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Optimal choices:

mean field games with controlled jumps and optimality in a stochastic volatility model

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The sun came up with no conclusion
Flowers sleeping in their beds
This city's cemetery's humming
I'm wide-awake, it's morning

Road to Joy
Bright Eyes

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Thesis Overview

"So, what should I do?" is the big question this thesis is concerned with. Indeed, this is the query each player asks himself when in the middle of a game he has to make his move, as in **Part I**. And then, again, this is the question that one needs to answer in **Part II** when considering a stochastic optimal control problem in the framework of stochastic volatility modeling.

Part I. A class of mean field games with state dynamics governed by jump-diffusion processes with controlled jumps

In a nutshell, game theory studies the behaviour of a bunch of decision makers, called players or agents, when interacting in strategic situations. This means that the outcome of this interaction, which may be different for every participant, depends not only on one's individual choices but also on the decisions taken by the other players. This connection ties together all the players, meaning that each agent cannot choose the strategy which maximises its preferences without considering the choices made by the others. Initially formalized by Von Neumann and Morgenstern in [VNM], over the past seven decades game theory results have been deeply and widely applied and extended to represent different situations in countless contexts. And the reason is straightforward since, as *social animals*, human beings are required to confront themselves into strategic decisions on a daily basis.

The distinctive trait of the class of non-cooperative, symmetric games with mean-field interactions we focus on is the fact that the state evolution of any player is given by a jump-diffusion process, where the size of the jumps is controlled by the player itself. Mean-field interaction refers to the fact that, by construction, both the dynamics of the private state and the possible outcomes of each player depend on the opposing players only through their overall distribution. This class of games is presented in **Chapter 1**. Considering non-cooperative games, the aim is to discuss the existence of a Nash equilibrium for them, and possibly to compute it. A Nash equilibrium, firstly introduced by Nash in [Nas+50; Nas51], is a set of strategies, one for each player, that are optimal for each of them when they are simultaneously played. In other words, none of the players has an incentive to unilaterally deviate from it, since no other strategy can improve his outcome if the strategies of the others remain unchanged. Unfortunately, this is *easier said than done*, being the computation of Nash equilibria a quite hard problem in general. However, since the games studied here are symmetric and charachterised by an interaction of mean-field type, it is possible to overcome (some of) these issues by considering their limiting game, that is the game arising when the number of players grows to infinity, which is in general easier to study. This is the main subject of the mean-field game theory, introduced by Lasry and Lions in [LL06a; LL06b; LL07] and, independently, by Huang, Malhamé, and Caines in [HMC06] combining ideas from the interacting particle system theory and results from the game theory. The key idea is that when the number of the intervening (homogeneous) players is large enough, the impact of one particular individual becomes morally negligible compared to the impact due to the overall population, and therefore it is possible to develop an efficient decision-rule by paying particular attention on the aggregate behaviour rather than on each individual player's choice. **Chapter 2** studies the existence of a mean-field game solution for the class of games introduced in the previous chapter by means of *relaxed controls*, introduced by Lacker in [Lac15a] and, independently, by Fischer in [Fis+17]. A mean-field game solution of the limiting game provides useful information also regarding the finite-player games and indeed it can be exploited to compute an approximate Nash equilibrium for them, at least when the number of agents is big enough, as examined in **Chapter 3**.

Lastly, **Chapter 4** presents a possible economic application of the class of games previously introduced: financial institutions, that are the players of this game, interact on an interbank lending market, aiming at controlling their level of reserves to balance their investment portfolio and, at the same time, to meet regulatory requirements. Assuming that this market is illiquid, each bank can access to it and therefore adjust its reserve level by borrowing or lending money only at some exogenously given instants, modeled as jump times of a Poisson process with constant intensity, which in turn represents a health indicator of the whole system.

Part II. An optimal control approach to stochastic volatility models

Stochastic control theory concerns dynamical systems whose evolution is modeled by stochastic differential equations depending on a control input which is chosen to reach the best possible outcome. **Chapter 5** presents a (very short) introduction to stochastic optimal control theory in the case of continuous-time Markov diffusion processes, collecting well-known results which are used in the subsequent chapter. If the deterministic case has been a classical topic since the 1600s, optimal control in stochastic systems is a more recent theory: introduced by Bellman in the mid '50s in [Bel58], it has been widely applied in finance since the late '60s, when Merton formulated his portfolio-consumption model in [Mer69; Mer75] and then Black and Scholes presented their mathematical model, which bears their names, representing a financial market containing derivative instruments and leading to determine the fair price of a European call option. Starting from this single query, the results in stochastic optimal control were used to solve several problems and to answer to disparate questions in different economic fields.

Chapter 6 formulates and discusses the stochastic optimization problem that is the main scope of Part II characterised by a system whose state evolves accordingly to a diffusion process with (partially) controlled stochastic volatility. Stochastic volatility models were introduced in the late '80s to overcome some biases of the most used models at that time, inter alia, the Black and Scholes pricing equation which nonetheless is still daily used, and therefore to better describe financial data series. The model considered here is a modification of the Heston model, presented in [Hes93]. Indeed, the state is described by a Heston process, except that a multiplicative control is added into its volatility component. The main objective is to consider the possible role of an external actor, whose exogenous contribution is summarised in the control itself.

Part I

A class of mean field games with state dynamics governed by jump-diffusion processes with controlled jumps

Chapter 1

A class of mean field games with controlled jumps in the state dynamics

Every human being faces the surrounding world everyday by taking decisions and making choices based on their personal preferences and values. Being in a *social environment*, their behaviuors and their choices cannot ignore the social and cultural structure where they take place. Reservations in fancy hotels depend on the feedback read on review sites, the car one decides to buy is conditioned by advertising and by the car models driven by their neighbours, political preferences depend on the education one receives, on the discussions with the coworkers, on the opinion polls reported in the media and much more. We all are affected by the choices made by our families, by our friends and by our colleagues.

But at the same time, we write reviews about the hotels where we have stayed overnight, we zip around the city on our brand-new cars and we try to convince our office mates that our political opinion is, quite obviously, the right one. So, even if affected, we influence the choices made by our families, by our friends and by our colleagues as well.

Furthermore, one's own behaviour and the one of other people influence not only the choices that every person makes, but also the outcomes of different situations one is in: which party will run the country after the next round of election? How much will the new *Audi* cost? Since, in principle, different people may have different wishes and preferences, the desired outcome may converge towards a same result or may diverge leading to a conflicting situation.

It would seem that as members of a economical, political and social life, each of us is actively *playing a game* whenever making a choice.

Consider, as an example, a number of firms producing a similar good and therefore competing in the same market (as in [GLL11],[CS15]). Each producer chooses its level of production knowing that the resulting aggregate supply, and therefore the resulting price, depends not only on his choice but also on the strategies adopted by its competitors. If the aim is to maximise one's own gain, which is the proper level each producer has to choose?

Or, consider the *consensus problem* (as in [OSFM07], [Nou+13]). A group of people is required to agree on a final decision concerning a certain subject. Clearly, regarding one's own preferences, each person would prefer an outcome rather than another ending and may try to convince the others of the goodness of its own beliefs. Then, the final agreement depends upon the preferences and the persuasive skills of all the people.

Decision-makers' interaction is the main subject of *game theory*, and in particular *mean field game theory* studies a class of differential decision problems characterised by a large (say huge, or better infinite) number of small and similar (say identical) players which are coupled together through their empirical average. Models with too many agents who mutually interact may be inefficient from a mathematical point of view, since it is not possible to consider simultaneously the dynamics of all the players, all their possible choices and all the ways these choices reflect on the other participants. Indeed, it would mean considering too many coupled equations and too many constraints at the same time, which may be not feasible. Actually, such a model would describe every detail of the reality, but it would be humanly and, even worse, computationally impossible to be solved. Therefore, the aim of mean field game theory is to *simplify* a (specific) class of large population games to make them more tractable, but without losing their meaning. Somehow, *mean field games look at the big picture.*

Mean Field Games (MFGs, henceforth) were introduced by Lasry and Lions in [LL06a; LL06b; LL07] and, independently, by Huang, Malhamé, and Caines in [HMC06] combining ideas from the interacting particle system theory and results from the game theory. Interacting particles become here interacting decision makers, i.e. rational players provided with preferences and goals who interact in a strategic situation, meaning that the outcome of the game for each of them depends on its own actions as well as on the strategies chosen by all the other players. Therefore, the behaviour of the peers becomes a crucial variable in computing one's own optimal strategy. Then, considering again the analogy with an interacting particle system, the outcome of a game is not *the sum of the forces* as in a physics model, but it is the sum of rational choices made by the players. In [GLL11], the authors strongly support that the primary purpose of MFGs is not (or not only) to compute and describe the *inevitable* result of a strategic game but it is to explain why the inevitable emergent phenomenon is a natural response of coherent behaviours.

The purpose of this chapter is to introduce the class of MFGs we will study in the following. Section 1.1 presents G_n: a non cooperative, symmetric, n-player game with mean-field interaction. Here we define how the state of each player evolves in time, what is his own personal objective and how the other participants may influence both his dynamics and his outcome. Particular attention will be paid to the characteristics of these games which guarantee the possibility of introducing the corresponding theoretical limiting game. This is the mean field game G_∞ arising when the number of players n grows to infinity, which is introduced in Section 1.2 along with the concept of mean field game solution.

1.1 The n-player game G_n

We present here the n-player game G_n, a non cooperative, symmetric game with mean-field coupling that is the main interest of the Part I of this Thesis.

Let $T > 0$ be a fixed and finite time horizon and let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ be a filtered probability space, supporting n independent Brownian motions W^i, for $i = 1, \dots, n$, and n independent Poisson processes N^i, for $i = 1, \dots, n$ with the same time-dependent intensity function $\nu(t)$.The filtration $(\mathcal{F}_t)_{t\in[0,T]}$ is assumed to satisfy the *usual conditions*, meaning that it is complete, i.e. $\mathcal{F}_0$ contains all the P-null sets, and it is right-continuous, i.e.

$$\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s \quad \text{for all } t \in [0, T] .$$

The state of each player i in the game, denoted by $X_t^{i,n}$ and belonging to the real space, evolves in time accordingly to the following stochastic differential equation

$$dX_t^{i,n} = b(t, X_t^{i,n}, \mu_t^n)\, dt + \sigma(t, X_t^{i,n})\, dW_t^i + \beta(t, X_{t-}^{i,n}, \mu_{t-}^n, \gamma_t^i)\, d\widetilde{N}_t^i, \tag{1.1}$$

subjected to a given initial condition

$$X_0^{i,n} = \xi^i.$$

Here, $\widetilde{N}_t^i$ denotes the compensated Poisson process $\widetilde{N}_t^i = \widetilde{N}_t^i - \int_0^t \nu(s)\, ds$ and μ^n stands for the empirical measure of the system $X^n = (X^{1,n}, \dots, X^{n,n})$, which is defined for any time $t \in [0, T]$ as

$$\mu_t^n = \frac{1}{n} \sum_{i=1}^n \delta_{X_t^{i,n}} , \tag{1.2}$$

where δ_x denotes the Dirac delta measure at the point x. In addition we assume that the initial conditions ξ^i, $i = 1, \dots, n$, are mutually independent real-valued random variables, all distributed according to the same distribution χ and that they are also independent from the noises W^i, N^i introduced before.

Observe that the functions b, σ and β appearing in the SDEs (1.1) do not depend on the specific player i, meaning that they are equal for any agent even if computed relative to the different players' positions/strategies.

Each player i has the chance to control his position, or better the dynamics defining his state, by choosing at any time $t \in [0, T]$ a control input γ_t^i. Each control process $\gamma^i = (\gamma_t^i)_{t\in[0,T]}$, also called the strategy chosen by player i, takes values in the *action space* $A \subset \mathbb{R}$ and it is assumed to be predictable and regular enough to assure the well definition of the SDE (1.1). This class of strategies is called the set of the admissible control processes and it is termed $\mathcal{A}^i$. We assume that the admissibility of a control does not depend on which specific agent is going to play it, i.e. $\mathcal{A}^i = \mathcal{A}$ for all $i = 1, \ldots, n$. Then, an admissible strategy profile γ for the game G_n, also called simply an admissible strategy, is an n-tuple $(\gamma^1, \ldots, \gamma^n)$ of admissible controls $\gamma^i \in \mathcal{A}$ for all $i = 1, \ldots, n$, collecting the control processes chosen by each player.

Observe that strategies can be distinguished between open-loop strategies, if they depend only on the time variable, i.e. $\gamma^i = \gamma^i(t)$, and feedback strategies, if the decision rule selects an action as a function also of the current state of the system x, i.e. $\gamma^i = \gamma^i(t, x)$. The more appropriate type depends on the context and in particular it is due to the information each player has: if any player has knowledge only of the initial state of the system and he cannot observe either the state of the system or the strategies chosen by the other players, it is natural to consider open-loop strategies, whereas, it is more suitable to consider feedback strategies if the players can observe the state of the system at any time.

Remark 1.1.1. In the following, we will require precise regularity conditions, both on the dynamics (1.1) and on the admissible strategies $\mathcal{A}$, so as to guarantee the existence of a unique strong solution to the SDEs (1.1).

Compared to the setting introduced in [Lac15a], which is a key reference for this Part I, the dynamics of all players in this game are given by jump-diffusion processes rather than continuous-time diffusion processes. This provides greater flexibility in the modeling of the players' dynamics.

In equation (1.1), the control of each player γ^i appears in the function β which multiplies the corresponding compensated Poisson Process $\widetilde{N}^i$, meaning that player i can affect the magnitude of the jumps appearing in his dynamics whereas, as formulated, no control is set on *when* these jumps occur. Indeed, the intensity function of the Poisson processes, $\nu(t)$, which is the same for all players, is not influenced by any control.

Remark 1.1.2. It should be pointed out that a control component could be also applied to the drift term b and to the diffusive component σ. This problem is already faced, and solved, in [Lac15a] and this is the main reason we skip it here. However, a more complete study is presented in [BCDP17a].

The dynamics of the n players are explicitly coupled together through the empirical distribution of their positions, μ^n. Although no strategy appears in the dynamics of player i but its own, the whole strategy profile $(\gamma^1, \ldots, \gamma^n)$ has an implicit impact on the dynamics $X^{i,n}$, i.e. $X^{i,n} = X^{i,n}(\gamma)$. Indeed, $X^{i,n}$ depends on the measure flow $\mu^n = (\mu^n_t)_{t\in[0,T]}$, whose evolution is in turn affected by the choices made by all the players in the game, since it depends on the state of all the agents. It will be relevant in the following observing that the empirical distribution is the only source of interaction among the evolution of the players' states, since this is the only way $X^{j,n}$ and γ^j may influence $X^{i,n}$ if $i \neq j$.

Each player takes part in the game in the hope of optimizing his outcome. The result of G_n relative to player i is given by the expected cost $J^{i,n}$, defined as

$$J^{i,n} = \mathbb{E}\left[\int_0^T f(t, X_t^{i,n}, \mu_t^n, \gamma_t^i)\, dt + g(X_T^{i,n}, \mu_T^n)\right], \tag{1.3}$$

and therefore player i aims to minimise it. As for the dynamics, also in the cost criterion, both the running cost function f and the terminal cost function g, which depend on the state and the strategy of the player and on the empirical distribution, are equal for each player, but then computed with respect to the different players' positions/strategies. Furthermore, by its definition, the expected cost faced by player i, $J^{i,n}$, depends on the opponents' choices, i.e. $J^{i,n} = J^{i,n}(\gamma^1, \dots, \gamma^n)$, due to (and only through) the empirical measure μ^n.

As pointed out before, by construction, both the dynamics of the private state, given in equation (1.1), and the cost functions, in equation (1.3), of each player depend on the opposing players only through the distribution of all the participants, μ^n. This kind of coupling is said to be of mean-field type. Inter alia, mean field interaction implies that the dependence on the opponents is anonymous: for each agent it is irrelevant which other particular player chooses which specific control but he cares only about the resulting aggregate state position. In other words, considering player i, a permutation of the other players' identities would lead unchanged the population distribution μ^n and therefore would not modify the game from his point of view since both his dynamics and his cost functions are invariant under such a permutation.

As defined, G_n is a non cooperative game, meaning that each agent pursues his own interest which in principle may conflict with the goals of the other players. In multi-person decision making problem, the meaning of *optimality* is not univocal, and in the following we will always consider Nash optimality.

Notation. Given an admissible strategy profile $\gamma = (\gamma^1, \dots, \gamma^n) \in \mathcal{A}^n$ and any admissible control $\eta \in \mathcal{A}$, (η, γ_{-i}) denotes a further admissible strategy where player i deviates from γ by playing η, wheres all the other players continue playing γ^j, $j \neq i$, i.e.

$$(\eta, \gamma_{-i}) = (\gamma^1, \dots, \gamma^{i-1}, \eta, \gamma^{i+1}, \dots, \gamma^n)\,.$$

Then,

Definition 1.1.1. *An admissible strategy profile $\gamma = (\gamma^1, \dots, \gamma^n) \in \mathcal{A}^n$ is a Nash equilibrium of the n-player game G_n if for each $i = 1, \dots, n$ and for any admissible strategy $\eta \in \mathcal{A}$*

$$J^{i,n}(\eta, \gamma_{-i}) \geq J^{i,n}(\gamma)\,. \tag{1.4}$$

This definition states that a strategy profile $\gamma = (\gamma^1, \dots, \gamma^n)$ is a Nash equilibrium for the game G_n if no player has the incentive to unilaterally deviate from it by playing any different admissible strategy. Indeed, considering any i, γ^i is a best response of player i if his opponents play accordingly to γ, i.e.

$$\gamma^i = \arg\min_{\eta \in \mathcal{A}} J^{i,n}(\eta, \gamma_{-i})$$

and a unilateral change would lead to a higher (or at least not lower) expected cost.

Sometimes, explicitly computing a Nash equilibrium of a game, or even proving its existence, is a too difficult task and therefore a slightly weaker equilibrium concept is introduced. Namely,

Definition 1.1.2. *For a given $\varepsilon \geq 0$, an admissible strategy profile $\gamma = (\gamma^1, \dots, \gamma^n) \in \mathcal{A}^n$ is an ε-Nash equilibrium of the n-player game G_n if for each $i = 1, \dots, n$ and for any admissible strategy $\eta \in \mathcal{A}$*

$$J^{i,n}(\eta, \gamma_{-i}) \geq J^{i,n}(\gamma) - \varepsilon \,. \tag{1.5}$$

Naturally, a Nash equilibrium of the game G_n is a 0-Nash equilibrium.

In other words, a strategy profile $(\gamma^1, \dots, \gamma^n)$ is an ε-Nash equilibrium if for each player in the game an unilateral change of his strategy when the others remain unchanged may lower the expected cost, but providing a maximum saving of ε.

Remark 1.1.3. In the two previous definitions, both Nash and ε-Nash equilibrium are defined with respect to open loop strategies. Analogously, these definitions can also be rewritten considering feedback strategies, but a clarification is necessary. Let $\gamma^i(t, x)$ be the feedback rule of player $i \neq 1$ and consider what happen when player 1 deviates: since the state processes depend on the mean measure of the system, they depend on the control of the deviating player and then, even if the feedback function γ^i is kept fixed, the resulting strategy $\gamma^i(t, X_t^{i,n})$ differs form the one in the initial scenario, violating the Definitions 1.1.1 and 1.1.2. In this case a different notion of equilibrium may be consider, the so called feedback Nash equilibrium.

By its definition, searching for a Nash equilibrium means solving, simultaneously, n optimization problems which are coupled together and, in turn, rely on n dynamic state processes that are also coupled together. Then, the difficulty of such a task is clear. Moreover, the complexity of this problem becomes larger and larger as the number of players increases. However MFG theory is a powerful tool to investigate the existence of a (approximate) Nash equilibrium at least for particular symmetric games when the number of the intervening agents is pretty large and the impact of one particular individual may be negligible compared to the influence of the overall population. We briefly present the fundamental underlying idea of this theory in the following section.

A crucial characteristic of these games G_n, which will allow tractability for the corresponding limiting game and it is a common requirement in MFG theory, is their *symmetry*. First, the population in the game is required to be of homogeneous players, meaning that the dynamics and the objectives of all the agents are provided by the same functions. This is our case since the functions b, σ and β appearing in the dynamics (1.1) and the cost functions f and g in equation (1.3) do not depend on the specific player, although then they are evaluated at the state of the related player. Second, as pointed out before, the interaction between the players is of mean-field type and therefore the opponents are *anonymous* since, for each player, a permutation of the other agents' identity does not modify the game.

1.2 The limiting game G_∞

In a game with a small number of players the position, and thus the strategy, of one single agent can significantly affect the distribution of the state across the players, μ^n, and therefore the outcome of the game. On the contrary, when the number of intervening agents in a homogeneous population grows to infinity, the behaviour of just one single player becomes morally negligible in the aggregate. In this case, large population condition can be exploited to develop efficient decision-rules by paying particular attention on the population behaviour rather than on each individual player's choice. See [HMC06]. Indeed, assuming that the population is distributed according to a given distribution, if the number of players is big enough, when a singular player deviates from his position in favor to a different one the population distribution does not move significantly. Therefore the deviation of just one player is not substantially felt by the other participants. This is the so called *decoupling effect*. Clearly, what has been said strongly depends on the fact that the interaction among the agents is of mean field type, otherwise this would not be true.

Therefore, in a symmetric game with a large homogeneous population and mean field interaction, it is possible to focus on just one representative player, say player p, and summarize the contribution of all his opponents through the population distribution, that is a measure flow $\mu = (\mu_t)_{t\in[0,T]}$, where μ_t is a probability distribution over the state space, in our case $\mu_t \in \mathcal{P}(\mathbb{R})$. The crucial point is that, since (at least theoretically) the impact of the choice of player p does not influence the population distribution, he can consider μ as a fixed deterministic function $\mu\colon [0,T] \to \mathcal{P}(\mathbb{R})$ when he searches for his optimal control among all the possible admissible strategies.

In the following we consider the naive, theoretical generalization of the previous game G_n to the the case when the number of players is *infinite*. We refer to this game as G_∞. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ be a filtered probability space satisfying the usual conditions, supporting a Brownian motion W and a Poisson process N with intensity $\nu(t)$. The state of the representative player p, $X = (X_t)_{t\in[0,T]}$, moves accordingly to

$$dX_t = b(t, X_t, \mu_t)\, dt + \sigma(t, X_t)\, dW_t + \beta(t, X_{t-}, \mu_{t-}, \gamma_t)\, d\widetilde{N}_t\,, \tag{1.6}$$

subjected to the initial condition

$$X_0 = \xi \sim \chi\,.$$

As before, assumptions granting the existence of a unique strong solution to the SDE (1.6) will be given in the following and for now we denote by $\mathcal{A}$ the admissible strategies that player p can choose from. Since different choices for the control process γ leads to different controlled dynamics, we will sometime stress this dependence by writing $X(\gamma)$ or X^γ to denote the solution to the SDE (1.6) under $\gamma \in \mathcal{A}$.

The effectiveness of an admissible strategy γ is evaluated accordingly to the expected cost of the game $J = J(\gamma)$, which is defined by

$$J = \mathbb{E}\left[\int_0^T f(t, X_t^\gamma, \mu_t, \gamma_t)\, dt + g(X_T^\gamma, \mu_T)\right]. \tag{1.7}$$

The definition of the expected cost J is equal to the one in the game G_n. However, since at this point the impact of the choice of player p no longer influences the population distribution being μ considered as fixed, then the outcome J depends directly only on the strategy γ chosen by the representative player p. Therefore, player p now faces a single-agent optimization problem and thus an admissible strategy $\hat{\gamma} \in \mathcal{A}$ is optimal if it attains the minimum of the expected cost, i.e.

$$J(\hat{\gamma}) = \inf_{\gamma \in \mathcal{A}} J(\gamma)\,.$$

Clearly, an optimal strategy $\hat{\gamma}$ depends by definition on the population distribution μ which is selected at the beginning. Therefore, it still depends on the choices of all the opponents that are summarised in this measure flow μ, but nevertheless, contrary to the previous case, they are kept fixed in this optimization step.

The subsequent step regards consistency for the choice of the measure flow μ that player p considers when optimizing. In a nutshell, due to the symmetry of the pre-limit game G_n the statistical properties of the representative player should approximate the empirical distribution generated by all the participants. Indeed, since all the infinite players in the game are identical (their dynamics solve the same SDE (1.6), their objectives matches and they interact symmetrically), being in the same situation, they would all act in the same way, meaning that they would all choose the same strategy. This means that an optimal strategy $\hat{\gamma}$ for player p is optimal also for all the other players when they are in place of p. Consequently, also the statistical distribution of the optimally controlled state $X^{\hat{\gamma}}$ would be the same for all the players, and therefore it must coincide with the population distribution, i.e.

$$\mathcal{L}(X^{\hat{\gamma}}) = \mu\,. \tag{1.8}$$

In other words, a fixed measure flow μ is consistent if the optimal behaviour of the representative player computed with respect to it, i.e. $\hat{\gamma}(\mu)$, generates this exactly measure flow μ. In economics, this is called the rational expectation hypothesis, here, in this game framework, we refer to condition (1.8) as the MFG consistency condition. Therefore,

Definition 1.2.1. *A mean field game solution for the game G_∞ is an admissible process $\hat\gamma \in \mathcal{A}$ that is optimal, meaning that $\hat\gamma \in \arg\min_{\gamma\in\mathcal{A}} J(\gamma)$, and, at the same time, it is such that the related controlled dynamics $X^{\hat\gamma}$ satisfies the MFG consistency condition $\mu_t = \mathcal{L}(X_t^{\hat\gamma})$ for all $t \in [0,T]$.*

A mean field game solution $\hat\gamma$ of G_∞ is said to be Markovian if there exists a measurable function $\hat\gamma\colon [0,T]\times\mathbb{R} \to A$ such that $\hat\gamma_t = \hat\gamma(t, X_{t-})$.

So, a MFG solution represents an *equilibrium relationship* between the individual strategies, required to be best responses to the infinite population behaviour, and the overall population distribution, required to be collectively determined by the players' strategies.

The existence of a mean field game solution for this game G_∞ is the main subject of Chapter 2.

In Chapter 3 we look for an approximate Nash equilibrium for the n-player game G_n, for any n large enough. The construction of these equilibria will strongly depend on the existence of a (regular enough) MFG solution of the limiting game G_∞. Indeed, there are two different ways to justify why G_∞ may be intended as the *limit* of the games G_n and therefore why we refer to G_∞ as *limiting game*: convergence or approximation results. The latter refers to the fact that a Markovian MFG solution for G_∞ may allow to construct approximate Nash equilibria for the corresponding n-player games, at least if the number of players n is large enough. In particular, we will show that if there exists a Markovian MFG solution $\hat\gamma = \hat\gamma(t, X_{t-})$ for G_∞, then the strategy profile $(\hat\gamma(t, X_{t-}^{1,n}(\hat\gamma)), \ldots, \hat\gamma(t, X_{t-}^{n,n}(\hat\gamma)))$ is a ε_n-Nash equilibrium for the corresponding game G_n, with the sequence ε_n satisfying $\varepsilon_n \to 0$ as $n \to \infty$. This approximation result is also practically relevant since a direct verification of the existence of Nash equilibria for n-player games when n is very large is usually not feasible. Furthermore, the computation of these possible equilibria is not even numerically feasible, due to the curse of dimensionality. See, e.g., [HMC06], [KLY11], [CD13a], [CD13b], [CL15] as well as the recent book [Car16] for further details.

On the other hand, the key question in the convergence approach is if and in which sense a sequence of Nash equilibria for the n-player games G_n converges towards a MFG solution of a limiting game. Assuming that for each n the game G_n admits a Nash equilibrium $\hat\gamma = (\hat\gamma^1, \ldots, \hat\gamma^n)$ then we expect that, at least heuristically, the empirical measure μ^n computed with respect to the optimally controlled processes $X^{i,n}(\hat\gamma)$ converges to a deterministic measure flow μ and that, in the light of the above observations, this measure μ coincides with the distribution of the optimally controlled state $X^{\hat\gamma}$. In other words, a mean field game solution $\hat\gamma$ for the game G_∞ should minimise

$$J = \mathbb{E}\left[\int_0^T f(t, X_t^{\hat\gamma}, \mathcal{L}(X_t^{\hat\gamma}), \hat\gamma_t)\, dt + g(X_T^{\hat\gamma}, \mathcal{L}(X_T^{\hat\gamma}))\right],$$

subjected to $X^{\hat\gamma}$ solving the McKean-Vlasov SDE

$$\begin{cases} dX_t^{\hat\gamma} = b(t, X_t^{\hat\gamma}, \mathcal{L}(X_t^{\hat\gamma})) + \sigma(t, X_t^{\hat\gamma})\, dW_t + \beta(t-, X_{t-}^{\hat\gamma}, \mathcal{L}(X_{t-}^{\hat\gamma}, \hat\gamma_{t-}))\, d\widetilde{N}_t\,, \\ X_0 = \xi\,. \end{cases}$$

These results are collected under the name *propagation of chaos*, see, e.g., [Szn91]. While the uncontrolled counter-part of MFG, that is particle systems and propagation of chaos for jump processes, has been thoroughly studied in the probabilistic literature (see, e.g., [Gra92; JMW08; ADPF]), MFGs with jumps have not attracted much attention so far. Indeed, most of the existing literature focuses on non-linear dynamics with continuous paths, with the exception of few papers such as [HAA14], [KLY11] and the more recent [CF17]. We do not address this problem for games G_n in the present work.

Chapter 2

Existence of a solution for the mean field game G_∞

In this chapter we study the stochastic differential game G_∞ introduced in the previous Chapter 1. Section 2.1 briefly recalls the mean field game G_∞, highlighting its main characteristics and the main difficulties involved in its study. To overcome these issues, the previous game is modified and re-written from the perspective of the relaxed controls. Section 2.2 contains the main result of this chapter, that is the existence of a relaxed mean field game solution for G_∞ whereas Section 2.3 investigates conditions guaranteeing that a relaxed Markovian mean field game solution can be built.

The novel contributions of what presented here are contained in [BCDP17a]. The main reference for this chapter is [Lac15a].

2.1 The relaxed MFG problem G_∞

As introduced in Chapter 1, the MFG G_∞ we are interest in is the following. Fixed a finite time horizon $T > 0$, let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ be a filtered probability space, which satisfies the usual conditions and supports a standard Brownian motion W and a Poisson process N with intensity function $\nu(t)$. These two processes W and N are assumed to be independent. The real-valued state variable X, which is controlled through the process γ, i.e. $X = X^\gamma$, follows the dynamics

$$dX_t = b(t, X_t, \mu_t)\, dt + \sigma(t, X_t)\, dW_t + \beta(t, X_{t-}, \mu_{t-}, \gamma_t)\, d\widetilde{N}_t, \quad t \in [0, T], \tag{2.1}$$

with initial condition $X_0 = \xi$ distributed according to a real-valued probability distribution $\chi \in \mathcal{P}(\mathbb{R})$. Here, μ represents a measure flow, meaning that $\mu : [0, T] \to \mu_t \in \mathcal{P}(\mathbb{R})$ is a given deterministic function, whose precise meaning will be explained in what follows.

An admissible control process, called also an admissible strategy, is any predictable control process $\gamma = (\gamma_t)_{t\in[0,T]}$ taking values in a fixed action space $A \subset \mathbb{R}$ and guaranteeing that the SDE (2.1) admits a unique strong solution. The set of all the admissible controls is denoted by $\mathcal{A}$. Then, a control process is chosen in order to minimise the expected cost

$$J(\gamma) = \mathbb{E}\left[\int_0^T f(t, X_t, \mu_t, \gamma_t)dt + g(X_T, \mu_T)\right],$$

and therefore $\hat{\gamma} \in \mathcal{A}$ is said to be optimal if

$$J(\hat{\gamma}) = \inf_{\gamma \in \mathcal{A}} J(\gamma). \tag{2.2}$$

According to Definition 1.2.1, an optimal strategy $\hat{\gamma}$ is a MFG solution for the game G_∞ if the related controlled state $\hat{X} = X(\hat{\gamma})$ satisfies the MFG consistency condition (1.8), that is $\mathcal{L}(\hat{X}_t) = \mu_t$ for all $t \in [0, T]$. Our aim is to prove the existence of such a solution.

The own definition of mean field game solution, Definition 1.2.1, provides a (possible) constructive *algorithm* to build a solution for the MFG G_∞. Indeed, this problem can be solved by splitting it into two parts:

1. **Optimization problem.** Consider a fixed measure flow μ and solve the stochastic minimisation problem $\inf_{\gamma\in\mathcal{A}} J(\gamma)$ finding the set of all the optimal strategies $\hat{\gamma} = \hat{\gamma}(\mu)$, say $\hat{\mathcal{A}}(\mu)$, which attain the minimum expected cost. Since μ is treated as an exogenous parameter, it is non affected by the choice of the control strategy $\hat{\gamma}$.
   Clearly, the existence of an optimal control is not granted for free and existence analysis has to be developed;

2. **Fixed point problem.** Find, if there exists, a fixed point of the correspondence

$$\Phi \colon \mu \mapsto \{\mathcal{L}(X_t^{\hat{\gamma}})_{t\in[0,T]} : \hat{\gamma} \in \hat{\mathcal{A}}(\mu)\}. \tag{2.3}$$

   If it exists, then the optimal control process $\gamma^* \in \hat{\mathcal{A}}(\mu)$ which provides the fixed point condition $\mathcal{L}(X^{\gamma^*}) = \mu$ is a MFG solution of the game G_∞.
   Observe that, at least theoretically, the optimization problem in the previous step should be solved for any measure flow μ.

The main difficulty in the present approach is to show the existence of a fixed point for the mapping Φ. Indeed, classical results require the continuity of Φ, which is hard to prove. Lacker in [Lac15a] and, independently, Fischer in [Fis+17] introduce a new powerful approach, the *martingale approach*, to avoid the direct study of the regularity of the correspondence Φ by means of the so called *relaxed controls*. The basic idea is to re-define the state variable and the controls on a suitable canonical space supporting all the randomness sources involved in the SDE (2.1), and identify the solution to the MFG G_∞ no longer with a stochastic process γ but with a probability measure P that can be seen as the joint law of the control-state pair. Therefore, finding a relaxed solution to the MFG above will boil down to finding a fixed point for a different suitably defined set-valued map, easier to study.

The rest of this section introduces the notation used throughout the chapter and sets up the main assumptions on the state variable and the cost functions as well as the precise definition of the relaxed mean field game G_∞.

2.1.1 Notation

A real-valued function defined on the time interval $[0,T]$, $x\colon [0,T]\to\mathbb{R}$, is said to be càdlàg if it is continuous from the right at all $t\in[0,T)$ and with finite left limit for all $t\in(0,T]$. The set of all real-valued càdlàg functions defined on $[0,T]$ will be denoted by $D=D([0,T];\mathbb{R})$. Then, it is well known that

Theorem 2.1.1. *Each $x\in D$ is a bounded, Borel measurable function with either finite or countably infinite discontinuities.*

Proof. See, e.g., [Whi07]. □

This space D can be endowed with the Skorohod topology J_1. Let Λ be the set of strictly increasing functions $\iota\colon[0,T]\to[0,T]$ such that ι, along with its inverse ι^{-1}, is continuous. Then for any $x,y\in D$, J_1-metric on D is defined by

$$d_{J_1}(x,y)=\inf_{\iota\in\Lambda}\{\|x\circ\iota-y\|_\infty\vee\|\iota-I\|_\infty\}\,,$$

where I denotes the identity map. J_1 denotes the topology induced by this metric. A peculiar property of the Skorohod J_1 topology is that whenever $x_n\to x$ with respect to J_1 then both the magnitudes and the locations of the jumps of x_n converge to those of x. Moreover, the space (D,J_1) is Polish. See [Whi07, Chapter 3-11-12] for further details on the càdlàg space and the Skorohod topology J_1.

Given any metric space (S,d), $\mathcal{B}(S)$ denotes the Borel σ-field of S induced by d. Then $\mathcal{P}(S)$ stands for the set of all probability measures defined on the measurable space $(S,\mathcal{B}(S))$. Furthermore, for any $p\geq 1$, $\mathcal{P}^p(S)\subset\mathcal{P}(S)$ denotes the set all probabilities on S such that $\int_S d(x,x_0)^pP(dx)<\infty$ for some (hence for all) $x_0\in S$. The space $\mathcal{P}^p(S)$ will be endowed with the Wasserstein metric $d_{W,p}$ that is defined for any $\mu,\eta\in\mathcal{P}^p(S)$ by

$$d_{W,p}(\mu,\eta)=\inf_{\pi\in\Pi(\mu,\eta)}\left(\int_{S\times S}|x-y|^p\,\pi(dx,dy)\right)^{\frac{1}{p}}\,, \tag{2.4}$$

where

$$\Pi(\mu,\eta)=\{\pi\in\mathcal{P}(S\times S):\pi\text{ has marginals }\mu,\eta\}\,.$$

More details on the Wasserstein metric can be found in [Vil08, Chapter 6].

