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On sound ranging in Hilbert space

Sergij V. Goncharov
Faculty of Mechanics and Mathematics, Oles Honchar Dnipro National University, 72 Gagarin Avenue, 49010 Dnipro, Ukraine goncharov@mmf.dnulive.dp.ua

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Abstract. We consider the sound ranging problem (SRP), which is to find the position of source-point from the moments when spherical wave reaches sensor-points, in the infinite-dimensional separable Hilbert space, and describe the solving methods, for entire space and for its unit sphere. In the former case, we give some sufficient conditions for solution's uniqueness. We also provide two examples with the sets of sensors being a basis: 1st, when SRP and so-called dual problem both have single solutions, and 2nd, when SRP has two distinct solutions.

Keywords: Hilbert space, infinite dimensional, sound ranging, TDOA, sphere, uniqueness

MSC 2000 classification: primary 46C05, secondary 40A05, 15A15

Introduction

By *sound ranging* (SR), we mean the following problem. Let $(X;\rho)$ be a metric space, i.e. the set X with metric $\rho\colon X \times X \to \mathbb{R}_+$. Let $\mathbf{s} \in X$ be an unknown point, "source". At unknown moment $t_e \in \mathbb{R}$ of time the source "emits the (sound) wave", which is the sphere

$$S\big(\mathbf{s}; v(t-t_e)\big) = \big\{\mathbf{x} \in X \mid \rho(\mathbf{x};\mathbf{s}) = v(t-t_e)\big\}$$

for any moment $t \geqslant t_e$. Here v is known "sound velocity", and we may assume, without loss of generality, that $v = 1$ (switching to scaled time $t \leftarrow vt$ if $v \neq 1$).

Let $R = \{\mathbf{r}^{(i)}\}_{i\in I}$, $\mathbf{r}^{(i)} \in X$, be an indexed set of "sensors", whose positions are known. Suppose that for each sensor we know the moment t_i when it was reached by the expanding wave; that is, $t_i = t_e + \rho(\mathbf{r}^{(i)};\mathbf{s})$ are known.

The problem is to find $\mathbf{s}$ and t_e, — from known moments when wave reaches known sensors, $(R;\{t_i\})$. We're also interested in uniqueness of the solution.

Retrospection. The researches on SR, — also called *passive location*, *sound triangulation*, *time-(difference-)of-arrival source localization*, — are considered to begin at the times of World War I, with the works of William L. Bragg, Lucien Bull, Erich Waetzmann among the others (see [3], [4], [5], and [23], [29] for a survey). Similar questions were studied in [2], [9], [11], [17] from that "geometrical" era. During the century that followed, along with military ([7], [12], [15], [26], [27]) and surveillance-related ([25]) applications, SR attracted acousticians and seismologists, to name a few, — [13], [21], [31], [37], [38, 5.7]. For a fairly long time, SR problems accompany the studies of (wireless) sensor networks ([1], [6], [10], [22], [24], [30], [32]; see also [40]).

Naturally, the majority of these researches relates to $\mathbb{R}^2$ or $\mathbb{R}^3$ (emphasized occasionally: [35], [37], [39]), though there are exceptions ([6], [22], [28], [30], [36]). The basic problem was generalized to take into account the "wind" or "flow" giving additional movement to the wave, the variations of sound velocity at different space regions, diffraction and reverberation, measurement inaccuracy and its influence on the solution(s), noise removal etc. ([16], [19], [21]), — generally speaking, the factors imposed by "physics" (as a result, sometimes there's the inclination towards practical applicability instead of rigour).

The uniqueness of the solution is analyzed in e.g. [8], [21], [28], [33], [34].

The substantial part of the studies in the field deals with the so-called overdetermined problems, when the data from each sensor contains some random error, and the position of the source is estimated in attempt to reduce the uncertainty ([6], [10], [13], [14], [19], [20], [36]).

Aim. The generalization here concerns only the infinite dimensionality of the (empty) space where source and sensors are placed, and omits "physical" factors (it is of much less "applicability" than most of papers in References).

Consider (associated) question about such problem: what limitations do we impose on the "procedure" — or "algorithm" — of obtaining the solution? There's always a "universal" one, $\mathcal{U}$: "go over all $\mathbf{s} \in X$ and select those satisfying $t_i = t_e + \rho(\mathbf{r}^{(i)}; \mathbf{s})$" (if we take $\mathbf{s}$, t_e can be found as $t_i - \rho(\mathbf{r}^{(i)}; \mathbf{s})$ for some i). But $\mathcal{U}$ seems to be too "heavy". If, in a sense, the verification of each $\mathbf{s}$ takes a non-zero amount of time τ, then for many kinds of spaces X we'll need an infinite, non-countable set of "verificating entities" to confine into a finite time.

Less heavy but still not appropriate (here) procedure $\mathcal{N}$ is as follows: take some point $\mathbf{b} \in X$ as the "origin" and, at each n-th step, make the "$\frac{1}{n}$-net N_n" in the ball $B(\mathbf{b}; n)$ ($\forall \mathbf{x} \in B(\mathbf{b}; n)\ \exists \mathbf{y} \in N_n$: $\rho(\mathbf{x}; \mathbf{y}) < \frac{1}{n}$). In turn, for each $\mathbf{y} \in N_n$ calculate the "defect" $\delta(\mathbf{y}) = \sup\limits_{i \in I} \left| t_e + \rho(\mathbf{r}^{(i)}; \mathbf{y}) - t_i \right|$ (or, when R is finite, let $\delta(\mathbf{y}) = \frac{1}{|R|} \sum\limits_{i \in I} \left[\frac{1}{|R|} \sum\limits_{j \in I} \left(t_j - \rho(\mathbf{r}^{(j)}; \mathbf{y}) \right) + \rho(\mathbf{r}^{(i)}; \mathbf{y}) - t_i \right]^2$), and select the $\mathbf{y}$ with defect not greater than $\inf\limits_{\mathbf{y} \in N_n} \delta(\mathbf{y}) + \frac{1}{n}$. $\rho(\mathbf{b}; \mathbf{s}) < \infty$ implies the existence of the sequence $\{\mathbf{y}'_n \in N_n\}$ such that $\mathbf{y}'_n \to \mathbf{s}$ as $n \to \infty$ (not unique, though).

We prefer more "countable", "aimed" methods; on the other hand, we "allow ourselves" the calculation of infinite sums in a finite time, as short as we want.

Well-knowns. Hereinafter, H is *the* separable infinite-dimensional Hilbert space over the field of reals $\mathbb{R}$. We denote by $<\mathbf{x}; \mathbf{y}>$ the scalar product of $\mathbf{x}, \mathbf{y} \in H$; $\|\mathbf{x}\|$ is the norm of $\mathbf{x}$. As usual, $<\mathbf{x}; \mathbf{x}> = \|\mathbf{x}\|^2$. Since the field is $\mathbb{R}$,

$<\mathbf{y};\mathbf{x}> = <\mathbf{x};\mathbf{y}>$ (the complex case reduces to the real one with "twice more dimensions", due to representability of distance between 2 points with complex coordinates through their real and imaginary parts). $\mathbf{x} \perp \mathbf{y}$ means $<\mathbf{x};\mathbf{y}> = 0$.

Some common properties of scalar product and norm (see e.g. [18]) are used without explicit reference:

• for any orthonormal basis $\{\mathbf{e_k}\}_{k\in\mathbb{N}}$ of H, if $\mathbf{x} = \sum\limits_{k=1}^{\infty} x_k \mathbf{e_k}$ and $\mathbf{y} = \sum\limits_{k=1}^{\infty} y_k \mathbf{e_k}$, then $<\mathbf{x};\mathbf{y}> = \sum\limits_{k=1}^{\infty} x_k y_k$, independent of basis $\{\mathbf{e_k}\}$.

• Cauchy-Bunyakowsky-Schwartz inequality (CBS): $|<\mathbf{x};\mathbf{y}>| \leqslant \|\mathbf{x}\| \cdot \|\mathbf{y}\|$, which becomes equality if and only if $\mathbf{x}$ and $\mathbf{y}$ are linearly dependent, that is, $\exists a, b \in \mathbb{R}$: $a^2 + b^2 \neq 0$ and $a\mathbf{x} + b\mathbf{y} = \theta$ (θ is the zero of H as linear vector space). Moreover, if $<\mathbf{x};\mathbf{y}> = \|\mathbf{x}\| \cdot \|\mathbf{y}\|$ and $\mathbf{y} \neq \theta$, then $\mathbf{x} = c\mathbf{y}$, where $c \geqslant 0$.

• $\|\mathbf{x}\pm\mathbf{y}\| = \|\mathbf{x}\|\pm\|\mathbf{y}\| \Rightarrow \mathbf{x}, \mathbf{y}$ are linearly dependent. • $\big|\|\mathbf{x}\|-\|\mathbf{y}\|\big| \leqslant \|\mathbf{x}-\mathbf{y}\|$.

Disclaimer. $X = H$ introduces nuances (mostly dealing with limits and convergency), however the basic method is that of $\mathbb{R}^n$ case. *We surmise many of subsequent results, — "auxiliary" ones especially, — to be already known, even as "folklore"*, perhaps; *if* so, *then* this is merely where they come together... once more (see also "Acknowledgements").

1 SR in Hilbert space

Hereinafter, the set $R \subset H$ of sensors is finite or countable, and the SRProblem is supposed to have at least one solution $(\mathbf{s}_0; t_{e;0})$, which may be unknown.

To simplify the notation, we move "the origin of space and time" to one of sensors at the moment when wave reaches it. So, $R = \{\mathbf{r}^{(0)}, \mathbf{r}^{(1)}, \ldots, \mathbf{r}^{(n)}, \ldots\}$ with $\mathbf{r}^{(0)} = \theta$, and the wave reaches these sensors at the moments $t_0 = 0$, t_1, t_2, $\ldots$, where $t_i = t_{e;0} + \|\mathbf{r}^{(i)} - \mathbf{s}_0\|$.

By $L(A)$ we denote the linear closure of the set $A \subseteq H$. Note that $L(R) = L(\{\mathbf{r}^{(i)}\}_{i\in\mathbb{N}})$, of all sensors but $\mathbf{r}^{(0)}$. We denote $\{\mathbf{r}^{(i)}\}_{i\in\mathbb{N}}$ by $\dot{R}$.

We begin by excluding the sets of sensors such that the solution, if it exists, is obviously not unique. If $L(\dot{R}) \neq H$ and the source $\mathbf{s}_0 \notin L(\dot{R})$, then by projection theorem $\mathbf{s}_0 = \mathbf{u}+\mathbf{h}$, where $\mathbf{u} \in L(\dot{R})$, $\mathbf{h} \perp L(\dot{R})$ and $\mathbf{h} \neq \theta$. Then for each sensor the square of "reaching time",

$$(t_i - t_{e;0})^2 = \|\mathbf{r}^{(i)} - \mathbf{s}_0\|^2 = \|\mathbf{r}^{(i)} - \mathbf{u} - \mathbf{h}\|^2 = \|\mathbf{r}^{(i)} - \mathbf{u}\|^2 + \|\mathbf{h}\|^2$$

($<\mathbf{r}^{(i)} - \mathbf{u}; \mathbf{h}> = 0$ since these elements are orthogonal) would be the same for $\mathbf{s_0'} = \mathbf{u} - \mathbf{h}$, — SRP has 2 solutions being non-distinguishable by the $\{t_i\}$.

