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Equivalence between distributional differential equations and periodic problems with state-dependent impulses

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Abstract. The paper points out a relation between the distributional differential equation of the second order and the periodic problem for differential equations with state-dependent impulses. The relation between these two problems is investigated and consequently the lower and upper functions method is extended to distributional differential equations. This enabled to get new existence results both for distributional equations and for non-autonomous periodic problems with state-dependent impulses.

Keywords: periodic solution, distributional differential equation, existence, state-dependent impulses, lower and upper functions.

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1 Introduction

Let $m \in \mathbb{N}$ and τ_i, $\mathcal{J}_i$, $i = 1, \dots, m$, be functionals defined on the set of 2π-periodic functions of bounded variation. We consider the distributional differential equation

$$D^2 z - f(\cdot, z) = \sum_{i=1}^{m} \mathcal{J}_i(z)\delta_{\tau_i(z)}, \tag{1.1}$$

where $D^2 z$ denotes the second distributional derivative of a 2π-periodic function z of bounded variation and $\delta_{\tau_i(z)}$, $i = 1, \dots, m$, are the Dirac 2π-periodic distributions which involve impulses at the state-dependent moments $\tau_i(z)$, $i = 1, \dots, m$. For more details see e.g. [12]. One of our aims is to find exact connections between a solution z of the distributional equation (1.1) and a solution (x, y) of the periodic boundary value problem with state-dependent impulses at the points $\tau_i(x) \in (0, 2\pi)$

$$x'(t) = y(t), \quad y'(t) = f(t, x(t)) \quad \text{for a.e. } t \in [0, 2\pi], \tag{1.2}$$

$$\Delta y(\tau_i(x)) = 2\pi \mathcal{J}_i(x), \quad i = 1, \dots, m, \tag{1.3}$$

$$x(0) = x(2\pi), \quad y(0) = y(2\pi), \tag{1.4}$$

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where x' and y' denote the classical derivatives of the functions x and y, respectively, $\Delta y(t) = y(t+) - y(t-)$. These connections make possible to transfer results reached for the classical impulsive periodic problem (1.2)–(1.4) to the distributional differential equation (1.1) and vice versa. In addition, methods and approaches developed for classical problems can be combined with those for distributional equations. We show it here and extend the lower and upper functions method to distributional equations. Consequently we obtain new existence results for both problems introduced above.

Earlier results on the existence of periodic solutions to distributional equations of the type (1.1) can be found in [5–7]. In [6] and [7] the authors reached interesting results for distributional equations which contain also first derivatives and delay. Their approach essentially depends on the global Lipschitz conditions for data functions in order to get a contractive operator corresponding to the problem. In [5] the distributional van der Pol equation with the term $\mu(x - x^3/3)'$ which does not satisfy the global Lipschitz condition is studied. For a sufficiently small value of the parameter μ and $m = 1$, the authors find a ball and a contractive operator on this ball, which yields a unique periodic solution.

In the literature there are also periodic problems where impulse conditions are given out of a differential equation as it is done in problem (1.2)–(1.4), see for example [18]. In particular, we can find a lot of papers studying impulsive periodic problems which are population or epidemic models. Differential equations in these models have mostly the form of *autonomous* planar differential systems [8,15–17,23–25,37,38,43]. On the other hand, non-autonomous population or epidemic models are investigated as well but only with *fixed-time impulses* which is a very special case of state-dependent ones [9, 10, 19–21, 35, 36, 41, 42, 44]. There are a few existence results for non-autonomous problems with state-dependent impulses. In particular, in [3], a scalar first order differential equation is studied provided lower and upper solutions exist, and a generalization to a system is done in [13] under the assumption of the existence of a solution tube. In [11] a linear system with delay and state-dependent impulses is transformed to a system with fixed-time impulses and then the existence of positive periodic solutions is reached. The monographs [1] and [34] investigate among other problems also periodic solutions of quasilinear systems with state-dependent impulses. In [40], a second order differential equation with state-dependent impulses is studied using lower and upper solutions method. For the case where the periodic conditions in state-dependent impulsive problems are replaced by other linear boundary conditions we can refer to the book [33] or to the papers [2, 4, 14, 26–32, 39].

In our present paper we get the existence of solutions to the distributional equation (1.1) as well as to problem (1.2)–(1.4). Let us emphasize that our differential equations are non-autonomous with state-dependent impulses and we need no global or local Lipschitz conditions, see Theorems 6.1 and 6.2. The novelty of our results is documented by Example 6.3, where no previously published theorem can be applied.

2 Preliminaries

In the paper we use the notion of 2π*-periodic distributions*, in short *distributions*. By $\mathcal{P}_{2\pi}$ we denote the complex vector space of all complex-valued 2π-periodic functions of one real variable having continuous derivatives of all orders on $\mathbb{R}$. Elements of $\mathcal{P}_{2\pi}$ are called *test functions*, and $\mathcal{P}_{2\pi}$ is equipped with locally convex topological space structure (see [12]). Its topological dual will be denoted by $(\mathcal{P}_{2\pi})'$. Elements of $(\mathcal{P}_{2\pi})'$ are called 2π*-periodic distributions* or just *distributions*. For a distribution $u \in (\mathcal{P}_{2\pi})'$ and a test function $\varphi \in \mathcal{P}_{2\pi}$, the symbol $\langle u, \varphi \rangle$

stands for a value of the distribution u at φ. The distributional derivative Du of $u \in (\mathcal{P}_{2\pi})'$ is a distribution which is defined by

$$\langle Du, \varphi \rangle = -\langle u, \varphi' \rangle \quad \text{for each} \quad \varphi \in \mathcal{P}_{2\pi}.$$

Let us take $n \in \mathbb{Z}$ and introduce a complex-valued function $e_n \in \mathcal{P}_{2\pi}$ by

$$e_n(t) := \mathrm{e}^{\mathrm{i}nt}, \quad t \in [0, 2\pi].$$

Then each distribution $u \in (\mathcal{P}_{2\pi})'$ can be uniquely expressed by the *Fourier series*

$$u = \sum_{n \in \mathbb{Z}} \widehat{u}(n) e_n, \tag{2.1}$$

where $\widehat{u}(n) \in \mathbb{C}$ are *Fourier coefficients* of u,

$$\widehat{u}(n) := \langle u, e_{-n} \rangle, \quad n \in \mathbb{Z}.$$

For a distribution $u \in (\mathcal{P}_{2\pi})'$ we define the *mean value* $\overline{u}$ as

$$\overline{u} := \widehat{u}(0),$$

and, for simplicity of notation, we write

$$\widetilde{u} := u - \overline{u}.$$

In general, the Fourier series in (2.1) need not be pointwise convergent and the equality in (2.1) is understood in the sense of distributions written as

$$\lim_{N \to \infty} \langle s_N, \varphi \rangle = \langle u, \varphi \rangle \in \mathbb{C} \quad \text{for each } \varphi \in \mathcal{P}_{2\pi}, \quad \text{where } s_N = \sum_{|n| \leq N} \widehat{u}(n) e_n.$$

In particular, the *Dirac* 2π*-periodic distribution* δ is defined by

$$\langle \delta, \varphi \rangle = \varphi(0) \quad \text{for each} \quad \varphi \in \mathcal{P}_{2\pi},$$

and it has the Fourier series

$$\delta = \sum_{n \in \mathbb{Z}} e_n. \tag{2.2}$$

The convolution $u * v$ of two distributions $u, v \in (\mathcal{P}_{2\pi})'$ has the Fourier series

$$u * v = \sum_{n \in \mathbb{Z}} \widehat{u}(n) \widehat{v}(n) e_n, \tag{2.3}$$

and the Fourier series for *distributional derivatives* Du and $D^2 u$ write as

$$Du = \sum_{n \in \mathbb{Z}, n \neq 0} \mathrm{i}n\, \widehat{u}(n) e_n \quad \text{and} \quad D^2 u = \sum_{n \in \mathbb{Z}, n \neq 0} (\mathrm{i}n)^2 \widehat{u}(n) e_n, \tag{2.4}$$

which immediately implies that

$$\overline{Du} = \overline{D^2 u} = 0, \quad D\widetilde{u} = Du, \quad D^2 \widetilde{u} = D^2 u. \tag{2.5}$$

Let us introduce distributions E_1 and E_2 by

$$E_1 := \sum_{n \in \mathbb{Z}, n \neq 0} \frac{1}{\mathrm{i}n} e_n, \quad E_2 := E_1 * E_1 = \sum_{n \in \mathbb{Z}, n \neq 0} \frac{1}{(\mathrm{i}n)^2} e_n, \tag{2.6}$$

and define linear operators $I, I^2 : (\mathcal{P}_{2\pi})' \to (\mathcal{P}_{2\pi})'$ by

$$\begin{aligned} Iu &:= E_1 * u = \sum_{n\in\mathbb{Z}, n\neq 0} \frac{1}{in}\widehat{u}(n)e_n, \\ I^2u &:= I(Iu) = E_1 * (E_1 * u) = E_2 * u = \sum_{n\in\mathbb{Z}, n\neq 0} \frac{1}{(in)^2}\widehat{u}(n)e_n. \end{aligned} \tag{2.7}$$

Using (2.3) and (2.4), we immediately get for every distribution $u \in (\mathcal{P}_{2\pi})'$

$$\begin{aligned} D(Iu) = I(Du) = \widetilde{u}, \qquad D^2(I^2u) = I^2(D^2u) = \widetilde{u}, \\ I^2(Du) = Iu = I\widetilde{u}, \qquad D^2(Iu) = Du = D\widetilde{u}. \end{aligned} \tag{2.8}$$

Due to these identities we see that I is an inverse to D on the set of all distributions having zero mean value and therefore we call I an *antiderivative operator*.

