POINT AND LIE BÄCKLUND SYMMETRIES OF CERTAIN PARTIAL DIFFERENTIAL EQUATIONS

A thesis presented in partial fulfilment
of the requirements
for the degree
of MA
in Mathematics at

MASSEY UNIVERSITY

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1994

ACKNOWLEDGEMENTS

I wish to record my gratitude for the support and encouragement given me by Associate Professor Dean Halford in the preparation of this thesis.

I also thank Mrs Gail Tyson for her sterling work in typing this thesis.

CONTENTS

CHAPTER 1: INTRODUCTION

The aim of this thesis is to:

(1) Explore the use of differential forms in obtaining point and contact symmetries of particular partial differential equations (PDEs) and hence their corresponding similarity solutions. [1] and [4].

(2) Explore the generalized or Lie-Bäcklund symmetries of particular PDEs with particular reference to the Korteweg-de Vries-Burgers (KdVB) equation [3].

Finding point symmetries of a PDE $H = 0$ with independent variables (x_1, x_2) which we take to represent space and time and dependent variable (u) means finding the **transformation group**

$$x_1' = x_1 + \varepsilon \xi_1 (x_1, x_2, u) + 0(\varepsilon^2)$$
$$x_2' = x_2 + \varepsilon \xi_2 (x_1, x_2, u) + 0(\varepsilon^2)$$

and

$$u' = u + \varepsilon\eta(x_1, x_2, u) + 0(\varepsilon^2)$$

that takes the variables (x_1, x_2, u) to the system (x_1', x_2', u') and maps solutions of $H = 0$ into solutions of the same equation. The form of $H = 0$ remains invariant. The transformation group is usually expressed in terms of its **infinitesimal generator** $(\mathbf{X})$ where

$$\begin{aligned}\mathbf{X} &= \xi_1 \partial_{x_1} + \xi_2 \partial_{x_2} + \eta\partial_u \\ &= \xi_i \partial_{x_i} + \eta \partial_u \qquad\qquad i = 1, 2\end{aligned}$$

using the tensor summation convention. $\mathbf{X}$ can be considered as a differential vector operator with components (ξ_1, ξ_2, η) operating in a three dimensional manifold (space) with coordinates (x_1, x_2, u). The invariance of $H = 0$ under the transformation group is expressed in terms of a suitable **prolongation** or **extension** of $\mathbf{X}$ (denoted by $\mathbf{X}^{(pr)}$) to cover the effect of the transformations on the derivatives of u in $H = 0$.

The **invariance condition** for $H = 0$ under the action of the transformation group is

$$\mathbf{X}^{(pr)}[H] = 0 \text{ whenever } H = 0.$$

We consider x_1, x_2, u and the derivatives of u to be independent variables.

In practical terms, finding point symmetries of $H = 0$ means finding the components (ξ_1, ξ_2, η) of the infinitesimal generator ($\mathbf{X}$). There are two general methods for finding ξ_1, ξ_2 and η.

1 THE CLASSICAL METHOD

The method we follow was developed mainly by Bluman and Cole [2]. Consider for example a k^{th} order PDE $H = 0$. H is regarded as a function of x_1, x_2 and u as well as the partial derivatives of u with respect to x_1 and/or x_2 up to and including the k^{th} order derivatives. The $\mathbf{k^{th}}$ **prolongation** of the infinitesimal generator is

$$\mathbf{X}^{(pr)} = \xi_i \partial_{x_i} + \eta \partial_u + \eta^{(1)}_{i_1}, \partial_{u_{i_1}} + \ldots$$
$$\ldots \ \eta^{(k)}_{i_1 i_2 \ldots i_k} \ \partial_{u_i i_2 \ldots i_k}$$

using again the tensor summation convention where $i_r = 1, 2$ and $r = 1, 2, \ldots k$.

Note: $i_r = 1 \equiv x_1$ and $i_r = 2 \equiv x_2$.

For example:

$$\eta^{(2)}_{i_1 i_2} \ \partial_{u_{i_1 i_2}} = \eta^{(2)}_{11} \ \partial_{u_{11}} + \eta^{(2)}_{12} \ \partial_{u_{12}} + \eta^{(2)}_{22} \ \partial_{u_{22}}$$
$$\equiv \eta^{(2)}_{xx} \partial_{u_{xx}} + \eta^{(2)}_{xt} \partial_{u_{xt}} + \eta^{(2)}_{tt} \partial_{u_{tt}}$$

where x represents space and t time.

