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ENTANGLEMENT ENTROPY IN EXCITED STATES

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Abstract

Negli ultimi anni l'entropia di entaglement è stata ampiamente studiata nel campo dell'integrabilità. Con l'introduzione del modello a replica è stato possibile portare alla luce le proprietà universali dell' entropia di entanglement di un sistema bipartito nello stato di vuoto.
In questa tesi si è investigato il problema dell'entropia di entanglement di un sistema bipartito in uno stato eccitato di singola particella. In particolare, si è considerata una teoria bosonica libera in un volume finito, in modo da sfruttare al meglio le tecniche dell'integrabilità. Nel corso di questa analisi, è stato possibile rielaborare il modello a replica in un volume finito grazie ad un raddoppiamento della teoria bosonica che ha indotto una simmetria U(1) su ogni copia del modello. Tale tecnica, nota in letteratura come *doubling trick* ha permesso di ricondurre il calcolo dell'entropia di Renyi a un'opportuna espansione in form factors dei campi U(1) implementanti tale simmetria e valutarne il contributo dominante nel limite in cui il volume è grande.
I risultati ottenuti per la *Second Rènyi entropy* mostrano che in tale limite, l'eccesso di entanglement dovuto allo stato eccitato rispetto a quello di vuoto è indipendente dall'energia dello stato stesso e può essere interpretato come quantità che misura la incertezza sulla localizzazione dell'eccitazione nelle due parti di cui è composto il sistema.

Contents

Chapter 1

Introduction

Entanglement is a peculiar phenomenon which occurs between two or more parts of a quantum system. Its most direct consequence is that a local measurement performed on a subsystem may affect the outcome of the other measurements potentially performed far away.

The concept of entanglement has played an important role in the development of quantum physics. In the early days, it was mainly perceived as the qualitative feature that most distinguished the quantum nature from the classical one, introducing a new form of correlation [1]. It is not surprising that the existence of entanglement has been at the centre of the scientific debate for some time (a well-known example is the *EPR paradox* [2]). The subsequent development of the *Bell's inequalites* [3] has definitively made these quantum correlations accessible to experimental verification. The reality of entanglement was confirmed by the experiments of Alain Aspect and collaborators [4] using pair of maximally entangled photons.

Entanglement has attracted attention in both theoretical and technological fields. In the last few decades, the technological progress has permitted to create controllable quantum correlations. From this point of view, entanglement has been viewed as a new resource that permits to deal with tasks that are impossible or inefficient by a classical analysis e.g. the quantum teleportation. This provided the basis for the modern quantum information science and its applications, as for example quantum computation and quantum cryptography [5].

This relevant role in so many branches of physics has raised the quest to develop efficient (theoretical) measures of entanglement. For instance, it is well known in the literature that the *von Neumann entropy* and the *Rényi entropy* provide a good measure of the amount of entanglement that a system in a pure state possesses.

In the context of many-body quantum systems, an active line of research is concerned with the bipartite entanglement entropy of a extended quantum system. A prominent example in one dimension is provided by a *quantum spin chain*: it consists of an infinitely long sequence of atoms that are characterized by their spin. These extended quantum systems in the scalar limit near a critical point show an asymptotic behaviour described by a *Quantum Field Theory (QFT)* (in the Infrared limit) or by a *Conformal Field Theory (CFT)* (in the Ultraviolet limit). From this point of view, the relevance of the entanglement entropy in quantum spin chains arises from the fact that its asymptotic behavior encodes universal properties.

The scaling universal properties of the entanglement entropy have been widely investigated in the ground state. In this context the work of Calabrese and Cardy [6] in CFT has played a crucial role. Beyond criticality, the entanglement entropy may be evaluated by means of the *branch-point twist fields* and by taking advantage of integral model techniques. This approach was introduced in the work of Cardy, Castro-Alvaredo and Doyon [7]. In contrast, the entanglement entropy in excited states is analyzed in the works of Sierra and collaborators [8, 9] that has provided CFT results in the large volume limit.

In this thesis we investigate the entanglement entropy in excited states in a finite-volume QFT. In particular, we consider a free boson theory. This choice is motivated by the simple structure of the theory that allows us to take full advantage of the integrable model techniques i.e. *form factor expansions*. The results can provide the basis on which other integrable models may be implemented.

In our investigation we employ the techniques developed in [7] and extended into finite volume. In order to do this, we take advantage of the *doubling trick* first introduced in the work of Fonseca and Zamolodchikov [10]. This enable us to deal with a doubled theory described by complex boson fields.

The thesis is organized as follows:

- In Chapter 2 we introduce entanglement in a bipartite quantum system, and, we discuss the von Neumann entropy and the Rényi entropy as entanglement measures in bipartite quantum systems in a pure state.

- In Chapter 3 we introduce the *replica trick* and show how this technique may be implemented in order to obtain the entanglement entropy in the ground state. Then we extend this approach to a finite volume where the entanglement entropy in excited states may be evaluated.

- In Chapter 4 we introduce integrability and we see how the form factor expansion

may be implemented to calculate the two-point correlation functions of local fields in both infinite and finite volume.

- In Chapter 5 we analyze the free boson theory by using the *doubling trick*. In particular we see that the doubling of the free boson theory induces a $U(1)$ symmetry to which a twist field can be associated, and, permits to derive the large-volume leading contribution to the Rényi entropy in a one-particle excited state. Furthermore we evaluate the large-volume leading contribution to the second Rényi entropy.

- In Chapter 6 we draw the conclusions of our work and we interpret the results. In addition, we discuss some possible future developments.

The most technical parts of the calculations have been collected in the appendices. Namely:

- In Appendix A we implement the form factor boostrap equations introduced in Chapter 4, in order to evaluate the two-particle form factors of the $U(1)$ fields.

- In Appendix B we calculate the large-volume leading contribution to the second Rényi entropy.

Chapter 2

Entanglement

2.1 Quantum Information

The measure of Entanglement has been of central importance in the field of Quantum Information. Essentially, the latter differs from the classic one by two relevant properties introduced by Quantum Mechanics:

- Superposition principle.
- Entanglement.

As is well-known, the *bit* is the fundamental unit of classical information, it may have two possible values described by the binary number 0 or 1. Quantum information is built upon an analogous concept that is *quantum bit* or *qubit* for short. Even for a qubit there are two states, namely $|0\rangle$ and $|1\rangle$, but now it lives in a *2-dimensional Hilbert space* $\mathcal{H}^{(2)}$ and, thanks to the superposition principle, it is possible to write it as:

$$|\phi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle \tag{2.1}$$

where $|0\rangle$ and $|1\rangle$ form an orthonormal basis that is known as *computational basis*, and, α and β are complex numbers related to the probability to get one of the two states once we measure a qubit. As result of this measurement we could either obtain 0, with probability $|\alpha|^2$, or the result 1, with probability $|\beta|^2$. For this probabilistic interpretation it is natural to assume the normalization $|\alpha|^2 + |\beta|^2 = 1$.

Despite the fact that qubit seems to be a mathematical prototype, it is a "real" object. Indeed, many different physical systems can be used to realize qubits [5]: the two different polarizations of a photon, the two states of an electron orbiting a single atom, the alignment of a nuclear spin in a uniform magnetic field, etc... In particular, in this thesis, we consider a single qubit as a system which represents one spin-$\frac{1}{2}$ particle e.g.

an electron. As consequence, in the previous expression (2.1), the state $|0\rangle$ (or $|1\rangle$) is the state with spin pointing up $|\uparrow\rangle$ (or down $|\downarrow\rangle$).

All we have discussed so far is a strict consequence of the superposition principle. We are interested in the other "innovation" of quantum information that is Entanglement. From an operational point of view [11], the usefulness of entanglement emerges because it permits to overcome the so-called *Local Operation with Classical Communication (LOCC) paradigm.* The latter is related to the technological motivation to exchange information between distantly separated laboratories. This context is known as *distance lab scenario* and involves a multipartite quantum system which consist of various parties that are restricted to act locally in their own sub-systems by performing measurements or , more general, quantum operations (*Local Operation*). Thus, these parties are spatially separated and able to communicate over long distance by using standard telecom technologies (*Classical Communication*) in order to coordinate the action of the different labs. Such implementation is called LOCC and can be thought as a special subset of all physically realizable operation on the global system.
The LOCCs allow us to differentiate classical from the quantum correlation. In fact, in the context of quantum information, classical correlations are defined as those that can be generated by LOCC. As a consequence, a first definition of *non-entangled* or *separable states* can be given as those states that can be generated exclusively by the action of LOCC. This leads to the crucial property that any measure of entanglement must satisfy: its expected value must not increase under LOCC. Thus, if in a quantum system we observe correlations that cannot be simulated classically, then we can label them *quantum correlations.* Such sub-systems that cannot be described by LOCC, can be defined as *entangled* and they require the implementation of non-local operations.
From a theoretical point of view, entanglement is strictly linked to some concepts of statistical mechanics that will be discussed further in the following sections e.g. pure and mixed states, reduced density matrix. The latter leads to a simple way to distinguish between entangled and separated state. It is important to stress that entanglement is a phenomenon that occurs among sub-systems of the same system, therefore it is relevant how these partitions are implemented. Indeed, we will see that the quality of a state being entangled or not may depend on the chosen partition. The elementary partition is provided by the qubit, thus it is necessary to introduce at least *two-qubit state* in order to treat entanglement.

2.2 Pure and Mixed states

Before focusing on entanglement, it will be useful to review some important concepts of quantum statistical mechanics. In particular the interest in the concept of pure/mixed states arises from its connection to that of separable/entangled states.

Essentially, when we look at quantum many-body systems, we have to deal with probability that acts on two levels [12]:

1. the probability due to the uncertainty on the initial state.
2. the probability due to the intrinsic nature of quantum mechanics, which is well described by the quantum mechanics postulates, whose effects cannot be neglected.

As consequence, the difference between pure and mixed states is that the former is not affected by the first type therefore its initial state is well-known with probability equal to 1. An efficient approach to show this is provided by the *density matrix*. The latter will be discussed in the following subsections.

2.2.1 The Density Operator

The *density operator* can be written for a mixed state as:

$$\rho = \sum_k p_k \left|\psi_k\right\rangle \left\langle\psi_k\right| =: \sum_k p_k \rho_k \tag{2.2}$$

where the mixed state is composed by the set $\{\left|\psi_k\right\rangle, p_k\}_k$ and p_k is the probability associated to each state $\left|\psi_k\right\rangle$. Obviously:

$$\sum_k p_k = 1 \tag{2.3}$$

A pure state is obtained when each probability p_k is 0 except for one state $\left|\psi\right\rangle$:

$$\rho = \left|\psi\right\rangle \left\langle\psi\right| \tag{2.4}$$

It is important to remark that a mixed state must not be confused with a superposition of states. The latter is normally defined by an eigenbasis of a certain observable $\mathbf{A}$, and, a coefficient a_j is associated for any components $\left|\phi_j\right\rangle$ of that basis. The physical interpretation of this coefficient a_j lies in the fact that $|a_j|^2$ represents the probability to find the system in $\left|\phi_j\right\rangle$ after a measurement. However the mixed states do not allow any effects of interference (generally associated to the $a_j^* a_j$ terms) that may arise in some amplitudes of the superposition.

The description by means of the density operator leads to many advantages. First of all, it permit to eliminate the ambiguity related to the existence of an arbitrary global phase factor for the quantum states. For instance, the two states $\left|\psi\right\rangle$ and $e^{i\theta}\left|\psi\right\rangle$ (where θ is a real number) are described by the same density operator. Another advantage is that the expectation value can be expressed by a linear formula with respect to the density operator.

2.2.2 Density Matrix of a pure state

Let us consider a normalized pure state $|\psi\rangle$ expanded in the complete orthonormal basis $\{|u_k\rangle\}_k$ (supposed to be discrete) :

$$|\psi\rangle = \sum_k c_k |u_k\rangle \tag{2.5}$$

The density operator can be represented by the *density matrix* with elements:

$$\rho_{ij} = \langle u_i| \rho |u_j\rangle = \langle u_i|\psi\rangle\langle\psi|u_j\rangle = c_i^* c_j \tag{2.6}$$

Properties of the density matrix of a pure state:

1. It is non-negative: $\rho_{ii} = |c_i|^2 \geqslant 0 \quad \forall i$
2. It is self-adjoint: $\rho^\dagger = \rho$
3. Its trace is one: $\mathbf{Tr}\rho = \sum_k \rho_{kk} = \sum_k |c_k|^2 = 1$
4. It acts as a projector operator therefore it satisfies the *idempotent relation*: $\rho^2 = \rho$

Expectation Values: we can now calculate the *expectation value* of a certain observable $\mathbf{A}$, thanks to (2.5):

$$\langle \mathbf{A} \rangle = \langle \psi| \mathbf{A} |\psi\rangle = \sum_{i,j} c_i^* c_j A_{ij} \tag{2.7}$$

where $A_{ij} := \langle u_i| \mathbf{A} |u_j\rangle$. Thus, the property 3 and (2.6) can be rewritten as:

$$\begin{aligned} \langle \mathbf{A} \rangle &= \sum_{i,j} \langle u_i| \rho |u_j\rangle\langle u_j| \mathbf{A} |u_i\rangle \\ &= \sum_i \langle u_i| \rho\mathbf{A} |u_i\rangle \\ &= \mathbf{Tr}\{\rho\mathbf{A}\} \end{aligned} \tag{2.8}$$

2.2.3 Density Matrix of a mixed state

Let us now consider the system described by the mixture of states $\{|\psi_k\rangle, p_k\}_k$. Introducing the orthonormal basis $\{|u_n\rangle\}_n$ we can expand each ψ_k in the adopted basis $\{|u_n\rangle\}$ by using the coefficients:

$$c_n^{(k)} = \langle u_n|\psi_k\rangle \tag{2.9}$$

The diagonal elements of the density matrix can be written as:

$$\rho_{nn} := \langle u_n | \, \rho \, | u_n \rangle = \sum_k p_k |c_n^{(k)}|^2 \tag{2.10}$$

$|c_n^{(k)}|^2$ is obviously a positive real number, whose physical interpretation is the following: if the state of the system is $|\psi_k\rangle$, $|c_n^{(k)}|^2$ represents the probability of finding the system in the state $|u_n\rangle$ after a measurement. However, before measuring, we cannot predict in which state $|\psi_k\rangle$ the system is collapsed. Thus ρ_{nn} can be interpreted as the average probability of finding the system in the state $|u_n\rangle$. For this reason, ρ_{nn} is called the *population* of the state $|u_n\rangle$: if we hypothetically performed the same measurement N times under the same initial conditions and N is a large number, then we would find $N\rho_{nn}$ systems in the state $|u_n\rangle$. It is evident from (2.10) that ρ_{nn} is equal to zero if and only if $|c_n^{(k)}|^2$ is zero for all k.
Similarly, the non-diagonal elements of the density matrix are:

$$\rho_{nm} := \langle u_n | \, \rho \, | u_m \rangle = \sum_k p_k c_n^{(k)} c_m^{(k)*} \tag{2.11}$$

$c_n^{(k)} c_m^{(k)*}$ expresses the interference effects between the state $|u_n\rangle$ and $|u_m\rangle$ which can occur when the state $|\psi_k\rangle$ is a coherent linear superposition of these states. Elements such as (2.11) are called *coherences* and if they are different from 0, it means that a certain coherence between the states of the basis subsists. In contrast to $|c_n^{(k)}|^2$, $c_n^{(k)} c_m^{(k)*}$ is a complex number then ρ_{nm} might be zero even if any $c_n^{(k)} c_m^{(k)*}$ are non-zero. In this case, the effects of interference cancell each other.

Properties of the density matrix of a mixed state:

1. It is non-negative: $\rho_{nn} = \sum_k p_k |c_n^{(k)}|^2 \geqslant 0 \quad \forall n$
2. It is self-adjoint: $\rho^\dagger = \rho$
3. Its trace is one: $\mathbf{Tr}\rho = \sum_n \rho_{nn} = \sum_n \sum_k p_k |c_n^{(k)}|^2 = 1$
4. It does not act as a projector operator therefore it does not satisfy the *idempotent relation*: $\rho^2 \neq \rho$

Thus, pure and mixed states differ by the property 4.

Expectation Value: as in the pure case, we can rewrite the expectation value of a certain observable $\mathbf{A}$ in terms of the density matrix:

$$\begin{aligned}\langle \mathbf{A} \rangle &= \sum_{n,m} \sum_k p_k c_n^{(k)} c_m^{(k)*} = \sum_{n,m} \rho_{n,m} A_{n,m} \\ &= \sum_{n,m} \langle u_m | \, \rho \, | u_n \rangle \langle u_n | \, \mathbf{A} \, | u_m \rangle = \sum_n \langle u_m | \, \rho \mathbf{A} \, | u_m \rangle \\ &= \mathbf{Tr}\{\rho \mathbf{A}\} \end{aligned} \tag{2.12}$$

2.3 A Relevant Example: The Two Qubit

The simplest example of a bipartite system is provided by a two qubit system. In particular we now work in a 4-dimensional Hilbert space $\mathcal{H}^{(4)} = \mathcal{H}_1^{(2)} \otimes \mathcal{H}_2^{(2)}$. Let us adopt the basis $\{|\uparrow_1\uparrow_2\rangle, |\uparrow_1\downarrow_2\rangle, |\downarrow_1\uparrow_2\rangle, |\downarrow_1\downarrow_2\rangle\}$ where:

$$|\uparrow_1\downarrow_2\rangle = |\uparrow_1\rangle \otimes |\downarrow_2\rangle \tag{2.13}$$

Considering the total angular momentum eigenstates it is possible to express *the singlet* and *triplet states* in terms of the basis thanks to *Clebsch-Gordan coefficients*:

$$\begin{aligned} |s_0\rangle &:= |0,0\rangle = \frac{1}{\sqrt{2}}(|\uparrow_1\downarrow_2\rangle - |\downarrow_1\uparrow_2\rangle) \\ |t_1\rangle &:= |1,1\rangle = |\uparrow_1\uparrow_2\rangle \\ |t_0\rangle &:= |1,0\rangle = \frac{1}{\sqrt{2}}(|\uparrow_1\downarrow_2\rangle + |\downarrow_1\uparrow_2\rangle) \\ |t_{-1}\rangle &:= |1,-1\rangle = |\downarrow_1\downarrow_2\rangle \end{aligned} \tag{2.14}$$

States like $|s_0\rangle$ or $|t_0\rangle$ show a peculiar behavior: if we measure the spin of the first particle and obtain a particular result, say $\langle \sigma_1^z \rangle = +\frac{1}{2}$, then the state of the second particle automatically collapses in $|\downarrow_2\rangle$, thus $\langle \sigma_2^z \rangle = -\frac{1}{2}$. States like these for which non-local correlations arise are called *entangled*. On the other hand, performing a measurement on the first particle of $|\uparrow_1\uparrow_2\rangle$ (as well as $|\downarrow_1\downarrow_2\rangle$) does not give any additional information about the second particle therefore states such as the triplets $|t_1\rangle$ and $|t_{-1}\rangle$ are *separable*.

Although this example appears simple and enlightening, this non-local correlation might not be so evident in general. For instance, it is known that [13] a pure bipartite state is not entangled if and only if it can be written as a tensor product of pure states of the parts. However when the partition is composed by an high number of states, it might not be easy to notice it. In addition, when the quantum system is more complex than the two qubits, we have to consider how the partition is realized. A clear example is provided by the three-spin state:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow_1\uparrow_2\downarrow_3\rangle - |\uparrow_1\downarrow_2\uparrow_3\rangle) \tag{2.15}$$

where $|\uparrow_1\uparrow_2\downarrow_3\rangle = |\uparrow_1\rangle \otimes |\uparrow_2\rangle \otimes |\downarrow_3\rangle$. Choosing 1 and $2 \cup 3$ as subsystem leads to a separated state while other partitions, for instance $1 \cup 2$ and 3, lead to an entangled state. Thus, it appears essential to develop a "criterion" in order to distinguish between entangled and separable states and then to quantify the amount of entanglement. This could be done by introducing the *Schmidt decomposition*.

2.4 The Bipartite System

We will consider, in general, a system $A \bigcup B$, described by a pure state such as (2.5), which is divided into two subsystems A and B. As already seen in the two qubit case, the Hilbert space of the total system may be factorized into those of the parts:

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \tag{2.16}$$

Thanks to (2.16), a state $|\psi\rangle \in \mathcal{H}$ can be expressed as:

$$|\psi\rangle = \sum_{i,j} C_{i,j} \, |\phi_i\rangle_A \otimes |\chi_j\rangle_B \tag{2.17}$$

where $\{|\phi_i\rangle_A\}_i$ and $\{|\chi_j\rangle_B\}_j$ are the complete orthonormal bases respectively of $\mathcal{H}_A$ and $\mathcal{H}_B$ and $C_{i,j}$s are the normalized complex coefficients:

$$\sum_{i,j} |C_{i,j}|^2 = 1 \tag{2.18}$$

2.4.1 The Reduced Density Matrix

In order to deal with the subsystems, it is useful to introduce the *reduced density operator* which acts non-trivially only on one of those two parts. For instance, if $\mathcal{O}$ is an operator that is defined only on the partition A, then it can be written as $\mathcal{O} = \mathcal{O}_A \otimes \mathbb{1}_B$. We are interested in the expectation value:

$$\langle\psi| \, \mathcal{O} \, |\psi\rangle = \mathbf{Tr}_{\mathcal{H}}\{\rho\mathcal{O}\} = \langle\psi| \, \mathcal{O}_A \otimes \mathbb{1}_B \, |\psi\rangle \tag{2.19}$$

$$\begin{aligned}\langle\psi|\,\mathcal{O}_A\otimes\mathbb{1}_B\,|\psi\rangle &= \sum_{i_1,i_2,j_1,j_2} C_{i_1,j_1}C^*_{i_2,j_2}\langle\phi_{i_2}|\,\mathcal{O}_A\,|\phi_{i_1}\rangle\langle\chi_{j_2}|\,\mathbb{1}_A\,|\chi_{j_1}\rangle \\ &= \sum_{i_1,i_2}\left(\sum_k C_{i_1,k}C^*_{i_2,k}\right)\langle\phi_{i_2}|\,\mathcal{O}_A\,|\phi_{i_1}\rangle\end{aligned} \tag{2.20}$$

Defining the *reduced density matrix* ρ_A:

$$\langle\phi_{i_1}|\,\rho_A\,|\phi_{i_2}\rangle := \sum_k C_{i_1,k}C^*_{i_2,k} \tag{2.21}$$

we can finally rewrite (2.20) as:

$$\begin{aligned}\langle\psi|\,\mathcal{O}_A\otimes\mathbb{1}_B\,|\psi\rangle &= \sum_{i_1,i_2}\langle\phi_{i_1}|\,\rho\,|\phi_{i_2}\rangle\langle\phi_{i_2}|\,\mathcal{O}_A\,|\phi_{i_1}\rangle \\ &= \sum_i\langle\phi_i|\,\rho_A\mathcal{O}_A\,|\phi_i\rangle = \mathbf{Tr}_{\mathcal{H}_A}\{\rho_A\mathcal{O}_A\}\end{aligned} \tag{2.22}$$

Thus, thanks to the reduced density matrix, it is possible to take into account only one partition. Furthermore ρ_A can be expressed as the trace of the density matrix acting on the Hilbert space $\mathcal{H}_B$:

$$\rho_A = \mathbf{Tr}_{\mathcal{H}_B}\rho \tag{2.23}$$

Indeed:

$$\begin{aligned}\mathbf{Tr}_{\mathcal{H}_B}\rho &= \sum_j\langle\chi_j|\,\rho\,|\chi_j\rangle = \sum_j\langle\chi_j|\psi\rangle\langle\psi|\chi_j\rangle \\ &= \sum_j\langle\chi_j|\left(\sum_{i_1,j_1}C_{i_1,j_1}\,|\phi_{i_1}\rangle\otimes|\chi_{j_1}\rangle\right)\left(\sum_{i_2,j_2}C^*_{i_2,j_2}\langle\phi_{i_2}|\otimes\langle\chi_{j_2}|\right)|\chi_j\rangle \\ &= \sum_{j,i_1,i_2,j_1,j_2}C_{i_1,j_1}C^*_{i_2,j_2}\langle\chi_j|\chi_{j_1}\rangle\langle\chi_{j_2}|\chi_j\rangle\,|\phi_{i_1}\rangle\langle\phi_{i_2}| \\ &= \sum_{i_1,i_2}\left(\sum_j C_{i_1,j}C^*_{i_2,j}\right)|\phi_{i_1}\rangle\langle\phi_{i_2}|\end{aligned} \tag{2.24}$$

Finally :

$$\langle\phi_{i_1}|\,\mathbf{Tr}_{\mathcal{H}_B}\rho\,|\phi_{i_2}\rangle = \sum_j C_{i_1,j}C^*_{i_2,j} = \langle\phi_{i_1}|\,\rho_A\,|\phi_{i_2}\rangle \tag{2.25}$$

Properties of the reduced density matrix ρ_A:

1. It is non-negative: $\langle\alpha|\,\rho_A\,|\alpha\rangle \geqslant 0 \quad \forall\,|\alpha\rangle \in \mathcal{H}_A$
2. It is self-adjoint: $\rho_A^\dagger = \rho_A$
3. Its trace on $\mathcal{H}_A$ is 1: $\mathbf{Tr}_{\mathcal{H}_A}\rho_A = 1$
4. if $\rho_A = |\psi_A\rangle\langle\psi_A|$ then it satisfies the *idempotent relation*: $\rho_A^2 = \rho_A$ (only if $|\psi_A\rangle$ is a pure state)

2.4.2 The Schmidt Decomposition

Let us introduce a new basis of the second partition:

$$|\tilde{\chi}_i\rangle = \sum_j C_{i,j}\,|\chi_j\rangle \tag{2.26}$$

As a consequence, a pure bipartite state such as (2.17) can be rewritten as:

$$|\psi\rangle = \sum_{i,j} C_{i,j}\,|\phi_i\rangle \otimes |\chi_j\rangle = \sum_i |\phi_i\rangle \otimes |\tilde{\chi}_i\rangle \tag{2.27}$$

The properties 1 and 2 of the reduced density matrix ensure that it is always possible to find a complete eigenbasis of ρ_A whose eigenvalues are non-negative. Let us suppose that $\{|\phi\rangle\}_i$ is such an eigenbasis for ρ_A with eigenvalues $\{p_i\}_i$ therefore we can express:

$$\rho_A = \sum_i p_i\,|\phi_i\rangle\langle\phi_i| \tag{2.28}$$

Using (2.21), we obtain:

$$\langle\phi_{i_1}|\,\rho_A\,|\phi_{i_2}\rangle = \sum_i p_i\,\langle\phi_{i_1}|\phi_i\rangle\langle\phi_i|\phi_{i_2}\rangle = p_{i_1}\delta_{i_1,i_2} = \sum_k C_{i_1,k}C^*_{i_2,k} \tag{2.29}$$

An important consequence of property 3 and (2.29) is:

$$\sum_i p_i = 1 \tag{2.30}$$

In addition

$$\langle\tilde{\chi}_{i_1}|\tilde{\chi}_{i_2}\rangle = \sum_{j_1,j_2} C_{i_1,j_1}C^*_{i_2,j_2}\,\langle\chi_{j_1}|\chi_{j_1}\rangle = \sum_k C_{i_1,k}C^*_{i_2,k} = p_{i_1}\delta_{i_1,i_2} \tag{2.31}$$

Thus, we can introduce the new normalized basis of $\mathcal{H}_B$

$$|\bar{\chi}_i\rangle = \frac{1}{\sqrt{p_i}} |\tilde{\chi}_i\rangle \tag{2.32}$$

and finally obtain the *Schimidt decomposition* of the state $|\psi\rangle$:

$$|\psi\rangle = \sum_i \sqrt{p_i} |\phi_i\rangle \otimes |\bar{\chi}_i\rangle \tag{2.33}$$

The reduced density matrix permits to deduce some statements that involve the whole bipartite system. Let us now introduce the number of non-zero eigenvalues p_i of ρ_A, which is called *Schmidt number* and plays an important role in distinguishing between entangled and separable states: **if the Schmidt number is equal to one then the pure bipartite state $|\psi\rangle$ is separable** because it is composed by an only term in (2.33) therefore $|\phi\rangle$ **and** $|\chi\rangle$ **are pure states** (since $p = 1$ from (2.30)).[1] On the other hand **if the Schimidt number is greater than 1 the state $|\psi\rangle$ is entangled** and **the subsystems are described by mixed states**.
It is important to stress that if we previously focused on the complete eigenbasis $|\chi_i\rangle$ of part B, ρ_B would take the same form as ρ_A, and, p_j would be the same coefficients as those in (2.28):

$$\rho_B = \sum_j p_j |\chi_j\rangle\langle\chi_j| \tag{2.34}$$

Indeed, the density operators ρ_A and ρ_B have common spectrum, this means that they are "equally mixed". If they have a different number of eigenvalues, then they can only differ by the number of zero eigenvalues.

Examples of entangled and separable states:

A simple example of maximally entangled states is provided by the *Bell states* [14] that form an orthonormal basis in the two-qubit Hilbert Space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ where 1 and 2 are the two qubits. The Bell states are quite popular in quantum information since they are widely used in both theoretical and experimental works. This basis is composed by the symmetric Bell states $|\Phi^{\pm}\rangle$, $|\Psi^{+}\rangle$ and the anti-symmetric Bell state $|\Psi^{-}\rangle$:[2]

$$\begin{aligned} \left|\Phi^{\pm}\right\rangle &= \frac{1}{\sqrt{2}}\big(|\downarrow_1\downarrow_2\rangle \pm |\uparrow_1\uparrow_2\rangle \big) \\ \left|\Psi^{\pm}\right\rangle &= \frac{1}{\sqrt{2}}\big(|\downarrow_1\uparrow_2\rangle \pm |\uparrow_1\downarrow_2\rangle \big) \end{aligned} \tag{2.35}$$

[1] we already mentioned the following statement: a pure bipartite state is not entangled if and only if it can be written as a tensor product of pure states of the parts.