For $\tau \in \mathbb{R}$ let us remind the definition of the translation operator $\mathcal{T}_\tau$ on test functions and distributions. For a function $\varphi \in \mathcal{P}_{2\pi}$ we define $\mathcal{T}_\tau\varphi \in \mathcal{P}_{2\pi}$ by

$$(\mathcal{T}_\tau\varphi)(t) := \varphi(t-\tau), \quad t \in \mathbb{R},$$

and for a distribution $u \in (\mathcal{P}_{2\pi})'$ we define a distribution $\mathcal{T}_\tau u \in (\mathcal{P}_{2\pi})'$ by

$$\langle \mathcal{T}_\tau u, \varphi\rangle := \langle u, \mathcal{T}_{-\tau}\varphi\rangle, \quad \varphi \in \mathcal{P}_{2\pi}.$$

Although, in the general theory, distributions are complex-valued functionals on the space $\mathcal{P}_{2\pi}$ of complex-valued test functions, we work with real-valued distributions and with real-valued test functions in next sections. To this aim functional spaces defined below consist of *real-valued* 2π*-periodic functions*. Clearly it suffices to prescribe their values on some semiclosed interval with the length equal to 2π.

- L^1 is the Banach space of Lebesgue integrable functions equipped with the norm $\|x\|_{\mathrm{L}^1} := \frac{1}{2\pi}\int_0^{2\pi} |x(t)|\,\mathrm{d}t$,
- BV is the space of functions of bounded variation; the total variation of $x \in \mathrm{BV}$ is denoted by $\mathrm{var}(x)$; for $x \in \mathrm{BV}$ we also define $\|x\|_\infty := \sup\{|x(t)| : t \in [0, 2\pi]\}$,
- NBV is the space of functions from BV normalized in the sense that $x(t) = \frac{1}{2}(x(t+) + x(t-))$,
- $\widetilde{\mathrm{NBV}}$ represents the Banach space of functions from NBV having zero mean value ($\overline{x} := \frac{1}{2\pi}\int_0^{2\pi} x(t)\,\mathrm{d}t = 0$), which is equipped with the norm equal to the total variation $\mathrm{var}(x)$,
- for an interval $J \subset [0, 2\pi]$ we denote by $\mathrm{AC}(J)$ the set of absolutely continuous functions on J, and if $J = [0, 2\pi]$ we simply write AC,
- $\mathrm{C}^\infty \subset \mathcal{P}_{2\pi}$ is the classical real Fréchet space of (real-valued) functions having derivative of an arbitrary order,
- for finite $\Sigma \subset [0, 2\pi)$ we denote by PAC_Σ the set of all functions $x \in \mathrm{NBV}$ such that $x \in \mathrm{AC}(J)$ for each interval $J \subset [0, 2\pi]$ for which $\Sigma \cap J = \emptyset$. For $\tau \in [0, 2\pi)$, we write $\mathrm{PAC}_\tau := \mathrm{PAC}_{\{\tau\}}$,
- $\widetilde{\mathrm{AC}} = \mathrm{AC} \cap \widetilde{\mathrm{NBV}}$; for finite $\Sigma \subset [0, 2\pi)$ we denote $\widetilde{\mathrm{PAC}}_\Sigma = \mathrm{PAC}_\Sigma \cap \widetilde{\mathrm{NBV}}$.

Further, Car designates the set of real functions $f(t,x)$ which are 2π-periodic in t and satisfy the Carathéodory conditions on $[0,2\pi]\times\mathbb{R}$. For $x\in\mathrm{BV}$ and $t\in\mathbb{R}$ we write

$$\Delta x(t) = x(t+) - x(t-).$$

We say that $u\in(\mathcal{P}_{2\pi})'$ is a *real-valued distribution* if

$$\langle u,\varphi\rangle\in\mathbb{R}\quad\text{for each}\quad\varphi\in\mathrm{C}^\infty.$$

A real-valued distribution u is characterized by the fact that its Fourier coefficients $\widehat{u}(n)$ and $\widehat{u}(-n)$ are complex conjugate for each $n\in\mathbb{Z}$. If u is a real-valued distribution and $\tau\in\mathbb{R}$, then Du, D^2u, Iu, I^2u and $\mathcal{T}_\tau u$ are also real-valued distributions. Similarly δ is a real-valued distribution, and for $\tau\in\mathbb{R}$ we work with a 2π-periodic real-valued Dirac distribution at the point τ which is defined as

$$\delta_\tau = \mathcal{T}_\tau\delta.$$

Since

$$\widehat{(\mathcal{T}_\tau u)}(n) = \langle\mathcal{T}_\tau u, e_{-n}\rangle = \langle u,\mathcal{T}_{-\tau}e_{-n}\rangle = \mathrm{e}^{-in\tau}\langle u,e_{-n}\rangle = \mathrm{e}^{-in\tau}\widehat{u}(n),\quad n\in\mathbb{Z},$$

it follows from (2.2) and (2.3) that

$$\delta_\tau = \sum_{n\in\mathbb{Z}}\mathrm{e}^{-in\tau}e_n\quad\text{and}\quad\overline{\delta_\tau}=1. \tag{2.9}$$

Moreover

$$u*\delta_\tau = \mathcal{T}_\tau u = \sum_{n\in\mathbb{Z}}\mathrm{e}^{-in\tau}\widehat{u}(n)e_n$$

and

$$I\delta_\tau = E_1*\delta_\tau = \mathcal{T}_\tau E_1,\quad I^2\delta_\tau = E_2*\delta_\tau = \mathcal{T}_\tau E_2. \tag{2.10}$$

We say that $u\in(\mathcal{P}_{2\pi})'$ is a *regular distribution* if u is a real-valued distribution and there exists $y\in\mathrm{L}^1$ such that

$$\langle u,\varphi\rangle = \frac{1}{2\pi}\int_0^{2\pi}y(s)\varphi(s)\,\mathrm{d}s\quad\text{for each}\quad\varphi\in\mathrm{C}^\infty. \tag{2.11}$$

Then we say that $u=y$ in the sense of distributions and write u in place of y in (2.11). Hence all functions from L^1 can be understood as regular distributions. For $u\in\mathrm{BV}$, we write u' as a classical derivative, which is defined a.e. on $\mathbb{R}$ and which is an element of L^1 and consequently a regular distribution. If $u\in\mathrm{AC}$, then $u'=Du$ in the sense of distributions.

Since the first series in (2.6) pointwise converges to the 2π-periodic function

$$\begin{cases}\pi - t & \text{for } t\in(0,2\pi),\\ 0 & \text{for } t=0,\end{cases}$$

we see that E_1 is a regular distribution and it can be considered as a function from $\widetilde{\mathrm{PAC}}_0$. The second series in (2.6) uniformly converges to the 2π-periodic function

$$\frac{t(2\pi-t)}{2}-\frac{\pi^2}{3}\quad\text{for } t\in[0,2\pi],$$

and so E_2 is a regular distribution which can be considered as a function from $\widetilde{\mathrm{AC}}$ and

$$\operatorname{var}(E_1) = 4\pi, \quad \|E_1\|_\infty = \pi, \quad \operatorname{var}(E_2) = \pi^2, \quad \|E_2\|_\infty = \pi^2/3. \tag{2.12}$$

Similarly for $\tau \in \mathbb{R}$,

$$\mathcal{T}_\tau E_1 \in \widetilde{\mathrm{PAC}}_\tau, \quad \mathcal{T}_\tau E_2 \in \widetilde{\mathrm{AC}}, \tag{2.13}$$

$$(\mathcal{T}_\tau E_2)' = \mathcal{T}_\tau E_1, \quad (\mathcal{T}_\tau E_1)' = -1 \quad \text{a.e. on } [0, 2\pi]. \tag{2.14}$$

