_3 \cap A_4 \neq \emptyset$. However, $A_2 \cap A_3 \neq \emptyset$ if and only if $A_3 \cap A_4 \neq \emptyset$. Therefore, we can only consider the case $A_1 \cap A_4 \neq \emptyset$ and the case $A_2 \cap A_3 \neq \emptyset$. By the assumption that $A_1 \cap A_2 = \emptyset$, if one is satisfied, the other fails to be satisfied.

We consider $\chi^0(M)$. By Theorem 2.0.15,

$$\chi^0(M) = \sum_{p \in M^{S^1}} \frac{1}{\prod_j (1 - t^{\xi_p^j})} = \sum_{p \in M^{S^1}} (-1)^{\frac{\lambda_p}{2}} \frac{\prod_{\xi_p^j < 0} t^{-\xi_p^j}}{\prod_j (1 - t^{|\xi_p^j|})} = 0.$$

Multiplying by $(1 - t^{a_1}) \cdots (1 - t^{a_5})$ on the equation above, we have

$$0 = \sum_{p \in M^{S^1}} (-1)^{\frac{\lambda_p}{2}} t^{\sum_{\xi_p^j < 0} (-\xi_p^j)}.$$

Assume first that $A_1 \cap A_4 \neq \emptyset$. Without loss of generality, let $a_5 = a_1 + a_2 + a_3 + a_4$. Suppose that there is a fixed point whose weights are $\{-a_1, a_2, a_3, a_4, a_5\}$. In the last expression, such a point has a summand $-t^{a_1}$. This term can only be cancelled by another term whose exponent is the sum of even elements. However, $a_5 = a_1 + a_2 + a_3 + a_4$ is the only equation among weights. Therefore, there cannot be a fixed point whose weights are $\{-a_1, a_2, a_3, a_4, a_5\}$. Similarly, one can show that we may only have fixed points whose weights are $\{a_1, a_2, a_3, a_4, -a_5\}$ and $\{-a_1, -a_2, -a_3, -a_4, a_5\}$. Therefore, we only have fixed points of index 2 and 8, which contradicts Corollary 3.0.28 below.

Next, suppose that $A_2 \cap A_3 \neq \emptyset$. Without loss of generality, let $a_1 + a_2 + a_3 = a_4 + a_5$. By using an argument similar to the case above, one can show that we may only have fixed points whose weights are $\{a_1, a_2, a_3, -a_4, -a_5\}$ and $\{-a_1, -a_2, -a_3, a_4, a_5\}$. Suppose that there are $k$ fixed points whose weights are $\{a_1, a_2, a_3, -a_4, -a_5\}$. By Theorem 2.0.15, $k = \chi^2(M) = -\chi^3(M)$ and there are $k$ fixed points of weights $\{-a_1, -a_2, -a_3, a_4, a_5\}$. Moreover,

$$\begin{aligned}
0 = \chi^1(M) = {} & k\frac{t^{a_1} + t^{a_2} + t^{a_3} + t^{-a_4} + t^{-a_5}}{(1-t^{a_1})(1-t^{a_2})(1-t^{a_3})(1-t^{-a_4})(1-t^{-a_5})} \\
& + k\frac{t^{-a_1} + t^{-a_2} + t^{-a_3} + t^{a_4} + t^{a_5}}{(1-t^{-a_1})(1-t^{-a_2})(1-t^{-a_3})(1-t^{a_4})(1-t^{a_5})} \\
= {} & k\frac{t^{a_4+a_5}(t^{a_1} + t^{a_2} + t^{a_3} + t^{-a_4} + t^{-a_5})}{(1-t^{a_1})(1-t^{a_2})(1-t^{a_3})(1-t^{a_4})(1-t^{a_5})} \\
& - k\frac{t^{a_1+a_2+a_3}(t^{-a_1} + t^{-a_2} + t^{-a_3} + t^{a_4} + t^{a_5})}{(1-t^{a_1})(1-t^{a_2})(1-t^{a_3})(1-t^{a_4})(1-t^{a_5})}.
\end{aligned}$$

Multiplying the equation above by $(1-t^{a_1})(1-t^{a_2})\cdots(1-t^{a_5})$, we have

$$\begin{aligned}
0 = {} & kt^{a_4+a_5}(t^{a_1} + t^{a_2} + t^{a_3} + t^{-a_4} + t^{-a_5}) \\
& - kt^{a_1+a_2+a_3}(t^{-a_1} + t^{-a_2} + t^{-a_3} + t^{a_4} + t^{a_5}) \\
= {} & k(t^{a_1+a_4+a_5} + t^{a_2+a_4+a_5} + t^{a_3+a_4+a_5} + t^{a_5} + t^{a_4}) \\
& - k(t^{a_2+a_3} + t^{a_1+a_3} + t^{a_1+a_2} + t^{a_1+a_2+a_3+a_4} + t^{a_1+a_2+a_3+a_5}).
\end{aligned}$$

By the assumption it follows that the term $kt^{a_4}$ cannot be cancalled out, which is a contradiction. $\square$

Another case where the condition that the sum of any three weights is never equal to zero guarantees that a symplectic action is Hamiltonian, is when the dimension of the manifold is six (Theorem 1.0.3). In fact, we prove a stronger result:

**Theorem 3.0.24.** *Consider a symplectic circle action on a six-dimensional closed symplectic manifold with non-empty isolated fixed points. Suppose that each negative weight at the fixed point of index 2 is never equal to the sum of negative weights at the fixed point of index 4. Then the action is Hamiltonian.*

*Proof.* Assume, on the contrary, that the action is not Hamiltonian. By Theorem 2.0.15, $\chi^0(M) = \chi^3(M) = 0$ and there are no fixed points of index 0 and 6. Moreover, the number of fixed points of index 2 and that of 4 are equal. Suppose that there are $k$ fixed points of index 2, and let $p_i, q_i$ be the fixed points of index 2,4, respectively, for $1 \leq i \leq k$. Let $\Sigma_{p_i} = \{-b_{p_i}^1, b_{p_i}^2, b_{p_i}^3\}, \Sigma_{q_i} = \{-c_{q_i}^1, -c_{q_i}^2, c_{q_i}^3\}$ be the weights at $p_i, q_i$, respectively, where $b_{p_i}^j, c_{q_i}^j$ are positive integers. By permuting $p_i$'s and $q_i$'s if neccesary, we may assume that $b_{p_1}^1 \leq b_{p_2}^1 \leq \cdots \leq b_{p_k}^1$ and $c_{q_1}^1 + c_{q_1}^1 \leq c_{q_2}^1 + c_{q_2}^1 \leq \cdots \leq c_{q_k}^1 + c_{q_k}^1$. By Theorem 2.0.15,

$$\chi^0(M)$$
$$= \sum_i \frac{1}{(1-t^{-b_{p_i}^1})(1-t^{b_{p_i}^2})(1-t^{b_{p_i}^3})} + \sum_i \frac{1}{(1-t^{-c_{q_i}^1})(1-t^{-c_{q_i}^2})(1-t^{c_{q_i}^3})}$$
$$= -\sum_i \frac{t^{b_{p_i}^1}}{(1-t^{b_{p_i}^1})(1-t^{b_{p_i}^2})(1-t^{b_{p_i}^3})} + \sum_i \frac{t^{c_{q_i}^1+c_{q_i}^2}}{(1-t^{c_{q_i}^1})(1-t^{c_{q_i}^2})(1-t^{c_{q_i}^3})} = 0.$$

Denote by $A = \{a_1, a_2, \cdots, a_l\}$ the collection of all the absolute values of the weights over all the fixed points counted with multiplicity, where for each positive integer $a$ the multiplicity of $a$ in $A$ is precisely $\max_{p \in M^{S^1}} |\{i | a = |w_p^i|\}|$. Let $B_i = A \setminus \{b_{p_i}^1, b_{p_i}^2, b_{p_i}^3\} = \{d_{p_i}^1, \cdots, d_{p_i}^{l-3}\}$, $C_i = A \setminus \{c_{q_i}^1, c_{q_i}^2, c_{q_i}^3\} = \{e_{q_i}^1, \cdots, e_{q_i}^{l-3}\}$ be the elements in $A$ minus the absolute values of weights at $p_i$,$q_i$, respectively.

Multiplying the equation above by $\prod_{a \in A}(1-t^a)$, we have

$$0 = -\sum_i t^{b_{p_i}^1} \prod_{a \in B_i}(1-t^a) + \sum_i t^{c_{q_i}^1+c_{q_i}^2} \prod_{a \in C_i}(1-t^a)$$
$$= \{-\sum_i t^{b_{p_i}^1} + \sum_{i,j} t^{b_{p_i}^1 + d_{p_i}^j} - \sum_{i,j_1<j_2} t^{b_{p_i}^1 + d_{p_i}^{j_1} + d_{p_i}^{j_2}} + \cdots\} + \{\sum_i t^{c_{q_i}^1+c_{q_i}^2} - \sum_{i,j} t^{c_{q_i}^1+c_{q_i}^2+e_{q_i}^j} + \sum_{i,j_1<j_2} t^{c_{q_i}^1+c_{q_i}^2+e_{q_i}^{j_1}+e_{q_i}^{j_2}} - \cdots\}.$$

In the equation, each summand in the exponent of any term is an element of $A$. a term has the coefficient -1 if its exponent is the sum of odd elements in $A$ and 1 if its exponent is the sum of even elements in $A$. Consider $-t^{b_{p_1}^1}$. Since $b_{p_1}^1 \leq b_{p_i}^1$ for $i \geq 2$, this term cannot be cancelled out by any summand in $-\sum_i t^{b_{p_i}^1} + \sum_{i,j} t^{b_{p_i}^1 + d_{p_i}^j} - \sum_{i,j_1<j_2} t^{b_{p_i}^1 + d_{p_i}^{j_1} + d_{p_i}^{j_2}}$. Therefore, it must be cancelled out by another summand in $\sum_i t^{c_{q_i}^1 + c_{q_i}^2} - \sum_{i,j} t^{c_{q_i}^1 + c_{q_i}^2 + e_{q_i}^j} + \sum_{i,j_1<j_2} t^{c_{q_i}^1 + c_{q_i}^2 + e_{q_i}^{j_1} + e_{q_i}^{j_2}} - \cdots$ for some $i$, whose exponent is the sum of even elements in $A$, where at least two elements of them are $c_{q_i}^1, c_{q_i}^2$. By the assumption, the exponent of such a summand cannot be $c_{q_i}^1 + c_{q_i}^2$. Hence, the exponent of the term must be the sum of at least four elements, say $b_{p_1}^1 = c_{q_i}^1 + c_{q_i}^2 + \alpha$. Next, consider $t^{c_{q_i}^1 + c_{q_i}^2}$. We have that $c_{q_i}^1 + c_{q_i}^2 < c_{q_i}^1 + c_{q_i}^2 + \alpha = b_{p_1}^1$. Since $c_{q_1}^1 + c_{q_1}^1 \leq c_{q_2}^1 + c_{q_2}^1 \leq \cdots \leq c_{q_k}^1 + c_{q_k}^1$, the term $t^{c_{q_i}^1 + c_{q_i}^2}$ cannot be cancelled out by any term in $\sum_i t^{c_{q_i}^1 + c_{q_i}^2} - \sum_{i,j} t^{c_{q_i}^1 + c_{q_i}^2 + e_{q_i}^j} + \sum_{i,j_1<j_2} t^{c_{q_i}^1 + c_{q_i}^2 + e_{q_i}^{j_1} + e_{q_i}^{j_2}} - \cdots$. On the other hand, $c_{q_i}^1 + c_{q_i}^2 < b_{p_1}^1$. Therefore, it cannot also be cancelled out by any term in $-\sum_i t^{b_{p_i}^1} + \sum_{i,j} t^{b_{p_i}^1 + d_{p_i}^j} - \sum_{i,j_1<j_2} t^{b_{p_i}^1 + d_{p_i}^{j_1} + d_{p_i}^{j_2}}$, which is a contradiction. $\square$

As a corollary, we recover the result by L. Godinho:

**Corollary 3.0.25.** *[G] Let the circle act symplectically on a six-dimensional closed symplectic manifold. Suppose that fixed points are isolated and their weights are $\{\pm a, \pm b, \pm c\}$, where $0 < a \leq b \leq c$ and $a + b \neq c$. If there is a fixed point, then the action is Hamiltonian.*

*Proof.* This follows from Theorem 3.0.20, Theorem 3.0.21, Theorem 3.0.23, or Theorem 3.0.24. $\square$

For a certain type of weights, we show that there is a restriction. To show the restriction, we introduce a terminology.

**Definition 3.0.26.** *Consider a circle action on a closed almost complex manifold. Suppose that the action preserves the almost complex structures and the fixed points are isolated. Denote by $A = \{a_1, a_2, \cdots, a_l\}$ the collection of all the absolute values of weights among all the fixed points counted with multiplicity, and $A_i = \{a_{j_1} + a_{j_2} + \cdots + a_{j_i}\}_{a_{j_1} < a_{j_2} < \cdots < a_{j_i}}$ the collection of sums of $i$ elements of $A$, for $1 \leq i \leq l$. A positive weight $w$ is called **primitive**, if $w \notin A_i$ for $i \geq 2$, i.e. $w$ is never equal to the sum of the absolute values of weights among all the fixed points, counted with multiplicity, other than $w$ itself.*

Note that the smallest positive weight is primitive. In [Ka], C. Kosniowski derives a certain formula for a holomorphic vector field on a complex manifold with only simple isolated zeros. We follow the idea of C. Kosniowski to find a restriction for a primitive weight of a circle action on a closed almost complex manifold with isolated fixed points. For the smallest positive weight, the Lemma is already given in [**?**] and the proof is almost identical, but we give a proof in details.

**Lemma 3.0.27.** *Consider a circle action on a $2n$-dimensional closed almost complex manifold. Suppose that the action preserves the almost complex structure and the fixed points are isolated. For each primitive weight $w$, the number of times the weight $-w$ occurs at fixed points of index $2i$ is equal to the number of times the weight $w$ occurs at fixed points of index $2i-2$, for all $i$.*

*Proof.* We first show that

$$\sum_{\lambda_p=2i}[N_p(w)+N_p(-w)]=\sum_{\lambda_p=2i-2}N_p(w)+\sum_{\lambda_p=2i+2}N_p(-w), (*)$$

where $N_p(w)$ is the number of times the weight $w$ occurs at $p$. The basic idea is to manipulate $\chi^i(M)$ and compare the coefficients of $t^w$-terms. By Theorem 2.0.15,

$$\chi^i(M)=\sum_{p\in M^{S^1}}\frac{\sigma_i(t^{\xi_p^1},\cdots,t^{\xi_p^n})}{\prod_{m=1}^n(1-t^{\xi_p^m})}$$

$$=\sum_{p\in M^{S^1}}(-1)^{\frac{\lambda_p}{2}}\frac{[\prod_{\xi_p^m<0}t^{-\xi_p^m}]\sigma_i(t^{\xi_p^1},\cdots,t^{\xi_p^n})}{\prod_{m=1}^n(1-t^{|\xi_p^m|})}.(**)$$

Denote by $J_p=[\prod_{\xi_p^m<0}t^{-\xi_p^m}]\sigma_i(t^{\xi_p^1},\cdots,t^{\xi_p^n})$ and $K_p=\prod_{m=1}^n(1-t^{|\xi_p^m|})$.

If $\lambda_p=2i$, then $J_p=1+f_p(t)$, where $f_p(t)$ is a polynomial that does not have a constant term and $t^w$-term.

If $\lambda_p=2i\pm 2$, then $J_p=N_p(\mp w)t^w+f_p(t)$, where $f_p(t)$ is a polynomial that does not have a constant term and $t^w$-term.

If $\lambda_p\neq 2i,2i\pm 2$, then $J_p=f_p(t)$, where $f_p(t)$ is a polynomial that does not have a constant term and $t^w$-term.

Multiplying $(**)$ by $\prod_{p\in M^{S^1}}K_p$ yields

$$\chi^i(M)[1 - \textstyle\sum_p (N_p(-w) + N_p(w))t^w] + g_1(t) =$$
$$\{(-1)^{i-1} \textstyle\sum_{\lambda_p=2i-2} N_p(w) + (-1)^{i+1} \sum_{\lambda_p=2i+2} N_p(-w) + (-1)^i \sum_{\lambda_p=2i} (N_p(w) + N_p(-w)) - \chi^i(M) \sum_p (N_p(w) + N_p(-w))\}t^w + \sum_{\lambda_p=2i} (-1)^i + g_2(t),$$

where $g_i(t)$ are polynomials without constant terms and $t^w$-terms. Comparing the coefficients of $t^w$-terms, the claim follows.

Applying $(*)$ for $i = 0$, we have

$$\sum_{\lambda_p=0} N_p(w) = \sum_{\lambda_p=2} N_p(-w).$$

Next, applying $(*)$ for $i = 1$, we have

$$\sum_{\lambda_p=2} [N_p(-w) + N_p(w)] = \sum_{\lambda_p=0} N_p(w) + \sum_{\lambda_p=4} N_p(-w).$$

Since $\sum_{\lambda_p=0} N_p(w) = \sum_{\lambda_p=2} N_p(-w)$, it follows that

$$\sum_{\lambda_p=2} N_p(w) = \sum_{\lambda_p=4} N_p(-w),$$

Continuing this, the Lemma follows. $\square$

As an application, there must be at least two fixed points whose indices are nearby. This is shown for a holomorphic vector field on a compact complex manifold with only simple isolated zeroes by C. Kosniowski [Ka].

**Corollary 3.0.28.** *Consider a circle action on a closed almost complex manifold. Suppose that the action preserves the almost complex structure and the fixed points are non-empty and isolated. Then there exist two fixed points whose indices differ by 2.*

*Proof.* Apply Lemma 3.0.27 to the smallest positive weight. $\square$

# Chapter 4

# The Case of Three Fixed Points

## 4.1 Preliminaries

In the introduction, we mentioned the following:

**Theorem 4.1.1.** *Let the circle act symplectically on a compact, connected symplectic manifold $M$. If there are exactly three fixed points, $M$ is equivariantly symplectomorphic to $\mathbb{CP}^2$.*

*Proof.* By quotienting out by the subgroup which acts trivially, without loss of generality we may assume that the action is effective. Then this is an immediate consequence of Proposition 4.2.3, Proposition 4.3.1, and Proposition 4.4.1 below. $\square$

We now give a brief overview of the proof. The proof is based on induction on the dimension of $M$. The main idea of the proof is to get restrictions on the weights at the three fixed points and show that if $\dim M > 4$, the weights cannot satisfy all the restrictions.

One of the key facts is that, as mentioned in Remark 2.0.17, for any nonzero integer $l$ such that $l \in \mathbb{Z} \setminus \{-1, 0, 1\}$, the subgroup $\mathbb{Z}_l \subset S^1$ also acts on $M$. Moreover, the set of points $M^{\mathbb{Z}_l}$ fixed by the $\mathbb{Z}_l$-action is a union of symplectic submanifolds, and the isotropy weights in $M^{\mathbb{Z}_l}$ are multiples of $l$.

Two important isotropy weights are the largest weight and two. First, let $d$ be the biggest weight among all the weights that occurs at the three fixed points. We show that $M^{\mathbb{Z}_d}$ is a union of a 2-sphere and a point. Moreover, we show that this gives significant restrictions on other weights. Second, we show that $M^{\mathbb{Z}_2}$ is $\mathbb{CP}^2$, provided that $\dim M > 4$. This implies that for manifolds of dimension greater than four, exactly two weights at each fixed point are even. If the largest weight $d$ is even, it is itself one of the weights in the isotropy submanifold fixed by the $\mathbb{Z}_2$-action. Because of this, we divide the theorem into two cases, depending on if the largest weight is odd or even. The former case is much easier and requires less work. Finally, if $\dim M > 4$, the fact that there are exactly three fixed points implies that the sum of the weights at any fixed point is zero. When combined with the restrictions described above, this determines the number of negative weights at each fixed point.

Finally, the ABBV localization formula describes the push-forward map from the equivariant cohomology of $M$ to the equivariant cohomology of a point in terms of a formula in the weights at the fixed points [AB], see Theorem 2.0.9. Since the push-forward of 1 is 0, this gives additional restrictions on the weights, we use these to complete the proof.

The classification of the case of three fixed points is organized in the following way. In section 4.1, we prove preliminary lemmas. In section 4.2, we prove the base case, that is, when $\dim M < 8$. In section 4.3, we consider the case where the largest weight is odd. Section 4.4 and 4.5 are preliminaries for section 4.6, in which we consider the case where the largest weight is even.

As mentioned in the overview of the proof above, one of the main ideas to prove the theorem is to look at the biggest weight among all the weights of fixed points. Consider a symplectic circle action on a $2n$-dimensional compact, connected symplectic manifold $M$ with exactly three fixed points. The key fact is that the largest weight occurs only once. From this it follows that if without loss of generality we assume that $\lambda_p \leq \lambda_q \leq \lambda_r$ where $p, q$, and $r$ are the three fixed points, then $\lambda_p = n-2$, $\lambda_q = n$, and $\lambda_r = n+2$.

**Definition 4.1.2.** *A weight $d$ is **the largest weight** if it is the biggest weight such that $\sum_{u \in M^{S^1}} N_u(d) > 0$.*

**Proposition 4.1.3.** *Let the circle act symplectically on a compact, connected symplectic manifold $M$ and suppose that there are exactly three fixed points. Let $d$ be the largest weight. Then the isotropy submanifold $M^{\mathbb{Z}_d}$ contains exactly two components that have fixed points: one isolated fixed point and one two-sphere that contains two fixed points.*

*Proof.* Consider the isotropy submanifold $M^{\mathbb{Z}_d}$. By Theorem 2.0.13, the only possible cases are:

1. The isotropy submanifold $M^{\mathbb{Z}_d}$ contains a 2-sphere with two fixed points. The third fixed point is another component of $M^{\mathbb{Z}_d}$.

2. The isotropy submanifold $M^{\mathbb{Z}_e}$ contains a 6-dimensional component with two fixed points. The third fixed point is another component of $M^{\mathbb{Z}_d}$.

3. The isotropy submanifold $M^{\mathbb{Z}_d}$ contains a component with the three fixed points.

The subset inclusions may not be equalities since $M^{\mathbb{Z}_d}$ may contain other components with no fixed points.

Suppose that the second case holds. By Theorem 2.0.13, the weights in the isotropy submanifold $M^{\mathbb{Z}_d}$ at two fixed points that lie in the 6-dimensional component are $\{a, b, -a-b\}$ and $\{-a, -b, a+b\}$ for some natural numbers $a$ and $b$. Moreover, $a, b$, and $a+b$ are multiples of $d$, which is impossible since $d$ is the largest weight.

Suppose that the third case holds. Let $Z$ be the component. Let $\dim Z = 2m$. Since all the weights in the isotropy submanifold $M^{\mathbb{Z}_d}$ are either $d$ or $-d$, by Theorem 2.0.9,

$$0 = \int_Z 1 = \frac{1}{\prod_{i=1}^m \pm d} + \frac{1}{\prod_{i=1}^m \pm d} + \frac{1}{\prod_{i=1}^m \pm d} = \pm\frac{1}{d^m} \mp \frac{1}{d^m} \pm \frac{1}{d^m} \neq 0,$$

which is a contradiction.

Hence the first case is the case and the weights in the isotropy submanifold $M^{\mathbb{Z}_d}$ at the two fixed points in the 2-sphere are $\{-d\}$ and $\{d\}$. $\square$

Considering $S^1$ as a subset of $\mathbb{C}$, denote a $S^1$-action on a manifold $M$ by $g \cdot p$ for $g \in S^1, p \in M$. For technical reasons, throughout the paper we sometimes reverse the $S^1$-action. By reversing the action, we mean a $S^1$-action on $M$ by $g^{-1} \cdot p$.

**Lemma 4.1.4.** *Let the circle act symplectically on a $2n$-dimensional compact, connected symplectic manifold $M$ and suppose that there are exactly three fixed points. Then we can label the fixed points $p, q$, and $r$ so that $\lambda_p = n-2, \lambda_q = n$, and $\lambda_r = n+2$. Moreover, if $\dim M \neq 4$, then after possibly reversing the circle action, we may assume that $-d \in \Sigma_p$ and $d \in \Sigma_q$, where $d$ is the largest weight.*

*Proof.* Let $p, q$, and $r$ be the fixed points. Without loss of generality, assume that $\lambda_p \leq \lambda_q \leq \lambda_r$. By Corollary 2.0.11, $n$ is even. Also, since the number of fixed points is odd, $\lambda_q = n$ by Lemma 2.0.10. Moreover, since $M$ is connected, $\dim M \neq 0$.