Moreover, if $H = L(\dot{R}) \oplus K$ and $\dim K \geqslant 2$, then $\exists \mathbf{w} \neq \theta$: $\mathbf{w} \perp L(\dot{R})$ and $\mathbf{w} \perp \mathbf{h}$. Consider normalized $\widetilde{\mathbf{h}} = \frac{\mathbf{h}}{\|\mathbf{h}\|}$, $\widetilde{\mathbf{w}} = \frac{\mathbf{w}}{\|\mathbf{w}\|}$, and let

$$\mathbf{s}(\varphi) = \mathbf{u} + \|\mathbf{h}\|(\cos\varphi \cdot \widetilde{\mathbf{h}} + \sin\varphi \cdot \widetilde{\mathbf{w}})$$

— it is easy to see that $\mathbf{s}(\varphi)$ is a solution of SRP for any $\varphi \in [0; 2\pi)$: we have an infinite, non-countable set of solutions.

Let $L(\dot{R}) = H$, and let $\dot{R}$ be a linearly independent set. In other words, let $\dot{R}$ be a basis of H.

We introduce the orthonormal basis $B = \{\mathbf{e}_i\}_{i\in\mathbb{N}}$, derived from $\dot{R}$ by Gram-Schmidt orthogonalization. With B, H is l_2 and

$$\mathbf{r}^{(i)} = \sum_{j=1}^{\infty} r_j^{(i)}\mathbf{e_j} = (r_1^{(i)}; r_2^{(i)}; \ldots; r_i^{(i)}; 0; 0; \ldots)$$

where $r_i^{(i)} \neq 0$. To find $\mathbf{s}$ is to find its coordinates $(s_1; s_2; \ldots)$.

The SRP is equivalent to the following set of equations:

$$t_i = t + \|\mathbf{r}^{(i)} - \mathbf{s}\|,\ i \in \mathbb{Z}_+ \tag{1}$$

Note that 0-th equation is actually $0 = t + \|\theta - \mathbf{s}\| \Leftrightarrow t = -\|\mathbf{s}\|$.

(For instance, take $H = L_2[a; b]$, and let $f \in L_2[a; b]$ be an unknown function. Suppose that for each $i \in \mathbb{N}$ we know $t_i = t + \big(\int_a^b |f(x) - x^{i-1}|^2 dx\big)^{\frac{1}{2}}$, where $t = -\big(\int_a^b f^2(x) dx\big)^{\frac{1}{2}}$ is unknown too.)

We now proceed to the implied set of equations

$$\|\mathbf{r}^{(i)} - \mathbf{s}\|^2 = (t_i - t)^2,\ i \in \mathbb{Z}_+ \tag{2}$$

which may have additional solutions $(\mathbf{s}; t)$. To distinguish them from those of (1), we verify that $\forall i \in \mathbb{Z}_+$: $t \leqslant t_i$ (in particular, $t \leqslant t_0 = 0$) — "the wave was emitted *before* it reached sensors".

Dual problem. The additional solutions of (2) such that $t \geqslant t_i$ are the solutions of the dual, "in-mission" problem (in contrast with the original "out-mission" one), where the wave is emitted from the source and propagates *backward* in time (being observed in "usual" time, it collapses into source): $t_i = t - \rho(\mathbf{r}^{(i)}; \mathbf{s})$. In reversed time $T = -t$ these problems are swapped.

If $(\mathbf{s}'; t')$ is a solution of SRP, and $(\mathbf{s}''; t'')$ is a solution of dual problem, then for any $\mathbf{r}^{(i)}$: $t_i = t' + \rho(\mathbf{r}^{(i)}; \mathbf{s}')$ and $t_i = t'' - \rho(\mathbf{r}^{(i)}; \mathbf{s}'')$, thus

$$\rho(\mathbf{r}^{(i)}; \mathbf{s}') + \rho(\mathbf{r}^{(i)}; \mathbf{s}'') = t'' - t' = const$$

which may be interpreted as: all sensors belong to the "ellipsoid" with $\mathbf{s}'$ and $\mathbf{s}''$ being its "focuses". The following example shows that it's possible in H.

Example 1. $\lhd$ Let $E = \big\{\mathbf{x} \in l_2 \mid \frac{x_1^2}{2} + \sum_{k=2}^{\infty} x_k^2 = 1\big\}$, and $\mathbf{s}' = (-1; 0; 0; \ldots)$, $\mathbf{s}'' = (1; 0; 0; \ldots)$. We claim that $\forall \mathbf{x} \in E$: $\|\mathbf{x} - \mathbf{s}'\| + \|\mathbf{x} - \mathbf{s}''\| = 2\sqrt{2}$. Indeed[1], $x_1^2 = 2\big(1 - \sum_{k=2}^{\infty} x_k^2\big) \leqslant 2$, thus $x_1 \in [-\sqrt{2}; \sqrt{2}] \subset [-2; 2]$ and

[1] This (simpler) proof was pointed out to us by the referee.

$$\|\mathbf{x}-\mathbf{s}'\|+\|\mathbf{x}-\mathbf{s}''\|=\sqrt{(x_1+1)^2+\sum_{k=2}^{\infty}x_k^2}+\sqrt{(x_1-1)^2+\sum_{k=2}^{\infty}x_k^2}=$$
$$=\sqrt{\tfrac{x_1^2}{2}+2x_1+2}+\sqrt{\tfrac{x_1^2}{2}-2x_1+2}=\tfrac{1}{\sqrt{2}}\big(|x_1+2|+|x_1-2|\big)=2\sqrt{2}$$

We place sensors in E as follows: $\mathbf{r}^{(0)}=(-\sqrt{2};0;0;\ldots)$, $\mathbf{r}^{(1)}=(\sqrt{2};0;0;\ldots)$,

$$\mathbf{r}^{(k)}=(\underbrace{0;\ldots;0;}_{k-1}1;0;0;\ldots)\text{ for } k\geqslant 2$$

(so $\mathbf{r}^{(1)}-\mathbf{r}^{(0)}=(2\sqrt{2};0;0;\ldots)$, $\mathbf{r}^{(k)}-\mathbf{r}^{(0)}=(\sqrt{2};0;\ldots;0;1;0;0;\ldots)$ for $k\geqslant 2$; $\dot{R}=\{\hat{\mathbf{r}}^{(k)}\}_{k\in\mathbb{N}}=\{\mathbf{r}^{(k)}-\mathbf{r}^{(0)}\}_{k\in\mathbb{N}}$ is a basis of H). Since $\forall k\in\mathbb{Z}_+$: $\|\mathbf{r}^{(k)}-\mathbf{s}'\|+\|\mathbf{r}^{(k)}-\mathbf{s}''\|=2\sqrt{2}$, we have for $t'=-\sqrt{2}$, $t''=\sqrt{2}$, and $t_k=t'+\|\mathbf{r}^{(k)}-\mathbf{s}'\|$: $t_k=(t''-2\sqrt{2})+(2\sqrt{2}-\|\mathbf{r}^{(k)}-\mathbf{s}''\|)=t''-\|\mathbf{r}^{(k)}-\mathbf{s}''\|$.

In other words, for sensors $R=\{\mathbf{r}^{(k)}\}_{k\in\mathbb{Z}_+}$ and moments $\{t_k\}_{k\in\mathbb{Z}_+}$, $(\mathbf{s}';t')$ is the solution of SRP, and $(\mathbf{s}'';t'')$ is the solution of dual problem. ▷

(It was enough to show that $\forall k\in\mathbb{Z}_+$: $\|\mathbf{r}^{(k)}-\mathbf{s}'\|+\|\mathbf{r}^{(k)}-\mathbf{s}''\|=2\sqrt{2}$, without resort to E.)

Now we return to solving SRP, with the wave propagating *forward* in time. Since $\|\mathbf{r}^{(i)}-\mathbf{s}\|^2=\sum_{j=1}^{\infty}(r_j^{(i)}-s_j)^2$, $r_j^{(0)}\equiv 0$, and $t_0=0$, we arrive to

$$\sum_{j=1}^{\infty}s_j^2=t^2,\qquad \forall i\in\mathbb{N}\colon \sum_{j=1}^{\infty}\big[(r_j^{(i)})^2+s_j^2-2r_j^{(i)}s_j\big]=t_i^2+t^2-2tt_i$$

Subtract 1st equation from others, transform and recall that $r_j^{(i)}=0$, $j>i$:

$$\begin{cases}\sum_{j=1}^{\infty}s_j^2=t^2,\\ \sum_{j=1}^{i}r_j^{(i)}s_j=\tfrac12\big[\|\mathbf{r}^{(i)}\|^2-t_i^2\big]+tt_i,\quad i\in\mathbb{N}\end{cases}\tag{3}$$

Let $b_i=\frac12\big[\|\mathbf{r}^{(i)}\|^2-t_i^2\big]$, $c_i=t_i$, so $\sum_{j=1}^{i}r_j^{(i)}s_j=b_i+tc_i$ for all $i\in\mathbb{N}$.

Let the infinite matrix $A=\|a_{ij}\|_{i,j\in\mathbb{N}}=\|r_j^{(i)}\|=\begin{pmatrix}r_1^{(1)}&0&0&\ldots\\ r_1^{(2)}&r_2^{(2)}&0&\ldots\\ \ldots&\ldots&\ldots&\ddots\end{pmatrix}$,

$S=\begin{pmatrix}s_1\\ s_2\\ \ldots\end{pmatrix}$, $G(t)=B+tC$, where $B=\begin{pmatrix}b_1\\ b_2\\ \ldots\end{pmatrix}$, $C=\begin{pmatrix}c_1\\ c_2\\ \ldots\end{pmatrix}$. Then $AS=G(t)$.

The way that we've specified $\{\mathbf{r}^{(i)}\}$ allows to express s_k through t from the first k equations of this set; if we "cut off" A, S, and $G(t)$ after first k rows and columns, the resulting matrix equation $A_kS_k=G_k(t)$ is equivalent to the set of k equations with k unknowns $s_1,\ldots,s_k$. By Cramer rule,

$$s_k = \det \begin{pmatrix} r_1^{(1)} & 0 & 0 & \dots & 0 & g_1(t) \\ r_1^{(2)} & r_2^{(2)} & 0 & \dots & 0 & g_2(t) \\ \dots & \dots & \dots & \ddots & \dots & \dots \\ r_1^{(k)} & r_2^{(k)} & r_3^{(k)} & \dots & r_{k-1}^{(k)} & g_k(t) \end{pmatrix} / \det A_k =$$

$$= \left\{ \begin{vmatrix} r_1^{(1)} & 0 & \dots & b_1 \\ r_1^{(2)} & r_2^{(2)} & \dots & b_2 \\ \dots & \dots & \ddots & \dots \\ r_1^{(k)} & r_2^{(k)} & \dots & b_k \end{vmatrix} + t \begin{vmatrix} r_1^{(1)} & 0 & \dots & c_1 \\ r_1^{(2)} & r_2^{(2)} & \dots & c_2 \\ \dots & \dots & \ddots & \dots \\ r_1^{(k)} & r_2^{(k)} & \dots & c_k \end{vmatrix} \right\} / \prod_{i=1}^{k} r_i^{(i)} = \widetilde{b_k} + t\widetilde{c_k} \quad (4)$$

$S = \widetilde{B} + t\widetilde{C}$ with $\widetilde{B} = \begin{pmatrix} \widetilde{b_1} \\ \widetilde{b_2} \\ \dots \end{pmatrix}$ and $\widetilde{C} = \begin{pmatrix} \widetilde{c_1} \\ \widetilde{c_2} \\ \dots \end{pmatrix}$; $A(\widetilde{B} + t\widetilde{C}) = B + tC \Rightarrow$ $A\widetilde{B} = B$, $A\widetilde{C} = C$.