Since

$$(u * v)(t) := \frac{1}{2\pi}\int_0^{2\pi} u(t-s)v(s)\,\mathrm{d}s \quad \text{for } u, v \in \mathrm{L}^1,$$

we have for $h \in \mathrm{L}^1$

$$(E_1 * h)(t) = \int_0^{2\pi} \frac{(1-t)s}{2\pi} h(s)\,\mathrm{d}s + \frac{1}{2}\left(\int_0^t h(s)\,\mathrm{d}s - \int_t^{2\pi} h(s)\,\mathrm{d}s\right), \quad t \in [0, 2\pi].$$

Therefore Ih is a regular distribution which is equal to the function $E_1 * h \in \mathrm{AC}$, and we conclude by (2.7),

$$h \in \mathrm{L}^1 \quad \Longrightarrow \quad Ih,\ I^2 h \in \widetilde{\mathrm{AC}}, \quad (Ih)'(t) = h(t) - \overline{h} = \widetilde{h} \quad \text{for a.e. } t \in [0, 2\pi]\,. \tag{2.15}$$

Further, for $u \in \mathrm{BV}$ we have that $\mathcal{T}_\tau u(t) = u(t-\tau)$ for $t \in \mathbb{R}$ which implies that

$$\operatorname{var}(\mathcal{T}_\tau u) = \operatorname{var} u \quad \text{and} \quad \|\mathcal{T}_\tau u\|_\infty = \|u\|_\infty \quad \text{for } u \in \mathrm{BV}. \tag{2.16}$$

Let us remind that the following inequalities hold

$$\operatorname{var}(x * y) \le \operatorname{var}(x)\|y\|_\infty, \quad x, y \in \mathrm{NBV}, \tag{2.17}$$

$$\operatorname{var}(x * f) \le \operatorname{var}(x)\|f\|_{\mathrm{L}^1}, \quad x \in \mathrm{NBV},\ f \in \mathrm{L}^1, \tag{2.18}$$

$$\|x\|_{\mathrm{L}^1} \le \|x\|_\infty \le \operatorname{var}(x), \quad x \in \widetilde{\mathrm{NBV}}. \tag{2.19}$$

Therefore, since

$$(\mathcal{T}_\tau E_1)(t) = \begin{cases} \pi - (t-\tau) & \text{for } t \in (\tau, \tau + 2\pi), \\ 0 & \text{for } t = \tau, \end{cases}$$

we see that for $\tau \in \mathbb{R}$

$$\Delta(\mathcal{T}_\tau E_1)(\tau) = (\mathcal{T}_\tau E_1)(\tau+) - (\mathcal{T}_\tau E_1)(\tau-) = \pi - (-\pi) = 2\pi, \tag{2.20}$$

and if we choose $\tau_1, \tau_2 \in \mathbb{R}$, we get by (2.7), (2.10), (2.18), (2.12) the inequality

$$\begin{aligned} \operatorname{var}(I^2\delta_{\tau_1} - I^2\delta_{\tau_2}) &= \operatorname{var}(I(I\delta_{\tau_1} - I\delta_{\tau_2})) = \operatorname{var}(E_1 * (\mathcal{T}_{\tau_1}E_1 - \mathcal{T}_{\tau_2}E_1)) \\ &\le \operatorname{var} E_1 \|\mathcal{T}_{\tau_1}E_1 - \mathcal{T}_{\tau_2}E_1\|_{\mathrm{L}^1} \le 8\pi|\tau_1 - \tau_2|. \end{aligned} \tag{2.21}$$

Finally, if Σ is a finite set, the symbol $\#\Sigma$ stands for the number of elements of Σ.

3 Equivalence of problems

In this section we assume that for $i \in \{1, \dots, m\}$

$$\begin{aligned} &\tau_i : \mathrm{NBV} \to [a,b] \subset (0, 2\pi) \text{ are continuous,} \\ &\mathcal{J}_i : \mathrm{NBV} \to \mathbb{R} \text{ are continuous and bounded,} \quad f \in \mathrm{Car} \end{aligned} \tag{3.1}$$

and for $z \in \mathrm{NBV}$ let us define a finite set

$$\Sigma_z = \{\tau_1(z), \tau_2(z), \dots, \tau_m(z)\}. \tag{3.2}$$

Remark 3.1. Let us emphasize that Σ_z is a finite subset of $(0, 2\pi)$ and it has *at most m elements*. Moreover, if $m > 1$ and $\tau_i(z) = \tau_j(z)$ for some i, j, $i \neq j$, then $\#\Sigma_z < m$.

Definition 3.2. A function $z \in \mathrm{NBV}$ is a *solution* of Eq. (1.1), if (1.1) is satisfied in the sense of distributions, i.e.

$$\langle D^2 z - f(\cdot, z), \varphi \rangle = \sum_{i=1}^{m} \mathcal{J}_i(z) \varphi(\tau_i(z)) \quad \text{for each } \varphi \in \mathrm{C}^\infty.$$

First of all, we consider Eq. (1.1) in the case where f, time instants τ_i and impulse functions $\mathcal{J}_i$ do not depend on z, which can be simply written as Eq. (3.3).

Lemma 3.3. *Let $z \in \mathrm{NBV}$, $h \in \mathrm{L}^1$, $\Sigma \subset [0, 2\pi)$ be a finite set and $a : \Sigma \to \mathbb{R}$. Then z is a solution of the distributional differential equation*

$$D^2 z = h + \sum_{s \in \Sigma} a(s) \delta_s \tag{3.3}$$

if and only if

$$\widetilde{z} = I^2 \left(h + \sum_{s \in \Sigma} a(s) \delta_s \right) \tag{3.4}$$

and

$$\overline{h} + \sum_{s \in \Sigma} a(s) = 0. \tag{3.5}$$

Proof. Let z be a solution of (3.3). Since $\overline{D^2 z} = 0$ and $\overline{\delta_s} = 1$ we get (3.5). Applying I^2 to (3.3) and using (2.8) we obtain (3.4). Conversely, let (3.4) and (3.5) be satisfied. Differentiating (3.4) and using (2.8) we get

$$\begin{aligned} D^2 z = D^2 \widetilde{z} = D^2 I^2 \left(h + \sum_{s \in \Sigma} a(s) \delta_s \right) &= \widetilde{h} + \sum_{s \in \Sigma} a(s) \widetilde{\delta_s} \\ = h + \sum_{s \in \Sigma} a(s) \delta_s - \left(\overline{h} + \sum_{s \in \Sigma} a(s) \right) &= h + \sum_{s \in \Sigma} a(s) \delta_s. \end{aligned}$$

The last equality follows from (3.5). □

The relation between the distributional equation (3.3) and a suitable impulsive problem with fixed-time impulses is pointed out in the next lemma.

Lemma 3.4. *Let $h \in \mathrm{L}^1$, $\Sigma \subset [0,2\pi)$ be a finite set and $a : \Sigma \to \mathbb{R}$. If $z \in \mathrm{NBV}$ is a solution of the distributional differential equation (3.3) then there exists unique $(x,y) \in \mathrm{AC} \times \widetilde{\mathrm{PAC}}_\Sigma$ such that $x = z$, $y = Dz$ a.e. on $[0,2\pi]$ and*

$$\begin{aligned} &x'(t) = y(t), \quad y'(t) = h(t) \quad \text{for a.e. } t \in [0,2\pi], \\ &\Delta y(s) = 2\pi a(s), \quad s \in \Sigma. \end{aligned} \tag{3.6}$$

Conversely, if a couple $(x,y) \in \mathrm{AC} \times \widetilde{\mathrm{PAC}}_\Sigma$ is a solution of (3.6), then $z = x$ is a solution of (3.3).

Proof. Let $z \in \mathrm{NBV}$ be a solution of (3.3). Then we get by (2.8) and (2.10)

$$Dz = I(D^2 z) = I\left(h + \sum_{s\in\Sigma} a(s)\delta_s\right) = Ih + \sum_{s\in\Sigma} a(s)\mathcal{T}_s E_1. \tag{3.7}$$

Using (2.15), we can put

$$y(t) = (Ih)(t) + \sum_{s\in\Sigma} a(s)(\mathcal{T}_s E_1)(t), \quad t \in [0,2\pi),$$

and get by (2.13) that $y \in \widetilde{\mathrm{PAC}}_\Sigma$ and $Dz = y$ a.e. on $[0,2\pi)$. According to (2.20) we see that

$$\Delta y(s) = 2\pi a(s), \quad s \in \Sigma.$$

Lemma 3.3 yields (3.5), and consequently by (2.14) and (2.15),

$$y'(t) = (Ih)'(t) + \sum_{s\in\Sigma} a(s)(\mathcal{T}_s E_1)'(t) = h(t) - \overline{h} - \sum_{s\in\Sigma} a(s) = h(t) \quad \text{for a.e. } t \in [0,2\pi).$$

Further put

$$x(t) = \overline{z} + (I^2 h)(t) + \sum_{s\in\Sigma} a(s)(\mathcal{T}_s E_2)(t), \quad t \in [0,2\pi).$$

Then by (2.13) and (2.15) we see that $x \in \widetilde{\mathrm{AC}}$, Lemma 3.3 yields (3.4) and so and $x = z$ a.e. on $[0,2\pi)$. The uniqueness of the couple (x,y) follows from the inclusions $x \in \mathrm{AC}$ and $y \in \widetilde{\mathrm{PAC}}_\Sigma$.