First, assume that $\dim M = 4$. Then by Lemma 2.0.10, either $\lambda_p = \lambda_q = \lambda_r = 2$, or $\lambda_p = 0, \lambda_q = 2$, and $\lambda_r = 4$. Suppose that $\lambda_p = \lambda_q = \lambda_r = 2$. Then by Theorem 2.0.9,

$$0 = \int_M 1 = \frac{1}{\prod_{i=1}^2 \xi_p^i} + \frac{1}{\prod_{i=1}^2 \xi_q^i} + \frac{1}{\prod_{i=1}^2 \xi_r^i} < 0,$$

which is a contradiction. Hence $\lambda_p = 0, \lambda_q = 2$, and $\lambda_r = 4$.

Next, assume that $\dim M \geq 8$. By Lemma 4.1.3, we can label the fixed points $\alpha$, $\beta$, and $\gamma$ such that $\alpha$ and $\beta$ lie in the same 2-dimensional connected component of $M^{\mathbb{Z}_d}$ such that $-d \in \Sigma_\alpha$, $d \in \Sigma_\beta$, and $N_\gamma(d) = N_\gamma(-d) = 0$. By reversing the circle action if necessary, we may assume that either $\lambda_\alpha \leq \lambda_\gamma$ or $\lambda_\beta \leq \lambda_\gamma$. Moreover, by Corollary 2.0.14, the first Chern class map is identically zero. By Lemma 4.1.5 below, $\lambda_\alpha + 2 = \lambda_\beta$. Together with Lemma 2.0.10, the above statements imply that $\lambda_p = n-2$, $\lambda_q = n$, $\lambda_r = n+2$, $-d \in \Sigma_p$, and $d \in \Sigma_q$. $\square$

To prove Lemma 4.1.4, we need the following technical Lemma.

**Lemma 4.1.5.** *Let the circle act on a $2n$-dimensional compact symplectic manifold $(M, \omega)$. Let $v$ and $w$ be fixed points in the same 2-dimensional component $Z$ of $M^{\mathbb{Z}_d}$, where $d$ is the largest weight. Also suppose that $-d \in \Sigma_v$, $d \in \Sigma_w$, and $c_1(M)|_v = c_1(M)|_w$. Then $\lambda_v + 2 = \lambda_w$.*

*Proof.* By Lemma 2.0.19, $\Sigma_v \equiv \Sigma_w \mod d$. Let $\xi_v^i, 1 \leq i \leq n$, and $\xi_w^i, 1 \leq i \leq n$, be the weights at $v$ and $w$, respectively, where $\xi_v^i, \xi_w^i \in \mathbb{Z} \setminus \{0\}$. By permuting if necessary, we can assume that $\xi_v^i \equiv \xi_w^i \mod d$, for all $i < n$, $\xi_v^n = -d$, and $\xi_w^n = d$. By Lemma 2.0.12, $d > |\xi_v^i|$ and $d > |\xi_w^i|$, for $i < n$. Then for all $i < n$, the following holds:

1. If $\xi_v^i > 0$ and $\xi_w^i > 0$, or if $\xi_v^i < 0$ and $\xi_w^i < 0$, then $\xi_v^i \equiv \xi_w^i \mod d$ implies $\xi_v^i - \xi_w^i = 0$.
2. If $\xi_v^i > 0$ and $\xi_w^i < 0$, then $\xi_v^i \equiv \xi_w^i \mod d$ implies $\xi_v^i - \xi_w^i = d$.
3. If $\xi_v^i < 0$ and $\xi_w^i > 0$, then $\xi_v^i \equiv \xi_w^i \mod d$ implies $\xi_v^i - \xi_w^i = -d$.

Moreover, there are $\frac{\lambda_v}{2} - 1$ negative weights in $\Sigma_v$ excluding $-d$ and $\frac{\lambda_w}{2}$ negative weights in $\Sigma_w$. Hence,

$$
\begin{aligned}
0 = c_1(M)|_v - c_1(M)|_w &= (\xi_v^1 + \cdots + \xi_v^{n-1} - d) - (\xi_w^1 + \cdots + \xi_w^{n-1} + d) \\
&= (\xi_v^1 - \xi_w^1) + \cdots + (\xi_v^{n-1} - \xi_w^{n-1}) - 2d = d\left(\frac{\lambda_w}{2} - \frac{\lambda_v}{2} + 1\right) - 2d \\
&= d\left(\frac{\lambda_w}{2} - \frac{\lambda_v}{2} - 1\right).
\end{aligned}
$$

Therefore, $\lambda_v + 2 = \lambda_w$. $\square$

**Remark 4.1.6.** *We can generalize Lemma 4.1.5 in the following way: let the circle act on a $2n$-dimensional compact symplectic manifold $(M, \omega)$. Let $v$ and $w$ be fixed points in the same component $Z$ of $M^{\mathbb{Z}_d}$, where $d$ is the largest weight. Then*

$$
\lambda_v(M) - \lambda_w(M) + \lambda_v(Z) - \lambda_w(Z) = -\frac{2}{d}\left(c_1(M)|_v - c_1(M)|_w\right).
$$

*The proof goes similarly to that of Lemma 4.1.5; by Lemma 2.0.19, their weights are equal modulo $d$. Then as a bijection between $\Sigma_v$ and $\Sigma_w$ modulo $d$, $\pm d$ at $v$ is paired with $\pm d$ at $w$. Consider other weights; a positive weight $\xi$ at $v$ is either paired with $\xi$ or $\xi - d$ at $w$, etc. Finally, we consider the difference of the first Chern class at $v$ and $w$ together with indices at $v$ and $w$.*

Finally, when the largest weight is odd, we need the following closely related technical lemma.

**Lemma 4.1.7.** *Let the circle act on a $2n$-dimensional compact symplectic manifold $(M, \omega)$. Suppose that fixed points $v$ and $w$ satisfy the conditions in Lemma 4.1.5. Let $d$ be the largest weight and assume that $d$ is odd. Suppose that $\Sigma_v$ and $\Sigma_w$ have $E_v^+$ and $E_w^+$ positive even weights and $E_v^-$ and $E_w^-$ negative even weights, respectively. Then $E_v^+ - E_v^- - E_w^+ + E_w^- = 2$.*

*Proof.* By Lemma 2.0.19, $\Sigma_v \equiv \Sigma_w \mod d$. Define $\xi_v^i$ and $\xi_w^i$ as in Lemma 4.1.5 and recall that the following hold:

(a) If $\xi_v^i > 0$ and $\xi_w^i > 0$, or if $\xi_v^i < 0$ and $\xi_w^i < 0$, then $\xi_v^i - \xi_w^i = 0$.

(b) If $\xi_v^i > 0$ and $\xi_w^i < 0$, then $\xi_v^i - \xi_w^i = d$.

(c) If $\xi_v^i < 0$ and $\xi_w^i > 0$, then $\xi_v^i - \xi_w^i = -d$.

Let $e_v^+$ be a positive even weight at $v$, $e_v^-$ a negative even weight at $v$, $o_v^+$ a positive odd weight at $v$, and $o_v^-$ a negative odd weight at $v$, and similarly for $w$. Then since the largest weight $d$ is odd, we have the following:

1. $e_v^+ \equiv e_w^+ \mod d$ implies that $e_v^+ = e_w^+$. Hence in $c_1(M)|_v - c_1(M)|_w$, this pair contributes 0. Suppose that there are $k_1$ such pairs.

2. $e_v^+ \equiv o_w^- \mod d$ implies that $e_v^+ - o_w^- = d$. Hence in $c_1(M)|_v - c_1(M)|_w$, this pair contributes $d$. There are $E_v^+ - k_1$ such pairs.

3. $e_v^- \equiv e_w^- \mod d$ implies that $e_v^- = e_w^-$. Hence in $c_1(M)|_v - c_1(M)|_w$, this pair contributes as 0. Suppose that there are $k_2$ such pairs.

4. $e_v^- \equiv o_w^+ \mod d$ implies that $e_v^- - o_w^+ = -d$. Hence in $c_1(M)|_v - c_1(M)|_w$, this pair contributes $-d$. There are $E_v^- - k_2$ such pairs.

5. $o_v^+ \equiv o_w^+ \mod d$ implies that $o_v^+ = o_w^+$. Hence in $c_1(M)|_v - c_1(M)|_w$, this pair contributes 0. Suppose that there are $k_3$ such pairs.

6. $o_v^+ \equiv e_w^- \mod d$ implies that $o_v^+ - e_w^- = d$. Hence in $c_1(M)|_v - c_1(M)|_w$, this pair contributes $d$. There are $E_w^- - k_2$ such pairs.

7. $o_v^- \equiv e_w^+ \mod d$ implies that $o_v^- - e_w^+ = -d$. Hence in $c_1(M)|_v - c_1(M)|_w$, this pair contributes $-d$. There are $E_w^+ - k_1$ such pairs.

8. $o_v^- \equiv o_w^- \mod d$ implies that $o_v^- = o_w^-$. Hence in $c_1(M)|_v - c_1(M)|_w$, this pair contributes 0. Suppose that there are $k_4$ such pairs.

Then

$$\begin{aligned} 0 &= c_1(M)|_v - c_1(M)|_w \\ &= d(E_v^+ - k_1) - d(E_v^- - k_2) + d(E_w^- - k_2) - d(E_w^+ - k_1) - 2d \\ &= d(E_v^+ - E_v^- - E_w^+ + E_w^- - 2). \end{aligned}$$

□

**Remark 4.1.8.** *We can also generalize Lemma 4.1.7 in the following way: Let the circle act on a $2n$-dimensional compact symplectic manifold $(M,\omega)$. Let $v$ and $w$ be fixed points in the same component $Z$ of $M^{\mathbb{Z}_d}$, where $d$ is the largest weight. Assume that the largest weight $d$ is odd. Then*

$$d(E_v^+ - E_v^- - E_w^+ + E_w^-) = c_1(M)|_v - c_1(M)|_w - c_1(Z)|_v + c_1(Z)|_w.$$

*The proof goes similarly to that of Lemma 4.1.7.*

## 4.2 Base Case

The proof of Theorem 4.1.1 is based on induction. The key fact to prove Theorem 4.1.1 is that an isotropy submanifold of a symplectic manifold is itself a smaller symplectic manifold.

To prove the base case, we need several theorems:

**Proposition 4.2.1.** *[MD] An effective symplectic circle action on a four dimensional compact, connected symplectic manifold is Hamiltonian if and only if the fixed point set is non-empty.*

Let the circle act symplectically on a 4-dimensional compact, connected symplectic manifold $M$ with isolated fixed points. Assume that the action is effective. Then we can associate a graph to $M$ in the following way: we assign a vertex to each fixed point. Label each fixed point by its moment image. Additionally, given two fixed points $p$ and $q$, we say that $(p, q)$ is an edge if there exists $k > 1$ such that $p$ and $q$ are contained in the same component of the isotropy submanifold $M^{\mathbb{Z}_k}$, where $k$ is the largest such. We label the edge by $k$. We say that two graphs $\Pi$ and $\Pi'$ are isomorphic, if there are one-to-one correspondence between vertices in $\Pi$ and vertices in $\Pi'$, and one-to-one correspondence between edges in $\Pi$ and edges in $\Pi'$ such that if $\sigma : \Pi \longrightarrow \Pi'$ is such a map and if $(p, q)$ is a $k$-edge, then $\sigma((p, q))$ is a $k$-edge.

**Theorem 4.2.2.** *(Uniqueness Theorem) [Ka] Let $(M, \omega, \Pi)$ and $(M', \omega', \Pi')$ be two compact four dimensional Hamiltonian $S^1$ spaces. Then any isomorphism between their corresponding graphs is induced by an equivariant symplectomorphism.*

We now prove the base case.

**Proposition 4.2.3.** *Let the circle act symplectically on a compact, connected symplectic manifold $M$ and suppose that there are exactly three fixed points. If $\dim M < 8$, then $M$ is equivariantly symplectomorphic to $\mathbb{CP}^2$.*

*Proof.* Suppose that $\dim M < 8$. By quotienting out by the subgroup which acts trivially, we may assume that the action is effective. Since the manifold $M$ is connected, $\dim M \neq 0$. Hence by Corollary 2.0.11, $\dim M = 4$. Let $p, q$, and $r$ denote the three fixed points and without loss of generality assume that $\lambda_p \leq \lambda_q \leq \lambda_r$. Then by Lemma 4.1.5, $\lambda_p = 0, \lambda_q = 2$, and $\lambda_r = 4$.

By a standard action on $\mathbb{CP}^2$ we mean that for each $g \in S^1 \subset \mathbb{C}$, $g$ acts on $\mathbb{CP}^2$ by

$$g \cdot [z_0 : z_1 : z_2] = [g^{a+b} z_0 : g^a z_1 : z_2]$$

for some positive integers $a$ and $b$. This action has three fixed points $[1 : 0 : 0]$, $[0 : 1 : 0]$, and $[0 : 0 : 1]$. And the weights at these points are $\{-a-b, -b\}$, $\{-a, b\}$, and $\{a, a+b\}$.

Since $\dim M = 4$, by Proposition 4.2.1, the action is Hamiltonian. Furthermore, by Lemma 2.0.12, there exist positive integers $a$, $b$, and $c$ such that the weights are $\Sigma_p = \{a, c\}$, $\Sigma_q = \{-a, b\}$, and $\Sigma_r = \{-b, -c\}$. By Theorem 2.0.9,

$$0 = \int_M 1 = \frac{1}{ac} - \frac{1}{ab} + \frac{1}{bc} = \frac{b - c + a}{abc}.$$

Thus $c = a + b$. It is straightforward to check that the corresponding graph is isomorphic to a graph corresponding to some standard action on $\mathbb{CP}^2$ where the action is given by $g \cdot [z_0 : z_1 : z_2] = [g^{a+b} z_0 : g^b z_1 : z_2]$, and hence this induces an equivariant symplectomorphism on manifolds by Theorem 4.2.2. □

Hence from now on we assume that $\dim M \geq 8$. Then note that, by Corollary 2.0.14, the Chern class map is identically zero.

**Lemma 4.2.4.** *Fix a natural number $n$ such that $n \geq 4$. Assume that Theorem 4.1.1 holds for all manifolds $M$ such that $\dim M < 2n$. Let the circle act symplectically on a $2n$-dimensional compact, connected symplectic manifold $M$ and suppose that there are exactly three fixed points. Assume that the action is effective. Then there exist even positive integers $a$ and $b$ such that the weights at the three fixed points in the isotropy submanifold $M^{\mathbb{Z}_2}$ are $\{a, c\}, \{-a, b\}$, and $\{-b, -c\}$, where $c = a + b$.*

*Proof.* Since the action is effective, the isotropy submanifold $\mathbb{Z}_2$ is a smaller manifold, i.e., for any component $Z$ of $M^{\mathbb{Z}_2}$, we have that $\dim Z < \dim M$. Then by the inductive hypothesis and Theorem 2.0.13, there are only four possible cases:

1. Each fixed point is a component of the isotropy submanifold $M^{\mathbb{Z}_2}$.

2. The isotropy submanifold $M^{\mathbb{Z}_2}$ contains a 2-sphere with two fixed points. The third fixed point is another component of $M^{\mathbb{Z}_2}$.

3. The isotropy submanifold $M^{\mathbb{Z}_2}$ contains a 4-dimensional component with the three fixed points.

4. The isotropy submanifold $M^{\mathbb{Z}_2}$ contains a 6-dimensional component with two fixed points. The third fixed point is another component of $M^{\mathbb{Z}_2}$.

Assume that the first case holds. Let $p, q$, and $r$ be the fixed points. The first case means that all the weights at $p, q$, and $r$ are odd. Let $A, B$, and $C$ be the products of the weights at $p, q$, and $r$, respectively. Then by Theorem 2.0.9,

$$\int_M 1 = \sum_{F \subset M^{S^1}} \int_F \frac{1}{e_{S^1}(N_F)} = \frac{1}{\prod \xi_p^i} t^{-n} + \frac{1}{\prod \xi_q^i} t^{-n} + \frac{1}{\prod \xi_r^i} t^{-n}$$
$$= \left(\frac{1}{A} + \frac{1}{B} + \frac{1}{C}\right) t^{-n} = 0.$$

Hence

$$\frac{1}{A} + \frac{1}{B} + \frac{1}{C} = 0.$$

Multiplying both sides by $ABC$ yields

$$BC + AC + AB = 0.$$

However, since $A, B$, and $C$ are odd,

$$BC + AC + AB \equiv 1 \mod 2,$$

which is a contradiction.

Assume that the second case holds. Then the two fixed points in the 2-sphere have one even weight and $n-1$ odd weights. By Corollary 2.0.11, $n$ is even. Then sums of the weights at these points are congruent to $1 \mod 2$, which contradicts Corollary 2.0.14 that the first Chern class map (the sum of the weights at a fixed point) is zero for all fixed points if $\dim M \geq 8$ and there are exactly three fixed points.

Assume that the fourth case holds. Then the two fixed points in the 2-sphere have three even weights and $n-3$ odd weights. By Corollary 2.0.11, $n$ is even. Again, the sums of weights at these points are congruent to 1 mod 2, which contradicts that the first Chern class map is zero for all fixed points by Corollary 2.0.14.

Hence the third case is the case. Thus as in the proof of Proposition 4.2.3, there are even natural numbers $a$ and $b$ such that the fixed points of the isotropy submanifold $M^{\mathbb{Z}_2}$ have weights $\{a+b, a\}$, $\{-a, b\}$, and $\{-b, -a-b\}$. $\square$

**Lemma 4.2.5.** *Fix a natural number $n$. Assume that Theorem 4.1.1 holds for all manifolds $M$ such that $\dim M < 2n$. Let the circle act symplectically on a $2n$-dimensional compact, connected symplectic manifold $M$ and suppose that there are exactly three fixed points. Assume that the action is effective. Given an integer $e \in \mathbb{Z} \setminus \{-1, 0, 1\}$, exactly one of the following holds:*

1. *Each fixed point is a component of the isotropy submanifold $M^{\mathbb{Z}_e}$.*

2. *The isotropy submanifold $M^{\mathbb{Z}_e}$ contains a 2-sphere with two fixed points, and the weights in the 2-sphere at these points are $\{a\}$ and $\{-a\}$ for some natural number $a$ that is a multiple of $e$. The third fixed point is another component of $M^{\mathbb{Z}_e}$.*

3. *The isotropy submanifold $M^{\mathbb{Z}_e}$ contains a 4-dimensional component with the three fixed points, and the weights in the isotropy submanifold at these points are $\{a+b, a\}$, $\{-a, b\}$, and $\{-b, -a-b\}$ for some natural numbers $a$ and $b$ that are multiples of $e$.*

4. *The isotropy submanifold $M^{\mathbb{Z}_e}$ contains a 6-dimensional component with two fixed points, and the weights in the isotropy submanifold at these points are $\{a, b, -a-b\}$ and $\{a+b, -a, -b\}$ for some natural numbers $a$ and $b$ that are multiples of $e$. The third fixed point is another component of $M^{\mathbb{Z}_e}$.*

*Proof.* Fix an integer $e \in \mathbb{Z} \setminus \{-1, 0, 1\}$. Since the action on $M$ is effective, for any component $Z$ of the isotropy submanifold $M^{\mathbb{Z}_e}$, we have that $\dim Z < \dim M$.

By ABBV Localization (Theorem 2.0.9), if any component of the isotropy submanifold $M^{\mathbb{Z}_e}$ only contains one fixed point, then the fixed point itself is the component.

If every fixed point is itself a component of the isotropy submanifold $M^{\mathbb{Z}_e}$, this is the first case of the Lemma.

Suppose instead that there exists a component $Z$ of the isotropy submanifold $M^{\mathbb{Z}_e}$ that contains exactly two fixed points. Then by Theorem 2.0.13, either

(a) The component is 2-sphere and the weights in the isotropy submanifold $M^{\mathbb{Z}_e}$ at these points are $\{a\}$ and $\{-a\}$ for some natural number $a$ that is a multiple of $e$. By the previous argument, the third fixed point is another component of $M^{\mathbb{Z}_e}$. This is the second case of the Lemma.

(b) The component is 6-dimensional and the weights in the isotropy submanifold $M^{\mathbb{Z}_e}$ at these points are $\{a, b, -a-b\}$ and $\{a+b, -a, -b\}$ for some natural numbers $a$ and $b$ that are multiples of $e$. The third fixed point is another component of $M^{\mathbb{Z}_e}$. This is the fourth case of the Lemma.

Finally, suppose that a component of the isotropy submanifold $M^{\mathbb{Z}_e}$ contains the three fixed points. Then by the inductive hypothesis, the component is 4-dimensional and the weights in the isotropy submanifold are $\{a+b, a\}$, $\{-a, b\}$, and $\{-b, -a-b\}$ for some natural numbers $a$ and $b$ that are multiples of $e$. This is the third case of the Lemma. $\square$

As particular cases of Lemma 4.2.5, we need the following Lemma.

**Lemma 4.2.6.** *Fix a natural number $n$. Assume that Theorem 4.1.1 holds for all manifolds $M$ such that $\dim M < 2n$. Let the circle act symplectically on a $2n$-dimensional compact, connected symplectic manifold $M$ and suppose that there are exactly three fixed points. Assume that the action is effective. Fix an integer $e \in \mathbb{Z} \setminus \{-1, 0, 1\}$.*

1. *Suppose that there exist distinct fixed points $\alpha$ and $\beta$ such that $N_\alpha(e) > 0$ and $N_\beta(-e) > 0$ such that $|e| > \frac{d}{2}$ where $d$ is the largest weight. Then $N_\alpha(e) = 1$, $N_\beta(-e) = 1$, $\Sigma_\alpha \equiv \Sigma_\beta \mod e$, and no additional multiples of $e$ appear as weights.*

2. *If there exist two distinct fixed points $\alpha$ and $\beta$ such that $N_\alpha(e) > 0$ and $N_\beta(e) > 0$, then after possibly switching $\alpha$ and $\beta$,*

$$\{2e, e\} \subset \Sigma_\alpha,\ \{-e, e\} \subset \Sigma_\beta,\ \text{and } \{-2e, -e\} \subset \Sigma_\gamma$$

*where $\gamma$ is the remaining fixed point. Moreover, no additional multiples of $e$ appear as weights.*

3. *If there exists a fixed point $\alpha$ such that $N_\alpha(e) > 1$, $\{-2e, e, e\} \subset \Sigma_\alpha$ and $\{2e, -e, -e\} \subset \Sigma_\beta$ for some fixed point $\beta \neq \alpha$. Moreover, no additional multiples of $e$ appear as weights.*

4. *Suppose that there exists a fixed point $\beta$ such that $N_\beta(e) > 0$ and $N_\beta(-e) > 0$. Then*

$$\{2e, e\} \subset \Sigma_\alpha,\ \{-e, e\} \subset \Sigma_\beta,\ \textit{and}\ \{-2e, -e\} \subset \Sigma_\gamma$$

*where $\alpha$ and $\gamma$ are the remaining two fixed points. Moreover, no additional multiples of $e$ appear as weights.*

*Proof.* Fix an integer $e \in \mathbb{Z}\backslash\{-1, 0, 1\}$ and consider the isotropy submanifold $M^{\mathbb{Z}_e}$.

1. By looking at the weights in the isotropy submanifold $M^{\mathbb{Z}_e}$, the second, the third, and the fourth cases of Lemma 4.2.5 are possible. In the third case or the fourth case, $a \geq |e|$ and $b \geq |e|$ hence $a + b \geq 2|e| > d$, which is a contradiction. Hence this must be the second case of Lemma 4.2.5 with $a = |e|$. Moreover, $\alpha$ and $\beta$ lie in the same 2-sphere of $M^{\mathbb{Z}_e}$. Hence by Lemma 2.0.19, $\Sigma_\alpha \equiv \Sigma_\beta \mod e$.

2. By looking at the weights in the isotropy submanifold $M^{\mathbb{Z}_e}$, this must be the third case of Lemma 4.2.5 with $a = b = |e|$.

3. By looking at the weights in the isotropy submanifold $M^{\mathbb{Z}_e}$, this must be the fourth case of Lemma 4.2.5 with $a = b = |e|$.

4. By looking at the weights in the isotropy submanifold $M^{\mathbb{Z}_e}$, this must be the third case of Lemma 4.2.5 with $a = b = |e|$.

□

## 4.3 The case where the largest weight is odd

Let the circle act symplectically on a compact, connected symplectic manifold $M$ with exactly three fixed points. In this section, we show that if $\dim M \geq 8$, the largest weight cannot be odd.