[2] this symmetry is understood under the exchange of the subsystems $1 \leftrightarrow 2$

where $|\uparrow_1\downarrow_2\rangle = |\uparrow\rangle_1 \otimes |\downarrow\rangle_2$. Let us consider, for example, $|\Phi^{\pm}\rangle$. The density operator ρ is given by:

$$\rho = \left|\Phi^{\pm}\right\rangle\left\langle\Phi^{\pm}\right| = \frac{1}{2}\big(|\downarrow_1\downarrow_2\rangle\langle\downarrow_1\downarrow_2| + |\uparrow_1\uparrow_2\rangle\langle\uparrow_1\uparrow_2| \pm |\uparrow_1\uparrow_2\rangle\langle\downarrow_1\downarrow_2| \pm |\downarrow_1\downarrow_2\rangle\langle\uparrow_1\uparrow_2| \big) \quad (2.36)$$

The reduced density matrix of 2 is obtained by:

$$\begin{aligned} \rho_2 &= \mathbf{Tr}_{\mathcal{H}_1}\rho = {}_1\langle\downarrow|\,\rho\,|\downarrow\rangle_1 + {}_1\langle\uparrow|\,\rho\,|\uparrow\rangle_1 \\ &= \frac{1}{2}\big(|\downarrow\rangle_2\,{}_2\langle\downarrow| + |\uparrow\rangle_2\,{}_2\langle\uparrow| \big) \end{aligned} \quad (2.37)$$

Thus, the subsystems are in a mixed state and $|\Phi^{\pm}\rangle$ is entangled. In an analogous way, we could obtain a similar result for $|\Psi^{\pm}\rangle$.
Another important example is the already mentioned singlet and triplet states. It is easy to show that states such as $|s_o\rangle$ and $|t_o\rangle$ are entangled while states like $|t_1\rangle$ have density matrix:

$$\rho = |t_1\rangle\langle t_1| = \langle\uparrow_1\uparrow_2| \quad (2.38)$$

and the reduced density matrix of 2:

$$\rho_2 = \mathbf{Tr}_{\mathcal{H}_1}\rho = |\uparrow\rangle_2\,{}_2\langle\uparrow| \quad (2.39)$$

Therefore, states like $|t_1\rangle$ and $|t_{-1}\rangle$ are separable.

In general, the Schmidt decomposition of a two-qubit state can be written as:

$$|\psi\rangle = \sqrt{p}\,|\downarrow_1\downarrow_2\rangle + \sqrt{1-p}\,|\uparrow_1\uparrow_2\rangle \quad (2.40)$$

Therefore the reduced density operator of one qubit becomes:

$$\rho_1 = p\,|\uparrow\rangle\langle\uparrow| + (1-p)\,|\downarrow\rangle\langle\downarrow| \quad (2.41)$$

It is clear from (2.41) that the state (2.40) is **maximally entangled** when $p = \frac{1}{2}$ while it is **completely separable** when $p = 0, 1$. Otherwise (2.40) and (2.41) describe the spectrum of all possible intermediate states which are simply entangled. The amount of entanglement these states possess can be quantified by the entanglement entropy that will be introduced in the next section.

2.5 Entanglement Entropy

Entanglement Entropy is a key concept in quantum information: it measures how much uncertainly there is in the state of a physical system. Starting from the *Shannon Entropy*, which is related to classical information, we will extend it to quantum systems by introducing the *Rènyi Entropy* and the *Von-Neumann Entropy*.

2.5.1 The Shannon Entropy

As already mentioned in section 2.2, a quantum system may present an uncertainly on its initial state therefore we can treat the state of the system as a random variable X. As a consequence, we can introduce the *Shannon entropy* which is a quantity that measures the amount of *uncertainty* about X before we learn its value [5]. This allow us to write the Shannon entropy as a function of a classical probability distribution $\mathcal{P}$ that is now described by the set $\{p_k\}_k$. The latter associates to each state $|\psi_k\rangle$ a probabilistic weight p_k (like in (2.2)) that must satisfy the following properties:

1. the sum over all p_k must be equal to one: $\sum_k p_k = 1$
2. the p_ks form a convex combination: $0 \leqslant p_k \leqslant 1 \quad \forall k$

As a result we can use this classical distribution $\mathcal{P}$ to express the expectation values of the observable $\mathcal{O}$:

$$\langle \mathcal{O} \rangle = \sum_k p_k \langle \mathcal{O} \rangle_k \tag{2.42}$$

.

and $\langle \mathcal{O} \rangle_k = \mathbf{Tr}\{\rho_k \mathcal{O}\}$. We can now define the Shannon entropy:

$$H[\mathcal{P}] = H(p_1, p_2, \ldots, p_n) = -\sum_k p_k \log p_k \tag{2.43}$$

where the logarithm is taken on base two therefore the corresponding unity of the Shannon entropy is the bit.[3] For instance, choosing a constant probability distribution $p_k = const$ lead to the maximal value of the Shannon entropy in contrast to the choice $p_k = \delta_{k,j}$ for which (2.43) vanishes.

Following [15], the Shannon entropy satisfies the set of postulates known as the *Fadeev's postulates*:

(a) $H(p_1, p_2, \ldots, p_n)$ is a symmetric function of its variable p_ks.

[3] If we choose another base of the logarithms, the unity changes i.e. for natural logarithms it is the *natural unit of information (nat)*. We will use this unity in the chapter 3 and 5.

(b) $H(p, 1-p)$ is a continuous function of p for $0 \leqslant p \leqslant 1$.

(c) $H(1/2, 1/2) = 1$

(d) $H(tp_1, (1-t)p_1, p_2, \ldots, p_k) = H(p_1, p_2, \ldots, p_k) + p_1 H(t, 1-t)$
for $\mathcal{P} = (p_1, \ldots p_k)$ and for all $\leqslant t \leqslant 1$

It is important to stress that the postulates (a)-(d) characterize the quantity (2.43) uniquely.

2.5.2 The Rènyi Entropy

Let us consider two independent probability distributions, namely $\mathcal{P} = (p_1, \ldots p_n)$ and $\mathcal{Q} = (p_1, \ldots p_m)$. Denoting by $\mathcal{P} * \mathcal{Q}$ the direct product of $\mathcal{P}$ and $\mathcal{Q}$, e.g. the probability distribution with weight $p_j q_k$ for all $j = 1, \ldots n$ and $k = 1, \ldots m$. In particular, it turns out from (2.43):[4]

$$\begin{aligned} H[\mathcal{P} * \mathcal{Q}] &= \sum_{j,k} p_j q_k \log p_j + \sum_{j,k} p_j q_k \log q_j \\ &= \left(\sum_k q_k \right) \sum_j p_j \log p_j + \left(\sum_j p_j \right) \sum_k q_k \log q_k \\ &= H[\mathcal{P}] + H[\mathcal{Q}] \end{aligned} \tag{2.44}$$

The latter is one of the most important properties of the entropy that is called *additivity*. It is essential to remark that one cannot replace the postulate (d) with (2.44) since the latter is much weaker. However it is possible to define another quantity known as *Rènyi entropy of order* α which satisfies the postulate (a), (b), (c) and (2.44):[5]

$$S_\alpha[\mathcal{P}] = S_\alpha(p_1, \ldots, p_n) = \frac{1}{1-\alpha} \log \left(\sum_k p_k^\alpha \right) \tag{2.45}$$

where α is a non-negative real number and $\alpha \neq 1$. Furthermore it can be thought as a generalization of the Shannon entropy since its limit leads to (2.43) when α tends to one:

$$H[\mathcal{P}] = \lim_{\alpha \to 1} S_\alpha[\mathcal{P}] \tag{2.46}$$

[4]It is important to stress that (2.44) holds only if $\mathcal{P}$ and $\mathcal{Q}$ are independent of each other while in general: $H(\mathbb{P} * \mathbb{Q}) \leqslant H(\mathbb{P}) + H(\mathbb{Q})$

[5](2.45) is not the only quantity defined by the postulates (a)-(c) and (2.44).

2.5.3 The Von Neumann Entropy

:

In the context of quantum mechanics, a more general picture is provided by the *von Neumann entropy*. In particular, it generalizes the Shannon entropy, replacing probability distributions with density operators:

$$S(\rho) = -\mathbf{Tr}\{\rho \log \rho\} \tag{2.47}$$

It is clear that (2.43) and (2.47) coincide when we consider a density operator such as (2.2). Furthermore it is possible to re-define the Rényi entropy in the density matrix picture:

$$S_\alpha(\rho) = \frac{1}{1-\alpha} \log \mathbf{Tr}\{\rho^\alpha\} \quad \alpha \in \mathbb{R}^+ \tag{2.48}$$

and, consequently, it can be related to the Von Neumann entropy:

$$S(\rho) = \lim_{\alpha \to 1} S_\alpha(\rho) \tag{2.49}$$

Another way to express (2.47) is provided by the set $\{\lambda_k\}_k$ that consists in the eigenvalues of ρ:

$$S(\rho) = -\sum_k \lambda_k \log \lambda_k \tag{2.50}$$

The Binary Entropy

A particular case of the von Neumann entropy that gains importance in the context of bipartite systems is the *binary entropy*. It describes a system that involves a two out-come random variable. A special example is provided by the two-qubit state such as (2.40) whose reduced density operator ρ_1 of one qubit is (2.41). The binary entropy may be calculated by (2.47) and written as:

$$S_{bin}(p) = S(\rho_1) = -p \log p - (1-p) \log(1-p) \tag{2.51}$$

where p and $1-p$ are the probabilities of the two outcomes e.g. the eigenvalues of ρ_1 in the considered case. It is important to stress that (2.51) is related to one qubit therefore it involves only one partition.

Analogously, we can introduce the *binary Rényi entropy*:

$$S_n^{bin}(p) = \frac{1}{1-n} \log \left(p^2 + (1-p)^2\right) \tag{2.52}$$

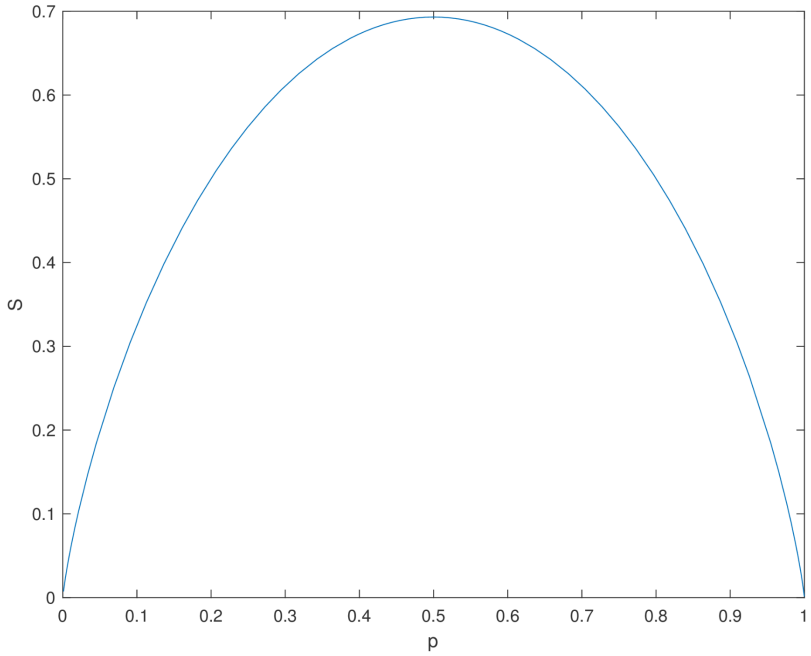

Figure 2.1: S is the binary entropy expressed in *nat*.

The figure 2.1 shows the form of S_{bin} with respect to p: it peaks at $p = \frac{1}{2}$ e.g. the state (2.40) is maximally entangled while it has zeros at $p = 0, 1$ when (2.40) is completely separable.
The equations (2.51) and (2.52) suggest that both the von Neumann entropy and Rényi entropy are good measures of entanglement.

2.6 Entanglement Measure Of A Pure State

Let us now list the main properties of the von Neumann Entropy [5, 16]:

(i) $S(\rho)$ is non-negative. The entropy is zero if and only if the state is pure.

(ii) The entropy is always bounded from above: $S \leqslant \log d$ where d is the dimension of the Hilbert space $\mathcal{H}$ in which ρ is defined. In particular $S(\rho)$ is equal to $\log d$ if and only if ρ describes a mixed state.

(iii) The entropy is invariant under unitary transformations: $S(\rho') = S(U^{-1}\rho U) = S(\rho)$

(iv) The entropy is concave: $S(t\rho_1 + (1-t)\rho_2) \geqslant tS(\rho_1) + (1-t)S(\rho_2)$ for $0 \leqslant t \leqslant 1$

(v) The entropy is sub-addictive: $S(\rho_{A\bigcup B}) \leqslant S(\rho_A) + S(\rho_A)$ (ρ_A is defined by (2.21))

We can compare (i)-(v) with the properties that are required in order to obtain a "good" measure of entanglement $E(\rho)$ [16, 17]:

1. An entanglement measure of a separable state is zero:
   $E(\rho) = E\big(\sum_i c_i \rho_A^{(i)} \otimes \rho_B^{(i)}\big) = 0$

2. A measure of entanglement is bounded from above by its values for maximally entangled states.

3. An entanglement measure is not increasing under LOCC[6]: $E(U\rho U^{-1}) \leqslant E(\rho)$

4. An entanglement measure is convex: $E(t\rho_1 + (1-t)\rho_2) \leqslant tE(\rho_1) + (1-t)E(\rho_2)$ for $0 \leqslant t \leqslant 1$

5. An entanglement measure is sub-addictive: $E(\rho_A \otimes \rho_B) \leqslant E(\rho_A) + E(\rho_A)$

It turns out that (i)-(v) and (1)-(5) are compatible except for the properties (iv) and (4), as a consequence the von Neumann entropy describes the amount of entanglement only if the whole system is in a pure state, whose Schmidt decomposition is unique.
As a result, considering the bipartite system $A \bigcup B$ we have seen in section 2.4, $S(\rho_A)$ provides a good measure of entanglement for the whole system (2.33). Furthermore it is easy to see that $S(\rho_A) = S(\rho_B)$ thus the amount of entanglement does not depend on which of the two partitions we choose to calculate (2.47).
Analogously, it is possible to extend these considerations to the Rényi entropy.

Many different ways of quantifying entanglement may be introduced for mixed states [13] e.g. the *entanglement cost*, the *distillable entanglement* [18] and the *entanglement of formation* E_F [19] although they may present many problems. For instance, the main difficulty behind the calculation of E_F lies in the infinite number of possible decomposition of a density matrix that introduce a sort of "ambiguity". Indeed, even knowing how to quantify entanglement in pure bipartite states, this knowledge is not directly applied to mixed states in order to obtain the latter as average over the mixture of the pure states. Thus, different decomposition may lead to different averages of entanglement. In order to overcome this problem the entropy of formation may be defined:

$$E_F(\rho) := min\{\sum_j p_j S(\rho_{A,j})\} \tag{2.53}$$

where the minimum is taken over all realizations of the mixed state $\rho_{AB} = \sum_j |\psi_j\rangle\langle\psi_j|$ and $S(\rho_{A,j})$ is the von Neumann entropy of the reduced density matrix $\rho_{A,j}$:

[6] *Local Operations and Classical Communication* are argued in section 2.1, in particular local operations can be implemented by unitary operators $U^\dagger = U^{-1}$.

$$\rho_{A,j} := \mathbf{Tr}_{\mathcal{H}_B}\{|\psi_j\rangle\langle\psi_j|\} \tag{2.54}$$

(2.53) may be obtained by the requirement that entanglement must not increase on average by means of local operatiors including LOCC.

Chapter 3

Replica Trick

3.1 The Bipartite Quantum Spin Chain

Let us focus on a particular case of a bipartite system provided by a *quantum spin chain* which consists in a infinitely long sequence of equidistant atoms characterized by their spin. Our aim is to analyze the entanglement entropy in the *scaling limit* where the connection with the (1+1) *Quantum Field Theories* emerges.
Hence, let us consider the whole chain in a pure state as we discussed in section 2.4. We initially choose the ground state $|gs\rangle$ that is always a pure state, then we will extend some results to excited states. Dividing the chain in two connected regions, say A and its complement $\bar{A}$ such that we can define two sets of local observables that act separately in each part, then we can express the Hilbert space of the whole system as $\mathcal{H} = \mathcal{A} \otimes \bar{\mathcal{A}}$. Since we are interested in calculating the von Neumann entropy, we focus in part A supposed to contain L sites. Thus, if we take the dimensionless *lattice scaling* $\tilde{a}$ equal to the unity, L measures the length of A.

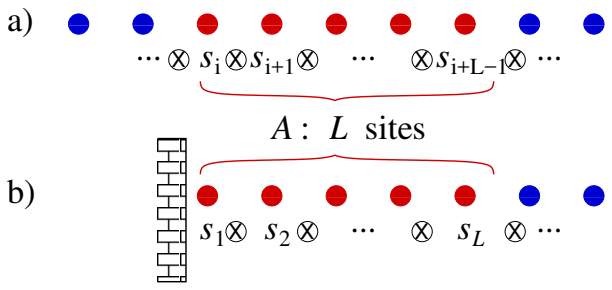

Figure 3.1: **a)** *bulk case* and **b)** *boundary case* (picture taken from [20])

There are many ways to implement this partition. We essentially look at two cases: the first one is the infinite chain in both directions where A is in contact with $\bar{A}$ at two

points i.e. two boundary points, (see **a)** in figure 3.1); the second one is the semi-infinite chain where A is a connected segment starting at the boundary of the chain, with only one true point of contact with $\bar{A}$, see **b)** in figure 3.1). The former is called *bulk case* while the latter is referred as the *boundary case*.

3.1.1 Scaling Limit

A relevant length-scale in the scaling limit is the correlation length ξ. The latter normally depends on the coupling constants of the theories g (e.g. temperature, external magnetic field, etc ...) and can be varied by varying them[1]. In general, the physical meaning of the *correlation length* is connected with the critical phenomena. Indeed, away from the critical point $g \neq g_c$, the *correlation length* is finite and provides a length-scale for the connected correlation functions [22]:

$$\langle s_i s_j \rangle_c := \langle s_i s_j \rangle - \langle s_i \rangle \langle s_j \rangle \sim \begin{cases} \exp\left[-\frac{|i-j|}{\xi(g)}\right] & \text{if } |i-j| \gg 1 \ , \ g \neq g_c \\ \frac{1}{|i-j|^{2d}} & \text{if } |i-j| \gg 1 \ , \ g = g_c \end{cases} \tag{3.1}$$

where i and j denotes the position of the sites and $\langle s_i s_j \rangle_c$ measures the mutual statistical dependence of the spin s_i and s_j. When the system approaches the criticality at $g = g_c$, ξ increases toward infinity. The divergence of the correlation length is one of the most important characteristic of the critical points.

In quantum spin chain such as **a)** and **b)** and close to the critical points, it turns out that $\xi \gg \tilde{a}$. In these conditions, it is natural to adopt a formalism based on continuous fields, but it is important how to take this limit. Indeed, the scaling limit is obtained by implementing an appropriate scaling transformation while at the same time taking the limit $\xi \to \infty$.

Let us consider, in general, an infinite quantum spin chain. If we take any k-point correlation functions $\langle \mathcal{O}_{n_1} \dots \mathcal{O}_{n_k} \rangle$ of the local operator $\mathcal{O}$ acting at the sites $n_1 \dots n_k$ and we scale up all positions n_j while we simultaneously send the correlation length ξ to infinity with constant ratio $\frac{n_j}{\xi}$, then the result of this limit is zero. In contrast, if we now take the same limit, where the correlation function is multiplied by ξ^{kd} for appropriate constant d (which uniquely characterizes $\mathcal{O}$), then the result of the limit is non-zero and it is described by QFT [23].
A way to exhibit this is by using the scaling transformation $n_j \mapsto x_j = \ell(\xi)^{-1} n_j$ for all

[1]Here, $g = (g_1, g_2, \dots)$ is thought as the vector, whose components are the coupling constants characterizing the theory. It lives in the *manifold of the coupling constants* where the procedure of the *Renormalization Group (RG)* is normally implemented [21] by means of a non-linear transformation $\tilde{\mathcal{R}}$. The latter has the effect of moving the system along the so-called *renormalization group trajectory*. The critical point g_c is a point of this trajectory that is invariant under $\tilde{\mathcal{R}}$ such that its corresponding correlation length $\xi(g_c)$ diverges.

$j = 1, \dots k$ such that $\ell(\xi) = \frac{m}{\tilde{a}}\xi$ for some dimensionful quantity m and where $\tilde{a}$ is the lattice spacing. This transformation leads to dimensionful coordinates x such that:

$$|x_i - x_j| = \frac{|n_i - n_j|}{\ell(\xi)} = \frac{\tilde{a}}{m}\frac{|n_i - n_j|}{\xi} \tag{3.2}$$

Thus, the scaling limit is obtained by looking at large distance $|n_i - n_j|$ while keeping n_j/ξ fixed in the limit:

$$\underbrace{\lim_{\xi\to\infty}}_{\frac{n_j}{\xi} \text{ fixed}} \ell^{kd} \langle \mathcal{O}_{\ell x_1}, \dots, \mathcal{O}_{\ell x_k} \rangle_{chain} \propto \langle \mathcal{O}(x_1) \dots \mathcal{O}(x_k) \rangle_{\text{QFT}} \tag{3.3}$$

where the correlation function on the r.h.s. is calculated in QFT and m represents the smallest mass of the theory. Furthermore, the unique (positive) number d making the limit (3.3) finite is called the *scaling dimension* of the local field $\mathcal{O}(x)$ and the QFT correlation function is referred to as a *scaling function*. It is important to stress that the result of (3.3) is largely independent of the details of the underlying quantum spin chain, in fact, different quantum chains may give rise to the same limit (3.3). This leads to the **universality of the scaling limit**: models may be organized in universality classes [24].

In addition, the scaling limit can be used to obtain the CFT by approaching directly the criticality $\xi = \infty$ and taking the scaling transformation $n_j \mapsto x_j = (\ell')^{-1} n_j$ looking at large distance $|n_i - n_j|$ i.e. $\ell' \gg 1$:

$$\lim_{\ell'\to\infty} \ell'^{kd} \langle \mathcal{O}_{\ell' x_1}, \dots, \mathcal{O}_{\ell' x_k} \rangle_{chain}|_{\xi=\infty} \propto \langle \mathcal{O}(x_1) \dots \mathcal{O}(x_k) \rangle_{\text{CFT}} \tag{3.4}$$

We can finally make a few comments on (3.3) and (3.4):

(i) We can obtain in most cases (but not all) the CFT by taking the limit of the QFT in (3.3) $m|x_i - x_j| \to 0$ therefore the scaling transformation at short distance $|n_i - n_j|$. This is known as *massless limit*.

(ii) (3.3) and (3.4) tell us that QFT and CFT correlation functions describe *asymptotic behaviours* of quantum chain correlation functions.

(iii) the scaling limit is independent of the state we choose to evaluate the correlation functions therefore both ground and excited states may be used.

Looking back at the quantum spin chain such as **a)** and **b)**, the scaling limit allows us to analyze the system in a QFT (which we assume to possess Poincarè invariance) or CFT in order to obtain information about the universal features of the Entanglement Entropy. The scaling limit is here obtained by approaching the critical point while letting

the length L of the the region A go to infinity such that the ratio $\frac{L}{\xi}$ is a constant. In this context, the end sites of the region L take an important role in the (Euclidean) QFT model, essentially they are the points involved in the boundary conditions. The scaling transformation leads to euclidean distance:

$$|a-b| = \frac{L}{m\xi} \tag{3.5}$$

where $(a,0)$ and $(b,0)$ are the dimensionful euclidean coordinates of the end points in the QFT model.

3.2 The Riemann Surface

Following [7], let us consider the partition function of a (1+1)-dimensional Quantum Field Theory defined on the Riemann Surface $\mathcal{R}$ and described by the local Lagrangian density $\mathcal{L}$. It can be written in the path integral formalism as:

$$\mathcal{Z}[\mathcal{L},\mathcal{R}] = \int \mathcal{D}_{\mathcal{R}}\phi \exp\Big[-\int_{\mathcal{R}} dxdy\mathcal{L}[\phi](x,y)\Big] \tag{3.6}$$

Focusing on the Riemann surface $\tilde{\mathcal{R}}$ with zero curvature everywhere except for a discrete set of points (let us call the latter I), it is clear from (3.6) that the Lagrangian density does not depend explicitly on the Riemann surface since it is local. As consequence, we can express the path integral (3.6) as an object evaluated on $\mathbb{R}^2$ where the structure of $\tilde{\mathcal{R}}$ is implemented by appropriate boundary conditions $\mathcal{C}(I)$ around the points of I:

$$\mathcal{Z}[\mathcal{L},\tilde{\mathcal{R}}] = \int_{\mathcal{C}(I)} \mathcal{D}_{\mathbb{R}^2}\phi \exp\Big[-\int_{\mathbb{R}^2} dxdy\mathcal{L}[\phi](x,y)\Big] \tag{3.7}$$

$\mathcal{C}(I)$ may be realized by some special fields[2] that are called *twist fields* and defined by (3.7).

In general, the twist fields exist whenever the QFT is equipped with a global internal symmetry σ.[3] It is important to stress that, although the symmetry does not affect the action

$$\int_{\mathbb{R}^2} dxdy\tilde{\mathcal{L}}[\phi](x,y) = \int_{\mathbb{R}^2} dxdy\tilde{\mathcal{L}}[\sigma\phi](x,y) \tag{3.8}$$

σ modifies the surface in which the field ϕ lives. A simple example is provided by classical statistical mechanics when one wants to evaluate the partition function of a

[2]the term "field" is taken in the most general QFT sense: it is an object which depends on x and y and such that we can evaluate its correlation functions.

[3]here, σ is thought as an operator that implements the symmetry transformation.

system composed of N indistinguishable particles in a three-dimensional space i.e. there is a symmetry under the permutation group. The partition function is not evaluated on the whole phase space that is isomorphic to $\mathbb{R}^{6N}$ but it is calculated on a restricted manifold in order to avoid the overvaluation when the multi-particle states differ by a permutation over the particles.
This modification can be implemented on the boundary condition. Introducing:

$$C_\sigma(\tilde{x},\tilde{y}) \quad : \quad \phi(x,\tilde{y}^+) = \sigma\phi(x,\tilde{y}^-) \quad x \in [\tilde{x},+\infty[\tag{3.9}$$

we can write the k-point correlation function of some fields $\mathcal{O}_1(z_1)\dots\mathcal{O}_k(z_k)$ as:

$$\langle \mathcal{O}_1(z_1)\dots\mathcal{O}_k(z_k)\rangle_{\tilde{\mathcal{R}},\tilde{\mathcal{L}}} \propto \int_{C_\sigma(0,0)} \mathcal{D}\phi\ \mathcal{O}_1(z_1)\dots\mathcal{O}_k(z_k)\exp\left[-\int_{\mathbb{R}^2} dxdy\tilde{\mathcal{L}}[\sigma\phi](x,y)\right] \tag{3.10}$$

A simple way to evaluate the l.h.s. of (3.10) on $\mathbb{R}^2$ is to involve the *twist field* $\mathcal{T}_\sigma$ that implements directly the symmetry σ without any boundary conditions:

$$\begin{aligned} &\langle \mathcal{O}_1(z_1)\dots\mathcal{O}_k(z_k)\rangle_{\tilde{\mathcal{R}},\tilde{\mathcal{L}}} = \langle \mathcal{T}_\sigma(\tilde{z})\mathcal{O}_1(z_1)\dots\mathcal{O}_k(z_k)\rangle_{\mathbb{R}^2,\tilde{\mathcal{L}}} \\ &\propto \int \mathcal{D}\phi\ \mathcal{T}_\sigma(\tilde{z})\mathcal{O}_1(z_1)\dots\mathcal{O}_k(z_k)\exp\left[-\int_{\mathbb{R}^2} dxdy\tilde{\mathcal{L}}[\sigma\phi](x,y)\right] \end{aligned} \tag{3.11}$$

where $\tilde{z} = (\tilde{x},\tilde{y})$. This procedure may be generalized, we can essentially insert in (3.11) as many $\mathcal{T}_\sigma$ as points belonging to I i.e. the number of boundary points (3.9) involved in (3.10).

3.2.1 The Replica Trick

We now concentrate attention on the Riemann surface $\mathcal{M}_{n,a,b}$ consisting of n sheets sequentially joined to each other on the segment described by $(x,y) = (p,0)$ with $p \in [a,b]$ whose a pictorial representation is provided by the figure 3.2.1. The implementation of (3.6) on $\mathbb{R}^2$ will involves certain fields acting on the coordinates $(a,0)$ and $(b,0) \in \mathbb{R}^2$. However these fields appear to be non-local in the new model on $\mathbb{R}^2$. Indeed, evaluating the Lagrangian density $\mathcal{L}$ (defined in $\mathcal{M}_{n,a,b}$) at the points $(a,0)$ and $(b,0)$ actually changes the value of the correlation function since it gets to a different Riemann sheet. In other words, the Lagrangian density $\mathcal{L}$ is not well-defined on $\mathbb{R}^2$.

In order to overcome this problem we will consider a multiple-copies model that is larger than the previous one described by (3.7) on $\mathbb{R}^2$: it is called the *replica model* and involves n independent copies of $\mathbb{R}^2$ (where n is the number of Riemann sheets whose

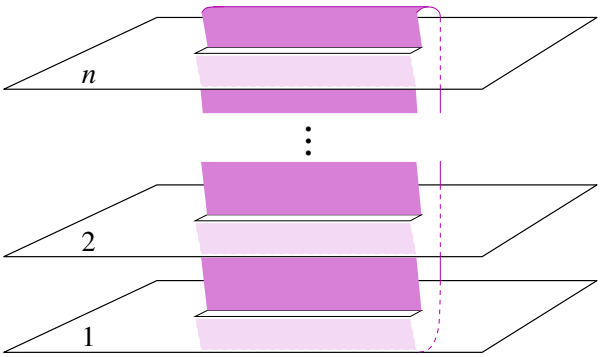

Figure 3.2: A representation of the Riemann surface $\mathcal{M}_{n,a,b}$ taken from [20].