Let $(x,y) \in \mathrm{AC} \times \widetilde{\mathrm{PAC}}_\Sigma$ be such that (3.6) is valid. Let us put $z = x$. Since $x \in \mathrm{AC}$, then $Dz = Dx = x' = y$ a.e. on $[0,2\pi)$. Let us denote $\Sigma = \{s_1, \dots, s_p\}$, $p \in \mathbb{N}$, where

$$0 = s_0 < s_1 < \cdots < s_p < s_{p+1} = 2\pi.$$

Then for $\varphi \in \mathrm{C}^\infty$ we have

$$\begin{aligned} \langle D^2 z, \varphi \rangle &= -\langle Dz, \varphi' \rangle = -\langle y, \varphi' \rangle = -\frac{1}{2\pi}\int_0^{2\pi} y(t)\varphi'(t)\,\mathrm{d}t = -\frac{1}{2\pi}\sum_{i=1}^{p+1}\int_{s_{i-1}}^{s_i} y(t)\varphi'(t)\,\mathrm{d}t \\ &= -\frac{1}{2\pi}\sum_{i=1}^{p+1}\left([y(t)\varphi(t)]_{s_{i-1}}^{s_i} - \int_{s_{i-1}}^{s_i} y'(t)\varphi(t)\,\mathrm{d}t\right) \\ &= \frac{1}{2\pi}\sum_{i=1}^{p+1}\left(y(s_{i-1}+)\varphi(s_{i-1}) - y(s_i-)\varphi(s_i)\right) + \frac{1}{2\pi}\int_0^{2\pi} y'(t)\varphi(t)\,\mathrm{d}t \\ &= \sum_{i=1}^{p}\frac{1}{2\pi}\Delta y(s_i)\varphi(s_i) + \langle y', \varphi \rangle \\ &= \left\langle \sum_{s\in\Sigma} a(s)\delta_s + h, \varphi \right\rangle. \end{aligned}$$

Hence z is a solution of (3.3). $\square$

Remark 3.5. Lemma 3.4 asserts that each solution $z \in \mathrm{NBV}$ of the distributional differential equation (3.3) is almost everywhere equal to absolutely continuous function and its distributional derivative is almost everywhere equal to a uniquely determined piecewise absolutely continuous function (which is almost everywhere equal to the classical derivative of z). Therefore, each solution z of (3.3) can be thought as absolutely continuous with a piecewise absolutely continuous derivative z'.

Now, let us turn our attention to the state-dependent case. We immediately obtain the next corollary from Lemma 3.3.

Corollary 3.6. *A function $z \in \mathrm{NBV}$ is a solution of the distributional differential equation (1.1) if and only if*

$$\widetilde{z} = I^2 \left(f(\cdot, z) + \sum_{i=1}^{m} \mathcal{J}_i(z) \delta_{\tau_i(z)} \right) \quad \textit{and} \quad \overline{f(\cdot, z)} + \sum_{i=1}^{m} \mathcal{J}_i(z) = 0. \tag{3.8}$$

Let us define a solution to the periodic state-dependent impulsive problem (1.2)–(1.4). As we can see, the condition (1.3) is not well-posed if $m > 1$ and there exist $i, j \in \{1, \dots, m\}$, $x \in \mathrm{NBV}$ such that $\mathcal{J}_i(x) \neq \mathcal{J}_j(x)$ and $\tau_i(x) = \tau_j(x)$. This case can be treated by assuming additional conditions on τ_i. Let us assume that

$$\tau_i(z) \neq \tau_j(z) \quad \text{for } z \in \mathrm{NBV},\ i, j = 1, \dots, m,\ i \neq j, \tag{3.9}$$

which is equivalent to the condition

$$\# \Sigma_z = m \quad \text{for } z \in \mathrm{NBV}. \tag{3.10}$$

Definition 3.7. Let us assume (3.9). A vector function $(x, y) \in \mathrm{AC} \times \widetilde{\mathrm{PAC}}_{\Sigma_x}$ is a solution of problem (1.2)–(1.4), if x and y fulfil (1.2) for a.e. $t \in [0, 2\pi]$ and the state-dependent impulse condition (1.3) is satisfied.

Remark 3.8.

1. The vector function (x, y) from Definition 3.7 satisfies the periodic boundary condition (1.4) because it belongs to the space of 2π-periodic functions.

2. Without any loss of generality, we can consider the component y as an element of $\widetilde{\mathrm{PAC}}_{\Sigma_x}$ due to the following considerations: By (1.2), if $x \in \mathrm{AC}$, then y can be chosen as absolutely continuous on each interval in $[0, 2\pi] \setminus \Sigma_x$ and we can define y on Σ_x such that it is normalized. So $y \in \mathrm{PAC}_{\Sigma_x}$. Finally, by (1.2), y has its mean value equal to zero, which follows from integrating $x' = y$ over $[0, 2\pi]$ and

$$\overline{y} = \frac{1}{2\pi} \int_0^{2\pi} y(t) \, \mathrm{d}t = \frac{1}{2\pi} \int_0^{2\pi} x'(t) \, \mathrm{d}t = \frac{1}{2\pi} (x(2\pi) - x(0)) = 0.$$

   Therefore $y \in \widetilde{\mathrm{PAC}}_{\Sigma_x}$.

Let us note that if (3.9) is not valid, we can say nothing about the relationship between Eq. (1.1) and problem (1.2)–(1.4), because the condition (1.3) is not well-posed. As we see in Theorem 3.11, if we do not assume (3.9), then Eq. (1.1) is equivalent to a periodic problem with a modified state-dependent impulse condition – let us define its solution.

Definition 3.9. A vector function $(x,y) \in \mathrm{AC} \times \widetilde{\mathrm{PAC}}_{\Sigma_x}$ is a solution of problem (1.2), (1.4), (3.11), if x and y fulfil (1.2) for a.e. $t \in [0,2\pi]$ and satisfy the state-dependent impulse condition

$$\Delta y(\tau) = \sum_{\substack{1\le j\le m\,: \\ \tau_j(x)=\tau}} 2\pi \mathcal{J}_j(x), \quad \text{for } \tau \in \Sigma_x. \tag{3.11}$$

Remark 3.10. Let $(x,y) \in \mathrm{AC} \times \widetilde{\mathrm{PAC}}_{\Sigma_x}$, where Σ_x is defined by (3.2). If $\#\Sigma_x = m$, i.e. y has m distinct impulse moments, then (1.3) is satisfied if and only if (3.11) is satisfied. It follows from the fact that if $\#\Sigma_x = m$, then

$$\sum_{\substack{1\le j\le m: \\ \tau_j(x)=\tau_i(x)}} \mathcal{J}_j(x) = \mathcal{J}_i(x), \quad i = 1,\dots,m.$$

If, for example, $m = 3$ and $\tau_1(x) \neq \tau_2(x) = \tau_3(x)$, then (3.11) yields

$$\Delta y(\tau_1(x)) = 2\pi \mathcal{J}_1(x), \quad \Delta y(\tau_2(x)) = 2\pi(\mathcal{J}_2(x) + \mathcal{J}_3(x)).$$

Theorem 3.11 (Equivalence I.). *If $z \in \mathrm{NBV}$ is a solution of the distributional equation (1.1), then there exists a unique $(x,y) \in \mathrm{AC} \times \widetilde{\mathrm{PAC}}_{\Sigma_x}$ such that $x = z$, $y = Dz$ a.e. on $[0,2\pi]$, (x,y) is a solution of the periodic problem with state-dependent impulses (1.2), (1.4), (3.11).*

Conversely, if $(x,y) \in \mathrm{AC} \times \widetilde{\mathrm{PAC}}_{\Sigma_x}$ is a solution of (1.2), (1.4), (3.11), then $z = x$ is a solution of (1.1).

