Parameters of the Two Generator Discrete Elementary Groups

A thesis presented in partial fulfillment of the requirements for the degree of

Master of Science

in

Mathematics

At Massey University, Albany,

New Zealand

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2006

Abstract

Let f, g be elements of $\mathcal{M}$, the group of Möbius transformations of the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \infty$. We identify each element of $\mathcal{M}$ with a 2×2 complex matrix with determinant 1. The three complex numbers,

$$\beta(f) = tr^2(f) - 4, \beta(g) = tr^2(g) - 4, \gamma(f, g) = tr[f, g] - 2,$$

define the group $\langle f, g \rangle$ uniquely up to conjugacy whenever $\gamma(f, g) \neq 0$; where $tr(f)$ and $tr(g)$ denote the traces of representive matrices of f and g respectively, $[f, g]$ denotes the multiplicative commutator $fgf^{-1}g^{-1}$. We call these three complex numbers the parameters of $\langle f, g \rangle$. This thesis is concerned with the parameters of discrete and elementary subgroups of $\mathcal{M}$.

Acknowledgments

I wish to express my gratitude to my supervisor Distinguished Professor Gaven Martin, for his support throughout this thesis. And also special thanks to him in patiently correcting my English.

I owe much to Professor David Gauld of Auckland University and my friend Haydn Cooper of Massey University for patiently proofreading this thesis.

I would like to thank Massey University for giving me the opportunity to learn and for the Scholarship I have received.

I am grateful to the Marsden Fund I have received for this thesis.

I would also like to thank my beloved cat Millie, who was born on 23rd of February 2006 and sadly killed on 20th of March 2007, for her lovely company throughout this thesis.

Last but not least I would like thank my husband Brian Pool, for his unconditional support and emotional encouragement.

Contents

Introduction

Möbius transformations were studied by the German mathematician A. F. Möbius in the 19th century. F. Klein proved the group of Möbius transformations acting on Euclidean n-space is isomorphic to the group of isometries of hyperbolic $(n+1)$-space (see [17] , page 147). This discovery leads to a deeper understanding of hyperbolic space and relations between conformal geometry of spheres, the models of hyperbolic space they bound and n-dimension geometry. Relevant references can be found in the works of Beardon [1], Ratcliffe [17], Thurston [20], Gehring and Martin (see for example [5], [6], [9], [10]) and references therein. In recent years, the study of the 3-dimensional hyperbolic orbifolds, which can be represented as H^3/G where H^3 is hyperbolic 3-space (discussed in Chapter 1) and G is a discrete non-elementary orientation preserving subgroup of the group of the isometry group, has attracted much attention. We are concerned here with such discrete subgroups G. We shall assume a basic knowledge of group theory in our discussion.

Let $\mathcal{M}$ denote the group of Möbius transformations of the form:

$$f(z) = \frac{az+b}{cz+d} \qquad a, b, c, d \in \mathbb{C}, ad - bc = 1, \tag{1}$$

which we associate with the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad a, b, c, d \in \mathbb{C}, ad - bc = 1. \tag{2}$$

There are two basic types of discrete subgroup of $\mathcal{M}$: *elementary* and *non-elementary*, whose definitions are given in Chapter 2. The discrete non-elementary groups are known as Kleinian groups in memory of the Mathematician F. Klein. All the discrete elementary groups are known and classified (see [1]), hence the study of Kleinian groups are of interest. But the discreteness or otherwise of a Kleinian group is not easy to establish. Klimenko and Kopteva gave a criterion for discreteness of Kleinian groups with an invariant plane (see [3]). While for the Kleinian groups without invariant plane, we have only necessary or only sufficient conditions for the discreteness of such groups.

Theorem 5.4.2 of [1] states that a non-elementary subgroup G of $\mathcal{M}$ is discrete if and only if for each f and g in G, $\langle f, g\rangle$ is discrete. Thus the problem of deciding the discreteness or otherwise of G boils down to consideration of the two generator subgroups. We shall study the discreteness of two generator groups $\langle f, g\rangle$. The advantage of studying a two generator group is that for every such group $\langle f, g\rangle$, there are three complex numbers corresponding to it, and the necessary or sufficient condition(s) for non-elementary $\langle f, g\rangle$ to be discrete can sometimes be described in terms of these numbers. These three complex numbers are

$$\beta(f) = tr^2(f) - 4, \beta(g) = tr^2(g) - 4, \gamma(f, g) = tr[f, g] - 2,$$

where $tr(f)$ and $tr(g)$ denote the traces of representive matrices of f and g respectively, and $[f, g]$ denotes the multiplicative commutator $fgf^{-1}g^{-1}$, see [7]. These three numbers are called the *parameters* of the two-generator group $\langle f, g\rangle$ and we write

$$par(\langle f, g\rangle) = (\gamma(f, g), \beta(f), \beta(g)).$$

These parameters are independent of the choice of representive matrices for f and g and define $\langle f, g\rangle$ uniquely up to conjugacy whenever $\gamma(f, g) \neq 0$. See [7]. Two subgroups G_0 and G_1 of G are conjugate if for some h in G, $G_0 = hG_1h^{-1}$. Conjugate subgroups are the same from a geometric point of view. For example, if there exists a unique point fixed by all $g_0 \in G_0$, then there exists a unique point fixed by all $g_1 \in G_1$. The volumes of H^3/G_1 and H^3/G_0 are the same and so forth.