For any measure $\mu\in\mathcal{P}^p(S)$ for S being either $\mathbb{R}$ or D we will use the notation

$$|\mu|^p=\int_{\mathbb{R}}|x|^p\mu(dx),$$
$$\|\mu\|_t^p=\int_D(|x|_t^*)^p\mu(dx),\quad |x|_t^*:=\sup_{s\in[0,t]}|x(s)|\,.$$

Moreover, unless otherwise stated, given two measurable spaces (S_1,Σ_1) and (S_2,Σ_2), the product space $S_1\times S_2$ will always be endowed with the product σ-fields $\Sigma_1\times\Sigma_2=\sigma(\{B_1\times B_2:B_1\in\Sigma_1\,,\,B_2\in\Sigma_2\})$.

2.1.2 Assumptions

In order to prove the existence of a solution for the MFG G_∞ the coefficient functions b, σ, β, the costs f, g, the initial distribution χ of the state process X and the intensity measure ν are required to be regular enough, that is to satisfy the following assumptions.

Assumption A. Let $p' > p \geq 1$ be given real numbers.

(A.1) The initial distribution χ belongs to $\mathcal{P}^{p'}(\mathbb{R})$.

(A.2) The intensity measure of the Poisson process $\nu\colon [0,T] \to \mathbb{R}^+$ is bounded, meaning that there exists a positive constant c_ν such that for all $t \in [0,T]$

$$|\nu(t)| \leq c_\nu \,.$$

(A.3) The coefficient functions $b\colon [0,T] \times \mathbb{R} \times \mathcal{P}^p(\mathbb{R}) \to \mathbb{R}$, $\sigma\colon [0,T] \times \mathbb{R} \to \mathbb{R}$ and $\beta\colon [0,T] \times \mathbb{R} \times \mathcal{P}^p(\mathbb{R}) \times A \to \mathbb{R}$, as well as the costs $f\colon [0,T] \times \mathbb{R} \times \mathcal{P}^p(\mathbb{R}) \times A \to \mathbb{R}$ and $g\colon \mathbb{R} \times \mathcal{P}^p(\mathbb{R}) \to \mathbb{R}$ are (jointly) continuous functions in all their variables.

(A.4) The functions b, σ and β are Lipschitz continuous with respect to the state variable and the mean measure, meaning that there exists a constant $c_1 > 0$ such that for all $t \in [0,T]$, $x, y \in \mathbb{R}$, $\mu, \eta \in \mathcal{P}^p(\mathbb{R})$ and $\alpha \in A$

$$\begin{aligned} |b(t,x,\mu) - b(t,y,\eta)| + |\sigma(t,x) - \sigma(t,y)| + |\beta(t,x,\mu,\alpha) - \beta(t,y,\eta,\alpha)| \\ \leq c_1 \left(|x-y| + d_{W,p}(\mu,\eta)\right) \end{aligned}$$

and in their whole domain satisfy the growth condition

$$|b(t,x,\mu)| + \left|\sigma^2(t,x)\right| + |\beta(t,x,\mu,\alpha)| \leq c_1 \left(1 + |x| + \left(\int_{\mathbb{R}} |z|^p \,\mu(dz)\right)^{\frac{1}{p}} + |\alpha|\right).$$

(A.5) The cost functions f and g satisfy the following growth conditions

$$-c_2 \left(1 + |x|^p + |\mu|^p\right) + c_3 \,|\alpha|^{p'} \leq f(t,x,\mu,\alpha) \leq c_2 \left(1 + |x|^p + |\mu|^p + |\alpha|^{p'}\right),$$

$$|g(x,\mu)| \leq c_2 \left(1 + |x|^p + |\mu|^p\right),$$

for each $t \in [0,T]$, $x \in \mathbb{R}$, $\mu \in \mathcal{P}^p(\mathbb{R})$ and $\alpha \in A$ for some positive constant $c_2 > 0$. Without loss of generality we can assume $c_1 = c_2 = c_\nu$.

(A.6) The control space A is a closed subset of $\mathbb{R}$.

The reason why Assumption A is required will be more clear in the proofs. However it is worth noting that conditions (A.2), (A.3) and (A.4) ensure the existence of a unique strong solution of the SDE (2.1) governing the evolution of the state variable. Furthermore, the (Lipschitz) continuity and the growth conditions are widely used in Lemma 2.2.1 and Lemma 2.2.2, which establish good compactness and continuity properties needed in the fixed point argument when the action space A is compact. Furthermore, along with Assumption (A.6), they are needed when extending the existence of a MFG solution to the unbounded case.

2.1.3 Relaxed controls and admissible laws

Let Γ be a measure on the set $[0,T]\times A$ equipped with the product σ-field $\mathcal{B}([0,T]\times A)$ such that its first marginal is given by the Lebesgue measure, meaning that $\Gamma([s,t]\times A) = t-s$ for all $0 \le s \le t \le T$, and its second marginal is a probability distribution over A. The set of all measures of this type and satisfying

$$\int_0^T \int_A |\alpha|^p \, \Gamma(dt, d\alpha) < \infty$$

will be denoted by $\mathcal{V}$, which is endowed with the *normalized* Wasserstein metric $d_{\mathcal{V}}$ defined by

$$d_{\mathcal{V}}(\Gamma, \Gamma') = d_{W,p}\left(\frac{\Gamma}{T}, \frac{\Gamma'}{T}\right).$$

Observe that, as soon as the action space A is compact, then also the complete separable metric space $\mathcal{V}$ is compact.

Let Ω^D denote the càdlàg space $D([0,T];\mathbb{R})$ and $\mathcal{F}^D$ the Borel σ-algebra induced on D by the Skorohod norm d_{J_1}. Then, the canonical map from this space $(\Omega^D, \mathcal{F}^D)$ into itself, which is given by

$$\begin{aligned} X \colon \Omega^D &\to D \\ \omega &\to X(\omega) = \omega\,, \end{aligned}$$

generates the canonical filtration $\mathcal{F}^X_t = \sigma\left(X_s, 0 \le s \le t\right)$. In the same way, the canonical map on $(\Omega^{\mathcal{V}}, \mathcal{B}([0,T]\times A)) = (\mathcal{V}, \mathcal{B}([0,T]\times A))$, defined as

$$\begin{aligned} \Gamma \colon \Omega^{\mathcal{V}} &\to \mathcal{V} \\ \omega &\to \Gamma(\omega) = \Gamma\,, \end{aligned}$$

provides the canonical filtration $\mathcal{F}^\Gamma_t = \sigma(\Gamma(F) : F \in \mathcal{B}([0,T]\times A))$. From now on, we refer to the product space $\mathcal{V}\times D$ endowed with the product σ-field $\mathcal{F}^X_t \otimes \mathcal{F}^\Gamma_t$ as the canonical filtered measurable space $(\hat{\Omega}, \hat{\mathcal{F}}, (\hat{\mathcal{F}}_t)_{t\in[0,T]})$. Then, a generic element of $\hat{\Omega}$ is a couple (Γ, X), and, with a slight abuse of notation, its projections onto $\mathcal{V}$ and D will still be denoted respectively by Γ and X.

Let L be the linear integro-differential operator defined on $C_0^\infty(\mathbb{R})$, i.e. the set of all infinitely differentiable functions $\phi \colon \mathbb{R}\to\mathbb{R}$ having compact support, by

$$\begin{aligned} L\phi(t,x,\mu,\Gamma) &= b(t,x,\mu)\phi'(x) + \frac{1}{2}\sigma^2(t,x)\phi''(x) \\ &\quad + \int_A [\phi(x+\beta(t,x,\mu,\alpha)) - \phi(x) - \beta(t,x,\mu,\alpha)\phi'(x)]\nu(t)\Gamma(d\alpha) \end{aligned} \tag{2.5}$$

for each $(t,x,\mu,\Gamma) \in [0,T]\times\mathbb{R}\times\mathcal{P}^p(\mathbb{R})\times\mathcal{P}(A)$. Moreover, for any $\phi \in C_0^\infty(\mathbb{R})$ and for any measure $\mu \in \mathcal{P}^p(D)$, let the operator $\mathcal{M}^{\mu,\phi}_t \colon \hat{\Omega} \to \mathbb{R}$ be defined by

$$\mathcal{M}^{\mu,\phi}_t(\Gamma, X) = \phi(X_t) - \int_0^t L\phi(s, X_{s-}, \mu_{s-}, \Gamma_s)\, ds\,, \quad t\in[0,T]\,. \tag{2.6}$$

Here $\mu_{t-} = \mu \circ \pi_{t-}^{-1}$ with $\pi_{t-} : D \to \mathbb{R}^d$ defined as $\pi_{t-}(x) = x_{t-}$ for $x \in D$. Notice that a.e. under the Lebesgue measure we have $\mu_{t-} = \mu_t$, where μ_t is defined similarly as the image of μ via the mapping $\pi_t : D \to \mathbb{R}^d$ given by $\pi_t(x) = x_t$, $x \in D$.

Definition 2.1.1. *Let μ be a measure in $(\mathcal{P}^p(D), d_{W,p})$ and P be a probability measure in $\mathcal{P}^p(\hat{\Omega})$ over the canonical filtered space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathcal{F}}_t)$. P is an admissible law with respect to μ if it satisfies the following conditions:*

1. $P \circ X_0^{-1} = \chi$*;*
2. $\mathbb{E}^P[\int_0^T |\Gamma_t|^p \, dt] < \infty$*;*
3. $\mathcal{M}^{\mu,\phi} = (\mathcal{M}_t^{\mu,\phi})_{t\in[0,T]}$ *is a P-martingale for each $\phi \in C_0^\infty(\mathbb{R})$.*

The set of all the admissible laws computed with respect to μ will be denoted by $\mathcal{R}(\mu)$.

Notation. Being X any random variable on the probability space $(\Omega, \mathcal{F}, P)$ with values in a measurable space $(S, \mathcal{B}(S))$, $P \circ X^{-1}$ represents the measure on $(S, \mathcal{B}(S))$ defined by

$$P \circ X^{-1}(B) = P(X \in B) \quad \forall B \in \mathcal{B}(S)\,.$$

Remark 2.1.1. According to Definition 2.1.1, $\mathcal{R}$ represents a correspondence which maps each probability measure $\mu \in \mathcal{P}^p(D)$ into the admissible probability measures P over $\hat{\Omega}$ which are consistent with it, i.e.

$$\begin{aligned} \mathcal{R}\colon \mathcal{P}^p(D) &\twoheadrightarrow \mathcal{P}^p(\hat{\Omega}) \\ \mu &\twoheadrightarrow \mathcal{R}(\mu) = \{P : P \text{ is an admissible law with respect to } \mu\}\,. \end{aligned} \tag{2.7}$$

Given any measure flow μ, $\mathcal{R}(\mu)$ is nonempty if the martingale problem (2.6) admits at least one solution. The latter is guaranteed by the fact that the SDE (2.1) has one strong solution due to the regularity Assumption A. Moreover, $\mathcal{R}(\mu)$ is a convex set. In fact, if Q is any convex combination of probability measures in $\mathcal{R}(\mu)$, that is $Q = aP_1 + (1-a)P_2$ with $a \in [0,1]$ and $P_1, P_2 \in \mathcal{R}(\mu)$, then Q is still an element of $\mathcal{R}(\mu)$. Indeed,

$$\begin{aligned} Q \circ X_0^{-1} &= (aP_1 + (1-a)P_2) \circ X_0^{-1} \\ &= aP_1 \circ X_0^{-1} + (1-a)P_2 \circ X_0^{-1} \\ &= a\chi + (1-a)\chi = \chi\,, \end{aligned}$$

and by linearity of the expectation, for any $0 \le s \le t \le T$

$$\begin{aligned} \mathbb{E}^Q\left[\mathcal{M}_t^{\mu,\phi} | \mathcal{F}_s\right] &= a\mathbb{E}^{P_1}\left[\mathcal{M}_t^{\mu,\phi} | \mathcal{F}_s\right] + (1-a)\mathbb{E}^{P_2}\left[\mathcal{M}_t^{\mu,\phi} | \mathcal{F}_s\right] \\ &= a\mathcal{M}_s^{\mu,\phi} + (1-a)\mathcal{M}_s^{\mu,\phi} = \mathcal{M}_s^{\mu,\phi}\,. \end{aligned}$$

Since the elements of $\mathcal{R}(\mu)$ are defined as solutions of the martingale problem associated to the operator L, an application of [EKM90, Theorem III-10] and [EKL77, Théorème 13] provides the following equivalent characterisation of P being an element of $\mathcal{R}(\mu)$.

Lemma 2.1.1. *Given a measure $\mu \in \mathcal{P}^p(D)$, the space of the admissible laws $\mathcal{R}(\mu)$ is the set of all probability measures Q on some filtered measurable space $(\Omega', \mathcal{F}', (\mathcal{F}'_t)_{t\in[0,T]})$ satisfying the usual conditions and supporting an $\mathcal{F}'_t$-adapted process X, an $\mathcal{F}'_t$-adapted Brownian motion B and a Poisson random measure N on $[0,T]\times A$ with mean measure $\nu(t)\,dt \times \Gamma_t(d\alpha)$, such that*

$$Q \circ X_0^{-1} = \chi$$

and the state process X satisfies the following equation

$$X_t = X_0 + \int_0^t b(s, X_s, \mu_s)ds + \int_0^t \sigma(s, X_s)dB_s + \int_0^t \int_A \beta(s, X_{s-}, \mu_{s-}, \alpha)\widetilde{N}(ds, d\alpha), \tag{2.8}$$

where, as usual, $\widetilde{N}$ denotes the compensated Poisson random measure, i.e. $\widetilde{N}(dt, d\alpha) = N(dt, d\alpha) - \nu(t)\Gamma_t(d\alpha)\,dt$.

Remark 2.1.2. In Lemma 2.1.1, the measurable space $(\Omega', \mathcal{F}')$ and the filtration $(\mathcal{F}'_t)_{t\in[0,T]}$ are not specified in advance. However, by definition, Γ is an element in $\mathcal{V}$ and the solution process X to equation (2.8) has càdlàg paths. Therefore, by considering the measurable map

$$\Omega' \ni \omega \mapsto (\Gamma(\omega), X(\omega)) \in \hat{\Omega} = \mathcal{V} \times D$$

we can induce a measure P' on the canonical space such that (Γ, X) has the same law under P' as it does under P. Thus, in the following, when we consider a $P \in \mathcal{R}(\mu)$, we may always assume that P is defined on the canonical space $(\hat{\Omega}, \hat{\mathcal{F}})$.

Any element $\Gamma \in \mathcal{V}$ is called a *relaxed control*. Indeed, in view of the previous lemma, choosing a probability $P \in \mathcal{R}(\mu)$ means choosing an intensity measure Γ for the Poisson random measure N. So, roughly speaking, the control is no longer a process γ in the function multiplying the Poisson Process in the state dynamics as in the classical game G_∞, see equation (2.1), but it is directly the intensity measure Γ of a Poisson measure, see equation (2.8). Hence the name relaxed control. Furthermore, a control $\Gamma \in \mathcal{V}$ is said to be strict if $\Gamma_t = \delta_{\gamma(t)}$ for some A-valued measurable stochastic process γ_t for $t \in [0,T]$, where δ_x denotes the Dirac delta function at the point x.

2.1.4 Relaxed mean-field game solutions

The next step is to generalize the optimization problem (2.2) to the new relaxed framework. For any measure $\mu \in \mathcal{P}^p(D)$, the corresponding cost function $\mathcal{C}^\mu : \hat{\Omega} \to \mathbb{R}$ is re-defined as

$$\mathcal{C}^\mu(\Gamma, X) = \int_0^T \int_A f(t, X_t, \mu_t, \alpha)\Gamma(dt, d\alpha) + g(X_T, \mu_T)\,. \tag{2.9}$$

Since at this point the measure μ is considered as fixed, the relaxed optimization problem consists in finding, for any $\mu \in \mathcal{P}^p(D)$, all the consistent admissible law $P^* \in \mathcal{R}(\mu)$ so that the expected cost under P^* is minimal, i.e.

$$\int_{\hat{\Omega}} \mathcal{C}^\mu \, dP^* = \inf_{P \in \mathcal{R}(\mu)} \int_{\hat{\Omega}} \mathcal{C}^\mu \, dP \,.$$

Then, in view of the previous discussion, an optimal measure $P^* \in \mathcal{R}(\mu)$ is a MFG relaxed solution if it guarantees that the corresponding state process X is distributed according to μ, that is the MFG consistency condition $P^* \circ X^{-1} = \mu$.

As for the classical setting, any relaxed MFG solution can be defined by a fixed point argument. Given a probability distribution $\mu \in \mathcal{P}^p(D)$, let the expected cost related to μ under $P \in \mathcal{P}(\hat{\Omega})$, $J(\mu, P)$, be defined by

$$\begin{aligned} J \colon P^p(D) \times \mathcal{P}(\hat{\Omega}) &\to \mathbb{R} \cup \{\infty\} \\ (\mu, P) &\mapsto J(\mu, P) = \mathbb{E}^P \left[\mathcal{C}^\mu\right] = \int_{\hat{\Omega}} \mathcal{C}^\mu \, dP \,, \end{aligned} \tag{2.10}$$

and let $\mathcal{R}^*$ be the correspondence which maps a measure flow μ into the set of the minimising probabilities consisted with it, i.e.

$$\begin{aligned} \mathcal{R}^* \colon P^p(D) &\twoheadrightarrow \mathcal{P}(\hat{\Omega}) \\ \mu &\twoheadrightarrow \mathcal{R}^*(\mu) = \arg \min_{P \in \mathcal{R}(\mu)} J(\mu, P) \,. \end{aligned} \tag{2.11}$$

Remark 2.1.3. Observe that $\mathcal{R}^*(\mu) \subset \mathcal{P}^p(\hat{\Omega})$ whenever $\mu \in \mathcal{P}^p(\mathcal{D})$. Indeed, by definition of the set $\mathcal{R}$, any $P \in \mathcal{R}$ satisfies $\mathbb{E}[\int_0^T |\Gamma_t|^p \, dt] < \infty$, hence $\mathbb{E}^P \left[(|X|_T^*)^p\right] < \infty$ in view of Lemma 2.2.3. Therefore $P \in \mathcal{P}^p(\Omega[A])$ and being $\mathcal{R}^* \subset \mathcal{R}$ the conclusion holds.

Therefore,

Definition 2.1.2. *A relaxed mean field game solution is a probability distribution $P \in \mathcal{P}(\hat{\Omega})$ providing a fixed point for the set-valued map*

$$\begin{aligned} \mathcal{E} \colon \mathcal{P}^p(D) &\twoheadrightarrow \mathcal{P}(D) \\ \mu &\twoheadrightarrow \mathcal{E}(\mu) = \{\, P \circ X^{-1} : P \in \mathcal{R}^*(\mu) \,\} \,. \end{aligned} \tag{2.12}$$

Or, in a more compact form, a relaxed MFG solution is any $P \in \mathcal{P}^p(\hat{\Omega})$ which satisfies

$$P \in \mathcal{R}^*(P \circ X^{-1}) \,.$$

A relaxed MFG solution is said to be Markovian if the $\mathcal{V}$-marginal of P, i.e. Γ, satisfies $P(\Gamma(dt, d\alpha) = dt\hat{\Gamma}(t, X_{t-})(d\alpha)) = 1$ for a measurable function $\hat{\Gamma} \colon [0,T] \times \mathbb{R} \to \mathcal{P}(A)$, whereas a relaxed MFG solution is said to be strict Markovian if $P(\Gamma(dt, d\alpha) = dt\delta_{\hat{\gamma}(t, X_{t-})}(d\alpha)) = 1$ for a measurable process $\hat{\gamma} \colon [0,T] \times \mathbb{R} \to A$.

Remark 2.1.4. The previous game can be generalized to the multidimensional case, meaning that the state X, the Brownian motion W and the Poisson N can be modeled as multidimensional processes. We present here the one dimensional case for simplicity.

2.2 Existence of a relaxed MFG solution

2.2.1 The bounded case

After introducing the relaxed setting, we are ready to prove the existence of such a relaxed solution for the limiting MFG G_∞ under the additional assumption of boundedness of the coefficients and compactness of the action space A. Namely,

Assumption B. The coefficients b, σ, β are bounded and the space of actions A is compact.

Then,

Theorem 2.2.1. *Under Assumptions A and B, there exists a relaxed solution for the relaxed mean field game G_∞.*

Due to Definition 2.1.2, proving the existence of a relaxed MFG solution to the relaxed MFG G_∞ means exhibiting a fixed point for the correspondence

$$\begin{aligned} \mathcal{E} \colon \mathcal{P}^p(D) &\twoheadrightarrow \mathcal{P}^p(D) \\ \mu &\twoheadrightarrow \mathcal{E}(\mu) = \{\, P \circ X^{-1} : P \in \mathcal{R}^*(\mu) \,\}\,. \end{aligned} \tag{2.13}$$

To this end, we will make use of the Kakutani-Fan-Glicksberg Theorem.

Theorem 2.2.2 (Kakutani-Fan-Glicksberg Theorem)**.** *Let K be a nonempty compact convex subset of a locally convex Hausdorff space, and let the correspondence $\varphi \colon K \twoheadrightarrow K$ have closed graph and nonempty convex values. Then the set of fixed points of φ is compact and nonempty.*

Proof. See, e.g., [AB06, Theorem 17.55]. □

In order to make the proof of Theorem 2.2.1 more readable, we break it into several parts. Some more technical results are collect in Subsection 2.2.3 at the end of this section.

As first step, we prove that

Lemma 2.2.1. *Under Assumption A and B, the set-valued correspondence $\mathcal{R}$ given in Definition 2.1.1 is continuous with relatively compact image $\mathcal{R}(\mathcal{P}(D)) = \bigcup_{\mu \in \mathcal{P}(D)} \mathcal{R}(\mu)$ in $\mathcal{P}(\hat{\Omega})$.*

Recall that

Definition 2.2.1. *A correspondence $\varphi \colon X \twoheadrightarrow Y$ between topological spaces is*

- *upper hemicontinuous at x if for every neighborhood U of $\varphi(x)$, there is a neighborhood V of x such that for each $z \in V$, $\varphi(z) \subset U$;*

- *lower hemicontinuous at x if for every open set U such that $\varphi(x) \cap U \neq \emptyset$ there is a neighborhood V of x such that if $z \in V$, then $\varphi(z) \cap U \neq \emptyset$;*
- *continuous at x if it is both upper and lower hemicontinuous at x. φ is continuous if it is continuous at each point $x \in X$.*

For further details see, e.g., [AB06, Chapter 17].

Before proving Lemma 2.2.1, we first show the relative compactness of a suitable set of probability measures which in turn will guarantee the relative compactness of the pushforward measures $\{P \circ X^{-1} : P \in \mathcal{R}(\mu)\}$, for any $\mu \in \mathcal{P}(D)$.

Proposition 2.2.1. *Let $c > 0$ be a given positive constant, $p' > p \geq 1$ and χ a probability law. $\mathcal{Q}_c \subset \mathcal{P}^p(\hat{\Omega})$ is defined as the set of laws $Q = P \circ (X, \Gamma)^{-1}$ of $\hat{\Omega}$-valued random variables defined on some filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, P)$ such that:*

1. *$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t + \int_A \beta(t, X_{t-}, \alpha)\widetilde{N}(dt, d\alpha)$, for a Brownian motion W and a random measure N with intensity $\Gamma_t(d\alpha)\nu(t)dt$, where ν is measurable and bounded by a constant c;*
2. *$P \circ X_0^{-1} \sim \chi$;*
3. *$b\colon [0, T] \times \mathbb{R} \to \mathbb{R}$, $\sigma\colon [0, T] \times \mathbb{R} \to \mathbb{R}$ and $\beta\colon [0, T] \times \mathbb{R} \times A \to \mathbb{R}$ are measurable functions such that*
$$|b(t, x)| + \left|\sigma^2(t, x)\right| + |\beta(t, x, \alpha)| \leq c(1 + |x| + |\alpha|)\,, \tag{2.14}$$
for all $(t, x, \alpha) \in [0, T] \times \mathbb{R} \times A$;
4. *$\mathbb{E}\left[|X_0|^{p'} + \int_0^T |\Gamma_t|^{p'}\,dt\right] \leq c$.*

Then $\mathcal{Q}_c$ is relatively compact in $\mathcal{P}^p(\hat{\Omega})$.

Proof. Since D is a Polish space under J_1 metric, Prokhorov's theorem (cf. [Bil13, Theorem 5.1, Theorem 5.2]) ensures that a family of probability measures on D is relatively compact if and only if it is tight. In order to prove the tightness, we will use the Aldous's criterion provided in [Bil13, Theorem 16.10]. By proceeding as in Lemma 2.2.3, there exists a constant $C = C(T, c, \chi)$ such that

$$\mathbb{E}^Q\left[(|X|_T^*)^2\right] \leq C\mathbb{E}^Q\left[1 + |X_0|^{p'} + \int_0^T |\Gamma_t|^{p'} dt\right],$$

which means that $\mathbb{E}^Q\left[(|X|_T^*)^2\right]$ is bounded by a constant which depends upon Q only through the initial distribution χ, which is the same for all $Q \in \mathcal{Q}_c$. Therefore

$$\sup_{Q\in\mathcal{Q}_c} \mathbb{E}^Q[(|X|_T^*)^p] < \infty\,. \tag{2.15}$$

Then we are left with proving that

$$\lim_{\delta \downarrow 0} \sup_{Q \in \mathcal{Q}_c} \sup_{\tau \in \mathcal{T}_T} \mathbb{E}^P \left[\left| X_{(\tau+\delta)\wedge T} - X_\tau \right|^p \right] = 0 \,, \tag{2.16}$$

where $\mathcal{T}_T$ denotes the family of all stopping times with values in $[0,T]$ almost surely. For each $Q \in \mathcal{Q}_c$ and each stopping time $\tau \in \mathcal{T}_T$, there exists a constant $\tilde{C}$ such that

$$\begin{aligned} \mathbb{E}^Q \left[\left| X_{(\tau+\delta)\wedge T} - X_\tau \right|^p \right] \leq \tilde{C} \mathbb{E}^Q & \left[\left| \int_\tau^{(\tau+\delta)\wedge T} b(t, X_t)\, dt \right|^p \right] \\ & + \tilde{C} \mathbb{E}^Q \left[\left| \int_\tau^{(\tau+\delta)\wedge T} \sigma(t, X_t)\, dW_t \right|^p \right] \\ & + \tilde{C} \mathbb{E}^Q \left[\left| \int_\tau^{\tau+\delta)\wedge T} \int_A \beta(t, X_{t-}, \alpha) \widetilde{N}(dt, d\alpha) \right|^p \right] . \end{aligned} \tag{2.17}$$

By applying Burkholder-Davis-Gundy inequality as in the proof of Lemma 2.2.3, there exists a constant $\bar{C}$ such that for any $Q \in \mathcal{Q}_c$ and $\tau \in \mathcal{T}_T$

$$\begin{aligned} \mathbb{E}^Q \left[\left| X_{(\tau+\delta)\wedge T} - X_\tau \right|^p \right] \leq \bar{C} \mathbb{E}^Q & \left[\left(\int_\tau^{(\tau+\delta)\wedge T} (1 + |X|_T^*) dt \right)^p \right] \\ & + \bar{C} \mathbb{E}^Q \left[\left(\int_\tau^{(\tau+\delta)\wedge T} (1 + |X|_T^*) dt \right)^{\frac{p}{2}} \right] \\ & + \bar{C} \mathbb{E}^Q \left[\left(\int_A \int_\tau^{(\tau+\delta)\wedge T} (1 + |X|_T^* + |\alpha|) \nu(t) \Gamma_t(d\alpha) dt \right)^{\frac{p}{2}} \right] . \end{aligned} \tag{2.18}$$

From this point onwards one can proceed as in the proof of [Lac15a, Proposition B.4], and exploiting the boundeded of the intensity ν and of $\sup_{Q \in \mathcal{Q}_c} \mathbb{E}^Q[(|X|_T^*)^p]$, see equation (2.15), and the regularity of Γ as assumed by condition (4) one has

$$\lim_{\delta \downarrow 0} \sup_{Q \in \mathcal{Q}_c} \sup_{\tau \in \mathcal{T}_T} \mathbb{E}^Q \left[\left| X_{(\tau+\delta)\wedge T} - X_\tau \right|^p \right] = 0 \,.$$

Hence Aldous' criterion applies and the proof is completed. □

Proof of Lemma 2.2.1. This proof is divided into three parts.

The image of $\mathcal{R}$ is relatively compact. Using [Lac15a, Lemma A.2], we prove that the range of the correspondence $\mathcal{R}$, i.e. $\mathcal{R}(\mathcal{P}^p(D))$, is relatively compact in $\mathcal{P}^p(\hat{\Omega})$ by proving that both $\{P \circ \Gamma^{-1} : P \in \mathcal{R}(\mathcal{P}^p(\hat{\Omega}))\}$ and $\{P \circ X^{-1} : P \in \mathcal{R}(\mathcal{P}^p(\hat{\Omega}))\}$ are relatively compact sets in $\mathcal{P}^p(\mathcal{V})$ and $\mathcal{P}^p(D)$ respectively. The compactness of $\{P \circ \Gamma^{-1} : P \in \mathcal{R}(\mathcal{P}^p(\hat{\Omega}))\}$ in $\mathcal{P}^p(\mathcal{V})$ equipped with the p-Wasserstein metric $d_{W,p}$ follows from the compactness of A, and therefore of $\mathcal{V}$. On the other hand, the compactness of $\{P \circ X^{-1} : P \in \mathcal{P}^p(\hat{\Omega})\}$ is due to Proposition 2.2.1 and the boundedness of b, σ and β.

$\mathcal{R}$ is upper hemicontinuous. In order to show that $\mathcal{R}$ is upper hemicontinuous, we prove that $\mathcal{R}$ is closed, i.e. its graph

$$\operatorname{Gr}\mathcal{R} = \{(\mu, \mathcal{R}(\mu)) : \mu \in \mathcal{P}^p(D)\}$$

is closed. Indeed, by the closed graph theorem, see, e.g., [AB06, Theorem 17.11], being closed and being upper hemicontinuous are equivalent properties for $\mathcal{R}$. By definition, $\mathcal{R}$ is closed if for each $\mu^n \to \mu \in \mathcal{P}^p(D)$ and for each convergent sequence $P^n \to P$ with $P^n \in \mathcal{R}(\mu^n)$, then $P \in \mathcal{R}(\mu)$. According to Definition 2.1.1, we have to show that $P \circ X_0^{-1} = \chi$ and that $\mathcal{M}^{\mu,\phi}$, defined as in (2.6) on the canonical filtered probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathcal{F}}_{t\in[0,T]}, P)$, is a P-martingale for all $\phi \in C_0^\infty(\mathbb{R})$.

The first condition is satisfied since convergence in probability implies convergence in distribution and therefore $X_0 \overset{d}{=} \lim_{n\to\infty} X_0^n$, whose law is given by χ.

Regarding the second condition, let $s, t \in [0,T]$ be such that $0 \le s \le t \le T$, and let h be any continuous, $\hat{\mathcal{F}}_s$-measurable, bounded function. Since for all n P^n belongs to $\mathcal{R}(\mu^n)$, $\mathcal{M}^{\mu^n,\phi}$ is a P^n-martingale on $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathcal{F}}_{t\in[0,T]})$ by construction, and therefore $\mathbb{E}^{P^n}[(\mathcal{M}_t^{\mu^n,\phi} - \mathcal{M}_s^{\mu^n,\phi})h] = 0$. Hence to prove that $\mathcal{M}^{\mu,\phi}$ is a P-martingale for all $\phi \in C_0^\infty(\mathbb{R})$ it suffices to prove that the following limit

$$\lim_{n\to\infty} \mathbb{E}^{P^n}[(\mathcal{M}_t^{\mu^n,\phi} - \mathcal{M}_s^{\mu^n,\phi})h] = E^P[(\mathcal{M}_t^{\mu,\phi} - \mathcal{M}_s^{\mu,\phi})h] \tag{2.19}$$

holds true for any h as before. By Taylor's theorem, the operator $L\phi$ is bounded on $C_0^\infty(\mathbb{R})$, since

$$\begin{aligned}
\|L\phi(t,x,\mu,\Gamma)\|_\infty &\le \left\|\phi'(x)b(t,x,\mu)\right\|_\infty + \left\|\frac{1}{2}\sigma^2(t,x)\phi''(x)\right\|_\infty \\
&\quad + \left\|\int_A \frac{|\phi''(x+\xi\beta(t,x,\mu,\alpha))|}{2}\beta^2(t,x,\mu,\alpha)\nu(t)\,\Gamma(d\alpha)\right\|_\infty \\
&\le C(\phi',\phi'')\left(\|b\|_\infty + \|\sigma\|_\infty^2 + \int_A \|\beta\|_\infty^2 \|\nu\|_\infty \Gamma(d\alpha)\right) \\
&\le C(c_1,\phi',\phi'') = C_\phi\,,
\end{aligned}$$

where ξ is a suitable parameter belonging $[0,1]$. This implies in turn that also $\mathcal{M}^{\mu^n,\phi}$ can be bounded, uniformly on n as

$$\left\|\mathcal{M}^{\mu^n,\phi}\right\|_\infty \le \max_{x\in\mathbb{R}} |\phi(x)| + TC_\phi = \bar{C}_\phi\,.$$

Furthermore, the global continuity of the functions b, σ, β and ν guarantees that also the function $L\phi$ is globally continuous for all test functions $\phi \in C_0^\infty(\mathbb{R})$, and , since, according to Assumptions A and B, all such coefficients are bounded, and b and β are Lipschitz continuous with respect to the variable μ uniformly in (t,x,Γ), then also $L\phi$

is Lipschitz with respect to $\mu \in \mathcal{P}(\mathbb{R})$. Indeed

$$
\begin{aligned}
&|L\phi(t,x,\mu,\Gamma) - L\phi(t,x,\eta,\Gamma)| \\
&\qquad \le |b(t,x,\mu) - b(t,x,\eta)| \left|\phi'(x)\right| \\
&\qquad\quad + \int_A \big[\phi(x+\beta(t,x,\mu,\alpha)) - \phi(x) - \beta(t,x,\mu,\alpha)\left|\phi'(x)\right| \\
&\qquad\qquad\quad - \phi(x+\beta(t,x,\eta,\alpha)) + \phi(x) + \beta(t,x,\eta,\alpha)\left|\phi'(x)\right|\big]\Gamma(d\alpha)\nu(t) \\
&\qquad \le |b(t,x,\mu) - b(t,x,\eta)| \left|\phi'(x)\right| \\
&\qquad\quad + \int_A \big[c_1 \left|\phi(x+\beta(t,x,\mu,\alpha)) - \phi(x+\beta(t,x,\eta,\alpha))\right| \\
&\qquad\qquad\quad + c_1 \left|\beta(t,x,\mu,\alpha) - \beta(t,x,\eta,\alpha)\right| \left|\phi'(x)\right|\big]\Gamma(d\alpha) \\
&\qquad \le |b(t,x,\mu,\alpha) - b(t,x,\eta,\alpha)| \left|\phi'(x)\right| \\
&\qquad\quad + c_1 \int_A |\beta(t,x,\mu,\alpha) - \beta(t,x,\eta,\alpha)| \big(\left|\phi'(x+\xi\beta(t,x,\mu,\alpha))\right| + \left|\phi'(x)\right|\big)\Gamma(d\alpha) \\
&\qquad \le C_{\phi'} d_{W,p}(\mu,\eta) + 2c_1 C_{\phi'} d_{W,p}(\mu,\eta) \le C(c_1, C_{\phi'}) d_{W,p}(\mu,\eta)
\end{aligned}
$$

where $\xi \in [0,1]$. Therefore we can conclude that

$$
(\mu, X, \Gamma) \mapsto \int L\phi(t, X_t, \mu_t, \Gamma_t)\, dt
$$

is continuous. Indeed, the continuity with respect to (X,Γ) is provided by Lemma 2.2.4, whereas the continuity with respect to μ is an application of Lemma 2.2.5 since by the previous computation it follows that

$$
|L\phi(t, X_t, \mu_t, \Gamma_t) - L\phi(t, X_t, \eta_t, \Gamma_t)| \le C d_{W,p}(\mu_t, \eta_t)\,.
$$

Therefore for each continuous, $\hat{\mathcal{F}}_s$-measurable, bounded function h

$$
\lim_{n\to\infty} \mathbb{E}^{P^n}\left[\left(\int L\phi(s, X^n_s, \mu^n_s, \alpha)\Gamma^n_s(d\alpha)ds\right) h\right] = \mathbb{E}^{P}\left[\left(\int L\phi(s, X_s, \mu_s, \alpha)\Gamma_s(d\alpha)ds\right) h\right]
$$

and moreover, since $P^n \to P$ by construction and ϕ is bounded and continuous, $P^n\phi \to P\phi$. Thus, we can conclude that for each continuous, $\hat{\mathcal{F}}_s$-measurable, bounded function h

$$
\mathbb{E}^P[(\mathcal{M}^{\mu,\phi}_t - \mathcal{M}^{\mu,\phi}_s)h] = \lim_{n\to\infty} \mathbb{E}^P[(\mathcal{M}^{\mu_n,\phi}_t - \mathcal{M}^{\mu_n,\phi}_s)h] = 0,
$$

which implies that $\mathbb{E}^P[\mathcal{M}^{\mu,\phi}_t - \mathcal{M}^{\mu,\phi}_s] = 0$, or, in other words, that $\mathcal{M}^{\mu,\phi}$ is a P-martingale.