**Proposition 4.3.1.** *Fix a natural number $n$ such that $n \geq 4$. Assume that Theorem 4.1.1 holds for all manifolds $M$ such that $\dim M < 2n$. Let the circle act symplectically on a $2n$-dimensional compact, connected symplectic manifold $M$ and suppose that there are exactly three fixed points. Assume that the action is effective. Then the largest weight is even.*

*Proof.* Assume on the contrary that the largest weight is odd. Then this is an immediate consequence of Lemma 4.3.2, Lemma 4.3.6, and Lemma 4.3.7 below. $\square$

**Lemma 4.3.2.** *Fix a natural number $n$ such that $n \geq 4$. Assume that Theorem 4.1.1 holds for all manifolds $M$ such that $\dim M < 2n$. Let the circle act symplectically on a $2n$-dimensional compact, connected symplectic manifold $M$ and suppose that there are exactly three fixed points $p, q$, and $r$, with $\lambda_p \leq \lambda_q \leq \lambda_r$. Assume that the action is effective and the largest weight $d$ is odd. Then after possibly reversing the circle action we may assume that $-d \in \Sigma_p$ and $d \in \Sigma_q$, and there exist even natural numbers $a$ and $b$ such that either*

1. *$\{a, c\} \subset \Sigma_p, \{-a, b\} \subset \Sigma_q$, and $\{-b, -c\} \subset \Sigma_r$; or*
2. *$\{-a, b\} \subset \Sigma_p, \{-b, -c\} \subset \Sigma_q$, and $\{a, c\} \subset \Sigma_r$,*

*where $c = a + b$. Moreover, these are the only even weights.*

*Proof.* By Lemma 4.1.3, $N_p(d) + N_q(d) + N_r(d) = 1$ and $N_p(-d) + N_q(-d) + N_r(-d) = 1$. By Lemma 4.1.4, $\lambda_p = n-2, \lambda_q = n$, and $\lambda_r = n+2$. Moreover, after possibly reversing the circle action, we may assume that $-d \in \Sigma_p$ and $d \in \Sigma_q$.

By Lemma 4.2.4, there exist even natural numbers $a$ and $b$ such that the weights at the three fixed points in the isotropy submanifold $M^{\mathbb{Z}_2}$ are $\{a, c\}, \{-a, b\}$, and $\{-b, -c\}$, where $c = a + b$. In the Lemma, the order is not specified. We have six possible cases. Other four cases are:

a. $\{a, c\} \subset \Sigma_p, \{-b, -c\} \subset \Sigma_q$, and $\{-a, b\} \subset \Sigma_r$.

b. $\{-a, b\} \subset \Sigma_p, \{a, c\} \subset \Sigma_q$, and $\{-b, -c\} \subset \Sigma_r$.

c. $\{-b, -c\} \subset \Sigma_p, \{-a, b\} \subset \Sigma_q$, and $\{a, c\} \subset \Sigma_r$.

d. $\{-b, -c\} \subset \Sigma_p, \{a, c\} \subset \Sigma_q$, and $\{-a, b\} \subset \Sigma_r$.

The fixed points $p$ and $q$ satisfy the conditions in Lemma 4.1.7. Therefore, $E_p^+ - E_p^- - E_q^+ + E_q^- = 2$ and the other cases are ruled out. $\square$

**Lemma 4.3.3.** *Under the assumption of Lemma 4.3.2, $a \neq b$.*

*Proof.* Assume on the contrary that $a = b$. Since $-d \in \Sigma_p$ and $d \in \Sigma_q$ where $d$ is the largest weight, by Lemma 4.2.6 part 1 for $d$, $\Sigma_p \equiv \Sigma_q \mod d$. As a result, we can find a bijection between the weights at $p$ and the weights at $q$ that takes each weight $\alpha$ at $p$ to a weight $\beta$ at $q$ such that $\alpha \equiv \beta \mod d$. Moreover, since $a = b$, we can take this bijection to take $a$ at $p$ to $b$ at $q$ in the first case, and we can take this bijection to take $-a$ at $p$ to $-b$ at $q$ in the second case.

Assume that the first case in Lemma 4.3.2 holds, i.e., $\{a, c\} \subset \Sigma_p, \{-a, b\} \subset \Sigma_q$, and $\{-b, -c\} \subset \Sigma_r$. Moreover, these are the only even weights. First, $-d$ at $p$ has to go to $d$ at $q$ since all the other weights are non-zero and have absolute values less than $d$. Next, $c$ at $p$ must go to $c - d$ at $q$ and $-a$ at $q$ must go to $d - a$ at $p$. If $l$ is any remaining positive odd weight at $p$, then it has to go to $l$ at $q$ since the largest weight $d$ is odd. Similarly, any negative odd weight $-k$ at $p$ must go to $-k$ at $q$.

By Corollary 2.0.11, $\frac{1}{2}\dim M$ is even. Since $\lambda_p = \frac{1}{2}\dim M - 2$ and $\lambda_q = \frac{1}{2}\dim M$ by Lemma 4.1.4, this implies that the weights at $p$ and $q$ are

$$\Sigma_p = \{-d, a, c, d-a\} \cup \{x_i\}_{i=1}^t \cup \{-y_i\}_{i=1}^t$$
$$\Sigma_q = \{d, b, c-d, -a\} \cup \{x_i\}_{i=1}^t \cup \{-y_i\}_{i=1}^t$$

for some odd natural numbers $x_i$'s and $y_i$'s where $\dim M = 8 + 4t$, for some $t \geq 0$.

Suppose that $x_i > 1$ for some $i$. Then by Lemma 4.2.6 part 2, $\{2x_i, x_i\} \subset \Sigma_p$, $\{-x_i, x_i\} \subset \Sigma_q$, and $\{-2x_i, -x_i\} \subset \Sigma_r$, or $\{-x_i, x_i\} \subset \Sigma_p$, $\{2x_i, x_i\} \subset \Sigma_q$, and $\{-2x_i, -x_i\} \subset \Sigma_r$. Moreover, no more multiples of $x_i$ should sppear as weights. This implies that $-x_i \neq -y_j$ for all $j$. Since $-y_j$'s are the only negative odd weights at $p$, this implies that the second case is impossible. Assume that the first case holds. Then we must have $c = 2x_i$. Also, since $\{-x_i, x_i\} \subset \Sigma_q$ but $-x_i \neq -y_j$ for all $j$, $-x_i = c - d$. However, this means that $2x_i - d = c - d = -x_i$ hence $d = 3x_i$, which is a contradiction since no more multiples of $x_i$ should appear.

Hence $x_i = 1$ for all $i$. Similarly, one can show that $y_i = 1$ for all $i$. Then $c_1(M)|_p = -d + a + c + d - a = c > 0$, which is a contradiction by Corollary 2.0.14 that the first Chern class map is identically zero.

Similarly, we get a contradiction of the second case of Lemma 4.3.2 with $a = b$ by a slight variation of this argument. $\square$

**Lemma 4.3.4.** *Assume that the first case in Lemma 4.3.2 holds. Then the weights are*

$$\begin{aligned} \Sigma_p &= \{-d, a, c, d-a, b-d, 1\} \cup \{-1, 1\}_{i=1}^{t} \\ \Sigma_q &= \{d, a-d, c-d, -a, b, 1\} \cup \{-1, 1\}_{i=1}^{t} \\ \Sigma_r &= \{-b, -c, \cdots\} \end{aligned}$$

*where the largest weight $d$ is odd, $a, b,$ and $c$ are even natural numbers such that $c = a + b$, and $\dim M = 12 + 4t$ for some $t \geq 0$. Moreover, $a \neq b$ and the remaining weights at $r$ are odd.*

*Proof.* Assume that the first case in Lemma 4.3.2 holds, i.e., there exist even natural numbers $a$, $b$, and $c$ such that $\{a, c\} \subset \Sigma_p, \{-a, b\} \subset \Sigma_q$, and $\{-b, -c\} \subset \Sigma_r$ where $c = a + b$. Moreover, these are the only even weights.

Since $-d \in \Sigma_p$ and $d \in \Sigma_q$ where $d$ is the largest weight, by Lemma 4.2.6 part 1 for $d$, $\Sigma_p \equiv \Sigma_q \mod d$. First $-d$ at $p$ has to go to $d$ at $q$ since all the other weights are non-zero and have absolute values less than $d$. Second, by Lemma 4.3.3, $a \neq b$. Hence $a$ at $p$ must go to $a - d$ at $q$ and $b$ at $q$ must go to $b - d$ at $p$. Next, $c$ at $p$ must go to $c - d$ at $q$ and $-a$ at $q$ must go to $d - a$ at $p$. If $l$ is any remaining positive odd weight at $p$, then it has to go to $l$ at $q$ since the largest weight $d$ is odd. Similarly, any remaining negative odd weight $-k$ at $p$ must go to $-k$ at $q$.

By Corollary 2.0.11, $\frac{1}{2}\dim M$ is even. Since $\lambda_p = \frac{1}{2}\dim M - 2$ and $\lambda_q = \frac{1}{2}\dim M$ by Lemma 4.1.4, this implies that the weights at $p$ and $q$ are

$$\begin{aligned} \Sigma_p &= \{-d, a, c, d-a, b-d\} \cup \{x_i\}_{i=1}^{t+1} \cup \{-y_i\}_{i=1}^{t} \\ \Sigma_q &= \{d, a-d, c-d, -a, b\} \cup \{x_i\}_{i=1}^{t+1} \cup \{-y_i\}_{i=1}^{t} \end{aligned}$$

for some odd natural numbers $x_i$'s and $y_i$'s where $\dim M = 12+4t$, for some $t \geq 0$. We also have

$$\Sigma_r = \{-b, -c, \cdots\}.$$

We show that $x_i = y_i = 1$ for all $i$.

1. $x_i = 1$ for all $i$.

   Suppose not. Without loss of generality, assume that $x_1 > 1$. Then by Lemma 4.2.6 part 2 for $x_1$, we have that $\{2x_1, x_1\} \subset \Sigma_p$, $\{-x_1, x_1\} \subset \Sigma_q$, and $\{-2x_i, -x_i\} \subset \Sigma_r$, or $\{-x_1, x_1\} \subset \Sigma_p$, $\{2x_1, x_1\} \subset \Sigma_q$, and $\{-2x_i, -x_i\} \subset \Sigma_r$. Moreover, no more multiples of $x_1$ should sppear as weights. This implies that $-x_1 \neq -y_j$ for all $j$. If the first case holds, we must have that $c = 2x_1$. Also, there must be a weight at $q$ that is equal to $-x_1$. If $-x_1 = a-d$, $\{2x_1, x_1, x_1\} = \{c, -a+d, x_1\} \subset \Sigma_p$, which is not possible by Lemma 4.2.6. If $-x_1 = c-d$, $-x_1 = c-d = 2x_1 - d$ implies that $d = 3x_1$, which contradicts that no more multiples of $x_1$ should appear as weights. If the second case holds, we must have that $b = 2x_1$. Also, there must be a weight at $p$ that is equal to $-x_1$. Since $-x_1 \neq -d$, $-x_1 = b-d$. However, $-x_1 = b-d = 2x_1 - d$ implies that $d = 3x_1$, which contradicts that no more multiples of $x_1$ should appear as weights.

2. $y_i = 1$ for all $i$.

   Suppose not. Without loss of generality, assume that $y_1 > 1$. Then by Lemma 4.2.6 part 2 for $y_1$, we must have that $\{2y_1, y_1\} \subset \Sigma_r$, which is a contradiction since $r$ has no positive even weight.

□

**Lemma 4.3.5.** *Assume that the second case in Lemma 4.3.2 holds. Then the weights are*

$$\Sigma_p = \{-d, -a, b, d-b, d-c, 1\} \cup \{-1, 1\}_{i=1}^{t}$$
$$\Sigma_q = \{d, -b, -c, d-a, b-d, 1\} \cup \{-1, 1\}_{i=1}^{t}$$
$$\Sigma_r = \{a, c, \cdots\},$$

*where the largest weight $d$ is odd, $a, b,$ and $c$ are even natural numbers such that $c = a + b$, and $\dim M = 12 + 4t$ for some $t \geq 0$. Moreover, $a \neq b$ and the remaining weights at $r$ are odd.*

*Proof.* Assume that the second case in Lemma 4.3.2 holds, i.e., there exist even natural numbers $a$, $b$, and $c$ such that $\{-a, b\} \subset \Sigma_p, \{-b, -c\} \subset \Sigma_q$, and $\{a, c\} \subset \Sigma_r$ where $c = a + b$. Moreover, these are the only even weights.

Since $-d \in \Sigma_p$ and $d \in \Sigma_q$ where $d$ is the largest weight, by Lemma 4.2.6 part 1 for $d$, $\Sigma_p \equiv \Sigma_q \mod d$. First $-d$ at $p$ has to go to $d$ at $q$ since all the other weights are non-zero and have absolute values less than $d$. Second, by Lemma 4.3.3, $a \neq b$. Hence $-a$ at $p$ must go to $d - a$ at $q$ and $-b$ at $q$ must go to $d - b$ at $p$. Next, $c$ at $p$ must go to $c - d$ at $q$ and $-a$ at $q$ must go to $d - a$ at $p$. If $l$ is any remaining positive odd weight at $p$, then it has to go to $l$ at $q$ since the largest weight $d$ is odd. Similarly, any negative odd weight $-k$ at $p$ must go to $-k$ at $q$.

By Corollary 2.0.11, $\frac{1}{2}\dim M$ is even. Since $\lambda_p = \frac{1}{2}\dim M - 2$ and $\lambda_q = \frac{1}{2}\dim M$ by Lemma 4.1.4, this implies that the weights at $p$ and $q$ are

$$\begin{aligned}\Sigma_p &= \{-d, -a, b, d-b, d-c\} \cup \{x_i\}_{i=1}^{t+1} \cup \{-y_i\}_{i=1}^{t}\\ \Sigma_q &= \{d, -b, -c, d-a, b-d\} \cup \{x_i\}_{i=1}^{t+1} \cup \{-y_i\}_{i=1}^{t}\end{aligned}$$

for some odd natural numbers $x_i$'s and $y_i$'s where $\dim M = 12 + 4t$, for some $t \geq 0$. We also have

$$\Sigma_r = \{a, c, \cdots\}.$$

We show that $x_i = y_i = 1$ for all $i$.

1. $x_i = 1$ for all $i$.

   Suppose not. Without loss of generality, assume that $x_1 > 1$. Then by Lemma 4.2.6 part 2 for $x_1$, we must have that $\{-2x_1, -x_1\} \subset \Sigma_r$, which is a contradiction since $r$ has no negative even weight.

2. $y_i = 1$, for all $i$.

   Suppose not. Without loss of generality, assume that $y_1 > 1$. Then by Lemma 4.2.6 part 2 for $y_1$, $\{-2y_1, -y_1\} \subset \Sigma_p$, $\{-y_1, y_1\} \subset \Sigma_q$, and $\{2y_1, y_1\} \subset \Sigma_r$, or $\{-y_1, y_1\} \subset \Sigma_p$, $\{-2y_1, -y_1\} \subset \Sigma_q$, and $\{2y_1, y_1\} \subset \Sigma_r$. Moreover, no more multiples of $y_1$ should appear as weights.

If the first case holds, we must have that $a = 2y_1$. Also, there must be a weight at $q$ that is equal to $y_1$. Thus, we have that $d - a = y_1$. However, this implies that $d - a = d - 2y_1 = y_1$ hence $d = 3y_1$, which is a contradiction since no more multiples of $y_1$ should appear as weights.

Suppose that the second case holds. Then we must have that $c = 2y_1$. Also, there must be a weight at $p$ that is equal to $y_1$. Hence $y_1 = d - b$. Then $\{-2y_1, -y_1, -y_1\} = \{-c, b - d, -y_2\} \subset \Sigma_q$, which is a contradiction.

□

**Lemma 4.3.6.** *The first case in Lemma 4.3.2 is not possible.*

*Proof.* By Lemma 4.3.4, the weights are

$$\Sigma_p = \{-d, a, c, d - a, b - d, 1\} \cup \{-1, 1\}_{i=1}^t$$
$$\Sigma_q = \{d, a - d, c - d, -a, b, 1\} \cup \{-1, 1\}_{i=1}^t$$
$$\Sigma_r = \{-b, -c, \cdots\},$$

where the largest weight $d$ is odd, $a, b$, and $c$ are even natural numbers such that $c = a + b$, and $\dim M = 12 + 4t$ for some $t \geq 0$. Moreover, $a \neq b$ and the remaining weights at $r$ are odd.

We consider Lemma 2.0.12 for each integer. Lemma 2.0.12 holds for $d$, $a$, $b$, and $c$. Since $d > c > b$, $b - d < -1$. Since $a \neq b$, by Lemma 2.0.12 for $b - d$, it is straightforward to show that $d - b \in \Sigma_r$.

First, suppose that $c - d \neq -1$. Then $N_p(1) = N_p(-1) + 1$ and $N_q(1) = N_q(-1) + 1$. Hence by Lemma 2.0.12 for 1, $N_r(1) + 2 = N_r(-1)$. Considering Lemma 2.0.12 for each integer, one can show that the weights are

$$\Sigma_p = \{-d, a, c, d - a, b - d, 1\} \cup \{-1, 1\}_{i=1}^t$$
$$\Sigma_q = \{d, a - d, c - d, -a, b, 1\} \cup \{-1, 1\}_{i=1}^t$$
$$\Sigma_r = \{-b, -c, d - b, -1, d - c, -1\} \cup \{-e_i, e_i\}_{i=1}^t$$

for some odd natural numbers $e_i$'s. We show that $e_i = 1$ for all $i$.

Suppose that $e_1 > 1$. Then by Lemma 4.2.6 part 4, either $\{-2e_1, -e_1\} \subset \Sigma_p$, $\{2e_1, e_1\} \subset \Sigma_q$, and $\{-e_1, e_1\} \subset \Sigma_r$, or $\{2e_1, e_1\} \subset \Sigma_p$, $\{-2e_1, -e_1\} \subset \Sigma_q$, and $\{-e_1, e_1\} \subset \Sigma_r$. However, since $p$ has no negative even weight, the first case is impossible. If the second case holds, we must have that

$-a = -2e_1$. Moreover, we must have that $a - d = -e_1$ or $c - d = -e_1$. If $a - d = -e_1$, $2e_1 - d = a - d = -e_1$ hence $3e_1 = d$, which is a contradiction since no additional multiples of $e_1$ should appear. Next, if $c - d = -e_1$, $\{d - c, -e_1, e_1\} = \{-e_1, -e_1, e_1\} \subset \Sigma_r$, which is also a contradiction. Hence $e_i = 1$, for all $i$. Then the weights are

$$\begin{aligned}
\Sigma_p &= \{-d, a, c, d-a, b-d, 1\} \cup \{-1, 1\}^t \\
\Sigma_q &= \{d, -a, b, a-d, -1, 1\} \cup \{-1, 1\}^t \\
\Sigma_r &= \{-b, -c, d-b, -1, 1, -1\} \cup \{-1, 1\}^t.
\end{aligned}$$

Second, suppose that $c - d = -1$. Then $N_p(1) = N_p(-1) + 1$ and $N_q(1) = N_q(-1)$. Hence by Lemma 2.0.12 for 1, $N_r(1) + 1 = N_r(-1)$. Considering Lemma 2.0.12 for each integer, one can show that the weights are

$$\begin{aligned}
\Sigma_p &= \{-d, a, c, d-a, b-d, 1\} \cup \{-1, 1\}_{i=1}^t \\
\Sigma_q &= \{d, a-d, -1, -a, b, 1\} \cup \{-1, 1\}_{i=1}^t \\
\Sigma_r &= \{-b, -c, d-b, -1\} \cup \{-e_i, e_i\}_{i=1}^{t+1}
\end{aligned}$$

for some odd natural numbers $e_i$'s. As above, $e_i = 1$ for all $i$.

Hence in either case the weights are

$$\begin{aligned}
\Sigma_p &= \{-d, a, c, d-a, b-d, 1\} \cup \{-1, 1\}^t \\
\Sigma_q &= \{d, -a, b, a-d, c-d, 1\} \cup \{-1, 1\}^t \\
\Sigma_r &= \{-b, -c, d-b, -1, d-c, -1\} \cup \{-1, 1\}^t.
\end{aligned}$$

Moreover, since $c_1(M)|_p = 0$ by Corollary 2.0.14, we have that $-d + a + c + d - a + b - d + 1 = 0$. Therefore, $d = c + b + 1$. Let $A = c_n(M)|_p = \prod \xi_p^j$, $B = c_n(M)|_q = \prod \xi_q^j$, and $C = c_n(M)|_r = \prod \xi_r^j$. Then

$$\begin{aligned}
(-1)^{t+1}(B + C) &= dab(d-a)(d-c) - bc(d-b)(d-c) \\
&= b(d-c)\{da(d-a) - c(d-b)\} \\
&= b(d-c)\{(c+b+1)a(c+b+1-a) - c(c+b+1-b)\} \\
&= b(d-c)\{(c+b+1)a(2b+1) - c(c+1)\} \\
&= b(d-c)\{(a+2b+1)a(2b+1) - (a+b)(a+b+1)\} \\
&= b(d-c)\{(a^2+2ab+a)(2b+1) - (a^2+2ab+b^2+a+b)\} \\
&= b(d-c)\{2a^2b + 4ab^2 + 2ab + a^2 + 2ab + a - (a^2+2ab+b^2+a+b)\} \\
&= b(d-c)\{(2a^2b - a^2) + (4ab^2 - b^2) + (2ab - 2ab) + (a^2 - a) + (2ab - b) + a\} > 0.
\end{aligned}$$

Hence $(-1)^{t+1}B > -(-1)^{t+1}C > 0$, i.e., $(-1)^{t+1}\left(\frac{1}{B} + \frac{1}{C}\right) < 0$. We also have that $(-1)^{t+1}\frac{1}{A} < 0$. Then, by Theorem 2.0.9,

$$0 = (-1)^{t+1} \int_M 1 = (-1)^{t+1} \left( \frac{1}{A} + \frac{1}{B} + \frac{1}{C} \right) < 0,$$

which is a contradiction. $\square$

**Lemma 4.3.7.** *The second case in Lemma 4.3.2 is not possible.*

*Proof.* By Lemma 4.3.5, the weights in this case are

$$\begin{aligned} \Sigma_p &= \{-d, -a, b, d-b, d-c, 1\} \cup \{-1, 1\}_{i=1}^t \\ \Sigma_q &= \{d, -b, -c, d-a, b-d, 1\} \cup \{-1, 1\}_{i=1}^t \\ \Sigma_r &= \{a, c, \cdots\}, \end{aligned}$$

where the largest weight $d$ is odd, $a, b$, and $c$ are even natural numbers such that $c = a + b$, and $\dim M = 12 + 4t$ for some $t \geq 0$.

Let $A = c_n(M)|_p = \prod \xi_p^j$, $B = c_n(M)|_q = \prod \xi_q^j$, and $C = c_n(M)|_r = \prod \xi_r^j$. Then

$$\begin{aligned} (-1)^{t+1}(B + A) &= dbc(d-a)(d-b) - dab(d-b)(d-c) \\ &= db(d-b)\{c(d-a) - a(d-c)\} > 0, \end{aligned}$$

since $c > a$ and $d - a > d - c$. Hence it follows that $(-1)^{t+1} \left( \frac{1}{A} + \frac{1}{B} \right) < 0$. Since $\lambda_r = \frac{1}{2} \dim M + 2$, we also have that $(-1)^{t+1} \frac{1}{C} < 0$. Then, by Theorem 2.0.9,

$$0 = (-1)^{t+1} \int_M 1 = (-1)^{t+1} \left( \frac{1}{A} + \frac{1}{B} + \frac{1}{C} \right) < 0,$$

which is a contradiction. $\square$

## 4.4 Preliminaries for the largest weight even case: part 1

Let the circle act symplectically on a compact, connected symplectic manifold $M$ with exactly three fixed points. Also assume that $\dim M \geq 8$ and the largest weight is even. The main idea to prove Theorem 4.1.1 is to rule out manfiolds such that $\dim M \geq 8$. In this section, we investigate properties that the manifold $M$ should satisfy, if it exists.