$\mathcal{M}_{n,a,b}$ is composed of) with a branch cut along $[a,b]$. This means we now deal with n fields ϕ_i and, consequently, with n Lagrangian densities $\mathcal{L}[\phi_i]$ such that

$$\mathcal{L}^{(n)}[\phi_1,\ldots,\phi_2](x,y) := \mathcal{L}[\phi_1](x,y) + \ldots \mathcal{L}[\phi_n](x,y) \tag{3.12}$$

is well-defined and local on the multiple copies model. Therefore (3.6) becomes:

$$\mathcal{Z}[\mathcal{L},\mathcal{M}_{n,a,b}] = \int_{\mathcal{C}(a,b)} \mathcal{D}\phi_1 \ldots \mathcal{D}\phi_n \exp\left[-\int_{\mathbb{R}^2} dxdy\{\mathcal{L}[\phi_1](x,y) + \ldots \mathcal{L}(x,y)[\phi_n]\}\right] \tag{3.13}$$

where $\mathcal{C}(a,b)$ denotes the boundary conditions on the fields ϕ_i:

$$\mathcal{C}(a,b) \quad : \quad \phi_i(x,0^+) = \phi_{i+1}(x,0^-) \quad \forall x \in [a,b], i = 1,\ldots,n \tag{3.14}$$

and we require $n+i \equiv i$. The local fields defined by (3.14) are examples of *twist fields*. Furthermore, the n-copy model presents a symmetry under exchange of the copies i.e. the cyclic permutation symmetries σ and σ^{-1}. This allows us to introduce the *branch-point twist fields* and *anti-twist fields* $\mathcal{T}$ and $\tilde{\mathcal{T}}$ acting at the branch points, respectively:

$$\begin{aligned} \mathcal{T} &:= \mathcal{T}_\sigma\,, & \sigma &: i \mapsto i+1 \\ \tilde{\mathcal{T}} &:= \mathcal{T}_{\sigma^{-1}}\,, & \sigma^{-1} &: i+1 \mapsto i \end{aligned} \tag{3.15}$$

Thus the fields $\mathcal{T}$ and $\tilde{\mathcal{T}}$ implement the symmetry transformations of the fields $\phi_1,\ldots,\phi_n$ when $z' = (x',y')$ and $z = (x,y)$ such that $y = y'$:

$$\begin{aligned} \phi_i(z)\mathcal{T}(z') &= \mathcal{T}(z')\phi_{i+1}(z) & x' &\geqslant x \\ \phi_i(z)\mathcal{T}(z') &= \mathcal{T}(z')\phi_i(z) & x' &\leqslant x \end{aligned} \tag{3.16}$$

Thanks to (3.10) and (3.11), (3.13) becomes

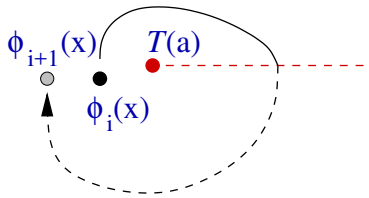

Figure 3.3: The representation of the semi-local field ϕ with respect to $\mathcal{T}(a,0)$ on the n-copy model. This picture is taken from [7].

$$\mathcal{Z}[\mathcal{L},\mathcal{M}_{n,a,b}] \propto \langle \mathcal{T}(a,0)\tilde{\mathcal{T}}(b,0)\rangle_{\mathcal{L}^{(n)},\mathbb{R}^2} \tag{3.17}$$

and more generally we can write the correlation functions in the model $\mathcal{L}$ on the Riemann surface $\mathcal{M}_{n,a,b}$ as:

$$\langle \mathcal{O}(x_{\mathcal{R}},y_{\mathcal{R}};\textit{sheet } i)\dots\rangle_{\mathcal{M}_{n,a,b},\mathcal{L}} = \frac{\langle \mathcal{T}(a,0)\tilde{\mathcal{T}}(b,0)\mathcal{O}_i(x,y)\dots\rangle_{\mathcal{L}^{(n)},\mathbb{R}^2}}{\langle \mathcal{T}(a,0)\tilde{\mathcal{T}}(b,0)\rangle_{\mathcal{L}^{(n)},\mathbb{R}^2}} \tag{3.18}$$

where $\mathcal{O}(x_{\mathcal{R}},y_{\mathcal{R}};\textit{sheet } i)$ is the field on the ith-Riemann sheet of the model $(\mathcal{M}_{n,a,b},\mathcal{L})$ while $\mathcal{O}_i(x,y)$ lives on the ith-copy of the replica model defined by $\mathcal{L}^{(n)}$.

It is important to stress that the branch-point twist fields $\mathcal{T}_\sigma$ and $\tilde{\mathcal{T}}_\sigma$ live only on the replica model and, in addition, they are not uniquely defined by (3.17). As we have already mentioned, implementing $\mathcal{T}_\sigma$ corresponds to perform a symmetry transformation to the field $\mathcal{O}$ in the l.h.s of (3.18). In the multi-sheet covering of $\mathbb{R}^2$ this means that $\mathcal{T}_\sigma$ leads to a clockwise turn of the field $\mathcal{O}_i$ around $[a,b]$ that is equivalent to the transformation $\mathcal{O}\mapsto\sigma\mathcal{O}$. If $\sigma\mathcal{O}\neq\mathcal{O}$ then the field $\mathcal{O}_i$ in the ith-copy crosses the branch cut and it is said *semi-local with respect to* $\mathcal{T}_\sigma$. This is essentially a consequence of (3.16) and can be illustrated as in figure 3.2.1. This property, along with the condition that $\mathcal{T}_\sigma$ has the lowest scaling dimension and be invariant under all symmetries that commute with σ, uniquely fixes the field up to a normalization [7].

Finally, an important consequence of the symmetry is that the correlation function (3.17) is invariant under continuous changes of the shape of the branch cut up to symmetry transformations of the field $\mathcal{T}_\sigma$ [20]. The reason why this holds it that we can always implement symmetry transformations without changing the result of the path integral (3.13) up to transformations of the local fields in (3.13). The result of these symmetry transformations is a change of the boundary condition and consequently it modifies the shape of the branch cut.

3.2.2 Branch Point Twist Fields in Conformal Field Theory

Let us now suppose $\mathcal{L}$ describes a *Conformal Field Theory (CFT)*. Choosing conformal maps and thanks to the conformal invariance of the theory, we can always define another CFT with local Lagrangian $\mathcal{L}^{(n)}$.
For instance, following [6, 7], let us consider the complex coordinates $\omega = x_{\mathcal{R}} + iy_{\mathcal{R}}$ and $\bar{\omega} = x_{\mathcal{R}} - iy_{\mathcal{R}}$ of $\mathcal{M}_{n,a,b}$, we can maps these coordinate to the z-plane $\mathbb{C}$ by making the conformal transformation:

$$z = \left(\frac{\omega - a}{\omega - b}\right)^{\frac{1}{n}}, \quad \mathcal{M}_{n,a,b} \ni \omega \mapsto z \in \mathbb{C} \tag{3.19}$$

Calling c the charge of the CFT described by $\mathcal{L}$, we can easily obtain the stress tensor $T(\omega)$ from $\mathcal{L}$. Furthermore, the transformation rule of the stress tensor is well-known by CFT [25]:

$$T(\omega) = \left(\frac{dz}{d\omega}\right)^2 T(z) + \frac{c}{12}\{z, \omega\} \tag{3.20}$$

where we have introduced the *Schwarzian derivative*:

$$\{z, \omega\} = \frac{z'''z' - \frac{3}{2}(z'')^2}{(z')^2} \tag{3.21}$$

Since $\langle T(z)\rangle_{\mathbb{C},\mathcal{L}} = 0$ in CFT, we obtain from (3.20) and (3.21):

$$\begin{aligned} \langle T(\omega)\rangle_{\mathcal{M}_{n,a,b},\mathcal{L}} &= \langle T(z)\rangle_{\mathbb{C},\mathcal{L}} + \frac{c}{12}\{z, \omega\} \\ &= \frac{c(n^2-1)}{24n^2}\frac{(a-b)^2}{(\omega-a)^2(\omega-b)^2} \end{aligned} \tag{3.22}$$

Furthermore, we can evaluate the l.h.s. of (3.22) on the n-copy model thanks to (3.18):

$$\langle T(\omega)\rangle_{\mathcal{M}_{n,a,b},\mathcal{L}} = \frac{\langle \mathcal{T}(a,0)\tilde{\mathcal{T}}(b,0)T^{(n)}(\omega)\rangle_{\mathcal{L}^{(n)},\mathbb{R}^2}}{\langle \mathcal{T}(a,0)\tilde{\mathcal{T}}(b,0)\rangle_{\mathcal{L}^{(n)},\mathbb{R}^2}} \tag{3.23}$$

and it is straightforward from (3.12) that

$$T^{(n)}(\omega) = \sum_{j=1}^{n} T_j(\omega) \tag{3.24}$$

where T_j is the copy of T on the jth-sheet. Since c is the central charge of the CFT on $\mathcal{M}_{n,a,b}$, it follows that the central charge of the n-copy model is equal to nc.
In order to define uniquely the twist fields $\mathcal{T}$ and $\tilde{\mathcal{T}}$, they must have the lowest scaling

dimension. In the language of CFT, this means they are suppose to be *primary fields*. As consequence, using the conformal ward identity, we can write:[4]

$$\langle \mathcal{T}(a,0)\tilde{\mathcal{T}}(b,0)T^{(n)}(\omega)\rangle_{\mathcal{L}^{(n)},\mathbb{R}^2} = \Big[\frac{1}{(\omega-a)}\frac{\partial}{\partial a}+\frac{1}{(\omega-b)}\frac{\partial}{\partial b}+\frac{\Delta_{\mathcal{T}}}{(\omega-a)^2}+\frac{\Delta_{\tilde{\mathcal{T}}}}{(\omega-b)^2}\Big]\langle \mathcal{T}(a,0)\tilde{\mathcal{T}}(b,0)\rangle_{\mathcal{L}^{(n)},\mathbb{R}^2} \tag{3.25}$$

where $\Delta_{\mathcal{T}}$ and $\Delta_{\tilde{\mathcal{T}}}$ are the conformal weight respectively of $\mathcal{T}$ and $\tilde{\mathcal{T}}$. Identifying the scaling dimension $d_n = 2\Delta_{\mathcal{T}}$, $\tilde{d_n} = 2\Delta_{\tilde{\mathcal{T}}}$ such that

$$\langle \mathcal{T}(a,0)\tilde{\mathcal{T}}(b,0)\rangle_{\mathcal{L}^{(n)},\mathbb{R}^2} = \frac{1}{|a-b|^{2d_n}} \tag{3.26}$$

then we can compare (3.22) and (3.23) with (3.25), and finally find:

$$d_n = \tilde{d_n} = \frac{c}{12}\Big(n-\frac{1}{n}\Big) \tag{3.27}$$

3.3 Entanglement Entropy And Replica Trick

Our interest in the replica trick is that the partition functions on the the replica model with branch-point twist fields provide a way to analyze the entanglement entropy of the original quantum spin chain (or more generally of the QFT associated to it by the scaling limit). Following [6, 20, 7], we will define the reduced density matrix ρ_A on 1 copy in the path integral formalism and then we will provide a connection with the n-copy model. As a result, we will obtain a formula similar to (2.48) that defines the Renyi entropy, but now the parameter $n \in \mathbb{N}$. As a consequence, in order to obtain the Renyi entropy, it is necessary to find an appropriate analytic continuation on the parameter n such that its limit $\mathbb{R}^+ \ni n \to 1^+$ gives a finite result.

3.3.1 The Reduced Density Matrix On The Replica Model

We are now interested in calculating the reduced density matrix ρ_A such that:

$$S_A = -\mathbf{Tr}_{\mathcal{A}}\{\rho_A \log \rho_A\}\ , \quad \rho_A = \mathbf{Tr}_{\tilde{\mathcal{A}}}\,|\psi\rangle\langle\psi| \tag{3.28}$$

where $|\psi\rangle$ is the state of the quantum chain. As we have already seen, the scaling limit of the quantum chain gives rise to a corresponding euclidean QFT model on $\mathbb{R}^2$. If we choose the ground state $|gs\rangle$, then the scaling limit maps it to the QFT vacuum state $|0\rangle$ while excited states of the quantum chain are mapped to QFT asymptotic states characterized by the particle content and their momenta[5].

[4] In this thesis we do not discuss in detail the fundamentals of Conformal Field Theory. The conformal ward identity and the primary fields may be found in [21]

[5] We will generally keep the notation $|\psi\rangle$ for these asymptotic states.

Following [20], let us consider the QFT Hilbert space of field configurations on $\mathbb{R}$ $\mathcal{H}_{\phi,\mathbb{R}} = \{\phi(x)\ ;\ x \in \mathbb{R}\}$. We can write the state $|\psi\rangle$ as the linear combination:

$$|\psi\rangle = \int \mathcal{D}_{\mathbb{R}}\phi\, |\phi\rangle\langle\phi|\psi\rangle \tag{3.29}$$

the coefficients $\langle\phi|\psi\rangle$ are evaluated by path integrals on the lower half of the $\mathbb{R}^2$-plane under appropriate boundary conditions ${\mathcal{C}^-}_{\phi,\psi}$:

$$\langle\phi|\psi\rangle = \frac{1}{\sqrt{Z_1}} \int_{{\mathcal{C}^-}_{\phi,\psi}} \mathcal{D}_{\mathbb{R}^2_-}\varphi\ e^{-S_{\mathbb{R}^2_-}[\varphi]} \tag{3.30}$$

where $\mathbb{R}^2_- = \{(x,y) \in \mathbb{R}^2\ ;\ x \in \mathbb{R},\ y \leqslant 0\}$, $S_\Omega[\dots] = \int_\Omega d^2x\mathcal{L}[\dots]$ is the action of the theory on Ω and

$${\mathcal{C}^-}_{\phi,\psi}\ :\ \begin{cases} \varphi(x,0^-) = \phi(x)\,, & x \in \mathbb{R} \\ \varphi(x, y \to -\infty) \sim f_\psi(x,y)\,, & x \in \mathbb{R} \end{cases} \tag{3.31}$$

where $f_\psi(x,y)$ represents the asymptotic condition of the state $|\psi\rangle$. If $f_\psi(x,y) = 0$ then $|\psi\rangle$ is the vacuum state otherwise $f_\psi(x,y)$ reproduces wave packets corresponding to asymptotic particles.
Analogously, we can write the state $\langle\psi|$ as the complex conjugation of (3.29) where now the coefficients $\langle\psi|\phi\rangle$ are evaluated by path integrals on the upper half of $\mathbb{R}^2$ namely $\mathbb{R}^2_+ = \{(x,y) \in \mathbb{R}^2\ ;\ x \in \mathbb{R},\ y \geqslant 0\}$ with the boundary condition ${\mathcal{C}^+}_{\bar{\phi},\psi}$:

$${\mathcal{C}^+}_{\phi',\psi}\ :\ \begin{cases} \varphi(x,0^+) = \phi'(x)\,, & x \in \mathbb{R} \\ \varphi(x, y \to +\infty) \sim f^*_\psi(x,y)\,, & x \in \mathbb{R} \end{cases} \tag{3.32}$$

Hence, the factor Z_1 emerges from the normalization $\langle\psi|\psi\rangle = 1$ and can be written as:

$$Z_1 = \int_{\mathcal{C}_\psi} \mathcal{D}_{\mathbb{R}^2}\varphi\ e^{-S_{\mathbb{R}^2}[\varphi]} \tag{3.33}$$

and $\mathcal{C}_\psi = {\mathcal{C}^+}_{\phi',\psi} \bigcup {\mathcal{C}^-}_{\phi,\psi}$.

Let us consider the system such as the bulk case in figure 3.1 and focus on one copy of the replica model. We can define the field configurations on A as $\mathcal{A}_{\phi_A} = \{\phi_A(x)\ ;\ x \in A\}$ and similarly $\bar{\mathcal{A}}_{\phi_{\bar{A}}} = \{\phi_{\bar{A}}(x)\ ;\ x \in \bar{A}\}$. The reduced density matrix ρ_A is obtained by the path integral on the whole $\mathbb{R}^2$, with continuity on $\bar{A}$ but with an open slit on $\mathrm{I} = \{(p,0) \in \mathbb{R}^2\ ;\ p \in [a,b]\}$:

$$\begin{aligned} \langle\phi_A|\,\rho_A\,|\phi'_A\rangle &= \langle\phi_A|\,\mathbf{Tr}_{\bar{\mathcal{A}}}\,\rho\,|\phi'_A\rangle = \langle\phi_A|\,\mathbf{Tr}_{\bar{\mathcal{A}}}\,\{|\psi\rangle\langle\psi|\}\,|\phi_A\rangle \\ &= \int \mathcal{D}_{\mathcal{A}}\phi_{\bar{A}}\ \langle\phi_{\bar{A}}, \phi'_A|\psi\rangle\langle\psi|\phi_{\bar{A}}, \phi_A\rangle \\ &= \frac{1}{Z_1}\int_{\mathcal{C}_{\mathrm{I},\psi}} \mathcal{D}_{\mathbb{R}^2\setminus\mathrm{I}}\,\varphi\ e^{-S_{\mathbb{R}^2\setminus\mathrm{I}}[\varphi]} \end{aligned} \tag{3.34}$$

where the boundary conditions $\mathcal{C}_{\mathrm{I},\psi}$:

$$\mathcal{C}_{\mathrm{I},\psi} \ : \ \begin{cases} \varphi(x,0^-) = \phi_A(x) \ , \ \varphi(x,0^+) = \phi'_A(x) & x \in [a,b] \\ \varphi(x, y \to -\infty) \sim f_\psi(x,y) \ , & x \in \mathbb{R} \\ \varphi(x, y \to +\infty) \sim f^*_\psi(x,y) \ , & x \in \mathbb{R} \end{cases} \tag{3.35}$$

We now assume that the state of the chain is the ground state. Let us calculate the n-th power of ρ_A:

$$\begin{aligned} \langle \phi_A | \, \rho_A^n \, | \phi'_A \rangle &= \langle \phi_A | \, \rho_A \cdot \mathbb{1} \ \ldots \ \rho_A \cdot \mathbb{1} \cdot \rho_A \, | \phi'_A \rangle \\ &= \int \mathcal{D}_A\phi_1 \ \mathcal{D}_A\phi_2 \ldots \mathcal{D}_A\phi_{n-1} \langle \phi_A | \, \rho_A \, | \phi_1 \rangle \langle \phi_1 | \, \rho_A \, | \phi_2 \rangle \ldots \langle \phi_{n-1} | \, \rho_A \, | \phi'_A \rangle \end{aligned} \tag{3.36}$$

Each factor $\langle \phi_i | \, \rho_A \, | \phi_j \rangle$ reproduces an independent path integral such as (3.34) over ϕ_j on a $\mathbb{R}^2$-sheet with the same slit on I. In order to have continuity we need to require, as consequence of (3.35), that $\phi_j = \phi_{j+1}$ and thus the n sheets appear to be sewed together. In particular, performing the trace of (3.36), the nth-sheet is connected to first one so that the n sheets are cyclically connected. It is easy to see from (3.34) that what we obtained is:

$$\mathbf{Tr}_{\mathcal{A}} \, \rho_A^n = \frac{Z_n(a,b)}{Z_1^n} \tag{3.37}$$

where Z_1 is given by (3.33) while $Z_n(a,b)$ by (3.13):

$$Z_n(a,b) := \mathcal{Z}[\mathcal{L}, \mathcal{M}_{n,a,b}] \tag{3.38}$$

3.3.2 The Analytic Continuation On The Parameter n

The next step consists of performing the analytic continuation of the quantity $Z_n(a,b)$ from $n \in \mathbb{N}$, where it is naturally associated to Riemann surfaces such as $\mathcal{M}_{n,a,b}$, to $\mathrm{Re}\, n \in]1, +\infty[$.
Looking back at the section 2.4.1, it is important to remark that ρ_A has non negative eigenvalues λ such that they are $0 \leqslant \lambda < 1$ and $\mathbf{Tr}_{\mathcal{A}} \, \rho_A = 1$. Thus we can rewrite the l.h.s. of (3.37) as:

$$\mathbf{Tr}_{\mathcal{A}} \, \rho_A^n = \sum_\lambda \lambda^n \tag{3.39}$$

the properties of ρ_A ensure that (3.39) is absolutely convergent and therefore analytic for all $\mathrm{Re}\, n > 1$. In addition, the derivative with respect to n also exists and it is analytic. If the entanglement entropy S_A is finite, we can use the identity $\rho_A \ \log \rho_A = \lim_{n \to 1^+} \frac{\partial}{\partial n} \rho_A^n$

$$S_A = -\mathbf{Tr}\{\rho_A \ \log \rho_A\} = - \lim_{n \to 1^+} \frac{\partial}{\partial n} \mathbf{Tr} \rho_A^n \tag{3.40}$$

where the trick consists in evaluating the trace with $n \in \mathbb{N}$ and then to take the analytic continuation. It follows from (3.37):

$$S_A = -\lim_{n\to 1^+} \frac{\partial}{\partial n} \frac{Z_n(a,b)}{Z_1^n} \tag{3.41}$$

when we evaluate (3.41) on the quantum chain, we have to take into account the scalar limit:

$$\frac{Z_n(a,b)}{Z_1^n} = \mathcal{Z}_n \epsilon^{2d_n} \langle 0| \mathcal{T}(a,0)\tilde{\mathcal{T}}(b,0) |0\rangle \tag{3.42}$$

where $\mathcal{Z}_n$ in an n-dependent non-universal constant, d_n is the scaling dimension of the twist and anti-twist fields, ϵ is the short distance cut-off which is scaled in such a way that $\frac{\partial \mathcal{Z}_n}{\partial n} = 0$ when $n = 1$.[6] This is known as the *CFT normalization* and it is built in such a way as to obtain exactly the CFT correlation function (3.26) when we look at the short distance behavior i.e. $m|a-b| \to 0$, the dependence of n is absorbed into $\mathcal{Z}_n$ and $\mathcal{Z}_1 = 1$ because the twist fields become simply the identity when we deal with one copy. Furthermore, ϵ is related to the correlation length ξ via $\epsilon = \frac{\tilde{c}}{m\xi}$ for some dimensionless non-universal constant $\tilde{c}$.[7]

3.4 Entanglement Entropy In The Ground State

As a result of section 3.3, the bulk entanglement entropy can be expressed as:

$$S_A^{\text{bulk}}(\ell) = -\lim_{n\to 1^+} \frac{\partial}{\partial n}\, \mathcal{Z}_n \epsilon^{2d_n} \langle 0| \mathcal{T}(a,0)\tilde{\mathcal{T}}(b,0) |0\rangle_{\mathcal{L}^{(n)},\mathbb{R}^2} \tag{3.43}$$

where $\ell = |a-b|$ and the correlation function on the l.h.s. is evaluated in the replica model described by (3.12).

The boundary case can be implemented by the bulk result. In particular we can consider the case where the system A lies completely on the region $\tilde{R} = \{(x,y) \in \mathbb{R}^2 \ ; \ x > 0, \ y \in \mathbb{R}\}$. This requires an additional boundary condition at $x = 0$ in (3.35) in order to fully define the model. A way to implement this condition is to insert an extra term $S_B = \int dy \mathcal{L}_B[\varphi](y)$ in the action of the theory such that this term is supported on $x = 0$. As consequence the new term affects the Hilbert space therefore the new ground state $|0\rangle_B$ will be different from the previous $|0\rangle$ and consequently even the excited states will be influenced by the boundary condition. In the previous Hilbert space on the full

[6] This choice basically makes the von Neumann entropy independent of the constant $\mathcal{Z}_n$.

[7] ϵ is essentially proportional to the inverse of scale parameter $\ell(\xi)$ introduced in section 3.1.1 and that makes a connection with the asymptotic behaviour of the quantum chain by (3.3) such that $n_j \mapsto x_j \propto \epsilon(\xi) n_j$. Looking at short distance behaviour (ultra-violet), ϵ tends to be very small. Thus, the expression (3.42) is obtained in the scaling limit.

line, this boundary condition corresponds to a boundary state $|B\rangle$ which depends on the particular model under consideration [20].

Choosing $z = (x, 0)$, $z_1 = (x_1, 0) \in \tilde{R}$, we can write:

$$S_A^{\text{bulk}}(z, z_1) = -\lim_{n\to 1^+} \frac{\partial}{\partial n}\, \mathcal{Z}_n \epsilon^{2d_n}\, {}_B\langle 0|\, \mathcal{T}(x,0)\tilde{\mathcal{T}}(x_1,0)\, |0\rangle_B \tag{3.44}$$

Now, taking the limit $x_1 \to +\infty$ the two point function in (3.44) becomes:

$${}_B\langle 0|\, \mathcal{T}(x,0)\tilde{\mathcal{T}}(x_1,0)\, |0\rangle_B \sim {}_B\langle 0|\, \mathcal{T}(x,0)\, |0\rangle_B\, {}_B\langle 0|\, \tilde{\mathcal{T}}(+\infty,0)\, |0\rangle_B \tag{3.45}$$

where the the last factor does not depend anymore on the boundary condition at 0 therefore it can be evaluated in the usual Hilbert space as $\langle 0|\, \mathcal{T}\, |0\rangle$ and it contributes to (3.45) as a constant. Dividing out this factor means to subtract the contribution evaluated at infinity to the entanglement entropy [20]. The l.h.s. of (3.45) may be expressed in the Hilbert space associated to $|0\rangle$:

$${}_B\langle 0|\, \mathcal{T}(x,0)\tilde{\mathcal{T}}(x_1,0)\, |0\rangle_B = \langle 0|\, \mathcal{T}(x,0)\tilde{\mathcal{T}}(x_1,0)\, |B\rangle \tag{3.46}$$

In (3.44), we essentially calculated $S_{\bar{A}}$. But remembering $S_A = S_{\bar{A}}$ and using an appropriate branch-point regularization, we can finally write:

$$S_A^{\text{boundary}}(\ell) = -\lim_{n\to 1^+} \frac{\partial}{\partial n}\, \mathcal{Z}_n \epsilon^{d_n}\, \langle 0|\, \mathcal{T}(\ell,0)\, |B\rangle_{\mathcal{L}^{(n)},\mathbb{R}^2} \tag{3.47}$$

where ℓ is the distance between the boundary at zero and the end point of A.

Thus, expressions such as (3.43) and (3.14) allow us to calculate the entanglement entropy once we manage to evaluate the correlation function on the r.h.s. in the replica model.

A special case is provided by considering the CFT correlation function (3.26) for the bulk case:

$$S_A^{\text{bulk}}|_{\text{CFT}} = -\lim_{n\to 1^+} \frac{\partial}{\partial n} \left(\frac{\epsilon}{|a-b|} \right)^{2d_n} = \frac{c}{3} \log \frac{\ell}{\epsilon} \tag{3.48}$$

where d_n is the scaling dimension of the twist field $\mathcal{T}$ and anti-twist field $\tilde{\mathcal{T}}$ given by (3.27), c is the central charge of the original CFT and and $\ell = |a - b|$.
Another way to express (3.48) is by using the Rènyi entropy $S_A^{(n)}$ defined by (2.48) and then taking the limit $S_A = \lim_{n\to 1^+} S_A^{(n)}$. The Rènyi entropy may be useful in order to build the Entanglement Entropy in the replica model. In CFT it turns out:

$$S_A^{(n)} = \frac{c(n+1)}{6n} \log \frac{\ell}{\epsilon} \tag{3.49}$$

An important consequence of (3.37) is that $\mathbf{Tr}_{\mathcal{A}}\rho_A^n$ transforms under a general conformal transformation as a two-point correlation function of primary fields i.e. $\mathcal{T}$ and $\tilde{\mathcal{T}}$. Thus we can consider the conformal mapping $z \mapsto \omega(z)$ [26] :

$$\langle \mathcal{T}(z_1, \bar{z_1})\tilde{\mathcal{T}}(z_2, \bar{z_2})\rangle = |w'(z_1)w'(z_1)|^{d_n}\langle \mathcal{T}(\omega_1, \bar{\omega_1})\tilde{\mathcal{T}}(\omega_2, \bar{\omega_2})\rangle \tag{3.50}$$

This allows us to evaluate $\mathbf{Tr}_{\mathcal{A}}\rho_A^n$ in other geometries. Particular interesting is the **finite size case** where the transformation $\omega \to z = (L/2\pi)\log\omega$ maps the system into a cylinder with the branch cut perpendicular to the axis which corresponds to the case where the subsystem of length ℓ belonging to a finite system of length L with boundary condition. It leads to the the Rènyi entropy

$$S_n^{|0\rangle}(\ell\,; L) = \frac{c(n+1)}{6n}\log\left(\frac{L}{\pi\epsilon}\sin\frac{\pi\ell}{L}\right) \tag{3.51}$$

where we changed the notation in order to remark we are evaluating entropy in the ground state. Consequently, the von Neumann entropy:

$$S_1^{|0\rangle}(\ell\,; L) = \lim_{n\to 1^+} S_n^{|0\rangle}(\ell) = \frac{c}{3}\log\left(\frac{L}{\pi\epsilon}\sin\frac{\pi\ell}{L}\right) \tag{3.52}$$

Notice the limit $\ell \ll L$ reproduces (3.48). It is important to stress that in finite volume the symmetry under $\ell \to L - \ell$ emerges in order to guarantee $S_A = S_{\bar{A}}$

3.4.1 The Asymptotic Behaviour and Form Factor Expansion

The CFT describes the short distance behaviour of S_A i.e. $m\ell$ small, also known as the *ultraviolet limit (UV)*. Essentially, in this limit, the leading term consists of one contribution $\frac{c}{6}\log\xi$ to S_A for each point in the correlation function.
In contrast, the large distance behaviour i.e. the *infrared limit (IR)* is established by QFT and it turn out that S_A saturates to a constant value in this limit. More precisely, we can sum up some results obtained in [27, 6] as in [7, 20]:

$$S_A^{\text{Bulk}}(\ell) = \begin{cases} \frac{c}{3}\log\left(\frac{\ell}{\epsilon}\right) + o(1) & \epsilon \ll \ell \ll m^{-1} \quad \text{UV limit} \\ -\frac{c}{3}\log(m\epsilon) + U^{\text{model}} + O((rm)^{-\infty}) & \epsilon \ll m^{-1} \ll \ell \quad \text{IR limit} \end{cases} \tag{3.53}$$

where U^{model} is a model-dependent constant. On the other hand, the boundary entropy asymptotic behaviour is given by:

$$S_A^{\text{boundary}}(\ell) = \begin{cases} \frac{c}{6}\log\left(\frac{2\ell}{\epsilon}\right) + V(\kappa) + o(1) & \epsilon \ll \ell \ll m^{-1} \quad \text{UV limit} \\ -\frac{c}{3}\log(m\epsilon) + \frac{U^{\text{model}}}{2} + O((rm)^{-\infty}) & \epsilon \ll m^{-1} \ll \ell \quad \text{IR limit} \end{cases} \tag{3.54}$$

where $V(\kappa)$ is related to the boundary state $|B\rangle$ via κ and the term $\frac{U_{\text{model}}}{2}$ is obtained by an appropriate choice of ϵ.[8]

In general, the way to compute the one- and the two-point correlation functions (respectively for the boundary and the bulk case) is by taking advantage of quantum integrable model techniques. As a consequence, we suppose that the original QFT which we handle in order to obtain the replica model is **integrable** and thus we deal with a *Integrable Quantum Field Theory (IQFT).*
In the scattering picture, integrability is normally associated to some scattering properties of the system we are taking into account (see section 4.2). The locality of the twist and anti-twist fields on the replica model allow us to use some mathematical objects knows as *Form Factors* that are defined as matrix elements of local operators at the origin between a "in" asymptotic multi-particle state and the vacuum:

$$F^{\mathcal{O}|_{\mu_1,\dots,\mu_k}}(\theta_1 \dots \theta_k) = \langle 0| \, \mathcal{O} \, |\theta_1, \dots, \theta_k\rangle^{\text{in}}_{\mu_1,\dots,\mu_k} \tag{3.55}$$

where the particles are characterized by the real rapidities θ_i and the quantum number μ_i. In case of the replica model they also depend on the copy in which they live, this dependence may be absorbed in μ_i. Furthermore, fixing the order of the rapidities $\theta_1 > \dots > \theta_k$ the multi-particle states provide a complete basis for the massive QFT considered identifying $|\theta_1, \dots, \theta_k\rangle$ as the incoming state. We will discuss in detail integrability and Form Factor in the next Chapter.