Proof. Let z be a solution of (1.1). Let us put

$$\Sigma := \Sigma_z, \quad h := f(\cdot, z), \quad a : \Sigma \to \mathbb{R}, \quad a(s) := \sum_{\substack{1\le j\le m: \\ \tau_j(z)=s}} \mathcal{J}_j(z),\ s \in \Sigma. \tag{3.12}$$

Since $f \in \mathrm{Car}$, it follows that $h \in \mathrm{L}^1$ and therefore according to Lemma 3.4 there exists a unique couple $(x,y) \in \mathrm{AC} \times \widetilde{\mathrm{PAC}}_{\Sigma}$ such that $z = x$ and (3.6) is valid. This means that (1.2) holds for a.e. $t \in [0,2\pi)$ and (3.11) is satisfied, as well. Conversely, let $(x,y) \in \mathrm{AC} \times \widetilde{\mathrm{PAC}}_{\Sigma}$ be a solution of (1.2), (1.4), (3.11) and $z = x$. If we use (3.12) and Lemma 3.4, we see that z is a solution of (1.1). □

From Remark 3.10 and Theorem 3.11 we infer the following assertion.

Corollary 3.12 (Equivalence II.). *Let (3.9) hold. If $z \in \mathrm{NBV}$ is a solution of the distributional equation (1.1), then there exists a unique $(x,y) \in \mathrm{AC} \times \widetilde{\mathrm{PAC}}_{\Sigma_x}$ such that $x = z$, $y = Dz$ a.e. on $[0,2\pi]$ and (x,y) is a solution of the periodic problem with state-dependent impulses (1.2)–(1.4). Conversely, if $(x,y) \in \mathrm{AC} \times \widetilde{\mathrm{PAC}}_{\Sigma_x}$ is a solution of (1.2)–(1.4), then $z = x$ is a solution of (1.1).*

4 Fixed point problem

We will construct a fixed point problem corresponding to the distributional differential equation (1.1). To this aim we choose $z \in \mathrm{NBV}$ and denote

$$r := \overline{z}, \quad u := \widetilde{z}.$$

By Corollary 3.6, z is a solution of (1.1) if and only if it satisfies (3.8), i.e.

$$u = I^2 \left(f(\cdot, u + r) + \sum_{i=1}^{m} \mathcal{J}_i(u + r)\delta_{\tau_i(u+r)} \right) \tag{4.1}$$

and

$$\overline{f(\cdot, u + r)} + \sum_{i=1}^{m} \mathcal{J}_i(u + r) = 0. \tag{4.2}$$

This fact motivates us to define operators $\mathcal{F}_1$ and $\mathcal{F}_2$ by

$$\mathcal{F}_1(u, r) = I^2 \left(f(\cdot, u + r) + \sum_{i=1}^{m} \mathcal{J}_i(u + r)\delta_{\tau_i(u+r)} \right), \quad (u, r) \in \widetilde{\mathrm{NBV}} \times \mathbb{R}, \tag{4.3}$$

$$\mathcal{F}_2(u, r) = r + \sum_{i=1}^{m} \mathcal{J}_i(u + r) + \overline{f(\cdot, u + r)}, \quad (u, r) \in \widetilde{\mathrm{NBV}} \times \mathbb{R}. \tag{4.4}$$

Having in mind (2.10), (2.13) and (2.15), we see that $\mathcal{F}_1(u, r) \in \widetilde{\mathrm{AC}} \subset \widetilde{\mathrm{NBV}}$ for $u \in \widetilde{\mathrm{NBV}}$ and $r \in \mathbb{R}$. Consequently,

$$\mathcal{F}_1 : \widetilde{\mathrm{NBV}} \times \mathbb{R} \to \widetilde{\mathrm{NBV}}, \quad \mathcal{F}_2 : \widetilde{\mathrm{NBV}} \times \mathbb{R} \to \mathbb{R}.$$

In what follows we will work with the Banach space $\mathbb{X} := \widetilde{\mathrm{NBV}} \times \mathbb{R}$ equipped by the norm

$$\|(u, r)\|_{\mathbb{X}} = \mathrm{var}(u) + |r|, \quad (u, r) \in \mathbb{X},$$

and with an operator $\mathcal{F} : \mathbb{X} \to \mathbb{X}$ defined by

$$\mathcal{F}(u, r) = (\mathcal{F}_1(u, r), \mathcal{F}_2(u, r)), \tag{4.5}$$

where $\mathcal{F}_1$ and $\mathcal{F}_2$ are introduced in (4.3) and (4.4), respectively. Simple relationship between the operator $\mathcal{F}$ and the distributional differential equation (1.1) follows immediately from the motivation and construction of $\mathcal{F}$. This is stated in Lemma 4.1.

Lemma 4.1. *Let (3.1) hold. If $(u, r) \in \mathbb{X}$ is a fixed point of the operator $\mathcal{F}$ given in (4.5), then the function*

$$z(t) = u(t) + r, \quad t \in [0, 2\pi], \tag{4.6}$$

is a solution of the distributional differential equation (1.1). Conversely, if $z \in \mathrm{NBV}$ is a solution of (1.1), then the couple $(\widetilde{z}, \overline{z})$ is a fixed point of $\mathcal{F}$.

Proof. If (u, r) is a fixed point of $\mathcal{F}$ then it satisfies equations (4.1) and (4.2). Let us consider z from (4.6). Since $u \in \widetilde{\mathrm{NBV}}$, then $\widetilde{z} = \widetilde{u} = u$. Hence (3.8) is satisfied. By Corollary 3.6 the function z is a solution of (1.1).

If z is a solution of the distributional differential equation (1.1), then by Corollary 3.6, z satisfies (3.8). Therefore (4.1) and (4.2) are fulfilled for $u = \widetilde{z}$ and $r = \overline{z}$. This means that the couple $(\widetilde{z}, \overline{z})$ is a fixed point of $\mathcal{F}$. □

According to Lemma 4.1, to obtain the existence of a solution of (1.1) it suffices to prove that $\mathcal{F}$ has a fixed point, which will be done by means of the Schauder fixed point theorem. To this goal we investigate properties of $\mathcal{F}$.

Lemma 4.2. *Let (3.1) hold. Then the operator $\mathcal{F}_1$ given in (4.3) is completely continuous on $\mathbb{X}$.*

Proof. Step 1. We prove that $\mathcal{F}_1$ is continuous on $\mathbb{X}$. In order to do it we consider a sequence $\{(u_n,r_n)\}_{n=1}^{\infty}$ from $\mathbb{X}$ converging in $\mathbb{X}$ to $(u,r)\in\mathbb{X}$. Then $\{(u_n,r_n)\}_{n=1}^{\infty}$ is bounded in $\mathbb{X}$ and by (2.19)

$$\lim_{n\to\infty}\|u_n-u\|_\infty=0,\quad \lim_{n\to\infty}|r_n-r|=0.$$

Denote

$$v_n:=\mathcal{F}_1(u_n,r_n),\quad v:=\mathcal{F}_1(u,r).$$

Then

$$\begin{aligned} v_n-v=I^2\big((f(\cdot,u_n+r_n)-f(\cdot,u+r))\big)+\sum_{i=1}^{m}\mathcal{J}_i(u_n+r_n)I^2\delta_{\tau_i(u_n+r_n)}\\ -\sum_{i=1}^{m}\mathcal{J}_i(u+r)I^2\delta_{\tau_i(u+r)}.\end{aligned}\tag{4.7}$$

Since $f\in\mathrm{Car}$, we have

$$\lim_{n\to\infty}|f(t,u_n(t)+r_n)-f(t,u(t)+r)|=0\quad\text{for a.e. } t\in[0,2\pi],$$

and there exists $h\in\mathrm{L}^1$ such that

$$|f(t,u_n(t)+r_n)|\le h(t)\quad\text{for a.e. } t\in[0,2\pi],\ n\in\mathbb{N}.$$

Therefore, by the Lebesgue convergence theorem, $f(\cdot,u+r)\in\mathrm{L}^1$ and

$$\lim_{n\to\infty}\|f(\cdot,u_n+r_n)-f(\cdot,u+r)\|_{\mathrm{L}^1}=0.\tag{4.8}$$

Using (2.18) we get from (2.7), (4.8) and from the fact that $E_2\in\widetilde{\mathrm{AC}}\subset\widetilde{\mathrm{NBV}}$,

$$\begin{aligned}&\lim_{n\to\infty}\operatorname{var}\left(I^2\left(f(\cdot,u_n+r_n)-f(\cdot,u+r)\right)\right)\\ &\quad=\lim_{n\to\infty}\operatorname{var}\left(E_2*\left(f(\cdot,u_n+r_n)-f(\cdot,u+r)\right)\right)=0.\end{aligned}\tag{4.9}$$