**Proposition 4.4.1.** *Fix a natural number $n$ such that $n \geq 4$. Assume that Theorem 4.1.1 holds for all manifolds $M$ such that $\dim M < 2n$. Let the circle act symplectically on a $2n$-dimensional compact, connected symplectic manifold $M$ and suppose that there are exactly three fixed points. Assume that the action is effective. Then the largest weight is odd.*

*Proof.* Assume on the contrary that the largest weight is even. Then this is an immediate consequence of Lemma 4.4.2, Lemma 4.5.1, Lemma 4.6.1, Lemma 4.6.2, Lemma 4.6.3, Lemma 4.6.4, Lemma 4.6.5, Lemma 4.6.6, Lemma 4.6.7, Lemma 4.6.8, Lemma 4.6.9, Lemma 4.6.10, and Lemma 4.6.11 below. ☐

**Lemma 4.4.2.** *Fix a natural number $n$ such that $n \geq 4$. Assume that Theorem 4.1.1 holds for all manifolds $M$ such that $\dim M < 2n$. Let the circle act symplectically on a $2n$-dimensional compact, connected symplectic manifold $M$ and suppose that there are exactly three fixed points $p, q$, and $r$, with $\lambda_p \leq \lambda_q \leq \lambda_r$. Assume that the action is effective and the largest weight $c$ is even. Then after possibly reversing the circle action we may assume that the weights are*

$$\Sigma_p = \{-c, -b\} \cup \{x_i\}_{i=1}^{t+3} \cup \{-y_i\}_{i=1}^{t} \cup \{1\} \cup \{-1, 1\}_{i=1}^{s}$$
$$\Sigma_q = \{c, a\} \cup \{x_i - c\}_{i=1}^{t+3} \cup \{c - y_i\}_{i=1}^{t} \cup \{1\} \cup \{-1, 1\}_{i=1}^{s}$$
$$\Sigma_r = \{-a, b, \cdots\}$$

*for some $s \geq 0$ and $t \geq 0$ such that $\dim M = 2n = 12 + 4t + 4s$, where $a$ and $b$ are even natural numbers such that $c = a + b$, and $x_i$'s and $y_i$'s are odd natural numbers for all $i$. Moreover, the remaining weights at $r$ are odd.*

*Proof.* Let $c$ be the largest weight. By Lemma 4.1.3, $N_p(c)+N_q(c)+N_r(c) = 1$ and $N_p(-c) + N_q(-c) + N_r(-c) = 1$. By Lemma 4.1.4, $\lambda_p = n - 2, \lambda_q = n$, and $\lambda_r = n + 2$. Moreover, after possibly reversing the circle action, we may assume that $-c \in \Sigma_p$ and $c \in \Sigma_q$.

By Lemma 4.2.4, there exist even natural numbers $a$ and $b$ such that the weights at the three fixed points in the isotropy submanifold $M^{\mathbb{Z}_2}$ are $\{a, d\}, \{-a, b\}$, and $\{-b, -d\}$, where $d = a + b$. In Lemma 4.2.4, the order is not specified. However, since we can assume without loss of generality that $-c \in \Sigma_p$ and $c \in \Sigma_q$, we can assume that $d = c$, hence $\{-c, -b\} \subset \Sigma_p$, $\{c, a\} \subset \Sigma_q$, and $\{-a, b\} \subset \Sigma_r$. Moreover, these are the only even weights.

Next, by Lemma 4.2.6 part 1 for $c$, $\Sigma_p \equiv \Sigma_q \mod c$. As a result, we can find a bijection between weights at $p$ and weights at $q$ that takes each weight $\alpha$ at $p$ to a weight $\beta$ at $q$ such that $\alpha \equiv \beta \mod c$.

First, $-c$ at $p$ has to go to $c$ at $q$ since all the other weights are non-zero and have absolute values less than $c$. Second, $-b$ at $p$ must go to $a$ at $q$. Next, if $l$ is any positive odd weight at $p$, then it either goes to $l$ or $l-c$ at $q$. Suppose that there are $t_0$ positive odd weights at $p$ that go to negative odd weights at $q$. Then since $\lambda_p = n-2$ and there is no positive even weight at $p$, there are $\frac{n}{2}+1-t_0$ positive odd weights $p$ that go to positive odd weights at $q$. Similarly, if $-k$ is any negative odd weight at $p$, either it has to go to $-k$ or $c-k$ at $q$. Suppose that there are $t_1$ negative odd weights at $p$ that go to positive odd weights at $q$. On the other hand, since $\lambda_q = n$ and $q$ has two positive even weights, the number of positive odd weights at $q$ that go to positive odd weights at $p$ is equal to $\frac{n}{2}-2-t_1$. Hence $\frac{n}{2}+1-t_0 = \frac{n}{2}-2-t_1$, i.e., $t_0 = t_1+3$. Let $t = t_1$ and $s = \frac{n}{2}-t-3$. By Corollary 2.0.11, $\frac{1}{2}\dim M$ is even. This implies that the weights are

$$\Sigma_p = \{-c,-b\} \cup \{x_i\}_{i=1}^{t+3} \cup \{-y_i\}_{i=1}^{t} \cup \{e_i\}_{i=1}^{s+1} \cup \{-f_i\}_{i=1}^{s}$$
$$\Sigma_q = \{c,a\} \cup \{x_i - c\}_{i=1}^{t+3} \cup \{c-y_i\}_{i=1}^{t} \cup \{e_i\}_{i=1}^{s+1} \cup \{-f_i\}_{i=1}^{s}$$
$$\Sigma_r = \{-a, b, \cdots\}$$

for some odd natural numbers $x_i$'s, $y_i$'s, $e_i$'s, and $f_i$'s, where $\dim M = 2n = 12+4t+4s$, for some $t \geq 0$ and $s \geq 0$.

Next, we show that $e_i = f_i = 1$ for all $i$.

1. $e_i = 1$ for all $i$.

   Assume on the contrary that $e_i > 1$ for some $i$. Denote $e = e_i$. Then by Lemma 4.2.6 part 3 for $e$, either $\{2e, e\} \subset \Sigma_p$, $\{-e, e\} \subset \Sigma_q$, and $\{-2e, -e\} \subset \Sigma_r$, or $\{-e, e\} \subset \Sigma_p$, $\{2e, e\} \subset \Sigma_q$, and $\{-2e, -e\} \subset \Sigma_r$. Moreover, no additional multiples of $e$ should appear as weights. Since $\{-2e, -e\} \subset \Sigma_r$ in either case, the only possibility is that $a = 2e$. Therefore, the latter is the case. Thus, we have that $\{-e, e\} \subset \Sigma_p$. Therefore, $-e = -y_i$ for some $i$ or $-e = -f_i$ for some $i$. Since no additional multiples of $e$ should appear as weights at $q$, $-f_i \neq -e$ for all $i$. Hence, $-y_i = -e$ for some $i$. Without loss of generality, let $y_1 = e$. Moreover, since no additional multiples of $e$ should appear as weights, $b \neq 2e$. In particular, $a = 2e \neq b$. Since $2e = a < a + b = c$, $e < \frac{c}{2}$. Thus $c - e > \frac{c}{2}$.

Next, we show that $e - c \in \Sigma_r$. We have that $c - e = c - y_1 \in \Sigma_q$. By Lemma 2.0.12 for $c - e$, either $e - c \in \Sigma_p$, $e - c \in \Sigma_q$, or $e - c \in \Sigma_r$.

(a) $e - c \notin \Sigma_p$.

Suppose that $e - c \in \Sigma_p$. Since $e - c$ is odd, either $e - c = -y_i$ for some $i$ or $e - c = -f_i$ for some $i$.

First, assume that $e - c = -y_i$ for some $i$. If $e - c = -y_1$, this implies that $e - c = -e$ hence $c = 2e$, which is a contradiction. Hence if $e - c = -y_i$ for some $i$, $i \neq 1$. Without loss of generality, let $e-c = -y_2$. Then we have that $\{-y_1, -y_2\} = \{e-c, e-c\} \subset \Sigma_p$. Then by Lemma 4.2.6 part 3 for $e - c$, $2(c - e) \in \Sigma_p$, which is a contradiction since $2(c - e) > c$.

Second, assume that $e - c = -f_i$ for some $i$. Then we have that $e-c = -f_i \in \Sigma_p$ and $e-c = -f_i \in \Sigma_q$. Then by Lemma 4.2.6 part 2 for $c-e$, $2(c-e) \in \Sigma_r$, which is a contradiction since $2(c-e) > c$. Hence $e - c \neq \Sigma_p$.

(b) $e - c \notin \Sigma_q$.

Suppose that $e - c \in \Sigma_q$. Then we have that $\{c - e, e - c\} = \{c - y_1, e - c\} \subset \Sigma_q$. Hence by Lemma 4.2.6 part 4 for $c - e$, either $2(c - e) \in \Sigma_p$ or $2(c - e) \in \Sigma_r$. However, $2(c - e) > c$, which is a contradiction.

Therefore $e - c \in \Sigma_r$. Then by Lemma 4.2.6 part 1 for $c - e$, $\Sigma_q \equiv \Sigma_r \mod c - e$. Consider $\{c, e\} \subset \Sigma_q$. We have that $c \notin \Sigma_r$ and $e \notin \Sigma_r$. Also, $e - (c - e) = 2e - c = a - a - b = -b$, but $-b \notin \Sigma_r$ since $-b$ is a negative even integer and $-a$ is the only negative even weight in $\Sigma_r$, but $a \neq b$. Since $|e + k(c - e)| > c$ for $k < -2$ or $k > 1$, $\Sigma_q \equiv \Sigma_r \mod c-e$ and $\{c, e\} \subset \Sigma_q$ imply that $N_r(e-2(c-e)) = N_r(3e-2c) = 2$. Then by Lemma 4.2.6 part 3 for $3e - 2c$, $2(2c - 3e) \in \Sigma_r$, which is a contradiction since $2(2c-3e) = 4c-6e = c+3c-6e = c+3a+3b-6e = c + 6e + 3b - 6e = c + 3b > c$, where $c$ is the largest weight. Therefore, $e_i = 1$ for all $i$.

2. $f_i = 1$ for all $i$.

Assume on the contrary that $f_i > 1$ for some $i$. Denote $f = f_i$. Then by Lemma 4.2.6 part 3 for $f$, either $\{-2f, -f\} \subset \Sigma_p$, $\{-f, f\} \subset \Sigma_q$, and $\{2f, f\} \subset \Sigma_r$, or $\{-f, f\} \subset \Sigma_p$, $\{-2f, -f\} \subset \Sigma_q$, and $\{2f, f\} \subset \Sigma_r$. Moreover, no additional multiples of $f$ should appear as weights. Since $\{2f, f\} \subset \Sigma_r$ in either case, the only possibility is that $b = 2f$. Therefore, the former is the case. Thus, we have that $\{-f, f\} \subset \Sigma_q$. Therefore, $f = c - y_i$ for some $i$. Without loss of generality, let $c - y_1 = f$. Moreover, since no additional multiples of $f$ should appear as weights, $a \neq 2f$. In particular, $b = 2f \neq a$. Since $2f = b < a + b = c$, $f < \frac{c}{2}$. Thus $c - f > \frac{c}{2}$.

Next, we show that $c - f \in \Sigma_r$. We have that $f - c = -y_1 \in \Sigma_p$. By Lemma 2.0.12 for $c - f$, either $c - f \in \Sigma_p$, $c - f \in \Sigma_q$, or $c - f \in \Sigma_r$.

(a) $c - f \notin \Sigma_p$.

Suppose that $c - f \in \Sigma_p$. Then we have that $\{c - f, f - c\} = \{c - f, -y_1\} \subset \Sigma_p$. Then by Lemma 2.0.12 part 4 for $c - f$, either $2(c - f) \in \Sigma_q$ or $2(c - f) \in \Sigma_r$, which is a contradiction since $2(c - f) > c$.

(b) $c - f \notin \Sigma_q$.

Suppose that $c - f \in \Sigma_q$. Then $c - f = c - y_i$ for some $i$. If $c - f = c - y_1$, then $c - f = c - y_1 = f$ hence $c = 2f < c$, which is a contradiction. Next, suppose that $c - f = c - y_i$ for some $i \neq 1$. Then we have that $\{f - c, f - c\} = \{-y_1, -y_i\} \subset \Sigma_p$. Hence, by Lemma 4.2.6 part 3 for $f - c$, $2(c - f) \in \Sigma_p$, which is a contradiction since $2(c - f) > c$.

Therefore $c - f \in \Sigma_r$. We also have that $f - c = -y_1 \in \Sigma_p$. Then by Lemma 4.2.6 part 1 for $c - f$, $\Sigma_p \equiv \Sigma_r \mod c - f$. Consider $\{-c, -f\} \subset \Sigma_p$. We have that $-c \notin \Sigma_r$ and $-f \notin \Sigma_r$. Also, $-f + (c - f) = c - 2f = a + b - 2f = a + 2f - 2f = a$, but $a \notin \Sigma_r$ since $a$ is a positive even integer and $b$ is the only positive even weight in $\Sigma_r$, but $a \neq b$. Since $|-f + k(c - f)| > c$ for $k < -1$ or $k > 2$, $\Sigma_p \equiv \Sigma_r \mod c - f$ and $\{-c, -f\} \subset \Sigma_p$ imply that $N_r(-f + 2(c - f)) = N_r(2c - 3f) = 2$. Then by Lemma 4.2.6 part 3 for $2c - 3f$, $-2(2c - 3f) \in \Sigma_r$, which is a contradiction since $-2(2c - 3f) = -4c + 6f = -c - 3c + 6f =$

$-c-3a-3b+6f=-c-3a-6f+6f=-c-3a<-c$, where $-c$ is the smallest weight. Therefore, $f_i=1$ for all $i$.

□

**Lemma 4.4.3.** *In Lemma 4.4.2, $x_i \neq c-y_j$, for all $i$ and $j$.*

*Proof.* Suppose not. Without loss of generality, assume that $x_1=c-y_1$. Then either $x_1>\frac{c}{2}$, $x_1-c<-\frac{c}{2}$, or $x_1=y_1=\frac{c}{2}$. If $x_1>\frac{c}{2}$, $x_1 \in \Sigma_p$ and $x_1=c-y_1 \in \Sigma_q$. Hence by Lemma 4.2.6 part 2 for $x_1$, $-2x_1 \in \Sigma_r$, which is a contradiction since $-2x_1<-c$ where $-c$ is the smallest weight. Next, assume that $x_1-c<-\frac{c}{2}$. Then $x_1-c=-y_1 \in \Sigma_p$ and $x_1-c \in \Sigma_q$. Again by Lemma 4.2.6 part 2 for $x_1-c$, $2(c-x_1) \in \Sigma_r$, which is a contradiction since $2(c-x_1)>c$ where $c$ is the largest weight. If $x_1=y_1=\frac{c}{2}$, $\{-2x_1,x_1,-x_1\}=\{-c,x_1,-y_1\} \subset \Sigma_p$, which is a contradiction by Lemma 4.2.6 part 4 for $x_1$. □

**Lemma 4.4.4.** *In Lemma 4.4.2, if $x_i=c-x_j$ for $i \neq j$, then $2x_i=2x_j=c$. Also, if $x_i=c-x_i$ for some $i$, then $2x_i=2x_j=c$ for some $j \neq i$. Moreover, there could be at most one such pair $(x_i,x_j)$ for $i \neq j$ such that $x_i=c-x_j$.*

*Proof.* First, suppose that $x_i=c-x_i$ for some $i$. Then $c=2x_1$. Thus, we have that $\{-2x_i,x_i\}=\{-c,x_i\} \subset \Sigma_p$ and $\{2x_i,-x_i\}=\{c,x_i-c\} \subset \Sigma_q$. By looking at the isotropy submanifold $M^{\mathbb{Z}_{x_i}}$, this must be the fourth case of Lemma 4.2.5. Hence, $\{-2x_i,x_i,x_i\} \subset \Sigma_p$ and $\{2x_i,-x_i,-x_i\} \subset \Sigma_q$. This implies that $x_i=x_j$ for some $j \neq i$.

Next, suppose that $x_i=c-x_j$ and $x_i \neq x_j$ for some $i \neq j$. Without loss of generality, let $x_1=c-x_2$ and $x_1 \neq x_2$. We can also assume that $x_1>x_2$. Then $x_1>\frac{c}{2}>x_2$. Since $x_1 \in \Sigma_p$ and $-x_1=x_2-c \in \Sigma_q$, by Lemma 4.2.6 part 1 for $x_1$, $\Sigma_p \equiv \Sigma_q \mod x_1$.

First, we can choose a bijection between $\Sigma_p$ and $\Sigma_q$ so that

$$\Sigma_p \supset \{1\} \cup \{-1,1\}^s \equiv \{1\} \cup \{-1,1\}^s \subset \Sigma_q \mod x_1.$$

Also, since $x_1+x_2=c$, we can also choose so that

$$\begin{aligned}\Sigma_p \supset\ &\{-c,x_1,x_2\}=\{-x_1-x_2,x_1,x_2\}\\ \equiv &\{-x_2,-x_1,x_1+x_2\}=\{x_1-c,x_2-c,c\} \subset \Sigma_q \mod x_1.\end{aligned}$$

We separate into two cases:

1. $t = 0$.

   In this case, we are left with

   $$\Sigma_p \supset \{-b, x_3\} \equiv \{a, x_3 - c = x_3 - x_1 - x_2\} \subset \Sigma_q \mod x_1.$$

   If $x_3 \equiv x_3 - x_1 - x_2 \mod x_1$, we have that $x_1 | x_2$, which is a contradiction since $x_1 > x_2$. Hence $x_3 \equiv a \mod x_1$. By Corollary 2.0.14, $c_1(M)|_p = -c - b + x_1 + x_2 + x_3 + 1 = -b + x_3 + 1 = 0$, hence $x_3 + 1 = b$. Then, since $a = c - b = x_1 + x_2 - x_3 - 1$, we have that $x_3 \equiv x_1 + x_2 - x_3 - 1 = a \mod x_1$, hence $2x_3 + 1 \equiv x_2 \mod x_1$. Since $2x_3 + 1$, $x_2$, and $x_1$ are odd, and $2x_1 > c$ where $c$ is the largest weight, this implies that $2x_3 + 1 = x_2$. Then we have that $a > x_1 > \frac{c}{2} > x_2 > b > x_3$. Therefore,

   $$\frac{-(-1)^s(B + A)}{x_1 x_2 (x_1 + x_2)} = a(x_1 + x_2 - x_3) - bx_3 > 0,$$

   hence $(-1)^{s+1}B > (-1)^s A$. Also, since $\lambda_r = \frac{1}{2}\dim M + 2$ by Lemma 4.1.4, $(-1)^s C > 0$. Then, by Theorem 2.0.9,

   $$0 = \int_M 1 = (-1)^s \left( \frac{1}{A} + \frac{1}{B} + \frac{1}{C} \right) > 0,$$

   which is a contradiction.

2. $t > 0$.

   In this case, we are left with

   $$\{-b\} \cup \{x_i\}_{i=3}^{t+3} \cup \{-y_i\}_{i=1}^{t} \equiv \{a\} \cup \{x_i - c\}_{i=3}^{t+3} \cup \{c - y_i\}_{i=1}^{t} \mod x_1.$$

   Without loss of generality, let $x_3 \le x_4 \le \cdots \le x_{t+3}$ and $-y_1 \le -y_2 \le \cdots \le -y_t$. Hence,

   $$\begin{gathered} \{-b, -y_1 \le \cdots \le -y_t < 0 < x_3 \le \cdots \le x_{t+3}\} \\ \equiv \{x_3 - c \le \cdots \le x_{t+3} - c < 0 < c - y_1 \le \cdots \le c - y_t, a\} \mod x_1 \end{gathered}$$

Recall that $x_1$ is odd and $2x_1 > c$ where $c$ is the largest weight.

Consider $x_3 \in \Sigma_p$. If $x_3 \equiv c - y_i \mod x_1$ for some $i$, then $x_3 = c - y_i$, which contradicts Lemma 4.4.3. If $x_3 \equiv x_i - c \mod x_1$ for some $i \neq 1$ and 2, we have that $x_3 - 2x_1 = x_i - c$, which is a contradiction since $x_3 - 2x_1 < x_i - c$ for $i \neq 1$ and 2. Hence, $x_3 \equiv a \mod x_1$. Then we can also choose so that $-b = a - c \equiv x_3 - c \mod x_1$. Then we are left with

$$\begin{aligned} &\{-y_1 \leq \cdots \leq -y_t < 0 < x_4 \leq \cdots \leq x_{t+3}\} \\ \equiv &\{x_4 - c \leq \cdots \leq x_{t+3} - c < 0 < c - y_1 \leq \cdots \leq c - y_t\} \mod x_1 \end{aligned}$$

Next, consider $-y_t \in \Sigma_p$. If $-y_t \equiv x_i - c \mod x_1$ for some $i \neq 1, 2,$ and 3, then $-y_t = x_i - c$, which is a contradiction by Lemma 4.4.3. If $-y_t \equiv c - y_i \mod x_1$ for some $i$, then $2x_1 - y_t = c - y_i$, which is a contradiction since $2x_1 - y_t > c - y_i$ for all $i$. Then $-y_t \in \Sigma_r$ is congruent to no element in $\Sigma_q$ modulo $x_1$, which is a contradiction.

Finally, without loss of generality, assume that $x_1 = c - x_2$ and $x_3 = c - x_i$ for some $i$. Then $2x_1 = 2x_2 = c$ and $2x_3 = 2x_i = c$ for some $i$. Then we have $\{-2x_1, x_1, x_1, x_1\} = \{-c, x_1, x_2, x_3\} \subset \Sigma_p$, which is a contradiction by Lemma 4.2.6 part 3 for $x_1$. □

**Lemma 4.4.5.** *In Lemma 4.4.2, $y_i \neq c - y_j$, if $i \neq j$.*

*Proof.* Suppose not. First, assume that $y_i = c - y_j$ and $y_i = y_j$ for some $i \neq j$ , i.e., $2y_i = 2y_j = c$. Then $\{-2y_i, -y_i, -y_i\} = \{-c, -y_i, -y_j\} \subset \Sigma_p$, which contradicts Lemma 4.2.6 part 3 for $y_i$.

Second, assume that $y_i = c - y_j$ and $y_i \neq y_j$ for some $i \neq j$. Without loss of generality assume that $y_1 = c - y_2$ and $y_1 \neq y_2$. We can also assume that $y_1 > \frac{c}{2} > y_2$. Then we have that $-y_1 \in \Sigma_p$ and $c - y_2 = y_1 \in \Sigma_q$. Hence by Lemma 4.2.6 part 1 for $y_1$, $\Sigma_p \equiv \Sigma_q \mod y_1$.

First, we can choose a bijection between $\Sigma_p$ and $\Sigma_q$ so that

$$\Sigma_p \supset \{-y_1, 1\} \cup \{-1, 1\}^s \equiv \{c - y_2 = y_1, 1\} \cup \{-1, 1\}^s \subset \Sigma_q \mod y_1.$$

Then we are left with

$$\begin{aligned} &\Sigma_p \supset \{-c, -b\} \cup \{x_i\}_{i=1}^{t+3} \cup \{-y_i\}_{i=2}^{t} \\ \equiv &\{c, a\} \cup \{x_i - c\}_{i=1}^{t+3} \cup \{c - y_1 = y_2\} \cup \{c - y_i\}_{i=3}^{t} \subset \Sigma_q \mod y_1. \end{aligned}$$

Without loss of generality, let $x_1 \leq x_2 \leq \cdots \leq x_{t+3}$ and $-y_3 \leq -y_4 \leq \cdots \leq -y_t$, i.e.,

$$\{-c, -b, -y_2, -y_3 \leq -y_4 \leq \cdots \leq -y_t < 0 < x_1 \leq x_2 \leq \cdots \leq x_{t+3}\}$$
$$\equiv \{x_1 - c \leq \cdots \leq x_{t+3} - c < 0 < c - y_3 \leq \cdots \leq c - y_t, c, a, c - y_2\} \mod x_1.$$

Recall that $y_1$ is odd and $2y_1 > c$ where $c$ is the largest weight.

Consider $x_1 \in \Sigma_p$. If $x_1 \equiv c - y_i \mod y_1$ for some $i \neq 1$ and 2, then $x_1 = c - y_i$, which contradicts Lemma 4.4.3. If $x_1 \equiv x_i - c$ for some $i$, $x_1 - 2y_1 = x_i - c$, which is a contradiction since $x_1 - 2y_1 < x_i - c$ for all $i$. If $x_1 \equiv c \mod y_1$, then $x_1 + y_1 = c$, which contradicts Lemma 4.4.3. Therefore, $x_1 \equiv a \mod y_1$.

Next, consider $-y_t \in \Sigma_p$. If $-y_t \equiv c - y_i$ for some $i \neq 1$ and 2, then $2y_1 - y_t = c - y_i$, which is a contradiction since $2y_1 - y_t > c - y_i$ for all $i$. If $-y_t \equiv x_i - c \mod y_1$ for some $i$, then $-y_t = x_i - c$, which contradicts Lemma 4.4.3. Therefore, we have that either $-y_t \equiv c \mod y_1$ or $-y_t \equiv c - y_2 \mod y_1$.

Suppose that $-y_t \equiv c \mod y_1$. This means that $-y_t + 3y_1 = c$. Then we have that $-y_t = c - 3y_1 = y_1 + y_2 - 3y_1 = y_2 - 2y_1 < y_1 - 2y_1 = -y_1 < -y_2$, hence $-y_t < -y_2$. Next, we consider $-y_2 \in \Sigma_p$. Using the same argument for $-y_t$, we have that $-y_2 \in \Sigma_p$ is congruent to no element in $\Sigma_q$ modulo $y_1$, which is a contradiction.