Therefore, satisfying Definition 2.1.1 we can conclude that P is an admissible law with respect to μ, i.e. $P \in \mathcal{R}(\mu)$ and thus $\mathcal{R}$ has closed graph as requested.

$\mathcal{R}$ is lower hemicontinuous. Let $\mu \in \mathcal{P}^p(D)$ and μ^n be a sequence in the same space converging to μ. Then, to show that $\mathcal{R}$ is lower hemicontinuous, we need to exhibit

a sequence $P_n \in \mathcal{R}(\mu^n)$ such that $P_n \to P$ in $\mathcal{P}^p(\hat{\Omega})$ for every $P \in \mathcal{R}(\mu)$. Consider any $P \in \mathcal{R}(\mu)$, than Lemma 2.1.1 ensures that there exist a filtered probability space $(\Omega', \mathcal{F}', \{\mathcal{F}'_t\}_{t\in[0,T]}, P')$, a $\mathcal{F}'$-Brownian motion W and a Poisson random measure N on $[0,T]\times A$ with intensity measure $\Gamma_t(d\alpha)\,\nu(t)\,dt$ such that $P' \circ (\Gamma, X)^{-1} = P$, where X is the unique strong solution of the following SDE:

$$dX_t = b(t, X_t, \mu_t)\,dt + \sigma(t, X_t)\,dW_t + \int_A \beta(t, X_{t-}, \mu_{t-}, \alpha)\,\widetilde{N}(dt, d\alpha) \tag{2.20}$$

subjected to an initial condition X_0. The existence and uniqueness of strong solution of equation (2.20) is once again guaranteed by Assumption A.

Then, for each n, let X^n be the process solving

$$dX_t^n = b(t, X_t^n, \mu_t^n)\,dt + \sigma(t, X_t^n)\,dW_t + \int_A \beta(t, X_{t-}^n, \mu_{t-}^n, \alpha), \quad X_0^n = X_0\,,$$

and define P^n as $P^n = P \circ (\Gamma, X^n)^{-1}$. We want to show that $\mathcal{R}(\mu^n) \ni P_n \to P$. To this end, since convergence in L^p implies convergence in distribution, we will prove that

$$\mathbb{E}^{P'}\left[(|X^n - X|_t^*)^p\right] \to 0\,. \tag{2.21}$$

Let $\bar{p} = \max\{2, p\}$. Then, for a suitable positive constant $C > 0$ we have

$$\begin{aligned}|X_t^n - X_t|^{\bar{p}} \le C\left|\int_0^t |b(s, X_s^n, \mu_s^n) - b(s, X_s, \mu_s)|\,ds\right|^{\bar{p}} + C\left|\int_0^t |\sigma(s, X_s^n) - \sigma(s, X_s)|\,dW_s\right|^{\bar{p}} \\ + C\left|\int_0^t \int_A \left|\beta(s, X_{s-}^n, \mu_{s-}^n, \alpha) - \beta(s, X_{s-}, \mu_{s-}, \alpha)\right|\,\widetilde{N}(ds, d\alpha)\right|^{\bar{p}}.\end{aligned} \tag{2.22}$$

Being b Lipschitz continuous in x and μ due to Ass. (A.4), it holds

$$\begin{aligned}\mathbb{E}^{P'}\left[\int_0^t |b(s, X_s^n, \mu_s^n) - b(s, X_s, \mu_s)|^{\bar{p}}\,ds\right] \le C(c_1, \bar{p}, t)\bigg(\int_0^t \mathbb{E}^{P'}\left[(|X^n - X|_s^*)^{\bar{p}}\right] ds \\ + \int_0^t d_{W,p}^{\bar{p}}(\mu_s^n, \mu_s)ds\bigg).\end{aligned}$$

Regarding the stochastic integrals in (2.22), we can apply Jensen and Burkholder-Davis-Gundy inequalities to obtain

$$\begin{aligned}\mathbb{E}^{P'}\left[\left|\left|\int_0^u |\sigma(s, X_s^n) - \sigma(s, X_s)|\;dW_s\right|^{\bar{p}}\right|_t^*\right] &\le k\mathbb{E}^{P'}\left[\int_0^t c_1\,|X_s^n - X_s|^2\;ds\right] \\ &\le C(c_1, \bar{p})\int_0^t \mathbb{E}^{P'}\left[(|X^n - X|_s^*)^2\right]\,ds.\end{aligned}$$

and

$$\mathbb{E}^{P'}\left[\left(\left|\int_0^{\cdot}\int_A \left|\beta(t,X^n_{t-},\mu^n_{t-},\alpha)-\beta(t,X_{t-},\mu_{t-},\alpha)\right|\widetilde{\mathcal{N}}(dt,d\alpha)\right|^*_t\right)^{\bar p}\right]$$
$$\leq C(\bar p)\mathbb{E}\left[\int_0^t\int_A \left|\beta(s,X^n_{s-},\mu^n_{s-},\alpha)-\beta(s,X_{s-},\mu_{s-},\alpha)\right|^{\bar p}\nu(s)\Gamma_s(d\alpha)\right]$$
$$\leq C(\bar p,c_1)\left(\int_0^t \mathbb{E}^{P'}\left[(|X^n-X|^*_s)^{\bar p}\right]ds+\int_0^t d^{\bar p}_{W,p}(\mu^n_s,\mu_s)^{\bar p}ds\right),$$

where we used also the boundedness assumption over ν and the Lipschitz continuity of σ and β. Notice that the integral $\int_0^t d^{\bar p}_{W,p}(\mu^n_s,\mu_s)ds$ in the estimates above converges to zero as $n\to\infty$ due to Lemma 2.2.5. Therefore, combining the previous results and applying the Gronwall's inequality, the validity of equation (2.21) follows.

At this point we have found a sequence such that $P^n\to P$ in $\mathcal{P}^p(\hat\Omega)$, and to conclude, we have to show that, for each n, P^n is an element of $\mathcal{R}(\mu^n)$. P^n satisfies condition (1) in Definition 2.1.1 by construction, and condition (3), by applying Itô's formula to $\phi(X^n)$, for each $\phi\in C_0^\infty(\mathbb{R})$. $\square$

A crucial hypothesis to apply Kakutani-Fan-Glicksberg Theorem is the closed graph property for the correspondence $\mathcal{E}$. Berge's Theorem states that

Theorem 2.2.3 (Berge Maximum Theorem)**.** *Let $\varphi : X \twoheadrightarrow Y$ be a continuous correspondence between topological spaces with nonempty compact values, and suppose that $\phi : Gr\,\varphi\to\mathbb{R}$ is continuous. Let the real-valued function $m\colon X\to\mathbb{R}$ be defined by*

$$m(x)=\max_{y\in\varphi(x)}\phi(x,y)$$

and the correspondence $\eta\colon X\twoheadrightarrow Y$ by

$$\eta(x)=\{y\in\varphi(x):\phi(x,y)=m(x)\}.$$

Then, m is continuous and η has nonempty compact values. Furthermore, if Y is Hausdorff, then η is upper hemicontinuous.

Therefore, since we have already proved that $\mathcal{R}$ is a closed continuous correspondence with relatively compact range, and thus with compact values, the continuity of the expected cost J would ensure that also $\mathcal{R}^*$ is a continuous correspondence with nonempty values, and that $\mathcal{E}$ is upper hemicontinuous, which means that $\mathcal{E}$ has closed graph, see, once again, [AB06, Theorem 17.11].

Lemma 2.2.2. *The operator J*

$$J\colon P(D)\times\mathcal{P}^p(\hat\Omega)\to\mathbb{R}\cup\{\infty\}$$
$$(\mu,P)\mapsto J(\mu,P)=\mathbb{E}^P\left[\mathcal{C}^\mu\right]=\int_{\hat\Omega}\mathcal{C}^\mu\,dP$$

is upper hemicontinuous under Assumption A. If Assumption B is also in force, then J is continuous.

Proof. The upper hemicontinuity is an easy consequence of Lemmas 2.2.4 and 2.2.5, while the continuity follows from the compactness of A. More precisely, Lemma 2.2.4 is used to prove the hemicontinuity of $\mathcal{C}^\mu(X,\Gamma)$ in (X,Γ), while the one in μ is granted by Lemma 2.2.5. $\square$

We are now ready to prove the existence of a relaxed MFG solution for the relaxed mean field game G_∞.

Proof of Theorem 2.2.1. The existence of a relaxed MFG solution for G_∞ is proved by showing that the correspondence $\mathcal{E}$ defined in equation (2.13) admits a fixed point. To apply the Kakutani-Fan-Glicksberg fixed point theorem, we need to consider a restriction of $\mathcal{E}$ to a suitably nonempty, compact, convex domain. Therefore we look for a convex compact subset $\mathcal{D} \subset \mathcal{P}^p(D)$ containing $\mathcal{E}(\mathcal{D})$, and then we consider the restriction of $\mathcal{E}$ on $\mathcal{D}$, which will be denoted by $\mathcal{E}_\mathcal{D}$.

To construct such a domain $\mathcal{D}$, define $\mathcal{Q}$ as the set of the probability measures P in $\mathcal{P}^p(\hat{\Omega})$ such that:

(i) $X_0 \sim \chi$;

(ii) $\mathbb{E}^P\left[(|X|^*_T)^p\right] \leq C$, where $C = C(T, c_1, \chi)$ denotes the constant appearing in equation (2.28) of Lemma 2.2.3, which depends upon P only through the initial distribution χ;

(iii) X is adapted to a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ and satisfies
$$\mathbb{E}^P\left[\left(X_{(t+u)\wedge T} - X_t\right)^p |\mathcal{F}_t\right] \leq \bar{C}\delta \tag{2.23}$$
for $t \in [0,T]$ and $u \in [0,\delta]$, with $\bar{C}$ defined in equation (2.18), independently of P.

Convexity of $\mathcal{Q}$ follows by construction: consider $\tilde{P} = aP_1 + (1-a)P_2$ for $a \in [0,1]$ where $P_1, P_2 \in \mathcal{Q}$ with corresponding filtration $\mathbb{F}^1$ and $\mathbb{F}^2$ as in condition (iii) above. Conditions (i) and (ii) are easily satisfied by $\tilde{P}$ since the initial distribution χ is the same for all the probabilities and the constant C depends on them only through χ. Condition (iii) for $\tilde{P}$ also holds with the same constant $\bar{C}$ as in equation (2.23) and the filtration $\tilde{\mathbb{F}} = \mathbb{F}^1 \wedge \mathbb{F}^2$. Clearly, being $\mathcal{Q}$ convex, also $\overline{\mathcal{Q}}$ is convex.

Furthermore, $\mathcal{Q}$ is relatively compact in $\mathcal{P}^p(\Omega[A])$. Observe that $\mathcal{Q}$ is tight since it satisfies the sufficient criterion for tightness given in [Whi07, Lemma 3.11]. Indeed since the constant $\bar{C}$ in (2.23) is independent of P, it suffices to choose (in the notation of [Whi07]) $Z(\delta) = \bar{C}\delta$, and the tightness follows. Therefore $\mathcal{Q}$ is relatively compact, and hence its closure $\overline{\mathcal{Q}}$ for the p-Wasserstein metric is compact in $\mathcal{P}^p(\Omega[A])$.

We can now define $\mathcal{D}$ as
$$\begin{aligned}\mathcal{D} &= \{\, \eta \in \mathcal{P}^p(D) : \text{there exists } P \in \overline{\mathcal{Q}} \text{ such that } \eta = P \circ X^{-1} \,\} \\ &= \{\, \eta = P \circ X^{-1} : P \in \overline{\mathcal{Q}} \,\} \subset \mathcal{P}^p(D).\end{aligned}$$

Since $\overline{\mathcal{Q}}$ is compact and convex and $P \mapsto P \circ X^{-1}$ is a continuous function, linear with respect to convex combinations, also $\mathcal{D}$ turns out to be a convex, compact set.

In order to prove that the range of $\mathcal{E}_{\mathcal{D}}$ is contained in $\mathcal{D}$ we show that $\mathcal{R}(\mu) \subset \mathcal{P}(\mathcal{Q})$ for each $\mu \in \mathcal{P}^p(D)$, so that $\mathcal{E}(\mu) \subset \mathcal{D}$ for all measures μ and therefore $\mathcal{E}_{\mathcal{D}}(\mathcal{P}^p(D)) \subset \mathcal{D}$. Let $P \in \mathcal{R}(\mu)$, then it satisfies conditions (i) and (ii) by construction, see Definition 2.1.1 and Lemma 2.2.3. The validity of condition (iii) can be proved arguing as in Proposition 2.2.1. Indeed, using the same notation therein, we have that for each $u \in [0, \delta]$

$$\mathbb{E}^P \left[\left(X_{(t+u)\wedge T} - X_t \right)^2 \mid \mathcal{F}_t \right] \leq \bar{C} u \leq \bar{C} \delta \,,$$

giving the same bound as in (2.23) with constant $\bar{C}$, which does not depend on P. Since $\mathcal{R}(\mu)$ is nonempty, as shown in Remark 2.1.1, then also $\mathcal{Q}$ and therefore $\mathcal{D}$ are nonempty sets.

The last condition to apply the Kakutani-Fan-Glicksberg Theorem is to show that $\mathcal{E}$ is an upper hemicontinuous correspondence with non-empty convex values. As pointed out above, since Lemma 2.2.2 implies the joint continuity of function J, as defined in (2.10), and Lemma 2.2.1 assures that $\mathcal{R}$ is continuous and has nonempty compact values, the Berge Maximum Theorem provides that the correspondence $\mathcal{R}^*$ is indeed upper hemicontinuous with nonempty compact values. By continuity and linearity with respect to convex combinations of $\mathcal{P}^p(\hat{\Omega}) \ni P \mapsto P \circ X^{-1} \in \mathcal{P}^p(D)$ also $\mathcal{E}$ is an upper hemicontinuous correspondence with nonempty compact values. Moreover, by Remark 2.1.1, $\mathcal{R}(\mu)$ is a convex set for each $\mu \in \mathcal{P}(D)$, and hence by linearity with respect to convex combinations and continuity of $P \mapsto J(\mu, P)$ and of $P \mapsto P \circ X^{-1}$, also $\mathcal{R}^*(\mu)$ and $\mathcal{E}(\mu)$ are convex sets.

Since all the hypotheses of the Kakutani-Fan-Glicksberg fixed point theorem are satisfied, we can conclude that there exists a fixed point for the correspondence $\mathcal{E}_{\mathcal{D}}$, and therefore for $\mathcal{E}$, meaning that there exists a relaxed MFG solution for the relaxed game G_∞, given in Section 2.1.3. □

2.2.2 The unbounded case

In the previous Section 2.2.1, the existence of a MFG solution when b, σ and β are bounded and the action space A is compact is proven. Then, the next goal is to prove the same result under weaker hypotheses, namely when the coefficient functions b, σ and β have linear growth and the action space A is not necessarily compact. Namely,

Theorem 2.2.4. *Under Assumption A, there exists a relaxed MFG solution.*

As in [Lac15a, Section 5], the basic idea is to work with a bounded approximation of the coefficient functions, their truncated version, and then, by a convergence argument, to show that the limit of the mean field game solutions found in the truncated setting is indeed a solution for the unbounded case.

Notation. For any $n \geq 1$, let (b_n, σ_n, β_n) be the truncated version of the coefficients (b, σ, β), i.e. $b_n = \min\{b, n\}$ and analogously for σ_n and β_n. Moreover, we denote A_n the intersection of A with the interval centered at the origin with length $2r_n = 2\sqrt{n/2c_1}$, where we recall that c_1 is the constant appearing in Ass. (A.4) granting Lipschitz continuity as well as growth conditions on the coefficients of the state variable. Since A is closed by assumption, there exists n_0 such that for all $n \geq n_0$ the set A_n is nonempty and compact, hence the truncated data set $(b_n, \sigma_n, \beta_n, f, g, A_n)$ satisfies Assumptions A and B with the same constants c_i $(i = 1, 2, 3)$ independent of n.

Let $\mathcal{V}[A_n]$ be the set of measures satisfying the same requirements as the measures belonging to $\mathcal{V}$, as in Section 2.1.3, but with A_n replacing A, and then let $\Omega[A_n]$ be the product space $\mathcal{V}[A_n] \times D$, endowed with the corresponding product σ-field. Due to Theorem 2.2.1 in the previous section, for all $n \geq 1$ there exists a relaxed MFG solution corresponding to the data set $(b_n, \sigma_n, \beta_n, f, g, A_n)$, which can be viewed as a probability measure on $\hat{\Omega} = \Omega[A]$ since $\mathcal{P}(\Omega[A_n])$ can be naturally embedded in $\mathcal{P}(\Omega[A])$ due to the inclusion $A_n \subset A$.

Let L^n be the operator defined as L in equation (2.5) with the truncated data (b_n, σ_n, β_n) replacing (b, σ, β). Now, for any n, we can define the set of admissible laws $\mathcal{R}_n(\mu)$ as the set of all measures $P \in \mathcal{P}(\hat{\Omega})$ such that

1. $P(\Gamma([0, T] \times A_n^c) = 0) = 1$;
2. $P \circ X_0^{-1} = \chi$;
3. for all functions $\phi \in C_0^\infty(\mathbb{R})$, the process
$$\mathcal{M}_t^{\mu,\phi,n} := \phi(X_t) - \int_0^t L^n \phi(s, X_{s-}, \mu_{s-}, \Gamma_s) ds, \quad t \in [0, T],$$
is a P-martingale.

Likewise, we also define $\mathcal{R}_n^*(\mu) = \arg\max_{P \in \mathcal{R}_n(\mu)} J(\mu, P)$. Due to the embedding of $\mathcal{P}(\Omega[A_n])$ in $\mathcal{P}(\hat{\Omega})$, we can identify $\mathcal{R}_n(\mu)$ and $\mathcal{R}_n^*(\mu)$ with the set of admissible laws and optimal laws of the MFG with data $(b_n, \sigma_n, \beta_n, f, g, A_n)$, respectively. Finally, any relaxed MFG solution for the n-truncated data can be viewed as a probability $P_n \in \mathcal{R}_n^*(\mu^n)$ with $\mu^n \in \mathcal{P}^p(\mathcal{D})$ satisfying the mean-field condition $\mu^n = P_n \circ X^{-1}$.

We are now ready to give the proof of Theorem 2.2.4.

Proof of Theorem 2.2.4. This proof follows closely [Lac15a, Section 5], hence we give more details only on those parts which are jump-specific whereas sketching the main arguments. As previously said, the basic idea is to show that the limit of the MFG solutions found in the truncated setting, whose existence is granted by Theorem 2.2.1, is indeed a solution for the unbounded case.

Let $(P_n)_{n \geq 1}$ be a sequence of relaxed MFG solutions for the corresponding game with data $(b_n, \sigma_n, \beta_n, f, g, A_n)$. Proposition 2.2.1 ensures that this sequence $(P_n)_{n \geq 1}$ is relatively compact in $\mathcal{P}^p(\hat{\Omega})$. Firstly we prove that any $P \in \mathcal{P}^p(\hat{\Omega})$ limit of a convergent subsequence $(P_{n_k})_{k\geq 1}$ of $(P_n)_{n\geq 1}$ is an admissible law for μ, i.e. $P \in \mathcal{R}(\mu)$, where $\mu = P \circ X^{-1}$. By construction one has $\mu = \lim_k \mu^{n_k} = \lim_k P_{n_k} \circ X^{-1} = P \circ X^{-1}$, and then, in order to prove that $P \in \mathcal{R}(\mu)$, we need to show that $\mathcal{M}^{\mu,\phi}$ is a P-martingale for all $\phi \in C_0^\infty(\mathbb{R})$. For all $t \in [0,T]$, consider

$$\begin{aligned}\mathcal{M}_t^{\mu^n,\phi,n}(\Gamma,X) - \mathcal{M}_t^{\mu^n,\phi}(\Gamma,X) = \int_0^t & \left(b_n(s,X_s,\mu_s^n) - b(s,X_s,\mu_s^n)\right)\phi'(X_s) \\ & + \frac{1}{2}\left(\sigma_n^2(s,X_s) - \sigma^2(s,X_s)\right)\phi''(X_s) \\ & + \Bigg(\int_A [\phi(X_s + \beta_n(s,X_s,\mu_s^n,\alpha)) \\ & \qquad - \phi(X_s + \beta(s,X_s,\mu_s^n,\alpha)) \\ & \qquad - \beta_n(s,X_s,\mu_s^n,\alpha)\phi'(X_s) \\ & \qquad + \beta(s,X_s,\mu_s^n,\alpha)\phi'(X_s)]\, d\alpha\Bigg)\nu(s)ds.\end{aligned}$$

Exploiting the linear growth of the involved functions and arguing as in [Lac15a, Lemma 5.2], we obtain the following bounds. Regarding the term containg b, one has

$$\int_0^t \left(b_n(s,X_s,\mu_s^n) - b(s,X_s,\mu_s^n)\right)\phi'(X_s)ds \leq 2\left\|\phi'\right\|_\infty c_1 t Z_1 \mathbf{1}_{\{2c_1 Z_1 > n\}}$$

where

$$Z_1 = 1 + |X|_T^* + \left(\sup_{n\geq 1}\int_D (|z|_T^*)^p \mu^n(dz)\right)^{1/p}, \tag{2.24}$$

and analogously

$$\int_0^t \left(\sigma_n^2(s,X_s) - \sigma^2(s,X_s)\right)\phi''(X_s)ds \leq 2\left\|\phi''\right\|_\infty c_1 t Z_1 \mathbf{1}_{\{2c_1 Z_1 > n\}}.$$

Lastly, regarding the term coming from the jump part, we have the following estimates

$$\begin{aligned}&\left|\phi(X_s + \beta_n(s,X_s,\mu_s^n,\alpha)) - \phi(X_s + \beta(s,X_s,\mu_s^n,\alpha)) - (\beta_n - \beta)(s,X_s,\mu_s^n,\alpha)\phi'(X_s)\right| \\ &\quad \leq |\phi(X_s + \beta_n(s,X_s,\mu_s^n,\alpha)) - \phi(X_s + \beta(s,X_s,\mu_s^n,\alpha))| \\ &\qquad\qquad + \left|\left(\beta_n(s,X_s,\mu_s^n,\alpha) - \beta(s,X_s,\mu_s^n,\alpha)\right)\phi'(X_s)\right| \\ &\quad \leq C\left(2 + |\beta_n(s,X_s,\mu_s^n,\alpha) - \beta(s,X_s,\mu_s^n,\alpha)|\right)\mathbf{1}_{\{|\beta(s,X_s,\mu_s^n,\alpha)|>n\}},\end{aligned}$$

for some constant $C \geq \|\phi\|_\infty + \|\phi'\|_\infty$. Then, due to Ass. (A.4) and by definition of Z_1

in (2.24), taking $n \geq 2c_1$ yields that P_n-a.s.

$$\begin{aligned}\int_0^t \int_A C\left(2 + |\beta_n(s, X_s, \mu_s^n, \alpha) - \beta(s, X_s, \mu_s^n, \alpha)|\right) \mathbf{1}_{\{|\beta(s,X_s,\mu_s^n,\alpha)|>n\}} \Gamma_s(d\alpha)ds \\ \leq 2C \int_0^t \int_A (1 + c_1(Z_1 + |\alpha|))\, \mathbf{1}_{\{c_1(Z_1+|\alpha|)>n\}} \Gamma_s(d\alpha)ds \\ \leq 2C\tilde{c}_1 \int_0^t \int_A (1 + Z_1 + |\alpha|))\, \mathbf{1}_{\{2c_1 Z_1>n\}} \Gamma_s(d\alpha)ds \\ \leq 2C\tilde{c}_1 \left(t(1 + Z_1) + \int_0^t |\Gamma_s| ds \right) \mathbf{1}_{\{2c_1 Z_1>n\}},\end{aligned}$$

where we set $\tilde{c}_1 = c_1 \vee 1$. Therefore, combining the bounds above we obtain that for all $t \in [0, T]$ and P_n-a.s.

$$\left| \mathcal{M}_t^{\mu^n,\phi,n}(\Gamma, X) - \mathcal{M}_t^{\mu^n,\phi}(\Gamma, X) \right| \leq 6C\tilde{c}_1 \left(t(1 + Z_1) + \int_0^t |\Gamma_s| ds \right) \mathbf{1}_{\{2c_1 Z_1>n\}}$$

for some constant $C \geq \|\phi\|_\infty + \|\phi'\|_\infty + \|\phi''\|_\infty$. Since, arguing as in [Lac15a, Lemma 5.1], one has

$$\sup_n \mathbb{E}^{P_n} \left[\int_0^T |\Gamma_t|^{p'} dt \right] < \infty, \quad \sup_n \mathbb{E}^{P_n} \left[(|X|_T^*)^{p'} \right] = \sup_n \|\mu^n\|_T^{p'} < \infty \tag{2.25}$$

a standard application of Fatou's lemma implies that

$$\mathbb{E}^P \left[\int_0^T |\Gamma_t|^{p'} dt \right] < \infty. \tag{2.26}$$

Then, the estimates above implies the convergence

$$\mathbb{E}^{P_n} \left[\left| \mathcal{M}_t^{\mu^n,\phi,n}(\Gamma, X) - \mathcal{M}_t^{\mu^n,\phi}(\Gamma, X) \right| \right] \to 0, \quad n \to \infty.$$

Finally using the continuity of $\mathcal{M}_t^{\mu,\phi}(\Gamma, X)$ in (μ, Γ, X), granted by Lemma 2.2.4 and Lemma 2.2.5, the previous convergence result implies that

$$\mathbb{E}^P \left[\left(\mathcal{M}_t^{\mu,\phi}(\Gamma, X) - \mathcal{M}_t^{\mu,\phi}(\Gamma, X) \right) h \right] = \lim_{n\to\infty} \mathbb{E}^{P_n} \left[\left(\mathcal{M}_t^{\mu^n,\phi,n}(\Gamma, X) - \mathcal{M}_t^{\mu^n,\phi,n}(\Gamma, X) \right) h \right] = 0$$

for any continuous, bounded and $\hat{\mathcal{F}}_s$-measurable function h, which in turn implies that $\mathcal{M}^{\mu,\phi}$ is a P-martingale, and therefore $P \in \mathcal{R}(\mu)$ as requested.

Lastly, to conclude the proof, we need to show that the limit point P is optimal, i.e. $P \in \mathcal{R}^*(\mu)$. First of all, let P' be any element of $\mathcal{R}(\mu)$ with $J(\mu, P') < \infty$. Then, one can show that there exists a sequence of probabilities $P'_n \in \mathcal{R}_n(\mu^n)$ such that

$$J_{n_k}(\mu^{n_k}, P'_{n_k}) \to J(\mu, P'), \quad k \to \infty, \tag{2.27}$$

where J_n denotes the objective corresponding to the truncated data. Indeed, this can be shown as in the proof of [Lac15a, Lemma 5.3] by using Burkholder-Davis-Gundy inequalities and Ass. (A.4) to estimate the jump part. Therefore, since P_n is optimal for each n we have

$$J_n(\mu^n, P'_n) \geq J_n(\mu^n, P_n),$$

so that thanks to (2.27) and to the fact that J is lower hemicontinuous (see Lemma 2.2.2) we obtain

$$J(\mu, P) \leq \liminf_{k\to\infty} J_{n_k}(\mu^{n_k}, P_{n_k}) \leq \lim_{k\to\infty} J_{n_k}(\mu^{n_k}, P'_{n_k}) = J(\mu, P').$$

The optimality of P follows since P' is arbitrary.

Then, the proof is completed since we have exhibited an admissible law P satisfying the mean-field condition which is optimal. Hence P is a relaxed MFG solution. □

2.2.3 Technical results

In the following, $|Y|^*_t$ is used as a shortcut for $\sup_{s\in[0,t]} |Y_s|$. Then, we study the moments of the controlled state process X, given as a solution to the SDE (2.8).

Lemma 2.2.3. *Let $\bar{p} \in [p, p']$. Under Assumption A, there exists a constant $C = C(T, c_1, \chi, \bar{p})$ such that for any $\mu \in \mathcal{P}^p(D)$ and $P \in \mathcal{R}(\mu)$*

$$\mathbb{E}^P \left[(|X|^*_T)^{\bar{p}} \right] \leq C \left(1 + \|\mu\|^{\bar{p}}_T + \mathbb{E}^P \int_0^T |\Gamma_t|^{\bar{p}}\, dt \right). \tag{2.28}$$

As a consequence, $P \in \mathcal{P}^p(\hat{\Omega})$. Furthermore, if $\mu = P \circ X^{-1}$, we have

$$\|\mu\|^{\bar{p}}_T = \mathbb{E}^P \left[(|X|^*_T)^{\bar{p}} \right] \leq C \left(1 + \mathbb{E}^P \left[\int_0^T |\Gamma_t|^{\bar{p}} dt \right] \right).$$

Proof. In what follows, the value of the constant C may change from line to line, however we will indicate what it depends on.

Consider a given measure $\mu \in \mathcal{P}^p(D)$ and any related admissible law $P \in \mathcal{R}(\mu)$. By Lemma 2.1.1,

$$\begin{aligned} |X_t|^{\bar{p}} \leq C_{\bar{p}} |X_0|^{\bar{p}} + C_{\bar{p}} \left| \int_0^t b(s, X_s, \mu_s)\, ds \right|^{\bar{p}} + C_{\bar{p}} \left| \int_0^t \sigma(s, X_s) dB_s \right|^{\bar{p}} \\ + C_{\bar{p}} \left| \int_0^t \int_A \beta(s, X_{s-}, \mu_{s-}, \alpha) \widetilde{N}(ds, d\alpha) \right|^{\bar{p}}. \end{aligned} \tag{2.29}$$

Assume that $\bar{p} \geq 2$. Then, by Jensen's inequality and boundedness of function b, see Ass. (A.4), for all $t \in [0, T]$,

$$\begin{aligned} \left| \int_0^t b(s, X_s, \mu_s)\, ds \right|^{\bar{p}} &\leq C_{t,\bar{p}} \int_0^t \sup_{u\in[0,s]} |b(u, X_u, \mu_u)|^{\bar{p}}\, ds \\ &\leq C_{t,\bar{p}} \int_0^t \sup_{0\leq u\leq s} c_1^{\bar{p}} \left(1 + (|X|^*_s)^{\bar{p}} + \|\mu\|^{\bar{p}}_s \right) ds\,. \end{aligned}$$

Burkholder-Davis-Gundy inequality, see e.g. [Pro90, Theorem 48, Ch. IV.4], ensures that there exists a positive constant $C_{\bar{p}}$, not depending on X , such that the expected supremum of the Itô integral in (2.29) can be bounded as follows

$$\mathbb{E}\left[\left(\left|\int_0^u \sigma(s,X_s)\,dB_s\right|_t^*\right)^{\bar{p}}\right] \leq C_{\bar{p}}\,\mathbb{E}\left[\left(\int_0^t \left|\sigma^2(s,X_s)\right|\,ds\right)^{\frac{\bar{p}}{2}}\right]$$
$$\leq C_{t,\bar{p}}\,\mathbb{E}\left[\int_0^t c_1^{\frac{\bar{p}}{2}}\,(1+|X|_s^*)^{\frac{\bar{p}}{2}}\right],$$

where the last inequality is due to the growth condition of the function σ, see Ass. (A.4). Lastly, we have to take into account the integral $I_t = \int_{[0,t]\times A}\beta(s,X_{s-},\mu_{s-},\alpha)\widetilde{N}(ds,d\alpha)$ appearing in (2.29). Using again the Burkholder-Davis-Gundy inequality, it follows that

$$\mathbb{E}\left[\left(\left|\int_0^\cdot \int_A \beta(s,X_{s-},\mu_{s-},\alpha)\widetilde{N}(ds,d\alpha)\right|\right)^{\bar{p}}\right]$$
$$\leq C_{\bar{p}}\mathbb{E}\left[\left(\int_0^t \int_A \sup_{u\in[0,s]} \left|\beta(u,X_{u-},\mu_{u-},\alpha)\right|^2 \nu(s)\,\Gamma_s(d\alpha)\,ds\right)^{\frac{\bar{p}}{2}}\right]$$

and since β has linear growth in (x,μ,α) uniformly in t, see Ass. (A.4), and ν is bounded, see Ass. (A.2), it is found that

$$\mathbb{E}^P\left[(|I|_t^*)^{\bar{p}}\right] \leq C\mathbb{E}\left[\int_0^t \int_A c_1^{\frac{\bar{p}}{2}+1}\left(1+|X|_s^*+\left(\int_D (|z|_s^*)^p\mu(dz)\right)^{\frac{1}{p}}+|\alpha|\right)^{\frac{\bar{p}}{2}}\Gamma_s(d\alpha)\,ds\right].$$

Observe that, if $\bar{p}\in[1,2)$, the conclusion still holds since $|y|^{\bar{p}/2}\leq 1+|y|^{\bar{p}}$ and then arguing as before.

Combining all the previous estimates, we get that there exists a positive constant $C = C(t,c_1,\chi,\bar{p})$ such that

$$\mathbb{E}^P\left[(|X|_t^*)^{\bar{p}}\right] \leq C\mathbb{E}^P\left[1+|X_0|^{\bar{p}}+\int_0^t \left(1+\|\mu\|_s^{\bar{p}}+|\Gamma_s|^{\bar{p}}\right)ds\right], \quad t\in[0,T].$$

Hence the estimates (2.28) follows from an application of Gronwall's lemma. As a consequence, when $\mu = P\circ X^{-1}$, we have

$$\|\mu\|_t^{\bar{p}} = \mathbb{E}^P[(|X|_t^*)^{\bar{p}}] \leq C\mathbb{E}^P\left[|X_0|^{\bar{p}}+\int_0^t (1+2\,\|\mu\|_s^{\bar{p}}+|\Gamma_s|^{\bar{p}})ds\right],$$

so that another application of Gronwall's lemma gives the second estimate. □

For completeness, we provide some continuity results. Lemma 2.2.4 is an extension of [Lac15a, Corollary A.5], where the space of continuous functions is replaced with the Skorokhod space $D([0,T],E)$ of all càdlàg functions taking values in some metric space (E,ρ).

Lemma 2.2.4. *Let (E, ρ) be a complete separable metric space. Let $\phi : [0,T] \times E \times A \to \mathbb{R}$ be a jointly measurable function in all its variables and jointly continuous in $(x, \alpha) \in E \times A$ for each $t \in [0,T]$. Assume that for some constant $c > 0$ and some $x_0 \in E$ one of the following two properties is satisfied*

1. *$\phi(t, x, \alpha) \leq c(1 + \rho^p(x, x_0) + |\alpha|^p)$, for all $(t, x, \alpha) \in [0,T] \times E \times A$;*
2. *$|\phi(t, x, \alpha)| \leq c(1 + \rho^p(x, x_0) + |\alpha|^p)$, for all $(t, x, \alpha) \in [0,T] \times E \times A$.*

Hence, if (1) *(resp.* (2)*) is fulfilled, the following function*

$$D([0,T]; E) \times \mathcal{V} \ni (x, q) \mapsto \int_0^T \int_A \phi(t, x(t), \alpha) q(dt, d\alpha) \tag{2.30}$$

is upper hemicontinuous (resp. continuous).