The coefficients $\eta^{(k)}_{i_1 i_2 \ldots i_k}$ are given by the expressions

$$\eta^{(1)}_{i_1} = D_{i_1}(\eta) - u_j D_{i_1}(\xi_j)$$

$$\eta^{(2)}_{i_1 i_2} = D_{i_2}\eta^{(1)}_{i_1} - (u_{i_1})_j \ D_{i_2}(\xi_j)$$

$$\eta^{(k)}_{i_1 i_2 \ldots i_k} = D_{i_k}\left(\eta^{(k-1)}_{i_1 i_2 \ldots i_{k-1}}\right) - \left(u_{i_1 i_2 \ldots i_{k-1}}\right)_j D_{i_k}\left(\xi_j\right)$$

$$i_r = 1, 2 \text{ or } (x, t) \quad r = 1, 2 \ldots k.$$

$$j = 1, 2 \text{ or } (x, t)$$

and D_{i_k} is the **Total Derivative Operator**

where
$$D_\alpha = \partial_\alpha + u_\alpha \partial_u + u_{\alpha_{i_1}} \partial_{u_{i_1}} + u_{\alpha_{i_1 i_2 \ldots i_r}} \partial_{u_{i_1 i_2 \ldots i_r}}$$

$$\alpha = 1, 2 \text{ or } (x, t).$$

For example

$$\eta^{(1)}_1 = \frac{\partial \eta}{\partial x_1} = \left\{\frac{\partial \eta}{\partial u} - \frac{\partial \xi_1}{\partial x_1}\right\}\left(\frac{\partial u}{\partial x_1}\right) - \left(\frac{\partial \xi_2}{\partial x_1}\right)\left(\frac{\partial u}{\partial x_2}\right) - \left(\frac{\partial \xi_1}{\partial u}\right)\left(\frac{\partial u}{\partial x_1}\right)^2 - \left(\frac{\partial \xi_2}{\partial u}\right)\left(\frac{\partial u}{\partial x_1}\right)\left(\frac{\partial u}{\partial x_2}\right)$$

$X^{(k)}[H] = 0$ consists of a polynomial in u and its derivatives and has to hold whenever $H = 0$ for all values of $u = u(x_1 \, x_2)$ that are solutions of the PDE $H = 0$. This implies that the coefficients of u and its derivatives must be identically equal to zero. This gives a set of linear partial differential equations called **determining equations** which can in principle be solved for ξ_1, ξ_2 and η. The general solution of $H = 0$ is a family of surfaces in $(x_1 \, x_2 \, u)$ space.

If $F(x_1, x_2, u) = 0$ defines such a surface then $F(x_1, x_2, u) = 0$ is an invariant of the transformation group

i.e.
$$\mathbf{X}[F] = 0$$

or
$$\xi_1 \frac{\partial F}{\partial x_1} + \xi_2 \frac{\partial F}{\partial x_2} + \eta \frac{\partial F}{\partial u} = 0$$

This is often referred to as the **invariant surface condition.**

A first step in obtaining similarity solutions is solving the subsidiary equations

$$\frac{dx_1}{\xi_1} = \frac{dx_2}{\xi_2} = \frac{du}{\eta}$$

The solution involves:

(1) $\zeta(x_1, x_2, u)$ called the **similarity variable** which becomes the independent variable;

(2) then the dependent variable is taken as $v = f(\zeta)$ where f is an arbitrary function of ζ.

The similarity form of the solution of the original PDE is;

$$u = u(x_1, x_2, f(\zeta))$$

Substitution of this form of u into $H = 0$ gives an ordinary differential equation (ODE) for $v = f(\zeta)$ which can in principle be solved for $f(\zeta)$ thus giving the similarity solution for $H = 0$.

2 THE USE OF DIFFERENTIAL FORMS

Harrison and Estabrook [1] used differential forms to formulate systems of partial differential equations and so obtain their point symmetries and similarity solutions.

An Introduction to Differential Forms [4]

We consider two geometrical objects, namely vectors (**V**) and differential forms (α), which exist in an n-dimensional differentiable manifold with coordinates x_i ($i = 1, 2 \ldots n$).

A **vector** (**V**) is a linear differential operator that at each point maps a differentiable, real valued function $f(x_i)$ into a real number. The vector is represented in the coordinate basis as

$$\mathbf{V} = v_a(x_i)\partial_{x_a} \quad (a = 1, 2, \ldots n)$$

using the tensor summation convention.

The functions $v_a(x_i)$ are the components of the vector. An example of a vector is the infinitesimal generator (**X**) of a transformation group which is also referred to as an **isovector**.