Our interest in Form Factors arises from the fact that the one- and two-point correlation functions such as (3.47) and (3.43) can be evaluated by using the form factor expansion of the twist and anti-twist fields [20].
For instance, let us consider the bulk case, we can use the completeness relation for the multi-particle states:

$$\mathbb{1} = \sum_{k=1}^{\infty} \sum_{\mu_1,\dots,\mu_k} \int_{\theta_1 > \dots > \theta_k} \frac{d\theta_1 \dots d\theta_k}{(2\pi)^k} \, |\theta_1, \dots, \theta_k\rangle_{\mu_1,\dots,\mu_k} \, {}_{\mu_1,\dots,\mu_k}\langle \theta_k, \dots, \theta_1| \tag{3.56}$$

Inserting (3.4.1) into the two-point correlation function, we obtain the form factor expansion:

$$\begin{aligned} &\langle 0| \, \mathcal{T}(a)\tilde{\mathcal{T}}(b) \, |0\rangle = \langle 0| \, \mathcal{T}(a) \times \mathbb{1} \times \tilde{\mathcal{T}}(b) \, |0\rangle = \\ &\sum_{k=1}^{\infty} \sum_{\mu_1,\dots,\mu_k} \int_{\theta_1 > \dots > \theta_k} \frac{d\theta_1 \dots d\theta_k}{(2\pi)^k} \exp\left[-\ell \sum_{i=1}^{k} m_{\mu_i} \cos\theta_i \right] | \langle 0| \, \mathcal{T}(0) \, |\theta_1, \dots, \theta_k\rangle_{\mu_1,\dots,\mu_k} |^2 \end{aligned} \tag{3.57}$$

where ℓ now expresses the space-like relativistic distance in Minkowsky space-time and $\tilde{\mathcal{T}}^\dagger = \mathcal{T}$. The exponential factor, which is due to covariance transformations, is crucial in

[8] In general, the short distance cut-off ϵ is different in the bulk and boundary cases.

the large-distance limit. Although the expansion (3.57) is over infinite terms, each term contributes at least with an exponential factor goes to $e^{-\ell k m_1}$ where k is the the particle number involved in the considered term and m_1 the lowest mass of the theory. Thus, the Form Factors involving large number of particles tend to be negligible in the large distance limit and the leading terms are obtained from the lowest particles contributions.

Thus, if we are interested in the large distance behaviour, we can assume the two-particle approximation for which we consider only the two-particle contributions to the Form Factors expansion in order to obtain the first correction in the IR limit.
This was successfully implemented in [7] for the bulk case:

$$S_A^{\text{Bulk}}(\ell) = -\frac{c}{3}\log(m\epsilon) + U^{\text{model}} + \sum_{k=1}^{\infty} e_k(\ell m) \tag{3.58}$$

where

$$e_k(\ell m) = \lim_{n\to 1^+}\frac{\partial}{\partial n}\sum_{\mu_1,\ldots,\mu_k}^{n}\int_{\theta_1>\cdots>\theta_k}\frac{d\theta_1\ldots d\theta_k}{(2\pi)^k}\exp\left[-\ell\sum_{i=1}^{k} m_{\mu_i}\cos\theta_i\right]|F^{\mathcal{T}|_{\mu_1,\ldots,\mu_k}}(\theta_1,\ldots,\theta_2)|^2$$

essentially U^{model} is seen as the 0-particle contribution that is proportional to $|\langle\mathcal{T}\rangle|^2$. (3.59) leads to the important result in the IR limit

$$S_A^{\text{Bulk}}(\ell) = -\frac{c}{3}\log(m_1\epsilon) + U^{\text{model}} - \frac{1}{8}\sum_{\alpha=1}^{N} K_0(2\ell m_\alpha) + O(e^{-3\ell m_1}) \tag{3.59}$$

providing the next-to-leading order correction to the large distance behaviour in (3.53). In particular, m_α is the mass the particle types involved in the IQFT and $K_0(z)$ is the *modified Bessel function*. The relevance of the new correction arises from the fact that it does not depend on the detail of the scattering content but just on the mass spectrum of the IQFT making a connection between the entanglement entropy and the universal properties.

3.5 Entanglement Entropy In Excited States

Since the relation (3.43) does not require any particular choices of the state in which to calculate the correlation function, the result in the ground state (3.43) and (3.47) may be generalized for excited states. Indeed, the choice of excited state basically modifies the boundary condition (3.35) we use to calculate the path integral (3.34) in particular the asymptotic behavior of the state $|\psi\rangle$. As a consequence, we now deal with correlation function evaluated in the excited state, such as:

$$\frac{\langle\psi|\,\mathcal{T}(a)\tilde{\mathcal{T}}(b)\,|\psi\rangle}{\langle\psi|\psi\rangle} \tag{3.60}$$

It is important to stress that now the normalization of the state $|\psi\rangle$ is not the standard one we introduced in order to define (3.33). Indeed, in the scattering picture, the numerator $\langle\psi|\,\mathcal{T}(a)\tilde{\mathcal{T}}(b)\,|\psi\rangle$ presents some infinite-volume divergences coming from coinciding rapidities of the asymptotic states on the left and the right. These infinite-volume divergences are expected to be cancelled out by similar divergences in the denominator $\langle\psi|\psi\rangle$. The most convenient approach in order to treat these divergences is to consider the QFT in finite volume and thus to adopt a finite size approach. This choice permits to obtain information about both finite and infinite volume behaviours. Indeed the result will be given as a series expansion in power of ℓ/L therefore it is possible to take the limit $\ell \ll L$ in order to obtain infinite volume corrections.
In general, we will consider a QFT in finite volume L and a region ℓ. In this case, it is still possible to write the Rényi entropy in the form:

$$S_n^{|\psi\rangle}(\ell) = \frac{1}{1-n}\log\left(\frac{{}_L\langle\psi|\,\mathcal{T}(0)\tilde{\mathcal{T}}(\ell)\,|\psi\rangle_L}{{}_L\langle\psi|\psi\rangle_L}\right) \tag{3.61}$$

where now (3.61) involves correlation functions evaluated in finite volume. It is important to remark that $|\psi\rangle$ is defined in each copy, thus:

$$|\psi\rangle = |\psi_o\rangle_1 \otimes \cdots \otimes |\psi_o\rangle_n = |\psi_o\rangle^{\otimes n} \tag{3.62}$$

where $|\psi_o\rangle$ is the state in the original model. In the chapter 5 we will deal with the finite-volume approach in the replica model by using the U(1) twist fields.

3.5.1 Excited States In Conformal Field Theory

A special case is provided in [8] considering a finite system $A\bigcup B$ of length L with boundary condition on the space coordinate x such that $x + L = x$. The partition is implemented at length ℓ.
The main idea is to focus on the simplest excited state we can construct in CFT i.e. the state generated by acting on the vacuum $|0\rangle$ with the primary field $\Upsilon(\zeta,\bar{\zeta})$ with conformal weight $(h,\bar{h})$:

$$|\psi\rangle = \lim_{\zeta\to-\infty}\Upsilon(\zeta,\bar{\zeta})\,|0\rangle \tag{3.63}$$

where $\zeta = x + iy$ and $\bar{\zeta} = x - iy$ (with $0 \leqslant x \leqslant L$ and $-\infty \leqslant y \leqslant \infty$) describes a cylinder of circumference L and infinite height. Essentially the construction of the reduced density matrix ρ_A in the path integral formalism is similar to that implemented in section 3.3 but now we have to consider boundary condition on the cylinder. In order to do that we can take advantage of the conformal invariance and use appropriate conformal transformations (for details see [8, 9]) such that in the end we can express the Rényi entropy as:

$$S_n^{|\psi\rangle} - S_n^{|0\rangle} = \frac{1}{1-n}\log F_\Upsilon^{(n)}(x) \tag{3.64}$$

where $x = \ell/L$ and $F_{\Upsilon}^{(n)}$ quantifies the excess of entanglement of the excited state $|\psi\rangle$ with respect to the ground state. The latter is obtained by implementing conformal techniques as in [8] and it turns out to be related to the $2n$-point correlations functions of the primary field Υ and its conjugate $\Upsilon^\dagger$ evaluated on a cylinder of circumference 2π

$$F_{\Upsilon}^{(n)}(x) = n^{-2n(h+\bar{h})} \frac{\langle \prod_{k=0}^{n-1} \Upsilon(\frac{2\pi j}{n}) \Upsilon^\dagger(\frac{2\pi(j+1)}{n}) \rangle_{cy}}{[\langle \Upsilon(0) \Upsilon^\dagger(2\pi x) \rangle_{cy}]^n} \tag{3.65}$$

Considering the limit $\ell \ll L$ i.e. $x \ll 1$ the product $\Upsilon\Upsilon^\dagger$ in (3.65) can be expressed by the *Operator Product Expansion (OPE)*[9]:

$$\Upsilon \times \Upsilon^\dagger = \mathbb{1} + \Psi + \dots \tag{3.66}$$

where Ψ is the field with the lowest conformal dimension Δ_Ψ of the theory. The OPE allows us to evaluate $F_{\Upsilon}^{(n)}$:

$$F_{\Upsilon}^{(n)} = 1 + \frac{1-n^2}{3n} \pi^2 (h + \bar{h}) x^2 + O(x^{4\Delta_\psi}) \tag{3.67}$$

and therefore the Rényi entropy:

$$S_n^{|\psi\rangle} - S_n^{|0\rangle} = \frac{1+n}{3n} \pi^2 (h + \bar{h}) \Big(\frac{\ell}{L}\Big)^2 + O\Big(\frac{\ell}{L}\Big)^{4\Delta_\psi} \tag{3.68}$$

Finally, we can obtain the von Neumann entropy:

$$S_1^{|\psi\rangle} - S_1^{|0\rangle} = -\frac{\partial}{\partial n} F_{\Upsilon}^{(n)} = \frac{2\pi^2}{3} (h + \bar{h}) \Big(\frac{\ell}{L}\Big)^2 + O\Big(\frac{\ell}{L}\Big)^{4\Delta_\psi} \tag{3.69}$$

[9] In this thesis we do not discuss in detail the fundamentals of Conformal Field Theory. The definition of OPE can be found in [22].

Chapter 4

Integrability And Form Factors

4.1 The Scattering Picture

In QFT, it is possible to implement the structure of the Hilbert Space without knowing the details of the Hamiltonian by means of the relativistic invariance. Indeed, we can introduce the eigenbasis of Hamiltionian $|\theta\rangle_m$ describing one-particle states of mass m such that

$$H\,|\theta\rangle_m = m\cosh\theta\,|\theta\rangle_m\ , \quad P\,|\theta\rangle_m = m\sinh\theta\,|\theta\rangle_m\,, \quad B\,|\theta\rangle_m = -\frac{\partial}{\partial\theta}\,|\theta\rangle_m \tag{4.1}$$

where θ is the relativistic parameter known as rapidity and H, P, B are respectively the Hamiltonian, the momentum and the boost operator that satisfy the algebra

$$[H,P]=0 \quad [B,P]=H \quad [B,H]=P \tag{4.2}$$

thanks to the relativistic invariance. In case of other internal symmetry we have to consider additional quantum numbers in order to describe correctly the states. Notice that (4.1) leads to the well-known *mass-shell relation*:

$$H^2 - P^2 = m^2 \tag{4.3}$$

If we want to describe the scattering process of a multi-particle system, we can develop the multi-particle basis by the tensor product of the one particle states :

$$|\theta_1,\theta_2,\ldots\rangle_{m_1,m_2,\ldots} = |\theta_1\rangle_{m_1}\otimes|\theta_2\rangle_{m_2}\ldots \tag{4.4}$$

where we do not specify the particle number. In this context, we assume that the particles are initially located at very large distance each other, then, they will be at finite distance for a finite period of time before and after the interaction and finally, waiting enough (ideally infinite time) they might be infinitely separated (free again) or

involved in bound states. Thus, it is convenient to consider the *in* and *out asymptotic states* describing respectively the infinite past and infinite future of the system. They both form a eigenbasis of the multi-particle Hamiltonian H:

$$H\,|\theta_1,\theta_2,\ldots\rangle^{in,out}_{m_1,m_2,\ldots}=\sum_j m_j\cosh\theta_j\,|\theta_1,\theta_2,\ldots\rangle^{in,out}_{m_1,m_2,\ldots} \tag{4.5}$$

Furthermore assuming to know the *in* state we have to deal with a linear combination over the *out* states. This is essentially expressed by the *scattering matrix* whose elements describe how *in* and *out* states are related each other [23]:

$$|\theta_1,\theta_2,\ldots\,;\star_{in}\rangle^{in}_{a_1,a_2,\ldots}=\sum_{b_1,b_2,\ldots}\int d\theta_1 d\theta_2\ldots S^{b_1,b_2,\ldots}_{a_1,a_2,\ldots}(\theta_1,\theta_1',\theta_2'\,;\star_{in},\star_{out})\,\big|\theta_1',\theta_2,\theta_2',\ldots\,;\star_{out}\rangle^{out}_{b_1,b_2,\ldots} \tag{4.6}$$

where in the last expression we generalize the mass of the particle to the index a_j which contains all information about the particle species e.g. mass and quantum numbers, and, $\star_{in}$ denotes the set of impact parameters involved in the interactions. Indeed, the scattering processes may be realized in many ways. For instance, if we consider the three states $|a\rangle^{in}$, $|b\rangle^{in}$, $|c\rangle^{in}$ with the rapidities $\theta_a<\theta_b<\theta_c$, they may collide simultaneously (at some time t) or in such a way that the whole scattering process involves three different two-particle collisions, e.g. first $|a\rangle^{in}$ and $|b\rangle^{in}$ crash at time t_1 and then $|b\rangle^{in}$ and $|c\rangle^{in}$ at time t_2. Essentially, the asymptotic state may be defined by the asymptotic limit of some wave packets that are solutions of the field equations established by the species. The amplitudes of these wave packets may leave a trace of the different ways how to implement the scattering process and these will be described by the impact parameters [23]. These amplitudes define the multi-particle states on the θ-space, therefore also the latter may depend on the impact parameters.

4.2 Integrability

QFTs can be expressed in terms of local fields. The connection between the local-field picture and the scattering picture is non trivial and hard to find in general. However this connection emerges in *Integrable Quantum Field Theory (IQFT)*. Integrability is related to another important concept in field theories that is the *locality*. In particular, introducing $h(x)$ and $p(x)$ such that

$$H=\int dx\;h(x)\;,\quad P=\int dx\;p(x) \tag{4.7}$$

and H and P are respectively the Hamiltonian and the momentum operator of the system, we can say that the operator $\mathcal{O}(x)$ is a *local field* if [23]

$$\begin{cases} [\mathcal{O}(x), h(y)] = 0 & \text{for } x \neq y \\ [\mathcal{O}(x), P] = -i \, \frac{d}{dx}\mathcal{O}(x) \end{cases} \tag{4.8}$$

We can express locality with respect to another operator $\mathcal{O}'(y)$. Two operators are *local fields with respect to each other* if

$$[\mathcal{O}(x), \mathcal{O}'(y)] = 0 \qquad \text{for } x \neq y \tag{4.9}$$

Furthermore a *conserved quantity* Q is said to be *local* if it can be expressed as

$$Q = \int dx \; q(x) \tag{4.10}$$

and the density $q(x)$ is a local field.
We can now introduce the definition of IQFT [23]: A QFT is said integrable if there exists a set of infinitely many local conserved charge Q_j such that $[Q_j, Q_k] = 0 \; \forall j, k$, and whose densities are local with respect to each other.

4.2.1 Elastic Scattering

A trivial example of these local conserved quantities is provided by the H and P in the scattering picture. It is convenient to define the new quantities

$$Q_1 = m^{-1}(H + P) \; , \quad Q_{-1} = m^{-1}(H - P) \tag{4.11}$$

which present better properties under boost transformations. In (4.11), m is an arbitrary mass of the theory that is used to normalize the operators, it is normal to assume it the lowest mass. Since $Q_{\pm 1}$ are local conserved charges, these operators obviously commute with the Hamiltonian and have eigenvalues:

$$Q_{\pm 1} \, |\theta_1, \theta_2, \ldots\rangle^{in,out}_{a_1,a_2,\ldots} = \sum_j \tilde{q}^{(1)}_{a_j} e^{\pm\theta_j} \, |\theta_1, \theta_2, \ldots\rangle^{in,out}_{a_1,a_2,\ldots} \tag{4.12}$$

where the species-dependent part is given by $\tilde{q}^{(1)}_{a_j} = m_{a_j}/m$. It is clear from (4.12) that $Q_{\pm 1}$ transform with spin $s = \pm 1$ under boost. Furthermore, we can easily prove that $[B, Q_{\pm 1}]$ are local conserved quantities too. Indeed, $[B, q_{\pm 1}(x)]$ are local since $Q_{\pm 1} = \int dx \; q_{\pm 1}(x)$ are local conserved charges. In addition, using the *Jacobi identity*[1], it turns out from (4.2) and (4.8)

$$[[B, Q_{\pm 1}], H] = [B, [Q_{\pm 1}, H]] + [Q_{\pm 1}, [B, H]] = [Q_{\pm 1}, P] = 0 \tag{4.13}$$

[1] $[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$

and

$$\begin{aligned}[[B, q_{\pm 1}(x)], P] &= [B, [q_{\pm 1}(x), P]] + [q_{\pm 1}(x), [B, P]] \\ &= [B, -i\frac{d}{dx} q_{\pm 1}(x)] + [q_{\pm 1}(x), H] = -i\frac{d}{dx}[B, q_{\pm 1}(x)]\end{aligned} \tag{4.14}$$

this may be generalize for all the local charges Q_j of the integrable theory by construction, and ,it allows us to label the infinite set of local charges Q_s by using the spin s:

$$Q_s \, |\theta_1, \theta_2, \ldots\rangle^{in,out}_{a_1,a_2,\ldots} = \sum_j \tilde{q}^{(s)}_{a_j} \; e^{s\theta_j} \, |\theta_1, \theta_2, \ldots\rangle^{in,out}_{a_1,a_2,\ldots} \tag{4.15}$$

The quantity $\tilde{q}^{(s)}$ depends on the particle species and it can be related to the internal symmetry. Indeed it is usual to assume that this number is the same for different particles belonging to the same symmetry multiplet, otherwise it is different.

Considering (4.15) for a given *in* state $|\theta_1, \theta_2, \ldots\rangle^{in}_{a_1,a_2,\ldots}$ and then inserting on its left the corresponding equation (4.15) for the conjugate *out* state ${}^{out}_{a'_1,a'_2,\ldots}\langle\theta'_1, \theta'_2, \ldots|$,

$$\begin{aligned}&{}^{out}_{a'_1,a'_2,\ldots}\langle\theta'_1, \theta'_2, \ldots| \, Q^\dagger_s \, Q_s \, |\theta_1, \theta_2, \ldots\rangle^{in}_{a_1,a_2,\ldots} = \\ &{}^{out}_{a'_1,a'_2,\ldots}\langle\theta'_1, \theta'_2, \ldots| \left(\sum_k \tilde{q'}^{(s)}_{a'_k} \; e^{s\theta'_k} \right)^\dagger \left(\sum_j \tilde{q}^{(s)}_{a_j} \; e^{s\theta_j} \right) |\theta_1, \theta_2, \ldots\rangle^{in}_{a_1,a_2,\ldots}\end{aligned} \tag{4.16}$$

Since $Q^\dagger_s \, Q_s = \mathbb{1}$, (4.16) is non zero if and only if:

$$\sum_j \tilde{q'}^{(s)}_{a'_j} \; e^{s\theta'_j} = \sum_j \tilde{q}^{(s)}_{a_j} \; e^{s\theta_j} \tag{4.17}$$

The (4.17) is a strict consequence of the conservation of the charges and it provides a set of infinite equations. These lead to the solution:

$$\{\theta_j\}_j = \{\theta'_j\}_j \; , \quad \tilde{q}^{(s)}_{a_j} = \tilde{q'}^{(s)}_{a'_j} \tag{4.18}$$

namely the set of the out-going particles must be the same as the set of the in-coming particles. The exchange of the rapidities is allowed only for particles belonging to the same symmetry multiplet (because they present the same number $\tilde{q}^{(s)}_{a_j}$) and in this case it is also possible the exchange of particle types as long as it is allowed by the internal symmetry (the particle number of mass m_{a_j} $\forall j$ remains the same before and after the collision). As a result, **the IQFT describing scattering processes are elastic** and phenomena such as annihilation and production of particles are absent.

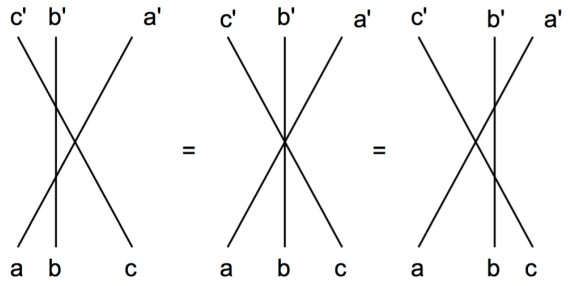

Figure 4.1: the three equivalent scattering processes of a three-particle state in IQFT.

4.2.2 The Factorized S-Matrix

The most important consequence of the elastic scattering in integrable models is that **the S-Matrix is independent of the choice of the impact parameters**.

Looking backward at the end of section 4.1, let us consider the example of the three one-particle states $|a\rangle^{in}$, $|b\rangle^{in}$ and $|c\rangle^{in}$ such that the rapidities $\theta_a < \theta_b < \theta_c$. We have already mentioned that each asymptotic one-particle state may be obtained by the asymptotic limit of the wave function derived by the field equation of the considered one-particle state. Each wave function is well-localized around the center of the packet (in the momentum space). In integral model, it is possible to shift differently for each particle the center of the wave packets by acting on it some operators e^{icQ_s} with $s \pm 1$ [21].[2] Since these operators commute with the Hamiltonian, their action must lead to equivalent physical situations [21], as a consequence there is no difference in the amplitudes of the asymptotic wave packets and the S-matrix is invariant under changes of the impact parameters.
Thus, we can always consider the scattering of three particles as described by the factorization of the three-particle S-matrix in three two-particle terms, whose element:

$$S_{a,b,c}^{a',b',c'}(\theta_a,\theta_b,\theta_c) = S_{a,b}^{\alpha,\beta}(\theta_a,\theta_b)S_{\alpha,c}^{a',\gamma}(\theta_a,\theta_c)S_{\beta,\gamma}^{b',c'}(\theta_b,\theta_c) \tag{4.19}$$

where the repeated indices are implicit summed.

The two equivalent ways how to realize this factorization lead to the so-called *Yang-Baxter equation*:

$$S_{a,b}^{\alpha,\beta}(\theta_a,\theta_b)S_{\alpha,c}^{a',\gamma}(\theta_a,\theta_c)S_{\beta,\gamma}^{b',c'}(\theta_b,\theta_c) = S_{b,c}^{\beta,\gamma}(\theta_b,\theta_c)S_{a,\gamma}^{\alpha,c'}(\theta_a,\theta_c)S_{\alpha,\beta}^{a',b'}(\theta_a,\theta_b) \tag{4.20}$$

where the figure 4.2.2 provides a pictorial representation.

[2] Q_1 shifts equally all the wave packets.

The previous result may be generalized for n-particle where the S-matrix factorizes into $n(n-1)/2$ two-particle terms. The properties of elasticity and factorization that the S-matrix show in IQFT allow us to focus on the two-particle S-matrix. In fact, **it is necessary to know only the structure of the two-particle S-matrix in order to analyze any other scattering processes**. In this respect, the two-particle S-matrix may be found as solution of the Yang-Baxer equation along with other requirements that we will discuss in the following subsection.

4.2.3 The Two-particle S-Matrix

In IQFT, we can essentially express the relation between the asymptotic two-particle states as:

$$|\theta_1, \theta_2\rangle^{(in)}_{a_1,a_2} = \sum_{b_1,b_2} S^{b_1,b_2}_{a_1,a_2}(\theta_1, \theta_2)\, |\theta_1, \theta_2\rangle^{(out)}_{b_1,b_2} \tag{4.21}$$

where $S^{b_1,b_2}_{a_1,a_2}(\theta_1, \theta_2)$ are the elements of the two-particle S-matrix.
We now list the main properties of the two-particle S-matrix. First of all, the Lorentz invariance implies:

$$S^{b_1,b_2}_{a_1,a_2}(\theta_1, \theta_2) = S^{b_1,b_2}_{a_1,a_2}(\theta_1 - \theta_2) \qquad \text{(Lorentz invariance)} \tag{4.22}$$

thus it depends only on the difference of the rapidities $\theta_1 - \theta_2$. Furthermore the orthonormality of both in and out states leads to the property:

$$\sum_{b_1,b_2} S^{b_1,b_2}_{a_1,a_2}(\theta)\big(S^{b_1,b_2}_{c_1,c_2}(\theta)\big)^* = \delta^{c_1}_{a_1}\delta^{c_2}_{a_2} \qquad \text{(unitarity relation)} \tag{4.23}$$

where we denotes $\theta = \theta_1 - \theta_2$. It is possible to identify by analytic arguments [28] a *physical sheet* on the θ-plane which provides the physical domain of the two particle S-matrix. This is provided by the strip $Im(\theta) \in [0, i\pi]$. The possible poles, i.e. the bound states, arise on the imaginary axis of this strip. An important relation that is valid on the physical sheet is :

$$\big(S^{b_1,b_2}_{a_1,a_2}(\theta)\big)^* = S^{a_2,a_1}_{b_2,b_1}(-\theta^*) \qquad \text{(Hermitian analyticity)} \tag{4.24}$$

combining (4.27) and (4.23) together:

$$\sum_{b_1,b_2} S^{b_1,b_2}_{a_1,a_2}(\theta) S^{c_2,c_1}_{b_2,b_1}(-\theta^*) = \delta^{c_1}_{a_1}\delta^{c_2}_{a_2} \tag{4.25}$$

If the system is invariant under parity:

$$S^{b_1,b_2}_{a_1,a_2}(\theta) = S^{b_2,b_1}_{a_2,a_1}(\theta) \qquad \text{(parity symmetry)} \tag{4.26}$$

the validity of (4.27) and (4.28) lead to:

$$\left(S_{a_1,a_2}^{b_1,b_2}(\theta)\right)^* = S_{a_1,a_2}^{b_1,b_2}(-\theta^*) \qquad \text{(real analyticity)} \tag{4.27}$$

Where the S-matrix is real for pure immaginary values of θ. Furthermore for time-independent Hamiltonian:

$$S_{a_1,a_2}^{b_1,b_2}(\theta) = S_{\bar{a}_2,\bar{a}_1}^{\bar{b}_2,\bar{b}_1}(\theta) \qquad \text{(time-reversal symmetry)} \tag{4.28}$$

where $\bar{a}_i$ and $\bar{b}_j$ denotes the corresponding anti-particles. Because of the Charge-Parity-Time-reversal symmetry:

$$S_{a_1,a_2}^{b_1,b_2}(\theta) = S_{b_1,b_2}^{a_1,a_2}(\theta) \qquad \text{(CPT symmetry)} \tag{4.29}$$

The last important property we mention is the *crossing symmetry*. This is a consequence of QFT, making a $\frac{\pi}{2}$ rotation of the the space and time axis the result must provide the same amplitudes. We essentially implement the transformation $\theta \to \frac{i\pi}{2} - \theta$ with θ on the physical sheet and then the time-reversal transformation. It turns out:

$$S_{a_1,a_2}^{b_1,b_2}(i\pi - \theta) = S_{\bar{b}_2,a_1}^{\bar{a}_2,b_1}(\theta) \qquad \text{(crossing symmetry)} \tag{4.30}$$

It is important to say a few words about bound states. Let us consider the case that the scattering between to incoming particles described by $|\theta_1, \theta_2\rangle_{i,j}$ leads to a bound state and this occurs at the rapidity $\theta = iu_{ij}^n$ with $u_{ij}^n \in [0, \pi]$. Since the bound state corresponds to a single pole of the S-matrix can be expressed:

$$S_{ij}^{kl}(\theta \sim iu_{ij}^n) \sim i\frac{(\ \Gamma_{i,j}^n\ )^2}{\theta - iu_{ij}^n} \tag{4.31}$$

where k, l are the indices characterizing the out-going particles and $\Gamma_{i,j}^n$ is the on-shell three-particle vertex functions [21]. In integrable model, it is usual to assume the so-called *nuclear democracy* namely the bound states are assumed to belonging to the asymptotic state spectrum [23]. As a result, the bound states are on the same footing of the asymptotic particles. This leads to additional requirements. A first evidence of them can be provided by the fact that this bound states occurs in the s-channel and it implies a relation among the masses and u_{ij}^n:

$$m_n^2 = m_i^2 + m_j^2 + 2m_i m_j \cos u_{ij}^n \tag{4.32}$$

Notice that (4.32) can be seen as the geometrical relation known as *Carnot's theorem* where the sides of the triangles are given by the masses m_i, m_j and m_n, this leads to a geometric relation among the external angles of this triangle:

$$u_{ij}^n + u_{in}^j + u_{jn}^i = 2\pi \tag{4.33}$$

where u^j_{in} and u^i_{jn} correspond to the poles in different channels.