Substituting (4) into $\sum_{j=1}^{\infty} s_j^2 = t^2$ gives $\sum_{j=1}^{\infty} (\widetilde{b_j} + t\widetilde{c_j})^2 = t^2$ (5)

Case 0: $t = 0$ is a root of (5). We claim that $\mathbf{s} = \theta$ is the unique solution of SRP then.

Proof. $t = 0$ turns (5) into equality: $\sum_{j=1}^{\infty} \widetilde{b_j}^2 = 0 \Leftrightarrow \widetilde{b_j} = 0$ for all $j \in \mathbb{N}$. Since $A\widetilde{B} = B$, it follows that $b_i = 0$ for any $i \in \mathbb{N}$: $\|\mathbf{r}^{(i)}\|^2 = t_i^2 \Leftrightarrow \|\mathbf{r}^{(i)}\| = |t_i|$. On the other hand, for any solution $(\mathbf{s}; t)$ of SRP $t_i = t + \|\mathbf{r}^{(i)} - \mathbf{s}\| = -\|\mathbf{s}\| + \|\mathbf{r}^{(i)} - \mathbf{s}\|$. Therefore $\|\mathbf{r}^{(i)}\| = \big|\|\mathbf{r}^{(i)} - \mathbf{s}\| - \|\mathbf{s}\|\big|$.

a) $\|\mathbf{r}^{(i)}\| = \|\mathbf{r}^{(i)} - \mathbf{s}\| - \|\mathbf{s}\| \Leftrightarrow \|\mathbf{r}^{(i)}\| + \| - \mathbf{s}\| = \|\mathbf{r}^{(i)} + (-\mathbf{s})\|$.

b) $\|\mathbf{r}^{(i)}\| = -\|\mathbf{r}^{(i)} - \mathbf{s}\| + \|\mathbf{s}\| \Leftrightarrow \|\mathbf{s}\| - \|\mathbf{r}^{(i)}\| = \|\mathbf{s} - \mathbf{r}^{(i)}\|$.

In any case, $\mathbf{r}^{(i)}$ and $\mathbf{s}$ are linearly dependent for any $i \in \mathbb{N}$. Since $\dot{R} = \{\mathbf{r}^{(i)}\}_{i\in\mathbb{N}}$ is linearly independent, it is only possible when $\mathbf{s} = \theta$. QED

Until now, the method had little relation with infinite dimensionality of H.

Case 1: $t = 0$ isn't a root of (5) (thus $\sum_{j=1}^{\infty} \widetilde{b_j}^2 \neq 0$, and $\mathbf{s} \neq \theta$).

We divide it by t: $\sum_{j=1}^{\infty} (\widetilde{c_j} + z\widetilde{b_j})^2 = 1$ (6)

where $z = 1/t < 0$. By assumption, SRP has at least 1 solution, so for some $z_{e;0} = 1/t_{e;0}$ (6) holds true, implying $\{\widetilde{c_j} + z_{e;0}\widetilde{b_j}\}_{j\in\mathbb{N}} = \mathbf{v} \in H$.

The relations $\widetilde{C} = \mathbf{v} - z_{e;0}\widetilde{B}$ and $\widetilde{B} = \frac{1}{z_{e;0}}(\mathbf{v} - \widetilde{C})$ show that $\widetilde{B}$ and $\widetilde{C}$ belong or don't belong to H simultaneously; the series $\sum_{j=1}^{\infty} \widetilde{b_j}^2$ and $\sum_{j=1}^{\infty} \widetilde{c_j}^2$ both converge or both diverge.

Subcase 1a (ruled out in $\mathbb{R}^n$)**:** $\sum_{j=1}^{\infty} \widetilde{b_j}^2$ and $\sum_{j=1}^{\infty} \widetilde{c_j}^2$ diverge. Yet for some $z_{e;0}$: $\sum_{j=1}^{\infty} (\widetilde{c_j} + z_{e;0}\widetilde{b_j})^2$ converges to 1. Assuming $\exists z' \neq z_{e;0}$ such that $\sum_{j=1}^{\infty} (\widetilde{c_j} + z'\widetilde{b_j})^2$ converges, we obtain from equality $\widetilde{b_j} = \frac{1}{z_{e;0}-z'}\big((\widetilde{c_j} + z_{e;0}\widetilde{b_j}) - (\widetilde{c_j} + z'\widetilde{b_j})\big)$ the convergence of $\sum_{j=1}^{\infty} \widetilde{b_j}^2$, which contradicts the assumption of the subcase.

Hence $z_{e;0}$ is the unique value not just satisfying (6), but providing convergence of the series in the left side of (6). How to obtain it? (Recall that we don't allow ourselves to "go over all $z < 0$ and select the one satisfying (6)").

$\exists n_0$: $\sum_{j=1}^{n_0} \widetilde{b_j}^2 > 0$, therefore for any $n \geqslant n_0$: $f_n(z) = \sum_{j=1}^{n} (\widetilde{c_j} + z\widetilde{b_j})^2 - 1 =$

$$= \big[\sum_{j=1}^{n} \widetilde{b_j}^2\big] z^2 + \big[2 \sum_{j=1}^{n} \widetilde{b_j}\widetilde{c_j}\big] z + \big[\sum_{j=1}^{n} \widetilde{c_j}^2 - 1\big] = \alpha_n z^2 + \beta_n z + \gamma_n$$

is a quadratic trinomial with $\alpha_n > 0$. $f_n(z) \leqslant f_{n+1}(z) \leqslant f_\infty(z)$, so $f_n(z_{e;0}) \leqslant 0$: the equation $f_n(z) = 0$ has at least one root. We know that $\{z \in \mathbb{R}\colon f_n(z) \leqslant 0\}$ is the segment $[z_-^{(n)}; z_+^{(n)}]$, whose center is $z_n = -\frac{\beta_n}{2\alpha_n}$.

For any $\varepsilon > 0$ the series diverges at $z_{e;0} - \varepsilon$ and $z_{e;0} + \varepsilon$, therefore $\exists n = n(\varepsilon)$: $f_n(z_{e;0}-\varepsilon) > 0$ and $f_n(z_{e;0}+\varepsilon) > 0$. Consequently, $[z_-^{(n)}; z_+^{(n)}] \subset (z_{e;0}-\varepsilon; z_{e;0}+\varepsilon)$; in particular, $z_n \in (z_{e;0} - \varepsilon; z_{e;0} + \varepsilon)$.

That is, $z_n = -\big[\sum_{j=1}^{n} \widetilde{b_j}\widetilde{c_j}\big] / \big[\sum_{j=1}^{n} \widetilde{b_j}^2\big] \xrightarrow[n\to\infty]{} z_{e;0}$.

Subcase 1b: $\sum_{j=1}^{\infty} \widetilde{b_j}^2$ and $\sum_{j=1}^{\infty} \widetilde{c_j}^2$ converge. From the common properties of series it follows that we can rewrite (6) as

$$\big[\sum_{j=1}^{\infty} \widetilde{b_j}^2\big] z^2 + \big[2 \sum_{j=1}^{\infty} \widetilde{b_j}\widetilde{c_j}\big] z + \big[\sum_{j=1}^{\infty} \widetilde{c_j}^2 - 1\big] = 0 \tag{7}$$

or $\alpha z^2 + \beta z + \gamma = 0$, where $\alpha > 0$; what's left to do is to solve it, $z_\pm = \frac{-\beta + \sqrt{D}}{2\alpha}$ ($D = \beta^2 - 4\alpha\gamma \geqslant 0$ since $z_{e;0}$ is a root), and select the root(s) z such that $z < 0$ and $t = 1/z \leqslant t_i$ for any $i \in \mathbb{N}$.

This concludes the description of the solving method for SRP in H.

(7) can have 2 distinct roots satisfying $t \leqslant t_i\big|_{i\in\mathbb{Z}_+}$; even when sensors make a basis, SRP in H can have 2 distinct solutions, as the following example indicates.

Example 2. $\lhd$ Let non-θ sensors, $\dot{R} = \{\mathbf{r}^{(k)}\}_{k\in\mathbb{N}}$, be $\mathbf{r}^{(k)} = \frac{1}{k}\mathbf{e_k}$, where $\{\mathbf{e_k}\}$ is an orthonormal basis of H. For the source $\mathbf{s}' = -\sum_{k=1}^{\infty} \frac{1}{k}\mathbf{e_k} = (-1; -\frac{1}{2}; -\frac{1}{3}; \ldots)$, which emits the wave at the moment $t' = -\|\mathbf{s}'\| = -\sqrt{\sum_{k=1}^{\infty} \frac{1}{k^2}} = -\frac{\pi}{\sqrt{6}}$, the

moments $\{t_k\}$ when k-th sensor is reached by this wave are such that
$(t_k - t')^2 = \|\mathbf{r}^{(k)} - \mathbf{s}'\|^2 = \sum\limits_{i\in\mathbb{N}, i\neq k} (0 - s'_i)^2 + (\frac{1}{k} - s'_k)^2 = \sum\limits_{i\in\mathbb{N}} \frac{1}{i^2} + \frac{3}{k^2} = \frac{\pi^2}{6} + \frac{3}{k^2}$
implying $t_k = -\frac{\pi}{\sqrt{6}} + \sqrt{\frac{\pi^2}{6} + \frac{3}{k^2}} = \frac{3}{k^2\left(\frac{\pi}{\sqrt{6}} + \sqrt{\frac{\pi^2}{6}+\frac{3}{k^2}}\right)} > 0$ (by construction, for $\mathbf{r}^{(0)} = \theta$, $t_0 = 0$).

Now we solve the corresponding SRP in accordance with the procedure described above, knowing that $(\mathbf{s}'; t')$ is a solution. The basis is $\{\mathbf{e_i}\}$; the equations from (3) take the form of

$$\frac{1}{i} s_i = \frac{1}{2}\left[\frac{1}{i^2} - \left\{\frac{3}{i^2\left(\frac{\pi}{\sqrt{6}} + \sqrt{\frac{\pi^2}{6}+\frac{3}{i^2}}\right)}\right\}^2\right] + t\frac{3}{i^2\left(\frac{\pi}{\sqrt{6}} + \sqrt{\frac{\pi^2}{6}+\frac{3}{i^2}}\right)} \Leftrightarrow s_k = \widetilde{b_k} + t\widetilde{c_k}$$

where $\widetilde{b_k} = \frac{1}{k}\cdot\frac{1}{2}\left[1 - \frac{9}{k^2\left(\frac{\pi}{\sqrt{6}} + \sqrt{\frac{\pi^2}{6}+\frac{3}{k^2}}\right)^2}\right]$, $\widetilde{c_k} = \frac{1}{k}\cdot\frac{3}{\frac{\pi}{\sqrt{6}} + \sqrt{\frac{\pi^2}{6}+\frac{3}{k^2}}}$.