Next, suppose that $-y_t \equiv c - y_2 \mod y_1$. This means that $2y_1 - y_t = c - y_2$. Then we have that $2y_1 - y_t = c - y_2 = y_1 + y_2 - y_2 = y_1$, hence $y_1 = y_t$. Hence, we have $\{-y_1, -y_1\} = \{-y_1, -y_t\} \subset \Sigma_p$. Then by Lemma 4.2.6 part 3 for $-y_1$, $2y_1 \in \Sigma_p$, which is a contradiction since $c < 2y_1$ where $c$ is the largest weight. $\square$

**Lemma 4.4.6.** *Fix a natural number $e$ such that $e \neq \frac{c}{2}$. In Lemma 4.4.2, at most one of $x_i$'s, $y_i$'s, $c - x_i$'s, and $c - y_i$'s can be $e$.*

*Proof.* Fix a natural number $e$ such that $e \neq \frac{c}{2}$.

1. $x_i = e$ for some $i$.

   First, by Lemma 4.4.3, $x_i \neq c - y_j$ for all $j$. Second, suppose that $x_i = c - x_j$ for some $j$. Then by Lemma 4.4.4, $2e = 2x_i = 2x_j = c$, which is a contradiction.

Third, suppose that $x_i = y_j$ for some $j$. Assume that $e > \frac{c}{2}$. Since $\{-e, e\} = \{-y_j, x_i\} \subset \Sigma_p$, either $2e \in \Sigma_q$ or $2e \in \Sigma_r$ by Lemma 4.2.6 part 4 for $e$, which is a contradiction since $2e > c$ where $c$ is the largest weight. Next, assume that $e < \frac{c}{2}$. Since $\{e-c, c-e\} = \{x_i-c, c-y_j\} \subset \Sigma_q$, either $2(c-e) \in \Sigma_p$ or $2(c-e) \in \Sigma_r$ by Lemma 4.2.6 part 4 for $c-e$, which is a contradiction since $2(c-e) > c$ where $c$ is the largest weight.

Last, suppose that $x_i = x_j$ for some $j \neq i$. Assume that $e > \frac{c}{2}$. Since $\{e, e\} = \{x_i, x_j\} \subset \Sigma_p$, $-2e \in \Sigma_p$ by Lemma 4.2.6 part 3 for $e$, which is a contradiction since $-2e < -c$ where $-c$ is the smallest weight. Next, assume that $e < \frac{c}{2}$. Since $\{e-c, e-c\} = \{x_i - c, x_j - c\} \subset \Sigma_q$, $2(c-e) \in \Sigma_q$ by Lemma 4.2.6 part 3 for $e-c$, which is a contradiction since $2(c-e) > c$ where $c$ is the largest weight.

2. $y_i = e$ for some $i$.

   As above, $y_i \neq x_j$ for all $j$. By Lemma 4.4.3, $y_i \neq c - x_j$ for all $j$.

   Next, suppose that $y_i = c - y_j$ for some $j$. Then by Lemma 4.4.4, $i = j$. Hence $c = 2y_i = 2e$, which is a contradiction by the assumption that $e \neq \frac{c}{2}$.

   Finally, Suppose that $y_i = y_j$ for some $j \neq i$. Assume that $e > \frac{c}{2}$. Since $\{-e, -e\} = \{-y_i, -y_j\} \subset \Sigma_p$, $2e \in \Sigma_p$ by Lemma 4.2.6 part 3 for $e$, which is a contradiction since $2e > c$ where $c$ is the largest weight. Next, assume that $e < \frac{c}{2}$. Since $\{c-e, c-e\} = \{c-y_i, c-y_j\} \subset \Sigma_q$, $-2(c-e) \in \Sigma_q$ by Lemma 4.2.6 part 3 for $c-e$, which is a contradiction since $-2(c-e) < -c$ where $-c$ is the smallest weight.

3. $c - x_i = e$ for some $i$.

   As above, $c - x_i \neq x_j$ for all $j$ and $c - x_i \neq y_j$ for all $j$. Since $x_j \neq y_k$ for all $j$ and $k$, $c - x_i \neq c - y_j$ for all $j$. Also, since $x_j \neq x_k$ for all $j$ and $k$, $c - x_i \neq c - x_j$ for all $j$.

4. $c - y_i = e$ for some $i$.

   As above, $c - y_i \neq x_j$, $c - y_i \neq y_j$, and $c - y_i \neq c - x_j$ for all $j$. Since $y_j \neq y_k$ for all $j$ and $k$ as above, $c - y_i \neq c - y_j$ for all $j$.

□

**Lemma 4.4.7.** *In Lemma 4.4.2, $x_i \neq y_j$ for all $i$ and $j$.*

*Proof.* Assume on the contrary that $x_i = y_j$ for some $i$ and $j$. Then by Lemma 4.4.6, $x_i = y_j = \frac{c}{2}$. Hence, $\{-2x_1, x_1, -x_1\} = \{-c, x_1, -y_1\} \subset \Sigma_p$, which contradicts Lemma 4.2.6 part 4 for $x_1$. $\square$

**Lemma 4.4.8.** *In Lemma 4.4.2, assume that $\{-f, f\} \subset \Sigma_r$ for some natural number $f$. If $f > 1$, then $c = 2f$, $\{-c = -2f, -f\} \subset \Sigma_p$, $\{2f = c, f\} \subset \Sigma_q$, and $\{-f, f\} \subset \Sigma_r$. Moreover, no additional multiples of $f$ should appear as weights.*

*Proof.* Assume that $f > 1$. By Lemma 4.2.6 part 4 for $f$, either $\{2f, f\} \subset \Sigma_p$, $\{-2f, -f\} \subset \Sigma_q$, and $\{-f, f\} \subset \Sigma_r$, or $\{-2f, -f\} \subset \Sigma_p$, $\{2f, f\} \subset \Sigma_q$, and $\{-f, f\} \subset \Sigma_r$. However, since $\Sigma_p$ does not have a positive even weight, the former case is impossible. Hence the latter must be the case. Then, $-2f \in \Sigma_p$ implies that $c = 2f$ or $b = 2f$. Suppose that $b = 2f$. Then we have that $\{b = 2f, -f, f\} \subset \Sigma_r$, which is a contradiction by Lemma 4.2.6 part 4 for $f$. Therefore, $c = 2f$. $\square$

**Lemma 4.4.9.** *In Lemma 4.4.2, if $N_p(1) > N_r(1)$ and $N_q(1) > N_r(1)$, then $N_p(1) < N_r(1)+3$ or $N_q(1) < N_r(1)+3$. Similarly, if $N_p(-1) > N_r(-1)$ and $N_q(-1) > N_r(-1)$, then $N_p(-1) < N_r(-1) + 3$ or $N_q(-1) < N_r(-1) + 3$.*

*Proof.* First we prove the former. For this suppose not, i.e., $N_p(1) \geq N_r(1)+3$ and $N_q(1) \geq N_r(1) + 3$. There are three cases:

1. $a > \frac{c}{2}$.

   Since $a \in \Sigma_q$ and $-a \in \Sigma_r$, by Lemma 4.2.6 part 1 for $a$, $\Sigma_q \equiv \Sigma_r \mod a$. With $N_q(1) \geq N_r(1) + 3$, this implies that $N_r(1 + a) \geq 2$ or $N_r(1 - a) \geq 2$, since $|1 + ka| > c$ for $|k| \geq 2$. If $N_r(1 + a) \geq 2$, $-2(1 + a) \in \Sigma_r$ by Lemma 4.2.6 part 3 for $1 + a$, but $2(1 + a) > c$, which is a contradiction. If $N_r(1 - a) \geq 2$, $2(a - 1) \in \Sigma_r$ by Lemma 4.2.6 part 3 for $1 - a$. However, $2(a - 1) \geq c$ but $c \notin \Sigma_r$.

2. $a < \frac{c}{2}$.

   Suppose that $a < \frac{c}{2}$. Then $b = c - a > \frac{c}{2}$. Since $-b \in \Sigma_p$ and $b \in \Sigma_r$, by Lemma 4.2.6 part 1 for $b$, $\Sigma_p \equiv \Sigma_r \mod b$. With $N_p(1) \geq N_r(1) + 3$, this implies that $N_r(1+b) \geq 2$ or $N_r(1-b) \geq 2$, since $|1+kb| > c$ for $|k| \geq 2$. If $N_r(1+b) \geq 2$, $-2(1+b) \in \Sigma_r$ by Lemma 4.2.6 part 3 for $1+b$. However, $2(1+b) > c$, which is a contradiction. If $N_r(1-b) \geq 2$, $2(b-1) \in \Sigma_r$ by Lemma 4.2.6 part 3 for $1-b$. However, $2(b-1) \geq c$ but $c \notin \Sigma_r$.

3. $a = \frac{c}{2}$.

   Since $a = \frac{c}{2}$, we have that $b = c - a = \frac{c}{2}$. Then the isotropy submanifold $M^{\mathbb{Z}_a}$ must be the third case of Lemma 4.2.5. This means that the three fixed point lie in the same component of $M^{\mathbb{Z}_a}$, hence $\Sigma_p \equiv \Sigma_q \equiv \Sigma_r \mod a$ by Lemma 2.0.19. With $N_q(1) \geq N_r(1) + 3$, $\Sigma_q \equiv \Sigma_r \mod a$ implies that $N_r(1+a) \geq 2$, $N_r(1-a) \geq 2$, $N_r(1-2a) \geq 2$, or $N_r(1+a) = N_r(1-a) = N_r(1-2a) = 1$, since $|1+ka| > c$ for $k \neq -2, -1, 0$, and 1.

   (a) $N_r(1+a) \geq 2$.

   Since $N_r(1+a) \geq 2$, $-2(1+a) \in \Sigma_r$ by Lemma 4.2.6 part 3 for $1+a$. However, $-2(1+a) < -c$ where $-c$ is the smallest weight, which is a contradiction.

   (b) $N_r(1-2a) \geq 2$.

   By Lemma 4.2.6 part 3 for $1-2a$, $2(2a-1) \in \Sigma_r$. However, $2(2a-1) = 2(c-1) > c$ where $c$ is the largest weight, which is a contradiction.

   (c) $N_r(1-a) \geq 2$.

   By Lemma 4.2.6 part 3 for $1-a$, $N_r(1-a) = 2$ and $2(a-1) \in \Sigma_r$. Since $b$ is the only positive even weight at $r$, this means that $2(a-1) = b$. Hence $a = b = 2$ and $c = a + b = 4$. Then the weights at $p$ and $q$ are

   $$\Sigma_p = \{-4, -2\} \cup \{x_i\}_{i=1}^{t+3} \cup \{-y_i\}_{i=1}^{t} \cup \{1\} \cup \{-1, 1\}^s$$
   $$\Sigma_q = \{4, 2\} \cup \{x_i - 4\}_{i=1}^{t+3} \cup \{4 - y_i\}_{i=1}^{t} \cup \{1\} \cup \{-1, 1\}^s.$$

   Since $c = 4$ is the largest weight, all of $x_i$'s and $y_i$'s are either 1 or 3. If at least two of $x_i$'s are 3, $N_p(3) \geq 2$ hence $-6 \in \Sigma_p$ by Lemma 4.2.6 part 3 for 3, which is a contradiction since $-4$ is the smallest weight. If at most one of $x_i$ is 1, then at least two $x_i - c$'s are -3. This means that $N_q(-3) \geq 2$ and hence $6 \in \Sigma_q$ by Lemma

4.2.6 part 3 for $-3$, which is a contradiction since 4 is the largest weight.

(d) $N_r(1+a) = N_r(1-a) = N_r(1-2a) = 1$.

Since $1-2a \in \Sigma_r$, by Lemma 2.0.12 for $1-2a$, there must be a weight of $2a-1$ for some fixed point. If it is $r$, $\{2a-1, 1-2a\} \subset \Sigma_r$. Hence $2(2a-1) \in \Sigma_p$ or $2(2a-1) \in \Sigma_q$ by Lemma 4.2.6 part 4 for $2a-1$, which is a contradiction since $2(2a-1) = 2(c-1) > c$ where $c$ is the largest weight. Hence either $2a-1 \in \Sigma_p$ or $2a-1 \in \Sigma_q$. Suppose that $2a-1 \in \Sigma_p$. Then by Lemma 4.2.6 part 1 for $2a-1$, $\Sigma_p \equiv \Sigma_r \mod 2a-1$. That $N_p(1) \geq N_r(1)+3$ implies that $N_r(2-2a) \geq 3$ since $|1+k(2a-1)| \geq c$ for $k \neq 0$ and $-1$, and the fixed point $r$ does not have a weights of $c$. However, $r$ has only one negative even weight $-a$, which is a contradiction. Similarly, $2a-1 \in \Sigma_q$ is also impossible.

With a slight variation of this argument, one can prove the latter. $\square$

## 4.5 Preliminaries for the largest weight even case: part 2

Let the circle act symplectically on a compact, connected symplectic manifold $M$ with exactly three fixed points. Also, assume that $\dim M \geq 8$ and the largest weight is even. In this section, for technical reasons, we consider $w = \min_{\alpha \in M^{S^1}} \min\{N_\alpha(-1), N_\alpha(1)\}$ and rewrite the weights in terms of $w$. And then we further investigate properties that the manifold $M$ should satisfy in terms of $w$, if such a manifold exists.

**Lemma 4.5.1.** *Let $w = \min_{\alpha \in M^{S^1}} \min\{N_\alpha(-1), N_\alpha(1)\}$. In Lemma 4.4.2, the weights are*

$$\Sigma_p = \{-c, -b\} \cup \{x_i\}_{i=1}^{t+3} \cup \{-y_i\}_{i=1}^{t} \cup \{1\} \cup \{-1, 1\}^{v+w}$$
$$\Sigma_q = \{c, a\} \cup \{x_i - c\}_{i=1}^{t+3} \cup \{c - y_i\}_{i=1}^{t} \cup \{1\} \cup \{-1, 1\}^{v+w}$$
$$\Sigma_r = \{-a, b, \cdots\} \cup \{-1, 1\}^{w}$$

*for some $t \geq 0$ and $v \geq 0$, where $a, b$, and $c$ are even natural numbers such that $c = a + b$ is the largest weight, and $x_i$'s and $y_i$'s are odd natural numbers for all $i$. Moreover, the remaining weights at $r$ are odd.*

*Proof.* In Lemma 4.4.2, the weights are

$$\Sigma_p = \{-c, -b\} \cup \{x_i\}_{i=1}^{t+3} \cup \{-y_i\}_{i=1}^{t} \cup \{1\} \cup \{-1, 1\}^s$$
$$\Sigma_q = \{c, a\} \cup \{x_i - c\}_{i=1}^{t+3} \cup \{c - y_i\}_{i=1}^{t} \cup \{1\} \cup \{-1, 1\}^s$$
$$\Sigma_r = \{-a, b, \cdots\}$$

for some $s \geq 0$ and $t \geq 0$ such that $\dim M = 2n = 12 + 4t + 4s$, where $a, b$, and $c$ are even natural numbers such that $c = a + b$ is the largest weight, and $x_i$'s and $y_i$'s are odd natural numbers for all $i$. Moreover, the remaining weights at $r$ are odd.

Let $w = \min_{\alpha \in M^{S^1}} \min\{N_\alpha(-1), N_\alpha(1)\}$. We rewrite the weights in terms of $w$. We show that $\{-c, -b\} \cup \{x_i\}_{i=1}^{t+3} \cup \{-y_i\}_{i=1}^{t} \cup \{1\}$ in $\Sigma_p$ and $\{c, a\} \cup \{x_i - c\}_{i=1}^{t+3} \cup \{c - y_i\}_{i=1}^{t} \cup \{1\}$ in $\Sigma_q$ do not contribute to $w$, i.e.,

$$\{-1, 1\} \nsubseteq$$
$$(\{-c, -b\} \cup \{x_i\}_{i=1}^{t+3} \cup \{-y_i\}_{i=1}^{t} \cup \{1\}) \cap (\{c, a\} \cup \{x_i - c\}_{i=1}^{t+3} \cup \{c - y_i\}_{i=1}^{t} \cup \{1\}).$$

First, $a$, $b$, and $c$ are even natural numbers. Second, by Lemma 4.4.6, at most one of $c - x_i$'s or $y_i$'s can be 1.

Suppose that $y_i = 1$ for some $i$. Then $c - x_j \neq 1$ for all $j$ by Lemma 4.4.6. Hence in $\Sigma_q$, $\{-1, 1\} \nsubseteq (\{c, a\} \cup \{x_i - c\}_{i=1}^{t+3} \cup \{c - y_i\}_{i=1}^{t} \cup \{1\})$.

Next, suppose that $c - x_i = 1$ for some $i$. Then $y_j \neq 1$ for all $j$ by Lemma 4.4.6. Hence in $\Sigma_p$, $\{-1, 1\} \nsubseteq (\{-c, -b\} \cup \{x_i\}_{i=1}^{t+3} \cup \{-y_i\}_{i=1}^{t} \cup \{1\})$.

Last, if $y_i \neq 1$ and $c - x_i \neq 1$ for all $i$, then $\{-1, 1\} \nsubseteq (\{-c, -b\} \cup \{x_i\}_{i=1}^{t+3} \cup \{-y_i\}_{i=1}^{t} \cup \{1\})$ in $\Sigma_p$ and $\{-1, 1\} \nsubseteq (\{c, a\} \cup \{x_i - c\}_{i=1}^{t+3} \cup \{c - y_i\}_{i=1}^{t} \cup \{1\})$ in $\Sigma_q$.

Therefore, we can rewrite the weights so that the weights are

$$\Sigma_p = \{-c, -b\} \cup \{x_i\}_{i=1}^{t+3} \cup \{-y_i\}_{i=1}^{t} \cup \{1\} \cup \{-1, 1\}^{v+w}$$
$$\Sigma_q = \{c, a\} \cup \{x_i - c\}_{i=1}^{t+3} \cup \{c - y_i\}_{i=1}^{t} \cup \{1\} \cup \{-1, 1\}^{v+w}$$
$$\Sigma_r = \{-a, b, \cdots\} \cup \{-1, 1\}^w$$

where $s = v + w$. □

**Lemma 4.5.2.** *In Lemma 4.5.1, for each $x_i$, either $x_i = c - x_j$ for some $j$ or $-x_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$.*

*Proof.* By Lemma 2.0.12, for each $x_i$, either $-x_i \in \Sigma_p$, $-x_i \in \Sigma_q$, or $-x_i \in \Sigma_r$.

First, assume that $x_i > 1$. By Lemma 4.4.6, $x_i \neq y_j$ for all $j$. Hence $-x_i \notin \Sigma_p$. Next, if $-x_i \in \Sigma_q$, then $-x_i = x_j - c$ for some $j$. If $-x_i \in \Sigma_r$, $-x_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$.

Second, assume that $x_i = 1$. By Lemma 4.4.6, at most one of $c - x_i$'s or $y_i$'s can be 1. Hence, either $N_p(1) > N_p(-1)$ and $N_q(1) \geq N_q(-1)$, or $N_p(1) \geq N_p(-1)$ and $N_q(1) > N_q(-1)$. By Lemma 2.0.12, this implies that $N_r(1) < N_r(-1)$. Therefore, $-x_i = -1 \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$. ☐

**Lemma 4.5.3.** *In Lemma 4.5.1, for each $c - y_i$, $y_i - c \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$.*

*Proof.* By Lemma 2.0.12, for each $c - y_i$, either $y_i - c \in \Sigma_p$, $y_i - c \in \Sigma_q$, or $y_i - c \in \Sigma_r$.

First, assume that $c - y_i > 1$. Suppose that $c - y_i \in \Sigma_q$. Then $c - y_i = c - x_j$ for some $j$, which is a contradiction since $c - y_i \neq c - x_j$ for all $j$ by Lemma 4.4.7. Next, suppose that $y_i - c \in \Sigma_p$. Then $y_i - c = -y_j$ for some $j$. By Lemma 4.4.5, $i = j$, i.e., $c = 2y_i$. Hence, $\{-2y_i, -y_i\} = \{-c, -y_i\} \subset \Sigma_p$ and $\{2y_i, y_i\} = \{c, c - y_i\} \subset \Sigma_q$. The isotropy submanifold $M^{\mathbb{Z}_{y_i}}$ must be the third case of Lemma 4.2.5. Therefore, we have that $\{-y_i, y_i\} \subset \Sigma_r$. In particular, $y_i - c = -y_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$, since $c - y_i = y_i = \frac{c}{2} \geq 2$ and $c - y_i$ is odd.

Second, assume that $c - y_i = 1$. By Lemma 4.4.6, at most one of $c - x_i$'s or $y_i$'s can be 1. Hence, either $N_p(1) > N_p(-1)$ and $N_q(1) \geq N_q(-1)$, or $N_p(1) \geq N_p(-1)$ and $N_q(1) > N_q(-1)$. By Lemma 2.0.12, this implies that $N_r(1) < N_r(-1)$. Therefore, $y_i - c = -1 \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$. ☐

**Lemma 4.5.4.** *In Lemma 4.5.1, suppose that $c - x_i \neq 1$. Then either $c - x_i = x_j$ for some $j$ or $c - x_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$.*

*Proof.* Suppose that $c - x_i \neq 1$. By Lemma 2.0.12, for each $c - x_i$, either $c - x_i \in \Sigma_p$, $c - x_i \in \Sigma_q$, or $c - x_i \in \Sigma_r$. First, assume that $c - x_i \in \Sigma_p$. Then $c - x_i = x_j$ for some $j$. Second, assume that $c - x_i \in \Sigma_q$. Then $c - x_i = c - y_j$ for some $j$, which is a contradiction by Lemma 4.4.7. Hence $c - x_i \notin \Sigma_q$. Last, assume that $c - x_i \in \Sigma_r$. Since $c - x_i \neq 1$, $c - x_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$.

□

**Lemma 4.5.5.** *In Lemma 4.5.1, suppose that $y_i \neq 1$. Then $y_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$.*

*Proof.* Suppose that $y_i \neq 1$. By Lemma 2.0.12, for each $y_i$, either $y_i \in \Sigma_p$, $y_i \in \Sigma_q$, or $y_i \in \Sigma_r$. First, assume that $y_i \in \Sigma_p$. Then $y_i = x_j$ for some $j$, which is a contradiction by Lemma 4.4.7. Hence $y_i \notin \Sigma_p$. Second, assume that $y_i \in \Sigma_q$. Then $y_i = c - y_j$ for some $j$. By Lemma 4.4.5, $i = j$, i.e., $c = 2y_i$. Hence, $\{-2y_i, -y_i\} = \{-c, -y_i\} \subset \Sigma_p$ and $\{2y_i, y_i\} = \{c, c - y_i\} \subset \Sigma_q$. The isotropy submanifold $M^{\mathbb{Z}_{y_i}}$ must be the third case of Lemma 4.2.5. Therefore, we have that $\{-y_i, y_i\} \subset \Sigma_r$. In particular, $y_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$, since $y_i = \frac{c}{2} \geq 2$ and $y_i$ is odd. Last, assume that $y_i \in \Sigma_r$. Since $y_i \neq 1$, $y_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$. □

**Lemma 4.5.6.** *In Lemma 4.5.1, if $x_i \neq c - x_j$ for all $i$ and $j$, then $t < v$.*

*Proof.* Assume on the contrary that $x_i \neq c - x_j$ for all $i$ and $j$, and $t \geq v$. By Lemma 4.5.2, $-x_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for all $i$. Also, by Lemma 4.5.3, $y_i - c \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for all $i$.

We show that $x_i \neq x_j$ and $y_i \neq y_j$ for $i \neq j$. Suppose that $x_i = x_j$ for some $i \neq j$. Then by Lemma 4.4.6, $2x_i = 2x_j = c$, hence $x_i = c - x_j$, which contradicts the assumption. Therefore, $x_i \neq x_j$ for $i \neq j$. Suppose that $y_i = y_j$ for some $i \neq j$. Then by Lemma 4.4.6, $2y_i = 2y_j = c$, hence $y_i = c - y_j$, which contradicts Lemma 4.4.5. Therefore, $y_i \neq y_j$ for $i \neq j$.

