



Blow-up problems for quasilinear reaction diffusion equations with weighted nonlocal source

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Abstract. In this paper, we investigate the following quasilinear reaction diffusion equations

$$\begin{cases} (b(u))_t = \nabla \cdot (\rho(|\nabla u|^2) \nabla u) + c(x)f(u) & \text{in } \Omega \times (0, t^*), \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, t^*), \\ u(x, 0) = u_0(x) \geq 0 & \text{in } \bar{\Omega}. \end{cases}$$

Here Ω is a bounded domain in \mathbb{R}^n ($n \geq 2$) with smooth boundary $\partial\Omega$. Weighted nonlocal source satisfies

$$c(x)f(u(x, t)) \leq a_1 + a_2 (u(x, t))^p \left(\int_{\Omega} (u(x, t))^\alpha dx \right)^m,$$


where a_2, p, α are some positive constants and a_1, m are some nonnegative constants. We make use of a differential inequality technique and Sobolev inequality to obtain a lower bound for the blow-up time of the solution. In addition, an upper bound for the blow-up time is also derived.

Keywords: blow-up problems, quasilinear reaction equation, weighted nonlocal source.

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1 Introduction

The blow-up problems to reaction diffusion equations has been extensively investigated by many researchers. Much of the work prior to the turn of the century is referenced in [1, 9, 10]. More recent work, we refer readers to [13–18, 21]. In practical situations, one would like to know whether the solutions blows up and if so, at which time blow-up occurs. Hence, finding bounds for blow-up time has become the focus of the researchers, especially the search for lower bounds of blow-up time. Since Payne and Schaefer [20] introduced a first-order inequality technique and obtained a lower bound for blow-up time, many authors are devoted to the lower bounds of blow-up time for various reaction diffusion problems, (see, for instance,

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[3–7]). We note that above mentioned studies mainly aimed at seeking lower bounds for blow-up time of local reaction-diffusion equations. In this paper, we concern the reaction diffusion equations with weighted nonlocal source

$$\begin{cases} (b(u))_t = \nabla \cdot (\rho(|\nabla u|^2) \nabla u) + c(x)f(u) & \text{in } \Omega \times (0, t^*), \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, t^*), \\ u(x, 0) = u_0(x) \geq 0 & \text{in } \bar{\Omega}. \end{cases} \quad (1.1)$$

In (1.1), Ω is a bounded domain of \mathbb{R}^n ($n \geq 2$) with smooth boundary $\partial\Omega$, ν represents the unit normal vector to $\partial\Omega$, $u_0(x) \in C^1(\bar{\Omega})$ is a nonnegative function satisfying the compatibility condition, t^* is the blow-up time if blow-up occurs, or else $t^* = \infty$. Weighted nonlocal source satisfies

$$c(x)f(u(x, t)) \leq a_1 + a_2(u(x, t))^p \left(\int_{\Omega} (u(x, t))^\alpha dx \right)^m,$$

where a_2, p, α are some positive constants and a_1, m are some nonnegative constants. Set $\mathbb{R}_+ = (0, \infty)$. Throughout this paper, we assume that b is a $C^2(\bar{\mathbb{R}}_+)$ function with $b'(s) > 0$ for $s > 0$, ρ is a positive $C^2(\bar{\mathbb{R}}_+)$ function satisfying $\rho(s) + 2s\rho'(s) > 0$ for $s > 0$, c is a positive $C(\bar{\Omega})$ function, and f is a nonnegative $C(\bar{\mathbb{R}}_+)$ function. By maximum principles [22], we know that the classical solution u of (1.1) is nonnegative in $\bar{\Omega} \times [0, t^*)$.

For the information about the nonlocal reaction diffusion equations, we refer readers to [2, 11, 12, 19, 23]. Fang and Ma [11] dealt with the following problems

$$\begin{cases} u_t = \sum_{i,j=1}^n (a^{ij}(x)u_{x_i})_{x_j} - c(x)f(u) & \text{in } \Omega \times (0, t^*), \\ \sum_{i,j=1}^n a^{ij}(x)u_{x_i}\nu_j = g(u) & \text{on } \partial\Omega \times (0, t^*), \\ u(x, 0) = u_0(x) & \text{in } \bar{\Omega}, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) is a bounded star-shaped domain with smooth boundary $\partial\Omega$, nonlocal source satisfies

$$f(u(x, t)) \geq a_2(u(x, t))^p \left(\int_{\Omega} (u(x, t))^\alpha dx \right)^m,$$

and a_2, p, α , and m are positive constants. They derived conditions which imply the solution blows up in finite time or exists globally. Furthermore, upper and lower bounds for blow-up time are obtained.

As far as we know, there is little information on the bounds for blow-up time of problem (1.1). Motivated by the above work [11], we study the problem (1.1). Our results of this paper are based on some Sobolev type inequalities and differential inequality technique. In Section 2, when $\Omega \subset \mathbb{R}^n$ ($n \geq 2$), we obtain a criterion for blow-up of the solution of (1.1) and get an upper bound for blow-up time. In Section 3, when $\Omega \subset \mathbb{R}^n$ ($n \geq 3$), we derive a lower bound for blow-up time. An example is presented in Section 4 to illustrate our abstract results derived in this paper.