It is clear that $t = 0$ isn't a root of $\sum\limits_{k=1}^{\infty} (\widetilde{b_k} + t\widetilde{c_k})^2 = t^2$, $\sum\limits_{k=1}^{\infty} \widetilde{b_k}^2$ converges and $\sum\limits_{k=1}^{\infty} \widetilde{c_k}^2$ converges. So we can switch to $z = 1/t$ and the equation $\alpha z^2 + \beta z + \gamma = 0$ from Subcase 1b. $D \geqslant 0$ because $z' = 1/t' = -\frac{\sqrt{6}}{\pi}$ is a root.

Let z'' be a second root; we claim that $z'' < 0$ and $z'' \neq z'$.

Proof. $z'z'' = \frac{\gamma}{\alpha}$. Since $\alpha > 0$, $\operatorname{sign} z'z'' = \operatorname{sign}\gamma$. From

$$\sum_{k=1}^{\infty} \widetilde{c_k}^2 = \sum_{k=1}^{\infty} \frac{9}{k^2\left(\frac{\pi}{\sqrt{6}} + \sqrt{\frac{\pi^2}{6}+\frac{3}{k^2}}\right)^2} > \frac{9}{\left(\frac{\pi}{\sqrt{6}} + \sqrt{\frac{\pi^2}{6}+3}\right)^2}\sum_{k=1}^{\infty}\frac{1}{k^2} =$$
$$= \frac{3\pi^2}{2\left(\frac{\pi}{\sqrt{6}} + \sqrt{\frac{\pi^2}{6}+3}\right)^2} = \frac{3}{2\left(\frac{1}{\sqrt{6}} + \sqrt{\frac{1}{6}+\frac{3}{\pi^2}}\right)^2} > \frac{3}{2\left(\frac{1}{\sqrt{6}}+\frac{1}{\sqrt{2}}\right)^2} = \frac{9}{(1+\sqrt{3})^2} > 1$$

it follows that $\gamma > 0$, hence $z'' < 0$.

Assume that $z' = z''$, then $\frac{\gamma}{\alpha} = (z')^2 = \frac{6}{\pi^2} \Leftrightarrow \sum\limits_{k=1}^{\infty} \widetilde{c_k}^2 = 1 + \frac{6}{\pi^2}\sum\limits_{k=1}^{\infty} \widetilde{b_k}^2 \Leftrightarrow \sum\limits_{k=1}^{\infty}\left(\widetilde{c_k}^2 - \frac{6}{\pi^2}\widetilde{b_k}^2\right) = 1$.

However, when $k \geqslant 3$, $0 < k\widetilde{b_k} \leqslant \frac{1}{2}\left[1 - \frac{9}{k^2\left(\frac{\pi}{\sqrt{6}} + \sqrt{\frac{\pi^2}{6}+\frac{1}{3}}\right)^2}\right] \leqslant \frac{1}{2} < 1 < \frac{3}{\frac{\pi}{\sqrt{6}} + \sqrt{\frac{\pi^2}{6}+\frac{1}{3}}} < k\widetilde{c_k} \Rightarrow 0 < \widetilde{b_k} < \widetilde{c_k} \Rightarrow 0 < \frac{\sqrt{6}}{\pi}\widetilde{b_k} < \widetilde{c_k} \Rightarrow \widetilde{c_k}^2 > \frac{6}{\pi^2}\widetilde{b_k}^2$. A little more numerical computation, and we get $\sum\limits_{k=1}^{3}\left(\widetilde{c_k}^2 - \frac{6}{\pi^2}\widetilde{b_k}^2\right) \approx 1.139918 > 1$, so $\sum\limits_{k=1}^{\infty}\left(\widetilde{c_k}^2 - \frac{6}{\pi^2}\widetilde{b_k}^2\right) > 1$; a contradiction. Therefore $z'' \neq z'$. QED

Then $t'' = 1/z'' < 0 < t_k$, and $\mathbf{s}'' = \{\widetilde{b_j} + t''\widetilde{c_j}\}$, is another solution, different from $(\mathbf{s}'; t')$. ▷

Remark. Non-uniqueness of SRP solution is a well known occasion in $\mathbb{R}^n$: in $\mathbb{R}^2$ we can place 3 sensors on half-hyperbola, and emit the wave at the moment t' from the focus $\mathbf{s}'$. Then another focus $\mathbf{s}''$, emitting at the moment

$$t'' = t' + \|\mathbf{r}^{(i)} - \mathbf{s}'\| - \|\mathbf{r}^{(i)} - \mathbf{s}''\| = t' + const$$

is a different solution of the SRP defined by $\{t_i\}_{i=1}^3$.

Some sufficient conditions for uniqueness of SRP solution follow (Prop. 1–4).

Proposition 1. *If the dual "in-mission" problem, "$t_i = t - \|\mathbf{r}^{(i)} - \mathbf{s}\|$ for any $i \in \mathbb{Z}_+$", also has a solution, then the solution of the original SRP is unique.*

Proof. The implied set of equations (2), when solved in t, has no more than 2 roots, and includes the solution $t' \leqslant 0$ of SRP, along with the solution $t'' \geqslant 0$ of dual problem. Note that $t'' \neq t'$, otherwise $t' = t'' = 0$, $\mathbf{s}' = \mathbf{s}'' = \theta$, and the wave reaches $\mathbf{r}^{(1)}$ at the moment $t_1 \neq 0$ when propagating both forward and backward in time, — $t_1 = \|\mathbf{r}^{(1)}\| > 0$ and $t_1 = -\|\mathbf{r}^{(1)}\| < 0$; a contradiction. In other words, $t'' > 0$ cannot be a solution of SRP, and t' is the unique solution. *QED*

Proposition 2. *If the solution* $\mathbf{s}'$ *of the SRP is identical to one of sensors, then this solution is unique.* (This conforms with Case 0 above.)

Proof. Without loss of generality we suppose $\mathbf{s}' = \mathbf{r}^{(0)} = \theta$, and $t' = 0$. Now, assume that $(\mathbf{s}''; t'')$ is another solution. Then for each $\mathbf{r}^{(i)}$, $i \in \mathbb{N}$, we have $\begin{cases} t_i = 0 + \|\mathbf{r}^{(i)} - \theta\|, \\ t_i = t'' + \|\mathbf{r}^{(i)} - \mathbf{s}''\|, \end{cases}$ and for $i = 0$: $0 = t'' + \|\theta - \mathbf{s}''\| \Leftrightarrow t'' = -\|\mathbf{s}''\|$. Thus $\|\mathbf{r}^{(i)}\| = -\|\mathbf{s}''\| + \|\mathbf{r}^{(i)} - \mathbf{s}''\| \Leftrightarrow \|\mathbf{r}^{(i)}\| + \|-\mathbf{s}''\| = \|\mathbf{r}^{(i)} + (-\mathbf{s}'')\|$; linear dependency of $\mathbf{r}^{(i)}$ and $\mathbf{s}''$ follows. $\mathbf{s}'' \neq \theta$ leads to $\mathbf{r}^{(i)} \in L(\{\mathbf{s}''\})$, for any i. This contradicts the linear independency of $\dot{R}$, so the assumption is wrong. *QED*

Of course, we prefer the conditions relating only to the set of sensors, so that for any position of the source the solution of SRP is that position and unique, — this is important when sensors must be placed *before* the source appears anywhere in space and emits the wave.

Proposition 3. *If SRP has a solution, and $\exists\{n_k\}_{k=1}^{\infty}$, $n_k < n_{k+1}$: $\mathbf{r}^{(n_k)} \perp \mathbf{r}^{(i)}$ for $1 \leqslant i < n_k$, and $\|\mathbf{r}^{(n_k)}\| \in [\lambda; \mu]$ with $\lambda > 0$, then this solution is unique.*

Proof. We denote the SRP solution by $(\mathbf{s}'; t')$. If $\mathbf{s}' = \theta = \mathbf{r}^{(0)}$, it is unique by Prop. 2. Consider $\mathbf{s}' \neq \theta$. $B = \{\mathbf{e}_k\}$ is made from $\dot{R} = \{\mathbf{r}^{(k)}\}$ by Gram-Schmidt orthogonalization: $\mathbf{e}_k = \mathbf{d}_k / \|\mathbf{d}_k\|$, where $\mathbf{d}_k = \mathbf{r}^{(k)} - \sum_{j=1}^{k-1} <\mathbf{r}^{(k)}; \mathbf{e}_j> \mathbf{e}_j$, so in B $\mathbf{r}_j^{(n_k)} = <\mathbf{r}^{(n_k)}; \mathbf{e}_j> = 0$ for $j < n_k$, $r_{n_k}^{(n_k)} = \|\mathbf{r}^{(n_k)}\|$. Let $n = n_k$, then by (4):

$$\widetilde{c_n} = \begin{vmatrix} r_1^{(1)} & 0 & 0 & \dots & 0 & t_1 \\ r_1^{(2)} & r_1^{(2)} & 0 & \dots & 0 & t_2 \\ \dots & \dots & \dots & \ddots & \dots & \dots \\ r_1^{(n-1)} & r_2^{(n-1)} & r_3^{(n-1)} & \dots & r_{n-1}^{(n-1)} & t_{n-1} \\ 0 & 0 & 0 & \dots & 0 & t_n \end{vmatrix} / \prod_{i=1}^{n} r_i^{(i)} =$$

$= t_n(-1)^{n+n} \det A_{n-1} / \prod_{i=1}^{n} r_i^{(i)} = t_n / r_n^{(n)} = t_n / \|r^{(n)}\|$, so $|\widetilde{c_n}| \geqslant |t_n|/\mu$.

In turn, $t_n = t' + \|\mathbf{r}^{(n)} - \mathbf{s}'\| = -\|\mathbf{s}'\| + \sqrt{<\mathbf{r}^{(n)} - \mathbf{s}'; \mathbf{r}^{(n)} - \mathbf{s}'>} =$
$= \sqrt{\|\mathbf{s}'\|^2 + \|\mathbf{r}^{(n)}\|^2 - 2<\mathbf{r}^{(n)}; \mathbf{s}'>} - \|\mathbf{s}'\| = \sqrt{\|\mathbf{s}'\|^2 + \|\mathbf{r}^{(n)}\|^2 - 2\|\mathbf{r}^{(n)}\| s'_n} - \|\mathbf{s}'\|$

$\|\mathbf{r}^{(n)}\| \leqslant \mu$, $\mathbf{s}' \in H \Rightarrow s'_n \xrightarrow[k\to\infty]{} 0$, implying $\|\mathbf{r}^{(n)}\| s'_n \xrightarrow[k\to\infty]{} 0$ ($n = n_k \to \infty$ as $k \to \infty$). Therefore $\varliminf_k t_n = \sup_{m\in\mathbb{N}} \inf_{k\geqslant m} t_n \geqslant \left[\sqrt{\|\mathbf{s}'\|^2 + \lambda^2} - \|\mathbf{s}'\|\right] > 0$, and $\varliminf_k |\widetilde{c_n}| > 0$. Consequently, $\lim_{k\to\infty} \widetilde{c_{n_k}} \neq 0$, so $\sum_{j=1}^{\infty} \widetilde{c_j}^2 = \infty$.