First, suppose that $c - x_i = 1$ for some $i$. Without loss of generality, let $c - x_1 = 1$. Then by Lemma 4.4.6, $c - x_i \neq 1$ for $i \neq 1$ and $y_j \neq 1$ for all $j$. Hence $c - x_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for $i \neq 1$ by Lemma 4.5.4. Also, $y_j \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for all $j$ by Lemma 4.5.5. Also, by Lemma 4.4.3, $x_i \neq c - y_j$ for all $i$ and $j$. Therefore, we have that $\{-x_i\}_{i=1}^{t+3} \cup \{c - x_i\}_{i=2}^{t+3} \cup \{y_i\}_{i=1}^{t} \cup \{c - y_i\}_{i=1}^{t} \subset \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$, which is a contradiction since $|\{-x_i\}_{i=1}^{t+3} \cup \{c - x_i\}_{i=2}^{t+3} \cup \{y_i\}_{i=1}^{t} \cup \{c - y_i\}_{i=1}^{t}| = 4t + 5$ and $|\Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)| = 2t + 4 + 2u + 2v$, but $4t + 5 = 2t + 2t + 5 > 2t + 4 + 2v$.

Second, suppose that $y_i = 1$ for some $i$. Without loss of generality, let $y_1 = 1$. Then by Lemma 4.4.6, $c - x_i \neq 1$ for all $i$ and $y_j \neq 1$ for $j \neq 1$. Hence $c - x_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for all $i$ by Lemma 4.5.4 and

$y_j \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for $j \neq 1$ by Lemma 4.5.5. Also, by Lemma 4.4.3, $x_i \neq c - y_j$ for all $i$ and $j$. Then we have that $\{-x_i\}_{i=1}^{t+3} \cup \{c - x_i\}_{i=1}^{t+3} \cup \{y_i\}_{i=2}^{t} \cup \{c - y_i\}_{i=1}^{t} \subset \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$, which is a contradiction.

Finally, suppose that $c - x_i \neq 1$ and $y_i \neq 1$ for all $i$. Then $c - x_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for all $i$ by Lemma 4.5.4 and $y_j \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for all $j$ by Lemma 4.5.5. Also, by Lemma 4.4.3, $x_i \neq c - y_j$ for all $i$ and $j$. Then we have that $\{-x_i\}_{i=1}^{t+3} \cup \{c - x_i\}_{i=1}^{t+3} \cup \{y_i\}_{i=1}^{t} \cup \{c - y_i\}_{i=1}^{t} \subset \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$, which is a contradiction. $\square$

## 4.6 The case where the largest weight is even

Let the circle act symplectically on a compact, connected symplectic manifold $M$ with exactly three fixed points. In this section, we show that if $\dim M \geq 8$, the largest weight cannot be even. We rule out case by case. In Lemma 4.5.1, we have the following cases:

1. $t = 0$ and $v = 0$.
2. $t = 0$ and $v = 1$.
3. $t = 0$ and $v = 2$.
4. $t = 1$ and $v = 0$.
5. $t = 1$ and $v = 1$.
6. $t = 1$ and $v = 2$.
7. $t = 2$ and $v = 1$.
8. $t = 2$ and $v = 2$.
9. $t = 3$ and $v = 2$.
10. $t \geq 2 + v$.
11. $v \geq 3$.

**Lemma 4.6.1.** *In Lemma 4.5.1, $t = 0$ and $v = 0$ are impossible.*

*Proof.* The weights in this case are

$$\Sigma_p = \{-c, -b, x_1, x_2, x_3, 1\} \cup \{-1, 1\}^w$$
$$\Sigma_q = \{c, a, x_1 - c, x_2 - c, x_3 - c, 1\} \cup \{-1, 1\}^w$$
$$\Sigma_r = \{-a, b, \cdots\} \cup \{-1, 1\}^w,$$

where $w = \min_{\alpha \in M^{S^1}} \min\{N_\alpha(-1), N_\alpha(1)\}$, $a, b$, and $c$ are even natural numbers such that $c = a + b$ is the largest weight, and $x_i$'s are odd natural numbers for all $i$. Moreover, and the remaining weights at $r$ are odd.

By Lemma 4.5.6, $x_i = c - x_j$ for some $i$ and $j$. Then by Lemma 4.4.4, there exist $x_i$ and $x_j$ where $i \neq j$ such that $2x_i = 2x_j = c$. Without loss of generality, let $2x_1 = 2x_2 = c$. Lemma 4.4.4 also implies that $x_3 \neq c - x_i$ for all $i$. Therefore, $-x_3 \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$. Note that $x_1 = c - x_1 = \frac{c}{2} \geq 2$.

First, suppose that $c - x_3 = 1$. Then we have that $x_3 = c - 1 > 1$. Hence, $N_p(1) = N_p(-1) + 1 = w + 1$ and $N_q(1) = N_q(-1) = w + 1$. Therefore, $N_r(1) + 1 = N_r(-1)$ by Lemma 2.0.12 for 1. Considering Lemma 2.0.12 for each integer, one can show that the weights are

$$\Sigma_p = \{-c, -b, x_1, x_1, x_3, 1\} \cup \{-1, 1\}^w$$
$$\Sigma_q = \{c, a, -x_1, -x_1, -1, 1\} \cup \{-1, 1\}^w$$
$$\Sigma_r = \{-a, b, -x_3, f, -f, -1\} \cup \{-1, 1\}^w$$

for some odd natural number $f$. If $f > 1$, by Lemma 4.4.8, we have that $c = 2x_1 = 2f$, which is a contradiction since no additional multiples of $x_1$ should appear by Lemma 4.2.6 part 3 for $x_1$. Hence $f = 1$.

Second, suppose that $c - x_3 \neq 1$. Then by Lemma 4.5.4, $c - x_3 \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$. Also, $N_p(1) \geq N_p(-1) + 1 = w + 1$ and $N_q(1) = N_q(-1) + 1 = w + 1$. Therefore, $N_r(1) + 2 \leq N_r(-1)$ by Lemma 2.0.12 for 1. Considering Lemma 2.0.12 for each integer, one can show that the weights are

$$\Sigma_p = \{-c, -b, x_1, x_1, x_3, 1\} \cup \{-1, 1\}^w$$
$$\Sigma_q = \{c, a, -x_1, -x_1, x_3 - c, 1\} \cup \{-1, 1\}^w$$
$$\Sigma_r = \{-a, b, -x_3, c - x_3, -1, -1\} \cup \{-1, 1\}^w$$

Therefore, in either case the weights are

$$\Sigma_p = \{-c, -b, x_1, x_1, x_3, 1\} \cup \{-1, 1\}^w$$
$$\Sigma_q = \{c, a, -x_1, -x_1, x_3 - c, 1\} \cup \{-1, 1\}^w$$
$$\Sigma_r = \{-a, b, -x_3, c - x_3, -1, -1\} \cup \{-1, 1\}^w$$

Let $A = c_n(M)|_q = \prod \xi_p^j$, $B = c_n(M)|_q = \prod \xi_q^j$, and $C = c_n(M)|_r = \prod \xi_r^j$. First, $2x_1 = c = a + b > b$. Since $c = a + b \geq 2 + 2 = 4$, $x_1 = \frac{c}{2} \geq 2$. Therefore, $c = 2x_1 > x_3$ implies that $x_1^2 > x_3$. Then

$$(-1)^w \frac{-B - C}{a(c - x_3)} = cx_1^2 - bx_3 > 0,$$

and this implies that $(-1)^{w+1} B > (-1)^w C$. Also, we have that $(-1)^w A > 0$. Then, by Theorem 2.0.9,

$$0 = (-1)^w \int_M 1 = (-1)^w \left( \frac{1}{A} + \frac{1}{B} + \frac{1}{C} \right) > 0,$$

which is a contradiction. $\square$

**Lemma 4.6.2.** *In Lemma 4.5.1, $t = 0$ and $v = 1$ are impossible.*

*Proof.* The weights in this case are

$$\Sigma_p = \{-c, -b, x_1, x_2, x_3, 1\} \cup \{-1, 1\}^{w+1}$$
$$\Sigma_q = \{c, a, x_1 - c, x_2 - c, x_3 - c, 1\} \cup \{-1, 1\}^{w+1}$$
$$\Sigma_r = \{-a, b, \cdots\} \cup \{-1, 1\}^w,$$

where $w = \min_{\alpha \in M^{S^1}} \min\{N_\alpha(-1), N_\alpha(1)\}$, $a, b$, and $c$ are even natural numbers such that $c = a + b$ is the largest weight, and $x_i$'s are odd natural numbers for all $i$. Moreover, and the remaining weights at $r$ are odd.

1. $x_i \neq c - x_j$ for all $i$ and $j$.

   By Lemma 4.5.2, $-x_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for all $i$. Assume that $x_i = x_j$ for some $i \neq j$. Then by Lemma 4.4.6, $2x_i = 2x_j = c$ hence $x_i = c - x_j$, which contradicts the assumption. Hence $x_i \neq x_j$ for $i \neq j$.

   First, assume that $c - x_i \neq 1$ for all $i$. Then by Lemma 4.5.4, $c - x_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for all $i$. Hence, the weights are

   $$\Sigma_p = \{-c, -b, x_1, x_2, x_3, 1, -1, 1\} \cup \{-1, 1\}^w$$
   $$\Sigma_q = \{c, a, x_1 - c, x_2 - c, x_3 - c, 1, -1, 1\} \cup \{-1, 1\}^w$$
   $$\Sigma_r = \{-a, b, -x_1, -x_2, -x_3, c - x_1, c - x_2, c - x_3\} \cup \{-1, 1\}^w$$

   Then we have that $\lambda_r = \frac{1}{2} \dim M$, which contradicts Lemma 4.1.4 that $\lambda_r = \frac{1}{2} \dim M + 2$.

Next, assume that $c - x_i = 1$ for some $i$. Without loss of generality, let $c - x_3 = 1$. Then by Lemma 4.4.6, $c - x_i \neq 1$ for $i \neq 3$. By Lemma 4.5.4, $c - x_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for $i \neq 3$. Also, by the assumption, $x_i \neq 1$ for $i \neq 3$. Then $N_p(1) = N_p(-1) + 1 = w + 2$ and $N_q(1) = N_q(-1) = w + 2$. Therefore, $N_r(1) + 1 = N_r(-1)$ by Lemma 2.0.12 for 1. Then the weights are

$$\Sigma_p = \{-c, -b, x_1, x_2, c - 1, 1, -1, 1\} \cup \{-1, 1\}^w$$
$$\Sigma_q = \{c, a, x_1 - c, x_2 - c, -1, 1, -1, 1\} \cup \{-1, 1\}^w$$
$$\Sigma_r = \{-a, b, -x_1, -x_2, 1 - c, c - x_1, c - x_2, -1\} \cup \{-1, 1\}^w$$

By Lemma 4.2.6 part 1 for $x_3 = c - 1$, $\Sigma_p \equiv \Sigma_r \mod c - 1$. Since $c - x_3 = 1$, $c - x_1 \neq 1$ and $c - x_2 \neq 2$ by Lemma 4.4.6. Hence $N_r(1) = w$. Then $N_p(1) \geq w + 2$, $N_r(1) = w$, and $\Sigma_p \equiv \Sigma_r \mod c - 1$ imply that $N_r(2 - c) \geq 2$ since $|1 + k(c - 1)| > c|$ for $|k| \geq 2$ and $c \notin \Sigma_r$. However, $r$ has only one negative even weight, which is a contradiction.

2. $x_i = c - x_j$ for some $i$ and $j$.

   By Lemma 4.5.6, $x_i = c - x_j$ for some $i$ and $j$. Then by Lemma 4.4.4, there exist $x_i$ and $x_j$ where $i \neq j$ such that $2x_i = 2x_j = c$. Without loss of generality, let $2x_1 = 2x_2 = c$. Lemma 4.4.4 also implies that $x_3 \neq c - x_i$ for all $i$. Therefore, by Lemma 4.5.2, $-x_3 \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$. Note that $x_1 = c - x_1 = \frac{c}{2} \geq 2$.

   First, assume that $c - x_3 = 1$. Then $N_p(1) = N_p(-1) + 1 = w + 2$ and $N_q(1) = N_q(-1) = w + 2$. Therefore, $N_r(1) + 1 = N_r(-1)$ by Lemma 2.0.12 for 1. Considering Lemma 2.0.12 for each integer, one can show that the weights are

$$\Sigma_p = \{-c, -b, x_1, x_1, c - 1, 1, -1, 1\} \cup \{-1, 1\}^w$$
$$\Sigma_q = \{c, a, -x_1, -x_1, -1, 1, -1, 1\} \cup \{-1, 1\}^w$$
$$\Sigma_r = \{-a, b, 1 - c, -1, -f, f, -g, g\} \cup \{-1, 1\}^w$$

   for some odd natural numbers $f$ and $g$. If $f > 1$, then by Lemma 4.4.8, $\{-2f, -f\} \subset \Sigma_p$, which is a contradiction since $p$ has no negative odd weight that is less than -1. Hence $f = 1$. However, this means that $\min_{\alpha \in M^{S^1}} \min\{N_\alpha(-1), N_\alpha(1)\} \geq w + 1$, which is a contradiction.

Second, assume that $c - x_3 \neq 1$. By Lemma 4.5.4, $c - x_3 \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$. Then $N_p(1) \geq N_p(-1) + 1 = w + 2$ and $N_q(1) = N_q(-1) + 1 = w + 2$. Therefore, $N_r(1) + 2 \leq N_r(-1)$ by Lemma 2.0.12 for 1. Considering Lemma 2.0.12 for each integer, one can show that the weights are

$$\Sigma_p = \{-c, -b, x_1, x_2, x_3, 1, -1, 1\} \cup \{-1, 1\}^w$$
$$\Sigma_q = \{c, a, -x_2, -x_1, x_3 - c, 1, -1, 1\} \cup \{-1, 1\}^w$$
$$\Sigma_r = \{-a, b, -x_3, c - x_3, -1, -1, -f, f\} \cup \{-1, 1\}^w$$

for some odd natural number $f$. As above, $f = 1$ and this means that $\min_{\alpha \in M^{S^1}} \min\{N_\alpha(-1), N_\alpha(1)\} \geq w + 1$, which is a contradiction.

□

**Lemma 4.6.3.** *In Lemma 4.5.1, $t = 0$ and $v = 2$ are impossible.*

*Proof.* The weights in this case are

$$\Sigma_p = \{-c, -b, x_1, x_2, x_3, 1\} \cup \{-1, 1\}^{w+2}$$
$$\Sigma_q = \{c, a, x_1 - c, x_2 - c, x_3 - c, 1\} \cup \{-1, 1\}^{w+2}$$
$$\Sigma_r = \{-a, b, \cdots\} \cup \{-1, 1\}^w,$$

where $w = \min_{\alpha \in M^{S^1}} \min\{N_\alpha(-1), N_\alpha(1)\}$, $a, b$, and $c$ are even natural numbers such that $c = a + b$ is the largest weight, and $x_i$'s are odd natural numbers for all $i$. Moreover, and the remaining weights at $r$ are odd.

1. $x_i \neq c - x_j$ for all $i$ and $j$.

   By Lemma 4.5.2, $-x_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for all $i$. Assume that $x_i = x_j$ for some $i \neq j$. Then by Lemma 4.4.6, $2x_i = 2x_j = c$ hence $x_i = c - x_j$, which contradicts the assumption. Hence $x_i \neq x_j$ for $i \neq j$.

   First, assume that $c - x_i \neq 1$ for all $i$. Then by Lemma 4.5.4, $c - x_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for all $i$. Also, $N_p(1) \geq N_p(-1) + 1 = w + 3$ and $N_q(1) = N_q(-1) + 1 = w + 3$. Therefore, $N_r(1) + 2 \leq N_r(-1)$ by Lemma 2.0.12 for 1. Considering Lemma 2.0.12 for each integer, one can show that the weights are

$$\Sigma_p = \{-c, -b, x_1, x_2, x_3, 1, -1, 1, -1, 1\} \cup \{-1, 1\}^w$$
$$\Sigma_q = \{c, a, x_1 - c, x_2 - c, x_3 - c, 1, -1, 1, -1, 1\} \cup \{-1, 1\}^w$$
$$\Sigma_r = \{-a, b, -x_1, -x_2, -x_3, c - x_1, c - x_2, c - x_3, -1, -1\} \cup \{-1, 1\}^w$$

Then we have that $N_p(1) \geq w + 3$, $N_q(1) \geq w + 3$, and $N_r(1) = w$, which contradict Lemma 4.4.9.

Next, assume that $c - x_i = 1$ for some $i$. Without loss of generality, let $c - x_3 = 1$. Then by Lemma 4.4.6, $c - x_i \neq 1$ for $i \neq 3$. Hence, by Lemma 4.5.4, $c - x_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for $i \neq 3$. Also, $N_p(1) \geq N_p(-1) + 1 = w + 3$ and $N_q(1) = N_q(-1) = w + 3$. Therefore, $N_r(1) + 1 \leq N_r(-1)$ by Lemma 2.0.12 for 1. Considering Lemma 2.0.12 for each integer, one can show that the weights are

$$\Sigma_p = \{-c, -b, x_1, x_2, c - 1, 1, -1, 1, -1, 1\} \cup \{-1, 1\}^w$$
$$\Sigma_q = \{c, a, x_1 - c, x_2 - c, -1, 1, -1, 1, -1, 1\} \cup \{-1, 1\}^w$$
$$\Sigma_r = \{-a, b, -x_1, -x_2, 1 - c, c - x_1, c - x_2, -1, -f, f\} \cup \{-1, 1\}^w$$

for some odd natural number $f$. If $f > 1$, then by Lemma 4.4.8, $\{-2f, -f\} \subset \Sigma_p$, which is a contradiction since $p$ has no negative odd weight that is less than -1. Hence $f = 1$. However, this means that $\min_{\alpha \in M^{S^1}} \min\{N_\alpha(-1), N_\alpha(1)\} \geq w + 1$, which is a contradiction.

2. $x_i = c - x_j$ for some $i$ and $j$.

   By Lemma 4.4.4, there exist $x_i$ and $x_j$ where $i \neq j$ such that $2x_i = 2x_j = c$. Without loss of generality, let $2x_1 = 2x_2 = c$. Lemma 4.4.4 also implies that $x_3 \neq c - x_i$ for all $i$. Therefore, $-x_3 \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$. Note that $x_1 = c - x_1 = \frac{c}{2} \geq 2$.

   First, assume that $c - x_3 = 1$. Then we have that $N_p(1) = N_p(-1) + 1 = w + 3$ and $N_q(1) = N_q(-1) = w + 3$. Therefore, $N_r(1) + 1 = N_r(-1)$ by Lemma 2.0.12 for 1. Considering Lemma 2.0.12 for each integer, one can show that the weights are

$$\Sigma_p = \{-c, -b, x_1, x_1, c - 1, 1, -1, 1, -1, 1\} \cup \{-1, 1\}^w$$
$$\Sigma_q = \{c, a, -x_1, -x_1, -1, 1, -1, 1, -1, 1\} \cup \{-1, 1\}^w$$
$$\Sigma_r = \{-a, b, 1 - c, -1, -f, f, -h, h, -k, k\} \cup \{-1, 1\}^w$$

for some odd natural numbers $f, h$, and $k$. As above, $f = 1$ and this means that $\min_{\alpha \in M^{S^1}} \min\{N_\alpha(-1), N_\alpha(1)\} \geq w + 1$, which is a contradiction.

Second, assume that $c - x_3 \neq 1$. By Lemma 4.5.4, $c - x_3 \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$. Also, $N_p(1) \geq N_p(-1) + 1 = w + 3$ and $N_q(1) = N_q(-1) + 1 = w + 3$. Therefore, $N_r(1) + 2 \leq N_r(-1)$ by Lemma 2.0.12 for 1. Considering Lemma 2.0.12 for each integer, one can show that the weights are

$$\Sigma_p = \{-c, -b, x_1, x_1, x_3, 1, -1, 1, -1, 1\} \cup \{-1, 1\}^w$$
$$\Sigma_q = \{c, a, -x_1, -x_1, x_3 - c, 1, -1, 1, -1, 1\} \cup \{-1, 1\}^w$$
$$\Sigma_r = \{-a, b, -x_3, c - x_3, -1, -1, -f, f, -h, h\} \cup \{-1, 1\}^w$$

for some odd natural numbers $f$ and $h$. As above, $f = 1$ and this means that $\min_{\alpha \in M^{S^1}} \min\{N_\alpha(-1), N_\alpha(1)\} \geq w + 1$, which is a contradiction.

$\square$

**Lemma 4.6.4.** *In Lemma 4.5.1, $t = 1$ and $v = 0$ are impossible.*

*Proof.* The weights in this case are

$$\Sigma_p = \{-c, -b, x_1, x_2, x_3, x_4, -y, 1\} \cup \{-1, 1\}^w$$
$$\Sigma_q = \{c, a, x_1 - c, x_2 - c, x_3 - c, x_4 - c, c - y, 1\} \cup \{-1, 1\}^w$$
$$\Sigma_r = \{-a, b, \cdots\} \cup \{-1, 1\}^w,$$

where $a, b$, and $c$ are even natural numbers such that $c = a + b$ is the largest weight, $x_i$'s and $y$ are odd natural numbers for all $i$, and $w = \min_{\alpha \in M^{S^1}} \min\{N_\alpha(-1), N_\alpha(1)\}$. Moreover, the remaining weights at $r$ are odd.

By Lemma 4.5.6, $x_i = c - x_j$ for some $i$ and $j$. Then by Lemma 4.4.4, there exist $x_i$ and $x_j$ where $i \neq j$ such that $2x_i = 2x_j = c$. Without loss of generality, let $2x_1 = 2x_2 = c$. Lemma 4.4.4 also implies that $x_i \neq c - x_j$ for $i \neq 1$ and 2, and for all $j$. Therefore, $-x_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for $i = 3$ and 4 by Lemma 4.5.2. Also, by Lemma 4.5.3, $y - c \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$. Moreover, by Lemma 4.2.6 part 3 for $x$, none of $x_i$'s, $y$, $c - x_i$'s and $c - y$

can be $x$ for $i \neq 1$ and 2. Hence, by Lemma 4.4.6, all of $x_i$'s, $y$, $c - x_i$'s and $c - y$ are different for $i \neq 1$ and 2.

First, suppose that $c - x_i \neq 1$ for all $i$ and $y \neq 1$. Then by Lemma 4.5.4, $x_i - c \in \Sigma_r \setminus (\{-a, b\} \cup \{-1,1\}^w)$ for $i = 3$ and 4. Also, by Lemma 4.5.5, $y \in \Sigma_r \setminus (\{-a, b\} \cup \{-1,1\}^w)$. Then the weights are

$$\begin{aligned}
\Sigma_p &= \{-c, -b, x_1, x_1, x_3, x_4, -y, 1\} \cup \{-1,1\}^w \\
\Sigma_q &= \{c, a, -x_1, -x_1, x_3 - c, x_4 - c, c - y, 1\} \cup \{-1,1\}^w \\
\Sigma_r &= \{-a, b, -x_3, -x_4, y, c - x_3, c - x_4, y - c\} \cup \{-1,1\}^w
\end{aligned}$$

Then we have that $\lambda_r = \frac{1}{2}\dim M$, which contradicts Lemma 4.1.4 that $\lambda_r = \frac{1}{2}\dim M + 2$.

Second, suppose that $y = 1$. Then by Lemma 4.4.6, none of $x_i$'s, $c - x_i$'s, and $c - y$ is 1 for all $i$. Hence, by Lemma 4.5.4, $x_i - c \in \Sigma_r \setminus (\{-a, b\} \cup \{-1,1\}^w)$ for $i = 3$ and 4. Moreover, $N_p(1) = N_p(-1) = w + 1$ and $N_q(1) - 1 = N_q(-1) = w$. By Lemma 2.0.12 for 1, this implies that $N_r(1) = N_r(-1) - 1$. Then the weights are

$$\begin{aligned}
\Sigma_p &= \{-c, -b, x_1, x_1, x_3, x_4, -1, 1\} \cup \{-1,1\}^w \\
\Sigma_q &= \{c, a, -x_1, -x_1, x_3 - c, x_4 - c, c - 1, 1\} \cup \{-1,1\}^w \\
\Sigma_r &= \{-a, b, -x_3, -x_4, c - x_3, c - x_4, 1 - c, -1\} \cup \{-1,1\}^w
\end{aligned}$$

By Corollary 2.0.14, $c_1(M)|_p = 0$ and this implies that $x_3 + x_4 = b$. Since $x_3 + x_4 = b < a + b = c = 2x_1$ and $x_1 = \frac{c}{2} \geq 2$, we have that $x_1^2 > x_3 x_4$. Therefore,

$$(-1)^w \frac{B + C}{a(c - x_3)(c - x_4)(c - 1)} = c x_1^2 - b x_3 x_4 > 0.$$

Also, $(-1)^w A < 0$. Then, by Theorem 2.0.9,

$$0 = \int_M 1 = (-1)^w \left( \frac{1}{A} + \frac{1}{B} + \frac{1}{C} \right) < 0,$$

which is a contradiction.