Proof. We prove first that the function

$$D([0,T]; E) \times \mathcal{V} \ni (x, q) \mapsto \eta(dt, d\alpha, de) := \frac{1}{T} q(dt, d\alpha) \delta_{x(t)}(de) \in \mathcal{P}^p([0,T] \times E \times A) \tag{2.31}$$

is jointly continuous. Using [Lac15a, Prop. A.1] it suffices to show that when $(x^n, q^n) \to (x, q)$ in $D([0,T]; E) \times \mathcal{V}[A]$ as $n \to \infty$, we have $\int \phi d\eta^n \to \int \phi d\eta$ for all continuous functions $\phi : [0,T] \times E \times A \to \mathbb{R}$ such that $|\phi(t, x, \alpha)| \leq c(1 + \rho^p(x, x_0) + |\alpha|^p)$, for all $(t, x, \alpha) \in [0,T] \times E \times A$. We use the notation η^n for the measure associated to (x^n, q^n) as in (2.31). Since $D([0,T]; E) \times \mathcal{V}$ is equipped with the product topology, it suffices to prove separately the continuity in x and q. Both are consequences of an application of dominated convergence theorem. We consider the continuity in x, the one in q can be easily showed using similar arguments. Let $x^n \to x$ in $D([0,T]; E)$, hence $x^n(t) \to x(t)$ for all $t \in [0,T]$ where the limit function x is continuous (see [EK09, Proposition 5.2]), hence for a.e. $t \in [0,T]$ with respect to the Lebesgue measure and hence for $q(dt, A)$ as well. Therefore, since ϕ is jointly continuous, we have $\phi(t, x^n(t), \alpha) \to \phi(t, x(t), \alpha)$ for a.e. $(t, \alpha) \in [0,T] \times A$ with respect to the measure $q(dt, d\alpha)$. Moreover, notice that for some (hence for all) $x_0 \in E$ there exists a constant $C > 0$, that might change from line to line, such that

$$\begin{aligned} |\phi(t, x^n(t), \alpha)| &\leq C(1 + \rho^p(x_0, x^n(t)) + |\alpha|^p) \\ &\leq C\left(1 + \rho^p(x_0, x(t)) + \rho^p(x(t), x^n(t)) + |\alpha|^p\right). \end{aligned}$$

Moreover by [EK09, Proposition 5.3] there exists a sequence of time changes $\tau_n(t)$, i.e. strictly increasing continuous functions mapping $[0,T]$ onto $[0,T]$, such that

$$\lim_{n \to \infty} \sup_{t \in [0,T]} \rho(x^n(t), x(\tau_n(t))) = 0.$$

Hence for all $\varepsilon > 0$ we can choose n large enough so that for all $t \in [0, T]$

$$
\begin{aligned}
|\phi(t, x^n(t), \alpha)| &\le C\left(1 + \rho^p(x_0, x(t)) + \rho^p(x(t), x(\tau_n(t))) + \rho^p(x(\tau_n(t)), x^n(t)) + |\alpha|^p\right) \\
&\le C\left(1 + \sup_{s\in[0,T]} \rho^p(x_0, x(s)) + \sup_{s\in[0,T]} \rho^p(x(s), x(\tau_n(s))) + \varepsilon + |\alpha|^p\right) \\
&\le C\left(1 + 3\sup_{s\in[0,T]} \rho^p(x_0, x(s)) + \varepsilon + |\alpha|^p\right).
\end{aligned}
$$

Therefore, since $\sup_{s\in[0,T]} \rho^p(x_0, x(s))$ is bounded, uniformly in n, we can apply dominated convergence and conclude that

$$\int \phi(t, x^n(t), \alpha) q(dt, d\alpha) \to \int \phi(t, x(t), \alpha) q(dt, d\alpha), \quad n \to \infty.$$

Similarly we have the continuity with respect to q, which gives the announced joint continuity of the function η in (2.31). Finally, the upper hemicontinuity (resp. continuity) of the function in equation (2.30) is obtained by applying [Lac15a, Corollary A.4] (resp. [Lac15a, Lemma A.3]). □

Lemma 2.2.5. *Let $\{\mu^n\} \subseteq \mathcal{P}^p(D)$ a convergent sequence to $\mu \in \mathcal{P}^p(D)$. Then, for any $q \ge 1$*

$$\int_0^T d_{W,p}(\mu_t^n, \mu_t)^q \, dt \to 0, \quad n \to \infty.$$

Proof. Since convergence with respect to $d_{W,p}$ implies also weak convergence, Skorokhod's representation theorem ensures that there exist D-valued random variables X^n and X, defined on a common probability space $(\Omega, \mathcal{F}, P)$ such that

$$\mu^n = P \circ (X^n)^{-1} \text{ and } \mu = P \circ X^{-1},$$

with

$$d_{J_1}(X^n, X) \to 0, \quad P \text{ a.s.}$$

By triangular inequality

$$|X_t^n - X_t|^p \le 2^p (d_{J_1}(X^n, 0)^p + d_{J_1}(X, 0)^p)$$

and

$$d_{J_1}(X^n, 0)^p + d_{J_1}(X, 0)^p \to 2 d_{J_1}(X, 0)^p, \quad n \to \infty,$$

where $d_{J_1}(X, 0)^p = (|X|_T^*)^p \in L^1(P)$, which does not depend on t. Then, since convergence $X^n \to X$ in J_1 implies convergence a.e $t \in [0, T]$, by applying a slightly more general version of dominated convergence (e.g. [Kal06, Theorem 1.21]), we have

$$\mathbb{E}\left[|X_t^n - X_t|^p\right] \to 0 \quad \text{a.e. } t \in [0, T].$$

Then, by applying dominated convergence once again in view of

$$\mathbb{E}^P \left[d_{J_1}(X^n, 0)^p + d_{J_1}(X, 0)^p\right] \to 2 \int_D d_{J_1}(x, 0)^p \mu(dx) < \infty\,,$$

it is found that

$$\int_0^T \left(\mathbb{E}\left[|X_t^n - X_t|^p\right]\right)^q\, dt \to 0\,, \quad n \to \infty. \qquad \square$$

2.3 Existence of a relaxed Markovian MFG solution

In the previous section, Theorem 2.2.4 ensures that, under suitable assumptions, there exists a relaxed MFG solution for the relaxed game G_∞. The following step is to prove that for any admissible law $P \in \mathcal{R}(\mu)$ it is possible to define a Markovian control $P^* \in \mathcal{R}(\mu)$ with a lower cost than P. This would imply that G_∞ admits also a Markovian MFG solution. To this end some further assumptions on the function β is required. Namely,

Theorem 2.3.1. *Assume that Assumption A holds true and that β satisfies for all $(t, x, \mu, \alpha) \in [0, T] \times \mathbb{R} \times \mathcal{P}^p(\mathbb{R}) \times A$*

$$|\beta(t, x, \mu, \alpha)| \leq c(1 + \psi(\alpha)), \tag{2.32}$$

for some continuous function $\psi : A \to (0, \infty)$ and constant $c > 0$. Then there exists a relaxed Markovian MFG solution to the relaxed game G_∞.

Proof. Let $P \in \mathcal{R}(\mu)$ be a relaxed MFG solution for G_∞, whose existence under Assumption A is guaranteed by Theorem 2.2.4. To show the existence of a Markovian MFG solution we build a (possibly different) probability measure $P^* \in \mathcal{R}(\mu)$, for the same measure flow $\mu \in \mathcal{P}(D)$, satisfying the following three properties:

Property MP. (MP.1) $J(\mu, P^*) \leq J(\mu, P)$;

(MP.2) $P^* \circ X_t^{-1} = P \circ X_t^{-1}$ for all $t \in [0, T]$;

(MP.3) $P^*(\Gamma(dt, d\alpha) = \hat{\Gamma}(t, X_{t-})(d\alpha)dt) = 1$ for a measurable function $\hat{\Gamma}\colon [0, T] \times \mathbb{R} \to \mathcal{V}$.

Being P optimal, i.e. $P \in \mathcal{R}^*(\mu)$, $J(\mu, P)$ is the minimum of the expected cost $J(\mu, \cdot)$ related to the flow measure μ, and therefore condition (MP.1) implies that also P^* attains its minimum, meaning that also P^* is an optimal admissible law, i.e. $P^* \in \mathcal{R}^*(\mu)$. The second property (MP.2) ensures that P^* satisfies the MFG consistency condition $\mu = P^* \circ X^{-1}$, and thus along with the previous one guarantees that also P^* is a relaxed MFG solution for G_∞. Condition (MP.3) is indeed the Markovian property for P^*.

The assumption (2.32) assures that the operator L defined in (2.5) satisfies assumptions (i)-(vi) in [KS98, pp. 611-612]. In [KS98], the authors establish that under these conditions for any solution to the controlled martingale problem for the generator L there exists another solution having a Markov control which has the same state and control distribution as the initially given one. Therefore, since by Definition 2.1.1 $P \in \mathcal{R}(\mu)$ means that P is a solution to the martingale problem for L, then [KS98, Corollary 4.9] guarantees the existence of a process Z, defined on some filtered probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t)_{t\in[0,T]}, Q)$ and a measurable function $\hat{\Gamma}\colon [0,T] \times \mathbb{R} \to \mathcal{V}$ such that

$$\mathcal{M}_t^{\mu,\phi}(\hat{\Gamma}, Z) = \phi(Z) - \int_0^t L\phi(s, Z_{s-}, \mu_{s-}, \hat{\Gamma}(s, Z_{s-}))\, ds$$

is a $\widetilde{\mathcal{F}}_t$-adapted Q-martingale for all $\phi \in C_0^\infty(\mathbb{R})$, and for all $t \in [0,T]$ it holds that

$$Q \circ (Z_t, \hat{\Gamma}(t, Z_t))^{-1} = P \circ (X_t, \mathbb{E}^P[\Gamma_t|X_t])^{-1}\,. \tag{2.33}$$

Define $P^* = Q \circ (\hat{\Gamma}(t, Z_t)dt, Z)^{-1}$. By construction, P^* belongs to $\mathcal{R}(\mu)$ and it satisfies conditions (MP.2) and (MP.3). Moreover it holds that

$$\begin{aligned}
J(\mu, P^*) &= \mathbb{E}^Q\left[\int_0^T \int_A f(t, Z_t, \mu_t, \alpha)\hat{\Gamma}(t, Z_t)(d\alpha)dt + g(Z_T, \mu_T)\right]\\
&\overset{(a)}{=} \mathbb{E}^P\left[\int_0^T \int_A f(t, X_t, \mu_t, \alpha)\mathbb{E}^P[\Gamma_t(d\alpha)|X_t]dt + g(X_T, \mu_T)\right]\\
&\overset{(b)}{=} \mathbb{E}^P\left[\int_0^T \int_A f(t, X_t, \mu_t, \alpha)\Gamma_t(d\alpha)dt + g(X_T, \mu_T)\right]\\
&= J(\mu, P)
\end{aligned}$$

Equality (a) follows from the equivalent distribution of the processes involved, i.e. $Q \circ Z_t^{-1} = P \circ X_t^{-1}$ and $Q \circ \hat{\Gamma}(t, Z_t)^{-1} = P \circ \mathbb{E}^P[\Gamma_t|X_t]^{-1}$ for any time $t \in [0,T]$, see equation (2.33), whereas equality (b) is just the tower property of conditional expectations. Therefore P^* satisfies also condition (MP.1) and the proof is complete. □

Theorem 2.2.4 and Theorem 2.3.1 guarantee the existence of a relaxed MFG solution and of a Markovian relaxed MFG solution, respectively, for the relaxed MFG G_∞ under suitable assumptions, namely Assumption A. Assume that, in addition to Assumption A, it is verified that

Assumption C. For all $(t, x, \mu) \in [0,T] \times \mathbb{R} \times \mathcal{P}(\mathbb{R})$, the set

$$K(t, x, \mu) = \{(\beta(t, x, \mu, \alpha), z) : \alpha \in A, z \leq f(t, x, \mu, \alpha)\} \subset \mathbb{R} \times \mathbb{R}$$

is convex.

Then it is possible to prove the existence also of a strict (strict Markovian, respectively) relaxed MFG by applying the same arguments as in the [Lac15a, Theorem 3.7]. More details on the jump-specific parts can be found in [BCDP17a].

Chapter 3

Existence of an ε-Nash equilibrium for the game G_n

In this chapter we exploit the existence of a MFG solution for the game G_∞, addressed in Chapter 2, to build ε-Nash equilibria for the corresponding prelimit games G_n. Denoting by $\gamma(t,x)$ a Markovian MFG solution for the limiting game G_∞, an ε_n-Nash equilibrium for the game G_n is obtained when each player i follows the same strategy γ but computed with respect his own state $\gamma(t, X^{i,n}_{t-})$, and this sequence approximates a (true) Nash equilibrium as $n \to \infty$, meaning that the sequence ε_n vanishes as $n \to \infty$.

In Section 3.1 we briefly recall how the games G_n are defined, introducing the notation and the main assumptions used throughout the whole chapter, whereas in Section 3.2 we prove the existence of an ε_n-Nash equilibrium to any G_n, with $\varepsilon_n \to 0$ as $n \to \infty$.

The original contributions of this chapter may be found in [BCDP17b]. The approximation scheme is inspired by [CF17].

3.1 Notation and Assumptions

As introduced more in details in Chapter 1, we recall the definition of the mean-field interaction game with n-player G_n and the infinite-player version G_∞ under study. Differently by the setting introduced in the previous chapters, here we assume that $\beta(t,x,\mu,\gamma) = \beta(\mu,\gamma)$ to simplify the computation in what follows.

The mean-field interaction game G_n. Each player $i = 1,\dots,n$ solves the optimization problem

$$\inf_{\gamma\in\mathcal{A}} \left\{ J^{i,n}(\gamma) = \mathbb{E}\left[\int_0^T f(t, X^{i,n}_t(\gamma), \mu^n_t(\gamma), \gamma^i_t)\, dt + g(X^{i,n}_T(\gamma), \mu^n_T(\gamma))\right]\right\}, \tag{3.1}$$

$$\text{s.t.} \begin{cases} dX^{i,n}_t(\gamma) = b(t, X^{i,n}_t, \mu^n_t)\, dt + \sigma(t, X^{i,n}_t)\, dW^i_t + \beta(\mu^n_{t-}, \gamma^i_t)\, d\widetilde{N}^i_t, \\ X^{i,n}_0 = \xi^i \sim \chi, \end{cases} \tag{3.2}$$

where the n Brownian motions, the n Poisson processes with the same intensity function $\nu(t)$ and the initial conditions ξ_i are mutually independent. γ^i represents the strategy of player i which is required to belong to the set of the admissible processes $\mathcal{A}$, that is the set of the A-valued predictable processes. Then, an admissible strategy profile γ for the game G_n is any n-tuple of admissible controls $\gamma^i \in \mathcal{A}$ for all i, i.e. $(\gamma^1, \dots, \gamma^n) \in \mathcal{A}^n$. Furthermore, μ^n stands for the empirical distribution of the state $X^n = (X^{1,n}, \dots, X^{n,n})$, meaning that at any time $t \in [0,T]$

$$\mu_t^n(\gamma) = \sum_{i=1}^{n} \frac{1}{n} \delta_{X_t^{i,n}(\gamma)} . \tag{3.3}$$

We sometimes write $X^{i,n}(\gamma)$, $\mu^n(\gamma)$ and $J^{i,n}(\gamma)$ to stress that the state, and thus the empirical distribution of the system and the expected cost of the game G_n of each player i depend not only on his own control γ^i but also on the decision rule of the other participants.

The mean field game G_∞. A process $\hat{\gamma} \in \mathcal{A}$ is a MFG solution to G_∞ if

$$\hat{\gamma} = \arg\min_{\gamma \in \mathcal{A}} \left\{ J(\gamma) = \mathbb{E}\left[\int_0^T f(t, Y_t(\gamma), \hat{\mu}_t, \gamma_t)\, dt + g(Y_T(\gamma), \hat{\mu}_T) \right] \right\} \tag{3.4}$$

$$\text{s.t.} \quad \begin{cases} dY_t(\gamma) = b(t, Y_t, \hat{\mu}_t)\, dt + \sigma(t, Y_t)\, dW_t + \beta(\hat{\mu}_{t-}, \gamma_t)\, d\widetilde{N}_t \\ Y_0 = \xi \sim \chi \end{cases} \tag{3.5}$$

and, at the same time, $\hat{\gamma}$ satisfies the MFG consistency condition

$$\hat{\mu}_t = \mathcal{L}(Y_t(\hat{\gamma})) \tag{3.6}$$

at any time $t \in [0,T]$. Furthermore, as stated in Definition 2.1.2, a mean-field solution $\hat{\gamma}$ of G_∞ is said to be Markovian if $\hat{\gamma}_t = \hat{\gamma}(t, Y_{t-})$ for a measurable function $\hat{\gamma} \colon [0,T] \times \mathbb{R} \to \mathbb{R}$.

For the games to be well-defined and to find an approximate Nash equilibrium for the n-player game G_n we have to require some integrability of the initial conditions of the state processes as well as some regularity on the functions

$$b \colon [0,T] \times \mathbb{R} \times \mathcal{P}^2(\mathbb{R}) \to \mathbb{R}\,, \;\; \sigma \colon [0,T] \times \mathbb{R} \to \mathbb{R}\,, \;\; \beta \colon \mathcal{P}^2(\mathbb{R}) \times A \to \mathbb{R}\,, \;\; \nu \colon [0,T] \to \mathbb{R}^+,$$
$$f \colon [0,T] \times \mathbb{R} \times \mathcal{P}^2(\mathbb{R}) \times A \to \mathbb{R}\,, \quad g \colon \mathbb{R} \times \mathcal{P}^2(\mathbb{R}) \to \mathbb{R}\,.$$

Assumption D. (D.1) The initial distribution χ belongs to $\mathcal{P}^q(\mathbb{R})$ for some $q > 2$, $q \neq 4$.

(D.2) A is a compact subset of $\mathbb{R}$. $\sup_{a \in A} |a|$ will be denoted by $\alpha_M < \infty$.

(D.3) The intensity function ν is bounded, i.e. there exists a positive constant M_ν satisfying $\|\nu\|_\infty \leq M_\nu$.

(D.4) b is a Lipschitz function both in x and μ, σ is Lipschitz in x, and β is Lipschitz in μ and γ. Namely, there exist positive constants L_b, L_σ and L_β such that for all $x, y \in \mathbb{R}$, $\mu, \eta \in \mathcal{P}^2(\mathbb{R})$, $\gamma, \lambda \in A$ and $t \in [0, T]$,

$$\begin{aligned} |b(t,x,\mu) - b(t,y,\eta)| &\le L_b\,|x-y| + L_b\,d_{W,2}(\mu,\eta)\,, \\ |\sigma(t,x) - \sigma(t,y)| &\le L_\sigma\,|x-y|\,, \\ |\beta(\mu,\gamma) - \beta(\eta,\lambda)| &\le L_\beta\,d_{W,2}(\mu,\eta) + L_\beta\,|\gamma-\lambda|\,. \end{aligned}$$

Moreover, b, σ and β are bounded, i.e. there exists a positive constant M satisfying

$$\|b\|_\infty + \|\sigma\|_\infty + \|\beta\|_\infty \le M_1\,.$$

Without loss of generality, we can assume $L_b = L_\sigma = L_\beta = L$ and $M_\nu = M_1 = M$.

(D.5) f and g are Lipschitz functions in both x and μ, i.e. there exist two positive constants L_f, L_g such that for all $x, y \in \mathbb{R}$, $\mu, \eta \in \mathcal{P}^2(\mathbb{R})$ and $t \in [0, T]$

$$\begin{aligned} |f(t,x,\mu) - f(t,y,\eta)| &\le L_f\,|x-y| + L_f\,d_{W,2}(\mu,\eta)\,, \\ |g(x,\mu) - g(y,\eta)| &\le L_g\,|x-y| + L_g\,d_{W,2}(\mu,\eta)\,. \end{aligned}$$

From now on, we shortly write d_W for the squared Wasserstein distance $d_{W,2}$, as defined in (2.4), and $\mathcal{P}(\mathbb{R})$ for $\mathcal{P}^2(\mathbb{R})$.

Remark 3.1.1. The technical assumption $q \neq 4$ is required to guarantee the applicability of [FG15, Theorem 1] to obtain the rate of convergence.

Remark 3.1.2. Ass. (D.1)-(D.4) will be used to construct approximate equilibria for the n-player game G_n under some further hypotheses on the existence and the regularity of a Markovian solution to the MFG G_∞. Sufficient conditions ensuring the existence of these Markovian MFG solutions are discussed in previous Chapter 2.

3.2 Markovian ε-Nash equilibrium

All the results of this section are proved under the following standing assumption on the limiting mean-field game G_∞:

Assumption E. Assume that there exists a Markovian MFG solution $\hat{\gamma}_t = \hat{\gamma}(t, Y_{t-})$ for the game G_∞, for some measurable function $\hat{\gamma} : [0, T] \times \mathbb{R} \to A$. Moreover, the function $\hat{\gamma}(t, x)$ is Lipschitz continuous in the state variable x, i.e.

$$|\hat{\gamma}(t,x) - \hat{\gamma}(t,y)| \le C_{\hat{\gamma}}\,|x-y| \quad \forall x, y \in \mathbb{R}\,, \forall t \in [0,T], \tag{3.7}$$

for some suitable constant $C_{\hat{\gamma}} > 0$.
Without loss of generality we can assume that $C_{\hat{\gamma}} = L$ as in Assumption D.

Consider the game G_n in the event that each agent i plays strategy $\hat{\gamma} = \hat{\gamma}(t, \hat{X}^{i,n}_{t-})$, i.e. each player follows the optimal (relative to game G_∞) strategy function $(t,x) \mapsto \hat{\gamma}(t,x)$ evaluated at the left-limit of his own state process $\hat{X}^{i,n}_{t-}$. In this case the state dynamics is the n-tuple $\hat{X}^n = (\hat{X}^{1,n}, \dots, \hat{X}^{n,n})$, defined as solution of the following system

$$\begin{cases} d\hat{X}^{i,n}_t = b(t, \hat{X}^{i,n}_t, \mu^n_t)\,dt + \sigma(t, \hat{X}^{i,n}_t)\,dW^i_t + \beta(\mu^n_{t-}, \hat{\gamma}(t, \hat{X}^{i,n}_{t-}))\,d\widetilde{N}^i_t, \\ \hat{X}^{i,n}_0 = \xi^i\,, \end{cases} \tag{3.8}$$

where μ^n is the empirical measure of $\hat{X}^n$. Assumption D ensures that there exists a unique strong solution to the previous SDEs. Moreover, for each player the strategy $\hat{\gamma}(t, \hat{X}^{i,n}_{t-})$ is admissible, i.e $(\hat{\gamma}(t, \hat{X}^{i,n}_{t-}))_{t\in[0,T]} \in \mathcal{A}$, being $\hat{\gamma}$ a (Borel)-measurable function by construction and $X^{i,n}_{t-}$ a predictable process as solution of the stochastic differential equation (3.8).

Then, the strategy profile $(\hat{\gamma}(t, \hat{X}^{1,n}_{t-}), \dots, \hat{\gamma}(t, \hat{X}^{n,n}_{t-}))$ is an ε-Nash equilibrium for the corresponding game G_n which approximates a (true) Nash equilibrium as $n \to \infty$. Namely,

Theorem 3.2.1. *Let Assumptions D and E be fulfilled. If $\hat{X}^n$ is the solution of the system* (3.8)*, the n-tuple $(\hat{\gamma}(t, \hat{X}^{1,n}_{t-}), \dots, \hat{\gamma}(t, \hat{X}^{n,n}_{t-}))$ is an ε_n-Nash equilibrium for the n-player game G_n, with $\varepsilon_n = O\left(n^{-\alpha/2}\right) \to 0$ as $n \to \infty$, where $\alpha = \min\left\{\frac{1}{2}, \frac{q-2}{2}\right\}$.*

As previously noted, all the players taking part in the game G_n are symmetric in their behaviour. For this reason in the following we will prove Theorem 3.2.1 considering without loss of generality deviations of player 1 only. Indeed the same arguments would apply to every other player in the game.

In the proof of Theorem 3.2.1, we will focus on two different scenarios: the case when all the players choose to play according to the optimal recipe suggested by G_∞, i.e. they all play $\hat{\gamma}(t, \hat{X}^{i,n}_{t-})$ as explained above, and the case when player 1 deviates by choosing any different strategy $\eta \in \mathcal{A}$, i.e.

$$(\eta, \hat{\gamma}^{\hat{X}^n}_{-1}) = (\eta_t, \hat{\gamma}(t, \hat{X}^{2,n}_{t-}), \dots, \hat{\gamma}(t, \hat{X}^{n,n}_{t-}))_{t\in[0,T]}\,.$$

Notation. From now on, the strategy profile $(\hat{\gamma}(t, \hat{X}^{1,n}_{t-}), \dots, \hat{\gamma}(t, \hat{X}^{n,n}_{t-}))$, $t \in [0,T]$, will be shortly denoted by $\hat{\gamma}^{\hat{X}^n}$ whereas strategy $(\eta, \hat{\gamma}^{\hat{X}^n}_{-1})$ by $\eta^{\hat{\gamma}}$. The corresponding state processes, that are the solutions of equation (3.2) under $\hat{\gamma}^{\hat{X}^n}$ and $\eta^{\hat{\gamma}}$, will be denoted by $\hat{X}$ and $\widetilde{X}$, respectively.

Remark 3.2.1. Concerning Remark 1.1.3, observe that in the following the deviating player is allowed to choose any admissible open-loop strategy $\eta \in \mathcal{A}$, whereas the (feedback) strategies of the other players are given by $\hat{\gamma}^i = \hat{\gamma}(t, \hat{X}^{i,n}_t)$, i.e. they are computed considering the state of the system as described in equation (3.8) and therefore understood as the related open-loop controls. Therefore, in order to prove Theorem 3.2.1 we will show that

$$J^{i,n}(\hat{\gamma}) \leq J^{i,n}((\eta, \hat{\gamma}^{\hat{X}^n}_{-i})) + \varepsilon_n$$

for each $i \in 1, \dots, n$ and $\eta \in \mathcal{A}$, and ε_n as requested.

In the following we will also make use of the auxiliary processes $Y^{i,n}$, with $i = 1, \dots, n$, and $\widetilde{Y}^{1,n}$, given as solutions of

$$dY_t^{i,n}(\hat{\gamma}) = b(t, Y_t^{i,n}, \hat{\mu}_t)\, dt + \sigma(t, Y_t^{i,n})\, dW_t^i + \beta(\hat{\mu}_{t-}, \hat{\gamma}(t, Y_{t-}^{i,n}))\, d\widetilde{N}_t^i\,, \quad Y_0^{i,n} = \xi^i \tag{3.9}$$

and of

$$d\widetilde{Y}_t^{1,n}(\eta) = b(t, \widetilde{Y}_t^{1,n}, \hat{\mu}_t)\, dt + \sigma(t, \widetilde{Y}_t^{1,n})\, dW_t^1 + \beta(\hat{\mu}_{t-}, \eta_t)\, d\widetilde{N}_t^1\,, \quad \widetilde{Y}_0^{1,n} = \xi^1\,, \tag{3.10}$$

respectively. Here $\hat{\mu}_t$ represents the distribution law of the state process (optimally) controlled by $\hat{\gamma}$ in the limiting game G_∞ , i.e. $Y(\hat{\gamma})$, solution to the SDE (3.5) under $\hat{\gamma}(t, Y_t)$, and therefore $\hat{\mu} = \mathcal{L}(Y_t(\hat{\gamma}))$.

Remark 3.2.2. For each $i = 1, \dots, n$, the process $Y^{i,n}$ is defined as the dynamics of a representative player in G_∞, given in equation (3.5), when the optimal strategy function $\hat{\gamma}$, given in Assumption E, is chosen as control process. Then, by definition, $Y^{i,n}$ satisfies the MFG consistency condition (3.6), meaning that $Y^{i,n}$ is distributed accordingly to $\hat{\mu}$, i.e.

$$\mathcal{L}(Y_t^{i,n}) = \hat{\mu}_t\,, \quad \forall t \in [0, T]\,. \tag{3.11}$$

Remark 3.2.3. The definition of processes $Y^{i,n}$ and $\widetilde{Y}^{1,n}$ differs from the one of $\hat{X}^{i,n}$ and $\widetilde{X}^{1,n}$ due to the different measure flow considered in the stochastic differential equations. Indeed in (3.2), the dynamics of $\hat{X}^{i,n}$ and $\widetilde{X}^{1,n}$ are computed taking into account the associated empirical distribution of the system $\hat{X}^n$ and $\widetilde{X}^n$, respectively, as defined in equation (3.3), while the dynamics of $Y^{i,n}$ and $\widetilde{Y}^{1,n}$ in (3.9) and (3.10) are computed with respect to $\hat{\mu}$. In particular, this implies that $Y^{i,n}$ and $\widetilde{Y}^{1,n}$ do no longer depend on the other players' choices (we will say that they do not depend on n for short) and therefore their dynamics are easier to study.

3.2.1 L^2-estimates for the state processes and the empirical mean process in G_n

The following lemma provides an estimate for the second moment of the processes $X^{i,n}$, defined as in equation (3.2).

Lemma 3.2.1. *Assume that Assumption D holds. Then, for each admissible strategy profile $\gamma \in \mathcal{A}^n$ the related controlled processes $X^{i,n}(\gamma)$ for $i = 1, \dots, n$, solving the related SDEs* (3.2)*, satisfy*

$$\mathbb{E}\left[\sup_{t\in[0,T]} |X_t^{i,n}|^2\right] \leq \hat{C}(\chi, T, M)\,, \tag{3.12}$$

where the constant $\hat{C}$ is independent of n and γ.

Proof. The proof follows the same steps as that of Lemma 2.2.3. By the equation (3.2),

$$\left|X_t^{i,n}\right|^2 \leq 4\left|\xi_i\right|^2 + 4\left|\int_0^t b(s, X_s^{i,n}, \mu_s^n)\, ds\right|^2 + 4\left|\int_0^t \sigma(s, X_s^{i,n})\, dW_s^i\right|^2 + 4\left|\int_0^t \beta(\mu_{s-}^n, \gamma_s^i)\, d\widetilde{N}_s^i\right|^2 .$$

Applying Jensen's and Burkholder-Davis-Gundy's inequalities, it follows that for a constant C (which may change from line to line)

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\in[0,T]}\left|X_t^{i,n}\right|^2\right] &\leq C\mathbb{E}\left[\left|\xi_i\right|^2\right] + Ct\mathbb{E}\left[\int_0^t \sup_{u\in[0,s]}\left|b(s, X_s^{i,n}, \mu_s^n)\right|^2\, ds\right] \\
&\quad + C\mathbb{E}\left[\int_0^t \sigma(s, X_s^{i,n})^2\, ds\right] + C\mathbb{E}\left[\int_0^t \beta^2(\mu_s^n, \gamma_s^i)\nu(s)\, ds\right] \\
&\leq C(\chi) + C\left\|b\right\|_\infty^2 t^2 + C\left\|\sigma\right\|_\infty^2 t + C\left\|\beta\right\|_\infty^2 \left\|\nu\right\|_\infty t \\
&\leq \hat{C}(\chi, T, M),
\end{aligned}$$

where we have used Ass. (D.1), (D.2) and (D.4) for the second inequality. □

An analogous result can be proved for the empirical distribution μ^n of the system X^n, given in equation (3.3).

Lemma 3.2.2. *Assume that Assumption D holds. Then, for any admissible strategy $\gamma \in \mathcal{A}^n$, the empirical distribution $\mu^n = \mu^n(\gamma)$ of the system X^n, solution of the SDE* (3.2), *satisfies*

$$\mathbb{E}\left[\sup_{t\in[0,T]} d_W^2(\mu_t^n, \delta_0)\right] \leq \hat{C}(\chi, T, M) \tag{3.13}$$

for a constant $\hat{C}$ independent of n and γ.

Proof. The constant $\hat{C}$ appearing in Lemma 3.2.1 provides the required bound, since

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\in[0,T]} d_W^2(\mu_t^n, \delta_0)\right] &\leq \mathbb{E}\left[\sup_{t\in[0,T]} \frac{1}{n}\sum_{i=1}^n \left|X_t^{i,n}\right|^2\right] \\
&\leq \frac{1}{n}\sum_{i=1}^n \mathbb{E}\left[\sup_{t\in[0,T]}\left|X_t^{i,n}\right|^2\right] \leq \hat{C}\,.
\end{aligned}$$ □

Arguing as in the previous Lemmas, or as in Lemma 2.2.3, but exploiting the stronger hypothesis on the initial distribution $\chi \in \mathcal{P}^q(\mathbb{R})$ with $q > 2$, we can prove that each solution Y to the SDE (3.5) satisfies

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|Y_t\right|^q\right] \leq \hat{C}_2(\chi, T, M)\,.$$

Furthermore, considering its distribution law at time $t \in [0,T]$, that is $\hat{\mu}_t$, the previous estimate guarantees that

$$\int_{\mathbb{R}} |y|^q \, \hat{\mu}_t(dy) < \hat{C}_2 < \infty \quad \text{for all } t \in [0,T]. \tag{3.14}$$

3.2.2 Approximation results

Let $Y^n = (Y^{1,n}, \ldots, Y^{n,n})$ be the system defined as in equation (3.9) and $\mu^{Y,n}$ its empirical measure, namely

$$\mu_t^{Y,n} = \frac{1}{n} \sum_{i=1}^{n} \delta_{Y_t^{i,n}} \quad \text{for all } t \in [0,T].$$

As first step we show that this measure $\mu^{Y,n}$ converges to $\hat{\mu}$ with respect to the squared Wasserstein distance as $n \to \infty$. Being $Y_t^{i,n}$ independent and identically distributed random variables with distribution $\hat{\mu}_t$, see Remark 3.2.2, [FG15, Theorem 1] ensures that

$$\mathbb{E}\left[d_W^2(\hat{\mu}_t, \hat{\mu}_t^{Y,n})\right] \leq C(q) M_q^{\frac{2}{q}}(\hat{\mu}) \left(\frac{1}{n^{\frac{1}{2}}} + \frac{1}{n^{\frac{q-2}{q}}} \right)$$

where C is a positive constant depending on q, and M_q is defined as

$$M_q(\mu) = \int_{\mathbb{R}} |x|^q \, \mu(dx) \, .$$

Since by previous considerations $M_q(\hat{\mu})$ is finite, see equation (3.14),

$$d_W(\hat{\mu}_t, \mu_t^{Y,n})^2 = O\left(n^{-\alpha}\right) \tag{3.15}$$

where $\alpha = \min\left\{\frac{1}{2}, \frac{q-2}{q}\right\}$, and being $q > 2$, this implies that

$$\lim_{n \to \infty} \mathbb{E}\left[d_W^2(\hat{\mu}_t, \mu_t^{Y,n})\right] = 0 \tag{3.16}$$

uniformly in time.

Now, we want to show that the process $Y^{i,n}(\hat{\gamma})$ *approximates* $\hat{X}^{i,n}$ as n grows to infinity, in a sense that will be specified later. In both the systems $\hat{X}^n$ and Y^n, all the n players choose the same strategy, or more precisely the same strategy form, i.e. $\hat{\gamma}(t, \hat{X}_{t-}^{i,n})$ and $\hat{\gamma}(t, Y_{t-}^{i,n})$, respectively, but the dynamics in $\hat{X}^n$ depend on the actual empirical distribution of this system, while the evolution of the state processes $Y^{i,n}$ are computed with respect to the measure flow $\hat{\mu}$.