Thus we are in Subcase 1a, where the solution of SRP is unique. *QED*

The trivial example of such $\dot{R}$ is *orthonormal* basis of H ($n_k = k$, $\lambda = \mu = 1$).

Now, for *arbitrary* basis $\dot{R}$ of H, let $\dot{R}'$ be the following "extension" of $\dot{R}$: $\dot{R}' = \dot{R} \cup \{\mathbf{r}^{(\omega+1)}\}$, where $\mathbf{r}^{(\omega+1)} = -\mathbf{r}^{(1)}$. Respectively, $R' = R \cup \{\mathbf{r}^{(\omega+1)}\}$. The wave reaches this additional sensor, opposite to $\mathbf{r}^{(1)}$, at the moment $t_{\omega+1}$.

Proposition 4. *If the SRP defined by R' and $\{t_i\}_{i\in\mathbb{Z}_+\cup\{\omega+1\}}$ has a solution, then it is unique.*

Proof. Assuming the contrary, let $(\mathbf{s}'; t')$ and $(\mathbf{s}''; t'')$ be the distinct solutions of such SRP. The reasonings above show that $\mathbf{s}$ is determined uniquely by t ($s_j = \widetilde{b_j} + t\widetilde{c_j}$), therefore $t' \neq t''$. From (3) for $i = 1$ and $i = \omega + 1$ (it is clear that $\mathbf{r}^{(\omega+1)} = (-r_1^{(1)}; 0; 0; \dots)$ and $\|\mathbf{r}^{(\omega+1)}\| = \|\mathbf{r}^{(1)}\|$):

$$\begin{cases} r_1^{(1)} s'_1 = \frac{1}{2}\left[\|\mathbf{r}^{(1)}\|^2 - t_1^2\right] + t' t_1, & (\mathbf{1}') \\ -r_1^{(1)} s'_1 = \frac{1}{2}\left[\|\mathbf{r}^{(1)}\|^2 - t_{\omega+1}^2\right] + t' t_{\omega+1} & (\mathbf{2}') \end{cases}$$
$$\begin{cases} r_1^{(1)} s''_1 = \frac{1}{2}\left[\|\mathbf{r}^{(1)}\|^2 - t_1^2\right] + t'' t_1, & (\mathbf{1}'') \\ -r_1^{(1)} s''_1 = \frac{1}{2}\left[\|\mathbf{r}^{(1)}\|^2 - t_{\omega+1}^2\right] + t'' t_{\omega+1} & (\mathbf{2}'') \end{cases}$$

Subtract $(1'')$ from $(1')$, and $(2'')$ from $(2')$:

$$\begin{cases} r_1^{(1)}(s'_1 - s''_1) = (t' - t'')t_1, \\ -r_1^{(1)}(s'_1 - s''_1) = (t' - t'')t_{\omega+1} \end{cases} \Rightarrow (t' - t'')(t_1 + t_{\omega+1}) = 0 \Rightarrow t_{\omega+1} = -t_1$$

Then add $(1')$ and $(2')$: $0 = \|\mathbf{r}^{(1)}\|^2 - t_1^2 \Leftrightarrow |t_1| = \|\mathbf{r}^{(1)}\|$. To be definite, suppose $t_1 \geqslant 0$. For any solution $(\mathbf{s}; t)$ of the SRP under study (that is, for $(\mathbf{s}'; t')$ and $(\mathbf{s}''; t'')$), we have $t = -\|\mathbf{s}\|$ and $\begin{cases} t_1 = t + \|\mathbf{r}^{(1)} - \mathbf{s}\|, \\ t_{\omega+1} = t + \|\mathbf{r}^{(\omega+1)} - \mathbf{s}\|, \end{cases}$ hence

$$\|\mathbf{r}^{(1)} - (-\mathbf{r}^{(1)})\| = 2t_1 = \|\mathbf{r}^{(1)} - \mathbf{s}\| - \| - \mathbf{r}^{(1)} - \mathbf{s}\| \Leftrightarrow$$
$$\Leftrightarrow \big\|(\mathbf{r}^{(1)} - \mathbf{s}) - (-\mathbf{r}^{(1)} - \mathbf{s})\big\| = \|\mathbf{r}^{(1)} - \mathbf{s}\| - \| - \mathbf{r}^{(1)} - \mathbf{s}\|$$

so $(\mathbf{r}^{(1)} - \mathbf{s})$ and $(-\mathbf{r}^{(1)} - \mathbf{s})$ are linearly dependent. $a(\mathbf{r}^{(1)} - \mathbf{s}) + b(-\mathbf{r}^{(1)} - \mathbf{s}) = \theta$ $\Leftrightarrow (a+b)\mathbf{s} = (a-b)\mathbf{r}^{(1)}$. $a + b \neq 0$, otherwise we divide the 1st equation by a and come to $\mathbf{r}^{(1)} - \mathbf{s} + \mathbf{r}^{(1)} + \mathbf{s} = \theta$, or $\mathbf{r}^{(1)} = \theta$, — a contradiction. Thus we can divide the 2nd equation by $(a+b)$: $\mathbf{s} = \frac{a-b}{a+b}\mathbf{r}^{(1)} \in L(\{\mathbf{r}^{(1)}\})$, therefore $\mathbf{s} = (s_1; 0; 0; \ldots)$, and $\|\mathbf{r}^{(1)} - \mathbf{s}\| = |r_1^{(1)} - s_1|$, $\|\mathbf{r}^{(\omega+1)} - \mathbf{s}\| = |r_1^{(1)} + s_1|$.

$t_1 \geqslant 0 \Rightarrow |r_1^{(1)} + s_1| \leqslant |r_1^{(1)} - s_1|$. $B = \{\mathbf{e}_i\}$ was made from $\dot{R}$ by Gram-Schmidt orthogonalization, so $r_1^{(1)} > 0$, $r_1^{(\omega+1)} = -r_1^{(1)} < 0$. Hence $s_1 \leqslant 0 \Rightarrow t = -\|\mathbf{s}\| = s_1$.

Now take into account other sensors, e.g. 2nd one: $\mathbf{r}^{(2)} = (r_1^{(2)}; r_2^{(2)}; 0; \ldots) = (p; h; 0; \ldots)$, where $h \neq 0$. $t_2 = t + \|\mathbf{r}^{(2)} - \mathbf{s}\| = t + \sqrt{(p - s_1)^2 + h^2}$ turns into equality for $(\mathbf{s}'; t')$ and $(\mathbf{s}''; t'')$: $t' + \sqrt{(p - t')^2 + h^2} = t'' + \sqrt{(p - t'')^2 + h^2}$.

But for $f(t) = t + \sqrt{(t-p)^2 + h^2}$: $f'(t) = 1 + \frac{t-p}{\sqrt{(t-p)^2+h^2}} > 0$, because $|t - p| < \sqrt{(t-p)^2 + h^2}$; $f(t)$ is strictly increasing, so $t' = t''$, — a contradiction.

Therefore the initial assumption is wrong; the solution is unique. *QED*

Proposition 5. *If SRP (1) has a solution, and* $(\mathbf{s}''; t'')$ *is a different solution of implied (2), then* $(\mathbf{s}''; t'')$ *is the solution of either SRP (1), or the dual problem.*

Proof. Assume the contrary, then $\exists m, k \in \mathbb{Z}_+$: $\begin{cases} t_m = t'' - \|\mathbf{r}^{(m)} - \mathbf{s}''\|, \\ t_k = t'' + \|\mathbf{r}^{(k)} - \mathbf{s}''\|. \end{cases}$ Let $(\mathbf{s}'; t')$ be the solution of SRP (1), so $\forall i \in \mathbb{Z}_+$: $t_i = t' + \|\mathbf{r}^{(i)} - \mathbf{s}'\|$. In particular, $\begin{cases} t_m = t' + \|\mathbf{r}^{(m)} - \mathbf{s}'\|, \\ t_k = t' + \|\mathbf{r}^{(k)} - \mathbf{s}'\|. \end{cases}$ Therefore

$$\|\mathbf{r}^{(m)} - \mathbf{s}'\| + \|\mathbf{r}^{(m)} - \mathbf{s}''\| = t'' - t' = \|\mathbf{r}^{(k)} - \mathbf{s}'\| - \|\mathbf{r}^{(k)} - \mathbf{s}''\|$$

By triangle inequality, $\|\mathbf{r}^{(m)} - \mathbf{s}'\| + \|\mathbf{r}^{(m)} - \mathbf{s}''\| \geqslant \|\mathbf{s}' - \mathbf{s}''\|$; contrariwise,

$$\big|\|\mathbf{r}^{(k)} - \mathbf{s}'\| - \|\mathbf{r}^{(k)} - \mathbf{s}''\|\big| \leqslant \|\mathbf{s}' - \mathbf{s}''\|$$

Hence $\begin{cases} \|\mathbf{s}' - \mathbf{r}^{(m)}\| + \|\mathbf{r}^{(m)} - \mathbf{s}''\| = \|\mathbf{s}' - \mathbf{s}''\|, \\ \|\mathbf{r}^{(k)} - \mathbf{s}'\| - \|\mathbf{r}^{(k)} - \mathbf{s}''\| = \|\mathbf{s}'' - \mathbf{s}'\|. \end{cases}$ From the 1st equality we obtain linear dependency of $\mathbf{s}' - \mathbf{r}^{(m)}$ and $\mathbf{r}^{(m)} - \mathbf{s}''$: $\exists a, b$: $a(\mathbf{s}' - \mathbf{r}^{(m)}) + b(\mathbf{r}^{(m)} - \mathbf{s}'') = \theta \Leftrightarrow (b - a)\mathbf{r}^{(m)} = b\mathbf{s}'' - a\mathbf{s}'$. $a \neq b$, otherwise we could divide by a and get $\mathbf{s}' - \mathbf{s}'' = \theta$; so $\mathbf{r}^{(m)} = \frac{1}{b-a}(b\mathbf{s}'' - a\mathbf{s}') \in L(\{\mathbf{s}'; \mathbf{s}''\})$.

From 2nd equality: $\mathbf{r}^{(k)} - \mathbf{s}'$ and $\mathbf{r}^{(k)} - \mathbf{s}''$ are linearly dependent, $a(\mathbf{r}^{(k)} - \mathbf{s}') + b(\mathbf{r}^{(k)} - \mathbf{s}'') = \theta \Leftrightarrow (a+b)\mathbf{r}^{(k)} = a\mathbf{s}' + b\mathbf{s}''$, $a + b \neq 0$ or it would be $\mathbf{s}' - \mathbf{s}'' = \theta$, thus $\mathbf{r}^{(k)} \in L(\{\mathbf{s}'; \mathbf{s}''\})$.