Finally, suppose that $c - x_i = 1$ for some $i$. Since $c \geq 4$, $c - x_1 = x_1 = \frac{c}{2} \geq 2$. Hence, $c - x_i = x_i \neq 1$ for $i = 1$ and 2. Therefore, without loss of generality, let $c - x_4 = 1$. By Lemma 4.4.6, none of $x_i$'s, $c - x_j$'s, $y$, and $c - y$ is 1 for all $i$ and for $j \neq 4$. Hence, by Lemma 4.5.4, $c - x_3 \in \Sigma_r \setminus (\{-a, b\} \cup \{-1,1\}^w)$. Also, by Lemma 4.5.5, $y \in \Sigma_r \setminus (\{-a, b\} \cup \{-1,1\}^w)$. Moreover, $N_p(1) = N_p(-1) + 1 = w + 1$ and $N_q(1) = N_q(-1) = w + 1$. By Lemma 2.0.12 for 1, this implies that $N_r(1) = N_r(-1) - 1$. Then the weights are

$$\Sigma_p = \{-c, -b, x_1, x_1, x_3, c-1, -y, 1\} \cup \{-1, 1\}^w$$
$$\Sigma_q = \{c, a, -x_1, -x_1, x_3 - c, -1, c - y, 1\} \cup \{-1, 1\}^w$$
$$\Sigma_r = \{-a, b, -x_3, 1 - c, c - x_3, y - c, y, -1\} \cup \{-1, 1\}^w$$

By Lemma 4.2.6 part 1 for $c - 1 = x_4$, $\Sigma_p \equiv \Sigma_r \mod c-1$. First, we can choose a bijection between $\Sigma_p$ and $\Sigma_q$ so that $\Sigma_p \supset \{c-1, -c\} \cup \{-1, 1\}^w \equiv \{1-c, -1\} \cup \{-1, 1\}^w \subset \Sigma_r \mod c-1$. Since $N_p(1) = w+1$ and $N_r(1) = w$, $\Sigma_p \equiv \Sigma_r \mod c-1$ implies that $2-c \in \Sigma_r$ since $|1+k(c-1)| > c$ for $|k| \geq 2$ and $c \notin \Sigma_r$. Since $-a$ is the only negative even weight at $r$, we have that $-a = 2 - c$, i.e., $a + 2 = c = a + b$. Hence, $b = 2$. Then we are left with

$$\{-2 = -b, x_1, x_1, x_3, -y\} \equiv \{2 = b, -x_3, c - x_3, y - c, y\} \mod c-1.$$

Since for $2 \in \Sigma_r$, $2 \neq -2, x_1$, and $x_3 \mod c-1$, the only possibility is that $2 \equiv -y \mod c-1$, i.e., $2 - c + 1 = -y$. Thus $y = c - 3$. By Corollary 2.0.14, $c_1(M)|_p = -c - 2 + x_1 + x_1 + x_3 + c - 1 - y + 1 = 0$. Hence, we have that $x_3 + c = y + 2$. However, $x_3 + c = y + 2 = c - 3 + 2 = c - 1$ and so $0 < x_3 = -1$, which is a contradiction. $\square$

**Lemma 4.6.5.** *In Lemma 4.5.1, $t = 1$ and $v = 1$ are impossible.*

*Proof.* The weights in this case are

$$\Sigma_p = \{-c, -b, x_1, x_2, x_3, x_4, -y, 1\} \cup \{-1, 1\}^{w+1}$$
$$\Sigma_q = \{c, a, x_1 - c, x_2 - c, x_3 - c, x_4 - c, c - y, 1\} \cup \{-1, 1\}^{w+1}$$
$$\Sigma_r = \{-a, b, \cdots\} \cup \{-1, 1\}^w,$$

where $a, b$, and $c$ are even natural numbers such that $c = a + b$ is the largest weight, $x_i$'s and $y$ are odd natural numbers for all $i$, and $w = \min_{\alpha \in M^{S^1}} \min\{N_\alpha(-1), N_\alpha(1)\}$. Moreover, the remaining weights at $r$ are odd.

By Lemma 4.5.6, $x_i = c - x_j$ for some $i$ and $j$. Then by Lemma 4.4.4, there exist $x_i$ and $x_j$ where $i \neq j$ such that $2x_i = 2x_j = c$. Without loss of generality, let $2x_1 = 2x_2 = c$. Denote $x = x_1$. Lemma 4.4.4 also implies that $x_i \neq c - x_j$ for $i \neq 1$ and 2, and for all $j$. Therefore, $-x_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for $i = 3$ and 4 by Lemma 4.5.2. Also, by Lemma 4.5.3, $y - c \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$. Moreover, by Lemma 4.2.6 part 3 for $x$, none of $x_i$'s, $y$, $c - x_i$'s, and $c - y$ can be $x$ for $i \neq 1$ and 2. Hence, by Lemma

4.4.6, all of $x_i$'s, $y$, $c-x_i$'s, and $c-y$ are different for $i \neq 1$ and 2. We have the following cases:

1. $y = 1$.

   By Lemma 4.4.6, none of $x_i$'s, $c-x_i$'s, and $c-y$ is 1 for all $i$. Then by Lemma 4.5.4, $c - x_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for $i = 3$ and 4. Moreover, $N_p(1) = N_p(-1) = N_q(1) = N_q(-1) + 1 = w + 2$. Hence, by Lemma 2.0.12 for 1, $N_r(-1) = N_r(1) + 1$. Considering Lemma 2.0.12 for each integer, one can show that the weights are

   $$\Sigma_p = \{-c, -b, x, x, x_3, x_4, -1, 1, -1, 1\} \cup \{-1, 1\}^w$$
   $$\Sigma_q = \{c, a, -x, -x, x_3 - c, x_4 - c, c - 1, 1, -1, 1\} \cup \{-1, 1\}^w$$
   $$\Sigma_r = \{-a, b, -x_3, -x_4, c - x_3, c - x_4, 1 - c, -1, -f, f\} \cup \{-1, 1\}^w$$

   for some odd natural number $f$. Suppose that $f > 1$. Then by Lemma 4.4.8, $c = 2x = 2f$, which is a contradiction since no additional multiples of $x$ should appear by Lemma 4.2.6 part 3 for $x$. Hence $f = 1$. However, this means that $\min_{\alpha \in M^{S^1}} \min\{N_\alpha(-1), N_\alpha(1)\} \geq w + 1$, which is a contradiction.

2. $c - x_i = 1$ for some $i$.

   Since $2x_1 = 2x_2 = c \geq 4$, $c - x_1 = c - x_2 = x_1 \geq 2$. Hence, without loss of generality, assume that $c - x_4 = 1$. By Lemma 4.4.6, none of $x_i$'s, $y$, $c-x_j$'s, and $c-y$ can be 1 for all $i$ and $j \neq 4$. Thus $y \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ by Lemma 4.5.5 and $c - x_3 \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ by Lemma 4.5.4. Moreover, $N_p(1) = N_p(-1) + 1 = N_q(1) = N_q(-1) = w+2$. Hence, by Lemma 2.0.12 for 1, $N_r(-1) = N_r(1)+1$. Considering Lemma 2.0.12 for each integer, one can show that the weights are

   $$\Sigma_p = \{-c, b, x, x, x_3, c - 1, -y, 1, -1, 1\} \cup \{-1, 1\}^w$$
   $$\Sigma_q = \{c, a, -x, -x, x_3 - c, -1, c - y, 1, -1, 1\} \cup \{-1, 1\}^w$$
   $$\Sigma_r = \{-a, b, -x_3, 1 - c, y, c - x_3, y - c, -1, -f, f\} \cup \{-1, 1\}^w$$

   for some odd natural number $f$. Then as above, $f = 1$, which is a contradiction since this means that $\min_{\alpha \in M^{S^1}} \min\{N_\alpha(-1), N_\alpha(1)\} \geq w + 1$.

3. $y \neq 1$ and $c - x_i \neq 1$, for all $i$.

   By Lemma 4.5.5, $y \in \Sigma_r \setminus (\{-a, b\} \cup \{-1,1\}^w)$. Also, $c - x_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1,1\}^w)$ for $i = 3$ and $4$ by Lemma 4.5.4. Moreover, $N_p(1) \geq N_p(-1) + 1$ and $N_q(1) \geq N_q(-1) + 1$. Hence, by Lemma 2.0.12 for 1, $N_r(-1) \geq N_r(1) + 2$. Considering Lemma 2.0.12 for each integer, one can show that the weights are

   $$\Sigma_p = \{-c, -b, x, x, x_3, x_4, -y, 1, -1, 1\} \cup \{-1,1\}^w$$
   $$\Sigma_q = \{c, a, -x, -x, x_3 - c, x_4 - c, c - y, 1, -1, 1\} \cup \{-1,1\}^w$$
   $$\Sigma_r = \{-a, b, -x_3, -x_4, c - x_3, c - x_4, y, y - c, -1, -1\} \cup \{-1,1\}^w$$

   By Lemma 4.2.6 part 3 for $x$, no additional multiples of $x$ appear. Hence, $a \neq x$ and $b \neq x$. Since $2x = c = a + b$, either $a > \frac{c}{2}$ or $b > \frac{c}{2}$.

   (a) $a > \frac{c}{2}$.

   By Lemma 4.2.6 part 1 for $a$, $\Sigma_q \equiv \Sigma_r \mod a$. First, we can choose so that

   $$\Sigma_q \supset \{c = a + b, a, -1\} \cup \{-1,1\}^w \equiv \{-a, b, -1\} \cup \{-1,1\}^w \subset \Sigma_r \mod a.$$

   Next, $\{-x, -x\} \subset \Sigma_q$ and $-x \notin \Sigma_r$ imply that $\{a - x, a - x\} \subset \Sigma_r \setminus (\{-a, b, -1\} \cup \{-1,1\}^w)$, $\{2a - x, 2a - x\} \subset \Sigma_r \setminus (\{-a, b, -1\} \cup \{-1,1\}^w)$, or $\{a - x, 2a - x\} \subset \Sigma_r \setminus (\{-a, b, -1\} \cup \{-1,1\}^w)$, since $|-x + ka| > c$ for $k < 0$ or $k > 2$. If the first case or the second case holds, it implies that two of $c - x_3$, $c - x_4$, and $y$ are equal since $c - x_3$, $c - x_4$, and $y$ are the only positive integers in $\Sigma_r \setminus (\{-a, b, -1\} \cup \{-1,1\}^w)$, which is a contradiction by Lemma 4.4.6. Hence we have that $\{a - x, 2a - x\} \subset \Sigma_r \setminus (\{-a, b, -1\} \cup \{-1,1\}^w)$. Similarly, $N_q(1) = w + 2$, $N_r(1) = w$, and $\Sigma_q \equiv \Sigma_r \mod a$ imply that either $\{1 + a, 1 + a\} \subset \Sigma_r \setminus (\{-a, b, -1\} \cup \{-1,1\}^w)$, $\{1 - a, 1 - a\} \subset \Sigma_r \setminus (\{-a, b, -1\} \cup \{-1,1\}^w)$, or $\{1 - a, 1 + a\} \subset \Sigma_r \setminus (\{-a, b, -1\} \cup \{-1,1\}^w)$, since $|1 + ka| > c$ for $|k| \geq 2$. If the first case holds, $-2(a + 1) \in \Sigma_r$ by Lemma 4.2.6 part 3 for $a + 1$, which is a contradiction since $-2(a + 1) < -c$ where $-c$ is the smallest weight. If the second case holds, $2(a - 1) \in \Sigma_r$ by Lemma 4.2.6 part 3 for $1 - a$, which is a contradiction since

$2(a-1) \geq 2x = c$ but $c \notin \Sigma_r$. Hence, the third case must be the case.

To sum up, we have $\{1-a, 1+a, a-x, 2a-x\} \subset \{-x_3, -x_4, y, c-x_3, c-x_4, y-c\} = \Sigma_r \setminus (\{-a, b, -1\} \cup \{-1, 1\}^w)$, i.e., $1-a \in \{-x_3, -x_4, y-c\}$ and $\{1+a, a-x, 2a-x\} = \{y, c-x_3, c-x_4\}$. For each $\alpha \in \{-x_3, -x_4, y-c\}$, we have that $\alpha + c \in \{y, c-x_3, c-x_4\}$. This implies that $1-a+c \in \{y, c-x_3, c-x_4\} = \{1+a, a-x, 2a-x\}$. If $1-a+c = 1+a$, then $1-a+c = 1-a+2x = 1+a$, hence $2x = 2a$, which contradicts that $a > x$. If $1-a+c = 2a-x$, then $1-a+c = 1-a+2x = 2a-x$, hence $3x+1 = 3a$, which is a contradiction since $a > x$. Hence $1-a+c = a-x$. Then we have that $c+1 = 2a-x$, which is a contradiction since $2a-x \in \{y, c-x_3, c-x_4\} \subset \Sigma_r$ but $2a-x = c+1 > c$, where $c$ is the largest weight.

(b) $b > \frac{c}{2}$.

By Lemma 4.2.6 part 1 for $b$, $\Sigma_p \equiv \Sigma_r \mod b$. First, we can choose so that

$$\begin{aligned} \Sigma_p &\supset \{-c = -a-b, -b, -1\} \cup \{-1, 1\}^w \\ &\equiv \{-a, b, -1\} \cup \{-1, 1\}^w \subset \Sigma_r \mod b. \end{aligned}$$

Next, as above, one can show that $\{b-x, 2b-x, b+1\} = \{y, c-x_3, c-x_4\}$. This implies that $b-x+2b-x+b+1 = 4b-2x+1 = 2c+y-x_3-x_4 = y+c-x_3+c-x_4$. Therefore, $4b-2x+1 = 2c+y-x_3-x_4 = 4x+y-x_3-x_4$, hence $4b+1 = 6x+y-x_3-x_4$. On the other hand, by Corollary 2.0.14, $c_1(M)|_p = b+x_3+x_4-y+1 = 0$. Thus $y-x_3-x_4 = b+1$. Then we have that $4b+1 = 6x+y-x_3-x_4 = 6x+b+1$, hence $3b = 6x$, which is a contradiction since $b < 2x = c$.

□

**Lemma 4.6.6.** *In Lemma 4.5.1, $t = 1$ and $v = 2$ are impossible.*

*Proof.* The weights in this case are

$$\begin{aligned} \Sigma_p &= \{-c, -b, x_1, x_2, x_3, x_4, -y, 1\} \cup \{-1, 1\}^{w+2} \\ \Sigma_q &= \{c, a, x_1-c, x_2-c, x_3-c, x_4-c, c-y, 1\} \cup \{-1, 1\}^{w+2} \\ \Sigma_r &= \{-a, b, \cdots\} \cup \{-1, 1\}^w, \end{aligned}$$

where $a, b$, and $c$ are even natural numbers such that $c = a + b$ is the largest weight, $x_i$'s and $y$ are odd natural numbers for all $i$, and $w = \min_{\alpha \in M^{S^1}} \min\{N_\alpha(-1), N_\alpha(1)\}$. Moreover, the remaining weights at $r$ are odd. By Lemma 4.5.3, $y - c \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$. We have the following cases:

1. $x_i = c - x_j$ for some $i$ and $j$.

   By Lemma 4.4.4, there exist $x_i$ and $x_j$ where $i \neq j$ such that $2x_i = 2x_j = c$. Without loss of generality, let $2x_1 = 2x_2 = c$. Denote $x = x_1$. Lemma 4.4.4 also implies that $x_i \neq c - x_j$ for $i \neq 1$ and 2, and for all $j$. Therefore, $-x_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for $i = 3$ and 4 by Lemma 4.5.2. Moreover, by Lemma 4.2.6 part 3 for $x$, none of $x_i$'s, $y$, $c - x_i$'s, and $c - y$ can be $x$ for $i \neq 1$ and 2. Hence, by Lemma 4.4.6, all of $x_i$'s, $y$, $c - x_i$'s, and $c - y$ are different for $i \neq 1$ and 2.

   First, assume that $y = 1$. Then none of $x_i$'s, $c - x_i$'s, and $c - y$ is 1 for all $i$. Hence $c - x_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for $i = 3$ and 4 by Lemma 4.5.4. Moreover, $N_p(1) = N_p(-1) = N_q(1) = N_q(-1) + 1 = w + 3$. Hence, by Lemma 2.0.12 for 1, $N_r(-1) = N_r(1) + 1$. Considering Lemma 2.0.12 for each integer, one can show that the weights are

   $$\Sigma_p = \{-c, -b, x, x, x_3, x_4, -1, 1, -1, 1, -1, 1\} \cup \{-1, 1\}^w$$
   $$\Sigma_q = \{c, a, -x, -x, x_3 - c, x_4 - c, c - 1, 1, -1, 1, -1, 1\} \cup \{-1, 1\}^w$$
   $$\Sigma_r = \{-a, b, -x_3, -x_4, c - x_3, c - x_4, 1 - c, -1, -f, f, -h, h\} \cup \{-1, 1\}^w$$

   for some odd natural numbers $f$ and $h$. Suppose that $f > 1$. Then by Lemma 4.4.8, $c = 2x = 2f$, which contradicts Lemma 4.2.6 part 3 for $x$. Hence $f = 1$, which is a contradiction since it means that $\min_{\alpha \in M^{S^1}} \min\{N_\alpha(-1), N_\alpha(1)\} \geq w + 1$.

   Second, assume that $c - x_i = 1$ for some $i$. Since $c - x_1 = c - x_2 = x_1 = \frac{c}{2} \geq 2$, without loss of generality, let $c - x_4 = 1$. Then none of $x_i$'s, $c - x_j$'s, $y$, and $c - y$ is 1 for all $i$ and $j \neq 4$. Thus $y \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ by Lemma 4.5.5 and $c - x_3 \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ by Lemma 4.5.4. Moreover, $N_p(1) = N_p(-1) + 1 = N_q(1) = N_q(-1) = w + 3$. Hence, by Lemma 2.0.12 for 1, $N_r(-1) = N_r(1) + 1$. Considering Lemma 2.0.12 for each integer, one can show that the weights are

$$\Sigma_p = \{-c, -b, x, x, x_3, c-1, -y, 1, -1, 1, -1, 1\} \cup \{-1, 1\}^w$$

$$\Sigma_q = \{c, a, -x, -x, x_3 - c, -1, c-y, 1, -1, 1, -1, 1\} \cup \{-1, 1\}^w$$

$$\Sigma_r = \{-a, b, -x_3, 1-c, y, c-x_3, y-c, -1, -f, f, -h, h\} \cup \{-1, 1\}^w$$

for some odd natural numbers $f$ and $h$. As above, $f = 1$, which is a contradiction since it means that $\min_{\alpha \in M^{S^1}} \min\{N_\alpha(-1), N_\alpha(1)\} \geq w+1$.

Last, assume that $y \neq 1$ and $c - x_i \neq 1$ for all $i$. By Lemma 4.5.5, $y \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$. Also, $c - x_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for $i = 3$ and $4$ by Lemma 4.5.4. Moreover, $N_p(1) \geq N_p(-1) + 1$ and $N_q(1) \geq N_q(-1) + 1$. Hence, by Lemma 2.0.12 for 1, $N_r(-1) \geq N_r(1) + 2$. Considering Lemma 2.0.12 for each integer, one can show that the weights are

$$\Sigma_p = \{-c, -b, x, x, x_3, x_4, -y, 1, -1, 1, -1, 1\} \cup \{-1, 1\}^w$$

$$\Sigma_q = \{c, a, -x, -x, x_3 - c, x_4 - c, c-y, 1, -1, 1, -1, 1\} \cup \{-1, 1\}^w$$

$$\Sigma_r = \{-a, b, -x_3, -x_4, y, c-x_3, c-x_4, y-c, -1, -1, -f, f\} \cup \{-1, 1\}^w$$

for some odd natural number $f$. As above, $f = 1$, which is a contradiction since it means that $\min_{\alpha \in M^{S^1}} \min\{N_\alpha(-1), N_\alpha(1)\} \geq w+1$.

2. $x_i \neq c - x_j$, for all $i$ and $j$.

   By Lemma 4.5.2, $-x_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for all $i$. Also, by Lemma 4.5.3, $y - c \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$.

   We show that $x_i \neq x_j$ for $i \neq j$. Suppose that $x_i = x_j$ for some $i \neq j$. Then by Lemma 4.4.6, $2x_i = 2x_j = c$, hence $x_i = c - x_j$, which contradicts the assumption. Therefore, $x_i \neq x_j$ for $i \neq j$. Moreover, by Lemma 4.4.3, $x_i \neq c - y$ for all $i$ and $j$.

   First, suppose that $y = 1$. By Lemma 4.4.6, none of $x_i$'s, $c - x_i$'s, and $c - y$ is 1. Then by Lemma 4.5.4, $c - x_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for all $i$. Moreover, $N_p(1) = N_p(-1) = N_q(1) = N_q(-1) + 1 = w + 3$. Hence, by Lemma 2.0.12 for 1, $N_r(-1) = N_r(1) + 1$. Hence, the weights are

$$\Sigma_p = \{-c, -b, x_1, x_2, x_3, x_4, -1, 1, -1, 1, -1, 1\} \cup \{-1, 1\}^w$$
$$\Sigma_q = \{c, a, x_1 - c, x_2 - c, x_3 - c, x_4 - c, c - 1, 1, -1, 1, -1, 1\} \cup \{-1, 1\}^w$$
$$\Sigma_r = \{-a, b\} \cup \{-x_i\}_{i=1}^4 \cup \{c - x_i\}_{i=1}^4 \cup \{1 - c, -1\} \cup \{-1, 1\}^w$$

Then we have that $N_p(1) \geq w + 3$, $N_q(1) \geq w + 3$, and $N_r(1) = w$, which is a contradiction by Lemma 4.4.9.

Second, suppose that $c - x_i = 1$ for some $i$. Without loss of generality, assume that $c - x_4 = 1$. By Lemma 4.4.6, none of $x_i$'s, $c - x_j$'s, $y$, and $c - y$ is 1 for all $i$ and $j \neq 4$. Thus $y \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ by Lemma 4.5.5 and $c - x_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for $i \neq 4$ by Lemma 4.5.4. Moreover, $N_p(1) = N_p(-1) + 1 = N_q(1) = N_q(-1) = w + 3$. Therefore, by Lemma 2.0.12 for 1, $N_r(-1) = N_r(1) + 1$. Hence, the weights are

$$\Sigma_p = \{-c, -b, x_1, x_2, x_3, x_4, -y, 1, -1, 1, -1, 1\} \cup \{-1, 1\}^w$$
$$\Sigma_q = \{c, a, x_1 - c, x_2 - c, x_3 - c, -1, c - y, 1, -1, 1, -1, 1\} \cup \{-1, 1\}^w$$
$$\Sigma_r = \{-a, b\} \cup \{-x_i\}_{i=1}^4 \cup \{c - x_i\}_{i=1}^3 \cup \{y, y - c, -1\} \cup \{-1, 1\}^w$$

Then we have that $N_p(1) \geq w + 3$, $N_q(1) \geq w + 3$, and $N_r(1) = w$, which is a contradiction by Lemma 4.4.9.

Finally, suppose that $y \neq 1$ and $c - x_i \neq 1$ for all $i$. By Lemma 4.5.5, $y \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$. Also, $c - x_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for all $i$ by Lemma 4.5.4. Hence, the weights are

$$\Sigma_p = \{-c, -b, x_1, x_2, x_3, x_4, -y, 1, -1, 1, -1, 1\} \cup \{-1, 1\}^w$$
$$\Sigma_q = \{c, a, x_1 - c, x_2 - c, x_3 - c, x_4 - c, c - y, 1, -1, 1, -1, 1\} \cup \{-1, 1\}^w$$
$$\Sigma_r = \{-a, b\} \cup \{-x_i\}_{i=1}^4 \cup \{y\} \cup \{c - x_i\}_{i=1}^4 \cup \{y - c\} \cup \{-1, 1\}^w$$

Then we have that $N_p(1) \geq w + 3$, $N_q(1) \geq w + 3$, and $N_r(1) = w$, which is a contradiction by Lemma 4.4.9.

□

**Lemma 4.6.7.** *In Lemma 4.5.1, $t = 2$ and $v = 1$ are impossible.*

*Proof.* The weights in this case are

$$\Sigma_p = \{-c, -b, x_1, x_2, x_3, x_4, x_5, -y_1, -y_2, 1\} \cup \{-1,1\}^{w+1}$$
$$\Sigma_q = \{c, a, x_1 - c, x_2 - c, x_3 - c, x_4 - c, x_5 - c, c - y_1, c - y_2, 1\} \cup \{-1,1\}^{w+1}$$
$$\Sigma_r = \{-a, b, \cdots\} \cup \{-1,1\}^{w},$$

where $a, b$, and $c$ are even natural numbers such that $c = a + b$ is the largest weight, $x_i$'s and $y_i$'s are odd natural numbers for all $i$, and $w = \min_{\alpha \in M^{S^1}} \min\{N_\alpha(-1), N_\alpha(1)\}$. Moreover, the remaining weights at $r$ are odd.