In the context of Integrability, it can be convenient to introduce the so-called *Zamolodchikov - Faddeev algebra* [28, 23]. It essentially consists in associating to each asymptotic basis a specific order of the set $\{\theta_1, \dots, \theta_k\}$ and an operator Z_{a_j} for each particle of the species a_j, which represents the asymptotic wave packet of the considered particle:

$$\begin{cases} Z_{a_1}(\theta_1) \dots Z_{a_k}(\theta_k) \; ; & \textit{in states} : \theta_1 > \dots > \theta_k \\ Z_{a_k}(\theta_k) \dots Z_{a_1}(\theta_1) \; ; & \textit{out states} : \theta_1 < \dots < \theta_k \end{cases} \tag{4.34}$$

the factorization of the k-particle S-matrix into 2-particle terms implies that the two-particle interactions are events quite separated to each other. These two-particle scattering processes may be thought as interaction involving some "intermediate states", which are neither *in* nor *out* states, whose basis can be implemented by any other orders of $\{\theta_1, \dots, \theta_k\}$. It is important to stress that this makes sense only in IQFT thanks to the independence of the k-particle S-matrix from the impact parameter that makes possible such factorization.
After introducing this notation, we can rewrite (4.21) as :

$$Z_{a_1}(\theta_1) Z_{a_2}(\theta_2) = S^{b_1,b_2}_{a_1,a_2}(\theta_1 - \theta_2) \; Z_{b_2}(\theta_2) Z_{b_1}(\theta_1) \qquad \text{for } \theta_1 > \theta_2 \tag{4.35}$$

where we omit the sum, and, the Zamolodchikov-Faddeev algebra may be defined by the relations:

$$Z_{a_1}(\theta_1) Z_{a_2}(\theta_2) - S^{b_1,b_2}_{a_1,a_2}(\theta_1 - \theta_2) \; Z_{b_2}(\theta_2) Z_{b_1}(\theta_1) = 0 \tag{4.36}$$

$$\bar{Z}^{a_1}(\theta_1) \bar{Z}^{a_2}(\theta_2) - S^{a_1,a_2}_{b_1,b_2}(\theta_1 - \theta_2) \; \bar{Z}^{b_2}(\theta_2) \bar{Z}^{b_1}(\theta_1) = 0 \tag{4.37}$$

$$Z_{a_1}(\theta_1) \bar{Z}^{a_2}(\theta_2) - S^{a_2,b_1}_{b_2,a_1}(\theta_2 - \theta_1) \; \bar{Z}^{b_2}(\theta_2) Z_{b_1}(\theta_1) = 2\pi \delta^{a_2}_{a_1} \; \delta(\theta_1 - \theta_2) \tag{4.38}$$

As a consequence, we can represent the Hilbert space described by the basis

$$|\theta_1, \dots, \theta_j\rangle^{in,out}_{a_1,\dots,a_j} \qquad \forall \; j = 1, \dots, k \tag{4.39}$$

as the Fock space over the algebra (4.36) with vacuum defined by

$$\begin{cases} \bar{Z}^{\alpha} \, |0\rangle = 0 & \forall \; \alpha\text{-species} \\ \bar{Z}^{\alpha} = (Z_{\alpha})^{\dagger} \end{cases} \tag{4.40}$$

where the "creation" of the one-particle state is now associated to the operator Z_{α}.

Additional simplifications may be provide by requiring the diagonal form of the two-particle S-matrix. This can occur in two cases [21]:

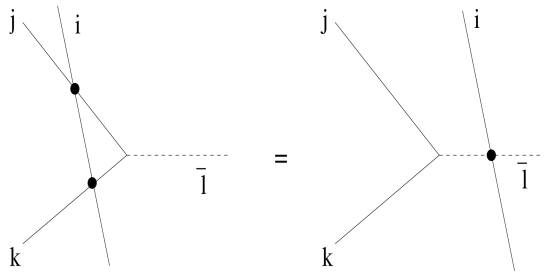

Figure 4.2: the boostrap principle: it is possible to shift the world-line of the particle Z_i thanks to integrability. This picture is taken from [21].

(i) the system has non-degenerate mass spectrum i.e each particle $Z_{a_j}(\theta)$ with different mass m_j.

(ii) the system has degenerate mass spectrum, but each particle is identified by the other quantum numbers i.e. a_j is uniquely associated to the particle $Z_{a_j}(\theta)$.

In these cases the two-particle S-matrix becomes:

$$S^{b_1,b_2}_{a_1,a_2}(\theta) = \delta^{b_1}_{a_1}\,\delta^{b_2}_{a_2}\,S_{a_1,a_2}(\theta) \qquad \text{for } \theta = \theta_1 - \theta_2 \tag{4.41}$$

we can sum up all the equations for the *diagonal two-particle S-matrix* for parity invariant system:

$$S_{i,\,l}(\theta) = S_{\,l,i}(\theta) \qquad \text{(parity symmetry)} \tag{4.42}$$

$$S_{i,\,l}(\theta) = \big(S_{i,\,l}(-\theta^*)\big)^* \qquad \text{(real analyticity)} \tag{4.43}$$

$$S_{i,\,l}(\theta)\;S_{i,\,l}(-\theta) = \mathbb{1} \qquad \text{(unitarity)} \tag{4.44}$$

$$S_{i,\,l}(\theta) = S_{i,\,\bar{l}}(i\pi - \theta) \qquad \text{(crossing symmetry)} \tag{4.45}$$

$$S_{i,\,\bar{l}}(\theta) = S_{i,j}(\theta + \bar{u}^k_{j,\,l})\;S_{i,k}(\theta - u^j_{\,l,k}) \qquad \text{(boostrap principle)} \tag{4.46}$$

where $\bar{l}$ denotes the anti-particle and $\bar{u}^k_{j,\,l} := \pi - u^k_{j,\,l}$. The boostrap principle is a consequence of the nuclear democracy whose pictorial representation is provided by figure 4.2. In many cases, these equations are sufficient to fix the S-matrix completely [28] and in general they provide the basis for the *S-matrix boostrap* namely the procedure implemented in order to find solution for the two-particle S-matrix. Sometimes the solution may be completely fixed up to some functions that do not involve new poles in the physical sheet (this is known as the *CDD ambiguity* [28]).

4.3 Form Factors Of Local Fields

Following the notation introduced in the previous subsection, we can express the asymptotic state by using the Zamolodchikov - Faddeev algebra for diagonal two-particle S-matrix. In particular, we assume the following notation for the asymptotic states:

$$\begin{aligned} |in\rangle &: |\theta_1, \theta_2 \ldots, \theta_k\rangle_{a_1,a_2,\ldots,a_k} \propto V_{a_1}(\theta_1)\, V_{a_2}(\theta_2) \ldots V_{a_k}(\theta_k)\, |0\rangle \\ |out\rangle &: |\theta_k, \theta_{k-1} \ldots, \theta_1\rangle_{a_1,a_2,\ldots,a_k} \propto V_{a_k}(\theta_k)\, V_{a_{k-1}}(\theta_{k-1}) \ldots V_{a_1}(\theta_1)\, |0\rangle \end{aligned} \tag{4.47}$$

where the order of θ_j into the kets denotes the asymptotic basis we are implementing and the $V_{a_j}(\theta_j)$ is the "creation operator" that satisfies the Zamolodchikov-Faddeev algebra (4.36). An important consequence of the algebra is [29]:

$$|\ldots, \theta_j, \theta_{j+1}, \ldots\rangle_{\ldots,a_j,a_{j+1},\ldots} = S_{a_j,a_{j+1}}(\theta_j - \theta_{j+1})\, |\ldots, \theta_{j+1}, \theta_j, \ldots\rangle_{\ldots,a_{j+1},a_j,\ldots} \tag{4.48}$$

namely each exchange over neighbour particles introduces a two-particle S-matrix. Another consequence it that the normalization of the one-particle states is fixed by the algebra:

$$_a\langle\theta_a|\theta_b\rangle_b = 2\pi\, \delta_{ab}\, \delta(\theta_a - \theta_b) \tag{4.49}$$

Form Factors of local operators can by thought as the building blocks of the corresponding correlation functions in some particular states. In particular the *elementary Form Factor*[3]:

$$F_k^{\mathcal{O}|_{a_1,\ldots,a_k}}(\theta_1, \ldots, \theta_k) = \langle 0|\, \mathcal{O}(0)\, |\theta_1, \ldots, \theta_k\rangle_{a_1,\ldots,a_k} \tag{4.50}$$

whose a graphical representation is provided in figure 4.3.

4.3.1 Form Factor Bootstrap

A first relevant property is provided by Lorentz invariance that establishes how (4.50) of a local operator $\mathcal{O}$ with spin s transforms under simultaneous translation of the rapidities [21]:

$$F_k^{\mathcal{O}|_{a_1,\ldots,a_k}}(\theta_1 + \Lambda, \ldots, \theta_k + \Lambda) = e^{s\Lambda}\, F_k^{\mathcal{O}|_{a_1,\ldots,a_k}}(\theta_1, \ldots, \theta_k) \tag{4.51}$$

this means that the elementary form factors of local scalar operators only depend on rapidity differences.
Like for the S-matrix, integrability constrains Form Factors so strongly that it is possible to find exact solution in many cases. The basis of the Form Factor bootstrap programm may be derived by the scattering properties of the states and written as [31]:

[3] we here follow the convention introduced in [29, 30] in which the Authors consider the matrix element of an arbitrary operator located in 0 as a form factor.

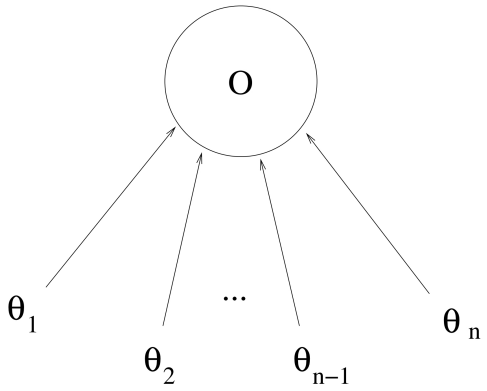

Figure 4.3: the n-particle elementary Form Factor of the local operator O placed at the origin. This graph is taken from [21].

I. Exchange :

$$F_k^{\mathcal{O}|_{a_1,\dots,a_j,a_{j+1},\dots,a_k}}(\theta_1,\dots,\theta_j,\theta_{j+1},\dots,\theta_k) = \tag{4.52}$$
$$S_{a_j,a_{j+1}}(\theta_j-\theta_{j+1})\; F_k^{\mathcal{O}|_{a_1,\dots,a_{j+1},a_j,\dots,a_k}}(\theta_1,\dots,\theta_{j+1},\theta_j,\dots,\theta_k)$$

II. Cyclic property :

$$F_k^{\mathcal{O}|_{a_1,a_2,\dots,a_k}}(\theta_1+2i\pi,\theta_2,\dots,\theta_k) = \tag{4.53}$$
$$F_k^{\mathcal{O}|_{a_2,\dots,a_k,a_1}}(\theta_2,\dots,\theta_k,\theta_1)$$

III. Kinematical poles :

$$-\,i\,\mathrm{Res}_{\bar\theta=\theta}\; F_{k+2}^{\mathcal{O}|_{\bar a,a,a_1,\dots,a_k}}(\bar\theta+i\pi,\theta,\theta_1,\dots,\theta_k) = \tag{4.54}$$
$$\Big(\,1-\prod_{j=1}^{k} S_{a,a_j}(\theta-\theta_j)\,\Big)\; F_k^{\mathcal{O}|_{a_1,\dots,a_k}}(\theta_1,\dots,\theta_k)$$

IV. Dynamical poles :

$$-\,i\,\mathrm{Res}_{\tilde\theta=\theta}\; F_{k+2}^{\mathcal{O}|_{\alpha,\beta,a_1,\dots,a_k}}(\theta+i\bar u^{\beta}_{\alpha\gamma}/2,\;\tilde\theta-i\bar u^{\alpha}_{\beta\gamma}/2,\;\theta_1,\dots,\theta_k) =$$
$$\Gamma^{\gamma}_{\alpha\beta}\; F_{k+1}^{\mathcal{O}|_{\gamma,a_1,\dots,a_k}}(\theta,\theta_1,\dots,\theta_k) \tag{4.55}$$

In order to have a power law behaviour of the two-point correlation of the field $\mathcal{O}(x)$, there is an additional condition for the corresponding form factors: these must behave at most as an exponential term in the follow limit [21].

$$\lim_{|\theta_i|\to\infty} F_k^{\mathcal{O}|_{a_1,a_2,\dots,a_k}}(\theta_1,\theta_2,\dots,\theta_k) \sim e^{c\,|\theta_i|} \qquad \forall i=1,\dots,k \tag{4.56}$$

and c is a (i-independent) constant related to the conformal weight of the field $\mathcal{O}$ i.e. $c < \Delta$.[4]

4.3.2 Correlation Functions

In general, we can define form factors by the matrix elements of the local Operator. Following [29, 30]:

$$F_{m,n}^{\mathcal{O}|_{a_1,\ldots,a_m;b_1,\ldots,b_n}}(\theta'_1,\ldots,\theta'_m|\ \theta_1,\ldots,\theta_n) = {}_{a_1,\ldots,a_m}\left\langle \theta'_1,\ldots,\theta'_m\right|\mathcal{O}(0)\left|\theta_1,\ldots,\theta_n\right\rangle_{b_1,\ldots,b_n} \quad (4.57)$$

Form factor such as (4.57) are related to each other thanks to the recursive equation:

$$\begin{aligned}
&F_{m,n}^{\mathcal{O}|_{a_1,\ldots,a_m;b_1,\ldots,b_n}}(\theta'_1,\ldots,\theta'_m|\ \theta_1,\ldots,\theta_n) = \qquad (4.58)\\
&F_{m-1,n+1}^{\mathcal{O}|_{a_1,\ldots,a_{m-1};\bar{a}_m,b_1,\ldots,b_n}}(\theta'_1,\ldots,\theta'_{m-1}|\ \theta'_m+i\pi,\theta_1,\ldots,\theta_n)\\
&+\sum_{j=1}^{n}\Big[\ 2\pi\delta_{a_j\,a_m}\ \delta(\theta'_m-\theta_j)\prod_{p=1}^{j-1}\ S_{a_p\,a_j}(\theta'_p-\theta_j)\ \times\\
&F_{m-1,n-1}^{\mathcal{O}|_{a_1,\ldots,a_{m-1};b_1,\ldots,b_{j-1},b_{j+1},\ldots,b_n}}(\theta'_1,\ldots,\theta'_{m-1}|\ \theta_1,\ldots,\theta_{j-1},\theta_{j+1},\ldots,\theta_n)\ \Big]
\end{aligned}$$

where $\bar{a}_m$ denotes the anti-particle of a_m. Indeed, using appropriate iterations of the equation (4.58), we are able to express the form factors in terms of the elementary form factor (4.50). Indeed, (4.58) may be derived by construction from the bootstrap equations.

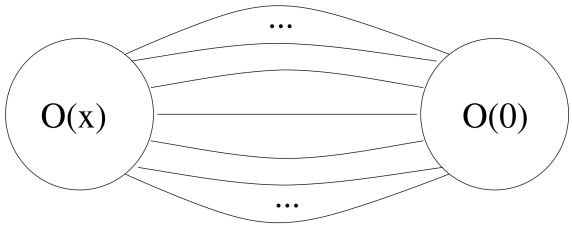

Figure 4.4: The effect of inserting a complete set of states into a correlation function. This graph is taken from [21].

Two-point correlation functions in the vacuum state may be given in terms of elementary form factors by the spectral decomposition in the euclidean space [29]:[5]

[4]The conformal weight dictates the short-distance behaviour of the correlation function $\langle\mathcal{O}(\ell)\mathcal{O}(0)\rangle \sim r^{-4\Delta}$ for $r \to 0$.

[5]we essentially use the completeness relation (3.4.1).

$$\langle 0|\,\mathcal{O}(x,y)\tilde{\mathcal{O}}(0,0)\,|0\rangle = \sum_{k=0}^{\infty}\sum_{a_1\ldots a_k}\int \frac{d\theta_1\ldots d\theta_k}{(2\pi)^k\,k!}F_k^{\mathcal{O}|_{a_1,\ldots,a_k}}(\theta_1,\ldots,\theta_k)\,\times \\ \left[F_k^{\tilde{\mathcal{O}}|_{a_1,\ldots,a_k}}(\theta_1,\ldots,\theta_k)\right]^*\,\exp\Big[-\ell\sum_{j=1}^{k}m_{a_j}\cosh\theta_j\Big] \tag{4.59}$$

where $\ell=\sqrt{x^2+y^2}$ the euclidean distance and we use the complex conjugate state:

$$\left[F_k^{\tilde{\mathcal{O}}|_{a_1,\ldots,a_k}}(\theta_1,\ldots,\theta_k)\right]^* := {}_{a_1,\ldots,a_k}\langle\theta_1,\ldots,\theta_k|\,\tilde{\mathcal{O}}(0,0)\,|0\rangle = F_k^{\tilde{\mathcal{O}}|_{\bar{a}_1,\ldots,\bar{a}_k}}(\theta_1+i\pi,\ldots,\theta_k+i\pi) \tag{4.60}$$

The equation (4.59) gives the large-distance behavior of the correlation function as the exponential rapidly converges in this limit. In this case, it is referred to as the *large-distance form factor expansion*. On the other hand, for the short-distance behaviour we need to consider other terms in the sum, although in same cases it has been observed a very fast convergence also for the short-distance limit. The short-distance properties has been analyzed in many works, for instance [32].

The n-point correlation functions may be generalized inserting $n-1$ times the relation (3.4.1) although they take a more complicated form [21].

It is worthwhile to say that the form factor expansion of the two-point correlation functions in excited states involve form factors such as (4.57). These will be discussed in section 4.5 when we employ a finite volume approach.

4.4 Form Factors Of Twist Fields

Because of the geometry, the form factor of twist fields (introduced in section 3.2.1) have different features and the set of equation (I-IV) showed the last section must be modified [7].
Let us considering an integrable model consisting in n copies of a (1+1)-IQFT. Furthermore, let us suppose the scattering processes in each copy are described by a diagonal S-matrix such as (4.41) and the bound states are absent, then the two-particle S-matrix in the n-copy model takes the following form:

$$S_{\mu_i\,\mu_j}(\theta_{ij}) = S_{a_i\,a_j}(\theta_{ij})^{\delta_{n_i,n_j}} = \begin{cases} S_{a_i\,a_j}(\theta_{ij}) & \text{for } n_i = n_j \\ 1 & \text{for } n_i \neq n_j \end{cases} \tag{4.61}$$

and we now label the ith-particle $\mu_i=(a_i,n_i)$ where a_i denotes the particle species and n_i the copy in which the particle lives. Thus, choosing an appropriate order of the rapidities

$\{\theta_i\}_i$, e.g. $\theta_1 > \dots \theta_k$ for the incoming states, the states

$$|\theta_1, \dots, \theta_k\rangle_{\mu_1 \dots \mu_k} \tag{4.62}$$

provide a basis of Hilbert space related to the replica model. S-matrix elements such as (4.61) tell us that particles localized in different copies do not interact each other.
As we have already discussed in section 3.2.1, the existence of the permutation symmetry among the copies implies the existence of the branch-point twist field $\mathcal{T}$ and anti-twist field $\tilde{\mathcal{T}}$. If ϕ represents the state of a fundamental particle of the original IQFT, the corresponding state in the replica model is $\Phi = (\phi_1, \dots, \phi_n)$ consisting in n copies of ϕ and the effects of the twist fields is to implement the symmetry as

$$\begin{cases} \phi_i(x,y)\,\mathcal{T}(x',y) = \mathcal{T}(x',y)\,\phi_{i+1}(x,y) & \text{for } x' > x,\ y' = y \\ \phi_i(x,y)\,\mathcal{T}(x',y) = \mathcal{T}(x',y)\,\phi_i(x,y) & \text{for } x' < x,\ y' = y \end{cases} \tag{4.63}$$

and

$$\begin{cases} \phi_i(x,y)\,\tilde{\mathcal{T}}(x',y) = \tilde{\mathcal{T}}(x',y)\,\phi_{i-1}(x,y) & \text{for } x' > x,\ y' = y \\ \phi_i(x,y)\,\tilde{\mathcal{T}}(x',y) = \tilde{\mathcal{T}}(x',y)\,\phi_i(x,y) & \text{for } x' < x,\ y' = y \end{cases} \tag{4.64}$$

for $i = 1, \dots, n$ such that $i + n = i$. A first consequence of these relations it that we have on the Hilbert space:

$$\tilde{\mathcal{T}} = \mathcal{T}^\dagger \tag{4.65}$$

We can define the elementary form factor of the twist field on a n-copy model as:

$$F_k^{\mathcal{T}|_{\mu_1,\dots,\mu_k}}(\theta_1, \dots, \theta_k\,; n) = \langle 0|\,\mathcal{T}(0)\,|\theta_1, \dots, \theta_k\rangle_{\mu_1,\dots,\mu_k} \tag{4.66}$$

An intrinsic assumption when we define Form Factors is the locality of the fields. Because of the properties (4.63) and (4.64), the fields creating the one-particle states on the n-copy model are semi-local with respect to the twist field $\mathcal{T}$ and $\tilde{\mathcal{T}}$ (see the subsection 3.2.1). Since both fields are involved in (4.66) it is necessary to extend the boostrap equations and it will lead to different cyclic property and the kinematical poles equations. However for the propose of this thesis we do not analyze this case.

In fact, in the chapter 5, we will consider a doubled boson theory obtained by doubling a free boson theory. In this case, it is possible to see that the branch-point twist field implementing the cyclic permutation symmetry can be decomposed into a direct product of $U(1)$ fields (induced by the doubling). We will see that these fields show simpler properties and allow us to adopt a finite-volume approach.
Fields such as the U(1) fields are defined by relations including the *factor of local commutativity* ω [33] which takes into account the semi-locality between the fields in the θ-space:[6]

$$V_{\mu_i}(\theta_i)\,\mathcal{O}(0) = \omega\,\mathcal{O}(0)\,V_{\mu_i}(\theta_i) \tag{4.67}$$

[6]This factor ω was first introduced in [34]. The locality may be implemented simply by requiring $\omega = 1$.

The modification of the equations (I-III) can be implemented by including ω and as a result the cyclic property (4.53) and the kinematical poles (4.55) become

$$\begin{aligned} F_k^{\mathcal{O}|_{\mu_1,\mu_2,\ldots,\mu_k}}(\theta_1+2i\pi,\theta_2,\ldots,\theta_k) &= \omega\, F_k^{\mathcal{O}|_{\mu_2,\ldots,\mu_k,\tilde{\mu}_1}}(\theta_2,\ldots,\theta_k,\theta_1) \\ &= \omega \prod_{i=2,k} S_{\mu_i,\mu_1}(\theta_i-\theta_1)\, F_k^{\mathcal{O}|_{\mu_1,\ldots,\mu_k}}(\theta_1,\ldots,\theta_k) \end{aligned} \tag{4.68}$$

and

$$-i\,\mathrm{Res}_{\bar{\theta}=\theta}\, F_{k+2}^{\mathcal{O}|_{\bar{\mu},\mu,\mu_1,\ldots,\mu_k}}(\bar{\theta}+i\pi,\theta,\theta_1,\ldots,\theta_k) = \left(1-\omega\prod_{i=1,k} S_{\mu,\mu_i}(\theta-\theta_i)\right) F_k^{\mathcal{O}|_{\mu_1,\ldots,\mu_k}}(\theta_1,\ldots,\theta_k) \tag{4.69}$$

The equations (4.68) and (4.69), along with the cyclic property (4.53) are implemented in appendix A in order to obtain the two-particle form factor of the U(1) fields.

4.5 Form Factors In Finite Volume

In section 3.5, we have seen we are interested in employing a the finite size approach. In general, in finite volume L a k-particle state $|\theta_1,\ldots,\theta_k\rangle_{a_1,\ldots,a_k}$ becomes quantized because the wave function defining the state is periodic along the space direction. This enables us to map the system into a cylinder of circumference L. The quantization conditions may expressed by the system of k non-linear equation known as the *Bethe-Yang equations*:

$$e^{im_{a_i}\,L\sinh\theta_i}\prod_{j\neq i}^{k} S_{a_i\,a_j}(\theta_i-\theta_j)=1 \qquad j=1,\ldots,k \tag{4.70}$$

where m_{a_j} is the mass of the particle a_j and $m_{a_i}\sinh\theta_i$ is the momentum of the single particle. (4.70) is a consequence of S-matrix factorization and holds in all integrable systems. Furthermore (4.70) may be seen as the analogy of the box quantization in a quantum integrable model. In large volume, the energy of the k-particle state may be expressed:

$$E_L(\{\theta_k\}) = E_o(L) + \sum_{j=1}^{k} m_{a_j}\cosh\theta_j + O\left(e^{-\mu L}\right) \tag{4.71}$$

where $E_o(L)$ is the *Casimir Energy* that is the finite volume ground state energy and depends on the normalization. For our propose it is reasonable to assume $E_o(L)=0$. The exponential correction come from virtual scattering process due to compactness on the cylinder and depends on a characteristic mass scale μ [35].

In Chapter 5 we will analyze a doubled free boson theory. Therefore for our proporse, we consider only free boson theories. In this case the S-matrix is simply:

$$S_{a_i\, a_j}(\theta) = 1 \qquad \forall i, j = 1, \ldots, k \tag{4.72}$$

and (4.70) provides the solutions:

$$Q_j = m_{a_j}\, L \sinh\theta_j = 2\pi I_j \qquad \forall j = 1, \ldots, k \tag{4.73}$$

where I_j are integers for boson theory. This implies that, in finite volume, the k-particle state $|\theta_1, \ldots, \theta_k\rangle_{a_1,\ldots,a_k}$ is described by the set of quantum numbers $\{I_1, \ldots, I_k\}$.

Denoting by $|\{I_1, \ldots, I_k\}\rangle_{a_1,\ldots,a_k,L}$ the finite volume k-particle states, if we consider a complete set of these, we can expand the two-point correlation function in finite volume [29]:

$$\begin{aligned}\langle 0|\, \mathcal{O}(\tau,0)\tilde{\mathcal{O}}(0,0)\, |0\rangle_L = \sum_{k=0}^{\infty} \sum_{a_1\ldots a_k} \sum_{I_1,\ldots,I_k} &\times \langle 0|\, \mathcal{O}(0,0)\, |\{I_1, \ldots, I_k\}\rangle_{a_1,\ldots,a_k,L} \\ &{}_{a_1,\ldots,a_k,L}\langle \{I_1, \ldots, I_k\}|\, \mathcal{O}(0,0)\, |0\rangle \exp\Big[-\tau \sum_{j=1}^{k} m_{a_j} \cosh\theta_j \Big]\end{aligned} \tag{4.74}$$

where for simplicity we have restricted the formula to separation in Euclidian time τ. Recalling the correlation function in infinite volume $\langle 0|\, \mathcal{O}(\tau,0)\tilde{\mathcal{O}}(0,0)\, |0\rangle$ (4.59), it is possible to argue, using the finite volume expansion developed by Lüscher in [35], that the finite-volume correlation function differs from (4.59) only for terms that are rapidly suppressed in large volume:[7]

$$\langle 0|\, \mathcal{O}(\tau,0)\tilde{\mathcal{O}}(0,0)\, |0\rangle - \langle 0|\, \mathcal{O}(\tau,0)\tilde{\mathcal{O}}(0,0)\, |0\rangle_L \sim O\left(e^{-\mu L}\right) \tag{4.75}$$

it follows:

$$\langle 0|\, \mathcal{O}(0,0)\, |\{I_1, \ldots, I_k\}\rangle_{a_1,\ldots,a_k,L} = \frac{F_k^{\mathcal{O}|_{a_1,\ldots,a_k}}(\theta_1, \ldots, \theta_k)}{\sqrt{\rho_{a_1,\ldots,a_k}(\theta_1, \ldots, \theta_k)}} + O\left(e^{-\mu L}\right) \tag{4.76}$$

where ρ is nothing but the Jacobi determinant corresponding to the change of variables $2\pi I_j \to \theta_j$. In free theories ρ takes a diagonal form:

$$\rho_{a_1,\ldots,a_k}(\theta_1, \ldots, \theta_k) = \prod_{j=1}^{n} m_{a_j} L \cosh\theta_j \tag{4.77}$$

[7] Lüscher's finite expansion is expected to hold non-perturbatively.

furthermore (4.78) may be understood as a relation between the two bases:

$$|\{I_1,\dots,I_k\}\rangle_{a_1,\dots,a_k,L} = \frac{1}{\sqrt{\rho_{a_1,\dots,a_k}(\theta_1,\dots,\theta_k)}}\,|\theta_1,\dots,\theta_k\rangle_{a_1,\dots,a_k} + O\left(e^{-\mu L}\right) \tag{4.78}$$

Considering the case when $\{I_1,\dots,I_k\}$ and $\{I'_1,\dots,I'_m\}$ are two disjoint sets it is possible to construct:

$$\begin{aligned} &{}_{a'_1,\dots,a'_m,L}\langle\{I'_1,\dots,I'_m\}|\,\mathcal{O}(0,0)\,|\{I_1,\dots,I_k\}\rangle_{a_1,\dots,a_k,L} = \\ &\frac{F^{\mathcal{O}|_{a'_1,\dots,a'_m,a_1,\dots,a_k}}_{m+k}(\theta'_m+\pi,\dots,\theta'_1+\pi,\theta_1,\dots,\theta_k)}{\sqrt{\rho_{a'_1,\dots,a'_m}(\theta'_1,\dots,\theta'_m)\,\rho_{a_1,\dots,a_k}(\theta_1,\dots,\theta_k)}} + O\left(e^{-\mu L}\right) \end{aligned} \tag{4.79}$$

these elements are called *off-diagonal form factors*. It is important to stress that this result holds only when the rapidities $\{\theta_j\}$ and $\{\theta'_i\}$ are all different from each other. Indeed, only in this case the disconnected pieces in (4.58) vanish. Otherwise, one has to take into account the *off-diagonal form factors* that occur when the state on the left and on the right are the same [30].

We will see later that for the double free boson only off-diagonal form factors contribute. As a result, the elements (4.79) provide the building blocks we need to construct the two-point correlation functions in excited states. In section 5.3.1 we will extend this result to the replica model.

Chapter 5

The Doubled Free Boson

5.1 The Doubling Trick

In this chapter we will adopt an approach known as the *doubling trick* and based on the work [10]. The authors developed this technique in order to find some differential equations known as *Painlevè equations* that are satisfied by appropriate combinations of the correlation functions in the *Ising model*[1].
The main idea is to double the theory in order to first analyze a complex version of the theory then to come back to the real case by requiring appropriate conditions. For real Majorana fermion, considering two copies of its fundamental fields namely $(\psi_a, \bar{\psi}_a)$ and $(\psi_b, \bar{\psi}_b)$ it is easy to construct a new field describing the Dirac fermion:

$$\Psi = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_a + i\bar{\psi}_b \\ \psi_a - i\bar{\psi}_b \end{pmatrix} \tag{5.1}$$

which possesses an internal continuous $U(1)$ symmetry under rotation.
Similarly, we are now interested in doubling the free boson field therefore let us consider two boson fields ϕ_a and ϕ_b , we can introduce the corresponding *complex free boson fields*:

$$\Phi = \frac{\phi_a + i\phi_b}{\sqrt{2}} \quad , \quad \Phi^\dagger = \frac{\phi_a - i\phi_b}{\sqrt{2}} \tag{5.2}$$

once again, the doubled theory possesses a $U(1)$ symmetry.