By (3.1) there exist $c_i\in(0,\infty)$, $i=1,\dots,m$, such that $|\mathcal{J}_i(u_n+r_n)|\le c_i$ for $i\in\{1,\dots,m\}$, $n\in\mathbb{N}$. Therefore by (2.21),

$$\begin{aligned}\operatorname{var}\big(\mathcal{J}_i(u_n+r_n)(I^2\delta_{\tau_i(u_n+r_n)}-I^2\delta_{\tau_i(u+r)})\big)&\le|\mathcal{J}_i(u_n+r_n)|\operatorname{var}(I^2\delta_{\tau_i(u_n+r_n)}-I^2\delta_{\tau_i(u+r)})\\ &\le 8\pi c_i|\tau_i(u_n+r_n)-\tau_i(u+r)|,\quad i=1,\dots,m,\ n\in\mathbb{N},\end{aligned}$$

and consequently the continuity of τ_i yields

$$\lim_{n\to\infty}\operatorname{var}\big(\mathcal{J}_i(u_n+r_n)(I^2\delta_{\tau_i(u_n+r_n)}-I^2\delta_{\tau_i(u+r)})\big)=0,\quad i=1,\dots,m.\tag{4.10}$$

Further, by (2.10), (2.12), (2.16),

$$\operatorname{var}\Big((\mathcal{J}_i(u_n+r_n)-\mathcal{J}_i(u+r))I^2\delta_{\tau_i(u+r)}\Big)\le\pi^2|\mathcal{J}_i(u_n+r_n)-\mathcal{J}_i(u+r)|,\quad i=1,\dots,m,\ n\in\mathbb{N},$$

and since $\mathcal{J}_i$ are continuous functionals, it holds

$$\lim_{n\to\infty}\operatorname{var}\Big((\mathcal{J}_i(u_n+r_n)-\mathcal{J}_i(u+r))I^2\delta_{\tau_i(u+r)}\Big)=0,\quad i=1,\dots,m.\tag{4.11}$$

To summarize (4.7), (4.9), (4.10) and (4.11) we see that

$$\lim_{n\to\infty} \operatorname{var}(v_n - v) = 0.$$

Step 2. We choose a bounded set $B \subset \mathbb{X}$ and prove that the set $\mathcal{F}_1(B)$ is relatively compact in $\widetilde{\mathrm{NBV}}$. To this aim we take an arbitrary sequence $\{v_n\}_{n=1}^{\infty} \subset \mathcal{F}_1(B)$. Then there exists a sequence $\{(u_n, r_n)\}_{n=1}^{\infty} \subset B$ such that

$$v_n = \mathcal{F}_1(u_n, r_n), \quad n \in \mathbb{N}.$$

Since B is bounded, there exists $\kappa > 0$ such that

$$\operatorname{var}(u_n) \le \kappa, \quad |r_n| \le \kappa,\ n \in \mathbb{N}. \tag{4.12}$$

By (3.1), τ_i maps B to $[a,b] \subset (0,2\pi)$ and $\mathcal{J}_i$ maps B to a bounded set in $\mathbb{R}$ for $i = 1,\dots,m$. Hence there exists $c_B \in (0,\infty)$ such that

$$|\mathcal{J}_i(u_n + r_n)| \le c_B, \quad i = 1,\dots,m,\ n \in \mathbb{N},$$

and we can choose a subsequence $\{(u_{n_k}, r_{n_k})\}_{k=1}^{\infty}$ such that

$$\lim_{k\to\infty} r_{n_k} = r, \quad \lim_{k\to\infty} \tau_i(u_{n_k} + r_{n_k}) = \tau_{0,i}, \quad \lim_{k\to\infty} \mathcal{J}_i(u_{n_k} + r_{n_k}) = J_{0,i}, \tag{4.13}$$

where $r \in [-\kappa,\kappa]$, $\tau_{0,i} \in (0,2\pi)$, $J_{0,i} \in [-c_B, c_B]$, $i = 1,\dots,m$. By (4.12) and the Helly's selection theorem (see e.g. [22, p. 222]) there exists a subsequence $\{u_{n_\ell}\}_{\ell=1}^{\infty} \subset \{u_{n_k}\}_{k=1}^{\infty}$ which is pointwise converging to a function $u \in \widetilde{\mathrm{NBV}}$. Using the same arguments as in Step 1, we get by the Lebesgue convergence theorem, that $f(\cdot, u + r) \in \mathrm{L}^1$ and

$$\lim_{\ell\to\infty} \|f(\cdot, u_{n_\ell} + r_{n_\ell}) - f(\cdot, u + r)\|_{\mathrm{L}^1} = 0. \tag{4.14}$$

Denote

$$v := I^2 \left(f(\cdot, u + r) + \sum_{i=1}^{m} J_{0,i} \delta_{\tau_{0,i}} \right).$$

Then, similarly as in Step 1,

$$\begin{aligned}\operatorname{var}(v_{n_\ell} - v) &\le \operatorname{var}\big(I^2(f(\cdot, u_{n_\ell} + r_{n_\ell}) - f(\cdot, u + r))\big) \\ &+ \sum_{i=1}^{m} \Big(\operatorname{var}\big(\mathcal{J}_i(u_{n_\ell} + r_{n_\ell})(I^2\delta_{\tau_i(u_{n_\ell}+r_{n_\ell})} - I^2\delta_{\tau_{0,i}})\big) + \operatorname{var}\big((\mathcal{J}_i(u_{n_\ell} + r_{n_\ell}) - \mathcal{J}_i(u + r))I^2\delta_{\tau_{0,i}}\big) \Big),\end{aligned}$$

and

$$\lim_{\ell\to\infty} \operatorname{var}(v_{n_\ell} - v) = 0.$$

Consequently we get that the sequence $\{v_{n_\ell}\}_{\ell=1}^{\infty}$ is convergent to v in $\widetilde{\mathrm{NBV}}$. This yields that $\mathcal{F}_1(B)$ is relatively compact in $\widetilde{\mathrm{NBV}}$. □

Lemma 4.3. *Let (3.1) hold. Then the operator $\mathcal{F}$ given in (4.5) is completely continuous on $\mathbb{X}$.*

Proof. Due to Lemma 4.2, the operator $\mathcal{F}_1 : \mathbb{X} \to \widetilde{\mathrm{NBV}}$ is completely continuous. Using (3.1) we get that the operator $\mathcal{F}_2 : \mathbb{X} \to \mathbb{R}$ is completely continuous, as well. This proves the assertion. □

5 Lower and upper functions method

In this section we extend the lower and upper functions method to the distributional differential equation

$$D^2 z - f(\cdot, z) = \sum_{i=1}^{m} J_i\big(z(\tau_i(z))\big)\delta_{\tau_i(z)}, \tag{5.1}$$

where functions J_i are considered instead of functionals $\mathcal{J}_i$, and the basic assumptions (3.1) for $i \in \{1, \dots, m\}$ are specified as

$$\begin{aligned} &\tau_i : \mathrm{NBV} \to [a,b] \subset (0, 2\pi) \text{ are continuous,} \\ &J_i : \mathbb{R} \to \mathbb{R} \text{ are continuous and bounded,} \quad f \in \mathrm{Car}. \end{aligned} \tag{5.2}$$

Definition 5.1. A function $\sigma_1 \in \mathrm{AC}$ is a *lower function* of Eq. (5.1), if there exist a finite (possibly empty) set $\Sigma_1 \subset [0, 2\pi)$, a nonnegative function $b_1 \in \mathrm{L}^1$ and a function $a_1 : \Sigma_1 \to (0, \infty)$ such that

$$D^2 \sigma_1 - f(\cdot, \sigma_1) = \sum_{t \in \Sigma_1} a_1(t)\delta_t + b_1. \tag{5.3}$$

Similarly, we define a dual notion – the upper function of Eq. (5.1).

Definition 5.2. A function $\sigma_2 \in \mathrm{AC}$ is an *upper function* of Eq. (5.1), if there exist a finite (possibly empty) set $\Sigma_2 \subset [0, 2\pi)$, a nonpositive function $b_2 \in \mathrm{L}^1$ and a function $a_2 : \Sigma_2 \to (-\infty, 0)$ such that

$$D^2 \sigma_2 - f(\cdot, \sigma_2) = \sum_{t \in \Sigma_2} a_2(t)\delta_t + b_2. \tag{5.4}$$

Simplest examples of lower and upper functions to Eq. (5.1) are constant functions

$$\sigma_1(t) = c, \quad \sigma_2(t) = d, \quad t \in [0, 2\pi], \tag{5.5}$$

where $c, d \in \mathbb{R}$, provided the inequalities

$$f(t, c) \le 0 \le f(t, d) \quad \text{for a.e. } t \in [0, 2\pi]$$

are fulfilled. It follows from the properties of constant functions

$$D\sigma_i = D^2\sigma_i = 0, \quad \Sigma_i = \emptyset, \quad i = 1, 2.$$

Essential properties of lower and upper functions of Eq. (5.1) are contained in the next lemma.