We start our consideration of differential forms by defining a **0-form** as a real valued function $f(x_i)$ ($i = 1, 2, \ldots n$) in the differentiable manifold. A **1-form** is then defined as a linear combination of the basis differentials dx_i ($i = 1, 2, \ldots n$)

$$\text{i.e. } \underset{(1)}{\alpha} = a_i\, dx_i \text{ where the } a_i\text{s are 0-forms.}$$

1-forms are combined by an operation denoted by $\wedge$ known as the **exterior** or **wedge product.** The exterior product of the 1-forms $\sigma = \sigma_a dx_a$ and $\omega = \omega_b dx_b$, denoted by $\sigma \wedge \omega$, is

$$\begin{aligned}\sigma\wedge\omega &= \sigma_a\omega_b dx_a \wedge dx_b \\ &= \frac{1}{2}(\sigma_a\,\omega_b - \sigma_b\omega_a)\, dx_a\wedge dx_b\end{aligned}$$

$\sigma\wedge\omega$ is a **2-form** and the most general 2-form is a linear superposition of the ${}^nC_2$ basis 2-forms $dx_a \wedge dx_b$, that is

$$\underset{(2)}{\alpha} \leq \alpha_{ab} dx_a \wedge dx_b \quad \text{where} \quad \alpha_{ab} = -\alpha_{ab} \text{ is skew symmetric.}$$

We now generalize. A **p-form** is the exterior product of $p(0 < p \leq n)$ 1-forms

$$\underset{(p)}{\alpha} = \frac{1}{p!}\,\alpha_{a_1 a_2 \ldots a_p} dx_{a_1} \wedge dx_{a_2} \wedge \ldots \wedge dx_{a_p}$$

where the coefficients $\alpha_{a_1 a_2 \ldots a_p}$ are completely skew-symmetric

i.e. $$\alpha_{a_1 a_2 \ldots a_p} = \alpha_{[a_1 a_2 \ldots a_p]}$$

where $[a_1 a_2 \ldots a_p]$ is an odd permutation of $a_1 a_2 \ldots a_p$.

Differential forms and the operation of exterior product form a Grassmann Algebra in the n-dimensional manifold with the following properties:

(1) Forms of the same degree may be added or subtracted.

(2) $\underset{(p)}{\alpha} \wedge \underset{(q)}{\beta}$ is a p + q-form which is zero if p + q > n.

(3) $\underset{(p)}{\alpha} \wedge \underset{(q)}{\beta} = (-1)^{pq} \underset{(q)}{\beta} \wedge \underset{(p)}{\alpha}$.

This implies that $dx_i \wedge dx_j = - dx_j \wedge dx_i \begin{cases} = 0 & i = j \\ \neq 0 & i \neq j \end{cases}$

(4) The exterior product is distributive, i.e. $(\alpha+\beta)\wedge\gamma = \alpha\wedge\gamma + \beta\wedge\gamma$

(5) The exterior product is associative, i.e. $(\alpha\wedge\beta)\wedge\gamma = \alpha\wedge(\beta\wedge\gamma)$

To construct a calculus of differential forms we need two differential operators namely

(1) **The exterior derivative** (d), and
(2) **The Lie derivative** ($£_v$) with respect to the vector V.

The exterior derivative acts on a p-form α to produce a (p+1)-form $d\alpha$ and is defined as

$$d\alpha = \frac{1}{p!} d\left(\alpha_{a_1 a_2 \dots a_p}\right) \wedge dx_{a_1} \wedge dx_{a_2} \wedge \dots \wedge dx_{a_p}$$

where $$d\left(\alpha_{a_1 a_2 \dots a_p}\right) = \frac{\partial\left(\alpha_{a_1 a_2 \dots a_p}\right)}{\partial x_j} dx_j$$

The exterior derivative has the following properties. Let α be a p-form, β a q-form and f a 0-form.

(1) $d(\alpha+\beta) = d\alpha + d\beta$ (linearity)
(2) $d(\alpha\wedge\beta) = d\alpha \wedge\beta + (-1)^p \alpha\wedge d\beta$ (Leibniz Rule)
(3) $d(d\alpha) = 0$ (Poincaré Lemma)
(4) $d(f\alpha) = df \wedge\alpha + fd\alpha$

The Lie derivative operator ($£_v$) is a linear differential operator associated with a vector field $V = v_a(x_i)\,\partial_{x_n}$ $(a \leq 1 \dots n)$ which can be applied to any geometrical object. We will confine it to vectors and differential forms.

