Systole of regular hyperbolic surfaces with an application to Delaunay triangulations

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Algorithms for Delaunay triangulations of point sets in Euclidean spaces are widely known. In the literature, the incremental algorithm, which inserts the points one by one, has been extended to periodic point sets. If we consider the Delaunay triangulation of a periodic point set in Euclidean space, and project this triangulation to the space of orbits, the result may be non-simplicial. Namely, if for example both endpoints of an edge project to the same point, then we obtain a loop. This is avoided when the inequality $\text{sys}(\mathbb{M}) > 2\delta_{\mathcal{P}}$ is satisfied, where $\text{sys}(\mathbb{M})$ denotes the systole of the orbit space $\mathbb{M}$, i.e. the length of the shortest non-contractible curve, and $\delta_{\mathcal{P}}$ the diameter of the largest disk not containing any points from the input set $\mathcal{P}$ in its interior. To make sure that this condition is satisfied, the triangulation can be initialized with a dummy point set that is added to the input set. At the end, the dummy points are removed (if possible).

We will consider periodic Delaunay triangulation in the hyperbolic plane. Hyperbolic translations do not commute, unlike Euclidean translations, which makes the situation more complicated. Furthermore, in general the systole of a hyperbolic surface is unknown. Even though there exists an upper bound of order $O(\log g)$, where g denotes the genus, lower bounds exist only for specific families of surfaces, often constructed using algebraic methods; a general lower bound does not exist. The exact value of the systole is known only for a few specific hyperbolic surfaces. This makes checking the condition mentioned above difficult.

One of the hyperbolic surfaces for which the systole is known, is the Bolza surface, the most symmetric hyperbolic surface of genus 2 to consider. Here the regular hyperbolic octagon is a fundamental region for the group of translations, i.e. it represents the space of orbits in a natural way. In [1] the incremental algorithm is extended to the Bolza surface. In this case, the triangulation is initialized with a dummy point set consisting of 14 points. An implementation is presented in [2].

In this talk, we will look at the surfaces $\mathbb{M}_g$ of genus g corresponding in a similar way to the regular hyperbolic $4g$-gon. We will show that the systole of $\mathbb{M}_g$ is given by $\cosh(\frac{1}{2}\,\text{sys}(\mathbb{M}_g)) = 1 + 2\cos(\frac{\pi}{2g})$ to be able to verify the inequality given above. The proof relies for the most part on hyperbolic trigonometry. As far as we know, this is the first instance where the exact value of the systole is computed for a family of hyperbolic surfaces. Secondly, we will present several strategies for finding a set of dummy points based on Delaunay refinement. By using Delaunay refinement directly we obtain a point set with cardinality of order $\Theta(g)$. Instead, we can use the symmetry of the fundamental region to duplicate points in the dummy point set, which increases the order to $\Theta(g \log g)$. A third strategy explicitly computes a point set and the corresponding Delaunay triangulation. For all strategies we will prove that the resulting point set satisfies the condition mentioned above.

Joint work with: Iordan Iordanov, Monique Teillaud, Gert Vegter.

This talk is closely related to the talk of I. Iordanov [3], who will discuss the algorithm for computing Delaunay triangulations in more detail.

References

[1] M. Bogdanov, M. Teillaud, G. Vegter. Delaunay triangulations on orientable surfaces of low genus. In *Proceedings of the Thirty-second International Symposium on Computational Geometry*, pages 20:1–20:15, 2016. doi:10.4230/LIPIcs.SoCG.2016.20

[2] I. Iordanov, M. Teillaud. Implementing Delaunay triangulations of the Bolza surface. In *Proceedings of the Thirty-third International Symposium on Computational Geometry*, pages 44:1–44:15, 2017. doi:10.4230/LIPIcs.SoCG.2017.44

[3] I. Iordanov. Delaunay triangulations of regular hyperbolic surfaces. *9th International Conference on Curves and Surfaces*, 2018.

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