In the context of the replica trick, we now deal with n copies of the complex boson:

$$\begin{pmatrix} \Phi_1 \\ \vdots \\ \Phi_n \end{pmatrix} \tag{5.3}$$

[1]The analogy between the content of the Ising model and the real Majorana Fermion fields is well-known [21] and discussed in [36].

where each complex field Φ_i is obtained by coupling the two free boson fields $\phi_a^{(i)}$ and $\phi_b^{(i)}$ in copy i.
In section 3.2, we have discussed that twist fields are present whenever the theory presents an internal symmetry. The advantage of doubling the theory arises from the presence of a $U(1)$ symmetry in each copy of the replica model. As a consequence, this $U(1)$ symmetry will be implemented by some twist fields defined on each copy. This permits to build the branch-point twist field $\mathcal{T}$ (associated to the permutation symmetry of the copies) in terms of those associated to the $U(1)$ symmetry on each copy.

5.1.1 The complex Free Boson

The free real boson fields ϕ_a and ϕ_b are described by (1+1) QFT and in particular they satisfy the *Klein-Goldon equation* therefore they may be expressed by the mode expansion:

$$\phi_a(x,t) = \int d\theta \ \left[a\,(\theta)\ e^{iP(\theta)\,x - iE(\theta)\,t} + a^\dagger(\theta)\ e^{-iP(\theta)\,x + iE(\theta)\,t}\right] \tag{5.4}$$

where we introduced the rapidity θ to parametrize the energy

$$E(\theta) = m\cosh\theta\ ; \quad P(\theta) = m\sinh\theta \tag{5.5}$$

and $a\,(\theta)$ and $a^\dagger(\theta)$ are respectively the annihilation and creation operators associated to the real free boson field ϕ_a. These operators define the Hilbert space $\mathcal{H}_a$:

$$\begin{aligned} a\,(\theta)\,|0\rangle_a &= 0 \\ a^\dagger(\theta)\,|0\rangle_a &= |a(\theta)\rangle_a \end{aligned} \tag{5.6}$$

where $|0\rangle_a$ is the vacuum and $a^\dagger(\theta)$ creates a one-particle state in the θ-space. Furthermore a and $a^\dagger$ satisfy the algebra:

$$[a(\theta), a(\theta')] = 0 = [a^\dagger(\theta), a^\dagger(\theta')] \tag{5.7}$$

$$[a(\theta), a^\dagger(\theta')] = \delta(\theta - \theta') \tag{5.8}$$

Similarly, we can express ϕ_b by equivalent relations where now the operators $a\,(\theta)$ and $a^\dagger(\theta)$ are replaced with $b\,(\theta)$ and $b^\dagger(\theta)$.

The complex free boson fields Φ and $\Phi^\dagger$ obey the Klein-Goldon equation whose solutions may be expressed:

$$\Phi(x,t) = \int d\theta \ \left[\alpha\,(\theta)\ e^{iP(\theta)\,x - iE(\theta)\,t} + \beta^\dagger(\theta)\ e^{-iP(\theta)\,x + iE(\theta)\,t}\right] \tag{5.9}$$

$$\Phi^\dagger(x,t) = \int d\theta \ \left[\beta\,(\theta)\ e^{iP(\theta)\,x - iE(\theta)\,t} + \alpha^\dagger(\theta)\ e^{-iP(\theta)\,x + iE(\theta)\,t}\right] \tag{5.10}$$

where now $\alpha^\dagger(\theta)$ and $\alpha(\theta)$ are understood as the creation and annihilaton operators of the particles associated to the fields Φ, while $\beta^\dagger(\theta)$ and $\beta(\theta)$ creates and destroys the corresponding anti-particles.
Inverting the relation (5.1) and comparing (5.4) of ϕ_a and ϕ_b with (5.9) and (5.10), we obtain relations among the creation and annihilation operators:

$$a(\theta) = \frac{\alpha(\theta) + \beta(\theta)}{\sqrt{2}} \; ; \quad a^\dagger(\theta) = \frac{\alpha^\dagger(\theta) + \beta^\dagger(\theta)}{\sqrt{2}} \tag{5.11}$$

and

$$b(\theta) = \frac{\alpha(\theta) - \beta(\theta)}{i\sqrt{2}} \; ; \quad b^\dagger(\theta) = -\frac{\alpha^\dagger(\theta) - \beta^\dagger(\theta)}{i\sqrt{2}} \tag{5.12}$$

In addition the commutation relations (5.7) (5.8) lead to a similar algebra for $\alpha, \alpha^\dagger, \beta, \beta^\dagger$:

$$[\alpha(\theta), \alpha^\dagger(\theta')] = \delta(\theta - \theta') \tag{5.13}$$
$$[\beta(\theta), \beta^\dagger(\theta')] = \delta(\theta - \theta') \tag{5.14}$$

all other commutation relations are zero, since ϕ_a and ϕ_b commute with each other. As a consequence, we can express an arbitrary states $|\Phi\rangle_c$ created by the operators $\alpha^\dagger(\theta)$ and $\beta^\dagger(\theta)$ as the tensor product of states on the Hilbert spaces $\mathcal{H}_a$ and $\mathcal{H}_b$:

$$|\Phi\rangle_c = |\phi\rangle_a \otimes |\phi\rangle_b \tag{5.15}$$

This factorization holds also for branch-point twist fields:

$$\mathcal{T}_{complex} = \mathcal{T}_a \otimes \mathcal{T}_b \tag{5.16}$$

where $\mathcal{T}_a$ and $\mathcal{T}_b$ are the twist fields involved in the single real theory. Consequently

$$\begin{aligned} {}_c\langle\Phi|\, \mathcal{T}_{complex}\, |\Phi\rangle_c &= {}_a\langle\phi| \otimes {}_b\langle\phi|\ \ \mathcal{T}_a \otimes \mathcal{T}_b\ \ |\phi\rangle_a \otimes |\phi\rangle_b \\ &= {}_a\langle\phi|\, \mathcal{T}_a\, |\phi\rangle_a \ \times\ {}_b\langle\phi|\, \mathcal{T}_b\, |\phi\rangle_b \end{aligned} \tag{5.17}$$

and similarly for the two-point correlation function

$${}_c\langle\Phi|\, \mathcal{T}(0)\ \tilde{\mathcal{T}}(\ell)\, |\Phi\rangle_c = {}_a\langle\phi|\, \mathcal{T}_a(0)\ \tilde{\mathcal{T}}_a(\ell)\, |\phi\rangle_a \ \times\ {}_b\langle\phi|\, \mathcal{T}_b(0)\ \tilde{\mathcal{T}}_b(\ell)\, |\phi\rangle_b \tag{5.18}$$

If we choose $|\Phi\rangle_c = |\phi\rangle_a \otimes |0\rangle_b$ in the end, we will be able to come back to the real boson, thanks to (5.18):

$${}_a\langle\phi|\, \mathcal{T}_a(0)\ \tilde{\mathcal{T}}_a(\ell)\, |\phi\rangle_a = \frac{{}_c\langle\Phi|\, \mathcal{T}(0)\ \tilde{\mathcal{T}}(\ell)\, |\Phi\rangle_c}{{}_b\langle 0|\, \mathcal{T}_b(0)\ \tilde{\mathcal{T}}_b(\ell)\, |0\rangle_b} \tag{5.19}$$

and we will get results for the free real boson up to the quantity ${}_b\langle 0|\, \mathcal{T}_b(0)\ \tilde{\mathcal{T}}_b(\ell)\, |0\rangle_b$. This choice is motivated by (3.68) where the result is expressed by the difference between the

entanglement entropy in the excited state and that in the vacuum.

Generalizing, let us consider the replica model with n copy of the previous (1+1)-QFT, the complex boson theory on the n-copy model is given by

$$|\Phi\rangle = |\phi\rangle_1 \otimes \cdots \otimes |\phi\rangle_n \tag{5.20}$$

where $|\phi\rangle_i$ is a state created by the operators $\alpha_i^\dagger(\theta)$ and $\beta_i^\dagger(\theta)$ on the ith-copy of the replica model. The vacuum state $|0\rangle$ defines the Hilbert space in which $|\Phi\rangle$ lives:

$$|0\rangle = |0\rangle_1 \otimes \cdots \otimes |0\rangle_n \tag{5.21}$$

such that the operators $\alpha_i(\theta)$ and $\beta_i(\theta)$ destroy the vacuum state in the ith-copy:

$$\alpha_i(\theta)\ |0\rangle_i = 0 = \beta_i(\theta)\ |0\rangle_i \tag{5.22}$$

while the operators $\alpha_j^\dagger(\theta)$ and $\beta_j^\dagger(\theta)$ create particles and anti-particles in the jth-copy and the corresponding algebra in the replica model is given by

$$[\alpha_i(\theta), \alpha_j^\dagger(\theta')] = \delta_{ij}\ \delta(\theta - \theta') \tag{5.23}$$

$$[\beta_i(\theta), \beta^\dagger(\theta')] = \delta_{ij}\ \delta(\theta - \theta') \tag{5.24}$$

while all other commutation relations are zero.

5.2 The $U(1)$ Twist Field Action

The action of the branch-point twist field $\mathcal{T}$ on the single-copy field may be represented in the matrix form by introducing the *T-matrix*. The latter act on the fields as:

$$T \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \vdots \\ \Phi_n \end{pmatrix} = \begin{pmatrix} \Phi_2 \\ \Phi_3 \\ \vdots \\ \Phi_1 \end{pmatrix} \tag{5.25}$$

as a consequence, the T-matrix takes the following form [37]:

$$T = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix} \tag{5.26}$$

It is easy to see that its eigenvalues are the n-th root of unity $\lambda_k = e^{i2\pi\frac{k}{n}}$ for $k = 0, \ldots, n-1$. Changing basis by a unitary transformation we can diagonalize T and the eigenvectors $\tilde{\Phi}_k$ of T are related to the fields Φ_j by[2]

$$\tilde{\Phi}_k = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} e^{i\,2\pi j\frac{k}{n}}\ \Phi_j = \begin{pmatrix} e^{i\,2\pi\frac{k}{n}}\Phi_1 \\ e^{i\,4\pi\frac{k}{n}}\Phi_2 \\ \vdots \\ \Phi_n \end{pmatrix} \tag{5.27}$$

Similarly we can diagonalize T^{-1} which represents the action of the $U(1)$ anti-twist field and we obtain the eigenvalues $\lambda_k = e^{-i2\pi\frac{k}{n}}$ for $k = 0, \ldots, n-1$. Then, we can write the complex conjugate fields of (5.27) as:

$$\tilde{\Phi}_k^\dagger = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} e^{-i\,2\pi j\frac{k}{n}}\ \Phi_j^\dagger = \begin{pmatrix} e^{-i\,2\pi\frac{k}{n}}\Phi_1^\dagger & e^{-i\,4\pi\frac{k}{n}}\,\Phi_2^\dagger & \ldots & \Phi_n^\dagger \end{pmatrix} \tag{5.28}$$

Both Φ_k and $\Phi_k^\dagger$ are complex boson fields $\forall k = 0, \ldots, n-1$ since their commutation relations are:

$$\begin{aligned} [\tilde{\Phi}_{k_1}, \tilde{\Phi}_{k_2}] &= \left[\frac{1}{\sqrt{n}} \sum_{j_1=1}^{n} e^{i\,2\pi j_1\frac{k_1}{n}}\ \Phi_{j_1}\ , \frac{1}{\sqrt{n}} \sum_{j_2=1}^{n} e^{i\,2\pi j_2\frac{k_2}{n}}\ \Phi_{j_2} \right] \\ &= \frac{1}{n} \sum_{j_1,j_2=1}^{n} e^{i\,2\pi(j_1\frac{k_1}{n}+j_2\frac{k_2}{n})}\ [\Phi_{j_1}, \Phi_{j_2}] = 0 \end{aligned} \tag{5.29}$$

Analogously $[\tilde{\Phi}_{k_1}^\dagger, \tilde{\Phi}_{k_2}^\dagger] = 0$.
In the θ-space, the fields $\tilde{\Phi}_k$ and $\tilde{\Phi}_k$ involve the operators $(\mathfrak{a}_k^+)^\dagger(\theta)\ \forall k$ which create particles $(+)$ and anti-particles $(-)$ associated to the new basis while the operators $\mathfrak{a}_k^\pm(\theta)$ destroy the vacuum $|0\rangle_k$:

$$\mathfrak{a}_k^\pm(\theta)\,|0\rangle_k = 0 \tag{5.30}$$

In particular, the creation operators are related to the old basis by (5.27) and (5.28):

$$(\mathfrak{a}_k^+)^\dagger(\theta) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} e^{i\,2\pi j\frac{k}{n}}\ \alpha_j^\dagger(\theta) \quad ; \quad (\mathfrak{a}_k^-)^\dagger(\theta) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} e^{i\,2\pi j\frac{k}{n}}\ \beta_j^\dagger(\theta) \tag{5.31}$$

$$\alpha_j^\dagger(\theta) = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} e^{-i\,2\pi j\frac{k}{n}}\ (\mathfrak{a}_k^+)^\dagger(\theta) \quad ; \quad \beta_j^\dagger(\theta) = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} e^{-i\,2\pi j\frac{k}{n}}\ (\mathfrak{a}_k^-)^\dagger(\theta) \tag{5.32}$$

[2]We here label the nth-copy of the model 0 for the new basis. This choice is always possible and does not change the results. Furthermore, this new notation is motivated by the trivial properties of the U(1)-fields on this copy.

As a consequence, $(\mathfrak{a}_k^{\pm})^\dagger(\theta)$ and $\mathfrak{a}_k^{\pm}(\theta)$ satisfy the algebra (5.23):

$$[\,\mathfrak{a}_i^{s_i}(\theta)\,,(\mathfrak{a}_k^{s_k})^\dagger(\theta')\,] = \delta^{s_i\,s_k}\;\delta_{i\,k}\;\delta(\theta-\theta') \tag{5.33}$$

$$[\,\mathfrak{a}_i^{s_i}(\theta)\,,\mathfrak{a}_k^{s_k}(\theta')\,] = 0 = [\,(\mathfrak{a}_i^{s_i})^\dagger(\theta)\,,(\mathfrak{a}_k^{s_j})^\dagger(\theta')\,] \tag{5.34}$$

The relevance of this basis arises from the exchange relation between twist fields and the fields $\tilde{\Phi}_k$. Since the T-matrix acts diagonally on the new basis, it holds:

$$\tilde{\Phi}_j(x,t)\;\mathcal{T}_{\frac{k}{n}}(y,t) = \left(e^{i2\pi\frac{k}{n}}\right)^{\delta_{jk}}\mathcal{T}_{\frac{k}{n}}(y,t)\;\tilde{\Phi}_j(x,t) \qquad \text{For } y>x \tag{5.35}$$

$$\tilde{\Phi}_j(x,t)\;\mathcal{T}_{\frac{k}{n}}(y,t) = \mathcal{T}_{\frac{k}{n}}(y,t)\;\tilde{\Phi}_j(x,t) \qquad \text{For } y<x \tag{5.36}$$

where we have introduced the $U(1)$ twist field $\mathcal{T}_{\frac{k}{n}}$ and $j,k=0,\dots,n-1$ such that $j+n=j$.

Following the section 3.2, the partition function of the QFT on the Riemann surface is related to the two-point correlation function of the twist fields on the replica model by (3.17). If we implement the $U(1)$ twist fields $\mathcal{T}_{\frac{k}{n}}$ and $\mathcal{T}_{-\frac{k}{n}}$ over all the replica model, we will obtain the original twist field and anti-twist field associated to the permutation symmetry as:

$$\mathcal{T} = \prod_{k=0}^{n-1}\mathcal{T}_{\frac{k}{n}} \quad ; \quad \tilde{\mathcal{T}} = \prod_{k=0}^{n-1}\mathcal{T}_{-\frac{k}{n}} \tag{5.37}$$

In particular, considering a state on the n-copy model such as (5.20), we can factorize the correlation function in the free boson excited state as:

$$Z^{|\Phi\rangle}[n] \propto \langle\Phi|\,\mathcal{T}(0)\tilde{\mathcal{T}}(\ell)\,|\Phi\rangle = \prod_{k=0}^{n-1}\;{}_k\langle\phi|\,\mathcal{T}_{\frac{k}{n}}(0)\mathcal{T}_{-\frac{k}{n}}(\ell)\,|\phi\rangle_k \tag{5.38}$$

These twist fields $\mathcal{T}_{\frac{k}{n}}$ are well known in literature [38, 39] and the corresponding conformal weight $\Delta_{k,n}$ is given by CFT:

$$\Delta_{k,\,n} = \frac{1}{2}\frac{k}{n}\left(1-\frac{k}{n}\right) \tag{5.39}$$

The sum reproduces (3.27) with $c=2$ i.e. complex free boson:

$$d_n = 2\sum_{k=0}^{n-1}\Delta_{k,\,n} = \frac{1}{6}\left(n-\frac{1}{n}\right) \tag{5.40}$$

Notice that the field $\mathcal{T}_0$ for $k=0$ is the Identity.

5.3 Entanglement Entropy In One-particle Excited State

Let us assume that the state $|\Phi\rangle$ in (5.38) is simply n copies pf a one-particle state. In order to obtain a real free boson state in the end, we choose $|\Phi\rangle = (|\phi\rangle_a \otimes |0\rangle_b)^n$ which is implemented by the operator $a_j^\dagger(\theta)$ in each copy. Using (5.11), we can write the state as:

$$|\Phi\rangle = \prod_{j=1}^{n} a_j^\dagger(\theta)\ |0\rangle_j = \frac{1}{\sqrt{2^n}} \prod_{j=1}^{n} (\,\alpha_j^\dagger(\theta) + \beta_j^\dagger(\theta)\,)\,|0\rangle_j \tag{5.41}$$

Changing basis in order to diagonalize the T-matrix, we obtain:

$$\begin{aligned} |\Phi\rangle &= \frac{1}{\sqrt{(2n)^n}} \prod_{j=1}^{n} \left(\sum_{k=0}^{n-1} e^{-i\,2\pi j\frac{k}{n}}\ (\mathfrak{a}_k^+)^\dagger\,(\theta) + \sum_{k=0}^{n-1} e^{-i\,2\pi j\frac{k}{n}}\ (\mathfrak{a}_k^-)^\dagger\,(\theta) \right) |0\rangle_j \\ &= \frac{1}{\sqrt{(2n)^n}} \sum_{s_1,\ldots,s_n=\pm} \sum_{k_1,\ldots,k_n=1}^{n} e^{-\frac{i2\pi(s_1k_1+2s_2k_2+\cdots+ns_nk_n)}{n}}\ (\mathfrak{a}_{k_1}^{s_1})^\dagger(\theta)\,(\mathfrak{a}_{k_2}^{s_2})^\dagger(\theta)\ldots(\mathfrak{a}_{k_n}^{s_n})^\dagger(\theta)\,|0\rangle \end{aligned} \tag{5.42}$$

Thus the state $|\Phi\rangle$ may be expanded by appropriate n-particle states. Particles and anti-particles are characterized by the same mass m. In this context, it is convenient to express equation (5.42) in terms of the population in each-copy, therefore we may define the set $\{N^\pm\} = \{N_0^+, N_0^-, \ldots, N_{n-1}^+, N_{n-1}^-\}$, where N_k^s is the number of particle ($s = +$) or anti-particles ($s = -$) that live in the kth-copy. Thus, (5.42) becomes:

$$|\Phi\rangle = \sum_{\{N^\pm\}} \mathcal{A}(\{N^\pm\}) \prod_{k=0}^{n-1} [(\mathfrak{a}_k^+)^\dagger(\theta)]^{N_k^+}\ [(\mathfrak{a}_k^-)^\dagger(\theta)]^{N_k^-}\ |0\rangle_k \tag{5.43}$$

where $\mathcal{A}(\{N^\pm\})$ contain the multiplicity associated to the configuration $\{N^\pm\}$ and the factor due to the phases in (5.43). It is clear from (5.42) that

$$N := \sum_k (N_k^+ + N_k^-) = n \qquad \forall\ \{N^\pm\} \tag{5.44}$$

We are interested in the correlation function (5.38):

$$\begin{aligned} \langle\Phi|\,\mathcal{T}(0)\tilde{\mathcal{T}}(\ell)\,|\Phi\rangle &= \sum_{\{N^\pm\},\{M^\pm\}} \mathcal{A}^*(\{N^\pm\})\,\mathcal{A}(\{M^\pm\}) \times \\ \prod_{k=0}^{n-1}\ {}_k\langle 0|\ [\mathfrak{a}_k^-\,(\theta)]^{N_k^-}\ [\mathfrak{a}_k^+\,(\theta)]^{N_k^+}\,\mathcal{T}_{\frac{k}{n}}(0)\ \mathcal{T}_{-\frac{k}{n}}(\ell)\ [(\mathfrak{a}_k^+)^\dagger(\theta)]^{M_k^+}\ [(\mathfrak{a}_k^-)^\dagger(\theta)]^{M_k^-}\ |0\rangle_k \end{aligned} \tag{5.45}$$

Since $\mathcal{T}_0(0) = \mathbb{1} = \mathcal{T}_0(\ell)$, their contribution is simply:

$$\begin{aligned} \mathcal{N}_0 &= {}_0\langle 0|\ [\mathfrak{a}_0^-\,(\theta)]^{N_0^-}\ [\mathfrak{a}_0^+\,(\theta)]^{N_0^+}\mathcal{T}_0(0)\ \mathcal{T}_0(\ell)\ [(\mathfrak{a}_0^+)^\dagger(\theta)]^{M_0^+}\ [(\mathfrak{a}_0^-)^\dagger(\theta)]^{M_0^-}\ |0\rangle_0 \\ &= {}_0\langle 0|\ [\mathfrak{a}_0^-\,(\theta)]^{N_0^-}\ [\mathfrak{a}_0^+\,(\theta)]^{N_0^+}[(\mathfrak{a}_0^+)^\dagger(\theta)]^{M_0^+}\ [(\mathfrak{a}_0^-)^\dagger(\theta)]^{M_0^-}\ |0\rangle_0 \end{aligned} \tag{5.46}$$

using the algebra (5.23), $\mathcal{N}_0$ constrains the states $\{N_k^\pm\}$ and $\{M_k^\pm\}$ in (5.45) to have the same number of particles and antiparticles on the 0th-copy:

$$N_0^+ - N_0^- = M_0^+ - M_0^- \tag{5.47}$$

In order to express (5.45) in terms of form factors, we now introduce a sum over a complete set of state between the fields $\mathcal{T}_{\frac{k}{n}}$ and $\mathcal{T}_{-\frac{k}{n}}$ in each copy except to 0:

$$\begin{aligned}\mathbb{1} = \sum_{p_k=0}^{\infty} \sum_{s_1,\ldots,s_{p_k}=\pm} \frac{1}{p_k!} \int_{-\infty}^{\infty} \frac{d\theta_1}{2\pi} \int_{-\infty}^{\infty} \frac{d\theta_2}{2\pi} \ldots \int_{-\infty}^{\infty} \frac{d\theta_{p_k}}{2\pi} \times \\ (\mathfrak{a}_k^{s_1})^\dagger(\theta_1)\,(\mathfrak{a}_k^{s_2})^\dagger(\theta_2)\ldots(\mathfrak{a}_k^{s_{p_k}})^\dagger(\theta_{p_k})\,|0\rangle_k \;\; {}_k\langle 0|\;\; \mathfrak{a}_k^{s_{p_k}}(\theta_{p_k})\,\mathfrak{a}_k^{s_{p_k}-1}(\theta_{p_k-1})\ldots\mathfrak{a}_k^{s_1}(\theta_{p_1})\end{aligned} \tag{5.48}$$

As a result, the correlation function may be written as

$$\begin{aligned}\langle\Phi|\,\mathcal{T}(0)\tilde{\mathcal{T}}(\ell)\,|\Phi\rangle = \sum_{\{N^\pm\},\{M^\pm\}} \mathcal{A}^*(\{N^\pm\})\,\mathcal{A}(\{M^\pm\})\,\mathcal{N}_0 \;\times \\ \prod_{k=1}^{n-1} \sum_{p_k=0}^{\infty} \sum_{s_1,\ldots,s_{p_k}=\pm} \int \frac{d\theta_1\ldots d\theta_{p_k}}{p_k!\,(2\pi)^{p_k}} \exp\left[m\ell\left(\sum_{i=1}^{p_k}\cosh\theta_i - (M_k^+ + M_k^-)\cosh\theta\right)\right] \\ F_{\frac{k}{n}}^{N_k^\pm\,;\,s_1,\ldots,s_{p_k}}(\theta\ldots|\theta_1,\ldots,\theta_{p_k})\;\left[F_{\frac{k}{n}}^{M_k^\pm\,;\,s_1,\ldots,s_{p_k}}(\theta\ldots|\theta_1,\ldots,\theta_{p_k})\right]^*\end{aligned} \tag{5.49}$$

where we introduce the form factor

$$\begin{aligned}F_{\frac{k}{n}}^{N_k^\pm\,;\,s_1,\ldots,s_{p_k}}(\theta\ldots|\theta_1,\ldots,\theta_{p_k}) = \\ {}_k\langle 0|\;\;[\mathfrak{a}_k^-(\theta)]^{N_k^-}\;\;[\mathfrak{a}_k^+(\theta)]^{N_k^+}\;\;\mathcal{T}_{\frac{k}{n}}(0)\;\;(\mathfrak{a}_k^{s_1})^\dagger(\theta_1)\,(\mathfrak{a}_k^{s_2})^\dagger(\theta_2)\ldots(\mathfrak{a}_k^{s_{p_k}})^\dagger(\theta_{p_k})\,|0\rangle_k\end{aligned} \tag{5.50}$$

which involves $g_k := N_k^+ + N_k^- + p_k$ particles. Although the equation (5.49) appears quite complicated, the $U(1)$ symmetry imposes many constrains:

(i) Only form factors with even particle number g_k are non-zero.

(ii) Defining $N_k^+ - N_k^-$ the charge of the state on the kth-copy, the overall charge in (5.50) must be neutral:

$$N_k^+ - N_k^- = \sum_{k=0}^{n-1} s_k \qquad \forall k = 1,\ldots,n-1 \tag{5.51}$$

As a result, only form factors involving the same amount of particles and antiparticles may be non-zero. Furthermore it turns out from (5.49):

$$N_k^+ - N_k^- = M_k^+ - M_k^- \qquad \forall k = 0,\ldots,n-1 \tag{5.52}$$

We can consequently rewrite (5.50) as:

$$F_{\frac{k}{n}}^{N_k^{\pm}\,;\,s_1,\ldots,s_{p_k}}(\theta\ldots|\theta_1,\ldots,\theta_{p_k}) = F_{\frac{k}{n}}^{+-+-\cdots+-}(\theta\ldots|\theta_1,\ldots,\theta_{p_k}) \tag{5.53}$$

The two-particle form factor may be obtained by solving the Form Factor Boostrap equations. We have implemented these in appendix A and the solution is given by (according with [39, 40]):

$$\begin{aligned} F_{\frac{k}{n}}^{+-}(\theta_+ - \theta_-) &= \frac{{}_k\langle 0|\ \mathcal{T}_{\frac{k}{n}}(0)\ (\mathfrak{a}_k^+)^\dagger(\theta_+)\ (\mathfrak{a}_k^-)^\dagger(\theta_-)\,|0\rangle_k}{\langle \mathcal{T}_{\frac{k}{n}}\rangle_k} \\ &= -\sin\left(\frac{\pi k}{n}\right)\frac{e^{(\frac{k}{n}-\frac{1}{2})(\theta_+-\theta_-)}}{\cosh\frac{(\theta_+-\theta_-)}{2}} \\ &= F_{\frac{k}{n}}^{-+}(\theta_- - \theta_+) \end{aligned} \tag{5.54}$$

where the latter expression is a consequence of the boson nature. Higher particle form factors may be obtained by the two-particle form factors (5.54), for instance, for form factors involving $2m$ particles :

$$\begin{aligned} &F_{\frac{k}{n}}^{+-\cdots+-}\left(\theta_+^{(1)},\ldots,\theta_+^{(m)},\theta_-^{(1)}\ldots,\theta_-^{(m)}\right) = \\ &\langle \mathcal{T}_{\frac{k}{n}}\rangle \sum_{\sigma,\kappa\,\in\,S_m} F_{\frac{k}{n}}^{+-}\left(\theta_+^{\sigma(1)}-\theta_-^{\kappa(1)}\right)\ldots F_{\frac{k}{n}}^{+-}\left(\theta_+^{\sigma(m)}-\theta_-^{\kappa(m)}\right) \end{aligned} \tag{5.55}$$

where σ and κ are permutations over the sets $\{\theta_1,\ldots,\theta_m\}$ and $\{\beta_1,\ldots,\beta_m\}$.

5.3.1 The Finite-Volume Approach On The Replica Model

As we have already discussed in section 3.5, we are interested in the finite size approach. In general, considering an euclidean (1+1) QFT in finite volume, we require the periodicity along the space axis $\Phi(0,t) = \Phi(L,t)$, where L is the length of the finite volume. This allows us to map the system into a cylinder with infinite height parametrized by τ while the space axis is compactified along the circumference L and described by the parameter σ. In this context, the periodicity condition may be expressed by the Bethe-Yang equations.

In the context of the replica trick, the n-copy model is composed of n copies of the cylinder where now the branch cut lies along the σ-direction (see **a)** in figure (5.1)). However, the presence of the branch cut along the circumference L may affect the quantization conditions.

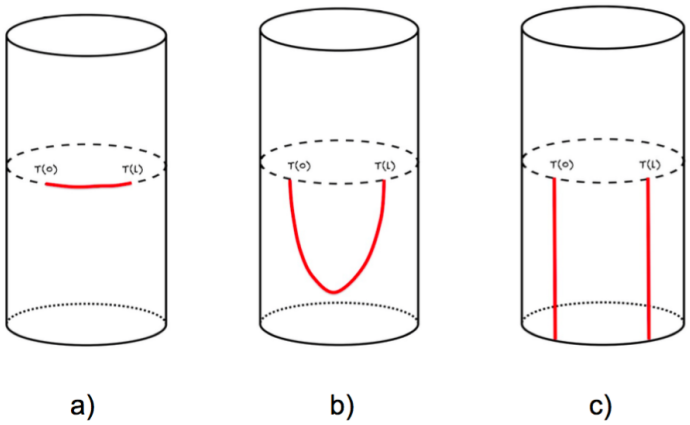

Figure 5.1: a single copy of the replica model in finite volume with the branch cut along the circumference is represented in **a)** . Deforming the branch cut as in **b)** it is possible to obtain two semi-infinite branch cuts along the time-direction as in **c)**.