The Lie derivative of a vector A with respect to V is the commutator of the two vectors

i.e. $$£_v(A) = [V, A] = -\, £_A(V)$$

or in coordinate form $$£_v\left(a_i \partial_{x_i}\right) = \left(v_k \frac{\partial a_i}{\partial_{x_k}} - a_k \frac{\partial v_i}{\partial_{x_k}} \right) \partial_{x_i}$$

Before considering the application of the Lie derivative to a differential form it is necessary to consider the **contraction** of a vector (V) and a p-form (α) which gives a (p–1) form β.

Notation contraction $\beta = \langle V, \alpha \rangle$

or $\beta = V \lrcorner\, \alpha$

In component notation $V \lrcorner\, \alpha$ is defined as

$$\begin{aligned} V \lrcorner\, \alpha &= \left(v_b \partial_{x_b}\right) \lrcorner \left(\alpha_{[a_1 a_2 \ldots a_p]} dx_{a_1} \wedge dx_{a_2} \wedge \ldots \wedge dx_{a_p}\right) \\ &= p!\, v_b \alpha_{[b a_2 \ldots a_p]} dx_{a_2} \wedge dx_{a_3} \wedge \ldots \wedge dx_{a_p} \end{aligned}$$

Note: The above definition implies that

$$\partial_{x_i} \lrcorner\ dx_j = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

δ_{ij} is the Kronecker delta.

Properties of the contraction of V and $\underset{(p)}{\alpha}$ include

(1) $$V \lrcorner \left(\underset{(p)}{\alpha_1} + \underset{(p)}{\alpha_2} \right) = V \lrcorner\ \underset{(p)}{\alpha_1} + V \lrcorner\ \underset{(p)}{\alpha_2}$$

(2) $$V \lrcorner \left(\underset{(p)}{\alpha} \wedge \underset{(q)}{\beta} \right) = \left(V \lrcorner\ \underset{(p)}{\alpha} \right) \wedge \underset{(q)}{\beta} + (-1)^p\, \underset{(p)}{\alpha} \wedge \left(V \lrcorner\ \underset{(q)}{\beta} \right)$$

(3) $Vf = V \lrcorner\ df$ f = 0-form

For a function f(0-form) the Lie derivative of f with respect to V is a 0-form

$$\pounds_v(f) = V \lrcorner df = V \lrcorner \left(\frac{\partial f}{\partial x_a} dx_a \right) = v_a \frac{\partial f}{\partial x_a}.$$

For a p-form (α) $(0 < p \leq n)$ the Lie derivative of α with respect to V is given in terms of the exterior derivative and the contraction as

$$\pounds_v(\alpha) = V \lrcorner\, d\alpha + d(V \lrcorner\, \alpha)$$

which is also a p-form.

Properties of the Lie derivative are

(1) $\pounds_v(d\alpha) = d(\pounds_v(\alpha))$ i.e. Lie derivative and exterior derivative commute

(2) $\pounds_v(\alpha \wedge \beta) = (\pounds_v(\alpha)) \wedge \beta + \alpha \wedge (\pounds_v(\beta))$

(3) $\pounds_v(W \lrcorner\, \alpha) = [V,W] \lrcorner\, \alpha + W \lrcorner\, \pounds_v(\alpha)$, where W is a vector.

The geometric approach of Harrison and Estabrook [1] involves the use of differential forms to find the isovector or infinitesimal generator (X) of the transformation group of a PDE $H = 0$. Here we shall consider a PDE of order k in one dependent and 2 independent variables, u, x_1, x_2 with derivatives of u up to degree k.

The PDE is first represented by a closed set of differential forms α_i $(i = 1, 2, \ldots)$ in an n-dimensional manifold M. This set of forms constitutes a closed differential ideal (I) on the manifold. I being closed means that if $\alpha_i = 0$ then $d\alpha_i = 0$ also.

An **integral manifold** is a submanifold of M on which the differential forms (α_i) are expressed in terms of the independent variables of the PDE and their differentials. The α_i are annulled (take zero values) on the integral manifold to give the following information:

(i) The original partial differential equation

(ii) The definition of the n–3 auxiliary variables in M. These are usually derivatives of u of degree less than k.

(iii) The integrability conditions on u.

Imposing independence of x_1 and x_2 and their differentials dx_1 and dx_2 puts the ideal (I) in involution with x_1 and x_2 (by definition). Cartan's geometric theory of PDEs [6, 7] implies that there exists a general or regular integral manifold that can be considered a solution manifold for the PDE. The Lie groups of point symmetries of PDEs is represented by the infinitesimal group generator, or isovector, X. It is suitably extended or prolonged as a linear differential vector operator $X^{(pr)}$ in the space of the variables x_1, x_2, u and derivatives of u. There are two differential operators that naturally arise in the ring of differential forms. These are

(1) the exterior derivative, and
(2) the Lie derivative.