Lemma 5.3. *A function $\sigma_1 \in \mathrm{AC}$ is a lower function of the distributional differential equation (5.1) if and only if there exist a finite set $\Sigma_1 \subset [0, 2\pi)$ such that $\sigma_1' \in \mathrm{PAC}_{\Sigma_1}$,*

$$\sigma_1''(t) \ge f(t, \sigma_1(t)) \qquad \textit{for a.e. } t \in [0, 2\pi], \tag{5.6}$$

$$\Delta\sigma_1'(t) > 0, \quad t \in \Sigma_1. \tag{5.7}$$

A function $\sigma_2 \in \mathrm{AC}$ is an upper function of the distributional differential equation (5.1) if and only if there exists a finite set $\Sigma_2 \subset [0, 2\pi)$ such that $\sigma_2' \in \mathrm{PAC}_{\Sigma_2}$,

$$\sigma_2''(t) \le f(t, \sigma_2(t)) \qquad \textit{for a.e. } t \in [0, 2\pi], \tag{5.8}$$

$$\Delta\sigma_2'(t) < 0, \quad t \in \Sigma_2. \tag{5.9}$$

Proof. Let $\sigma_1 \in \mathrm{AC}$ be a lower function of Eq. (5.1). By (5.3) and Corollary 3.12 this is equivalent to the fact that $(\sigma_1, \sigma_1') \in \mathrm{AC} \times \widetilde{\mathrm{PAC}}_{\Sigma_1}$ satisfies

$$\sigma_1'(t) = (D\sigma_1)(t), \quad \sigma_1''(t) = f(t, \sigma_1(t)) + b_1(t) \quad \text{for a.e. } t \in [0, 2\pi],$$

$$\Delta\sigma_1'(t) = 2\pi a_1(t), \quad t \in \Sigma_1.$$

Since b_1 is nonnegative and a_1 is positive, we get (5.6) and (5.7). Similarly for σ_2. □

The lower and upper functions method for Eq. (5.1) is based on the following construction. We assume that the functions σ_1 and σ_2 are well ordered

$$\sigma_1(t) \leq \sigma_2(t), \quad t \in [0, 2\pi], \tag{5.10}$$

construct the auxiliary functions

$$\sigma(t,x) = \begin{cases} \sigma_1(t), & x < \sigma_1(t), \\ x, & \sigma_1(t) \leq x \leq \sigma_2(t), \\ \sigma_2(t), & \sigma_2(t) < x, \end{cases} \qquad \sigma^*(x) = \begin{cases} \overline{\sigma_1}, & x < \overline{\sigma_1}, \\ x, & \overline{\sigma_1} \leq x \leq \overline{\sigma_2}, \\ \overline{\sigma_2}, & \overline{\sigma_2} < x, \end{cases} \tag{5.11}$$

$t \in [0, 2\pi]$, $x \in \mathbb{R}$,

$$f^*(t,x) = f(t, \sigma(t,x)) + \frac{x - \sigma(t,x)}{|x - \sigma(t,x)| + 1}, \quad \text{a.e. } t \in [0, 2\pi], \ x \in \mathbb{R}, \tag{5.12}$$

and the auxiliary functionals

$$\mathcal{J}_i^*(x) = J_i\big(\sigma(\tau_i(x), x(\tau_i(x)))\big), \quad x \in \mathrm{NBV}, \ i = 1, \dots, m. \tag{5.13}$$

Now, consider the auxiliary distributional differential equation

$$D^2 z - f^*(\cdot, z) - \overline{z} + \sigma^*(\overline{z}) = \sum_{i=1}^{m} \mathcal{J}_i^*(z) \delta_{\tau_i(z)}. \tag{5.14}$$

Theorem 5.4 (Lower and upper functions method)**.** *Let* (5.2) *hold and let σ_1, σ_2 be lower and upper functions of the distributional differential equation* (5.1) *such that $\sigma_1 \leq \sigma_2$ on $[0, 2\pi]$. Further, let*

$$J_i(\sigma_1(t)) \leq 0, \quad J_i(\sigma_2(t)) \geq 0, \quad t \in [0, 2\pi], \ i = 1, \dots, m. \tag{5.15}$$

Then each solution z of the auxiliary equation (5.14) *is also a solution of Eq.* (5.1) *and in addition*

$$\sigma_1(t) \leq z(t) \leq \sigma_2(t), \quad t \in [0, 2\pi]. \tag{5.16}$$

Proof. Let z be a solution of (5.14) and σ_k, $k = 1, 2$ be lower and upper functions of (5.1).
Step 1. Let us prove that

$$\overline{\sigma_1} \leq \overline{z} \leq \overline{\sigma_2}.$$

We prove the first inequality by contradiction and assume that $\overline{\sigma_1} > \overline{z}$. Define an auxiliary function v by

$$v := z - \sigma_1. \tag{5.17}$$

Then v satisfies

$$D^2 v = f^*(\cdot, z) - f(\cdot, \sigma_1) - b_1 + \overline{z} - \sigma^*(\overline{z}) + \sum_{i=1}^{m} \mathcal{J}_i^*(z) \delta_{\tau_i(z)} - \sum_{t \in \Sigma_1} a_1(t) \delta_t. \tag{5.18}$$

For Σ_1 from Definition 5.1 and Σ_z from (3.2), define

$$\Sigma := \Sigma_1 \cup \Sigma_z.$$

According to Remark 3.5 we can assume that $v \in AC$, $v' \in PAC_\Sigma$. Due to Lemma 3.4, the inequality

$$\overline{z} - \sigma^*(\overline{z}) = \overline{z} - \overline{\sigma_1} \le 0$$

and the nonnegativity of b_1, we see that

$$v''(t) \le f^*(t, z(t)) - f(t, \sigma_1(t)) \quad \text{for a.e. } t \in \mathbb{R}. \tag{5.19}$$

The continuity of v and the assumption $\overline{v} < 0$ yield that the function v has its negative minimum, i.e. there exists $t_0 \in [0, 2\pi)$ such that

$$v(t_0) = \min_{t \in \mathbb{R}} v(t) < 0. \tag{5.20}$$

Therefore there exists $\delta > 0$ such that $v < 0$ on the neighborhood $(t_0 - \delta, t_0 + \delta)$. According to the definition of f^* we get

$$\begin{aligned} v''(t) &\le f(t, \sigma_1(t)) + \frac{v(t)}{|v(t)| + 1} - f(t, \sigma_1(t)) \\ &= \frac{v(t)}{|v(t)| + 1} < 0 \quad \text{for a.e. } t \in (t_0 - \delta, t_0 + \delta). \end{aligned} \tag{5.21}$$

On the other hand, $v \in AC$, $v' \in AC(t_0 - \delta, t_0)$ and $v' \in AC(t_0, t_0 + \delta)$. Hence the minimality of $v(t_0)$ and the Lagrange mean value theorem imply that there exist $a \in (t_0 - \delta, t_0)$ and $b \in (t_0, t_0 + \delta)$ such that

$$v'(a) = \frac{v(t_0) - v(t_0 - \delta)}{\delta} \le 0 \quad \text{and} \quad v'(b) = \frac{v(t_0 + \delta) - v(t_0)}{\delta} \ge 0. \tag{5.22}$$

Consequently

$$\int_a^b v''(s)\,\mathrm{d}s = v'(b) - v'(a) - \Delta v'(t_0) \ge -\Delta v'(t_0). \tag{5.23}$$

Let us determine $\Delta v'(t_0)$. There are several cases.

Case A. If $t_0 \notin \Sigma$, then $v' \in AC(t_0 - \delta, t_0 + \delta)$ and so $\Delta v'(t_0) = 0$.

Case B. If $t_0 \in \Sigma_1$ and $t_0 \ne \Sigma_z$, then according to (5.18) and Lemma 3.4 we have $\Delta v'(t_0) = -2\pi a_1(t_0) < 0$.

Case C. If $t_0 \notin \Sigma_1$ and $t_0 \in \Sigma_z$, then using (5.18) and Lemma 3.4 we get as in the proof of Theorem 3.11

$$\Delta v'(t_0) = \sum_{\substack{1 \le i \le m: \\ t_0 = \tau_i(z)}} 2\pi \mathcal{J}_i^*(z) = \sum_{\substack{1 \le i \le m: \\ t_0 = \tau_i(z)}} 2\pi J_i(\sigma_1(t_0)) \le 0,$$

where the last inequality follows from (5.15) and (5.20).