Since the correlation function is invariant under continuous changes of the shape of the branch cut, in finite volume we may understand equation (5.49) as the insertion of intermediate states has been deforming the branch cut along the space direction to basically get two semi-infinite branch cuts along the time direction, a pictorial representation is provided by figure 5.1. This is a crucial point in order to obtain the right periodic condition in the replica model.
In particular the periodicity condition of the intermediate states must be modified because of (5.35). If a particle $\alpha_j^{\pm}$ performs a trip around the space direction, it will surely gain a phase due to (5.35):

$$e^{imL\sinh\theta_j} = e^{\mp\frac{i2\pi\, k_j}{n}} \tag{5.56}$$

where the sign $+/-$ denotes the particle/anti-particles associated to the field $\tilde{\Phi}_j$. Then the quantization condition for the finite-size intermediate state becomes:

$$Q_j^{\pm} = mL\sinh\theta_j \pm \frac{2\pi\, k_j}{n} = 2\pi\, I_j^{\pm} \quad I_j \in \mathbb{Z} \tag{5.57}$$

where the integer number I_i is because of the boson nature. In contrast, the periodicity condition of the original states Φ_i are not affected by the action of the U(1) twist fields $\mathcal{T}_{\frac{k}{n}}$:

$$Q = mL\,\sinh\theta = 2\pi\, I \quad I \in \mathbb{Z} \tag{5.58}$$

Since the matrix $\frac{\partial Q_j^{\pm}}{\partial \theta_l}$ is diagonal

$$\frac{\partial Q_j^{\pm}}{\partial \theta_l} = \delta_{j\,l}\; mL\cosh(\theta_j) \tag{5.59}$$

the Bethe-Yang Jacobi determinant can be expressed simply as the product of the diagonal elements of (5.59). As a result, the finite-volume intermediate states may be related to the infinite-volume states by:

$$|\theta_1,\ldots,\theta_{p_k}\rangle_{s_1,\ldots,s_p\,;\,L} = \frac{(\mathfrak{a}_k^{s_1})^\dagger(\theta_1)\,(\mathfrak{a}_k^{s_2})^\dagger(\theta_2)\ldots(\mathfrak{a}_k^{s_{p_k}})^\dagger(\theta_{p_k})}{\sqrt{(mL)^{p_k}\cosh\theta_1\ldots\cosh\theta_{p_k}}}\;|0\rangle_k \tag{5.60}$$

where $\theta_1,\ldots,\theta_{p_k}$ are related to the finite volume quantum number $\{I^{s_1}\ldots I^{s_{p_k}}\}$ by (5.57). While the external states of (5.49) may be expressed by:

$$\left|\theta;N_k^+,N_k^-\right\rangle_L = \frac{\left[(\mathfrak{a}_k^-)^\dagger\,(\theta)\right]^{N_k^-}\,\left[(\mathfrak{a}_k^+)^\dagger\,(\theta)\right]^{N_k^+}}{\sqrt{\left[mL\cosh\theta\right]^{N_k^++N_k^-}}}\;|0\rangle_k \tag{5.61}$$

Thus the correlation function (5.49) in finite volume takes the form:

$$\begin{aligned}
{}_L\langle\Phi|\,\mathcal{T}(0)\tilde{\mathcal{T}}(\ell)\,|\Phi\rangle_L = &\sum_{\{N^\pm\},\{M^\pm\}} \mathcal{A}^*(\{N^\pm\})\,\mathcal{A}(\{M^\pm\})\,\mathcal{N}_0\;\times \\
&\prod_{k=1}^{n-1}\sum_{p_k=0}^{\infty}\sum_{s_1,\ldots,s_{p_k}=\pm}\sum_{I_1^{s_1},\ldots,I_{p_k}^{s_{p_k}}\in\mathbb{Z}}\frac{1}{p_k!\,(2\pi)^{p_k}}\;\exp\left[m\ell\left(\sum_{i=1}^{p_k}\cosh\theta_i-(M_k^++M_k^-)\cosh\theta\right)\right] \\
&F_{\frac{k}{n}}^{N_k^\pm,p_k\,;L}(\theta\ldots|\theta_1,\ldots,\theta_{p_k})\;\left[F_{\frac{k}{n}}^{M_k^\pm,p_k\,;L}(\theta\ldots|\theta_1,\ldots,\theta_{p_k})\right]^*
\end{aligned} \tag{5.62}$$

where only $\{s_j\}$ compatible with (5.51) gives non-zero form factors in finite volume:

$$\begin{aligned}
F_{\frac{k}{n}}^{N_k^\pm,p_k\,;L}(\theta\ldots|\theta_1,\ldots,\theta_{p_k}) &= {}_L\left\langle\theta;N_k^+,N_k^-\right|\;\mathcal{T}_{\frac{k}{n}}(0)\;|\theta_1,\ldots,\theta_{p_k}\rangle_{s_1,\ldots,s_p\,;\,L} \\
&= \frac{F_{\frac{k}{n}}^{+-+-\cdots+-}(\theta,\ldots|\theta_1,\ldots,\theta_{p_k})}{\sqrt{(mL)^{g_k}(\cosh\theta)^{N_k^++N_k^-}\ldots\cosh\theta_1\ldots\cosh\theta_{p_k}}}
\end{aligned} \tag{5.63}$$

which involves $g_k := N_k^+ + N_k^- + p_k$ particles and all (i-ii) are still valid in finite volume.

5.3.2 Large-Volume Leading Contribution to the Rényi Entropy S_n

In order to evaluate the leading contribution to (5.62) we consider the two-particle form factors

$$\begin{aligned} F^{+-}_{\frac{k}{n}}(\theta + i\pi - \theta_i) &= -\sin\left(\frac{\pi k}{n}\right) \frac{e^{\left(\frac{k}{n}-\frac{1}{2}\right)(\theta-\theta_i+i\pi)}}{\cosh\left(\frac{\theta-\theta_i+i\pi}{2}\right)} \\ &= e^{i\pi\frac{k}{n}} \sin\left(\frac{\pi k}{n}\right) \frac{e^{\left(\frac{k}{n}-\frac{1}{2}\right)(\theta-\theta_i)}}{\sinh\left(\frac{\theta-\theta_i}{2}\right)} \end{aligned} \tag{5.64}$$

where θ satisfies the quantization condition (5.58) while θ_i is associated to one particle of an intermediate state therefore it satisfies (5.57). It is easy to see that the leading contribution in the large volume expansion occurs when the rapidities are very close to each other. Indeed, we can write:

$$\sinh\left(\frac{\theta-\theta_i}{2}\right) = \frac{\sinh\theta \ -\sinh\theta_i}{2\,\cosh\left(\frac{\theta+\theta_i}{2}\right)} \sim_{\theta\approx\theta_i} \frac{\pi}{mL}\frac{(I - I_i^{\pm} \pm \frac{k}{n})}{\cosh\theta} \tag{5.65}$$

In this case, the two-particle form factors are dominated by their behaviour around the kinematic pole:

$$F^{+-}_{\frac{k}{n}}(\theta + i\pi - \theta_i) \sim_{\theta\approx\theta_i} e^{i\pi\frac{k}{n}} \sin\left(\frac{\pi k}{n}\right) \frac{mL\cosh\theta}{\pi\,(I - I_i^{\pm} \pm \frac{k}{n})} \tag{5.66}$$

In (5.62) the leading contribution comes from terms such that $\{N_k^{\pm}\} = \{M_k^{\pm}\}$ and thus when the product of the form factors becomes a square.
We have already mentioned that the g_k-particle form factors may be obtained by appropriate combination of two-particle form factors by using (5.55). The latter involves $n_k = N_k^+ + N_k^-$ factors like (5.64) for each term of the sum and the large volume limit may be obtained when the n_k different θ_is involved in the g_k-particle form factor are close to θ. Under this condition,it is possible to show that the only L-dependence may come from the exponential in (5.62). In particular, it is necessary to keep the ratio $\frac{\ell}{L}$ finite in order to obtain non-trivial leading terms when we only consider the form factors contribution around the poles. An example is provided in appendix B for the second Rènyi entropy.
Once we evaluate the leading contribution to the correlation function (5.62), the leading behaviour of the Rényi entropy is obtained by:

$$S_n^{|\Phi\rangle}\left(\frac{\ell}{L}\right) = \underbrace{\lim_{L\to\infty}}_{\frac{\ell}{L}\text{ finite}} \frac{1}{1-n}\,\log\,{}_L\langle\Phi|\ \mathcal{T}(0)\,\mathcal{T}(\ell)\ |\Phi\rangle_L \tag{5.67}$$

where we supposed the state $|\Phi\rangle$ to be normalized. In particular we are interested in obtaining the quantity $S_n^{|\Phi\rangle} - S_n^0$ which measures the excess of entanglement given by the excited state.

We will see this in detail for the specific case of the second Rényi entropy in the following section.

5.4 The Second Rényi Entropy S_2 In One-particle state

Let us consider a replica model consisting of $n = 2$ copies, then the twist and anti-twist fields associated to the cyclic permutation symmetry coincide $\mathcal{T} = \tilde{\mathcal{T}}$.
In this case the eigenvalues of the T-matrix take the values $\lambda_0 = +1$ and $\lambda_1 = -1$. The U(1) fields $\mathcal{T}_0$ and $\mathcal{T}_{\frac{1}{2}}$ act diagonally on the new basis $\tilde{\Phi}_k$ with $k = 0, 1$ and such that $\mathcal{T}_0$ corresponds to the identity $\mathbb{1}$. In the θ-space, the relations with the old basis can be written:

$$\alpha_1^\dagger(\theta) = \frac{1}{\sqrt{2}} \left((\mathfrak{a}_0^+)^\dagger(\theta) - (\mathfrak{a}_1^+)^\dagger(\theta) \right) \; ; \qquad \beta_1^\dagger(\theta) = \frac{1}{\sqrt{2}} \left((\mathfrak{a}_0^-)^\dagger(\theta) - (\mathfrak{a}_1^-)^\dagger(\theta) \right) \tag{5.68}$$

$$\alpha_2^\dagger(\theta) = \frac{1}{\sqrt{2}} \left((\mathfrak{a}_0^+)^\dagger(\theta) + (\mathfrak{a}_1^+)^\dagger(\theta) \right) \; ; \qquad \beta_2^\dagger(\theta) = \frac{1}{\sqrt{2}} \left((\mathfrak{a}_0^-)^\dagger(\theta) + (\mathfrak{a}_1^-)^\dagger(\theta) \right) \tag{5.69}$$

We may expand the state (5.41) in the new basis. In particular, it turns out that the state (5.41) takes a very simple form when $n = 2$ and it can be written as a sum of two states that live separately in copy 0 and copy 1:

$$|\Phi\rangle = \frac{1}{2} \prod_{j=1}^{2} (\alpha_j^\dagger(\theta) + \beta_j^\dagger(\theta)) \, |0\rangle_j = |\chi_0\rangle + |\psi_1\rangle \tag{5.70}$$

where:

$$\begin{aligned} |\chi_0\rangle &= \frac{1}{4} (\mathfrak{a}_0^+)^\dagger(\theta) \, (\mathfrak{a}_0^+)^\dagger(\theta) \, |0\rangle + \frac{1}{4} (\mathfrak{a}_0^-)^\dagger(\theta) \, (\mathfrak{a}_0^-)^\dagger(\theta) \, |0\rangle + \frac{1}{2} (\mathfrak{a}_0^+)^\dagger(\theta) \, (\mathfrak{a}_0^-)^\dagger(\theta) \, |0\rangle \\ |\psi_1\rangle &= -\frac{1}{4} (\mathfrak{a}_1^+)^\dagger(\theta) \, (\mathfrak{a}_1^+)^\dagger(\theta) \, |0\rangle - \frac{1}{4} (\mathfrak{a}_1^-)^\dagger(\theta) \, (\mathfrak{a}_1^-)^\dagger(\theta) \, |0\rangle - \frac{1}{2} (\mathfrak{a}_1^+)^\dagger(\theta) \, (\mathfrak{a}_1^-)^\dagger(\theta) \, |0\rangle \end{aligned} \tag{5.71}$$

Thus the case $n = 2$ shows many simplifications. First of all, it is easy to see that $\mathcal{T}_{-\frac{1}{2}} = \mathcal{T}_{\frac{1}{2}}$. Furthermore, thanks to the U(1) constrains, the correlation function gives non-zero contribution only when (5.62) is such that $\{N_0^\pm, N_1^\pm\} = \{M_0^\pm, M_1^\pm\}$.[3] In fact,

[3] Although in the n-copy model $\{N_k^\pm\} = \{M_k^\pm\}$ holds only in the large-volume leading behaviour, when $n = 2$ the U(1) constrains are so strong that this expression holds in general. This is essentially a consequence of the low particle number involved in (5.62).

the only other combination the U(1) symmetry allows is zero:

$$\begin{aligned}&\langle 0|\ \mathfrak{a}_0^+(\theta)\,\mathfrak{a}_0^-(\theta)\,\mathcal{T}(0)\,\mathcal{T}(\ell)\,(\mathfrak{a}_1^-)^\dagger(\theta)\,(\mathfrak{a}_1^+)^\dagger(\theta)\ |0\rangle = \\ &{}_0\langle 0|\,\mathfrak{a}_0^+(\theta)\,\mathfrak{a}_0^-(\theta)\,|0\rangle_0 \times {}_1\langle 0|\ \mathcal{T}_{\frac{1}{2}}(0)\,\mathcal{T}_{\frac{1}{2}}(\ell)\,(\mathfrak{a}_1^-)^\dagger(\theta)\,(\mathfrak{a}_1^+)^\dagger(\theta)\ |0\rangle_1 = 0\end{aligned} \tag{5.72}$$

As a result, we can express the correlation function:

$$\langle\Phi|\,\mathcal{T}(0)\,\mathcal{T}(\ell)\,|\Phi\rangle = \langle\chi_0|\,\mathcal{T}(0)\,\mathcal{T}(\ell)\,|\chi_0\rangle + \langle\psi_1|\,\mathcal{T}(0)\,\mathcal{T}(\ell)\,|\psi_1\rangle \tag{5.73}$$

The first term can be easily evaluated and its calculation involves the norm of the two-particle states. The latter may be obtained by implementing the algebra (5.33) and (5.34), and, it turns out that the two-particle states are normalized to one when they are composed by one particle and one antiparticle:

$$\mathcal{N}_0^{\pm\mp} := {}_0\langle 0|\ \mathfrak{a}_0^\mp(\theta)\,\mathfrak{a}_0^\pm(\theta)\,\mathfrak{a}_0^\pm(\theta)\,\mathfrak{a}_0^\mp(\theta)\ |0\rangle_0 = 1 \tag{5.74}$$

otherwise:

$$\mathcal{N}_0^{\pm\pm} := {}_0\langle 0|\ \mathfrak{a}_0^\pm(\theta)\,\mathfrak{a}_0^\pm(\theta)\,(\mathfrak{a}_0^\pm)^\dagger(\theta)\,(\mathfrak{a}_0^\pm)^\dagger(\theta)\ |0\rangle_0 = 2 \tag{5.75}$$

Notice that the state (5.70) is normalized $\langle\Phi|\Phi\rangle = 1$. Since the state $|\chi_0\rangle$ involves no particles in the 1st-copy and the U(1) twist field $\mathcal{T}_0$ acts trivially on the copy 0, we can use directly (5.38) and write:

$$\langle 0|\ \mathfrak{a}_0^{s_1}(\theta)\,\mathfrak{a}_0^{s_2}(\theta)\,\mathcal{T}(0)\,\mathcal{T}(\ell)\,(\mathfrak{a}_0^{s_2})^\dagger(\theta)\,(\mathfrak{a}_0^{s_1})^\dagger(\theta)\ |0\rangle = \mathcal{N}_0^{s_1 s_2} \times {}_1\langle 0|\,\mathcal{T}_{\frac{1}{2}}(0)\,\mathcal{T}_{\frac{1}{2}}(\ell)\,|0\rangle_1 \tag{5.76}$$

for $s_1, s_2 = \pm$. Then:

$$\begin{aligned}\langle\chi_0|\,\mathcal{T}(0)\,\mathcal{T}(\ell)\,|\chi_0\rangle &= \left(\frac{\mathcal{N}_0^{++}}{16} + \frac{\mathcal{N}_0^{--}}{16} + \frac{\mathcal{N}_0^{+-}}{4}\right) {}_1\langle 0|\,\mathcal{T}_{\frac{1}{2}}(0)\,\mathcal{T}_{\frac{1}{2}}(\ell)\,|0\rangle_1 \\ &= \frac{1}{2}\ {}_1\langle 0|\,\mathcal{T}_{\frac{1}{2}}(0)\,\mathcal{T}_{\frac{1}{2}}(\ell)\,|0\rangle_1\end{aligned} \tag{5.77}$$

The evaluation of the term $\langle\psi_1|\,\mathcal{T}(0)\,\mathcal{T}(\ell)\,|\psi_1\rangle$ is non-trivial and requires the form factor expansion (5.62). Its leading contribution in the large-volume limit $L \gg 1$ keeping the ratio $\frac{\ell}{L}$ finite is calculated in Appendix B and it is given by (B.20).

As a result we obtain the large-volume behaviour of the correlation function:

$$\frac{{}_L\langle\Phi|\,\mathcal{T}(0)\,\mathcal{T}(\ell)\,|\Phi\rangle_L}{{}_1\langle 0|\ \mathcal{T}_{\frac{1}{2}}(0)\,\mathcal{T}_{\frac{1}{2}}(\ell)\ |0\rangle_1} = r^2 + (1-r)^2 + O\left(L^{-1}\right) \tag{5.78}$$

where $r = \frac{\ell}{L}$.

Reminding we doubled the free real boson theory in order to induce the $U(1)$ symmetry, we now want to get results for the real state $|\phi\rangle_a$. As we have discussed in section

5.1.1, we chose the state (5.70) in such a way that $|\Phi\rangle = |\phi\rangle_a \otimes |0\rangle_b$. This allows us to relate the correlation function of the complex boson state to those of the real states:

$$\langle\Phi|\,\mathcal{T}(0)\mathcal{T}(\ell)\,|\Phi\rangle = {}_a\langle\phi|\,\mathcal{T}_a(0)\tilde{\mathcal{T}}_a(\ell)\,|\phi\rangle_a \times {}_b\langle 0|\,\mathcal{T}_b(0)\tilde{\mathcal{T}}_b(\ell)\,|0\rangle_b \tag{5.79}$$

where we now omit the index L. In the replica model $n = 2$, since $\mathcal{T}_0$ acts trivially, the ground states is given by:

$$\langle 0|\,\mathcal{T}(0)\mathcal{T}(\ell)\,|0\rangle = {}_1\langle 0|\,\mathcal{T}_{\frac{1}{2}}(0)\mathcal{T}_{\frac{1}{2}}(\ell)\,|0\rangle_1 = ({}_a\langle 0|\,\mathcal{T}_a(0)\mathcal{T}_a(\ell)\,|0\rangle_a)^2 = ({}_b\langle 0|\,\mathcal{T}_b(0)\tilde{\mathcal{T}}_b(\ell)\,|0\rangle_b)^2 \tag{5.80}$$

As a consequence, we can easily express the result (5.78) by the real state $|\phi\rangle_a$ thanks to:

$$\frac{\langle\Phi|\,\mathcal{T}(0)\mathcal{T}(\ell)\,|\Phi\rangle}{\langle 0|\,\mathcal{T}(0)\mathcal{T}(\ell)\,|0\rangle} = \frac{\langle\Phi|\,\mathcal{T}(0)\mathcal{T}(\ell)\,|\Phi\rangle}{{}_1\langle 0|\,\mathcal{T}_{\frac{1}{2}}(0)\mathcal{T}_{\frac{1}{2}}(\ell)\,|0\rangle_1} = \frac{{}_a\langle\phi|\,\mathcal{T}_a(0)\mathcal{T}_a(\ell)\,|\phi\rangle_a}{{}_a\langle 0|\,\mathcal{T}_a(0)\mathcal{T}_a(\ell)\,|0\rangle_a} \tag{5.81}$$

5.4.1 The Large-Volume Leading Behavior of S_2

We can finally write the leading contribution to the second Rényi entropy in the large volume limit:

$$(S_2^{|\phi\rangle} - S_2^{|0\rangle})\,(\ell,\, L) = \Delta S_2^1(r) + O\left(L^{-1}\right) = -\log\left[r^2 + (1-r)^2\right] + O\left(L^{-1}\right) \tag{5.82}$$

where $r = \frac{\ell}{L}$ is kept finite in this limit. In fact, it is easy to see that if one chooses $r \ll 1$, the formula (5.82) gives zero. This means that when $\ell \ll L$ there i no entanglement difference at leading order in $1/L$, but there will be some subleading corrections we have not yet studied.

However, keeping r finite, the second Rényi entropy shows an interesting behavior. First of all, the result is independent of the energy of the excitation (the quantum number I). Secondly, it satisfies manifestly the symmetry $r \to 1 - r$ that is necessary to be a measure of entanglement in finite volume.
Furthermore it has two zeros at $r = 0, 1$ and a peak at $r = 1/2$ with the value of $\log 2$ which corresponds to one bit of Entanglement. In fact, it is easy to see that the expression (5.82) takes the same form of the binary Rényi entropy (2.52) where the two outcomes are r and $1 - r$.

This additional contribution $\Delta S_2^1(r)$ given by the one-particle state to the Rényi entropy S_2 in the vacuum may be understood as follows: it describes the two possible outcomes for the excitation to be allocated on the sub-system ℓ or in its complement $L - \ell$. This excitation contribution maximizes at 1 *bit* when both partitions are $L/2$ thus identical and the uncertainty about the allocation is consequently maximal.

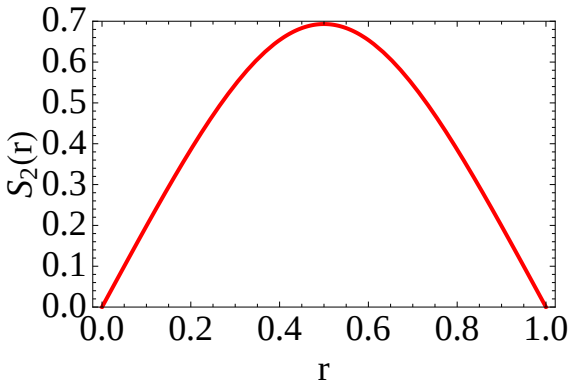

Figure 5.2: the leading contribution to the second rényi Entropy in one-particle excitation in the limit $L \gg 1$ while $r = \frac{\ell}{L}$ is chosen finite. The entropy is expressed in *nat* where $1\ bit = \log 2\ nat\ \sim\ 0.693\ nat$.

Chapter 6

Conclusion

The aim of this thesis has been to investigate the bipartite entanglement entropy in excited states in a free boson theory. The thesis is a part of a wider research project which will lead to a publication [41]. In particular, in this thesis, we have discussed the outcomes obtained by January 2018. The investigation has been done by taking advantage of the techniques we have presented in this thesis.

We first employed the *replica trick* [6]. We have considered a multi-copy model of the original theory. Thanks to the cyclic permutation symmetry over the copies it has been possible to define the branch point twist fields $\mathcal{T}$ and $\tilde{\mathcal{T}}$ whose correlation functions are directly related to the entanglement entropy. The evaluation of these correlation functions normally requires integrable model techniques e.g. form factor expansions. The replica trick was already used in order to obtain the asymptotic behaviour of the entanglement entropy in the vacuum state [7] and in a CFT excited state [8, 9] and it may be easily extended to our case.
We have adopted a *finite-volume approach* in order to avoid infinite-volume divergences of the correlation function. However this approach has shown to be non-trivial in the replica model because of the twist action of the fields $\mathcal{T}$ and $\tilde{\mathcal{T}}$.

In order to overcome this problem, we have considered a new model based on the *doubling trick* [10]. We have essentially doubled a free boson theory in order to work with complex boson fields. This doubling induces a U(1) symmetry on each copy of the replica model, allowing us to define the corresponding twist fields $\mathcal{T}_{\frac{k}{n}}$. The U(1) symmetry lies at the heart of our work. In particular:

(i) we can introduce a new particle basis such that the U(1) fields have diagonal action on this basis and are semi-local with respect to the fundamental fields on each copy,

namely:

$$\tilde{\Phi}_j(x,t)\,\mathcal{T}_{\frac{k}{n}}(y,t) = \left(e^{i2\pi\frac{k}{n}}\right)^{\delta_{jk}}\mathcal{T}_{\frac{k}{n}}(y,t)\,\tilde{\Phi}_j(x,t) \qquad \text{For } y > x \tag{6.1}$$

$$\tilde{\Phi}_j(x,t)\,\mathcal{T}_{\frac{k}{n}}(y,t) = \mathcal{T}_{\frac{k}{n}}(y,t)\,\tilde{\Phi}_j(x,t) \qquad \text{For } y < x \tag{6.2}$$

$\forall j,k = 0,\dots,n-1$. In addition, it turns out that the twist fields associated to the cyclic permutation symmetry may be expressed in terms of the U(1) fields:

$$\mathcal{T} = \prod_{k=0}^{n-1}\mathcal{T}_{\frac{k}{n}} \quad ; \quad \tilde{\mathcal{T}} = \prod_{k=0}^{n-1}\mathcal{T}_{-\frac{k}{n}} \tag{6.3}$$

where $\mathcal{T}_0$ is simply the identity. These considerations allowed us to evaluate the correlation function $\langle\Phi|\,\mathcal{T}(0)\,\tilde{\mathcal{T}}(\ell)\,|\Phi\rangle$ by introducing a complete set of intermediate states for each copy therefore we were able to find a form factor expansion.

(ii) It has been possible to establish a finite volume approach. Here we worked on the cylinder. The effect of inserting the intermediate state has been to deform the branch cut along the circumference to basically obtain two semi-infinite branch cut along the time direction (see figure 5.1). The quantitation conditions associated to the intermediate states have changed because of (6.1):

$$Q_j^{\pm} = mL\sinh\theta_j \pm \frac{2\pi\,k_j}{n} = 2\pi\,I_j^{\pm} \quad I_j \in \mathbb{Z}\ ,\ k_j = 1,\dots,n-1 \tag{6.4}$$

where $I_j^{\pm}$ denotes the quantum number of the particle $(+)$ or anti-particle $(-)$ associated to the complex boson field Φ_j, θ_j is the rapidity and k_j the copy in which the field lives.

(iii) The factorization (6.3) constrains a lot the form factor expansion. For instance, it imposes the same number of particles and antiparticles in the k-particle form factors.

The points (i-iii) allowed us to evaluate the large-volume leading contribution to the correlation function in a replica model with $n = 2$.
In this limit, the two-particle form factors are dominated by their behaviour around the kinematic pole. The calculation of this large-volume leading contribution have been reported in appendix B. Here, we imposed the limit $L \gg 1$ keeping the ratio ℓ/L finite, where L and ℓ denote the length of the finite bipartite system and the sub-system chosen to evaluate the entanglement entropy.
In the conclusion of the thesis, we have been able to give an expression for the large-volume leading contribution to the second Rényi entropy:

$$(S_2^{|\phi\rangle} - S_2^{|0\rangle})\,(\ell,L) = \Delta S_2^1(r) + O\left(L^{-1}\right) = -\log\left[r^2 + (1-r)^2\right] + O\left(L^{-1}\right) \tag{6.5}$$

This result is surprising for its simplicity: it is independent of the energy of the excitation and symmetric under the transformation $r \to 1-r$. The latter is, in fact, an important requirement for the entanglement entropy: the amount of entanglement measured on ℓ must be the same if we perform the measurement on its complement subsystem $L-\ell$. In addition, we have noticed (6.5) takes the same form as for the *binary Rényi entropy* we have introduced in section 2.5.3, where now the random variable is provided by r: it essentially measures the chance of finding the boson excitation in ℓ or its complement $L-\ell$. Consequently, ΔS_2^1 measures the uncertainly associated with the two possible out-comes: it has two zeros at $r=0,1$ and it maximizes at $r=1/2$ when the two subsystems are equal to $L/2$ and thus the uncertainly about the allocation of the excitation reaches its maximum value (see the figure 5.2).

The result (6.5) may provide the basis to construct the large-volume leading behaviour of the Rényi entropy S_n in a k-particle excited state. This has been recently implemented and will be shown in details in [41]. Some results so far generalize (6.5). This is possible when we consider *distinct rapidities*, then the additional contribution to the Rényi entropy in the large volume limit may be written as:

$$\Delta S_n^{k_d}(r) = \frac{k \log(r^n + (1-r)^n)}{1-n} \qquad \text{for } k \text{ distinct rapidities} \tag{6.6}$$

For instance, the leading contribution from the two-particle excited state to the second Rényi entropy in the vacuum is twice that of the one-particle state. For the von Neumann entropy the leading contribution can be written:

$$\lim_{n\to 1} \Delta S_n^{k_d}(r) = -k\, r \log r - k\,(1-r)\log(1-r) \tag{6.7}$$

and shows a pick at $r=1/2$ corresponding to k qubits. In addition, the boson nature allows coinciding rapidities among the state and in case all the rapidities are *equal*, the large-volume leading contribution is given by the following formula:

$$\lim_{n\to 1} \Delta S_n^{k_e}(r) = \frac{1}{n-1} \log \sum_{q=0}^{k} \left[\binom{k}{q} r^q (1-r)^{k-q} \right]^n \qquad \text{for } k \text{ equal rapidities} \tag{6.8}$$

and the von Neumann entropy:

$$\Delta S_n^{k_e}(r) = \sum_{q=0}^{k} \left[\binom{k}{q} r^q (1-r)^{k-q} \right] \log \left[\binom{k}{q} r^q (1-r)^{k-q} \right] \tag{6.9}$$

this essentially implies that when the rapidities coincide there are less ways in which they can combine into the two partitions.
Even for k-particle excited states, the leading contribution takes a very simple form and

is independent of the particle energies. These features have been confirmed numerically and will be studied further. The outcomes will be shown in [41].

Having understood the free boson case, there are a few further projects that are of interest:

(a) to consider the free fermion theory, where similar techniques may be used in order to have a complete understanding of this large-volume leading contribution in free theories.

(b) to investigate entanglement entropy in excited states beyond the free theories. In fact, the results we presented mainly followed from the structure of the two-particle form factor. This suggests that this large-volume behavior could hold more generally in interacting theories.

(c) to analyze higher order corrections, which are expected to depend on the rapidities.

(d) to employ similar techniques in order to investigate other measures of entanglement e.g. the logarithmic negativity.