Proposition 3.2.1. *Let $\hat{X}^{i,n}$ and $Y^{i,n}$ be defined as in equation* (3.8) *and* (3.9)*, respectively, and μ^n the empirical distribution of the system $\hat{X}^n$. Then we have*

$$\sup_{t \in [0,T]} \mathbb{E}\left[d_W^2(\mu_t^n, \hat{\mu}_t)\right] = O\left(n^{-\alpha}\right), \tag{3.17}$$

$$\sup_{t \in [0,T]} \mathbb{E}\left[\left|\hat{X}_t^{i,n} - Y_t^{i,n}\right|^2\right] = O\left(n^{-\alpha}\right) . \tag{3.18}$$

Proof. For each $t \in [0,T]$

$$\left|\hat{X}_t^{i,n} - Y_t^{i,n}\right|^2 \leq 3\left(\int_0^t b(s,\hat{X}_s^{i,n},\mu_s^n) - b(s,Y_s^{i,n},\hat{\mu}_s)\,ds\right)^2 + 3\left(\int_0^t \sigma(s,\hat{X}_s^{i,n}) - \sigma(s,Y_s^{i,n})\,dW_s^i\right)^2 + 3\left(\int_0^t \beta(\mu_{s-}^n,\hat{\gamma}(s,\hat{X}_{s-}^{i,n})) - \beta(\hat{\mu}_{s-},\hat{\gamma}(s,Y_{s-}^{i,n}))d\widetilde{N}_s^i\right)^2 .$$

Then,

$$\mathbb{E}\left[\left|\hat{X}_t^{i,n} - Y_t^{i,n}\right|^2\right] \leq 3t\mathbb{E}\left[\int_0^t \left|b(s,\hat{X}_s^{i,n},\mu_s^n) - b(s,Y_s^{i,n},\hat{\mu}_s)\right|^2 ds\right] + 3\mathbb{E}\left[\int_0^t \left|\sigma(s,\hat{X}_s^{i,n}) - \sigma(s,Y_s^{i,n})\right|^2 ds\right] + 3\mathbb{E}\left[\int_0^t \left(\beta(\mu_{s-}^n,\hat{\gamma}(s,\hat{X}_{s-}^{i,n})) - \beta(\hat{\mu}_{s-},\hat{\gamma}(s,Y_{s-}^{i,n}))\right)^2 \nu(s)ds\right],$$

where we used again Jensen's and Burkholder-Davis-Gundy's inequalities. Using the Lipschitz continuity of functions b, σ and β, given by Ass. (D.4), and of $\hat{\gamma}(\cdot,x)$, as explained in equation (3.7), as well as the finiteness of $\mathbb{E}[\sup_{t\in[0,T]} d_W(\mu_t^n,\delta_0)^2]$, as in (3.13), we obtain

$$\begin{aligned}\mathbb{E}\left[\left|\hat{X}_t^{i,n} - Y_t^{i,n}\right|^2\right] &\leq 6tL^2\int_0^t \mathbb{E}\left[\left|\hat{X}_s^{i,n} - Y_s^{i,n}\right|^2\right] + \mathbb{E}\left[d_W^2(\mu_s^n,\hat{\mu}_s)\right]\,ds \\ &\quad + 3L^2\int_0^t \mathbb{E}\left[\left|\hat{X}_s^{i,n} - Y_s^{i,n}\right|^2\right] ds \\ &\quad + 6L^2\left\|\nu\right\|_\infty \int_0^t \mathbb{E}\left[\left|\hat{X}_s^{i,n} - Y_s^{i,n}\right|^2\right] + \mathbb{E}\left[d_W^2(\mu_s^n,\hat{\mu}_s)\right]\,ds \\ &\leq \widetilde{C}(T,L,M)\int_0^t \mathbb{E}\left[\left|\hat{X}_s^{i,n} - Y_s^{i,n}\right|^2\right] + \mathbb{E}\left[d_W^2(\mu_s^n,\hat{\mu}_s)\right]\,ds\end{aligned} \tag{3.19}$$

for a suitable constant $\widetilde{C}$. Moreover, by the previous inequality (3.19), we get

$$\begin{aligned}\mathbb{E}\left[d_W^2(\mu_t^n,\mu_t^{Y,n})\right] &\leq \frac{1}{n}\sum_{i=1}^n \mathbb{E}\left[\left|\hat{X}_t^{i,n} - Y_t^{i,n}\right|^2\right] \\ &\leq \widetilde{C}\int_0^t \frac{1}{n}\sum_{i=1}^n \mathbb{E}\left[\left|\hat{X}_s^{i,n} - Y_s^{i,n}\right|^2\right] + \mathbb{E}\left[d_W^2(\mu_s^n,\hat{\mu}_s)\right]\,ds\,.\end{aligned}$$

Then, it holds that

$$\begin{aligned}
&\mathbb{E}\left[d_W^2(\mu_t^n,\hat{\mu}_t)\right]+\frac{1}{n}\sum_{i=1}^n\mathbb{E}\left[\left|\hat{X}_t^{i,n}-Y_t^{i,n}\right|^2\right]\\
&\quad\le 2\mathbb{E}\left[d_W^2(\mu_t^n,\mu_t^{Y,n})\right]+2\mathbb{E}\left[d_W^2(\mu_t^{Y,n},\hat{\mu}_t)\right]+\frac{1}{n}\sum_{i=1}^n\mathbb{E}\left[\left|\hat{X}_t^{i,n}-Y_t^{i,n}\right|^2\right]\\
&\quad\le 2\mathbb{E}\left[d_W^2(\mu_t^{Y,n},\hat{\mu}_t)\right]+3\widetilde{C}\left(\int_0^t\mathbb{E}\left[d_W^2(\mu_s^n,\hat{\mu}_s)\right]+\frac{1}{n}\sum_{i=1}^n\mathbb{E}\left[\left|\hat{X}_s^{i,n}-Y_s^{i,n}\right|^2\right]ds\right).
\end{aligned}\tag{3.20}$$

Therefore, by equation (3.16), we have

$$\begin{aligned}
&\mathbb{E}\left[d_W^2(\mu_t^n,\hat{\mu}_t)\right]+\frac{1}{n}\sum_{i=1}^n\mathbb{E}\left[\left|\hat{X}_t^{i,n}-Y_t^{i,n}\right|^2\right]\\
&\quad\le O\left(n^{-\alpha}\right)+2\widetilde{C}\left(\int_0^t\mathbb{E}\left[d_W^2(\mu_s^n,\hat{\mu}_s)\right]+\frac{1}{n}\sum_{i=1}^n\mathbb{E}\left[\left|\hat{X}_s^{i,n}-Y_s^{i,n}\right|^2\right]ds\right)
\end{aligned}$$

and an application of the Gronwall's inequality implies the desired results, i.e. equations (3.17) and (3.18). □

In the previous estimates, we have considered the case when all the n players are choosing the same strategic plan, $\hat{\gamma}(t,x)$. We now investigate what happen to the players' dynamics when one of them, namely player 1, deviates from the strategy profile $\hat{\gamma}^{\hat{X}^n}$ by playing any other admissible strategy $\eta\in\mathcal{A}$. In this case the strategy profile adopted in the game G_n becomes $\eta^{\hat{\gamma}}$ and the dynamics of each player in G_n are given by $\widetilde{X}^{i,n}$. The following proposition shows that the empirical distributions of the two systems $\hat{X}$ and $\widetilde{X}$ converge with respect to d_W as the number of players grows to infinity and that the dynamics of the deviating player $\widetilde{X}^{1,n}$ can be *approximated* by $\widetilde{Y}^{1,n}$, which does not depend on n as pointed out in Remark 3.2.3.

Proposition 3.2.2. *Let $\hat{X}$ and $\widetilde{X}$ be the solutions of the system* (3.2) *when the strategy profile is given by $\hat{\gamma}^{\hat{X}^n}$ and by $\eta^{\hat{\gamma}}$, respectively. We denote by μ^n and $\widetilde{\mu}^n$ the empirical measure of the two systems. Then,*

$$\sup_{t\in[0,T]}\mathbb{E}\left[d_W^2(\mu_t^n,\widetilde{\mu}_t^n)\right]=O\left(n^{-1}\right).$$

Moreover, considering $\widetilde{Y}^{1,n}$, as defined in equation (3.10)*, it holds that*

$$\sup_{t\in[0,T],\,\eta\in\mathcal{A}}\mathbb{E}\left[\left|\widetilde{X}_t^{1,n}-\widetilde{Y}_t^{1,n}\right|^2\right]=O\left(n^{-\alpha}\right).\tag{3.21}$$

Proof. Firstly, we compare the dynamics of player 1 in the two different settings. Letting $\hat{C}$ be the constant appearing in Lemma 3.2.1, that is independent of n, Ass. (D.4) implies

$$\begin{aligned}
\mathbb{E}\left[\left|\hat{X}_t^{1,n} - \widetilde{X}_t^{1,n}\right|^2\right] &\leq 6tL^2 \int_0^t \mathbb{E}\left[\left|\hat{X}_s^{1,n} - \widetilde{X}_s^{1,n}\right|^2\right] + \mathbb{E}\left[d_W^2(\mu_s^n, \widetilde{\mu}_s^n)\right] \, ds \\
&\quad + 3L^2 \int_0^t \mathbb{E}\left[\left|\hat{X}_s^{1,n} - \widetilde{X}_s^{1,n}\right|^2\right] ds \\
&\quad + 3\mathbb{E}\left[\int_0^t \left(L d_W\left(\mu_s^n, \widetilde{\mu}_s^n\right) + L\left|\hat{\gamma}(s, \hat{X}_s^{1,n}) - \eta\right|\right)^2 \nu(s) ds\right] \\
&\leq 12L^2\hat{C}(2t^2 + t + 2\left\|\nu\right\|_\infty t) + 12L^2\alpha_M^2 \left\|\nu\right\|_\infty t \\
&\leq 12L^2\hat{C}(2T^2 + T + 2MT) + 12L^2\alpha_M^2 MT = C_1
\end{aligned}$$

where C_1 is again independent of n, and furthermore, by construction, does not depend on η either.

On the other hand, the other players for $i = 2, \ldots, n$ play the strategy $\hat{\gamma}(t, \cdot)$ in both cases, even if computed with respect to the different states, and then to estimate $\mathbb{E}\left[\left|\hat{X}_t^{i,n} - \widetilde{X}_t^{i,n}\right|^2\right]$ we can argue as in (3.19).

Finally, following the same idea as to obtain (3.20) but taking into account the different role of player 1, we have that

$$\begin{aligned}
\mathbb{E}\left[d_W^2(\mu_t^n, \widetilde{\mu}_t^n)\right] &\leq \frac{1}{n}\mathbb{E}\left[\left|\hat{X}_t^{1,n} - \widetilde{X}_t^{1,n}\right|^2\right] + \frac{1}{n}\sum_{i=2}^n \mathbb{E}\left[\left|\hat{X}_t^{i,n} - \widetilde{X}_t^{i,n}\right|^2\right] \\
&\leq \frac{C_1}{n} + \frac{\widetilde{C}}{n}\sum_{i=2}^n \int_0^t \mathbb{E}\left[\left|\hat{X}_s^{i,n} - \widetilde{X}_s^{i,n}\right|^2\right] + \mathbb{E}\left[d_W^2(\mu_s^n, \widetilde{\mu}_s^n)\right] \, ds
\end{aligned}$$

and therefore

$$\begin{aligned}
&\mathbb{E}\left[d_W^2(\mu_t^n, \widetilde{\mu}_t^n)\right] + \frac{1}{n}\sum_{i=2}^n \mathbb{E}\left[\left|\hat{X}_t^{i,n} - \widetilde{X}_t^{i,n}\right|^2\right] \\
&\qquad \leq \frac{C_1}{n} + \frac{2\widetilde{C}}{n}\sum_{i=2}^n \int_0^t \mathbb{E}\left[\left|\hat{X}_s^{i,n} - \widetilde{X}_s^{i,n}\right|^2\right] + \mathbb{E}\left[d_W^2(\mu_s^n, \widetilde{\mu}_s^n)\right] \, ds \, .
\end{aligned}$$

Applying again Gronwall's lemma, it is found that

$$\mathbb{E}\left[d_W^2(\mu_t^n, \widetilde{\mu}_t^n)\right] + \frac{1}{n}\sum_{i=2}^n \mathbb{E}\left[\left|\hat{X}_t^{i,n} - \widetilde{X}_t^{i,n}\right|^2\right] \leq \frac{1}{n} K_1(\hat{C}, C_1) \, , \tag{3.22}$$

with K_1 independent of n, t and η since $\hat{C}$ and C_1 are so. Therefore,

$$\sup_{t \in [0,T], \eta \in \mathcal{A}} \mathbb{E}\left[d_W^2(\mu_t^n, \widetilde{\mu}_t^n)\right] = O\left(n^{-1}\right) \, .$$

This, together with (3.17), implies also that

$$\sup_{t\in[0,T],\eta\in\mathcal{A}} \mathbb{E}\left[d_W^2(\hat{\mu}_t, \widetilde{\mu}_t^n)\right] = O\left(n^{-\alpha}\right). \tag{3.23}$$

Lastly, considering $\widetilde{Y}^{1,n}$ as defined in equation (3.10) and arguing as in the proof of Proposition 3.2.1, we have

$$\begin{aligned}
\mathbb{E}\left[\left|\widetilde{X}_t^{1,n} - \widetilde{Y}_t^{1,n}\right|^2\right] &\leq 3(2t+1)L^2 \int_0^t \mathbb{E}\left[\left|\widetilde{X}_s^{i,n} - \widetilde{Y}_s^{i,n}\right|^2\right] ds \\
&\qquad + 3(2t + \|\nu\|_\infty)L^2 \int_0^t \mathbb{E}\left[d_W^2(\widetilde{\mu}_s^n, \hat{\mu}_s)\right] ds \\
&\leq \widetilde{K}(T,L,M) \int_0^t \mathbb{E}\left[\left|\widetilde{X}_s^{1,n} - \widetilde{Y}_s^{1,n}\right|^2\right] + \mathbb{E}\left[d_W^2(\widetilde{\mu}_s^n, \hat{\mu}_s)\right] ds
\end{aligned}$$

so that by the previous convergence result (3.23)

$$\mathbb{E}\left[\left|\widetilde{X}_t^{1,n} - \widetilde{Y}_t^{1,n}\right|^2\right] \leq \widetilde{K}(T,L,M) \int_0^t \mathbb{E}\left[\left|\widetilde{X}_s^{1,n} - \widetilde{Y}_s^{1,n}\right|^2\right] ds + T\widetilde{K}(T,L,M)O\left(n^{-\alpha}\right).$$

Hence Gronwall's lemma implies

$$\mathbb{E}\left[\left|\widetilde{X}_t^{1,n} - \widetilde{Y}_t^{1,n}\right|^2\right] \leq \bar{K}(T,L,M)O\left(n^{-\alpha}\right) \tag{3.24}$$

for a suitable constant $\bar{K}$ independent of n, t and η, and therefore

$$\sup_{t\in[0,T],\,\eta\in\mathcal{A}} \mathbb{E}\left[\left|\widetilde{X}_t^{1,n} - \widetilde{Y}_t^{1,n}\right|^2\right] = O\left(n^{-\alpha}\right). \qquad \square$$

Remark 3.2.4. It is crucial here and in what follows that the two constants K_1 and $\bar{K}$ appearing in the estimates (3.22) and (3.24) do not depend on how player 1 deviates from the strategy profile $\hat{\gamma}^{\hat{X}^n}$.

In order to prove Theorem 3.2.1, we will make use of the following two operators: $\widetilde{J}_n \colon \mathcal{A}^n \to \mathbb{R}$ and $\widetilde{J} \colon \mathcal{A} \to \mathbb{R}$, defined by

$$\widetilde{J}_n(\gamma) = \mathbb{E}\left[\int_0^T f(t, X_t^{1,n}(\gamma), \hat{\mu}_t, \gamma_t^1)\, dt + g(X_T^{1,n}(\gamma), \hat{\mu}_T)\right] \tag{3.25}$$

and

$$\widetilde{J}(\eta) = \mathbb{E}\left[\int_0^T f(t, Y_t^{1,n}, \hat{\mu}_t, \eta_t)\, dt + g(Y_T^{1,n}, \hat{\mu}_T)\right] \tag{3.26}$$

respectively, where $X^{1,n}(\gamma)$ and $Y^{1,n}(\eta)$ are given as in (3.2) and (3.10). It is worth observing that since both $Y^{1,n}$ and $\hat{\mu}$ do not depend on the number of players in the game, then also $\widetilde{J}$ does not depend on n. Furthermore, since $Y^{1,n}$ follows the dynamics of a representative player in the mean-field game G_∞, $\widetilde{J}$ is exactly the expected cost of the strategy η in G_∞ with respect to the measure flow $\hat{\mu}$, as given in equation (3.4). Therefore, since $\hat{\gamma}(t, Y_t^{1,n})$ is by construction one of the minimising strategies, i.e. $\hat{\gamma} \in \arg\min_{\gamma \in \mathcal{A}} J(\gamma)$, we have that

$$\widetilde{J}((\hat{\gamma}(t, Y_{t-}^{1,n}))_{t \in [0,T]}) \leq \widetilde{J}(\eta) \,, \tag{3.27}$$

for any admissible strategy $\eta \in \mathcal{A}$.

As first step, we show that the value of player 1 in the game G_n when he deviates from the candidate approximate Nash equilibrium $\hat{\gamma}^{\hat{X}^n}$ to any different admissible strategy $\eta \in \mathcal{A}$, that is $J^{1,n}(\eta^{\hat{\gamma}})$ given in equation (3.1), can be approximated (when n is large) with $\widetilde{J}_n(\eta^{\hat{\gamma}})$, that is the expected cost computed under the same strategy profile $\eta^{\hat{\gamma}}$, but evaluated with respect to the measure $\hat{\mu}$.

Proposition 3.2.3. *Let $(t,x) \mapsto \hat{\gamma}(t,x)$ be as in Assumption E. Consider the strategy profile*

$$\hat{\gamma}_t^{\hat{X}^n} = (\hat{\gamma}(t, \hat{X}_{t-}^{1,n}), \dots, \hat{\gamma}(t, \hat{X}_{t-}^{n,n})), \quad t \in [0,T],$$

and let η be an admissible strategy in $\mathcal{A}$. Then

$$\sup_{\eta \in \mathcal{A}} \left| J^{1,n}(\eta^{\hat{\gamma}}) - \widetilde{J}_n(\eta^{\hat{\gamma}}) \right| = O\left(n^{-\frac{\alpha}{2}}\right). \tag{3.28}$$

Proof. By definition (3.25)-(3.26) and Ass. (D.5), the distance between the two operators $J^{1,n}$ and $\widetilde{J}_n$ can be bounded as follows:

$$\begin{aligned}\left| J^{1,n}(\eta^{\hat{\gamma}}) - \widetilde{J}_n(\eta^{\hat{\gamma}}) \right| &\leq \mathbb{E}\left[\int_0^T \left| f(t, \widetilde{X}_t^{1,n}, \widetilde{\mu}_t^n, \eta_t) - f(t, \widetilde{X}_t^{1,n}, \hat{\mu}_t, \eta_t) \right| dt\right] \\ &\quad + \mathbb{E}\left[\left| g(\widetilde{X}_T^{1,n}, \widetilde{\mu}_T^n) - g(\widetilde{X}_T^{1,n}, \hat{\mu}_T) \right|\right] \\ &\leq L \int_0^T \mathbb{E}\left[d_W(\widetilde{\mu}_t^n, \hat{\mu}_t)\right] dt + L\mathbb{E}\left[d_W(\widetilde{\mu}_T^n, \hat{\mu}_T)\right] .\end{aligned}$$

Then, previous results in Proposition 3.2.1 and in Proposition 3.2.2 imply that

$$\mathbb{E}\left[d_W(\widetilde{\mu}_t^n, \hat{\mu}_t)\right] \leq \left(\mathbb{E}\left[d_W(\widetilde{\mu}_t^n, \hat{\mu}_t)^2\right]\right)^{\frac{1}{2}} = O\left(n^{-\frac{\alpha}{2}}\right),$$

and by Lemma 3.2.2 and the dominate convergence theorem, the limit in equation (3.28) is obtained. □

As second step, we approximate $\widetilde{J}_n(\eta^{\hat{\gamma}})$ with $\widetilde{J}(\eta)$, that is the expected cost for playing η in the MFG G_∞.

Proposition 3.2.4. *Let $(t,x) \mapsto \hat{\gamma}(t,x)$ represent the Markovian structure of a mean-field game solution of the game G_∞, $\hat{\gamma}_t^{\hat{X}^n} = (\hat{\gamma}(t, \hat{X}_{t-}^{1,n}), \dots, \hat{\gamma}(t, \hat{X}_{t-}^{n,n}))$, for $t \in [0,T]$, and let $\eta \in \mathcal{A}$ be any admissible strategy. It holds that*

$$\sup_{\eta \in \mathcal{A}} \left| \widetilde{J}_n(\eta^{\hat{\gamma}}) - \widetilde{J}(\eta) \right| = O\left(n^{-\frac{\alpha}{2}}\right). \tag{3.29}$$

Proof. Arguing as in Proposition 3.2.3, we have that

$$\begin{aligned}\left|\widetilde{J}_n(\eta^{\hat{\gamma}}) - \widetilde{J}(\eta)\right| &\leq \mathbb{E}\left[\int_0^T \left|f(t, \widetilde{X}_t^{1,n}, \hat{\mu}_t, \eta_t) - f(t, \widetilde{Y}_t^{1,n}, \hat{\mu}_t, \eta_t)\right| dt\right] \\ &\qquad + \mathbb{E}\left[\left|g(\widetilde{X}_T^{1,n}, \hat{\mu}_T) - g(\widetilde{Y}_T^{1,n}, \hat{\mu}_T)\right|\right] \\ &\leq L\int_0^T \mathbb{E}\left[\left|\widetilde{X}_t^{1,n} - \widetilde{Y}_t^{1,n}\right|\right] dt + L\mathbb{E}\left[\left|\widetilde{X}_T^{1,n} - \widetilde{Y}_T^{1,n}\right|\right].\end{aligned}$$

Since by Proposition 3.2.2, $\mathbb{E}\left[\left|\widetilde{X}_t^{1,n}(\eta^{\hat{\gamma}}) - \widetilde{Y}_t^{1,n}(\eta)\right|\right] = O\left(n^{-\frac{\alpha}{2}}\right)$, then

$$\sup_{\eta\in\mathcal{A}}\left|\widetilde{J}_n(\eta^{\hat{\gamma}}) - \widetilde{J}(\eta)\right| = O\left(n^{-\frac{\alpha}{2}}\right), \tag{3.30}$$

as claimed. □

Thanks to all the previous approximation results, we are ready to prove Theorem 3.2.1.

Proof of Theorem 3.2.1. Given any admissible strategy $\eta \in \mathcal{A}$, let

$$\varepsilon_{1,n} = 4\sup_{\eta\in\mathcal{A}}\left|J^{1,n}(\eta^{\hat{\gamma}}) - \widetilde{J}_n(\eta^{\hat{\gamma}})\right|, \ \varepsilon_{2,n} = 4\sup_{\eta\in\mathcal{A}}\left|\widetilde{J}_n(\eta^{\hat{\gamma}}) - \widetilde{J}(\eta)\right| \text{ and } \varepsilon_n = \varepsilon_{1,n} + \varepsilon_{2,n}.$$

Then

$$J^{1,n}(\eta^{\hat{\gamma}}) \geq -\frac{\varepsilon_n}{2} + \widetilde{J}(\eta) \geq -\frac{\varepsilon_n}{2} + \widetilde{J}(\hat{\gamma}) \geq -\varepsilon_n + J^{1,n}(\hat{\gamma}),$$

meaning that the deviating player saves at most ε_n. More in detail, the first and the third inequalities are guaranteed by Proposition 3.2.3 and Proposition 3.2.4 respectively, whereas the second inequality is justified in equation (3.27). The symmetry of G_n guarantees that $(\hat{\gamma}(t, X_{t-}^{1,n}), \ldots, \hat{\gamma}(t, X_{t-}^{n,n}))$, for $t \in [0, T]$, is an ε-Nash equilibrium of this game. See Definition 1.1.2.

The rate convergence, i.e. $\varepsilon_n = O\left(n^{-\frac{\alpha}{2}}\right)$, is also granted by the previous approximations in Propositions 3.2.3 and 3.2.4. □

Chapter 4

An illiquid interbank market model

We illustrate the relevance of the class of mean field games introduced in the previous Chapter 1 by means of an illiquid interbank market model. Inspired by the systemic risk model proposed by Carmona and co-authors in [CFS15], we consider n banks interacting in an illiquid interbank lending market. Each bank controls its level of reserves to meet its financial obligations and its reserve requirements. However, being this market illiquid, the banks can access it only at some exogenously given instants, modeled as jump times of a Poisson process. The intensity ν of these Poisson processes, which does not depend on the specific bank, can be viewed as a health indicator of the whole system: the lower the intensity, hence the lower the probability of controlling the reserves, the higher the illiquidity of the system.

In Section 4.1, the mathematical structure of this game is introduced and an open-loop Nash equilibrium is computed explicitly, whereas in Section 4.2 the resulting limiting MFG is investigated. Furthermore, we perform some numerical experiments showing the role of illiquidity in driving the evolution over time of the optimal controls and the related state variables. These results are summarised in Section 4.3.

The original results are collected in [BCDP17a].

4.1 The n-bank case

Consider n banks, the players of this game, which lend to and borrow from a central bank in an interbank lending market. This is the market where banks can ask or extend loans to one another and therefore it is their primary source to manage liquidity. Determining an appropriate level for its reserves is a crucial task for any financial institutions since after the financial crisis of 2008, they are required to store an adequate amount of liquid assets, like cash, to manage possible market stress by international regulations, like Basel III or Solvency II. But at the same time, also holding more cash than needed is costly due to its low return.

We model illiquidity phenomenon by not allowing the banks to control their reserves continuously over time. On the contrary, each bank can adjust its reserve level by borrowing or lending money only at some exogenous random instants modeled as jump times of a Poisson process.

Let n be a strictly positive natural number, representing the number of banks intervening in the interbank market under study, and let $X^i = (X^i_t)_{t\in[0,T]}$ denote the monetary log-reserves of each bank i, for $i = 1, \ldots n$, over the finite time interval $[0, T]$, with $0 < T < \infty$. The evolution of these processes is given by

$$dX^i_t = \frac{a}{n}\sum_{j=1}^{n}(X^j_t - X^i_t)\,dt + \sigma\,dW^i_t + \gamma^i_{t-}dP^i_t\,, \quad i = 1, \ldots, n, \tag{4.1}$$

subjected to an initial condition $X^i_0 = \xi_i$ for all $i = 1, \ldots, n$. Here $(W^1, \ldots, W^n)$ is an n-dimensional Brownian motion and $(P^1, \ldots, P^n)$ is an n-dimensional Poisson process, each component with a constant intensity $\nu > 0$. ξ_i are i.i.d. random variables such that $\mathbb{E}[\xi^i] = 0$ for all $i = 1, \ldots, n$. Furthermore, initial conditions, Brownian motions W^i and Poisson processes P^i are all mutually independent.

Each bank can control its reserves by means of the control γ^i, which multiplies the corresponding Poisson Process P^i. This implies that the institution i actively, or deliberately, modifies its state only at the jump times of P^i.

Through the parameter ν, that is the intensity of each independent Poisson process P^i, we represent the market liquidity, like it is an health indicator of the whole system. For instance, when ν is low, the system becomes very illiquid, meaning that each bank can intervene in the market, and therefore adjust its reserve level, more rarely. Observe that while the banks can borrow or lend money at different times, since the processes P^i are mutually independent, ν does not depend on the particular bank i, since it measures the depth of the market and not a specific characteristic of each bank.

In the following, let $X = (X^1, \ldots, X^n)$ and $\gamma = (\gamma^1, \ldots, \gamma^n)$. We denote by $\bar{X}_t$ the empirical mean of the monetary log-reserves X at time t, that is

$$\bar{X}_t = \frac{1}{n}\sum_{i=1}^{n} X^i_t\,. \tag{4.2}$$

Therefore, the dynamics given in the previous SDE (4.1) can be rewritten in the mean field form as

$$dX^i_t = [a(\bar{X}_t - X^i_t) + \nu\gamma^i_t]dt + \sigma\,dW^i_t + \gamma^i_{t-}d\tilde{P}^i_t\,, \quad i = 1, \ldots, n\,,$$

which clearly show that the dynamics of the monetary reserves are coupled together through their drifts by means of the average state of the system. Differentiating the formula (4.2), the dynamics of the average state $\bar{X}_t$ follows

$$d\bar{X}_t = \frac{1}{n}\sum_{k=1}^{n} dX^k_t = \frac{\nu}{n}\sum_{k=1}^{n}\gamma^k_t\,dt + \frac{\sigma}{n}\sum_{k=1}^{n} dW^k_t + \frac{1}{n}\sum_{k=1}^{n}\gamma^k_{t-}\,d\tilde{P}^k_t. \tag{4.3}$$

Any bank i controls its level of reserves through the control process γ^i in order to minimise the cost functional J^i, defined by

$$J^i(\gamma) = J^i(\gamma^1, \dots, \gamma^n) = \mathbb{E}\left[\int_0^T \nu f^i(X_t, \gamma_t^i)\, dt + g^i(X_T)\right].$$

This cost J^i depends directly on the strategy chosen by player i, and indirectly also on the choices of the opponents, since X does. The running cost $f^i\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ and the terminal cost $g^i\colon \mathbb{R}^n \to \mathbb{R}$ are the following quadratic functions

$$f^i(x, \gamma^i) = \frac{1}{2}(\gamma^i)^2 - \theta\gamma^i(\bar{x} - x^i) + \frac{\varepsilon}{2}(\bar{x} - x^i)^2\,, \tag{4.4}$$

$$g^i(x) = \frac{c}{2}(\bar{x} - x^i)^2\,, \tag{4.5}$$

where $\bar{x} = \frac{1}{n}\sum_{i=1}^n x^i$. Note that both the cost functions of player i depend on the other players' strategies only through the mean of the states $\bar{x}$, i.e. $f^i(x, \gamma^i) = f(\bar{x}, x^i, \gamma^i)$ and $g^i(x) = g(\bar{x}, x^i)$ and that these cost functions are the same for each player, even if computed with respect to the own strategy/state.

As mention before, if, on the one hand, banks need to maintain a certain level of reserves due to regulators, on the other hand, holding too much cash is costly. Then we assume that banks try to keep their reserve level away from critical values, both from above and from below, by using as a benchmark the average value of reserves in the system. With this reason in mind, both the cost functions penalize departures from the average $\bar{x}$. Then, the parameter $\theta > 0$ is to control the incentive to borrowing or lending: each bank i wants to increase its reserves (i.e. borrow: $\gamma_t^i > 0$) if its state X_t^i is smaller than the empirical mean $\bar{X}_t$ and decrease them (i.e. lend: $\gamma_t^i < 0$) if X_t^i is greater than $\bar{X}_t$. Also the parameters ε and c are strictly grater than 0, so that the quadratic term $(\bar{x} - x^i)^2$ in both costs punishes deviations from the average. Moreover we assume that

$$\theta^2 \leq \varepsilon\,,$$

which guarantees the convexity of $f^i(x, \gamma)$ in (x, γ).

Remark 4.1.1. Differently from what we have studied in the previous chapters, both the dynamics and the cost functions depend on the empirical state distribution only through its first moment, that is the empirical mean of the system.

4.1.1 The open-loop problem

We look for a Nash equilibrium among all admissible open-loop strategies $\gamma_t = \{\gamma_t^i, i = 1, \dots, n\}$. A game is open-loop if no player obtains any dynamic information during the decision process. Therefore an open-loop strategy is any adapted and càdlàg process $\eta = (\eta_t)_{t\in[0,T]}$ with values in a fixed action set $A \subset \mathbb{R}$ satisfying the integrability condition $\mathbb{E}\big[\int_0^T |\eta_t|^2\, dt\big] < \infty$. We denote the set of all these admissible open-loop controls by $\mathcal{A}$ and we will consider as control space A the whole real line $\mathbb{R}$.

Since due to the formulation of the model the game under study is symmetric, we can focus on a representative player, say player i, implying that the following holds for each player $i = 1, \dots, n$.

Let $\hat{\gamma} = (\hat{\gamma}^i, \dots, \hat{\gamma}^n)$ be an admissible strategy profile. To prove that it is a Nash equilibrium for this game, we need to show that, when all the opponents j, with $j \neq i$, are following $\hat{\gamma}^j$, $\hat{\gamma}^i$ is a best response for player i, meaning that $\hat{\gamma}^i$ is a minimising control. We will solve the optimization problem faced by player i via the Pontryagin's maximum principle, developed in the stochastic framework in [ØS05]. The Hamiltonian related to the minimisation problem of bank i is the function

$$H^i(t, x, \gamma, y^i, q^i, r^i) : [0, T] \times \mathbb{R}^n \times A^n \times \mathbb{R}^n \times \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n} \to \mathbb{R}$$

defined by

$$\begin{aligned} H^i(t, x, \gamma, y^i, q^i, r^i) &= \nu f^i(x, \gamma) + (a(\bar{x} - x) + \nu\gamma) \cdot y^i + \sigma \mathrm{tr}(q^i) + \gamma \cdot \mathrm{diag}(r^i) \\ &= \nu \left(\frac{(\gamma^i)^2}{2} - \theta(\bar{x} - x^i)\gamma^i + \frac{\varepsilon}{2}(\bar{x} - x^i)^2 \right) + \sum_{k=1}^{n} \left[a(\bar{x} - x^k) + \nu\gamma^k \right] y^{i,k} \\ &\qquad + \sigma \sum_{k=1}^{n} q^{i,k,k} + \sum_{k=1}^{n} \gamma^k r^{i,k,k}. \end{aligned} \tag{4.6}$$

The processes $Y_t^i = \{Y_t^{i,k} : k = 1, \dots, n\}$, $Q_t^i = \{Q_t^{i,k,j} : k, j = 1, \dots, n\}$ and $R_t^i = \{R_t^{i,k,j} : k, j = 1, \dots, n\}$ appearing in the definition of the Hamiltonian H^i are the so called adjoint processes, which are defined as the solutions of the following BSDEs with jumps (see, e.g., [Del13, Theorem 3.1.1])

$$\begin{cases} dY_t^{i,k} = -\frac{\partial H^i(t, X_t, \gamma_t, Y_t^i, Q_t^i, R_t^i)}{\partial x^k} \, dt + \sum_{j=1}^{n} Q_t^{i,k,j} \, dW_t^j + \sum_{j=1}^{n} R_{t-}^{i,k,j} \, d\tilde{P}_t^j \\ Y_T^{i,k} = \frac{\partial g^i}{\partial x^k}(X_T) \,, \end{cases} \tag{4.7}$$

for $k = 1, \dots, n$.

To compute an optimal strategy, we will exploit the following Theorem.

Theorem 4.1.1. *Let $\hat{\gamma} = (\hat{\gamma}^1, \dots, \hat{\gamma}^n)$ be an admissible strategy profile, $\hat{\gamma} \in \mathcal{A}^n$, and $\hat{X} = X^{\hat{\gamma}}$ the corresponding controlled state. Suppose that there exists a solution $(\hat{y}^i, \hat{q}^i, \hat{r}^i)$ of the corresponding adjoint SDE* (4.7) *such that*

$$\mathbb{E} \left[\int_0^T \hat{q}_t^i (\hat{q}_t^i)^\top + \sum_{j,k=1}^{n} \left| \hat{r}_t^{i,j,k} \right|^2 \, dt \right] < \infty \,. \tag{4.8}$$

Moreover, suppose that for all $t \in [0, T]$

$$H^i(t, \hat{X}_t, \hat{\gamma}_t, \hat{y}_t^i, \hat{q}_t^i, \hat{r}_t^i) = \inf_{\alpha \in A} H^i(t, \hat{X}_t, \hat{\gamma}_t^1, \dots, \hat{\gamma}_t^{i-1}, \alpha, \hat{\gamma}_t^{i+1}, \dots, \hat{\gamma}_t^n, \hat{y}_t^i, \hat{q}_t^i, \hat{r}_t^i)$$

and that

$$\hat{H}^i(x) = \min_{\alpha \in A} H^i(t, x, \hat{y}_t^1, \dots, \hat{\gamma}_t^{i-1}, \alpha, \hat{\gamma}_t^{i+1}, \dots, \hat{\gamma}_t^n, \hat{y}^i, \hat{q}^i, \hat{r}^i)$$

is a well-defined convex function, for all $t \in [0,T]$. Then, $\hat{\gamma}^i$ is optimal, meaning that it minimises J^i.

Proof. See Theorem 3.4 in [ØS05] □

The previous theorem suggests to consider as candidate for the optimal control $\hat{\gamma}^i$ the process which minimise the Hamiltonian H^i with respect to all possible control values, that is

$$\hat{\gamma}^i = \theta(\bar{x} - x^i) - y^{i,i} - \frac{1}{\nu} r^{i,i,i} \,. \tag{4.9}$$

Observe that the convexity of f^i in (x, γ) provides convexity also for H^i and $\hat{H}^i$ in (x, γ) and x, respectively. Then, the following step is to explicitly solve the BSDEs given in equation (4.7).

Note that the drift term in the definition of process Y^i as adjoint process, that is equation (4.7), is given by

$$-\frac{\partial H^i(t, x, \gamma, y^i, q^i, r^i)}{\partial x^k} = \nu\theta \left(\frac{1}{n} - \delta_{i,k}\right) \gamma_t^i - \nu\varepsilon \left(\frac{1}{n} - \delta_{i,k}\right) (\bar{x} - x^i) - \frac{a}{n} \sum_{j=1}^n (y^{i,j} - y^{i,k}) \,.$$

Due to its linearity with respect to $\left(\frac{1}{n} - \delta_{i,k}\right)(\bar{x} - x^i)$, it is natural to consider as ansatz for $Y_t^{i,k}$ a linear process in the same difference, i.e.

$$Y_t^{i,k} = \left(\frac{1}{n} - \delta_{i,k}\right) (\bar{X}_t - X_t^i) \phi_t, \tag{4.10}$$

where $\delta_{i,j}$ is the Kronecker delta and ϕ is a deterministic scalar function of class $C^1([0,T])$. To guarantee the final condition of process $Y^{i,k}$ as requested in (4.7), that is

$$Y_T^{i,k} = c \left(\frac{1}{n} - \delta_{i,k}\right) \left(\bar{X}_T - X_T^i\right) \frac{\partial g^i}{\partial x^k}(X_T) \,,$$

we require that ϕ satisfies the terminal condition $\phi_T = c$. By applying Itô's Lemma to the ansatz (4.10) and exploiting the SDEs (4.1) and (4.3), it is found that $Y^{i,k}$ solves the following SDE:

$$\begin{aligned} dY_t^{i,k} &= \left(\frac{1}{n} - \delta_{i,k}\right) d(\bar{X}_t - X_t^i)\phi_t + \left(\frac{1}{n} - \delta_{i,k}\right) (\bar{X}_t - X_t^i) \dot{\phi}_t \, dt \\ &= \left(\frac{1}{n} - \delta_{i,k}\right) \left[\nu\phi_t(\bar{\gamma}_t - \gamma_t^i) + (\dot{\phi}_t - a\phi_t)(\bar{X}_t - X_t^i)\right] dt \\ &\qquad + \left(\frac{1}{n} - \delta_{i,k}\right) \frac{\phi_t \sigma}{n} \sum_{j=1}^n \left(dW_t^j - dW_t^i\right) \\ &\qquad + \left(\frac{1}{n} - \delta_{i,k}\right) \frac{\phi_t}{n} \sum_{j=1}^n \left(\gamma_{t-}^j \, d\tilde{P}_t^j - \gamma_{t-}^i \, d\tilde{P}_t^i\right) \end{aligned} \tag{4.11}$$

where $\bar{\gamma}_t = \frac{1}{n}\sum_{k=1}^n \gamma_t^k$ denotes the average value of all control processes at time t. Comparing (4.7) and (4.11) under the ansatz (4.10) yields

$$Q_t^{i,k,j} = \sigma\left(\frac{1}{n} - \delta_{i,k}\right)\left(\frac{1}{n} - \delta_{i,j}\right)\phi_t, \tag{4.12}$$

$$R_t^{i,k,j} = \left(\frac{1}{n} - \delta_{i,k}\right)\left(\frac{1}{n} - \delta_{i,j}\right)\phi_t\gamma_t^j, \tag{4.13}$$

for all indeces $k, j = 1, \ldots, n$. Being ϕ a bounded function and $\gamma^j \in \mathcal{A}$ for all j, these processes satisfy the desired regularity condition (4.8).