The study of the discreteness of two generator groups has a rich history, see all of our references except [15] and [16]. For example in [1], Beardon studies necessary conditions for a two generator Kleinian group by considering the displacement function

$$z \longmapsto \sinh\frac{1}{2}\rho(z, gz).$$

Gehring and Martin obtain conditions for $\langle f, g\rangle$ to be discrete by examining the distances of f, g from the identity element in $\mathcal{M}$ in [6]. They also obtain some sharp estimates for the distance between the axes of elliptic elements in a discrete group in [12]. Klimenko and Kopteva found criteria for discreteness of two generator Kleinian groups generated by a hyperbolic element and an elliptic element of even order with intersection axes in [3]. The most well-known necessary theorem in the subject is due to Jørgensen (see [1]):

Theorem 0.1. *(Jørgensen's inequality) Suppose that the Möbius transformations f and g generate a discrete non-elementary group with $\gamma(f, g) = \gamma$ and $\beta(f) = \beta$, then*

$$|\gamma| + |\beta| \geq 1. \tag{3}$$

This inequality was studied by Troels Jørgensen in [19]. He proved the inequality by the iteration of the relation

$$B_0 = B, \qquad B_{n+1} = B_nAB_n^{-1}$$

where A and B are the matrices representing f and g respectively. Another inequality was studied by Delin Tan in [2]:

Theorem 0.2. *Suppose that the Möbius transformations f and g generate a discrete group with $\gamma(f, g) = \gamma$ and $\beta(f) = \beta$. If $\gamma \neq -1$, then*

$$|\gamma + 1| + |\beta + 2| \geq 1. \tag{4}$$

If $\gamma = -1$ *and* $\beta \neq -2$, *then*

$$|\beta + 2| > \frac{1}{2}.$$

Tan used *Lemma 2* in his paper to prove (4); This *Lemma* was proved by the iteration of the relation

$$B_0 = B, \qquad B_{n+1} = [A_n, B_n],$$

which is essentially the same as Jøgensen's iteration scheme. Gehring and Martin proved (3) and (4) independently by investigating the two fixed points 0 and $\beta + 1$ of the polynomial trace $\gamma(f, gfg^{-1}) = \gamma(\gamma - \beta)$ in [4].

The inequalities (3) and (4) and Gehring and Martin's approach to them give a different perspective to look at the conditions for discreteness of $\langle f, g\rangle$. The fact is that in the space of two generator discrete groups, all two generator Kleinian groups form a closed set. This has essentially been proved by Jørgensen in [19]. We claim that all the elementary groups are isolated from the set of Kleinian groups in this space. This claim and precise bounds to describe this isolation in terms of geometric quantities as well as the complex parameters are investigated in this and future research. As we know that every two generator group $\langle f, g\rangle$ has three complex numbers as its parameters, we can therefore view $\langle f, g\rangle$ as a point in $\mathbb{C}^3$, the three dimensional complex space. Let D^3 be the subset of $\mathbb{C}^3$ which contains all the parameters of two generator discrete groups. We prove that whenever $(a, b, b') \in D^3$ corresponds to a discrete elementary group, it is isolated from the points corresponding to Kleinian groups. We establish the isolation of (a, b, b') by proving an inequality of the form

$$|\gamma + a| + |\beta + b| \geq c \tag{5}$$

where c is a real positive number and (γ, β, β') are the parameters for any Kleinian group. The reason that the isolation of (a, b, b') only depends on a, b will be explained in Chapter 5, but note here that (5) also implies immediately that

$$|\gamma + a| + |\beta' + b'| \geq c$$

by interchanging the order of the generators. Note also that the inequality (5) also indicates a necessary condition for a Kleinian $\langle f, g\rangle$ to be discrete. This is the main reason for looking at the isolation of discrete elementary groups.

The main concern of the first part of this thesis is to determine all the possible parameters for discrete elementary groups. These are the points in $\mathbb{C}^3$ that we shall show to be isolated. We then go on to give estimates on this isolation using some of the ideas discussed above (iteration). This recovers some known results and also generates some new ones. Baribeau and Ransford have given a general description of these parameters in [18]. Gehring and Martin have discussed some of them in many of their papers, see for example [4], 5], [13],or [14]. We consider all the parameters for all the discrete elementary groups more specified in this thesis. To this end we start with some preliminary topics such as the spherical and hyperbolic geometries, Möbius transformations, Triangle groups through Chapter 1 to Chapter 3. The results in these three Chapters are a matter of rewriting known facts. Our main results are stated in Chapter 4. In this Chapter we investigate the parameters under question using a combination of two methods: geometric and algebraic methods. We omit the lengthy but purely elementary computation process and state our final results in three tables. As we will see in these tables, for some elementary groups, we are able to find the exact parameters; while for others we can only give a general description similar to the results in [18] as there are parametrised families of these groups. For those whose exact parameters are known, we shall investigate their isolation from Kleinian groups by using the inequality of the form (5). For instance in Chapter 5, we consider several examples to show how to derive such inequalities for $(-1, -2, b_1)$, $(-2, -3, b_2)$, $(-1, -3, b_3)$.