The Lie derivative, defined in terms of an isovector of the symmetry group, is used in formulating the invariant conditions. Point symmetries of a PDE H = 0 are defined by the action of $X^{(pr)}$ on the PDE,

i.e. $$X^{(pr)}[H] = 0 \text{ whenever } H = 0.$$

With differential forms this is equivalent to saying that the Lie derivative in the direction of X of all differential forms $\alpha_i \in I$ are in the ideal and should vanish if $\alpha_i = 0$. That is $£_x(\alpha_i) = 0$ if $\alpha_k = 0$ or equivalently $£_x(\alpha_i)$ is a linear combination of the differential forms $\underset{(q)}{\alpha_k}$ where $q \leq p$ i.e. $£_x(\alpha_i) = \lambda_i^k \wedge \alpha_k$ (sum over k). The λ_i^k are arbitrary differential forms, including in some cases 0-forms or functions. In such cases $\lambda_i^k \wedge \alpha_k$ is usually written as $\lambda_i^k \alpha_k$. After eliminating the λ_i^k forms the symmetry or invariant condition can be reduced to a set of determining equations which can be solved for the components ξ_1, ξ_2 and η of the infinitesimal generator of the invariant transformation group. These components, besides being functions of x_1, x_2 and u also contain a number of arbitrary integration constants.

The finding of **similarity solutions** involves augmenting the ideal of differential forms and imposing the condition that the augmented forms be annulled on the integral manifold as well as the ideal. One way to augment the ideal is by contracting the differential forms (α_i) in the ideal with the isovector (now denoted by V).

That is $$\sigma_i = V \lrcorner \alpha_i .$$

Now $$£_v(\sigma_i) = £_v(V \lrcorner \alpha_i) = V \lrcorner £_v(\alpha_i) = V \lrcorner \left(\lambda_i^k \wedge \alpha_k\right)$$
$$= \left(V \lrcorner \lambda_i^k\right)\alpha_k + (-1)^{p-q}\lambda_i^k \wedge \alpha_k$$

which means the augmented ideal $\{\alpha_i, \sigma_i\}$ is invariant under the action of V. Annulling certain forms in the augmented ideal should then produce similarity solutions of the PDE.

Appendix A shows in detail the use of the classical method of Bluman & Cole for finding similarity solutions of the nonlinear diffusion equation $\phi_{x_2} = \left(K(\phi)\phi_{x_1}\right)_{x_1}$ and the use of differential forms for finding the point symmetries of the Korteweg-deVries (KdV) equation

$$u_{x_2} + uu_{x_1} + \varepsilon u_{x_1x_1x_1} = 0$$

where ε is a constant.

Comparing the two methods as far as hand computation is concerned, I find that they are of comparable difficulty, although some PDEs might be more easily processed by one or other of the two methods. In general the Harrison-Estabrook method using differential forms gives simpler determining equations which are however, usually obtained by more involved manipulations.

The use of differential forms seems to be the preferred method for the various computer packages, for example MACSYMA [8] that are used to find the determining equations. In my opinion the classical method of Bluman & Cole has the characteristics of an algorithm and gives comparatively little insight into the process of finding symmetries and similarity solutions. The use of differential forms on the other hand involves the manipulation of geometric objects in an n dimensional manifold and the process of finding symmetries can be given a geometrical interpretation.

Lie-Bäcklund Symmetries [3, 4, 11 and 12]

We begin by considering a partial differential equation $H(\mathbf{x}, \mathbf{u}^{(n)}) = 0$ with n independent variables $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and a single dependent variable u.

$\mathbf{u}^{(N)} = (u, \underset{1}{u}, \underset{2}{u}, \dots \underset{N}{u})$ where $\underset{i}{u}$ are the i^{th} order partial derivatives of u with respect to the components of $\mathbf{x}$. One-parameter Lie point symmetries of such a PDE are transformations of the form

$$x_i' = x_i + \varepsilon\, \xi_i\, (\mathbf{x}, u) + 0(\varepsilon^2)\ \ (i = 1, \dots, n)$$