Moreover, by (4.10) and (4.13), it follows that the optimal process $\hat{\gamma}^i$ given in (4.9) solves

$$\hat{\gamma}_t^i = \theta(\bar{X}_t - X_t^i) - \left(\frac{1}{n} - 1\right)\phi_t(\bar{X}_t - X_t^i) - \frac{1}{\nu}\left(\frac{1}{n} - 1\right)^2\phi_t\hat{\gamma}_t^i\,,$$

and therefore the optimal best response $\hat{\gamma}^i$ turns out to be

$$\hat{\gamma}_t^i = \frac{\theta + \left(1 - \frac{1}{n}\right)\phi_t}{1 + \frac{1}{\nu}\left(1 - \frac{1}{n}\right)^2\phi_t}(\bar{X}_t - X_t^i)\,. \tag{4.14}$$

Then, at any time $t \in [0, T]$, the optimal strategy of player i is proportional to the distance between his state position and the average state with *rate*

$$\psi_t = \frac{\theta + \left(1 - \frac{1}{n}\right)\phi_t}{1 + \frac{1}{\nu}\left(1 - \frac{1}{n}\right)^2\phi_t}\,. \tag{4.15}$$

Furthermore, it should be noted that even if in principle we were looking for an open-loop optimal strategy, it turned out to have a closed-loop structure, since $\hat{\gamma}_t^i = \hat{\gamma}^i(t, X_t)$.

To complete the description of the optimal open-loop strategy $\hat{\gamma}^i$, we need to provide a characterisation of the function ϕ appearing in the definition of the adjoint process $Y^{i,k}$, given in equation (4.10). From the definition of the Hamiltonian H^i, equation (4.6), the ansatz (4.10) and the related implications, the SDE (4.7) becomes

$$dY_t^{i,k} = \left(\frac{1}{n} - \delta_{i,k}\right)\left[\nu\theta\psi_t - \nu\varepsilon + a\phi_t\right](\bar{X}_t - X_t^i)\,dt + \sum_{j=1}^n\left(Q_t^{i,k,j}\,dW_t^j + R_{t-}^{i,j,k}\,d\tilde{P}_t^j\right). \tag{4.16}$$

Since both equations (4.11) and (4.16) hold simultaneously, we have that the following equality must hold

$$\dot{\phi} - a\phi - \nu\psi_t\phi = \nu\theta\psi_t - \nu\varepsilon + a\phi_t$$

and in turn this implies that ϕ_t need to solve the ODE

$$\begin{aligned}\left(1 + \frac{1}{\nu}\left(1 - \frac{1}{n}\right)^2\phi_t\right)\dot{\phi}_t &= \left[\nu + \frac{2a}{\nu}\left(1 - \frac{1}{n}\right)\right]\left(1 - \frac{1}{n}\right)\phi_t^2 \\ &\quad + \left[\nu\theta\left(2 - \frac{1}{n}\right) - \varepsilon\left(1 - \frac{1}{n}\right)^2 + 2a\right]\phi_t + \nu(\theta^2 - \varepsilon)\,,\end{aligned} \tag{4.17}$$

with terminal condition $\phi_T = c$.

Remark 4.1.2. Observe that a solution ϕ to the ODE (4.17) exists and it can be computed at least in implicit form.

ϕ solves the following final value Cauchy problem

$$\begin{cases} \dot{\phi}(t) = F(\phi(t))\,, \\ \phi(T) = c > 0\,, \end{cases}$$

where F is given by

$$F(u) = \frac{Au^2 + Bu + C}{1 + ku},$$

and k, A, B and C are fixed constants depending on the model parameters, namely

$$\begin{aligned} k &= \frac{1}{\nu}\left(1 - \frac{1}{n}\right)^2, \\ A &= \left(\nu + \frac{2a}{\nu}\left(1 - \frac{1}{n}\right)\right)\left(1 - \frac{1}{n}\right) > 0\,, \\ B &= \nu\theta\left(2 - \frac{1}{n}\right) - \varepsilon\left(1 - \frac{1}{n}\right)^2 + 2a\,, \\ C &= \nu(\theta^2 - \varepsilon) < 0\,. \end{aligned}$$

The parameter k is strictly positive as soon as $\nu > 0$ and $n \geq 2$, which we can safely assume to rule out trivialities. Performing the time reversal $\tau = T - t$ we can consider the equivalent Cauchy problem

$$\begin{cases} \dot{\phi}(\tau) = -F(\phi(\tau)) \\ \phi(0) = c > 0\,, \end{cases} \tag{4.18}$$

which is uniquely solvable in the domain $D_k = [0, T] \times \left(-\frac{1}{k}, \infty\right)$. Indeed, since $-F$ and $-\dot{F}$ are continuous functions in this domain D_k, standard results assures the existence of a unique C^1 solutions to problem (4.18) for any initial condition $c > -\frac{1}{k}$.

4.2 The limiting game

We continue the study of this model by computing the solution of the MFG obtained from the game introduced in Section 4.1 when the number of players grows to infinity. In particular, we will study the convergence of the n-player Nash equilibria towards the related MFG solution. Clearly, this is a theoretical argument since the financial interpretation of the model gets lost, being the participants of a lending market finite and rather few.

Firstly, we introduce the corresponding limiting game as explained in Chapter 1. Let $m\colon [0, T] \to \mathbb{R}$ be a given càdlàg function, $m \in D$, representing a candidate for the evolution of the expected average state of the system, that is $\mathbb{E}[\bar{X}_t(n)]$, when $n \to \infty$. In this limit case, a representative player aims at minimising the expected cost

$$J(\gamma) = \mathbb{E}\left[\int_0^T \nu\left(\frac{\gamma_t^2}{2} - \theta\gamma_t(m(t) - X_t) + \frac{\varepsilon}{2}(m(t) - X_t)^2\right) dt + \frac{c}{2}(m(T) - X_T)^2\right]$$

among all the admissible strategies $\gamma \in \mathcal{A}$, subject to the dynamics

$$dX_t = [a(m(t) - X_t) + \nu\gamma_t]\, dt + \sigma\, dW_t + \gamma_{t-}\, d\tilde{P}_t, \quad X_0 = \xi. \tag{4.19}$$

As before, $\mathcal{A}$ represents the set of the càdlàg and adapted processes with values in A and such that $\mathbb{E}\left[\int_0^T |\gamma_t|^2\, dt\right] < \infty$, W and P denote a standard Brownian motion and a Poisson process with constant intensity $\nu > 0$, respectively, and ξ is the initial condition of the state process. W, P and ξ are assumed to be independent. In view of Definition 1.2.1, a MFG solution is any admissible strategy $\hat{\gamma} \in \mathcal{A}$ which minimises the objective function J, i.e.

$$\hat{\gamma} = \arg\inf_{\gamma\in\mathcal{A}} J(\gamma)\,,$$

and at the same time satisfies the MFG consistency condition (1.8), that in this case reduces to

$$\mathbb{E}[X_t^{\hat{\gamma}}] = m_t \quad \forall t \in [0, T]\,.$$

As for the game with a finite number of players, the problem is solved via the Pontryagin maximum principle. In this case, the Hamiltonian turns out to be

$$H(t, x, \gamma, y, q, r) = \nu\left(\frac{\gamma^2}{2} - \theta\gamma(m(t) - x) + \frac{\varepsilon}{2}(m(t) - x)^2\right) + [a(m(t) - x) + \nu\gamma]y + \sigma q + \gamma r$$

where (y, q, r) are the adjoint processes, defined as the triple (Y, Q, R) solving

$$\begin{cases} dY_t = -\frac{\partial H(t,X_t,Y_t,Q_t,R_t,\gamma_t)}{\partial x}\, dt + Q_t\, dW_t + R_{t-}\, d\tilde{P}_t\,, \\ Y_T = c(X_T - m(T))\,. \end{cases} \tag{4.20}$$

Note that this time, contrary to what happen in the n-player game, the optimization problem is one-dimensional and therefore Y, Q and R are real-valued stochastic processes. Then, the Hamiltonian $H(t, x, \cdot, y, q, r)$ attains its minimum when

$$\hat{\gamma} = \theta(m(t) - x) - y - \frac{1}{\nu}r\,.$$

In this case the dynamics of the related control state $X = X^{\hat{\gamma}}$ is given by

$$\begin{cases} dX_t = [(a + \nu\theta)(m(t) - X_t) - \nu Y_t - R_t]\, dt + \sigma\, dW_t + [\theta(m(t-) - X_{t-}) \\ \qquad\qquad -Y_{t-} - \frac{1}{\nu}R_{t-})]\, d\tilde{P}_t \\ X_0 = \xi\,, \end{cases} \tag{4.21}$$

whereas the triple (Y, Q, R) solves

$$\begin{cases} dY_t = \big[(a+\nu\theta)Y_t + \theta R_t + \nu(\varepsilon-\theta^2)(m(t)-X_t)\big]\ dt + Q_t\, dW_t + R_{t-}\, d\tilde{P}_t\,, \\ Y_T = c(X_T - m(T))\,. \end{cases} \tag{4.22}$$

The two stochastic differential equations (4.21)-(4.22) are linear in X and Y, and therefore we can firstly solve for their expected value, i.e. $\mathbb{E}[X_t]$ and $\mathbb{E}[Y_t]$. In fact, by taking expectation in both sides of equation (4.21) and using the martingale property of the integrals with respect to Brownian motion and compensated Poisson process, it is found that $\mathbb{E}[X_t]$ solves

$$d\mathbb{E}[X_t] = [(a+\nu\theta)(m(t)-\mathbb{E}[X_t]) - \nu\mathbb{E}[Y_t] + \mathbb{E}[R_t]]\ dt.$$

Since a MFG solution $\hat{\gamma}$ is required to guarantee the consistency condition $\mathbb{E}[X_t^{\hat{\gamma}}] = m_t$, to find a solution it must be the case that m evolves according to

$$dm(t) = -\nu\mathbb{E}[Y_t] + \mathbb{E}[R_t]\, dt\,. \tag{4.23}$$

Now, in order to solve the BSDE (4.22) and therefore to find explicitly the adjoint processes, we make the *usual* ansatzes, namely

$$Y_t = -\phi_t(m(t)-X_t)\,,\quad Q_t = \sigma\phi_t\,,\quad R_t = \frac{\theta+\phi_t}{1+\frac{1}{\nu}\phi_t}\phi_t(m(t)-X_t)\,, \tag{4.24}$$

for some deterministic function ϕ of class $C^1([0,T])$ with final value $\phi_T = c$. As before, this final condition over ϕ assures that $Y_T = c(X_T - m(T))$ as requested. Observe that, defined as in equation (4.24), the adjoint processes Y, R and Q are the limit of the processes consider in the n-bank case given in equations (4.10), (4.12) and (4.13), when n tends to infinity.

Due to the fact that both processes Q and R are proportional to the difference $m(t) - \mathbb{E}[X_t]$ and that $m(t) = \mathbb{E}[X_t]$ for all $t \in [0,T]$ when a MFG solution is chosen as control process, it follows that in the optimal case $\mathbb{E}[Q_t] = \mathbb{E}[R_t] = 0$ for all $t \in [0,T]$.

By plugging the ansatzes in the BSDE (4.22), it is found that the process Y solves

$$\begin{aligned} dY_t = &\left(-(a+\nu\theta)\phi_t + \theta\frac{\theta+\phi_t}{1+\frac{1}{\nu}\phi_t}\phi_t + \nu(\varepsilon-\theta^2)\right)(m(t)-X_t)dt + \sigma\phi_t\, dW_t \\ &+ \frac{\theta+\phi_t}{1+\frac{1}{\nu}\phi_t}\phi_t(m(t-)-X_{t-})\, d\tilde{P}_t \end{aligned} \tag{4.25}$$

and, at the same time, by differentiating its definition, we have that

$$
\begin{aligned}
dY_t &= \Big(-\dot{\phi}_t(m(t)-X_t) + \phi_t\left((a+\nu\theta)(m(t)-X_t) - \nu Y_t - R_t + \nu\mathbb{E}[Y_t] + \mathbb{E}[R_t]\right)\Big)\, dt \\
&\qquad\qquad\qquad\qquad + \sigma\phi_t\, dW_t + \frac{\theta+\phi_t}{1+\frac{1}{\nu}\phi_t}\phi_t(m(t-)-X_{t-})\, d\tilde{P}_t \\
&= \left(-\dot{\phi}_t + \phi_t\left(a+\nu\theta+\nu\phi_t - \frac{\theta+\phi_t}{1+\frac{1}{\nu}\phi_t}\phi_t\right)\right)(m(t)-X_t)\, dt + \sigma\phi_t\, dW_t \\
&\qquad\qquad\qquad\qquad + \frac{\theta+\phi_t}{1+\frac{1}{\nu}\phi_t}\phi_t(m(t-)-X_{t-})\, d\tilde{P}_t,
\end{aligned}
$$

where once again we used the identity $m(t) = \mathbb{E}[X_t]$ and its implications, namely $\mathbb{E}[Y_t] = \mathbb{E}[R_t] = 0$ and equation (4.23).

Therefore (Y, Q, R) as defined in equation (4.24) is the solution to the BSDE (4.22) as soon as ϕ_t solves the following Cauchy problem

$$
\begin{cases}
\left(1+\frac{1}{\nu}\phi_t\right)\dot{\phi}_t = \left(\nu+\frac{2a}{\nu}\right)\phi_t^2 + (2(a+\nu\theta)-\varepsilon)\phi_t - \nu(\varepsilon-\theta^2)\,, \\
\phi(T) = c\,,
\end{cases}
\tag{4.26}
$$

and then the optimal control strategy turns out to be

$$
\hat{\gamma}_t = \frac{\theta+\phi_t}{1+\frac{1}{\nu}\phi_t}(\mathbb{E}[X_t]-X_t)\,.
$$

Observe that this can also be obtained as limit for $n \to \infty$ of the Nash equilibrium computed before in the n-player game (see equations (4.14) and (4.17)). Figure 4.1 displays the behaviour of ϕ, solution of the ODE (4.17), for different values of players' number n. As n increases, the graph of $\phi = \phi(n)$ quickly converges to the solution we found in the game with an infinite number of players, given in equation (4.26).

4.3 Simulations

We conclude this chapter by performing some numerical analysis on the n-player game introduced in the Section 4.1. In particular, we examine the dependence of the open-loop Nash equilibrium on the intensity of the Poisson processes, ν.

First, in Figure 4.2, we consider a general possible scenario of the model. Figure 4.2a shows the dynamic of each bank, with the one of player 1 marked in bold, and the evolution of the average level of the reserves in red, whereas Figure 4.2b shows the corresponding optimal strategy for player 1. This optimal strategy is such that at each jump time in his own dynamic, that is when bank 1 can adjust its reserves by borrowing or lending money, the corresponding state process moves closer to the average level $\bar{X}$. This is clearly expected due to the form of the cost functions, being such an average a benchmark for each bank. Then, since both the cost functions penalize each deviation

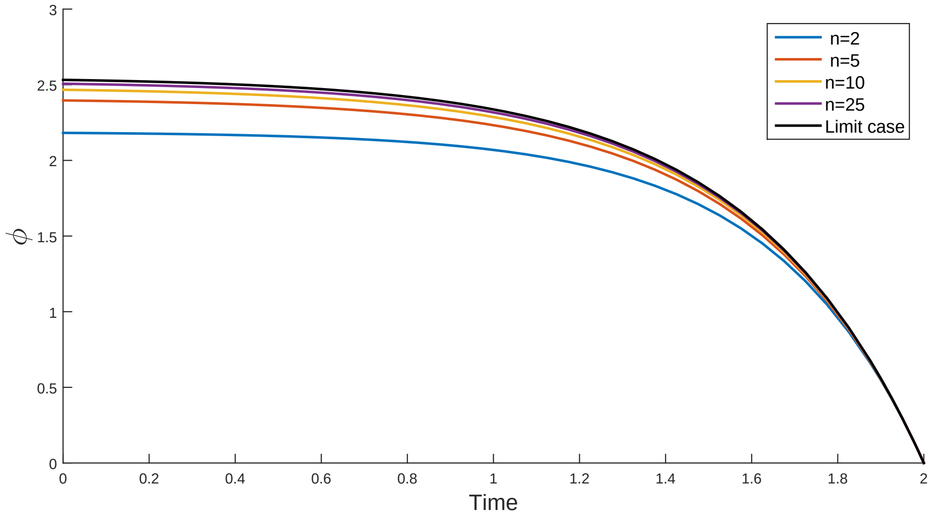
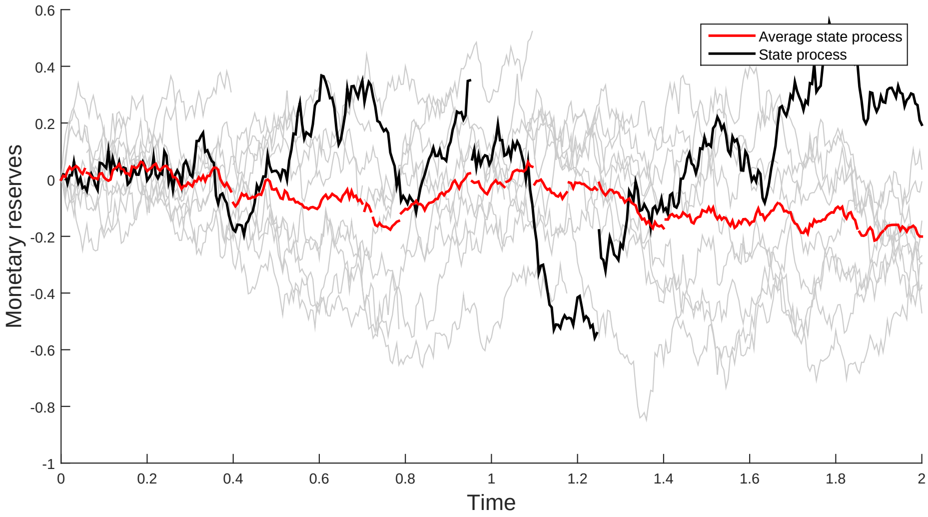


Figure 4.1: Plots of ϕ, solution of the ODE (4.17) for different values of n. Model's parameter: $T = 2$, $a = 1$, $\theta = 1$, $\varepsilon = 10$, $\nu = 0.7$, $c = 0$

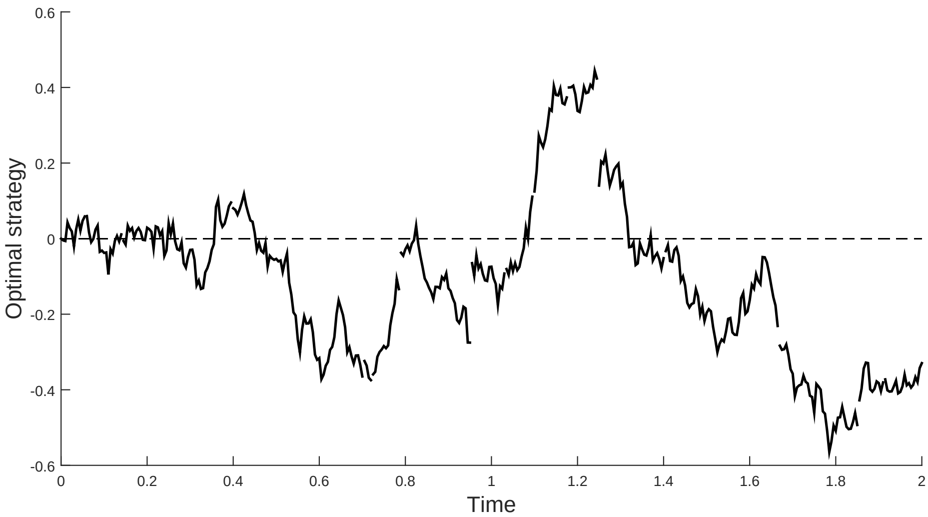
of the players' state from the average state $\bar{X}$, each player try to replicate $\bar{X}$ for its own reserve level. Therefore, as it can be seen in the figures, when the reserve level of bank 1 is below the average value $\bar{X}$, the optimal strategy is positive, meaning that bank 1 wishes to raise its reserves up, and on the contrary when X^1 is above $\bar{X}$, the optimal strategy is negative, meaning that bank 1 wants to decrease its reserves.

It could be noted that, even if after a jump, the state of player 1 moves closer to the benchmark $\bar{X}_t$, these two values do not (always) match exactly. This depends on two reasons. First, reaching $\bar{X}$ can be too costly due to the quadratic cost of the control in the running cost function. Second, the choice of each bank, say bank 1, at time $t-$ depends on the difference between its reserve X^1_{t-} and the average reserves $\bar{X}_{t-}$ computed immediately before the jump time t. But at the same time, $\bar{X}$ might have a jump at the same time t, as a consequence of the jump in the reserves of bank 1. So even if $X^1_t = \bar{X}_{t-}$, it may occur that $X^1_t \neq \bar{X}_t$.

Now, we focus on the variation of the equilibrium strategy due to changes in the intensity ν of the Poisson processes, representing the liquidity parameter of the interbank market. It is more convenient for the analysis to consider the function $\psi\colon [0,T] \to \mathbb{R}$, as defined in equation (4.15), so that the open-loop optimal strategy can be expressed as $\hat{\gamma}^i_t = \psi_t(\bar{X}_t - X^i_t)$, for $t \in [0,T]$. Hence, whenever one of the Poisson processes, say P^i, jumps, bank i would modify its reserves by an amount $\hat{\gamma}^i_t$ which is proportional to the difference $\bar{X}_{t-} - X^i_{t-}$ just before the jump, with a proportionality factor ψ_t. Routine



(a) Monetary reserves



(b) Optimal strategy

Figure 4.2: Possible scenario. Model's parameters: $n = 10$, $T = 2$, $a = 1$, $\sigma = 0.8$, $X_0 = 0$, $\theta = 1$, $\varepsilon = 10$, $c = 0$, $\nu = 0.7$.

computation shows that ψ solves the following ODE

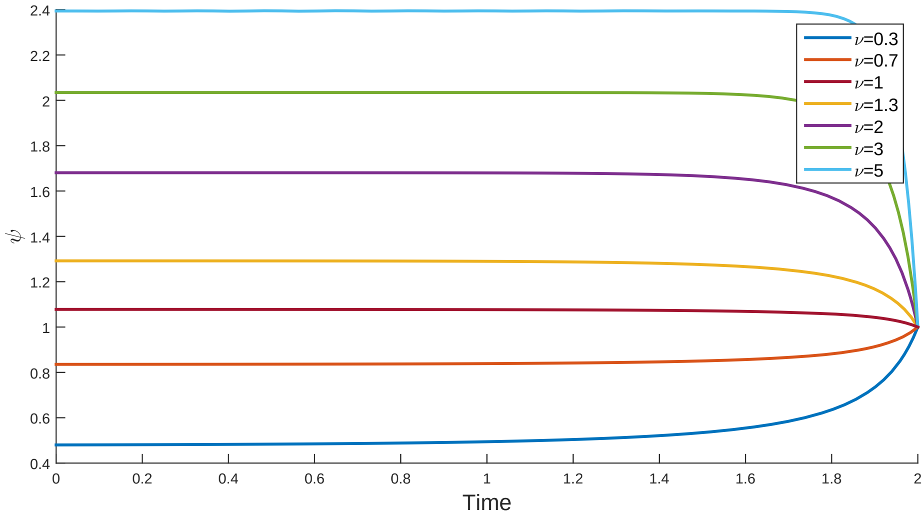
$$\begin{aligned}\left(1-\frac{1}{\nu}\left(1-\frac{1}{n}\right)\theta\right)^2\dot{\psi}_t &= \left[\nu+\frac{2a}{\nu}\left(1-\frac{1}{n}\right)\right]\left(1-\frac{1}{\nu}\left(1-\frac{1}{n}\right)\psi_t\right)(\psi_t-\theta)^2\\ &+\left[\nu\theta\left(2-\frac{1}{n}\right)-\varepsilon\left(1-\frac{1}{n}\right)^2+2a\right]\left(1-\frac{1}{\nu}\left(1-\frac{1}{n}\right)\psi_t\right)^2(\psi_t-\theta)\\ &+\left[\left(1-\frac{1}{n}\right)\nu(\theta^2-\varepsilon)\right]\left(1-\frac{1}{\nu}\left(1-\frac{1}{n}\right)\psi_t\right)^3,\end{aligned}$$

with final value $\psi_T = \dfrac{\theta+\left(1-\frac{1}{n}\right)c}{1+\frac{1}{\nu}\left(1-\frac{1}{n}\right)^2 c}$, which is increasing in ν.

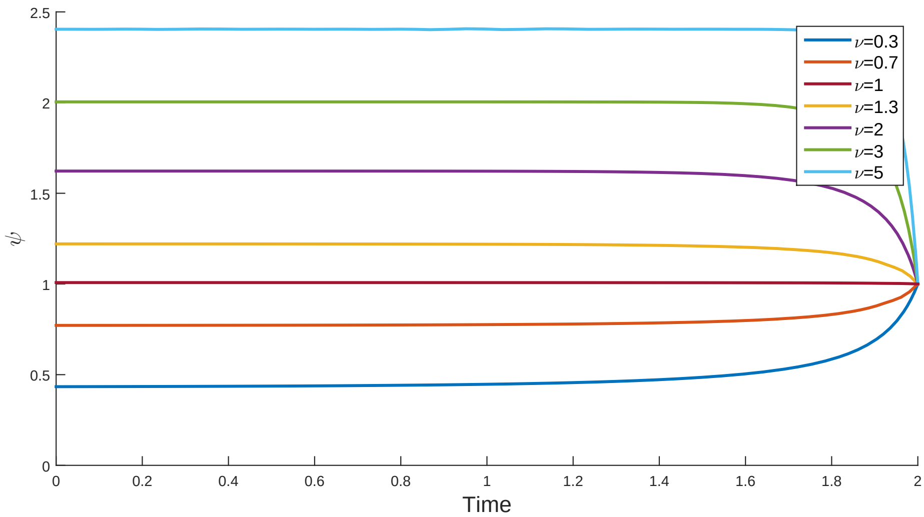
All our numerical experiments revealed that such a monotonicity behaviour propagates to the whole time interval, i.e. the proportionality factor ψ_t is increasing in the intensity ν for all $t \in [0,T]$. Here, in figures 4.3 and 4.4, we show only the behaviour of ψ as function of time $t \in [0,T]$ with $T = 2$, $n \in \{10,100\}$, $c \in \{0,1\}$, and more importantly for different values of ν. Moreover, observe that (see Fig 4.4) the final value ψ_T depends on the parameter ν whenever c is different than zero.

In the Nash equilibrium we have found, when ν is small, hence the interbank market is very illiquid in the sense that banks will have (in expectation) very few possibilities to change their reserves, the reserves will change very little proportionally to $(\bar{X}_{t-} - X^i_{t-})$. On the other hand, when ν is large, so that in expectation banks will have many occasions to lend/borrow money from the central bank, changes in their reserves will be very big proportionally to $(\bar{X}_{t-} - X^i_{t-})$. Therefore, focusing on the first case, we notice that instead of compensating the lack of liquidity (ν small), banks seem to amplify it by borrowing and lending very little.

Another interesting feature one can notice from the figures is that when ν is small, the proportionality factor ψ_t is increasing in time. When the market is very illiquid, there are very few possibility for the banks to change their reserves during the time period, so that when the maturity T is approaching, the banks knowing that they are running out of time to move their reserves closer to the average reserve $\bar{X}$, amplify their efforts, whence an increasing ψ_t. An analogue interpretation can be provided for the opposite situation of a time-decreasing ψ_t when ν is large.

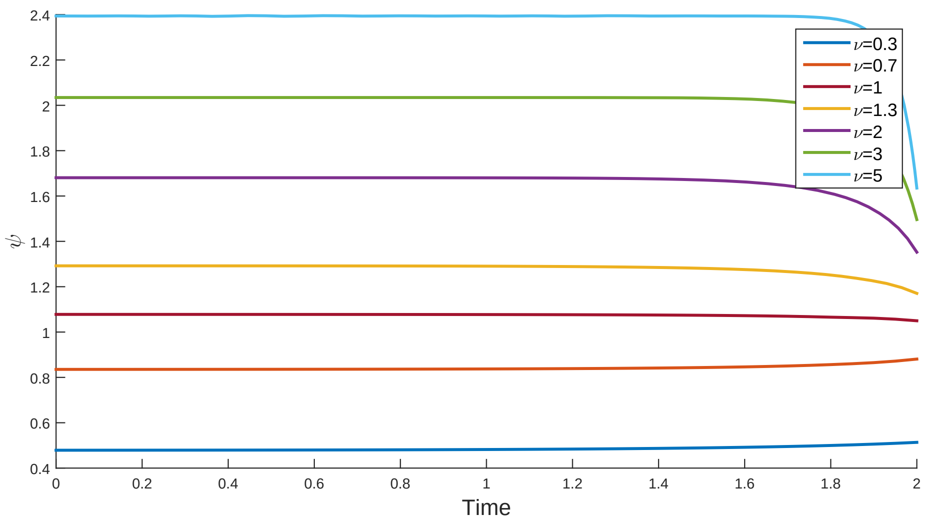


(a) $n = 10$

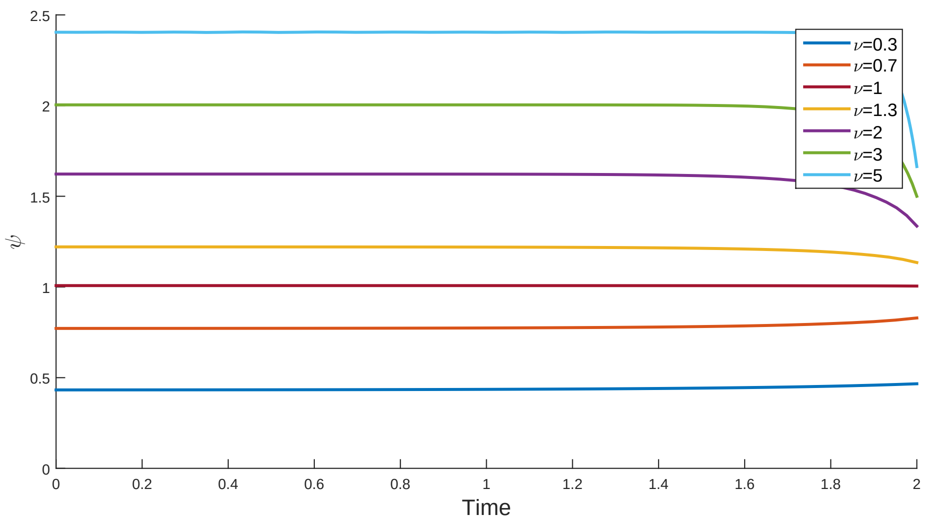


(b) n=100

Figure 4.3: Evolution of ψ for different values of ν. Model's parameters: $T = 2$, $a = 1$, $\theta = 1$, $\varepsilon = 10$, $c = 0$.



(a) $n = 10$



(b) n=100

Figure 4.4: Evolution of ψ for different values of ν. Model's parameters: $T = 2$, $a = 1$, $\theta = 1$, $\varepsilon = 10$, $c = 1$.

Part II

An optimal control approach to stochastic volatility models

Chapter 5

Stochastic optimal control theory

The objective of this chapter is to collect some useful results on stochastic optimal control theory, needed in Chapter 6. Stochastic control theory concerns controlled dynamical systems when subjected to random perturbations. In the following we consider diffusion models, meaning that the state of the system evolves over time according to an Itô's stochastic differential equation, which depends on a controlled input chosen to achieve the best possible outcome.

Here the presentation is restricted to the key results, but a more general and detailed study of these topics can be found in [Pha09; FR12; Tou12; YZ99] which are the main references for this chapter.

5.1 Stochastic optimal control problems

Let t_0 and T be two fixed times in the interval $(0, \infty)$ such that $t_0 < T$ and let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [t_0,T]}, P)$ be a filtered probability space. Consider a dynamical system whose random state $X = X(\omega) \in \mathbb{R}^n$ evolves according to the stochastic differential equation

$$dX_t = b(t, X_t, u(t))\, dt + \sigma(t, X_t, u(t)) dW_t\,, \quad t \in [t_0, T]\,, \tag{5.1}$$

subjected to an initial condition $X_{t_0} = x$, where W_t stands for a d-dimensional Brownian motion. The dynamic $t \mapsto X_t$ depends on the controlled input u, where the value $u(t)$ represents the control applied in the system at time t, chosen with respect to the available information.

The drift function $b \colon [t_0, T] \times \mathbb{R}^n \times U \to \mathbb{R}^n$ and the diffusion function $\sigma \colon [t_0, T] \times \mathbb{R}^n \times U \to \mathbb{R}^{n \times d}$ are assumed to be continuous and Lipschitz continuous with respect to the state variable x, that is for all $x, y \in \mathbb{R}^n$, $t \in [t_0, T]$ and $v \in U$, it holds that

$$|b(t, x, v) - b(t, y, v)| + |\sigma(t, x, v) - \sigma(t, y, v)| \leq C\, |x - y|$$

for a suitable positive constant C.

To be admissible, a control $u = \{u(t) : t_0 \leq t \leq T\}$ is required to be an $\mathcal{F}_t$-adapted process, taking values in a given set of $U \subset \mathbb{R}^k$, and such that

$$\mathbb{E}\left[\int_{t_0}^{T} |b(t,0,u(t))|^2 + |\sigma(t,0,u(t))|^2 \; dt\right] < \infty \,.$$

In the following, the set of all admissible control processes will be denoted by $\mathcal{U}(t_0, x)$. Furthermore, an admissible control $u \in \mathcal{U}(t,x)$ is called feedback (or Markov) control if it is adapted to the natural filtration generated by the state process and can be written as $u_s = \nu(s, X_s^{t,x})$ for a measurable function $\nu\colon [t,T] \times \mathbb{R}^n \to U$.

Given the previous assumptions regarding the drift and the diffusion function, u being in $\mathcal{U}$ ensures that there exists a unique strong solution to the stochastic differential equation (5.1) for each initial data $(t,x) \in [t_0,T] \times \mathbb{R}^n$. Given an admissible control process $u \in \mathcal{U}(t,x)$, the unique solution of equation (5.1) starting at time t from value x will be denoted with $X^{t,x,u} = \{X_s^{t,x,u}\}_{s\in[t,T]}$, or simply with $X^{t,x} = \{X_s^{t,x}\}_{s\in[t,T]}$ if there is no ambiguity.

The performance of any control process is measured by a given cost criterion J, and the objective is to minimise it. Considering a finite horizon problem, given an initial data $(t,x) \in [t_0,T] \times \mathbb{R}^n$, the functional J is defined as

$$\begin{aligned} J(t,x;\cdot)\colon \mathcal{U} &\to \mathbb{R} \\ u \to J(t,x;u) &= \mathbb{E}\left[\int_t^T f(s, X_s^{t,x,u}, u(s))\, ds + g(T, X^{t,x,u}(T))\right], \end{aligned}$$

where f and g are two measurable functions. We refer to $J(t,x;u)$ as the expected cost associated to the control u. In order to be J well defined, $\mathcal{U}(t,x)$ is restricted to the admissible processes providing $\mathbb{E}\big[\int_t^T |f(s, X_s^{t,x,u}, u(s))|\, ds\big] < \infty$ and g is required to be is lower-bounded or of sub-quadratic growth.

Therefore, given an initial data $(t,x) \in [t_0,T] \times \mathbb{R}^n$ for the state process X, the stochastic control problem (P) considered here is to find (if it exists) an admissible control process u^* attaining the minimum of $J(t,x;\cdot)$ over all the admissible control processes $\mathcal{U}(t,x)$:

(P) Search $u^* \in \mathcal{U}(t,x)$ such that $J(t,x;u^*) = \min_{u\in\mathcal{U}(t,x)} J(t,x;u)$.

5.2 The Hamilton-Jacobi-Bellman equation

Let $V(t,x)$ be the value function associated to the minimisation problem (P), that is the infimum value of the objective function J given as a function of the initial data (t,x). Then, for all $(t,x) \in [t_0,T] \times \mathbb{R}^n$, $V(t,x)$ is defined as

$$V(t,x) = \inf_{u\in\mathcal{U}(t,x)} J(t,x;u)\,. \tag{5.2}$$

Thus, a control u^* is optimal for problem (P) if $u^* \in \mathcal{U}(t,x)$ and $V(t,x)$ equals the expected cost corresponding to u^*, that is $V(t,x) = J(t,x;u^*)$.