Moreover, for any $j \in \mathbb{Z}_+$ such that $t_j = t'' + \|\mathbf{r}^{(j)} - \mathbf{s}''\|$ we can repeat these reasonings for the same m, but taking j instead of k. Consequently,

$$R_+ = \{\mathbf{r}^{(j)} \mid t_j = t'' + \|\mathbf{r}^{(j)} - \mathbf{s}''\|\} \subseteq L(\{\mathbf{s}'; \mathbf{s}''\})$$

Similarly, keeping k and going over suitable m,

$$R_- = \{\mathbf{r}^{(j)} \mid t_j = t'' - \|\mathbf{r}^{(j)} - \mathbf{s}''\|\} \subseteq L(\{\mathbf{s}'; \mathbf{s}''\})$$

Since $R = R_+ \cup R_-$, we have $R \subseteq L(\{\mathbf{s}'; \mathbf{s}''\})$, which is impossible, because $\dot{R} = R \backslash \{\theta\}$ is a basis of H, while $\dim L(\{\mathbf{s}'; \mathbf{s}''\}) \leqslant 2$. This contradiction proves that the assumption is wrong. QED

In other words, when a solution of SRP exists, the transition from (1) to (2) may add only the solution of dual problem, not some "mixed" one.

When we have the countable set R of sensors and corresponding moments $\{t_i\}_{i \in \mathbb{Z}}$, we may "downdimension" the original SRP by taking into account only the sensors from 0-th to n-th, $R_n = \{\mathbf{r}^{(i)}\}_{i=0}^n$. Since $\mathbf{r}^{(0)} = \theta$, we have $L_n := L(R_n) = L(\dot{R}_n)$ and is isomorphic to $\mathbb{R}^n$.

Further, we seek the solution $(\mathbf{s}; t)$ of the problem "$t_i = t + \|\mathbf{r}^{(i)} - \mathbf{s}\|$ for any $i = \overline{0, n}$" inside L_n. We denote this "downdimensioned" problem by SRP_n.

Proposition 6. *If SRP has a solution, then $\forall n \in \mathbb{N}$: SRP_n has a solution.*

Proof. Denote the solution of original SRP by $(\mathbf{s}^{(\infty)}; t^{(\infty)})$. By projection theorem, $\mathbf{s}^{(\infty)} = \mathbf{u} + \mathbf{h}$, where $\mathbf{u} \in L_n$ and $\mathbf{h} \perp L_n$. If $\mathbf{h} = \theta$, then $\mathbf{s}^{(\infty)}$ is the solution of SRP_n. We consider another case, $\mathbf{h} \neq \theta$. Let $h = \|\mathbf{h}\|$.

Let $L'_n = L_n \oplus L(\{\mathbf{h}\}) = L(R_n \cup \{\mathbf{h}\})$. It is isomorphic to $\mathbb{R}^{n+1}$; $\mathbf{x} \in L'_n$ is $(x_1; \ldots; x_n; x_{n+1})$ in the basis made, using Gram-Schmidt orthogonalization, from $\dot{R}_n \cup \{\mathbf{h}\}$. In particular, for $i = \overline{1, n}$ the sensor $\mathbf{r}^{(i)}$ has the coordinates $\{r_j^{(i)}\}_j$, $r_{n+1}^{(i)} = 0$. Also, $\mathbf{s}^{(\infty)} \in L'_n$ and $\mathbf{s}^{(\infty)} = (s_1^{(\infty)}; \ldots; s_n^{(\infty)}; s_{n+1}^{(\infty)})$, $s_{n+1}^{(\infty)} = h$.

We now consider the SRP defined by $(R_n; \{t_i\}_{i=0}^n)$ in L'_n; it has (at least one) solution $(\mathbf{s}^{(\infty)}; t^{(\infty)})$. Following the way of (1)-(2)-(3)-(4)-(5) (now there's a finite sum instead of series),

$$s_j = \widetilde{b_j} + t\widetilde{c_j} \text{ for } j = \overline{1, n}; \ \sum_{j=1}^{n} (\widetilde{b_j} + t\widetilde{c_j})^2 + s_{n+1}^2 = t^2$$

We rewrite the latter equation, in t, as $\alpha t^2 + \beta t + \gamma + s_{n+1}^2 = 0$. Note that $\gamma = \sum_{j=1}^{n} \widetilde{b_j}^2 \geqslant 0$. There's a solution $t = t^{(\infty)} \leqslant t_i$, $i = \overline{0, n}$, when $s_{n+1} = \pm h$ (hence this SRP has at least 2 solutions in L'_n, symmetrical with respect to L_n).

We claim that it has a solution $t^{(n)} \leqslant t_i$ when $s_{n+1} = 0$. Consider the cases:

Case $\alpha > 0$: $f_{h^2}(t) = \alpha t^2 + \beta t + \gamma + h^2 = 0$ has a root $t^{(\infty)} \leqslant t_i$. If t' is its lesser root, then all the more $t' \leqslant t_i$. Since $f_{h^2}(t)$ is a quadratic trinomial, for h^2 replaced by 0 it has 2 roots, with the lesser one $t'' < t'$. Let $t^{(n)} = t''$.

Case $\alpha < 0$: $f_0(t)$ has a root(s) because $D = \beta^2 - 4\alpha(\gamma + 0) \geqslant \beta^2 \geqslant 0$ (perhaps the root is multiple). Its roots are $t_\pm = \frac{-\beta \pm \sqrt{D}}{2\alpha}$, and $t_+ t_- = \frac{\gamma}{\alpha} \leqslant 0$, thus $t_+ \leqslant 0$ (and $t_- \geqslant 0$). In other words, there's only 1 root satisfying $t \leqslant 0$, which distinguishes the solution of SRP from the solution of dual problem.

Now, if we repeat the solving method after re-enumerating the sensors so that i-th sensor ($i = \overline{1,n}$) becomes $\mathbf{r}^{(0)}$, and moving "the origin of space and time" to this new $\mathbf{r}^{(0)}$, then we come to essentially the same SRP, L_n, L'_n, ... in different reference frame. And we obtain the single root $T_+ \leqslant 0$. But T_+ *and* t_+ *are the same moment* of time, only in different temporal reference frames. Therefore, $T_+ \leqslant 0$ means $t_+ \leqslant t_i$. Let $t^{(n)} = t_+$.

Case $\alpha = 0$, $\beta \neq 0$: $f_{h^2}(t) = \beta t + \gamma + h^2 = 0 \Leftrightarrow t = t^{(\infty)} = -\frac{\gamma + h^2}{\beta}$. $t^{(\infty)} \leqslant 0 \Rightarrow \beta > 0$, thus $f_0(\hat{t}) = 0$ for $\hat{t} = -\frac{\gamma}{\beta} \leqslant 0$. Similarly, the symmetry implies $\hat{t} \leqslant t_i$ for any $i = \overline{1,n}$. Let $t^{(n)} = \hat{t}$.

Case $\alpha = 0$, $\beta = 0$: impossible, because $\gamma + h^2 > 0$.

Anyway, $\exists t^{(n)} \leqslant t_i$ for any $i = \overline{0,n}$: $\sum\limits_{j=1}^{n} (\widetilde{b_j} + t^{(n)} \widetilde{c_j})^2 = (t^{(n)})^2$. It determines the solution $\mathbf{s}^{(n)} = \{\widetilde{b_j} + t^{(n)} \widetilde{c_j}\}_{j=1}^{n} \in L_n$ of SRP_n. QED

The statement of Prop. 6 remains true for any finite $\widehat{R} = \{\mathbf{r}^{(i_j)}\}_{j=1}^{n} \subset \dot{R}$, if we seek the solution of 'truncated" SRP "$t_{i_j} = t + \|\mathbf{r}^{(i_j)} - \mathbf{s}\|$ for $j = \overline{0,n}$" in $L(\widehat{R})$, — just re-enumerate elements of $\dot{R}$ so that $\widehat{R} = \{\mathbf{r}^{(1)}; \ldots; \mathbf{r}^{(n)}\}$, $\dot{R} \backslash \widehat{R} = \{\mathbf{r}^{(n+1)}; \mathbf{r}^{(n+2)}; \ldots\}$ to get SRP_n.

However, this statement is false for infinite $\widehat{R} \subset \dot{R}$, in general case. Consider

Example 3. $\lhd$ Let $\dot{R} = B$ be an orthonormal basis of H, $\mathbf{r}^{(i)} = \mathbf{e}_i$, thus $r_j^{(i)} = \delta_{ij}$; also, let $\mathbf{s}' = \mathbf{r}^{(1)} = (1; 0; 0; \ldots)$ and $t' = -1$. Then $t_0 = t' + \|\mathbf{s}'\| = 0$, $t_1 = t' + \|\mathbf{r}^{(1)} - \mathbf{s}'\| = -1$, and $\forall i \geqslant 2$: $t_i = t' + \|\mathbf{r}^{(i)} - \mathbf{s}'\| = -1 + \sqrt{2}$.

Obviously, $(\mathbf{s}'; t')$ is the solution of the SRP defined by $\big(R; \{t_i\}_{i \in \mathbb{Z}_+}\big)$. We claim that for any infinite $\widehat{R} \subset \dot{R}$ such that $\mathbf{r}^{(1)} \notin \widehat{R}$ the truncated SRP "$t_i = t + \|\mathbf{r}^{(i)} - \mathbf{s}\|$ for any $\mathbf{r}^{(i)} \in \widehat{R} \cup \{\mathbf{r}^{(0)}\}$" has no solution in $\widehat{L} = L(\widehat{R})$.

Proof. Assume the contrary and enumerate the elements of $\widehat{R}$ as the subsequence of $\dot{R}$, ascending: $\widehat{R} = \{\hat{\mathbf{r}}^{(1)}; \hat{\mathbf{r}}^{(2)}; \ldots\}$. $\widehat{R}$ is the orthonormal basis of $\widehat{L}$, in itself $\hat{r}_j^{(i)} = \delta_{ij}$ as well, and $\dim \widehat{L} = \infty$. We denote the solution of truncated SRP in $\widehat{L}$ by $(\mathbf{s}; t)$, with $\mathbf{s} = (s_1; s_2; \ldots)$.

(1)-(2)-(3)-(4)-(5) implies

$$\sum_{k=1}^{\infty} s_k^2 = t^2, \qquad \forall k \in \mathbb{N}\colon \sum_{j=1}^{k} \hat{r}_j^{(k)} s_j = s_k = \tfrac{1}{2}(\|\hat{\mathbf{r}}^{(k)}\|^2 - \hat{t}_k^2) + t\hat{t}_k$$

(that is, $b_k = \widetilde{b_k}$, $c_k = \widetilde{c_k}$). Hence $s_k \equiv \frac{1}{2}\big(1 - (\sqrt{2} - 1)^2\big) + t(\sqrt{2} - 1) = (\sqrt{2} - 1)(t + 1)$, therefore $\mathbf{s} \in H$ only if $s_k \equiv 0 \Leftrightarrow t = -1$. Then for $\hat{\mathbf{r}}^{(0)} = \theta = \mathbf{s}$: $t_0 = t = -1 \neq 0$, — a contradiction. QED $\rhd$

Proposition 7. *If the solution* $(\mathbf{s}^{(\infty)};t^{(\infty)})$, $\mathbf{s}^{(\infty)}\neq\theta$, *of SRP is unique, and for each* $n\in\mathbb{N}$ *the solution* $(\mathbf{s}^{(n)};t^{(n)})$, $\mathbf{s}^{(n)}\neq\theta$, *of SRP*$_n$ *is unique, and* $\sum_{j=1}^{\infty}\widetilde{c_j}^2<\infty$, *then* $t^{(n)}\xrightarrow[n\to\infty]{}t^{(\infty)}$ *and* $\mathbf{s}^{(n)}\xrightarrow[n\to\infty]{}\mathbf{s}^{(\infty)}$.