Appendix A

Two-particle Form Factors Of The U(1) Fields

The bootstrap equation for the two-particle form factor are:

$$F^{\pm\mp}_{j,n}(\theta) = F^{\mp\pm}_{j,n}(-\theta) \qquad (\textit{Exchange}) \tag{A.1}$$

$$F^{\pm\mp}_{j,n}(\theta + 2\pi i) = \gamma_{\pm} F^{\mp\pm}_{j,n}(-\theta) = \gamma_{\pm} F^{\pm\mp}_{\substack{j \\ n}}(\theta) \qquad (\textit{Cyclic property}) \tag{A.2}$$

$$\mathrm{Res}_{\theta=0}\, F^{\pm\mp}_{j,n}(\theta + i\pi) = i(1-\gamma_{\pm}) F_{j,n} \qquad (\textit{Kinematical poles}) \tag{A.3}$$

here $j = 0, \ldots, k-1$, the S-matrix is $S_{\pm\mp}(\theta) = S_{\pm\pm}(\theta) = 1$ because of the boson nature, $F_{j,n} = \langle \mathcal{T}_{j,n} \rangle_j$ is the vacuum expectation value and $\gamma_{\pm}$ are the factors of local commutativity associated to the bosons $\pm$. If we assume these to be phases and to be conjugate to each other we have that $\gamma_+ = \gamma_-^{-1}$.

Other two-particle form factors are zero because of the $U(1)$ symmetry. We can make a general ansatz for the equation above:

$$F^{+-}_{j,n}(\theta) = \frac{A e^{a\theta}}{\cosh \frac{\theta}{2}} \tag{A.4}$$

where A and a are constants to be determined. The cosh-function guarantees the presence of a pole at $\theta = i\pi$. From (A.2) we have that

$$\frac{A e^{a(\theta+2\pi i)}}{\cosh \frac{\theta+2\pi i}{2}} = \gamma_+ \frac{A e^{a\theta}}{\cosh \frac{\theta}{2}} \tag{A.5}$$

that is

$$-e^{2\pi i a} = \gamma_+ \tag{A.6}$$

and from (A.3) we have

$$\lim_{\theta \to 0} \theta \frac{A e^{a(\theta+i\pi)}}{\cosh \frac{\theta+i\pi}{2}} = -2iAe^{ai\pi} = i(1-\gamma_{\pm}) F_{j,n} \tag{A.7}$$

Let us now assume that $\gamma_+ = e^{\frac{2\pi ij}{n}}$. Then we have from (A.6)

$$e^{\pm\frac{2\pi ij}{n}} = -e^{2\pi ia} \tag{A.8}$$

thus introducing $-1 = e^{\pm i\pi}$

$$a_\pm = \frac{j}{n} \pm \frac{1}{2} \tag{A.9}$$

there are two possible choices of a all satisfying (A.6). We now have that

$$1 - \gamma_+ = 1 - e^{\frac{2\pi ij}{n}} = -2ie^{\frac{\pi ij}{n}} \sin\frac{\pi j}{n} \tag{A.10}$$

The kinematic residue equation gives

$$-2iAe^{i\pi\left(\frac{j}{n}\pm\frac{1}{2}\right)} = 2e^{\frac{\pi ij}{n}} \sin\frac{\pi j}{n} F_{j,n} \tag{A.11}$$

that is

$$A(-i)\, e^{\pm\frac{i\pi}{2}} = \sin\frac{\pi j}{n} F_{j,n} \tag{A.12}$$

thus $A_\pm$ shows two possible values corresponding to the choices of $a_\pm$:

$$A_\pm = \pm F_{j,n} \sin\frac{\pi j}{n} \tag{A.13}$$

As a result we can write the solutions:

$$\text{solution (+)} \quad : \qquad F^{+-}_{j,n}(\theta) = F^j \sin\frac{\pi j}{n} \frac{e^{\left(\frac{j}{n}+\frac{1}{2}\right)\theta}}{\cosh\frac{\theta}{2}} \tag{A.14}$$

$$\text{solution (-)} \quad : \qquad F^{+-}_{j,n}(\theta) = -F^j \sin\frac{\pi j}{n} \frac{e^{\left(\frac{j}{n}-\frac{1}{2}\right)\theta}}{\cosh\frac{\theta}{2}} \tag{A.15}$$

The second solution is in accord with the conformal result (5.39) therefore we will assume this one.

Appendix B

Computation Of The Large-volume Leading Contribution to S_2

We are interested in calculating the correlation function $\langle\Phi|\,\mathcal{T}(0)\,\mathcal{T}(\ell)\,|\Phi\rangle$ in the large-volume limit. In section 5.4 we expressed it by the sum of two terms (5.73). The first term $\langle\chi_0|\,\mathcal{T}(0)\,\mathcal{T}(\ell)\,|\chi_0\rangle$ gives trivial contribution and it was already calculated in (5.77). We are now going to evaluate the second term which, according with (5.71), it can be written as:

$$\langle\psi_1|\,\mathcal{T}(0)\tilde{\mathcal{T}}(\ell)\,|\psi_1\rangle = {}_0\langle 0|0\rangle_0 \times \sum_{s_1,s_2=\pm} \mathcal{A}_{s_1,s_2}\ \langle s_1,s_2|\ \mathcal{T}_{\frac{1}{2}}(0)\,\mathcal{T}_{\frac{1}{2}}(\ell)\ |s_2,s_1\rangle \tag{B.1}$$

where we redefine the two-particles state living in copy 1 as

$$|s_2,s_1\rangle = (\mathfrak{a}_1^{s_2})^\dagger\,(\theta)\ (\mathfrak{a}_1^{s_1})^\dagger(\theta)\ |0\rangle_1 \tag{B.2}$$

and the coefficients are given by (5.71)

$$\mathcal{A}_{s_1,s_2} = \begin{cases} \frac{2}{16} & \text{if } s_1 = s_2 \\ \frac{1}{16} & \text{if } s_1 \neq s_2 \end{cases} \tag{B.3}$$

since the terms $(s_1,s_2)=(+,-)$ and $(s_1,s_2)=(+,-)$ in (B.1) are equal to each other.

First of all, we consider the term $(s_1,s_2)=(+,+)$. Since we have two positive particles on the considered state, the U(1) constrains imply that the intermediate states that give non-zero contribution in (5.62) are composed by $2p+2$ particles, where $p+2$ are the particles and p the antiparticles necessary to keep the neutral total charge on the form factor. In finite volume the quantization conditions for the states are:

$$mL\sinh\theta_i = \pi\,(2J_j - 1) \qquad \text{for particles} \quad j = 1,\dots,p+2 \tag{B.4}$$

$$mL\sinh\beta_i = \pi\,(2I_i + 1) \qquad \text{for antiparticles} \quad i = 1,\dots,p \tag{B.5}$$

$$mL\sinh\theta = 2\pi\, I \qquad \text{for the particles in copy 1} \quad (+,-) \tag{B.6}$$

In finite volume the correlation function of the twist fields in the state $|+,+\rangle_L$ is given by (5.62):

$$_L\langle +,+|\ \mathcal{T}_{\frac{1}{2}}(0)\,\mathcal{T}_{\frac{1}{2}}(\ell)\ |+,+\rangle_L = \tag{B.7}$$

$$\sum_{p=0}^{\infty}\frac{1}{p!\,(p+2)!}\sum_{I_1\dots I_p,\,J_1\dots J_{p+2}\in\mathbb{Z}}\left[\frac{F^{(1,2)}_{2p+4}\,(\theta_1,\dots,\theta_{p+2};\theta+i\pi,\theta+i\pi,\beta_1,\dots,\beta_p)}{(mL)^{p+2}\cosh\theta\sqrt{\cosh\theta_{p+1}\cosh\theta_{p+2}\prod_{j=1}^{p}\cosh\theta_j\cosh\beta_j}}\right]^2\times$$

$$\exp\left[i\ell\left(p(\theta_{p+1})+p(\theta_{p+2})-2p(\theta)+\sum_{j=1}^{p}p_j(\theta)\right)\right]$$

where the momenta are given by the quantization conditions and the form factors on the l.h.s are evaluated in infinite volume and they are written in the form:

$$F^{(k,n)}_{2m}\,(\theta_1,\dots,\theta_m;\beta_1,\dots,\beta_m) = F^{+-\cdots+-}_{\frac{k}{n}}\,(\theta_1,\dots,\theta_m,\beta_1,\dots,\beta_m) \tag{B.8}$$

In free theories, the latter may be expressed in terms of the two-particle form factors:

$$F^{(k,n)}_{2m}\,(\theta_1,\dots,\theta_m;\beta_1,\dots,\beta_m) = \langle\mathcal{T}_{\frac{k}{n}}\rangle\sum_{\sigma,\kappa\,\in\,S_m}F_2^{(k,n)}(\theta_{\sigma(1)}-\beta_{\kappa(1)})\ \dots\ F_2^{(k,n)}(\theta_{\sigma(m)}-\beta_{\kappa(m)}) \tag{B.9}$$

where σ and κ are permutations over the sets $\{\theta_1,\dots,\theta_m\}$ and $\{\beta_1,\dots,\beta_m\}$ and the two-particle form factor is given by (5.54).

Focus on the leading contribution in the large volume limit, we have seen that this occurs when the structure of the two-particle form factor is dominated by their poles:

$$F_2^{(k,n)}(\theta'_\pm-\theta-i\pi) = F_2^{(k,n)}(\theta-\theta'_\pm+i\pi)\ \sim_{\theta\approx\theta'}\ -\frac{i}{\pi}\frac{mL\cosh\theta}{(I'_\pm-I\ \mp\ \frac{1}{2})} \tag{B.10}$$

namely when we have $\theta_j\approx\theta\ \forall j=1,\dots,p+2$ and $\beta_i\approx\theta\ \forall i=1,\dots,p$. Since the sum over all quantum numbers I_i and J_j makes these quantities essentially equivalent to each other, in the large-volume limit it is reasonable to write:

$$\sum_{I_1\dots I_p,\,J_1\dots J_{p+2}\in\mathbb{Z}}\left[F^{(1,2)}_{2p+4}\,(\theta_1,\dots,\theta_{p+2};\theta+i\pi,\theta+i\pi,\beta_1,\dots,\beta_p)\right]^2 = \tag{B.11}$$

$$\sum_{I_1\dots I_p,\,J_1\dots J_{p+2}\in\mathbb{Z}}\left[F_2^{(k,n)}(\theta_a-\theta-i\pi)\,F_2^{(k,n)}(\theta_b-\theta-i\pi)\sum_{\sigma\in\,S_m}\prod_{j=1}^{p}F_2^{(k,n)}(\theta_{j_{\{a,b\}}}-\beta_{\sigma(j)})\right]^2$$

where the approximation arise from considering the rapidities θ in pairs with the same two rapidities θ_a and θ_b where $a, b \in \{1, \ldots, p+2\}$ over all sum, this is motivated by the domain of the poles that puts the two-particle form factors on the same footing once we choose an appropriate order among the rapidities:

$$j_{\{a,b\}} = \begin{cases} j & \text{for } j < \min\{a,b\} \\ j+1 & \text{for } \min\{a,b\} \leqslant j < \max\{a,b\} \\ j+2 & \text{for } \max\{a,b\} \leqslant j \end{cases} \tag{B.12}$$

Fixing θ_a and θ_b at θ_{p+2} and θ_{p+1}, we obtain

$$\sum_{I_1 \ldots I_p,\, J_1 \ldots J_{p+2} \in \mathbb{Z}} \left[F_{2p+4}^{(1,2)}(\theta_1, \ldots, \theta_{p+2}; \theta + i\pi, \theta + i\pi, \beta_1, \ldots, \beta_p) \right]^2 = \tag{B.13}$$

$$2(p+2)(p+1) \left[\sum_{J_{p+1} J_{p+2} \in \mathbb{Z}} \left[F_2^{(k,n)}(\theta_{p+1} - \theta - i\pi)\, F_2^{(k,n)}(\theta_{p+2} - \theta - i\pi) \right]^2 \right] \times$$

$$\left[\sum_{I_1 \ldots I_p,\, J_1 \ldots J_p \in \mathbb{Z}} \left[\sum_{\sigma \in S_m} \prod_{j=1}^{p} F_2^{(k,n)}(\theta_{j_{\{a,b\}}} - \beta_{\sigma(j)}) \right]^2 \right]$$

where the factor $2(p+2)(p-2)$ is due to the $p+2$ ways to choose θ_a in (B.11) and, once we do this there are $p+1$ remaining possibilities for θ_b. Finally, the factor 2 comes from the square on the r.h.s of (B.11), in fact, there are essentially two way to extract a similarly set of $\{a', b'\}$ from the other form factor in the square: one is $a' = a$ and $b' = b$ another one is $a' = b$ and $b' = a$. It is important to notice that these two options may occur only if the particles a and b must be of the same type i.e. two particles or two antiparticles. As a consequence the factor 2 will disappear if we consider the case $(s_1, s_2) = (+, -)$.

These considerations allows us to factorize the correlation function (B.7) into two terms, where one is the correlation function in the vacuum state, indeed:

$$\sum_{p=0}^{\infty} \frac{1}{(p!)^2} \sum_{I_1 \ldots I_p,\, J_1 \ldots J_p \in \mathbb{Z}} \left[\frac{\prod_{j=1}^{p} F_2^{(k,n)}(\theta_{j_{\{a,b\}}} - \beta_{\sigma(j)})}{(mL)^p \sqrt{\prod_{j=1}^{p} \cosh\theta_j \cosh\beta_j}} \right]^2 \exp\left[i\ell \sum_{j=1}^{p} p_j(\theta) \right] = {}_1\langle \mathcal{T}_{\frac{1}{2}}(0)\, \mathcal{T}_{\frac{1}{2}}(\ell) \rangle_1 \tag{B.14}$$

As a result, we successfully obtain:

$$\frac{{}_L\langle +,+|\ \mathcal{T}_{\frac{1}{2}}(0)\,\mathcal{T}_{\frac{1}{2}}(\ell)\ |+,+\rangle_L}{{}_1\langle 0|\ \mathcal{T}_{\frac{1}{2}}(0)\,\mathcal{T}_{\frac{1}{2}}(\ell)\ |0\rangle_1} = \tag{B.15}$$

$$2\sum_{J_1,J_2\in\mathbb{Z}}\left[\frac{F_2^{(k,n)}(\theta_1-\theta-i\pi)\,F_2^{(k,n)}(\theta_2-\theta-i\pi)}{(mL)^2\cosh\theta\sqrt{\cosh\theta_1\cosh\theta_2}}\right]^2\exp[i\ell\,(p(\theta_1)+p(\theta_2)-2p(\theta))] =$$

$$2\left[\sum_{J'\in\mathbb{Z}}\frac{\left(F_2^{(k,n)}(\theta'-\theta-i\pi)\right)^2}{(mL)^2\cosh\theta\,\cosh\theta'}\exp\big[i\ell\,\big(p(\theta')-p(\theta)\big)\big]\right]^2$$

we are now interested in evaluating the large-volume leading contribution to the correlation function using (B.15). The large volume limit $L\gg 1$ is taken in such a manner as to keep the ratio $\frac{\ell}{L}$ finite. The exponential factor may be rewritten thanks to the quantizazion condition:

$$i\ell\,(p(\theta')-p(\theta)) = \frac{2\pi\ell}{L}\left(J'-I-\frac{1}{2}\right) \tag{B.16}$$

Implementing (5.62), the finite-volume leading behaviour of (B.15) can be written as:

$$\begin{aligned}\frac{{}_L\langle +,+|\ \mathcal{T}_{\frac{1}{2}}(0)\,\mathcal{T}_{\frac{1}{2}}(\ell)\ |+,+\rangle_L}{{}_1\langle 0|\ \mathcal{T}_{\frac{1}{2}}(0)\,\mathcal{T}_{\frac{1}{2}}(\ell)\ |0\rangle_1} &= 2\left[\sum_{J'\in\mathbb{Z}}\frac{\left(F_2^{(k,n)}(\theta'-\theta-i\pi)\right)^2}{(mL)^2\cosh\theta\,\cosh\theta'}\exp\big[i\ell\,\big(p(\theta')-p(\theta)\big)\big]\right]^2\\ &\sim\frac{2}{\pi^4}\left[\sum_{J'\in\mathbb{Z}}\frac{1}{(J'-I-\frac{1}{2})^2}\exp\left[2\pi i\frac{\ell}{L}\left(J'-I-\frac{1}{2}\right)\right]\right]^2\\ &\sim\frac{2}{\pi^4}\,e^{-2\pi ir(2I+1)}\left[\sum_{J'\in\mathbb{Z}}\frac{e^{2\pi irJ'}}{(J'-I-\frac{1}{2})^2}\right]^2\end{aligned} \tag{B.17}$$

where in the last expression we introduced the new variable $r=\ell/L$.

The procedure may be iterated giving similar results. Indeed, in order to evaluated other terms in (B.1) we analogously may extrapolate two two-particle form factors like in (B.11) where now the particles a and b we choose are dictated by the $U(1)$ constrains. For instance, since the state $|-,-\rangle$ has two antiparticles, then we will extract two antiparticles with rapidities β_a and β_b in order to factorize the correlation function in two terms whose one is the correlation function in the vacuum state. This will lead to similar formulas for the other terms where now some signs in (B.17) will be changed because we are considering different quantizazion conditions for the integer J_1 and J_2 in (B.15). In

other words, we may extend the result (B.17) to the other terms:

$$\frac{{}_L\langle s_1, s_2|\ \mathcal{T}_{\frac{1}{2}}(0)\,\mathcal{T}_{\frac{1}{2}}(\ell)\ |s_2, s_1\rangle_L}{{}_1\langle 0|\ \mathcal{T}_{\frac{1}{2}}(0)\,\mathcal{T}_{\frac{1}{2}}(\ell)\ |0\rangle_1} \sim \frac{2^{|\lambda_1-\lambda_2|}}{\pi^4}\ \prod_{j=1}^{2} e^{-2\pi i r(I+\lambda_j)} \left[\sum_{J_j} \frac{e^{2\pi i J_j\, r}}{(J_j - I - \lambda_j)^2}\right] \tag{B.18}$$

where we denotes $s_j = \pm$, $\lambda_j = s_j \frac{1}{2} = \pm\frac{1}{2}$, I the quantum number associated to $|s_2, s_1\rangle$, and, $2^{|\lambda_1+\lambda_2|}$ a symmetry factor that does not appear when we extract a particle and an antiparticle in (B.11). Finally we obtain:

$$\begin{aligned}
\frac{{}_L\langle \psi_1|\,\mathcal{T}(0)\tilde{\mathcal{T}}(\ell)\,|\psi_1\rangle_L}{{}_1\langle 0|\ \mathcal{T}_{\frac{1}{2}}(0)\,\mathcal{T}_{\frac{1}{2}}(\ell)\ |0\rangle_1} &\sim \frac{e^{-2\pi i r(2I+1)}}{8\pi^4}\ \left[\sum_{J_1} \frac{e^{2\pi i J_2\, r}}{(J_1 - I - \frac{1}{2})^2}\right]^2 + \frac{e^{-2\pi i r(2I-1)}}{8\pi^4}\ \left[\sum_{J_2} \frac{e^{2\pi i J_2\, r}}{(J_2 - I + \frac{1}{2})^2}\right]^2 \\
&\quad + \frac{e^{-4\pi i r I}}{4\pi^4}\ \left[\sum_{J} \frac{e^{2\pi i J\, r}}{(J - I - \frac{1}{2})^2}\right]\left[\sum_{J'} \frac{e^{2\pi i J'\, r}}{(J' - I + \frac{1}{2})^2}\right]. \\
&\sim \frac{e^{-4\pi i r I}}{8\pi^4}\ \left[e^{-\pi i r}\ \sum_{J_1} \frac{e^{2\pi i J_1\, r}}{(J_1 - I - \frac{1}{2})^2} + e^{\pi i r}\ \sum_{J_2} \frac{e^{2\pi i J_2\, r}}{(J_2 - I + \frac{1}{2})^2}\right]^2
\end{aligned} \tag{B.19}$$

shifiting $J_1 = J + 1$ we can write:

$$\begin{aligned}
\frac{{}_L\langle \psi_1|\,\mathcal{T}(0)\tilde{\mathcal{T}}(\ell)\,|\psi_1\rangle_L}{{}_1\langle 0|\ \mathcal{T}_{\frac{1}{2}}(0)\,\mathcal{T}_{\frac{1}{2}}(\ell)\ |0\rangle_1} &\sim \frac{e^{-2\pi i r(2I-1)}}{2\pi^4}\ \left[\sum_{J} \frac{e^{2\pi i J\, r}}{(J - I + \frac{1}{2})^2}\right]^2 \\
&\sim \frac{1}{2\pi^4}\ \left[\sum_{J} \frac{e^{2\pi i r(J-I+\frac{1}{2})}}{(J - I + \frac{1}{2})^2}\right]^2
\end{aligned} \tag{B.20}$$

In the replica model $n = 2$, (B.20) provides the r-dependent contribution to the correlation function $\langle\Phi|\,\mathcal{T}(0)\tilde{\mathcal{T}}(\ell)\,|\Phi\rangle$:

$$\begin{aligned}
\frac{{}_L\langle \Phi|\,\mathcal{T}(0)\,\mathcal{T}(\ell)\,|\Phi\rangle_L}{{}_1\langle 0|\ \mathcal{T}_{\frac{1}{2}}(0)\,\mathcal{T}_{\frac{1}{2}}(\ell)\ |0\rangle_1} &= \frac{\langle \chi_0|\,\mathcal{T}(0)\,\mathcal{T}(\ell)\,|\chi_0\rangle}{{}_1\langle 0|\ \mathcal{T}_{\frac{1}{2}}(0)\,\mathcal{T}_{\frac{1}{2}}(\ell)\ |0\rangle_1} + \frac{{}_L\langle \psi_1|\,\mathcal{T}(0)\,\mathcal{T}(\ell)\,|\psi_1\rangle_L}{{}_1\langle 0|\ \mathcal{T}_{\frac{1}{2}}(0)\,\mathcal{T}_{\frac{1}{2}}(\ell)\ |0\rangle_1} \\
&\sim \frac{1}{2} + \frac{1}{2\pi^4}\ \left[\sum_{J\in\mathbb{Z}} \frac{e^{2\pi i r(J'+\frac{1}{2})}}{(J' + \frac{1}{2})^2}\right]^2
\end{aligned} \tag{B.21}$$

where $\langle \chi_0|\,\mathcal{T}(0)\,\mathcal{T}(\ell)\,|\chi_0\rangle$ was already calculated in section 5.4 and given by (5.77). It is possible to show that the sum (B.21) is a polynomial in r. As a result:

$$\frac{{}_L\langle \Phi|\,\mathcal{T}(0)\,\mathcal{T}(\ell)\,|\Phi\rangle_L}{{}_1\langle 0|\ \mathcal{T}_{\frac{1}{2}}(0)\,\mathcal{T}_{\frac{1}{2}}(\ell)\ |0\rangle_1} \sim 2r^2 - 2r + 1 \tag{B.22}$$

(B.22) allows us to evaluate the second Rényi entropy in one-particle excited state.

Bibliography

[1] E. Schrödinger, "Discussion of probability relations between separated systems," in *Proceedings of the Cambridge Philosophical Society*, vol. 31, pp. 555–563. 1935.

[2] A. Einstein, B. Podolsky, and N. Rosen, "Can quantum-mechanical description of physical reality be considered complete?," *Phys. Rev.* **47** (1935) 777–780.

[3] J. S. Bell, "On the einstein-podolsky-rosen paradox," *Physics* **1** (1964) 195–200.

[4] A. Aspect, P. Grangier, and G. Roger, "Experimental realization of einstein-podolsky-rosen-bohm gedankenexperiment: A new violation of bell's inequalities," *Phys. Rev. Lett.* **49** (1982) 91–94.

[5] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information*. Cambridge University Press, 10th ed., 2011.

[6] P. Calabrese and J. Cardy, "Entanglement entropy and quantum field theory," *Journal of Statistical Mechanics: Theory and Experiment* **2004** no. 06, (2004) P06002.

[7] J. L. Cardy, O. A. Castro-Alvaredo, and B. Doyon, "Form factors of branch-point twist fields in quantum integrable models and entanglement entropy," *Journal of Statistical Physics* **130** no. 1, (Jan, 2008) 129–168.

[8] F. C. Alcaraz, M. I. Berganza, and G. Sierra, "Entanglement of low-energy excitations in conformal field theory," *Phys. Rev. Lett.* **106** (May, 2011) 201601.

[9] M. I. Berganza, F. C. Alcaraz, and G. Sierra, "Entanglement of excited states in critical spin chains," *Journal of Statistical Mechanics: Theory and Experiment* **2012** no. 01, (2012) P01016.

[10] P. Fonseca and A. Zamolodchikov, "Ward identities and integrable differential equations in the ising field theory,".

[11] M. B. Plenio and S. Virmani, "An introduction to entanglement measures," *Quantum Info. Comput.* **7** no. 1, (Jan., 2007) 1–51.

[12] C. Cohen-Tannoudji, B. Diu, and F. Laloe, *Quantum mechanics, volume 1*. Wiley,, 1978.

[13] L. Amico, R. Fazio, A. Osterloh, and V. Vedral, "Entanglement in many-body systems," *Rev. Mod. Phys.* **80** (May, 2008) 517–576.

[14] D. Sych and G. Leuchs, "A complete basis of generalized bell states," *New Journal of Physics* **11** no. 1, (2009) 013006.

[15] A. Rényi, "On measures of entropy and information," in *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, Volume 1: Contributions to the Theory of Statistics*, pp. 547–561. University of California Press, Berkeley, Calif., 1961.

[16] C. D. E. Boschi, "Introduzione alla teoria dell'informazione quantistica*[23,1]." Https://www.bo.imm.cnr.it/users/degliesposti/tiq.pdf, 2017.

[17] D. Bianchini, "Entanglement entropy in restricted integrable spin-chains," Master's thesis. http://amslaurea.unibo.it/5627/.

[18] C. H. Bennett, H. J. Bernstein, S. Popescu, and B. Schumacher, "Concentrating partial entanglement by local operations," *Phys. Rev. A* **53** (Apr, 1996) 2046–2052.

[19] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, and W. K. Wootters, "Mixed-state entanglement and quantum error correction," *Phys. Rev. A* **54** (Nov, 1996) 3824–3851.

[20] O. A. Castro-Alvaredo and B. Doyon, "Bi-partite entanglement entropy in massive (1+1)-dimensional quantum field theories," *Journal of Physics A: Mathematical and Theoretical* **42** no. 50, (2009) 504006.

[21] G. Mussardo, *Statistical Field Theory: an introduction to exactly solved models in statistical physics*. Oxford University Press, 1th ed., 2010.

[22] D. S. P. Di Francesco, P. Matrieu, *Conformal Field Theory*. Springer, 1997.

[23] O. A. C. Alvaredo and B. Doyon, "The bologna lectures on entanglement and non-equilibrium physics in extended quantum systems." Https://thebolognalectures.weebly.com, 2016.

[24] J. Cardy, *Scaling and Renormalization in Statistical Physics*. Cambridge Lecture Notes in Physics. Cambridge University Press, 1996.

[25] A. Belavin, A. Polyakov, and A. Zamolodchikov, "Infinite conformal symmetry in two-dimensional quantum field theory," *Nuclear Physics B* **241** no. 2, (1984) 333 – 380.

[26] P. Calabrese and J. Cardy, "Entanglement entropy and conformal field theory," *Journal of Physics A: Mathematical and Theoretical* **42** no. 50, (2009) 504005.

[27] C. Holzhey, F. Larsen, and F. Wilczek, "Geometric and renormalized entropy in conformal field theory," *Nuclear Physics B* **424** no. 3, (1994) 443 – 467.

[28] A. B. Zamolodchikov and A. B. Zamolodchikov, "Factorized s-matrices in two dimensions as the exact solutions of certain relativistic quantum field theory models," *Annals of Physics* **120** no. 2, (1979) 253 – 291.

[29] B. Pozsgay and G. Takács, "Form factors in finite volume i: Form factor bootstrap and truncated conformal space," *Nuclear Physics B* **788** no. 3, (2008) 167 – 208.

[30] B. Pozsgay and G. Takács, "Form factors in finite volume ii: Disconnected terms and finite temperature correlators," *Nuclear Physics B* **788** no. 3, (2008) 209 – 251.

[31] M. Karowski and P. Weisz, "Exact form factors in (1 + 1)-dimensional field theoretic models with soliton behaviour," *Nuclear Physics B* **139** no. 4, (1978) 455 – 476.

[32] A. Fring, G. Mussardo, and P. Simonetti, "Form factors for integrable lagrangian field theories, the sinh-gordon model," *Nuclear Physics B* **393** no. 1, (1993) 413 – 441.

[33] O. A. C. Alvaredo, *Bootstrap Methods in (1+1)-Dimensional Quantum Field Theories: the Homogeneous Sine-Gordon Models*. PhD thesis, University of Santiago de Compostela.

[34] V. P. Yurov and A. B. Zamolodchikov, "Correlation functions of integrable 2d models of the relativistic field theory: Ising model," *Int. J. of Mod. Phys. A* **06** no. 19, (1991) 3419–3440.

[35] M. LÃ$\frac{1}{4}$scher, "Volume dependence of the energy spectrum in massive quantum field theories. i. stable particle states," *Comm. Math. Phys.* **104** no. 2, (1986) 177–206.

[36] T. D. Schultz, D. C. Mattis, and E. H. Lieb, "Two-dimensional ising model as a soluble problem of many fermions," *Rev. Mod. Phys.* **36** (1964) 856–871.

[37] H. Casini, C. D. Fosco, and M. Huerta, "Entanglement and alpha entropies for a massive dirac field in two dimensions," *Journal of Statistical Mechanics: Theory and Experiment* **2005** no. 07, (2005) P07007.

[38] L.Dixon, D. Friedan, E. Martinec, and S. Shenker, "The conformal field theory of orbifolds," *Nuclear Physics B* **282** (1987) 13 – 73.

[39] O. Blondeau-Fournier and B. Doyon, "Expectation values of twist fields and universal entanglement saturation of the free massive boson," *Journal of Physics A: Mathematical and Theoretical* **50** no. 27, (2017) 274001.

[40] M. Sato, T. Miwa, and M. Jimbo, "Studies on holonomic quantum fields, iv," *Proc. Japan Acad. Ser. A Math. Sci.* **53** no. 6, (11, 1977) 183–185.

[41] O. Castro-Alvaredo, C. De Fazio, B. Doyon, and I. Szécsényi, "Entanglement entropy of finite desity excited states in the massive free boson theory (*work in progress*),".