$$u' = u + \varepsilon\eta(\mathbf{x}, u) + 0(\varepsilon^2)$$

that leave $H(\mathbf{x}, \mathbf{u}^{(N)}) = 0$ invariant. These point symmetries are usually expressed in terms of their infinitesimal generator $X = \xi_i \partial_{x_i} + \eta \partial_u$ suitably prolonged to cover the action of derivatives of u. A possible generalization of this would be to transformations where the coefficients ξ_i and η are also functions of derivatives of u. i.e. $\xi_i = \xi_i(\mathbf{x}, u, \underset{1}{u}, \ldots \underset{N}{u})$ and $\eta = \eta(\mathbf{x}, u, \underset{1}{u}, \ldots \underset{N}{u})$. If N is finite these so called generalized transformations are either prolonged point transformations or contact transformations [4]. Contact transformations only occur in situations involving a single dependent variable (u) and are characterized by $\xi_i = \xi_i(\mathbf{x}, u, \underset{1}{u})$ and $\eta = \eta(\mathbf{x}, u, \underset{1}{u})$. For broader generalizations we have to consider transformations where the coefficients ξ_i and η contain derivatives of u of arbitrarily high order. The prolongation of $X = \xi_i \partial_{x_i} + \eta \partial_u$ to cover the effects of derivatives of u has to be in general an infinite prolongation.

$$X^{(\infty)} = X + \sum_J \eta_J (\mathbf{x}, u, \underset{1}{u}, \ldots) \partial_{u_J}$$

where

$$\eta_J = D_J(\eta - \xi_i u_i) + \xi_i u_{Ji}$$

and $J = j_i \ldots j_k$ where j_k is a suitable integer of $\mathbf{x}$ and $k \geq 0$.

In an analogous way to point symmetries X is a **generalized symmetry** of $H(\mathbf{x}, \mathbf{u}^{(N)}) = 0$ if and only if $X^{(\infty)} [H(\mathbf{x}, \mathbf{u}^{(N)})] = 0$ for every smooth* solution $u = f(\mathbf{x})$ of the PDE. In practice $H(\mathbf{x}, \mathbf{u}^{(N)}) = 0$ depends only on a finite number of derivatives of u so only a finite number of terms of $X^{(\infty)}$ are required in any given instance. This means that the question of convergence of $X^{(\infty)}$ does not arise. Generalized symmetries of this type are commonly called **Lie-Bäcklund symmetries.** (Olver [3] uses the term generalized symmetry) and include point and contact symmetries as special cases.

In this thesis we shall deal exclusively with time-evolution equations in two independent variables x and t of the form $H \equiv u_t + K(x, u, u_1, \ldots, u_n) = 0$ where $u_i = \dfrac{\partial^i u}{\partial x^i}$ $(i = 0, 1, \ldots, n)$ and $u_0 = u$ is the (only) dependent variable.

* smooth means that u and its derivatives are continuous in the domain of applicability.

Bluman and Kumei [12] and others [3, 4, 10, 11] prove that, for a time evolution equation, a Lie-Bäcklund transformation of the form

$$x' = x + \varepsilon\, \xi_1(x, t, u, u_1 \ldots) + 0(\varepsilon^2)$$
$$t' = t + \varepsilon\, \zeta_2(x, t, u, u_1 \ldots) + 0(\varepsilon^2)$$

and $$u' = u + \varepsilon\, \eta(x, t, u, u_1 \ldots) + 0(\varepsilon^2)$$

acts on a solution surface $F(x, t, u) = 0$ of the PDE in the same manner as

$$x' = x$$
$$t' = t$$

and $$u' = u + \varepsilon Q + 0(\varepsilon^2)$$

where $$Q = \eta - \xi_1 u_1 - \xi_2 u_t$$

This means that the infinitesimal generator can now be expressed in the simpler form $X(Q) = Q(x, t, u, u_1 \ldots)\, \partial_u$. $X(Q)$ is called the **evolutionary infinitesimal generator** and Q is referred to as its **characteristic**. The infinite prolongation of $X(Q)$ now takes the form

$$X^{(\infty)}(Q) = \sum_J D_J[Q]\, \partial_{u_J}$$

From the equivalence of the two Lie-Bäcklund transformations detailed above, the following result can be easily proved [3]; **An infinitesimal generator X is a Lie-Bäcklund symmetry of a PDE if and only if its evolutionary form X(Q) is a Lie-Bäcklund symmetry.**

For a time-evolution equation

$$u_t + K(x, u, u_1 \ldots u_n) = 0$$

the infinite prolongation $X^{(\infty)}(Q)$ of the infinitesimal generator takes the form

$$X^{(\infty)}(Q) = Q\partial_u + (D_t[Q])\partial_{u_t} + \sum_{j=1}^{\infty} D_x^j [Q]\partial_{u_j}$$

where $Q = Q(x, t, u, u_1 \ldots u_N)$ N arbitrary and D_i is the total derivative operator with respect to x_i. Thus

$$D_x \equiv \partial_x + u_x\partial_u + u_{xt}\partial_{u_t} + u_{xx}\partial_{u_x} + \ldots$$

and
$$D_t \equiv \partial_t + u_t\partial_u + u_{tx}\partial_{u_x} + u_{tt}\partial_{u_t} + \ldots$$

In considering Lie-Bäcklund symmetries of a PDE it is convenient to use two operators namely:

(i) the Fréchet derivative, and

(ii) the recursion operator.