Proof. (4) gives $\mathbf{s}^{(\infty)}=(\widetilde{b_1}+t^{(\infty)}\widetilde{c_1};\ldots;\widetilde{b_n}+t^{(\infty)}\widetilde{c_n};s_{n+1}^{(\infty)};s_{n+2}^{(\infty)};\ldots)$ and $\mathbf{s}^{(n)}=(\widetilde{b_1}+t^{(n)}\widetilde{c_1};\ldots;\widetilde{b_n}+t^{(n)}\widetilde{c_n};0;0;\ldots)$, consequently

$$\|\mathbf{s}^{(n)}-\mathbf{s}^{(\infty)}\|^2=(t^{(n)}-t^{(\infty)})^2\sum_{j=1}^{n}\widetilde{c_j}^2+\sum_{j=n+1}^{\infty}(s_j^{(\infty)})^2$$

$\sum_{j=n+1}^{\infty}(s_j^{(\infty)})^2\xrightarrow[n\to\infty]{}0$ and $\sum_{j=1}^{n}\widetilde{c_j}^2\xrightarrow[n\to\infty]{}\sum_{j=1}^{\infty}\widetilde{c_j}^2$; it remains to prove that

$$t^{(n)}\xrightarrow[n\to\infty]{}t^{(\infty)}\Leftrightarrow z^{(n)}\xrightarrow[n\to\infty]{}z^{(\infty)}$$

where $z^{(\infty)}=1/t^{(\infty)}$, $z^{(n)}=1/t^{(n)}$.

From the assumptions of this proposition it follows that, speaking of SRP, we're in Subcase 1b, where $z^{(\infty)}$ is one of two roots, $z_{\pm}^{(\infty)}=\frac{-\beta\pm\sqrt{\beta^2-4\alpha\gamma}}{2\alpha}$, of (7). Using the symbols α_n, β_n and γ_n from Subcase 1a (this is different from notation in Prop. 6), we state that, similarly, $z^{(n)}$ is one of two roots, $z_{\pm}^{(n)}=\frac{-\beta_n\pm\sqrt{\beta_n^2-4\alpha_n\gamma_n}}{2\alpha_n}$, of the equation $\alpha_n z^2+\beta_n z+\gamma_n=0$, which appears while solving SRP$_n$. $\alpha_n\to\alpha>0$, $\beta_n\to\beta$, $\gamma_n\to\gamma$ as $n\to\infty$, therefore $z_{-}^{(n)}\to z_{-}^{(\infty)}$, $z_{+}^{(n)}\to z_{+}^{(\infty)}$, and the selection of the root $z^{(n)}$ in SRP$_n$ ($z_{-}^{(n)}$ or $z_{+}^{(n)}$) becomes the same as the selection of the root $z^{(\infty)}$ in SRP (perhaps for $n\geqslant n_0$). In any case, $z^{(n)}\to z^{(\infty)}$ as $n\to\infty$. *QED*

This proposition shows another method, one of Galerkin kind, to obtain the SRP solution.

2 SR on unit sphere in Hilbert space

Let it be $S=\{\mathbf{x}\in H\colon\|\mathbf{x}\|=1\}$. Instead of "embracing-space-induced" $\|\mathbf{x}-\mathbf{y}\|$, we consider the so-called geodesic metric

$$d\colon S\times S\to\mathbb{R}_+\colon d(\mathbf{x};\mathbf{y})=\arccos{<}\mathbf{x};\mathbf{y}{>}\in[0;\pi]$$

which (we remind) is really a metric. *Proof.* Obviously, $d(\mathbf{x};\mathbf{x})=\arccos 1=0$ and $d(\mathbf{x};\mathbf{y})=d(\mathbf{y};\mathbf{x})$. If $d(\mathbf{x};\mathbf{y})=0$, then ${<}\mathbf{x};\mathbf{y}{>}=1=\|\mathbf{x}\|\cdot\|\mathbf{y}\|\Rightarrow\mathbf{x}$ and $\mathbf{y}$ are linearly dependent with $\mathbf{y}=a\mathbf{x}$, $a\geqslant 0$; $1={<}\mathbf{x};\mathbf{y}{>}=a\|\mathbf{x}\|^2=a\Rightarrow\mathbf{x}=\mathbf{y}$.

The triangle inequality $\forall\mathbf{x},\mathbf{y},\mathbf{z}\in S$: $d(\mathbf{x};\mathbf{z})\leqslant d(\mathbf{x};\mathbf{y})+d(\mathbf{y};\mathbf{z})$ can be established as follows. It is equivalent to

$$\left[\begin{array}{l} d(\mathbf{x};\mathbf{y})+d(\mathbf{y};\mathbf{z})\geqslant\pi,\\ \cos d(\mathbf{x};\mathbf{z})\geqslant\cos\big(d(\mathbf{x};\mathbf{y})+d(\mathbf{y};\mathbf{z})\big),\quad d(\mathbf{x};\mathbf{y})+d(\mathbf{y};\mathbf{z})\leqslant\pi;\end{array}\right.$$

we rewrite the inequality in the 2nd case as

$$<\mathbf{x};\mathbf{z}> \geqslant <\mathbf{x};\mathbf{y}><\mathbf{y};\mathbf{z}> - \sqrt{1-<\mathbf{x};\mathbf{y}>^2}\cdot\sqrt{1-<\mathbf{y};\mathbf{z}>^2} \Leftrightarrow$$
$$\Leftrightarrow \left[\begin{array}{l} <\mathbf{x};\mathbf{y}><\mathbf{y};\mathbf{z}> \leqslant <\mathbf{x};\mathbf{z}>, \\ \left(1-<\mathbf{x};\mathbf{y}>^2\right)\left(1-<\mathbf{y};\mathbf{z}>^2\right) \geqslant \left(<\mathbf{x};\mathbf{y}><\mathbf{y};\mathbf{z}> - <\mathbf{x};\mathbf{z}>\right)^2; \end{array}\right.$$

The 2nd inequality here, being rearranged,

$$1 + 2<\mathbf{x};\mathbf{y}><\mathbf{y};\mathbf{z}><\mathbf{x};\mathbf{z}> \geqslant <\mathbf{x};\mathbf{y}>^2 + <\mathbf{y};\mathbf{z}>^2 + <\mathbf{x};\mathbf{z}>^2 \tag{8}$$

Using projection theorem (and $\dim H = \infty$), we represent $\mathbf{y} = y_1\mathbf{x} + y_2\mathbf{h}$, where $\mathbf{h} \in S$, $\mathbf{h} \perp \mathbf{x}$, and $\mathbf{z} = z_1\mathbf{x} + z_2\mathbf{h} + z_3\mathbf{w}$, where $\mathbf{w} \in S$, $\mathbf{w} \perp \mathbf{x}$, $\mathbf{w} \perp \mathbf{h}$. And $y_1^2 + y_2^2 = 1$, $z_1^2 + z_2^2 + z_3^2 = 1$, so (8) $\Leftrightarrow 1 + 2y_1(y_1z_1 + y_2z_2)z_1 \geqslant y_1^2 + (y_1z_1 + y_2z_2)^2 + z_1^2 \Leftrightarrow 1 - y_1^2 \geqslant z_1^2(1-y_1^2) + y_2^2z_2^2 \Leftrightarrow y_2^2z_3^2 \geqslant 0$. QED

The set of sensors $R = \{\mathbf{r}^{(i)}\}_i \subset S$ (obviously, $\theta \notin R$). As before, we assume the existence of at least one solution $(\mathbf{s}_0; t_{e;0})$, $\mathbf{s}_0 \in S$, of the SRP "$t_i = t + d(\mathbf{r}^{(i)};\mathbf{s})$ for any i".

Remark. We may consider the wave to "oscillate forever" on S, from $\mathbf{s}_0$ to antipodal $-\mathbf{s}_0$ ($d(\mathbf{s}_0;-\mathbf{s}_0) = \pi$), then back to $\mathbf{s}_0$, and so forth. Then t_i is the *first* time when the wave reaches $\mathbf{r}^{(i)}$. However, the wave as the sphere of increasing radius $t - t_{e;0}$ vanishes at $-\mathbf{s}_0$.

The reasonings we've used for the entire H show that if $L(R) \neq H$ and $\mathbf{s}_0 \notin L(R)$, then the solution is certainly not unique: for $\mathbf{s}_0 = \mathbf{u_0} + \mathbf{u_1}$ with $\mathbf{u_0} \in L(R)$, $\mathbf{u_1} \perp L(R)$, the "$\mathbf{s}(\varphi) = \mathbf{u_0} + \|\mathbf{u_1}\|(\cos\varphi\cdot\widetilde{\mathbf{u_1}} + \sin\varphi\cdot\widetilde{\mathbf{u_2}})$" (where $\widetilde{\mathbf{u_2}} \perp L(R), \mathbf{u_1}$) construction works as well, since $\mathbf{s}(\varphi) \in S$ and $d(\mathbf{r}^{(i)};\mathbf{s}(\varphi)) =$

$$= \arccos\left[<\mathbf{r}^{(i)};\mathbf{u_0}> + \|\mathbf{u_1}\|\cos\varphi<\mathbf{r}^{(i)};\widetilde{\mathbf{u_1}}> + \|\mathbf{u_1}\|\sin\varphi<\mathbf{r}^{(i)};\widetilde{\mathbf{u_2}}>\right] =$$
$$= \arccos<\mathbf{r}^{(i)};\mathbf{u_0}> = \arccos<\mathbf{r}^{(i)};\mathbf{u_0}+\mathbf{u_1}> = d(\mathbf{r}^{(i)};\mathbf{s}_0)$$

Therefore, let $R = \{\mathbf{r}^{(i)}\}_{i\in\mathbb{N}}$ be a basis of H, and let B be the orthonormal basis of H, derived from R by Gram-Schmidt orthogonalization; thus, in B, $\mathbf{r}^{(i)} = (r_1^{(i)};\ldots;r_i^{(i)};0;0;\ldots)$ (and $r_1^{(1)} = 1$).

Then $\mathbf{s}$ can be written in the form of $(s_1;s_2;\ldots)$.