By Lemma 4.5.6, $x_i = c - x_j$ for some $i$ and $j$. Then by Lemma 4.4.4, there exist $x_i$ and $x_j$ where $i \neq j$ such that $2x_i = 2x_j = c$. Without loss of generality, let $2x_1 = 2x_2 = c$. Denote $x = x_1$. Lemma 4.4.4 also implies that $x_i \neq c - x_j$ for $i \neq 1$ and 2, and for all $j$. Therefore, $-x_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1,1\}^w)$ for $i \neq 1$ and 2 by Lemma 4.5.2. Also, by Lemma 4.5.3, $y_i - c \in \Sigma_r \setminus (\{-a, b\} \cup \{-1,1\}^w)$ for all $i$. Moreover, by Lemma 4.2.6 part 3 for $x$, none of $x_i$'s, $y_j$'s, $c - x_i$'s, and $c - y_j$'s can be $x$ for $i \neq 1$ and 2, and for all $j$. Hence, by Lemma 4.4.6, all of $x_i$'s, $y_j$'s, $c - x_i$'s, and $c - y_j$'s are different for $i \neq 1$ and 2, and for all $j$.

First, suppose that $y_i = 1$ for some $i$. Without loss of generality, let $y_2 = 1$. Then none of $x_i$'s, $c - x_i$'s, $y_1$, and $c - y_i$'s is 1 for all $i$. Then we have that $y_1 \in \Sigma_r \setminus (\{-a, b\} \cup \{-1,1\}^w)$ by Lemma 4.5.5 and $c - x_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1,1\}^w)$ for $i \neq 1$ and 2 by Lemma 4.5.4. Moreover, $N_p(1) = N_p(-1)$ and $N_q(1) = N_q(-1) + 1$. Hence, by Lemma 2.0.12 for 1, $N_r(1) + 1 = N_r(-1)$. Then the weights are

$$\Sigma_p = \{-c, -b, x, x, x_3, x_4, x_5, -y_1, -1, 1, -1, 1\} \cup \{-1,1\}^{w}$$
$$\Sigma_q = \{c, a, -x, -x, x_3 - c, x_4 - c, x_5 - c, c - y_1, c - 1, 1, -1, 1\} \cup \{-1,1\}^{w}$$
$$\Sigma_r = \{-a, b, -x_3, -x_4, -x_5, y_1, c - x_3, c - x_4, c - x_5, y_1 - c, 1 - c, -1\} \cup \{-1,1\}^{w}$$

Then $\Sigma_q \equiv \Sigma_r \mod c-1$ by Lemma 4.2.6 part 1 for $c-1$. First, $|1+k(c-1)| > c$ for $|k| \geq 2$. Also, $c \notin \Sigma_r$. Then, $N_q(1) = w+2$, $N_r(1) = w$, and $\Sigma_q \equiv \Sigma_r \mod c-1$ imply that $N_r(2-c) = 2$, which is a contradiction since $r$ has only one negative even weight.

Second, suppose that $c - x_i = 1$ for some $i$. Since $c - x_1 = c - x_2 = x_1 = \frac{c}{2} \geq 2$, without loss of generality, let $c - x_5 = 1$. Then none of $x_i$'s, $c - x_j$'s, $y_i$'s, and $c - y_i$'s is 1 for all $i$ and $j \neq 5$. Therefore, we have that $y_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1,1\}^w)$ for all $i$ by Lemma 4.5.5 and $c - x_j \in \Sigma_r \setminus (\{-a, b\} \cup \{-1,1\}^w)$

for $j = 3$ and $4$ by Lemma 4.5.4. Moreover, $N_p(1) = N_p(-1) + 1$ and $N_q(1) = N_q(-1)$. Hence, by Lemma 2.0.12 for 1, $N_r(1) + 1 = N_r(-1)$. Then the weights are

$$\Sigma_p = \{-c, -b, x, x, x_3, x_4, c-1, -y_1, -y_2, 1, -1, 1\} \cup \{-1, 1\}^w$$
$$\Sigma_q = \{c, a, -x, -x, x_3 - c, x_4 - c, -1, c - y_1, c - y_2, 1, -1, 1\} \cup \{-1, 1\}^w$$
$$\Sigma_r = \{-a, b, -x_3, -x_4, 1-c, y_1, y_2, c-x_3, c-x_4, y_1 - c, y_2 - c, -1\} \cup \{-1, 1\}^w$$

Then $\Sigma_p \equiv \Sigma_r \mod c-1$ by Lemma 4.2.6 part 1 for $c-1$. As above, $N_p(1) = w+2$, $N_r(1) = w$, and $\Sigma_p \equiv \Sigma_r \mod c-1$ imply that $N_r(2-c) = 2$, which is a contradiction since $r$ has only one negative even weight.

Last, suppose that $c - x_i \neq 1$ and $y_i \neq 1$ for all $i$. Then we have that $y_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for all $i$ by Lemma 4.5.5 and $c - x_j \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for $j \neq 1$ and 2 by Lemma 4.5.4. Then the weights are

$$\Sigma_p = \{-c, -b, x, x, x_3, x_4, x_5, -y_1, -y_2, 1, -1, 1\} \cup \{-1, 1\}^w$$
$$\Sigma_q = \{c, a, -x, -x, x_3 - c, x_4 - c, x_5 - c, c - y_1, c - y_2, 1, -1, 1\} \cup \{-1, 1\}^w$$
$$\Sigma_r = \{-a, b, -x_3, -x_4, -x_5, y_1, y_2, c-x_3, c-x_4, c-x_5, y_1 - c, y_2 - c\} \cup \{-1, 1\}^w$$

Then we have that $\lambda_r = \frac{1}{2}\dim M$, which contradicts Lemma 4.1.4 that $\lambda_r = \frac{1}{2}\dim M + 2$. $\square$

**Lemma 4.6.8.** *In Lemma 4.5.1, $t = 2$ and $v = 2$ are impossible.*

*Proof.* In this case, the weights are

$$\Sigma_p = \{-c, -b, x_1, x_2, x_3, x_4, x_5, -y_1, -y_2, 1\} \cup \{-1, 1\}^{w+2}$$
$$\Sigma_q = \{c, a, x_1 - c, x_2 - c, x_3 - c, x_4 - c, x_5 - c, c - y_1, c - y_2, 1\} \cup \{-1, 1\}^{w+2}$$
$$\Sigma_r = \{-a, b, \cdots\} \cup \{-1, 1\}^w,$$

where $a, b$, and $c$ are even natural numbers such that $c = a + b$ is the largest weight, $x_i$'s and $y_i$'s are odd natural numbers for all $i$, and $w = \min_{\alpha \in M^{S^1}} \min\{N_\alpha(-1), N_\alpha(1)\}$. Moreover, the remaining weights at $r$ are odd.

By Lemma 4.5.6, $x_i = c - x_j$ for some $i$ and $j$. Then by Lemma 4.4.4, there exist $x_i$ and $x_j$ where $i \neq j$ such that $2x_i = 2x_j = c$. Without loss of generality, let $2x_1 = 2x_2 = c$. Denote $x = x_1$. Lemma 4.4.4 also implies that $x_i \neq c - x_j$ for $i \neq 1$ and 2, and for all $j$. Therefore, $-x_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for $i \neq 1$ and 2 by Lemma 4.5.2. Also, by Lemma 4.5.3, $y_i - c \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for all $i$. Moreover, by Lemma 4.2.6 part

3 for $x$, none of $x_i$'s, $y_j$'s, $c-x_i$'s, and $c-y_j$'s can be $x$ for $i \neq 1$ and 2, and for all $j$. Hence, by Lemma 4.4.6, all of $x_i$'s, $y_j$'s, $c-x_i$'s, and $c-y_j$'s are different for $i \neq 1$ and 2, and for all $j$.

First, suppose that $y_i = 1$ for some $i$. Without loss of generality, let $y_2 = 1$. Then none of $x_i$'s, $c-x_i$'s, $y_1$, and $c-y_i$'s is 1 for all $i$. Therefore, we have that $y_1 \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ by Lemma 4.5.5 and $c - x_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for $i \neq 1$ and 2 by Lemma 4.5.4. Moreover, $N_p(1) = N_p(-1)$ and $N_q(1) = N_q(-1) + 1$. Hence, by Lemma 2.0.12 for 1, $N_r(1) + 1 = N_r(-1)$. Considering Lemma 2.0.12 for each integer, the weights are

$$\Sigma_p = \{-c, -b, x, x, x_3, x_4, x_5, -y_1, -1, 1, -1, 1, -1, 1\} \cup \{-1, 1\}^w$$
$$\Sigma_q = \{c, a, -x, -x, x_3 - c, x_4 - c, x_5 - c, c - y_1, c - 1, 1, -1, 1, -1, 1\} \cup \{-1, 1\}^w$$
$$\Sigma_r = \{-a, b\} \cup \{-x_i\}_{i=3}^5 \cup \{y_1\} \cup \{c - x_i\}_{i=3}^5 \cup \{y_i - c\}_{i=1}^2 \cup \{-1, -f, f\} \cup \{-1, 1\}^w$$

for some odd natural number $f$. If $f > 1$, by Lemma 4.4.8, $c = 2x = 2f$, which is a contradiction that no additional multiples of $x$ should appear as weights by Lemma 4.2.6 part 3 for $x$. Hence $f = 1$, which is a contradiction since it means that $\min_{\alpha \in M^{S^1}} \min\{N_\alpha(-1), N_\alpha(1)\} \geq w + 1$.

Second, suppose that $c - x_i = 1$ for some $i$. Since $c - x_1 = c - x_2 = x_1 = \frac{c}{2} \geq 2$, without loss of generality, let $c - x_5 = 1$. Then none of $x_i$'s, $c - x_j$'s, $y_i$'s, and $c - y_i$'s is 1 for all $i$ and $j \neq 5$. Therefore, we have that $y_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for all $i$ by Lemma 4.5.5 and $c - x_j \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for $j = 3$ and 4 by Lemma 4.5.4. Moreover, $N_p(1) = N_p(-1)$ and $N_q(1) = N_q(-1) + 1$. Hence, by Lemma 2.0.12 for 1, $N_r(1) + 1 = N_r(-1)$. Then the weights are

$$\Sigma_p = \{-c, -b, x, x, x_3, x_4, c - 1, -y_1, -y_2, 1, -1, 1, -1, 1\} \cup \{-1, 1\}^w$$
$$\Sigma_q = \{c, a, -x, -x, x_3 - c, x_4 - c, -1, c - y_1, c - y_2, 1, -1, 1, -1, 1\} \cup \{-1, 1\}^w$$
$$\Sigma_r =$$
$$\{-a, b\} \cup \{-x_i\}_{i=3}^5 \cup \{y_i\}_{i=1}^2 \cup \{c - x_i\}_{i=3}^4 \cup \{y_i - c\}_{i=1}^2 \cup \{-1, -f, f\} \cup \{-1, 1\}^w$$

for some odd natural number $f$. As above, $f = 1$, which is a contradiction since it means that $\min_{\alpha \in M^{S^1}} \min\{N_\alpha(-1), N_\alpha(1)\} \geq w + 1$.

Last, suppose that $c - x_i \neq 1$ and $y_i \neq 1$ for all $i$. Then we have that $y_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for all $i$ by Lemma 4.5.5 and $c - x_j \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for $j \neq 1$ and 2 by Lemma 4.5.4. Since $N_p(1) \geq N_p(-1) + 1$ and $N_q(1) \geq N_q(-1) + 1$, by Lemma 2.0.12 for 1, $N_r(1) + 2 \leq N_r(-1)$. Then the weights are

$$\Sigma_p = \{-c, -b, x, x, x_3, x_4, x_5, -y_1, -y_2, 1, -1, 1, -1, 1\} \cup \{-1, 1\}^w$$
$$\Sigma_q = \{c, a, -x, -x, x_3-c, x_4-c, x_5-c, c-y_1, c-y_2, 1, -1, 1, -1, 1\} \cup \{-1, 1\}^w$$
$$\Sigma_r = \{-a, b\} \cup \{-x_i\}_{i=3}^{5} \cup \{y_i\}_{i=1}^{2} \cup \{c-x_i\}_{i=3}^{5} \cup \{y_i-c\}_{i=1}^{2} \cup \{-1, -1\} \cup \{-1, 1\}^w$$

Then we have that $N_p(1) \geq w+3$, $N_q(1) \geq w+3$, and $N_r(1) = w$, which is a contradiction by Lemma 4.4.9. $\square$

**Lemma 4.6.9.** *In Lemma 4.5.1, $t = 3$ and $v = 2$ are impossible.*

*Proof.* In this case, the weights

$$\Sigma_p = \{-c, -b\} \cup \{x_i\}_{i=1}^{6} \cup \{-y_i\}_{i=1}^{3} \cup \{1\} \cup \{-1, 1\}^{w+2}$$
$$\Sigma_q = \{c, a\} \cup \{x_i - c\}_{i=1}^{6} \cup \{c - y_i\}_{i=1}^{3} \cup \{1\} \cup \{-1, 1\}^{w+2}$$
$$\Sigma_r = \{-a, b, \cdots\} \cup \{-1, 1\}^w,$$

where $a, b$, and $c$ are even natural numbers such that $c = a + b$ is the largest weight, $x_i$'s and $y_i$'s are odd natural numbers for all $i$, and $w = \min_{\alpha \in M^{S^1}} \min\{N_\alpha(-1), N_\alpha(1)\}$. Moreover, the remaining weights at $r$ are odd.

By Lemma 4.5.6, $x_i = c - x_j$ for some $i$ and $j$. Then by Lemma 4.4.4, there exist $x_i$ and $x_j$ where $i \neq j$ such that $2x_i = 2x_j = c$. Without loss of generality, let $2x_1 = 2x_2 = c$. Denote $x = x_1$. Lemma 4.4.4 also implies that $x_i \neq c - x_j$ for $i \neq 1$ and 2, and for all $j$. Therefore, $-x_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for $i \neq 1$ and 2 by Lemma 4.5.2. Also, by Lemma 4.5.3, $y_i - c \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for all $i$. Moreover, by Lemma 4.2.6 part 3 for $x$, none of $x_i$'s, $y_j$'s, $c - x_i$'s, and $c - y_j$'s can be $x$ for $i \neq 1$ and 2, and for all $j$. Hence, by Lemma 4.4.6, all of $x_i$'s, $y_j$'s, $c - x_i$'s, and $c - y_j$'s are different for $i \neq 1$ and 2, and for all $j$.

First, suppose that $y_i = 1$ for some $i$. Without loss of generality, let $y_3 = 1$. Then none of $x_i$'s, $c - x_i$'s, $y_j$'s, and $c - y_i$'s is 1 for all $i$ and $j \neq 3$. Therefore, $y_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for $i \neq 3$ by Lemma 4.5.5 and $c - x_j \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for $j \neq 1$ and 2 by Lemma 4.5.4. Moreover, $N_p(1) = N_p(-1)$ and $N_q(1) = N_q(-1) + 1$. Hence, by Lemma 2.0.12 for 1, $N_r(1) + 1 = N_r(-1)$. Then the weights are

$$\Sigma_p = \{-c, -b, x, x, x_3, x_4, x_5, x_6, -y_1, -y_2, -1, 1\} \cup \{-1, 1\}^{w+2}$$
$$\Sigma_q = \{c, a, -x, -x, x_3-c, x_4-c, x_5-c, x_6-c, c-y_1, c-y_2, c-1, 1\} \cup \{-1, 1\}^{w+2}$$
$$\Sigma_r = \{-a, b\} \cup \{-x_i\}_{i=3}^{6} \cup \{y_i\}_{i=1}^{2} \cup \{c-x_i\}_{i=3}^{6} \cup \{y_i-c\}_{i=1}^{3} \cup \{-1\} \cup \{-1, 1\}^w$$

Then we have that $N_p(1) = w + 3$, $N_q(1) = w + 3$, and $N_r(1) = w$, which is a contradiction by Lemma 4.4.9.

Second, suppose that $c - x_i = 1$ for some $i$. Since $c - x_1 = c - x_2 = x_1 = \frac{c}{2} \geq 2$, without loss of generality, let $c - x_6 = 1$. Then none of $x_i$'s, $c - x_j$'s, $y_i$'s, and $c - y_i$'s is 1 for all $i$ and $j \neq 6$. Therefore, $y_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for all $i$ by Lemma 4.5.5 and $c - x_j \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for $j = 3, 4,$ and 5 by Lemma 4.5.4. Moreover, $N_p(1) = N_p(-1) + 1$ and $N_q(1) = N_q(-1)$. Hence, by Lemma 2.0.12 for 1, $N_r(1) + 1 = N_r(-1)$. Then the weights are

$$\Sigma_p = \{-c, -b, x, x, x_3, x_4, x_5, c - 1, -y_1, -y_2, -y_3, 1\} \cup \{-1, 1\}^{w+2}$$
$$\Sigma_q = \{c, a, -x, -x, x_3 - c, x_4 - c, x_5 - c, -1, c - y_1, c - y_2, c - y_3, 1\} \cup \{-1, 1\}^{w+2}$$
$$\Sigma_r = \{-a, b\} \cup \{-x_i\}_{i=3}^{6} \cup \{y_i\}_{i=1}^{3} \cup \{c - x_i\}_{i=3}^{5} \cup \{y_i - c\}_{i=1}^{3} \cup \{-1\} \cup \{-1, 1\}^w$$

Then we have that $N_p(1) = w + 3$, $N_q(1) = w + 3$, and $N_r(1) = w$, which is a contradiction by Lemma 4.4.9.

Last, suppose that $c - x_i \neq 1$ and $y_i \neq 1$ for all $i$. Then we have that $y_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for all $i$ by Lemma 4.5.5 and $c - x_j \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for $j \neq 1$ and 2 by Lemma 4.5.4. Then the weights are

$$\Sigma_p = \{-c, -b, x, x, x_3, x_4, x_5, x_6, -y_1, -y_2, -y_3, 1\} \cup \{-1, 1\}^{w+2}$$
$$\Sigma_q = \{c, a, -x, -x, x_3 - c, x_4 - c, x_5 - c, x_6 - c, c - y_1, c - y_2, c - y_3, 1\} \cup \{-1, 1\}^{w+2}$$
$$\Sigma_r = \{-a, b\} \cup \{-x_i\}_{i=3}^{6} \cup \{y_i\}_{i=1}^{3} \cup \{c - x_i\}_{i=3}^{6} \cup \{y_i - c\}_{i=1}^{3} \cup \{-1, 1\}^w$$

Then we have that $N_p(1) \geq w + 3$, $N_q(1) \geq w + 3$, and $N_r(1) = w$, which is a contradiction by Lemma 4.4.9. $\square$

**Lemma 4.6.10.** *In Lemma 4.5.1, $t \geq v + 2$ is impossible.*

*Proof.* By Lemma 4.5.6, $x_i = c - x_j$ for some $i$ and $j$. Then by Lemma 4.4.4, there exist $x_i$ and $x_j$ where $i \neq j$ such that $2x_i = 2x_j = c$. Without loss of generality, let $2x_1 = 2x_2 = c$. Lemma 4.4.4 also implies that $x_i \neq c - x_j$ for $i \neq 1$ and 2, and for all $j$. Therefore, $-x_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for $i \neq 1$ and 2 by Lemma 4.5.2. Also, by Lemma 4.5.3, $y_i - c \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for all $i$. Moreover, by Lemma 4.2.6 part 3 for $x$, none of $x_i$'s, $y_j$'s, $c - x_i$'s, and $c - y_j$'s can be $x$ for $i \neq 1$ and 2, and for all $j$. Hence, by Lemma 4.4.6, all of $x_i$'s, $y_j$'s, $c - x_i$'s, and $c - y_j$'s are different for $i \neq 1$ and 2, and for all $j$.

First, suppose that $c - x_i = 1$ for some $i$. Since $c - x_1 = c - x_2 = x_1 = \frac{c}{2} \geq 2$, without loss of generality, let $c - x_3 = 1$. Then, by Lemma 4.4.6, $c - x_i \neq 1$ for

$i \neq 3$ and $y_j \neq 1$ for all $j$. Then $c - x_i \in \Sigma_r(\{-a, b\} \cup \{-1, 1\}^w)$ for $i \neq 1, 2,$ and 3 by Lemma 4.5.4, and $y_j \in \Sigma_(\{-a, b\} \cup \{-1, 1\}^w)$ for all $j$ by Lemma 4.5.3. Therefore, we have that $\{-x_i\}_{i=3}^{t+3} \cup \{y_i\}_{i=1}^{t} \cup \{c - x_i\}_{i=4}^{t+3} \cup \{y_i - c\}_{i=1}^{t} \subset \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$, which is a contradiction since there are $2t+4+2u+2v$ spaces in $\Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ but $4t + 1 > 4t \geq 2t + 4 + 2u + 2v$ by the assumption.

Second, suppose that $y_i = 1$ for some $i$. Without loss of generality, let $y_1 = 1$. Then, by Lemma 4.4.6, $c - x_i \neq 1$ for all $i$ and $y_j \neq 1$ for $j \neq 1$. Then $c - x_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for $i \neq 1$ and 2 by Lemma 4.5.4, and $y_j \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for $j \neq 1$ by Lemma 4.5.3. Therefore, we have that $\{-x_i\}_{i=3}^{t+3} \cup \{y_i\}_{i=2}^{t} \cup \{c - x_i\}_{i=3}^{t+3} \cup \{y_i - c\}_{i=1}^{t} \subset \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$, which is a contradiction since there are $2t+4+2u+2v$ spaces in $\Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ but $4t + 1 > 4t \geq 2t + 4 + 2u + 2v$ by the assumption.

Last, suppose that $c - x_i \neq 1$ and $y_i \neq 1$ for all $i$. Then $c - x_i \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for $i \neq 1$ and 2 by Lemma 4.5.4, and $y_j \in \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ for all $j$ by Lemma 4.5.3. Therefore, we have that $\{-x_i\}_{i=3}^{t+3} \cup \{y_i\}_{i=1}^{t} \cup \{c - x_i\}_{i=3}^{t+3} \cup \{y_i - c\}_{i=1}^{t} \subset \Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$, which is a contradiction since there are $2t+4+2u+2v$ spaces in $\Sigma_r \setminus (\{-a, b\} \cup \{-1, 1\}^w)$ but $4t + 2 > 4t \geq 2t + 4 + 2u + 2v$ by the assumption. $\square$

**Lemma 4.6.11.** *In Lemma 4.5.1, $v \geq 3$ is impossible.*

*Proof.* First, $\min\{N_p(-1), N_p(1)\} \geq w+3$ and $\min\{N_q(-1), N_q(1)\} \geq w+3$. If $\min\{N_r(-1), N_r(1)\} > w$, then $\min_{\alpha \in M^{S^1}} \min\{N_\alpha(-1), N_\alpha(1)\} \geq w + 1$, which is a contradiction. Hence $\min\{N_r(-1), N_r(1)\} = w$. Then either $N_r(-1) = w$ or $N_r(1) = w$. However, neither case is possible by Lemma 4.4.9. $\square$

# Chapter 5

# References


[AB] M. Atiyah and R. Bott: The moment map and equivariant cohomology. *Topology* **23** (1984) 1-28.

[F] T. Frankel: Fixed points and torsion on Kähler manifolds. *Ann. Math.* **70** (1959), 1–8.

[G] L. Godinho: On certain symplectic circle actions, *J. Symplectic Geom.* **3** (2005), 357-383. MR2198781 (2006k:57074)

[Ka] Y. Karshon: Periodic Hamiltonian flows on four dimensional manifolds. *Mem. Amer. Math. Soc.* **672** (1999). MR 2000c:53113.

[Ko] C. Kosniowski: Holomorphic vector fields with simple isolated zeros. *Math. Ann.* **208** (1974) 171-173.

[L] P. Li: The rigidity of Dolbeault-type operators and symplectic circle actions. *Proc. Amer. Math. Soc.* **251** (2011) 1987-1995.

[MD] D. McDuff: The moment map for circle actions on symplectic manifolds. *J. Geom. Phys.* **5** (1988), no. 2, 149-160.

[PT] A. Pelayo and S. Tolman: Fixed points of symplectic periodic flows. *Ergodic Theory and Dynamical Systems* **31** (2011), 1237-1247.

[TW] S. Tolman and J. Weitsman: On semifree symplectic circle actions with isolated fixed points. *Topology* **39** (2000), no. 2, 299–309.

[T1] S. Tolman: On a symplectic generalization of Petrie's conjecture. *Trans. Amer. Math. Soc.* **362** (2010), no.8, 3963–3996. MR2638879

[T2] S. Tolman: Non-Hamiltonian actions with isolated fixed points. In preparation.