The **Fréchet derivative** of a smooth differential function $H[u] = H[x, t, u, u_t, u_1, \ldots u_n]$ is defined as

$$D_H(Q) = \left|\frac{d}{d\varepsilon} H[u + \varepsilon Q]\right|_{\varepsilon=0}$$

It can be readily shown that this is equivalent to

$$D_H(Q) = \left(\frac{\partial H}{\partial u} + \frac{\partial H}{\partial u_t} D_t + \sum_{j=1}^{\infty} \frac{\partial H}{\partial u_j} D_x^j\right)[Q]$$

Comparison with $X^{(\infty)}(Q)[H]$ shows that

$$X^{(\infty)}(Q)[H] = D_H(Q)$$

The invariance condition for the PDE $H = 0$ under the action of $X^{(\infty)}(Q)$ can be written as $D_H(Q) = 0$ whenever $H = 0$. Either form of the invariance condition can be used as an algorithm for finding Q as the solution of a system of determining equations.

Definition

The operator $R = R(u, u_t, u_1, \ldots u_n)$ **is a recursion operator of the time evolution equation $H \equiv u_t + K(x, u, u_1, \ldots u_n) = 0$ if and only if $[D_H, R]_{H=0} = 0$.** From this definition Fokas [9] and others prove that if R is a recursion operator of $H = 0$ and

$Q = Q(x, t, u, u_1, \dots u_N)$, N arbitrary, is an Lie-Bäcklund symmetry of $H = 0$, then $X(R^j[Q])_j$ for j = 1, 2 ... are also Lie-Bäcklund symmetries of the PDE.

That is, the recursion operator can generate an infinite sequence of Lie-Bäcklund symmetries depending on higher order derivatives of u.

The method of determining all Lie-Bäcklund symmetries of a PDE $H(\mathbf{x}, \mathbf{u}^{(u)}) = 0$ is to start with the evolutionary form $X(Q) = Q\partial_u$ and to decide on some arbitrary order of derivatives for Q. We then use the invariance condition $D_H(Q) = 0$ whenever $H = 0$ to generate an equation involving derivatives of Q and u.

A significant calculational feature is that for time - evolution equations the PDE can be used to substitute for any t derivatives of u which implies that Q involves only x derivatives of u

i.e. $$Q = Q(x, t, u, u_1 \dots u_N)$$

As the invariance condition holds for any solution $u = u(x, t)$ of $H = 0$ we can equate coefficients of the derivatives of u in descending order to zero and find the general form of Q. Bluman and Kumei [10] use this method to find two finite order Lie-Bäcklund symmetries of the non-linear diffusion equation $\{a(u+b)^{-2}u_x]_x - u_t = 0$ and then obtain the recursion operator by inspection. In this way they can generate the entire sequence of Lie-Bäcklund symmetries for this equation.

To find a Lie-Bäcklund symmetry we must **assume** the order (N) of the highest derivative in Q. **What value of N do we start with?** If a recursion operator exists, then in almost all known cases the point symmetry operator for invariance of $H = 0$ under a t translation is generated (by the recursion operator) from that expressing invariance under a x translation [9].

For a time-evolution equation of the form $u_t + u_n + G(x, u, u_1 \dots u_{n-1}) = 0$ we have two Lie point symmetries $Q_1 = u_1$ and $Q_2 = u_n + G[u]$. If there is a recursion operator R such that $Q_2 = R[Q_1]$ then $R = D_x^{n-1} + \dots$. Therefore the first Lie-Bäcklund symmetry is $Q_3 = R[Q_2] = u_{2n-1} + g(x, u, u_1, \dots, u_{2n-2})$, which implies that $N = 2n - 1$.

Similarity (invariant) solutions

As with point symmetries, similarity or invariant solutions can be found from a given Lie-Bäcklund symmetry [12]. A solution $u = u(x, t)$ of a time-evolution equation $H \equiv u_t + K(x, u, u_1, \dots, u_n) = 0$ is invariant under the action of a Lie-Bäcklund symmetry if and only if $u = u(x, t)$ satisfies the invariant surface condition $Q(x, t, u, u_1, \dots, u_N) = 0$. $Q(x, t, u, u_1, \dots u_N) = 0$ is regarded as an N^{th} order ordinary differential equation in the independent variable x with t as a parameter. The solution of the ODE is a similarity form

$$\phi(x, t, u, c_1(t), \dots, c_N(t)) = 0$$

with the arbitrary functions $c_1(t), \dots, c_N(t)$ acting as integration "constants". These integration "constants" can be determined by substitution of the similarity form into the time-evolution equation.