Case D. If $t_0 \in \Sigma_1 \cap \Sigma_z$, then according to (5.18) and Lemma 3.4 we get similarly as before

$$\Delta v'(t_0) = -2\pi a_1(t_0) + \sum_{\substack{1 \le i \le m: \\ t_0 = \tau_i(z)}} 2\pi J_i(\sigma_1(t_0)) < 0.$$

As we can see, in all cases $\Delta v'(t_0) \le 0$, which implies that the integral in (5.23) is nonnegative. This is in contradiction with (5.21). We have proved that $\overline{\sigma_1} \le \overline{z}$. Using dual arguments we can prove that $\overline{z} \le \overline{\sigma_2}$. Therefore z is a solution of the distributional differential equation

$$D^2 z - f^*(\cdot, z) = \sum_{i=1}^{m} \mathcal{J}_i^*(z)\delta_{\tau_i(z)}.$$

Step 2. Now to prove that z is a solution of (5.1) it suffices to prove (5.16). It can be done in a similar way as in Step 1 . We denote $v := z - \sigma_1$ and assume that there exists t_0 such that (5.20) holds. The function v satisfies (5.18) with $\overline{z} - \sigma^*(\overline{z}) = 0$. Therefore, (5.19) is satisfied (even equality). The rest of the proof is the same. □

6 Existence results

We are ready to prove our main existence results for the distributional differential equation (5.1) and for the periodic problem with state dependent impulses

$$x'(t) = y(t), \quad y'(t) = f(t, x(t)), \tag{6.1}$$

$$\Delta y(\tau_i(x)) = 2\pi J_i(\tau_i(x)), \quad i = 1, \dots, m, \tag{6.2}$$

$$x(0) = x(2\pi), \quad y(0) = y(2\pi), \tag{6.3}$$

where the impulse condition (6.2) is a special case of (1.3).

Theorem 6.1. *Let* (5.2) *hold and let σ_1, σ_2 be lower and upper functions of the distributional differential equation* (5.1) *such that $\sigma_1 \le \sigma_2$ on $[0, 2\pi]$. Further, assume that* (5.15) *is fulfilled, that is*

$$J_i(\sigma_1(t)) \le 0, \quad J_i(\sigma_2(t)) \ge 0, \quad t \in [0, 2\pi], \; i = 1, \dots, m.$$

Then there exists a solution z of Eq. (5.1) *and in addition*

$$\sigma_1(t) \le z(t) \le \sigma_2(t), \quad t \in [0, 2\pi].$$

In addition, if (3.9) *holds, then there exists a solution (x, y) of the periodic problem with state dependent impulses* (6.1), (6.2), (6.3) *such that $x = z$.*

Proof. Consider the operator $\mathcal{F}^* = (\mathcal{F}_1^*, \mathcal{F}_2^*) : \mathbb{X} \to \mathbb{X}$, where

$$\mathcal{F}_1^*(u, r) = I^2 \left(f^*(\cdot, u + r) + \sum_{i=1}^{m} \mathcal{J}_i^*(u + r)\delta_{\tau_i(u+r)} \right), \quad (u, r) \in \widetilde{\mathrm{NBV}} \times \mathbb{R}, \tag{6.4}$$

$$\mathcal{F}_2^*(u, r) = \sigma^*(r) - \sum_{i=1}^{m} \mathcal{J}_i^*(u + r) - \overline{f^*(\cdot, u + r)}, \qquad (u, r) \in \widetilde{\mathrm{NBV}} \times \mathbb{R}. \tag{6.5}$$

If we compare (4.3) and (4.4) with (6.4) and (6.5) respectively, we see that by Lemma 4.3 the operator $\mathcal{F}^*$ is completely continuous on $\mathbb{X}$. Since there exist $h^* \in \mathrm{L}^1$ and $c^* \in (0, \infty)$ such that $|f^*(t, x)| \le h^*(t)$ for a.e. $t \in [0, 2\pi]$ and all $x \in \mathbb{R}$ and $|\mathcal{J}_i^*(x)| \le c^*$ for $x \in \mathbb{R}$, $i = 1, \dots, m$, we use (2.7), (2.18) and have

$$\mathrm{var}\big(I^2 f^*(\cdot, u + r)\big) = \mathrm{var}\big(E_2 * f^*(\cdot, u + r)\big) \le \pi^2 \|h^*\|_{\mathrm{L}^1},$$

and similarly by (2.10), (2.12), (2.16),

$$\operatorname{var}\left(I^2\left(\sum_{i=1}^m \mathcal{J}_i^*(u+r)\delta_{\tau_i(u+r)}\right)\right) = \operatorname{var}\left(\sum_{i=1}^m \mathcal{J}_i^*(u+r)\mathcal{T}_{\tau_i(u+r)}E_2\right) \le mc^*\pi^2,$$

$$|\mathcal{F}_2^*(u,r)| \le \max\{\|\sigma_1\|_{L^1}, \|\sigma_2\|_{L^1}\} + mc^* + \|h^*\|_{L^1}.$$

Thus we can find a ball $\Omega \subset \mathbb{X}$ such that $\mathcal{F}^*(\mathbb{X}) \subset \Omega$ and, by the Schauder fixed point theorem, the operator $\mathcal{F}^*$ has a fixed point $(u,r) \in \Omega$. Let us put $z = u + r$. Since $u \in \widetilde{\mathrm{NBV}}$, it follows that $r = \overline{z}$ and $u = \widetilde{z}$, and by (6.4), (6.5), it holds

$$\widetilde{z} = I^2\left(f^*(\cdot,z) + \sum_{i=1}^m \mathcal{J}_i^*(z)\delta_{\tau_i(z)}\right) \tag{6.6}$$

and

$$\overline{z} = \sigma^*(\overline{z}) - \sum_{i=1}^m \mathcal{J}_i^*(z) - \overline{f^*(\cdot,z)}. \tag{6.7}$$

Due to (2.8) we have $Iz = I\widetilde{z}$ and hence $I\overline{z} = 0$. Similarly $I\sigma^*(\overline{z}) = 0$. Therefore equations (6.6) and (6.7) are equivalent to

$$\widetilde{z} = I^2\left(f^*(\cdot,z) + \overline{z} - \sigma^*(\overline{z}) + \sum_{i=1}^m \mathcal{J}_i^*(z)\delta_{\tau_i(z)}\right) \tag{6.8}$$

and

$$\overline{f^*(\cdot,z) + \overline{z} - \sigma^*(\overline{z})} + \sum_{i=1}^m \mathcal{J}_i^*(z) = 0. \tag{6.9}$$

By Corollary 3.6 and (6.8), (6.9) it follows that z is a solution of the distributional differential equation (5.14), which writes as

$$D^2 z - f^*(\cdot,z) - \overline{z} + \sigma^*(\overline{z}) = \sum_{i=1}^m \mathcal{J}_i^*(z)\delta_{\tau_i(z)}.$$

By Theorem 5.4, z is also a solution of Eq. (5.1) and (5.16) holds. The last assertion follows from Corollary 3.12. □

As a consequence of Theorem 6.1 we get new existence result with simple effective sufficient conditions for the distributional differential equation (5.1) as well as for the periodic problem (6.1), (6.2), (6.3) with state-dependent impulses.

Theorem 6.2. *Let (5.2) hold and let there exist $c, d \in \mathbb{R}$, $c < 0 < d$ such that*

$$f(t,c) \le 0 \le f(t,d) \quad \textit{for a.e. } t \in [0,2\pi] \quad \textit{and} \quad J_i(c) \le 0 \le J_i(d), \quad i = 1,\dots,m.$$

Then there exists a solution z of the distributional differential equation (5.1) satisfying

$$c \le z(t) \le d \quad \textit{for } t \in [0,2\pi]. \tag{6.10}$$

In addition, if (3.9) holds, then the vector-function (x,y), where $x = z$, $y = Dz$ a.e. on $[0,2\pi]$ is a solution of the periodic problem (6.1), (6.2), (6.3) with state-dependent impulses.

Proof. It is sufficient to put

$$\sigma_1(t) = c < 0, \quad \sigma_2(t) = d > 0, \quad t \in [0, 2\pi],$$

and the assertion follows from Theorem 6.1. □

Example 6.3. Note that no Lipschitz continuity is required for f and J_i in Theorem 6.2. Therefore we can consider Eq. (5.1) with $m = 1$,

$$f(t,x) = c_1 t^\alpha + c_2 \sqrt[n]{x}, \quad t \in (0, 2\pi],\ x \in \mathbb{R} \quad \text{and} \quad J_1(x) = c_3 \sqrt[k]{x},\ x \in \mathbb{R},$$

where $\alpha, c_j \in \mathbb{R}$, $j = 1, 2, 3$, $c_2, c_3 > 0$, $\alpha > -1$ and n, k are positive odd integers. Then, if we choose $c, d \in \mathbb{R}$, $c < 0 < d$ such that

$$c < -\left(\frac{|c_1|}{c_2}(2\pi)^\alpha\right)^n \quad \text{and} \quad d > \left(\frac{|c_1|}{c_2}(2\pi)^\alpha\right)^n,$$

we can easily check that all the assumptions of Theorem 6.2 are satisfied.

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