Since $t \leqslant t_i$ and $t_i - t = d(\mathbf{r}^{(i)};\mathbf{s}) \leqslant \pi$, we have $t \in [\sup\{t_i\} - \pi; \inf\{t_i\}] = \mathbf{\Delta}$ ($|\mathbf{\Delta}| \leqslant \pi$). The equations of SRP are equivalent to $\cos(t_i - t) = <\mathbf{r}^{(i)};\mathbf{s}>$. Adding "$\mathbf{s} \in S$", we have

$$\sum_{j=1}^{\infty} s_j^2 = 1, \qquad \forall i \in \mathbb{N}\colon \sum_{j=1}^{i} r_j^{(i)}s_j = \cos t\cos t_i + \sin t\sin t_i \tag{9}$$

Similarly to (3)-(4)-(5), we obtain $s_j = \widetilde{p_j}\cos t + \widetilde{q_j}\sin t$, where

$$\widetilde{p_k} = \begin{vmatrix} r_1^{(1)} & 0 & \ldots & \cos t_1 \\ r_1^{(2)} & r_2^{(2)} & \ldots & \cos t_2 \\ \ldots & \ldots & \ddots & \ldots \\ r_1^{(k)} & r_2^{(k)} & \ldots & \cos t_k \end{vmatrix} \Big/ \prod_{i=1}^{k} r_i^{(i)},\ \widetilde{q_k} = \begin{vmatrix} r_1^{(1)} & 0 & \ldots & \sin t_1 \\ r_1^{(2)} & r_2^{(2)} & \ldots & \sin t_2 \\ \ldots & \ldots & \ddots & \ldots \\ r_1^{(k)} & r_2^{(k)} & \ldots & \sin t_k \end{vmatrix} \Big/ \prod_{i=1}^{k} r_i^{(i)} \tag{10}$$

and $\sum_{j=1}^{\infty}(\widetilde{p_j}\cos t + \widetilde{q_j}\sin t)^2 = 1$ (11)

Case 1a: $\sum\limits_{j=1}^{\infty} \widetilde{p_j}^2$ converges, $\sum\limits_{j=1}^{\infty} \widetilde{q_j}^2$ diverges. If the series in the left side of (11) converges for t such that $\sin t \neq 0$, then $\{\widetilde{q_j}\} = \frac{1}{\sin t}\big(\{\widetilde{p_j}\cos t + \widetilde{q_j}\sin t\} - \{\widetilde{p_j}\cos t\}\big) \in H$, which contradicts the assumption. Thus, if t satisfies (11), then $\sin t = 0$ (in particular, $\sin t_{e;0} = 0$, so $\sum\limits_{j=1}^{\infty} \widetilde{p_j}^2 = 1$).

All $t = \pi m \in \mathbf{\Delta}$, where $m \in \mathbb{Z}$ (there's 1 or 2 such values), satisfy (11). Then $s_j = \widetilde{p_j}\cos t + \widetilde{q_j}\sin t = \widetilde{p_j}\cos t$, $j \in \mathbb{N}$.

Case 1b: $\sum\limits_{j=1}^{\infty} \widetilde{p_j}^2$ diverges, $\sum\limits_{j=1}^{\infty} \widetilde{q_j}^2$ converges. Similarly, the convergence of the series in the left side of (11) leads to $\cos t = 0$ ($\sum\limits_{j=1}^{\infty} \widetilde{q_j}^2 = 1$ since it converges when $t = t_{e;0}$), and we take $t \in (\frac{\pi}{2} + \pi\mathbb{Z}) \cap \mathbf{\Delta}$, satisfying (11).

Case 2: $\sum\limits_{j=1}^{\infty} \widetilde{p_j}^2$, $\sum\limits_{j=1}^{\infty} \widetilde{q_j}^2$ diverge. (11) is true for $t' = t_{e;0}$; if the series in the left side converges for $t'' \in \mathbf{\Delta}$, $t'' \neq t'$, then $\begin{cases} \cos t'\{\widetilde{p_j}\} + \sin t'\{\widetilde{q_j}\} = \mathbf{v}' \in S, \\ \cos t''\{\widetilde{p_j}\} + \sin t''\{\widetilde{q_j}\} = \mathbf{v}'' \in H \end{cases}$

$$\Rightarrow \begin{cases} \cos t' \sin t''\{\widetilde{p_j}\} + \sin t' \sin t''\{\widetilde{q_j}\} = \sin t'' \cdot \mathbf{v}' \in H, \\ \cos t'' \sin t'\{\widetilde{p_j}\} + \sin t'' \sin t'\{\widetilde{q_j}\} = \sin t' \cdot \mathbf{v}'' \in H \end{cases} \Rightarrow$$

$\{\widetilde{p_j}\} = \frac{1}{\sin(t'-t'')}(\sin t' \cdot \mathbf{v}'' - \sin t'' \cdot \mathbf{v}') \in H$ ($\sin(t'-t'') \neq 0$, because $t', t'' \in \mathbf{\Delta}$, $|\mathbf{\Delta}| \leqslant \pi$), — a contradiction. So, t' is the unique value providing the convergence of series in the left side of (11).

We obtain t' using the method analogous to that of Subcase 1a (Section 1).

$f_n(t) = \sum\limits_{j=1}^{n} (\widetilde{p_j}\cos t + \widetilde{q_j}\sin t)^2 - 1$ is non-decreasing relative to n, $f_n(t) \leqslant f_{n+1}(t)$. Rearranging, $f_n(t) = (\alpha_n - 1)\cos^2 t + (\beta_n - 1)\sin^2 t + \gamma_n \cos t \sin t$.

$\cos t \neq 0$, otherwise $\sum\limits_{j=1}^{\infty} \widetilde{q_j}^2 = 1$. Consequently, to solve $f_n(t) = 0$, we can divide it by $\cos t$: $(\beta_n - 1)\tan^2 t + \gamma_n \tan t + (\alpha_n - 1) = 0$.

This quadratic trinomial (in $\tan t$) has no more than 2 roots ($\beta_n \to +\infty$ as $n \to \infty$), hence $f_n(t)$ has a finite number of zeroes in $\mathbf{\Delta}$. Between them, the sign of continuous $f_n(t)$ is constant. We consider it "easy enough" to determine, for each $n \in \mathbb{N}$, the set $U_n = \{t \in \mathbf{\Delta} \mid f_n(t) \leqslant 0\}$. It is clear that $U_n \supseteq U_{n+1}$. Moreover, $\{t'\} = \bigcap\limits_{n \in \mathbb{N}} U_n$, since $f_n(t') \leqslant f_\infty(t') = 0$ and $\forall t \neq t'\ \exists n_0$: $f_{n_0}(t) > 0$.

Case 3: $\sum\limits_{j=1}^{\infty} \widetilde{p_j}^2$, $\sum\limits_{j=1}^{\infty} \widetilde{q_j}^2$ converge. Then we denote $\alpha = \sum\limits_{j=1}^{\infty} \widetilde{p_j}^2$, $\beta = \sum\limits_{j=1}^{\infty} \widetilde{q_j}^2$, $\gamma = 2\sum\limits_{j=1}^{\infty} \widetilde{p_j}\widetilde{q_j}$, and rewrite (11) as $(\alpha - 1)\cos^2 t + (\beta - 1)\sin^2 t + \gamma \sin t \cos t = 0$

$$\Leftrightarrow \left[\begin{array}{l} \beta = 1, \quad \cos t = 0, \\ (\beta - 1)\tan^2 t + \gamma \tan t + (\alpha - 1) = 0, \quad \cos t \neq 0. \end{array} \right.$$

Subcase 3a: $(\beta - 1)^2 + \gamma^2 + (\alpha - 1)^2 > 0$. Then again (11) has a finite number of "easy to obtain" (in accordance with our allowances) roots in $\mathbf{\Delta}$.

Subcase 3b: $\beta - 1 = \gamma = \alpha - 1 = 0$, therefore $\{\widetilde{p_j}\} \in S$, $\{\widetilde{q_j}\} \in S$, and $\{\widetilde{p_j}\} \perp \{\widetilde{q_j}\}$. Then any $t \in \mathbf{\Delta}$ satisfies (11).

Each root t, in turn, determines $\mathbf{s} = \{\widetilde{p_j} \cos t + \widetilde{q_j} \sin t\}_j$.

This concludes the description of the solving method for SRP on S with d.

Again, there's the question about the conditions providing the uniqueness of the solution, especially in Subcase 3b with the most "ambiguity". We restrict our attention to the finiteness of the set of solutions.

Proposition 8. *If* $\mathbf{r}^{(1)} \perp \mathbf{r}^{(2)} \perp \mathbf{r}^{(3)} \perp \mathbf{r}^{(1)}$*, then Subcase 3b is impossible.*

Proof. In the basis B, not only $\mathbf{r}^{(1)} = (1; 0; 0; \ldots)$, but $\mathbf{r}^{(2)} = (0; 1; 0; 0; \ldots)$ and $\mathbf{r}^{(3)} = (0; 0; 1; 0; 0; \ldots)$ then. From (10) it follows that $\widetilde{p_k} = \cos t_k$ and $\widetilde{q_k} = \sin t_k$ for $k = 1, 2, 3$; therefore $\sum\limits_{j=1}^{\infty} (\widetilde{p_j}^2 + \widetilde{q_j}^2) \geqslant \sum\limits_{j=1}^{3} (\widetilde{p_j}^2 + \widetilde{q_j}^2) = 3$.

Meanwhile, in Subcase 3b we have $\sum\limits_{j=1}^{\infty} \widetilde{p_j}^2 + \sum\limits_{j=1}^{\infty} \widetilde{q_j}^2 = \alpha + \beta = 2 < 3$. *QED*

The orthogonality constraint in Prop. 8 can be weakened: $\sum\limits_{j=1}^{3} (\widetilde{p_j}^2 + \widetilde{q_j}^2) > 2$ would suffice.

Example 4. (analogous to Ex. 3). $\lhd$ Let R be an orthonormal basis of H, $r_j^{(i)} = \delta_{ij}$, and let $\mathbf{s}' = \mathbf{r}^{(1)}$, $t' = 0$. Then $t_1 = t' = 0$, $t_k = t' + d(\mathbf{r}^{(k)}; \mathbf{s}') = \arccos 0 = \frac{\pi}{2}$ for any $k \geqslant 2$. $(\mathbf{s}'; t')$ is the solution, on S, of the SRP $(R; \{t_i\}_{i \in \mathbb{N}})$.

Meanwhile, for any infinite $\widehat{R} \subset R$ such that $\mathbf{r}^{(1)} \notin \widehat{R}$, the truncated SRP "$t_i = t + d(\mathbf{r}^{(i)}; \mathbf{s})$ for $\mathbf{r}^{(i)} \in \widehat{R}$" has no solution on $\widehat{S} = \{\mathbf{x} \in L(\widehat{R}) \mid \|\mathbf{x}\| = 1\}$.
Proof. $\widehat{R}$ is the orthonormal basis of $L(\widehat{R})$, and we enumerate the elements of $\widehat{R}$ as they follow in R: $\widehat{R} = \{\hat{\mathbf{r}}^{(1)}; \hat{\mathbf{r}}^{(2)}; \ldots\}$; then, decomposing $\widehat{R}$ in itself, $\hat{r}_j^{(i)} = \delta_{ij}$. Assuming $(\mathbf{s}; t)$, $\mathbf{s} \in \widehat{S}$, to be the solution of truncated SRP, we have $\frac{\pi}{2} \equiv \hat{t}_k = t + \arccos \langle\hat{\mathbf{r}}^{(k)}; \mathbf{s}\rangle = t + \arccos s_k$, thus $s_k \equiv \cos(\frac{\pi}{2} - t) = \sin t$. $\dim L(\widehat{R}) = \infty$, so $\mathbf{s} \in H$ implies $s_k \equiv 0 \Leftrightarrow \mathbf{s} = \theta \notin \widehat{S}$, — a contradiction. *QED* $\rhd$

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