In chapter 3, I intend to study possible Lie-Bäcklund symmetries and similarity solutions of the **Korteweg-de Vries-Burgers** (KdVB) equation

$$u_t + auu_x + bu_{xx} + cu_{xxx} = 0, \text{ where a, b, and c are constants.}$$

This equation is the simplest form of a wave equation that incorporates nonlinearity (the auu_x term), dispersion (cu_{xxx}) and attenuation (bu_{xx}). In a wave equation $u = u(x, t)$ is a perturbation of the medium through which the wave is travelling and can be either perpendicular to (transverse waves) or parallel to (compressional waves) the direction of wave propagation. The KdVB equation has been widely used to model many types of nonlinear wave motion including for example:

(i) the propagation of waves in liquid filled elastic tubes [14]
(ii) tidal bores [21]
(iii) magneto-hydrodynamic shock waves in plasmas [20]
(iv) the propagation of acoustic waves in liquids containing small bubbles [13].

Johnson [14], using phase plane analysis on the steady state (constant wave velocity) form of the KdVB equation, obtained soliton progressive wave, and shock wave solutions by using various values of a, b and c, particularly b (the constant governing the degree of attenuation of the wave by the medium). Exact solutions of the KdVB equation have been obtained by several authors [17 - 19], however Vlieg-Hulstman and Halford [16] demonstrated that these solutions are essentially equivalent to a single exact solution that is a linear combination of particular solutions of the KdV equation and Burgers equation. Lakshmanan and Kaliappan [15] found that the KdVB equation has the following point symmetries

$$\xi_1 = ak_1 t + k_2, \quad \xi_2 = k_3 \quad \text{and} \quad \eta = k_1,$$

where k_1, k_2 and k_3 are integration constants. This implies that the KdVB equation is invariant under the following transformations

$$\begin{aligned} k_1 &= 1 \quad X_1 = at\partial_x + \partial_u \text{ (Galilean transformation)}, \\ k_2 &= 1 \quad X_2 = \partial_x \text{ (x-translation)}, \\ k_3 &= 1 \quad X_3 = \partial_t \text{ (t-translation)}. \end{aligned}$$

These lead to a similarity variable

$$\zeta = \frac{k_1 x}{c} - \frac{ak_2 t^2}{2} + k_3 t$$

and similarity solution

$$u = \left(\frac{k_2 c}{k_1}\right) t + f(\zeta),$$

where $f(\zeta)$ is an arbitrary functional of ζ. Substitution in the KdVB equation gives the ODE

$$c\frac{d^3f}{d\zeta^3} + b\frac{d^2f}{d\zeta^2} + \left[\frac{k_3 c}{k_1} + af\right]\frac{df}{d\zeta} - \frac{k_2 c}{k_1} = 0$$

which on integrating gives

$$c\frac{d^2f}{d\zeta^2} + b\frac{df}{d\zeta} + \frac{a}{2}f^2 + \left(\frac{k_3 c}{k_1}\right)f + \left(\frac{k_2 c}{k_1}\right)\zeta + c_1 = 0,$$

where c_1 is an integration constant.

A suitable Ince transformation [22]

$$z = \left(\frac{-25ac}{12b^2}\right)^{1/2} \exp\left(-\frac{b\zeta}{5c}\right)$$

and $$f = W(z) \exp\left(-\frac{2b\zeta}{5c}\right) + \frac{1}{a}\left(\frac{6b^2}{25c} - \frac{k_3 c}{k_1}\right)$$

gives the invariant ODE

$$\frac{d^2W}{dZ^2} = 6W^2 + S(Z)$$

which is free from movable critical points only if $S(Z) = pZ + q$ (p and q are constants) [22]. Hence the invariant ODE has in general movable critical points. Ablowitz and others [23] suggest that this implies that the KdVB equation is not in general exactly solvable. However, Fokas [9] defines exact solvability of a PDE in terms of it admitting a Lax formulation. That is the PDE can be expressed in the form

$$L_t = [A, L]$$

where A is the Fréchet derivative of the t independent part of the PDE and L is the recursion operator.

The motivation to investigate the Lie-Bäcklund symmetries of the KdVB equation is twofold:

(i) to obtain, if possible, more generalized similarity solutions that could extend the use of the KdVB equation to other cases involving nonlinear wave propagation.

(ii) to gain insight into questions of the exact solvability of the KdVB equation.