The bahaviour of the value function $V(t,x)$ is studied in the dynamic programming principle (DPP), a fundamental result in stochastic control theory. The DPP states that

Theorem 5.2.1 (Dynamic programming principle). *Let $\mathcal{T}_{t,T}$ be a family of finite stopping times with value in $[t,T]$. Then,*

$$\begin{aligned} V(t,x) &= \inf_{u\in\mathcal{U}} \sup_{\theta\in\mathcal{T}_{t,T}} \mathbb{E}\left[\int_t^\theta f(s,X_s^{t,x},u(s))\,ds + V(\theta, X^{t,x}(\theta))\right] \\ &= \inf_{u\in\mathcal{U}} \inf_{\theta\in\mathcal{T}_{t,T}} \mathbb{E}\left[\int_t^\theta f(s,X_s^{t,x},u(s))\,ds + V(\theta, X^{t,x}(\theta))\right]. \end{aligned} \tag{5.3}$$

The DPP asserts that the optimization problem (P), which takes under consideration the whole time interval $[t,T]$, can be split into two (or more) minimisation problems:

> Let $\theta \in [t,T]$. First, look for an admissible control process u^* which is optimal over the (possibly) shorter time window $[\theta,T]$ when the state process X starts at time θ from value $X_\theta^{t,x}$, meaning that
>
> $$V(\theta, X_\theta^{t,x}) = J(\theta, X_\theta^{t,x}; u^*)\,.$$
>
> Secondly, search for an admissible control process $u \in \mathcal{U}(t,x)$ which attains the minimum of the expectation
>
> $$\mathbb{E}\left[\int_t^\theta f(s,X_s^{t,x},u(s))\,ds + V(\theta, X_\theta^{t,x})\right].$$
>
> Naively, this is an analogous minimisation problem to (P) on the time interval $[t,\theta]$ when the terminal cost is given by $V(\theta, X_\theta^{t,x})$. This counts the infimum expected cost on the remaining time $[\theta,T]$ when the initial state is $X_\theta^{t,x,u}$.

From the DPP, also the local behaviour of the value function V can be derived, and it is described in the so called dynamic programming equation, better known as the Hamilton-Jacobi-Bellman (HJB) equation. We shortly, and heuristically, recall how the HJB equation can be essentialy obtained by the DPP.

Let $h > 0$ and consider the optimization problem (P) over the time window $[t,t+h] \subset [t_0,T]$. Considering the constant control process $\nu(s) = \nu$ for all $s \in [t,t+h]$, with $\nu \in U$, and the corresponding state process $X^{t,x,\nu}$, it holds that

$$V(t,x) \le \mathbb{E}\left[\int_t^{t+h} f(s,X_s^{t,x,\nu},\nu)\,ds + V(t+h, X_{t+h}^{t,x,\nu})\right], \tag{5.4}$$

being the value function V, by definition, the infimum over all the admissible processes of the expected value in the RHS of the previous inequality. If V is a smooth enough function, let say $V \in C^{1,2}([t_0,T] \times \mathbb{R}^n)$, applying Itô's Lemma to the stochastic process $V(s, X_s^{t,x,\nu})$ with $s \in [t, t+h]$, it follows that

$$V(t+h, X_{t+h}^{t,x}) = V(t,x) + \int_t^{t+h} \left(\frac{\partial V}{\partial t} + \mathcal{L}^\nu V \right) (s, X_s^{t,x})\, ds + Q_t\,. \tag{5.5}$$

Here $Q_t = \int_t^{t+h} \sigma(s, X_s^{t,x,\nu}, \nu) \cdot D_x V(s, X_s^{t,x,\nu})\, dW_s$ is a (local) martingale, whereas the linear second order operator $\mathcal{L}^\nu$ associated to the controlled process X^ν, where ν is the constant control process $\nu(s) \equiv \nu$, is defined by

$$\mathcal{L}^\nu V = b(t,x,\nu) \cdot D_x V + \frac{1}{2} \mathrm{tr}\left[a(t,x,\nu) \cdot D_x^2 V \right].$$

Moreover D_x and D_x^2 stand respectively for the gradient and the Hessian operator with respect to x, the trace operator $\mathrm{tr}[\cdot]$, which is defined on the set of symmetric, positive semidefinite $m \times m$ matrices $\mathcal{S}_m$, is given by

$$\mathrm{tr}[M] = \sum_{i=1}^m M_{ii}\,, \quad \text{for all } M \in \mathcal{S}_m\,,$$

and $a(t,x,u) = \sigma(t,x,u)\sigma'(t,x,u)$. Substituting equation (5.5) into the inequality (5.4) implies

$$\mathbb{E}\left[\int_t^{t+h} \left(\frac{\partial V}{\partial t} + \mathcal{L}^\nu V \right) (s, X_s^{t,x,\nu}) + f(s, X_s^{t,x,\nu}, \nu)\, ds \right] \geq 0$$

and letting $h \to 0$, the mean value Theorem ensures that

$$\left(\frac{\partial V}{\partial t} + \mathcal{L}^\nu V \right)(t,x) + f(t,x,\nu) \geq 0\,. \tag{5.6}$$

Here we strongly use the continuity of the value function V and of its derivatives $\partial_t V$, $D_x V$ and $D_x^2 V$. Since equation (5.6) holds for each constant admissible control $\nu \in \mathcal{U}$, that is for each $\nu \in U$, then

$$\frac{\partial V}{\partial t}(t,x) + \inf_{\nu \in U} \left\{ \mathcal{L}^\nu V(t,x) + f(t,x,\nu) \right\} \geq 0\,. \tag{5.7}$$

At the same time, by definition, an optimal control process u^* satisfies

$$V(t,x) = \mathbb{E}\left[\int_t^{t+h} f(s, X_s^{t,x,u^*}, u^*(s))\, ds + V(t+h, X_{t+h}^{t,x,u^*}) \right]$$

and, arguing as before, this time it is found that

$$\frac{\partial V}{\partial t}(t,x) + \mathcal{L}^{u^*(t)} V(t,x) + f(t,x,u^*(t)) = 0\,.$$

Combining the two equations (5.6) and (5.7), the value function $V(t,x)$ is required to satisfy the following partial differential equation

$$\frac{\partial V}{\partial t}(t,x) + \inf_{\nu \in U} \{\mathcal{L}^{\nu} V(t,x) + f(t,x,\nu)\} = 0\,, \tag{5.8}$$

called the Hamilton-Jacobi-Bellman equation associated to the optimization problem (P). Furthermore, considering the final horizon T, the definition of the value function implies that its natural terminal condition is

$$V(T,x) = g(x)\,, \quad \forall x \in \mathbb{R}^n\,.$$

Usually, the HJB equation (5.8) is shortly written as

$$\begin{cases} -\frac{\partial V}{\partial s}(s,x) - H(s,x,D_x V(s,x), D_x^2 V(s,x)) = 0\,, & \forall (s,x) \in [t,T) \times \mathbb{R}^n\,, \\ V(T,x) = g(x)\,, \quad \forall x \in \mathbb{R}^n \end{cases} \tag{5.9}$$

where, for $(t,x,p,M) \in [t_0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{S}_n$, the Hamiltonian function H is defined by

$$H(t,x,p,M) = \inf_{u \in U} \left[b(t,x,u) \cdot p + \frac{1}{2} \mathrm{tr}\left[a(t,x,u)M\right] + f(t,x,u) \right]. \tag{5.10}$$

5.2.1 The verification Theorem

Therefore, if the value function V exists smooth enough, it solves the HJB equation (5.9). Then, the following natural and crucial question is if also the *converse* holds true, that is when, or under which conditions, a solution to the HJB equation coincides with the value function of the corresponding optimization problem. An answer to this query is provided by the Verification Theorem.

Theorem 5.2.2 (Verification Theorem)**.** *Let v be a function in $C^{1,2}([t_0,T) \times \mathbb{R}^n) \cap C^0([t_0,T] \times \mathbb{R}^n)$ satisfying for a suitable constant C the quadratic growth condition*

$$|v(t,x)| \leq C(1+|x|^2)\,, \quad \textit{for all } (t,x) \in [t_0,T) \times \mathbb{R}^n.$$

Assume that v is a classical solution to the HJB equation

$$\begin{cases} -\frac{\partial v}{\partial s}(s,x) - \inf_{u \in U} \left[b(t,x,u) \cdot D_x v + \frac{1}{2} \mathrm{tr}\left[a(t,x,u) D_x^2 v\right] + f(t,x,u) \right] = 0\,, \\ V(T,x) = g(x)\,, \end{cases}$$

for all $(s,x) \in [t,T) \times \mathbb{R}^n$. Then, $v \leq V$ on $[0,T] \times \mathbb{R}^n$.

Assume further that there exists a process $u^* = u^*(t,x) \in \mathcal{U}$ for a measurable function $u^* \colon [0,T] \times \mathbb{R}^n \to U$, which along with the related controlled state $X^* = X^{t,x,u^*}$ satisfies

$$\begin{aligned} &\inf_{u \in U} \left[b(t,X_t^*,u) \cdot D_x v + \frac{1}{2} \mathrm{tr}[a(t,X_t^*,u) D_x^2 v] + f(t,X_t^*,u) \right] \\ &\qquad = b(t,X_t^*,u^*) \cdot D_x v + \frac{1}{2} \mathrm{tr}(a(t,X_t^*,u^*) D_x^2 v) + f(t,X_t^*,u^*)\,. \end{aligned}$$

Then, v coincides with the value function V, $v(t,x) = V(t,x)$, and therefore $u^* = u^*(t,x)$ is an optimal feedback control process.

Remark 5.2.1. Here, we have assume for simplicity that the SDE (5.1) admits a unique strong solution. However, the optimization problem (P) can be solves analogously if the state equation admits a unique weak solution. Indeed, in this case, the optimization problem can be analogously solved by re-defining the control as the 5-tupla $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}, w, u)$ where

(i) $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ is a complete probability space;

(ii) $w = (w_t)_{t\in[t_0,T]}$ is a Brownian motion defined on the previous probability space, whose natural filtration augmented by the $\tilde{P}$-null sets is denoted by $(\tilde{\mathcal{F}}_t^w)_{t\in[t_0,T]}$;

(iii) u is a $\tilde{\mathcal{F}}_t^w$-adapted process such that the SDE (5.1) admits a unique solution, for any initial data.

See [YZ99] for further details.

5.2.2 Weak generalized solutions

In the previous, we consider a *smooth enough* value function V to ensure that it is a classical solution of the Hamilton-Jacobi-Bellman equation (5.9). When this is not the case, to still apply this methodology a weak formulation of solution is required. For this purpose, Crandall and Lions introduced in [CL83] the concept of viscosity solution, which properly suits for a large class of control problems since the value function V is indeed the unique viscosity solution of equation (5.9) under weaker assumptions regarding its regularity relative to the classical setting.

Another possibility is to describe the value functions as a weak generalized solution to the HJB equation, see [FS06, Chapter IV.10]. In this case $V_x \in L^1_{\text{loc}}([t_0,T]\times\mathbb{R})$ is said to be a generalized partial derivative of V with respect to variable x if

$$\int_{[t_0,T]\times\mathbb{R}} V_x\psi\, dxdt = -\int_{[t_0,T]\times\mathbb{R}} V\psi_x\, dxdt\,,$$

and analogously the generalized partial derivative of V_t and V_{xx} are given, if they exists, as the $L^1_{\text{loc}}([t_0,T]\times\mathbb{R})$ functions such that

$$\int_{[t_0,T]\times\mathbb{R}} V_t\psi\, dxdt = -\int_{[t_0,T]\times\mathbb{R}} V\psi_t\, dxdt\,,$$
$$\int_{[t_0,T]\times\mathbb{R}} V_{xx}\psi\, dxdt = \int_{[t_0,T]\times\mathbb{R}} V\psi_{xx}\, dxdt\,.$$

Then the value function V is proved to be a generalized subsolution to the HJB equation (5.9).

This is the concept of weak solution we will use in the following chapter. See Definition 6.3.1 for further details.

Chapter 6

Optimality in a controlled Heston model

This chapter studies a stochastic optimization problem where the evolution of the state process is modeled as in the Heston model, but with a further multiplicative control input in the volatility of the state. The basics of the stochastic volatility models, with particular attention on the Heston model, are presented in Section 6.1. Section 6.2 formally introduces the stochastic optimal control problem under investigation, whereas in Section 6.3 the Hamilton-Jacobi-Bellman equation associated to this optimal control problem is introduced and the existence of solutions is discussed. Lastly, in Section 6.4 we construct optimal feedback controls for a class of problems which approximates the original one.

The results of this chapter are collected in [BBDP17a].

6.1 The Heston model

Stochastic volatility models (SVMs) are widely used in a large number of different financial settings, as in the risk sector, in the interest rate policy or in insurance problems. In fact, SVMs allow for a careful analysis of relevant time series as they appear in the real financial world. Daily return data series show two peculiarities among different types of assets in different markets and in different periods, namely the *volatility clustering phenomenon* and the *fat-tailed and highly peaked distribution* relative to the normal distribution. Volatility clustering refers to the fact that large changes in prices tend to be followed by large changes, regardless of their sign, whereas small changes tend to be followed by small changes, see [Man97]. This means that, over a significant time window, it can be observed the presence of both high volatility periods and low volatility ones, separately, rather than a constant average level of volatility persisting over time.

The two above-mentioned peculiarities can be captured by the so called SVMs, a class of models characterised by the fact that the volatility of the state is a stochastic process itself.

One of the most popular SVMs is the Heston model, which we shortly introduce here recalling its main properties. The Heston model, named after his inventor, was firstly introduced in [Hes93] to price European bond and currency options, aiming at generalize and overcome (some of) the biases of the Black and Scholes model, starting with the assumption of normal distributed returns. In this case, the asset price moves according to the diffusive dynamics

$$\frac{dS_t}{S_t} = \mu\, dt + \sqrt{\nu_t} dW_1(t)\,, \tag{6.1}$$

subjected to an initial condition $S_0 = s$, where the volatility ν_t is modeled by a stochastic process satisfying

$$d\sqrt{\nu_t} = -\beta\sqrt{\nu_t}\, dt + \delta dW_2(t)\,,$$

with $\nu_0 \geq 0$. Then, the volatility of the price is no longer a deterministic function of S, but it is itself randomly distributed. Here, W_1 and W_2 are two Brownian motions which may be possibly correlated, and in the following their correlation will be denoted by ρ. By applying Itô's Lemma, it follows that ν solves the Ornstein-Uhlenbeck SDE

$$d\nu_t = 2\beta\left(\frac{\delta^2}{2\beta} - \nu_t\right) dt + 2\delta\sqrt{\nu_t}\, dW_2(t)\,. \tag{6.2}$$

As a matter of fact, the Heston model generalizes the Black and Scholes one, since if δ is identically zero, then ν becomes a deterministic function of time and the latter model is recovered. Furthermore, a crucial advantage of the Heston model is that there exists a closed-form option pricing formula for a European call written on an asset whose dynamics is given by (6.1)-(6.2).

Different choices for modeling the volatility ν allow for a multitude of models that properly represent different financial data, see, e.g, [BNS02; CIJR05; HW87; Sco87; Wig87].

Beside the fact that the volatility of returns of the underlying asset S varies stochastically over time, the key difference between the Heston and the Black and Scholes model is the correlation between the volatility of the price, ν, and the price itself, S. The natural consequence of this correlation has an impact in the skewness of the asset return distribution. Indeed a positive correlation $\rho > 0$ provides a higher variance when the asset price rises and therefore it leads to a fat right-tailed distribution. Moreover, the mean reversion of the volatility ν can explain the clustering effect: even in periods of high volatility, ν is expected to eventually return to *normal* values.

In the following of this chapter we study an optimization problem where the state dynamics is modeled by an Heston model when a multiplicative control component is added into its volatility term. Details on the mathematical setting will be specified in Section 6.2. The aim is to take under consideration the possible exogenous role of an external actor. Consider, as an example, a market sector where a relevant number of banks are exposed simultaneously. Then, to preserve stability, a Central Bank may intend to prevent abrupt changes in this market by actively intervening. Recently, a similar measure has been implemented by the European Central Bank with the so called quantitative easing monetary policy. Buying a predetermined amount of financial assets emitted mainly from (national) commercial banks, the ECB managed to rise the price of (some) interested financial assets, lower their yield and increase the money supply and therefore it provides a radical reduction of the volatility as a final result.

6.2 The controlled Heston model

Let $T \in (0, \infty)$ be a fixed time horizon and let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a filtered probability space. Let $X = (X_1, X_2) \in \mathbb{R}^2$ be a stochastic process defined by

$$\begin{cases} dX_1 = \mu X_1 \, dt + X_1 \sqrt{u X_2} \, dW_1 \,, & t \in (0,T)\,, \\ dX_2 = k(\theta - X_2) \, dt + \sigma \sqrt{X_2} \, dW_2 \,, & t \in (0,T)\,, \\ X_1(0) = X_1^0 \,, \quad X_2(0) = X_2^0 \,, \end{cases} \tag{6.3}$$

where the initial conditions X_1^0, X_2^0 are both positive. Here W_1 and W_2 are two $\mathcal{F}_t$-adapted Brownian motions, whereas μ, κ, σ and θ are fixed positive parameters. This system (6.3) extends the classical Heston model introduced in the previous section in equations (6.1)-(6.2) by adding a control u in the volatility term of the stochastic process X_1.

Let U denote the real interval $[a, b]$, with $0 < a < b < \infty$. Then, an admissible control process u is any $\mathcal{F}_t$-adapted stochastic process $u : [0, T] \to \mathbb{R}$, taking values in U. The class of all these stochastic processes will be denoted by $\mathcal{U}$. The effectiveness of any control $u \in \mathcal{U}$ is measured by the cost criterion J defined by

$$J(X_1^0, X_2^0; u) = \mathbb{E}\left[\int_0^T X_1^2(t) f(X_1(t), u(t)) \, dt + g(X_1(T))\right]. \tag{6.4}$$

whose form is mainly inspired by [FP11]. The function $f : \mathbb{R} \times U \to \mathbb{R}$ is required to satisfy the following hypotheses.

(i) f is a continuous function on $\mathbb{R} \times U$. Moreover, for each $x \in \mathbb{R}$, $u \mapsto f(x, u)$ is convex and $\inf\{f(x, u); u \in [a, b]\} = 0$.

Then, the stochastic optimal control problem (P) we consider here is:

(P) Minimise $J(X_1^0, X_2^0; u)$ over the set of all admissible control processes $u \in \mathcal{U}$.

A first problem regards the well-posedness of the state system (6.3), that is the existence of a solution X to this system for any admissible process $u \in \mathcal{U}$. It is clear that by its definition a solution X_2, if exists, should be found in the class of non negative processes on $[0, T]$. However since the diffusion term in the SDE of X_2 $x \mapsto \sigma\sqrt{x}$ is not

Lipschitz, it is not clear if such a solution exists for any initial data $X_2^0 \geq 0$. The following theorem states suitable conditions on the model parameters to ensure the existence of a weak solution $X = (X_1, X_2)$ to the system (6.3).

Theorem 6.2.1. *Assume that*

$$k\theta \geq \frac{1}{2}\sigma^2 . \tag{6.5}$$

Then, there is at least one weak solution $X = (X_1, X_2)$ for the system (6.3) *belonging to $(L^2(\Omega; C([0,T])))^2$ such that X is non negative almost surely, i.e.*

$$X_1(t) \geq 0\,,\; X_2(t) \geq 0\,, \quad \forall t \in [0,T]\,, \quad \mathbb{P}\text{-a.s.} \tag{6.6}$$

Moreover

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left(|X_1(t)|^2 + |X_2(t)|^2\right)\right] \leq \bar{C}\left(\left|X_1^0\right|^2 + \left|X_2^0\right|^2\right) + CT \tag{6.7}$$

for suitable positive constants $\bar{C}$ and C.

Proof. As first step, to show the positiveness of X_2, we approximate the second equation in (6.3) by a different SDE, namely

$$\begin{cases} dX_2^\varepsilon = k(\theta - X_2^\varepsilon)\,dt + \sigma \dfrac{X_2^\varepsilon}{\sqrt{|X_2^\varepsilon| + \varepsilon}}\,dW_2\,, \quad t \in (0,T)\,, \\ X_2^\varepsilon(0) = X_0^2\,, \end{cases} \tag{6.8}$$

where ε is an arbitrary positive constant. Since for each fixed $\varepsilon > 0$ the map $x \mapsto \frac{x}{\sqrt{|x|+\varepsilon}}$ is Lipschitz continuous, standard results, see, e.g. [Pha09, Theorem 1.3.15], ensures that the SDE (6.8) has a unique strong solution $X_2^\varepsilon \in L^2(\Omega; C([0,T]))$. After proving that the solution process $X_2^\varepsilon(t)$ is non negative for all $t \in [0,T]$, we will show that also X_2 is non negative by approximation results.

Assume that the assumption (6.5) holds, then the solution to (6.8) satisfies

$$X_2^\varepsilon(t) \geq 0\,, \quad \forall t \in [0,T]\,, \quad \mathbb{P}\text{-a.s.} \tag{6.9}$$

Indeed, consider the function $\varphi \colon \mathbb{R} \to \mathbb{R}^+$ defined by $\varphi(x) = \frac{1}{2}(x^-)^2$, where x^- is the negative part of x, namely $x^- = \max\{0, -x\}$, and its *essential* derivatives

$$\varphi'(x) = -x^-\,, \quad \varphi''(x) = H(-x), \quad \forall x \in \mathbb{R}\,,$$

where H is the Heaviside function, i.e. $H(x) = \mathbb{1}_{(0,\infty)}(x)$. Applying the Itô's Lemma, or better, a generalization for convex function of the Itô's Lemma, see [KS12, theorem 6.22], to $\varphi(X_2^\varepsilon(t))$ and exploiting the equality $x^- x = -(x^-)^2$, it follows that

$$\begin{aligned} \frac{1}{2}d\left|(X_2^\varepsilon)^-(t)\right|^2 = -k\left|(X_2^\varepsilon)^-(t)\right|^2\,dt - k\theta(X_2^\varepsilon)^-(t)\,dt + \sigma\frac{(X_2^\varepsilon(t)^-)^2}{\sqrt{|X_2^\varepsilon(t)| + \varepsilon}}\,dW_2(t) \\ + \frac{\sigma^2}{2}\frac{((X_2^\varepsilon)^-(t))^2}{(X_2^\varepsilon)^-(t) + \varepsilon}H(-X_2^\varepsilon(t))\,dt\,. \end{aligned}$$

Since $(X_2^\varepsilon)^-$ is non negative by definition and $\frac{x^2}{x+\varepsilon} \leq x$ for all $x \geq 0$ and $\varepsilon > 0$, the previous SDE implies that

$$\frac{1}{2}\mathbb{E}\left[\left|(X_2^\varepsilon)^-(t)\right|^2 + k\int_0^t \left|(X_2^\varepsilon)^-(s)\right|^2 ds\right] \leq \mathbb{E}\left[\int_0^t \left(\frac{\sigma^2}{2} - k\theta\right)(X_2^\varepsilon)^-(t)\, ds\right] \leq 0$$

for all $t \in [0,T]$, where the last inequality is an immediate result of condition (6.5). Therefore, $(X_2^\varepsilon)^-(t) = 0$ on $(0,T)\times\Omega$ which implies (6.9) as claimed.

As second step, we associate to each equation (6.8), the SDE

$$\begin{cases} dX_1^\varepsilon = \mu X_1^\varepsilon\, dt + X_1^\varepsilon\sqrt{uX_2^\varepsilon}\, dW_1\,, \quad t \in (0,T)\,, \\ X_1^\varepsilon(0) = X_1^0\,, \end{cases} \tag{6.10}$$

for the corresponding value of ε, and we study its solvability. Given a fixed $\varepsilon > 0$, let X_2^ε be the strong, non negative solution to equation (6.8), then X_1^ε can be represented as

$$X_1^\varepsilon(t) = \exp\left(\int_0^t \sqrt{u(s)X_2^\varepsilon(s)}dW_1(s)\right) y_\varepsilon(t)\,.$$

Under this representation, by Itô's formula, X_1^ε solves the following SDE

$$\begin{aligned} dX_1^\varepsilon(t) &= \exp\left(\int_0^t \sqrt{u(s)X_2^\varepsilon(s)}dW_1(s)\right) dy_\varepsilon(t) \\ &\quad + \sqrt{u(t)X_2^\varepsilon(t)}\exp\left(\int_0^t \sqrt{u(s)X_2^\varepsilon(s)}dW_1(s)\right) y_\varepsilon(t)\, dW_1(t) \\ &\quad + \frac{1}{2}u(t)X_2^\varepsilon(t)\exp\left(\int_0^t \sqrt{u(s)X_2^\varepsilon(s)}dW_1(s)\right) y_\varepsilon(t)dt \\ &= \exp\left(\int_0^t \sqrt{u(s)X_2^\varepsilon(s)}dW_1(s)\right) dy_\varepsilon(t) + \frac{1}{2}u(t)X_2^\varepsilon(t)X_1^\varepsilon dt \\ &\quad + \sqrt{u(t)X_2^\varepsilon(t)}X_1^\varepsilon\, dW_1(t) \end{aligned}$$

which coincides with (6.10) when y_ε solves the random differential equation

$$\begin{aligned} &\frac{dy_\varepsilon}{dt} = \left(\mu - \frac{1}{2}uX_2^\varepsilon\right) y_\varepsilon\,, \quad t \in (0,T) \\ &y(0) = X_1^0\,, \end{aligned}$$

which has a unique $\mathcal{F}_t$-adapted solution y_ε, namely

$$y_\varepsilon(t) = X_1^0 \exp\left(\int_0^t \left(\mu - \frac{1}{2}u(s)X_2^\varepsilon(s)\right) ds\right).$$

Therefore,

$$X_1^\varepsilon(t) = X_1^0\exp\left(\int_0^t \left(\mu - \frac{1}{2}u(s)X_2^\varepsilon(s)\right) ds + \int_0^t \sqrt{u(s)X_2^\varepsilon(s)}dW_1(s)\right) \tag{6.11}$$

together with X_2^ε, is a strong solution to the system (6.8)-(6.10). The uniqueness of such a solution is immediate by construction. Moreover, by its definition in equation (6.11), also $X_1^\varepsilon \geq 0$ for each $\varepsilon > 0$, as claimed.

Regarding the bound given in equation (6.7), it follows by Itô's formula that

$$\begin{aligned}\frac{1}{2}\left|X_2^\varepsilon(t)\right|^2 = \frac{1}{2}\left|X_2^0\right|^2 + \int_0^t k(\theta - X_2^\varepsilon(s))X_2^\varepsilon(s)\,ds + \frac{\sigma^2}{2}\int_0^t \frac{|X_2^\varepsilon(s)|^2}{X_2^\varepsilon(s)+\varepsilon}\,ds \\ + \int_0^t \frac{\sigma\,|X_2^\varepsilon(s)|^2}{\sqrt{X_2^\varepsilon(s)+\varepsilon}}\,dW_2(s)\,.\end{aligned}$$

Applying Burkholder-Davis-Gundy Theorem (see e.g. [DPZ14]) to the martingale

$$I_t = \int_0^t \frac{\sigma(X_2^\varepsilon(s))^2}{\sqrt{X_2^\varepsilon(s)+\varepsilon}}\,dW_2(s)\,,$$

it is found

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,T]} I_t\right] &\leq c_1\mathbb{E}\left[\left(\int_0^T \frac{\sigma^2(X_2^\varepsilon(s))^4}{X_2^\varepsilon(s)+\varepsilon}\,ds\right)^{\frac{1}{2}}\right] \\ &\leq \mathbb{E}\left[\left(\sup_{t\in[0,T]}|X_2^\varepsilon(t)|^2\int_0^T c_1^2\sigma^2 X_2^\varepsilon(s)\,ds\right)^{\frac{1}{2}}\right] \\ &\leq \frac{1}{4}\mathbb{E}\left[\sup_{t\in[0,T]}|X_2^\varepsilon(t)|^2\right] + \mathbb{E}\left[\int_0^T c_1^2\sigma^2X_2^\varepsilon(s)\,ds\right] \\ &\leq \frac{1}{4}\mathbb{E}\left[\sup_{t\in[0,T]}|X_2^\varepsilon(t)|^2\right] + \mathbb{E}\left[\int_0^T c_1^2\left(1+\sigma^2\left|X_2^\varepsilon(s)\right|^2\right)\,ds\right] \\ &\leq \frac{1}{4}\mathbb{E}\left[\sup_{t\in[0,T]}|X_2^\varepsilon(t)|^2\right] + \mathbb{E}\left[\int_0^T c_1^2\left(1+\sigma^2\sup_{u\in[0,s]}\left|X_2^\varepsilon(u)\right|^2\right)\,ds\right].\end{aligned}$$

Moreover,

$$\begin{aligned}\sup_{t\in[0,T]}\int_0^t k(\theta - X_2^\varepsilon(s))X_2^\varepsilon(s) + \frac{\sigma^2}{2}\frac{(X_2^\varepsilon(s))^2}{X_2^\varepsilon(s)+\varepsilon}\,ds &\leq \int_0^T \bar{c}(1+|X_2^\varepsilon(s)|^2)\,ds \\ &\leq \bar{c}T + \int_0^T \bar{c}\sup_{u\in[0,s]}|X_2^\varepsilon(u)|^2\,ds\,,\end{aligned}$$

where $\bar{c} = \max\left\{k(\theta+1), \frac{\sigma^2}{2}\right\}$. Summing up the previous estimates,

$$\mathbb{E}\left[\sup_{t\in[0,T]}|X_2^\varepsilon(t)|^2\right] \leq \bar{c}_1 + \bar{c}_2\int_0^T \mathbb{E}\left[\sup_{u\in[0,s]}|X_2^\varepsilon(s)|^2\right]\,ds$$

and applying Gronwall's Lemma it follows that for each $\varepsilon > 0$

$$\mathbb{E}\left[\sup_{t\in[0,T]} |X_2^\varepsilon(t)|^2\right] \le C_1\left(1+\left|X_2^0\right|^2\right),$$

for a suitable constant C_1. This is the desired bound (6.7) for the process X_2^ε. Furthermore, thanks to Jensen's inequality and Itô isometry, the SDE (6.8) provides that

$$\begin{aligned}\mathbb{E}\left[X_2^\varepsilon(t) - X_2^\varepsilon(s)\right]^2 &= \mathbb{E}\left[\left(\int_s^t k(\theta - X_2^\varepsilon(r))\,dr + \sigma \frac{X_2^\varepsilon(r)}{\sqrt{|X_2^\varepsilon(r)|+\varepsilon}}\,dW_2(r)\right)^2\right] \\ &\le 2T\,\mathbb{E}\left[\int_s^t k^2(\theta - X_2^\varepsilon(r))^2\,dr\right] + 2\sigma^2\mathbb{E}\left[\int_s^t \frac{|X_2^\varepsilon(r)|^2}{X_2^\varepsilon(r)+\varepsilon}\,dr\right]\end{aligned}$$

for all $s \le t$ belonging to $[0,T]$. Then, arguing as before, for suitable constants C and C_2

$$\mathbb{E}\left[|X_2^\varepsilon(t) - X_2^\varepsilon(s)|^2\right] \le C\,\mathbb{E}\left[\int_s^t (1+|X_2^\varepsilon(r)|^2)\,dr\right] \le C_2(t-s) \quad \forall s \le t \in [0,T]\,.$$

The previous consideration can be analogously repeated for process X_1^ε, solution to the SDE (6.10), obtaining that

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,T]} |X_1^\varepsilon(t)|^2\right] &\le C_3\left(1+\left|X_1^0\right|^2\right), \\ \mathbb{E}\left[|X_1^\varepsilon(t) - X_1^\varepsilon(s)|^2\right] &\le C_4(t-s) \quad \forall s,t \in [0,T]\,.\end{aligned}$$

Hence, combining the previous estimation, it is found that

$$\mathbb{E}\left[\sup_{t\in[0,T]} |X^\varepsilon(t)|^2\right] \le C_5\left(1+\left|X_1^0\right|^2+\left|X_2^0\right|^2\right), \tag{6.12}$$

$$\mathbb{E}\left[|X^\varepsilon(t) - X^\varepsilon(s)|^2\right] \le C_6(t-s) \quad \forall s,t \in [0,T]\,, \tag{6.13}$$

for any $X^\varepsilon = (X_1^\varepsilon, X_2^\varepsilon)$.

Let $\nu_\varepsilon = \mathcal{L}(X^\varepsilon)$, that is $\nu_\varepsilon(\Gamma) = \mathbb{P}(X^\varepsilon \in \Gamma)$ for each Borel set $\Gamma \subset (C([0,T];\mathbb{R}))^2$. Then, the sequence $\{\nu_\varepsilon\}_{\varepsilon>0}$ is tight in $C([0,T];\mathbb{R}^2)$. To prove its tightness, we need to exhibit for each $\delta > 0$ a compact set $\Gamma \subset (C([0,T];\mathbb{R}^2))$ such that $\nu_\varepsilon(\Gamma^c) \le \delta$ for all $\varepsilon > 0$. Consider the set $\Gamma_{r,\gamma}$ defined as

$$\begin{aligned}\Gamma_{r,\gamma} = \Big\{y \in C\left([0,T];\mathbb{R}^2\right) \,:\, &|y(t)| \le r\,, \forall t \in [0,T]\,, \\ &|y(t)-y(s)| \le \gamma\,|t-s|^{\frac{1}{2}}\,, \forall t,s \in [0,T]\Big\}\,,\end{aligned}$$

which is compact in $C([0,T];\mathbb{R}^2)$ in view of the Ascoli-Arzela theorem. Then, due to the bounds given in equations (6.12)-(6.13) and the well known inequality

$$\rho\mathbb{P}(|Y|\geq\rho)\leq\mathbb{E}\,|Y|\ ,\quad\forall\rho>0\,,$$

it follows that there exist two constants r and γ independent of ε such that $\nu_\varepsilon(\Gamma^c_{r,\gamma})\leq\delta$ as claimed.

Therefore, being $\{\nu_\varepsilon\}_\varepsilon$ tight and thus relatively compact, by the Skorohod's theorem there exists a probability space $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}})$ and random variables $\widetilde{X},\widetilde{X}_\varepsilon$ such that $\mathcal{L}(\widetilde{X}_\varepsilon)=\mathcal{L}(X_\varepsilon)$ and satisfying

$$\widetilde{X}_\varepsilon\to\widetilde{X}\quad\text{in } C([0,T];\mathbb{R}^2),\ \tilde{\mathbb{P}}\text{-a.e. }\omega\in\tilde{\Omega}.$$

Then we may pass to limit in (6.8)-(6.10) and observe that $\widetilde{X}=(\widetilde{X}_1,\widetilde{X}_2)$ satisfies system (6.15) in the space $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}})$ for a new pair $\tilde{W}=(\tilde{W}_1,\tilde{W}_2)$ of Wiener processes in this space, see [BBT16]. This completes the proof of existence of a weak solution.

Clearly the bounds given in equations (6.6) and (6.7) hold also for this solution $\widetilde{X}=(\widetilde{X}_1,\widetilde{X}_2)$ □

6.3 The dynamic programming equation

The following step in order to compute an optimal control for problem (P) is to study the solvability of the corresponding HJB equation.

Let $V:[0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be the optimal value function associated to problem (P), see equations (5.2) and (6.4), that is

$$V(t,x,y)=\inf_{u\in\mathcal{U}}\left\{\mathbb{E}\left[\int_t^T X_1^2(s)f(X_1(s),u(s))\,ds+g(X_1(T))\right]\right\}\tag{6.14}$$

subject to the controlled system

$$\begin{cases}dX_1(s)=\mu X_1(s)\,ds+X_1\sqrt{u(s)X_2(s)}\,dW_1(s)\,, & s\in(t,T)\,,\\ dX_2(s)=k(\theta-X_2(s))\,ds+\sigma\sqrt{X_2(s)}\,dW_2(s)\,, & s\in(t,T)\,,\\ X_1(t)=x\,,\quad X_2(t)=y\,.\end{cases}\tag{6.15}$$

We shall assume that the initial value of the system is positive, namely $x\geq0$ and $y\geq0$, and that conditions (6.5) holds, so that Theorem 6.2.1 ensures the existence of a weak solution $X=(X_1,X_2)$ to the SDEs (6.15) such that $X_1(s)$ and $X_2(s)$ are non negative processes for all $s\in[t,T]$, $\mathbb{P}$-a.s. The HJB equation associated with problem (P), as defined in (5.8), becomes

$$\begin{cases}-\varphi_t(t,x,y)-\inf\limits_{u\in U}\Big\{\mu x\varphi_x(t,x,y)+k(\theta-y)\varphi_y(t,x,y)+\dfrac{1}{2}\sigma^2y\varphi_{yy}(t,x,y)\\ \qquad\qquad+\dfrac{1}{2}x^2uy\varphi_{xx}(t,x,y)+x^2f(x,u)\Big\}=0\,,\quad t\in[0,T],\quad x,y\in\mathbb{R}\\ \varphi(T,x,y)=g(x)\,,\quad\forall x,y\in\mathbb{R}\,,\end{cases}$$

which can be rewritten as

$$\begin{cases} \varphi_t(t,x,y) + \mu x \varphi_x(t,x,y) + k(\theta - y)\varphi_y(t,x,y) + \dfrac{1}{2}\sigma^2 y \varphi_{yy}(t,x,y) \\ \qquad\qquad +x^2 G(x,y,\varphi_{xx}(t,x,y)) = 0\,, \quad t \in [0,T], \quad x,y \in \mathbb{R}\,, \\ \varphi(T,x,y) = g(x)\,, \quad \forall x,y \in \mathbb{R}\,, \end{cases} \tag{6.16}$$

where $G\colon \mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is given by

$$G(x,y,z) = \min_{u\in[a,b]} \left\{ \frac{1}{2} uyz + f(x,u) \right\}\,, \quad \forall x,y,z \in \mathbb{R}\,. \tag{6.17}$$

For all $t \in [0,T]$ and $x, y \in \mathbb{R}$, let p be defined as

$$p(t,x,y) = \varphi_x(t,x,y)\,. \tag{6.18}$$

Then, differentiating with respect to x equation (6.16), it is found that, if φ is solution to the previous partial differential equation (6.16), p solves

$$\begin{cases} p_t(t,x,y) + \mu(xp(t,x,y))_x + k(\theta - y)p_y(t,x,y) + \dfrac{1}{2}\sigma^2 y p_{yy}(t,x,y) \\ \qquad\qquad +(x^2 G(x,y,p_x(t,x,y)))_x = 0\,, \quad \forall t \in [0,T], \quad x,y \in \mathbb{R}\,, \\ p(T,x,y) = g_x(x)\,, \quad \forall x,y \in \mathbb{R}\,, \end{cases} \tag{6.19}$$

with natural boundary conditions at $x = \pm\infty$ and $y = \pm\infty$. Therefore, the HJB equation (6.16) can be reduced to a second order nonlinear parabolic equation. Of course, the two equations (6.16) and (6.19) are not equivalent since in general the former has not a strong smooth solution φ, and thus p may be not well-defined. However, if the PDE (6.19) is well posed we can recover a solution φ to (6.16) by the solution to (6.19) as better explained in the following.

Taking into account that by (6.6) the state X_2 is in the positive half plane $\{y \geq 0\}$, we see that the flow $t \mapsto (X_1(t), X_2(t))$ leaves invariant the domain

$$Q^0 = \{\, (x,y) \in \mathbb{R}^2; 0 \leq y < \infty \,\}$$

and so equation (6.19) can be treated on this narrow domain. For simplicity we shall restrict the domain Q^0 to

$$Q = \{\, x \in \mathbb{R}, \rho < y < M \,\} = \mathbb{R} \times (\rho, M)$$

where M is sufficient large, but finite, and ρ is arbitrarily small but strictly positive. In other words, we shall consider the equation (6.19) on the domain $(0,T) \times Q$ with boundary value condition on ∂Q given by

$$p(t,x,\rho) = 0 \quad \text{and} \quad p(t,x,M) = 0 \quad \forall x \in \mathbb{R}\,, \quad t \in [0,T]\,. \tag{6.20}$$

The domain Q is not invariant for the stochastic flow $t \to (X_1(t), X_2(t))$, but we infer that, for M large enough and ρ extremely small, this is a convenient approximation for problem (6.19) on Q^0.

We set $H = L^2(Q)$ with the standard norm $\|\cdot\|_H$ and define the space V as

$$V = \{\, z \in H \cap H^1_{\mathrm{loc}}(Q)\,;\; xz_x, z_y \in L^2(Q)\,;\; z(x,\rho) = z(x,M) = 0\,, \quad \forall x \in \mathbb{R}\,\} \tag{6.21}$$

where derivatives z_x, z_y are taken in sense of distributions on Q. This space V is an Hilbert space with the norm

$$\|z\|_V = \left(\int_Q z^2 + x^2 z_x^2 + z_y^2 \, dxdy\right)^{\frac{1}{2}}\,, \quad \forall z \in V$$

such that $V \subset H$ algebraically and topologically. Moreover we denote by $V^\star$ the dual space of V having H as pivot space and by $\|\cdot\|_{V^\star}$ the dual norm of $V^\star$. Then,

Definition 6.3.1. *A function p defined on $[0,T] \times Q$ is called weak solution to problem* (6.19)-(6.20) *if the following conditions hold*

$$\begin{aligned}
&p \in C([0,T];H) \cap L^2([0,T];V)\,, \quad \frac{dp}{dt} \in L^2([0,T];V^\star)\,,\\
&\frac{d}{dt}\int_Q p(t,x,y)\psi(x,y)\,dx\,dy + \int_Q (\mu(xp(t,x,y))_x + k(\theta - y)p_y(t,x,y))\psi(x,y)\,dx\,dy\\
&\qquad\qquad - \frac{\sigma^2}{2}\int_Q p_y(t,x,y)(y\psi(x,y))_y\,dx\,dy\\
&\qquad\qquad - \int_Q x^2 G(x,y,p_x(t,x,y))\psi_x(x,y)\,dx\,dy = 0 \quad \forall \psi \in V\,, \quad a.e.\ t \in [0,T]\,,\\
&p(T,x,y) = g_x(x)\,, \quad \forall (x,y) \in Q\,.
\end{aligned}$$

Considering the relation between φ and p, given by equation (6.18) and the previous Definition 6.3.1, we say that

Definition 6.3.2. *A function φ is a weak solution to* (6.16) *on $[0,T] \times Q$ if*

$$\begin{aligned}
&\varphi \in L^2([0,T];L^2_{loc}(\mathbb{R}\times\mathbb{R})),\ \varphi_x \in C([0,T];H) \cap L^2([0,T];V)\,,\\
&\frac{d\varphi_x}{dt} \in L^2([0,T];V^\star)\,,\\
&\frac{d}{dt}\int_Q \varphi_x(t,x,y)\psi(x,y)\,dx\,dy + \int_Q (\mu(x\varphi_x(t,x,y))_x + k(\theta - y)\varphi_{xy}(t,x,y))\psi(x,y)\,dx\,dy\\
&\qquad\qquad - \frac{\sigma^2}{2}\int_Q \varphi_{xy}(t,x,y)(y\psi(x,y))_y\,dx\,dy\\
&\qquad\qquad - \int_Q x^2 G(x,y,\varphi_{xx}(t,x,y))\psi_x(x,y)\,dx\,dy = 0 \quad \forall \psi \in V\,, \quad a.e.\ t \in [0,T]\,,\\
&\varphi(T,x,y) = g(x)\,, \quad \forall (x,y) \in Q\,,\\
&\varphi_x(t,x,\rho) = \varphi_x(t,x,M) = 0 \quad \forall x \in \mathbb{R}\,, \quad t \in [0,T]\,.
\end{aligned}$$

Clearly, if p is a weak solution to (6.19)-(6.20) then by equation (6.18) the function

$$\varphi(t,x,y) = \int_{-\infty}^{x} p(t,\xi,y)\,d\xi\,, \quad (t,x,y) \in [0,T] \times \mathbb{R} \times [0,M]$$

is a weak solution to (6.16). Conversely, when φ is a weak solution to (6.16) then p, as defined in equation (6.18) is a weak solution to (6.19). It should be said that φ is unique up to an additive function $\tilde{\varphi} = \tilde{\varphi}(t,y)$.

Under some conditions on the regularity of g_x, the existence of a weak solution to problem (6.19)-(6.20) is ensured. In particular,

Theorem 6.3.1. *Let $g_x \in L^2(\mathbb{R})$. Then, there is a unique weak solution p to problem (6.19)-(6.20).*

Proof. Let A,

$$\begin{aligned} A : V &\to V^\star \\ z &\to Az\,, \end{aligned}$$

be the nonlinear operator defined by

$$\begin{aligned} (Az,\psi) = -\int_Q (\mu(xz)_x + k(\theta - y)z_y)\psi(x,y)\,dx\,dy + \frac{\sigma^2}{2}\int_Q z_y(y\psi(x,y))_y\,dx\,dy \\ + \int_Q x^2 G(x,y,z_x)\psi_x(x,y)\,dx\,dy\,, \quad t \in [0,T]\,, \quad z,\psi \in V\,. \end{aligned} \tag{6.22}$$

Here $V^\star$ is the dual space of V, whereas $(v^\star, v)$ denotes the value of a functional $v^\star \in V^\star$ at point $v \in V$. Then, problem (6.19)-(6.20) can be shortly rewritten as the backward infinite dimensional Cauchy problem

$$\begin{cases} \frac{d}{dt}p(t) - Ap(t) = 0\,, & \text{a.e. } t \in (0,T)\,, \\ p(T) = g_x\,. \end{cases} \tag{6.23}$$

In order to apply standard results on the existence of solutions for the Cauchy problem (6.23) we need to show that the operator A satisfies the three following properties.

(I) There exists $\alpha_1 \geq 0$ such that

$$(Az - A\bar{z}, z - \bar{z}) \geq -\alpha_1 \|z - \bar{z}\|_H^2\,, \quad \forall z, \bar{z} \in V\,. \tag{6.24}$$

(II) There exists $\alpha_2 > 0$ such that

$$\|Az\|_{V^\star} \leq \alpha_2 \|z\|_V\,, \quad \forall z \in V. \tag{6.25}$$

(III) There exist $\alpha_3 > 0$ and $\alpha_4 \geq 0$ such that

$$(Az, z) \geq \alpha_3 \|z\|_V^2 - \alpha_4 \|z\|_H^2\,, \quad \forall z \in V\,. \tag{6.26}$$

Property (I).

Let z and $\bar{z}$ be two elements in V. We want to provide a lower bound for $(Az - A\bar{z}, z-\bar{z}) = (Az, z-\bar{z}) - (A\bar{z}, z-\bar{z})$. We consider the three integrals appearing in the definition of the operator A, equation (6.22), separately.

First consider

$$-\int_Q \left(\mu\left((xz)_x - (x\bar{z})_x\right) + k(\theta - y)(z_y - \bar{z}_y)\right)(z-\bar{z})\,dxdy\,.$$

The integral

$$\begin{aligned}-\int_Q k\theta(z_y - \bar{z}_y)(z-\bar{z})\,dxdy &= -\frac{k\theta}{2}\int_Q \left((z-\bar{z})^2\right)_y\,dxdy\\ &= -\frac{k\theta}{2}\int_{\mathbb{R}} (z(x,\cdot) - \bar{z}(x,\cdot))^2|_\rho^M\,dx = 0\end{aligned}$$

in view of the boundary conditions (6.21). For the same reason, by integration by parts

$$\begin{aligned}\int_Q ky(z_y - \bar{z}_y)(z-\bar{z})\,dxdy &= \frac{k}{2}\int_Q y\left((z-\bar{z})^2\right)_y\,dxdy\\ &= -\frac{k}{2}\int_Q (z-\bar{z})^2\,dxdy = -\frac{k}{2}\left\|z-\bar{z}\right\|_H^2\end{aligned}$$

and

$$\begin{aligned}-\int_Q \mu\left(x(z-\bar{z})\right)_x\,dxdy &= -\mu\int_Q (z-\bar{z})^2\,dxdy - \mu\int_Q x(z-\bar{z})(z-\bar{z})_x\,dxdy\\ &= -\frac{\mu}{2}\left\|z-\bar{z}\right\|_H^2\,,\end{aligned}$$

where the previous equalities hold in the sense of distributions.

Moreover, the second component satisfies

$$\begin{aligned}\frac{\sigma^2}{2}\int_Q z_y(y(z-\bar{z}))_y - \bar{z}_y(y(z-\bar{z}))_y\,dxdy &= \frac{\sigma^2}{2}\int_Q (z_y - \bar{z}_y)(y(z-\bar{z}))_y\,dxdy\\ &= \frac{\sigma^2}{2}\int_Q \frac{1}{2}\left((z-\bar{z})^2\right)_y + y\left((z-\bar{z})_y\right)^2 dxdy \geq 0\end{aligned}$$

since for all $(x,y)\in Q$, $\int_Q y\left((z-\bar{z})_y\right)^2\,dxdy \geq 0$ due to the fact that $y>0$ in Q, and, as before, $\int_Q \frac{1}{2}\left((z-\bar{z})^2\right)_y\,dxdy = 0$.

Lastly, recalling its definition in equation (6.17) that is

$$G(x,y,z) = -\sup_{u\in[a,b]}\left\{-\frac{1}{2}uyz - f(x,u)\right\}\,,$$

the function G can be equivalently written as

$$G(x,y,z) = -\tilde{f}^\star\left(x, -\frac{1}{2}yz\right) \quad \forall (x,y)\in Q, z\in\mathbb{R}\,, \tag{6.27}$$

where $\tilde{f}^\star(x,v)$ is the convex conjugate of function $v \overset{\tilde{f}}{\to} f(x,v)+I_{[a,b]}(v)$, that is $\tilde{f}^\star(x,q) = \sup_v \{ qv - \tilde{f}(x,v);\, v \in [a,b] \}$. Here $I_{[a,b]}$ represents the characteristic function of the interval $U = [a,b]$, i.e.

$$I_{[a,b]}(v) := \begin{cases} 0 & \text{if } v \in [a,b], \\ +\infty & \text{if } v \in]-\infty, a[\cup]b, +\infty[. \end{cases}$$

Given a convex, lower semicontinuous function $h\colon \mathbb{R} \to]-\infty,+\infty]$, the subdifferential $\partial h(v)$ of h at v is the set

$$\partial h(v) = \{ \eta \in \mathbb{R} : \eta(v-\bar{v}) \geq h(v) - h(\bar{v}) \quad \forall \bar{v} \in \mathbb{R} \},$$

which we will simply denote as $h_v(v)$. Since $\partial [h(\alpha v)] = \alpha \partial h(\alpha v)$ for all $\alpha, v \in \mathbb{R}$, it follows that the subdifferential of function $z \mapsto G(x,y,z)$, i.e. G_z, satisfies

$$G_z(x,y,z) = \frac{1}{2} y \tilde{f}^\star_v \left(x, -\frac{1}{2} yz \right), \quad \forall (x,y) \in Q, z \in \mathbb{R}, \tag{6.28}$$

Moreover, if $h^\star$ is the conjugate of h, then its subdifferential is the inverse of ∂h in the sense of multivalued mappings, i.e. $\partial h^\star(q) = (\partial h)^{-1}(q)$, $\forall q \in \mathbb{R}$. See [Roc15] for further details. Then, for all $v \in \mathbb{R}$ we have

$$\begin{aligned} \tilde{f}^\star_v(x,v) &= \left(\tilde{f}_u(x,\cdot) \right)^{-1}(v) \\ &= \left(f_u(x,\cdot) + \partial I_{[a,b]}(\cdot) \right)^{-1}(v) \\ &= \left(f_u(x,\cdot) + N_{[a,b]}(\cdot) \right)^{-1}(v), \end{aligned} \tag{6.29}$$

where $N_{[a,b]}(v) \subset 2^{\mathbb{R}}$ is the normal cone to $[a,b]$ in v, that is

$$N_{[a,b]}(v) = \begin{cases} \mathbb{R}^- & \text{if } v = a, \\ 0 & \text{if } a < v < b, \\ \mathbb{R}^+ & \text{if } v = b, \end{cases}$$

and therefore, for all $x, v \in \mathbb{R}$

$$\tilde{f}^\star_v(x,v) \in [a,b]. \tag{6.30}$$

Since $y \in [\rho, M]$ and $a > 0$, $G_z(x,y,z) \geq \frac{1}{2} a\rho > 0$ whenever $(x,y) \in Q$, meaning that the function $z \to G(x,y,z)$ is monotonically increasing. This provides

$$\begin{aligned} &\int_Q x^2 G(x,y,z_x)(z-\bar{z}_x) - x^2 G(x,y,\bar{z}_x)(z_x - \bar{z}_x)\, dxdy = \\ &\qquad\qquad = \int_Q x^2 \left(G(x,y,z_x) - G(x,y,\bar{z}_x) \right)(z_x - \bar{z}_x)\, dxdy \geq 0. \end{aligned}$$

Summing up all the previous estimates, it is found

$$(Az - A\bar{z}, z - \bar{z}) \geq -\frac{k+\mu}{2} \left\| z - \bar{z} \right\|_H^2 ,$$

that is the desired bound given in equation (6.24).

Property (II).

Let $z \in V$. By definition of the dual norm $\left\| \cdot \right\|_{V^\star}$,

$$\left\| Az \right\|_{V^*} = \sup \left\{ |(Az, \psi)| : \left\| \psi \right\|_V = 1 \right\} .$$

Then, considering any $\psi \in V$, Hölder's inequality implies

$$\begin{aligned} \int_Q x^2 G(x,y,z_x)\psi_x(x,y)\, dx\, dy &\leq \left(\int_Q x^2 \psi_x^2 \, dx\, dy \right)^{\frac{1}{2}} \left(\int_Q x^2 G(x,y,z)^2 \, dx\, dy \right)^{\frac{1}{2}} \\ &\leq \left\| \psi \right\|_V \left(\int_Q |xG(x,y,x)|^2 \, dx\, dy \right)^{\frac{1}{2}} . \end{aligned}$$

Hypothesis (i) over the function f implies that $G(x,y,0) = \inf_u f(x,u) = 0$ for all $x \in \mathbb{R}$. Therefore, $|G(x,y,v)| = |vG_z(x,y,\xi_v)|$ for a suitable $\xi_v \in [0,v]$, and the previous estimates (6.28) and (6.30) guarantee that

$$|G(x,y,v)| \leq \frac{1}{2} Mb \, |v| ,$$

for all $(x,y) \in Q$ and $v \in \mathbb{R}$. Hence for all $z, \psi \in V$

$$\left| \int_Q x^2 G(x,y,z_x)\psi_x(x,y)\, dx\, dy \right| \leq \frac{1}{2} Mb \left\| \psi \right\|_V \left(\int_Q x^2 \left| z_x \right|^2 \, dx\, dy \right)^{\frac{1}{2}} \leq \frac{1}{2} Mb \left\| \psi \right\|_V \left\| z \right\|_V .$$

Similarly, we have that for all $z, \psi \in V$

$$\begin{aligned} \left| \int_Q z_y (y\psi)_y \, dx\, dy \right| &\leq \int_Q |z_y(\psi + y\psi_y)| \, dx\, dy \\ &\leq \left(\int_Q z_y^2 \, dx\, dy \right)^{\frac{1}{2}} \left[\left(\int_Q \psi^2 \, dxdy \right)^{\frac{1}{2}} + \left(\int_Q y^2 \psi_y^2 \, dx\, dy \right)^{\frac{1}{2}} \right] \\ &\leq (1+M) \left\| \psi \right\|_V \left\| z \right\|_V \end{aligned}$$

and

$$\begin{aligned} \left| \int_Q (\mu(xz)_x + k(\theta - y)z_y)\psi \, dx\, dy \right| &\leq \left(\int_Q \psi^2 \, dx\, dy \right)^{\frac{1}{2}} \left[\sqrt{2}\mu \left(\int_Q z^2 + x^2 z_x^2 \, dx\, dy \right)^{\frac{1}{2}} \right. \\ &\qquad \left. + k\bar{\theta} \left(\int_Q z_y^2 \, dx\, dy \right)^{\frac{1}{2}} \right] \\ &\leq (\sqrt{2}\mu + k\bar{\theta}) \left\| \psi \right\|_V \left\| z \right\|_V , \end{aligned}$$

where $\bar{\theta} = \max\{\theta, |\theta - M|\}$. Therefore, combining the previous considerations, for all z and $\psi \in V$

$$|(Az, \psi)| \leq C \, \|\psi\|_V \, \|z\|_V$$

where $C = \left(\frac{1}{2}Mb + \frac{\sigma^2}{2}(M+1) + \sqrt{2}\mu + k\bar{\theta}\right)$, that is Property (II) given in equation (6.25).

Property (III).

As before, by previous considerations on G, i.e. equations (6.28)-(6.30), $G_z(x,y,z) \geq \frac{1}{2}ay > \frac{1}{2}a\rho$ on $Q \times \mathbb{R}$. Therefore,

$$\int_Q x^2 G(x,y,z_x) z_x \, dx \, dy = \int_Q x^2 G_z(x,y,\xi_z) z_x^2 \, dx \, dy$$

for some ξ_z and thus

$$\int_Q x^2 G(x,y,z_x) z_x \, dx \, dy \geq \frac{1}{2} a\rho \int_Q x^2 z_x^2 \, dx \, dy \, . \tag{6.31}$$

Moreover, by integration by parts

$$\begin{aligned}\int_Q z_y (yz)_y \, dx \, dy &= \int_Q \left(y|z_y|^2 + \frac{1}{2}(z^2)_y \right) \, dx \, dy \\ &\geq \rho \int_Q |z_y|^2 \, dx \, dy\end{aligned}$$

and

$$\begin{aligned}-\int_Q \left(\mu(xz)_x + k(\theta - y)z_y\right) z \, dx \, dy &= -\int_Q \frac{\mu}{2} x (z^2)_x + \mu z^2 + \frac{k}{2}(\theta - y)(z^2)_y \, dx \, dy \\ &= -\frac{\mu + k}{2} \, \|z\|_H^2 \, .\end{aligned}$$

Together with (6.31) the latter implies

$$\begin{aligned}(Az, z) &\geq \frac{1}{2} a\rho \left(\int_Q x^2 z_x^2 \, dxdy \right) + \rho \left(\int_Q z_y^2 \, dxdy \right) - \left(\frac{\mu + k}{2} \right) \|z\|_H^2 \\ &\geq \alpha_3 \left(\int_Q z^2 + x^2 z_x^2 + z_y^2 \, dxdy \right) - \left(\alpha_3 + \frac{\mu + k}{2} \right) \|z\|_H^2 \, ,\end{aligned}$$

where $\alpha_3 = \min\left\{\frac{1}{2}a\rho, \rho\right\} > 0$, that is the bound (6.26) required by Property (III).

Then we infer, see e.g. [Bar10, Theorem 4.10], that the Cauchy problem (6.23) has a unique solution p as in Definition 6.3.1 and this completes the proof of the theorem. □

6.3.1 Semigroup approach

An alternative methodology to study the Cauchy Problem given in equation (6.23) is the semigroup approach.

Let $q\colon [0,T] \to H$ be the function defined by $q(t) = p(T-t)$. Then, the previous problem (6.23) can be rewritten as the forward Cauchy problem

$$\begin{cases} \frac{dq}{dt} + Aq = 0\,, & t \in (0,T)\,, \\ q(0) = g_x\,, \end{cases} \tag{6.32}$$

where A is the same operator introduced before in equation (6.22).

Let A_H be the restriction of the operator A over the functions $q \in V$ such that $Aq \in H$, namely $\mathcal{D}(A_H) = \{q \in V : Aq \in H\} \subset H$ and

$$\begin{aligned} A_H : \mathcal{D}(A_H) &\to H \\ q &\to A_H q = Aq\,. \end{aligned}$$

A_H is quasi-m-accretive in $H \times H$, meaning that there exists a real parameter $\eta_0 \in \mathbb{R}$ such that

$$(A_H q - A_H \bar{q}, q - \bar{q}) \geq -\eta_0 \left\| q - \bar{q} \right\|_H^2\,, \quad \forall q, \bar{q} \in \mathcal{D}(A_H)\,.$$

Indeed, A_H still satisfies Property (I), given in (6.24), and therefore by choosing $\eta_0 > \alpha_1$ the previous inequality follows straightforwardly. Moreover Property (III), given in equation (6.26), implies that

$$((\eta I + A_H)q, q) = (A_H q, q) + \eta \left\| q \right\|_H^2 \geq \alpha_3 \left\| q \right\|_V^2 + (\eta - \alpha_4) \left\| q \right\|_H^2 > (\eta - \alpha_4) \left\| q \right\|_H^2$$

since $\alpha_3 > 0$, and therefore, if $\eta \geq \alpha_4$, $\eta I + A_H$ is coercive. Here I denotes the identity operator. Therefore, via Minty-Browder theory, see e.g. [Bar10, Theorem 2.2], it follows that $\eta I + A_H \colon \mathcal{D}(A_H) \to H$ is surjective for all $\eta > \eta_0 = \max\{\alpha_0, \alpha_3\}$, meaning that its range $R(\eta I + A_H)$ coincides with the space H, i.e.

$$R(\eta I + A_H) = H\,, \quad \forall \eta > \eta_0\,.$$

The existence theorem for the Cauchy problem associated with non-linear quasi-m-accretive operators in Hilbert spaces [Bar10, Theorem 4.5] ensures that when $g_x \in \mathcal{D}(A_H)$ Problem (6.32) admits a unique strong solution

$$q \in W^{1,\infty}([0,T];H) = \left\{ q \in L^\infty([0,T],H);\, \frac{dq}{dt} \in L^\infty([0,T],H) \right\},$$

that is

$$\begin{aligned} & q \in L^2([0,T],V)\,, \quad Aq(t) \in L^\infty([0,T];H)\,, \\ & \frac{d^+}{dt} q(t) + A_H(q(t)) = 0\,, \quad t \in [0,T[\,, \\ & q(0) = g_x\,. \end{aligned}$$

And this means,

Corollary 6.3.1. *Let* $g_x \in L^2(\mathbb{R})$. *Then equation* (6.19)-(6.20) *has a unique solution in the sense of Definition 6.3.1.*

Moreover, this unique solution q is given by the Crandall-Liggett exponential formula, that is

$$q(t) = \lim_{n\to\infty} \left(1 + \frac{t}{n}A\right)^{-n} q_0 \,, \tag{6.33}$$

where the limit is uniformly in t over $[0, T]$. See, e.g., [Bar10, Theorem 4.3]. Let $q_0 = g_x$ and q_h be an h-approximate solution to the previous Cauchy problem given by the following finite difference scheme:

$$q_h(t) = q_h^i \in \mathbb{R} \quad \text{if } t \in [ih, (i+1)h)$$

where

$$\begin{cases} q_h^0 = q_0 \,, \\ q_h^{i+1} + hAq_h^{i+1} = q_h^i \,, \quad i = 0, 1, 2, \dots, N = \left[\frac{T}{h}\right] - 1 \,. \end{cases}$$

Then, the Crandall-Liggett formula (6.33) means that q is given as the limit of these approximate solution q_h, i.e.

$$q(t) = \lim_{h\to 0} q_h(t) \quad \forall t \in [0, T] \,.$$

This scheme may be useful to numerically compute an approximation of the solution to equation (6.19). Moreover, this reveals that under regular assumptions over the terminal function g, the solution p of equation (6.19) is locally in $H^2(Q)$.

Then, coming back to the equation (6.16),

Corollary 6.3.2. *Let $g_x \in L^2(\mathbb{R})$. Then, equation* (6.16) *has a weak solution in the sense of Definition 6.3.2. This solution is unique up to an additive function $\tilde{\varphi} \equiv \tilde{\varphi}(t, y)$.*

Remark 6.3.1. The main advantage of Theorem 6.3.1 and respectively of Corollary 6.3.2 is the regularity properties of a weak solution φ.

6.4 A sub-optimal feedback control

In the previous Section 6.3, we have found suitable conditions to ensure the existence of a weak solution to the HJB equation (6.16) related to the optimization problem (P). Such a solution may be exploited to construct an optimal control in feedback form. Indeed, Theorem 5.2.2 suggests that the feedback controller $u^\star$ defined by

$$u^\star(t) = \phi(t, X_1(t), X_2(t)) \quad t \in (0, T) \,,$$

where

$$\phi(t, x, y) = \arg\min_{u\in[a,b]} \left\{ \frac{1}{2} uyp_x(t, x, y) + f(x, u) \right\}, \quad \forall (x, y) \in Q$$

and p is a weak solution to equation (6.19)-(6.20), may be optimal in problem (P) for $(X_1, X_2) \in (0,T) \times Q$. In order to conclude, one should consider the regularity of the system under this candidate optimal control.

Let $g_x \in L^2(\mathbb{R})$, then Corollary 6.3.1 ensures that the exists a unique weak solution p, as in Definition 6.3.1, to the PDE (6.19). Then, in view of equations (6.27) and (6.29), the map ϕ is defined by

$$\phi(t,x,y) = \left(f_u(x,\cdot) + N_{[a,b]}\right)^{-1}\left(-\frac{1}{2}yp_x(t,x,y)\right), \quad \forall t \in [0,T], (x,y) \in Q \tag{6.34}$$

where $f_u(x,\cdot)$ is the subdifferential of function $u \mapsto f(x,u)$. Function ϕ is well defined since [LSU88, Theorem 6.1] provides a bound for p_x, as solution of a quasilinear parabolic equation. Thus, the corresponding closed loop system (6.3) becomes

$$\begin{cases} dX_1 = \mu X_1\, dt + X_1\sqrt{\phi(t,X_1,X_2)X_2}\, dW_1\,, \quad t \in (0,T)\,, \\ dX_2 = k(\theta - X_2)\, dt + \sigma\sqrt{X_2}\, dW_2\,, \quad t \in (0,T)\,, \\ X_1(0) = X_1^0\,, \quad X_2(0) = X_2^0\,. \end{cases} \tag{6.35}$$

The existence of a strong solution (X_1, X_2) to (6.35) would imply by standard computations that the map $u^\star = \phi(t, X_1, X_2)$ is indeed an optimal feedback controller for problem (P). However the existence of a strong solution for (6.35) is a delicate problem and the best one can expect in this case is a martingale solution.

Assume in addition to the above hypotheses that

(ii) $u \mapsto f_u(x,u)$ is strictly monotone, for all $x \in \mathbb{R}$.

Then,

Theorem 6.4.1. *Assume that* (i),(ii) *and* (6.5) *hold. Then there is a weak solution* (X_1, X_2) *to stochastic system* (6.35).

Proof. The proof follows similarly to the one of Theorem 6.2.1.

Consider the random map $\psi\colon [0,T] \times \mathbb{R} \to \mathbb{R}$ given by

$$\psi(t,x) = x\sqrt{\phi(t,x,X_2(t))X_2(t)}\,, \quad t \in [0,T],\, x \in \mathbb{R}\,.$$

By construction, the control $\phi(t,x,y)$ given in equation (6.34) belongs to the real interval $[a,b]$ for all $(t,x,y) \in [0,T] \times Q$, with $a > 0$, and $X_2(t) \geq 0$ for all $t \in [0,T]$, and thus ψ is well defined. Moreover, ψ is upper bounded by

$$|\psi(t,x)| \leq |x|\sqrt{bX_2(t)}\,. \tag{6.36}$$

By hypothesis (ii), f_u is monotone, and therefore invertible. Moreover, the map $\left(f_u(x,\cdot) + N_{[a,b]}\right)^{-1}$ is Lipschitz continuous on $\mathbb{R}$ in x, i.e. $|\phi(t,x,y) - \phi(t,\bar{x},y)| \leq$

$L_\phi |x - \bar{x}|$ for a suitable constant L_ϕ. Also p_x is Lipschitz in x on $[0,T] \times Q_R$, where $Q_R = (0,R) \times (\rho, M)$, namely

$$|p_x(t,x,y) - p_x(t,\bar{x},y)| \leq L_R |x - \bar{x}| \,, \quad \text{for } |x| + |\bar{x}| \leq R \,,$$

for a suitable constant L_R. The latter is a consequence of high order regularity of solutions to quasilinear parabolic equation (see again [LSU88, Theorem 6.1]). Therefore, it follows that for all $R > 0$, $|x| + |\bar{x}| \leq R$, $\mathbb{P}$-a.s.

$$\begin{aligned} |\psi(t,x) - \psi(t,\bar{x})| &= \left| x\sqrt{\phi(t,x,X_2(t))X_2(t)} - \bar{x}\sqrt{\phi(t,\bar{x},X_2(t))X_2(t)} \right| \\ &\leq \sqrt{X_2(t)}\Big(|x-\bar{x}|\sqrt{\phi(t,x,X_2(t))} + |\bar{x}|\sqrt{|\phi(t,x,X_2(t)) - \phi(t,\bar{x},X_2(t))|}\Big) \\ &\leq \sqrt{bX_2(t)}\,|x-\bar{x}| + \frac{1}{2}RX_2(t)\sqrt{L_\phi\,|p_x(t,x,X_2(t)) - p_x(t,\bar{x},X_2(t))|} \\ &\leq \sqrt{bX_2(t)}\,|x-\bar{x}| + \frac{1}{2}RX_2(t)\sqrt{L_\phi L_R\,|x-\bar{x}|}\,. \end{aligned} \tag{6.37}$$

Then, let

$$\begin{cases} dX_1^\varepsilon = \mu X_1^\varepsilon\,dt + X_1^\varepsilon\sqrt{\phi(t,X_1^\varepsilon,X_2^\varepsilon)X_2^\varepsilon}\,dW_1\,, \\ dX_2^\varepsilon = k(\theta - X_2^\varepsilon)\,dt + \sigma\frac{X_2^\varepsilon}{\sqrt{|X_2^\varepsilon| + \varepsilon}}\,dW_2\,, \\ X_1^\varepsilon(0) = X_1^0\,, \quad X_2^\varepsilon(0) = X_2^0\,, \end{cases} \tag{6.38}$$

approximate the original system (6.35). Arguing as in the proof of Theorem 6.2.1 it follows by (6.36)-(6.37) that (6.38) has a unique solution $(X_1^\varepsilon, X_2^\varepsilon)$, $X_1^\varepsilon, X_2^\varepsilon \geq 0$, $\mathbb{P}$-a.s., for any $\varepsilon > 0$. Moreover, one obtains also in this case estimates (6.12)-(6.13) and so by the Skorohod theorem it follows as above the existence of a weak solution $(\tilde{X}_1, \tilde{X}_2)$ satisfying system (6.35) in a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \tilde{W}_1, \tilde{W}_2)$. □

Remark 6.4.1. Roughly speaking Theorem 6.4.1 amounts to saying that there is a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \tilde{W}_1, \tilde{W}_2)$ where the closed loop system (6.35) has a solution $(\tilde{X}_1, \tilde{X}_2)$. This means that the feedback controller u^* is admissible in problem (P) though it is not clear if it is optimal. However this is a suboptimal feedback controller. Indeed, if one constructs in a similar way the feedback controller $u^*_\varepsilon = \phi_\varepsilon(X_1^\varepsilon, X_2^\varepsilon)$ for problem (P), but with state system (6.38), then u^*_ε is optimal for the approximating optimal control problem and it is convergent in law to a controller u^* as found above.

We conclude this section considering two possible examples.

Example 1. Consider the simple case when $f(x,u) = 0$ for all $x \in \mathbb{R}$ and $u \in U$. Then equation (6.19) reduces to

$$\begin{cases} p_t + \mu(xp)_x + k(\theta - y)p_y + \dfrac{\sigma^2}{2}yp_{yy} + y\left(x^2(aH(p_x) + bH(-p_x))\right)_x = 0\,, \\ \qquad\qquad\qquad\qquad\qquad\qquad x \in \mathbb{R}\,,\ y \in (\rho, M)\,, \\ p(T,x,y) = g_x(x)\,, \quad \forall x \in \mathbb{R}. \end{cases} \tag{6.39}$$

where H is the Heaviside function. Therefore, in this case the optimal feedback control $u^\star$, given in equation (6.34), becomes

$$u^\star(t) = \begin{cases} a & \text{if } p_x(t, X_1(t), X_2(t)) > 0, \\ \in (a,b) & \text{if } p_x(t, X_1(t), X_2(t)) = 0, \\ b & \text{if } p_x(t, X_1(t), X_2(t)) < 0. \end{cases}$$

However, in this case Theorem 6.4.1 does not apply since hypothesis (ii) is not verified by a constant function. Therefore, even if the parabolic equation (6.39) admits a weak solution p in the sense of Definition 6.3.1, p is not sufficiently regular to assume the existence of a solution to the closed loop system (6.39) even in the weak sense.

Nevertheless, observe that if for almost all $\omega \in \Omega$ the set

$$\Sigma = \{\, t : p_x(t, X_1^\star(t), X_2^\star(t)) = 0 \,\}$$

is finite, then the control $u^\star$ is a bang-bang controller with Σ as set of switch points. This fact might be lead to a simplification of control problem (P) by replacing the set $\mathcal{U}$ of admissible control process $u\colon [0,T] \to \mathbb{R}$ by

$$\tilde{\mathcal{U}_0} = \{\, u\colon [0,T] \to \mathbb{R}\,, \mathcal{F}_t - \text{adapted, } u(t) = \sum_{i=0}^{N-1} v_i \chi_{[t_i,t_{i+1}]}(t) \,\}\ .$$

Here $t_0 = 0 < t_1 < t_2 < \cdots < t_N = T$ is a given partition of interval $[0,T]$ while $v_i\colon \Omega \to \mathbb{R}$ are $\mathcal{F}_{t_i}$-measurable functions.

Example 2. Now, let $f\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be

$$f(x,u) = f(u) = \frac{1}{2}(u - u_0)^2\,,$$

where $u_0 \in [a,b]$. The function f satisfies hypotheses (i) and (ii), since f is continuous and convex, $\min_u f = 0$ and $f_u = (u - u_0)$ is strictly increasing. Then Theorem 6.4.1 ensures that there exists a candidate optimal feedback control $u^\star(t) = \phi(t, X_1(t), X_2(t))$, where ϕ is given by

$$\phi(t,x,y) = \big(I(\cdot) - u_0 + N_{[a,b]}(\cdot)\big)^{-1}\left(\frac{1}{2} y p_x(t,x,y)\right)\,, \quad \forall t \in [0,T]\,, (x,y) \in \mathbb{R}\,,$$

where p is the solution to (6.19)-(6.20). In this case $u^\star$ reads as

$$u^\star(t) = \begin{cases} a & \text{if } X_2(t) p_x(t, X_1(t), X_2(t)) > 2(b - u_0) \\ b & \text{if } X_2(t) p_x(t, X_1(t), X_2(t)) < 2(a - u_0) \\ u_0 - \frac{X_2(t)}{2} p_x(t, X_1(t), X_2(t)) \in (a,b) & \text{otherwise.} \end{cases}$$

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