2 Blow-up solution

In this section, we establish conditions on data to ensure that the solution blows up at t^* and obtain an upper bound for t^* . To accomplish these tasks, we introduce the following auxiliary functions

$$D(t) = \int_{\Omega} G(u(x,t))dx, \quad E(t) = - \int_{\Omega} P(|\nabla u|^2)dx + 2 \int_{\Omega} c(x)F(u)dx, \quad t \geq 0, \quad (2.1)$$

$$G(u) = 2 \int_0^u sb'(s)ds, \quad P(|\nabla u|^2) = \int_0^{|\nabla u|^2} \rho(s)ds, \quad F(u) = \int_0^u f(s)ds, \quad (2.2)$$

where u is the classical solution of (1.1). Our main result of this section is the following Theorem 2.1

Theorem 2.1. *Let u be a classical solution of (1.1). We suppose that functions b, c, ρ , and f satisfy*

$$\begin{aligned} b''(s) < 0, \quad s\rho(s) \leq (1 + \beta)P(s), \\ \int_{\Omega} c(x)s(x,t)f(s(x,t))dx \geq 2(1 + \beta) \int_{\Omega} c(x)F(s(x,t))dx, \quad s \geq 0, \end{aligned} \quad (2.3)$$

where β is a nonnegative constant. In addition, initial data are assumed to satisfy

$$E(0) = - \int_{\Omega} P(|\nabla u_0|^2)dx + 2 \int_{\Omega} c(x)F(u_0)dx > 0. \quad (2.4)$$

Then u must blow up at $t^* \leq T$ in measure $D(t)$ with

$$T = \begin{cases} \frac{D(0)}{2\beta(1 + \beta)E(0)}, & \beta > 0, \\ \infty, & \beta = 0. \end{cases}$$

Proof. It follows from Green's formula and (2.3) that

$$\begin{aligned} D'(t) &= \int_{\Omega} G'(u)u_t dx = 2 \int_{\Omega} ub'(u)u_t dx \\ &= 2 \int_{\Omega} u [\nabla \cdot (\rho(|\nabla u|^2)\nabla u) + c(x)f(u)] dx \\ &= 2 \int_{\Omega} \nabla \cdot (u\rho(|\nabla u|^2)\nabla u) dx - 2 \int_{\Omega} \rho(|\nabla u|^2)|\nabla u|^2 dx + 2 \int_{\Omega} c(x)uf(u) dx \\ &= 2 \int_{\partial\Omega} u\rho(|\nabla u|^2)\frac{\partial u}{\partial \nu} dS - 2 \int_{\Omega} \rho(|\nabla u|^2)|\nabla u|^2 dx + 2 \int_{\Omega} c(x)uf(u) dx \\ &\geq -2(1 + \beta) \int_{\Omega} P(|\nabla u|^2)dx + 4(1 + \beta) \int_{\Omega} c(x)F(u)dx \\ &= 2(1 + \beta) \left[- \int_{\Omega} P(|\nabla u|^2)dx + 2 \int_{\Omega} c(x)F(u)dx \right] = 2(1 + \beta)E(t). \end{aligned} \quad (2.5)$$

Differentiating $E(t)$, we get

$$\begin{aligned}
E'(t) &= -2 \int_{\Omega} \rho(|\nabla u|^2) (\nabla u \cdot \nabla u_t) \, dx + 2 \int_{\Omega} c(x) f(u) u_t \, dx \\
&= 2 \int_{\partial\Omega} \rho(|\nabla u|^2) u_t \frac{\partial u}{\partial \nu} \, dS - 2 \int_{\Omega} \rho(|\nabla u|^2) (\nabla u \cdot \nabla u_t) \, dx + 2 \int_{\Omega} c(x) f(u) u_t \, dx \\
&= 2 \int_{\Omega} \nabla \cdot (\rho(|\nabla u|^2) u_t \nabla u) \, dx - 2 \int_{\Omega} \rho(|\nabla u|^2) (\nabla u \cdot \nabla u_t) \, dx + 2 \int_{\Omega} c(x) f(u) u_t \, dx \\
&= 2 \int_{\Omega} u_t [\nabla \cdot (\rho(|\nabla u|^2) \nabla u) + c(x) f(u)] \, dx \\
&= 2 \int_{\Omega} b'(u) u_t^2 \, dx \geq 0,
\end{aligned} \tag{2.6}$$

which with (2.4) imply $E(t) > 0$ and $D'(t) > 0$ for all $t \in (0, t^*)$. By the Hölder inequality, (2.5) and $b'(s) > 0$ for $s > 0$, we obtain

$$\begin{aligned}
2(1 + \beta)E(t)D'(t) &\leq (D'(t))^2 = \left(2 \int_{\Omega} b'(u) u u_t \, dx \right)^2 \\
&\leq 4 \left(\int_{\Omega} b'(u) u^2 \, dx \right) \left(\int_{\Omega} b'(u) u_t^2 \, dx \right).
\end{aligned} \tag{2.7}$$

Using (2.3) and integrating by part, we lead to

$$G(u) = 2 \int_0^u s b'(s) \, ds = \int_0^u b'(s) \, ds^2 = b'(u) u^2 - \int_0^u s^2 b''(s) \, ds \geq b'(u) u^2. \tag{2.8}$$

We combine (2.7) and (2.8) to derive

$$(1 + \beta)E(t)D'(t) \leq 2 \left(\int_{\Omega} G(u) \, dx \right) \left(\int_{\Omega} b'(u) u_t^2 \, dx \right) = D(t)E'(t);$$

that is

$$\left(E(t) D^{-(1+\beta)}(t) \right)' \geq 0. \tag{2.9}$$

Integrating (2.9) over $[0, t]$, we have

$$E(t) D^{-(1+\beta)}(t) \geq E(0) D^{-(1+\beta)}(0).$$

By (2.5), we can deduce

$$D'(t) D^{-(1+\beta)}(t) \geq 2(1 + \beta) E(0) D^{-(1+\beta)}(0). \tag{2.10}$$

If $\beta > 0$, integrating (2.10) over $[0, t]$, we derive

$$D^{-\beta}(t) \leq D^{-\beta}(0) - 2\beta(1 + \beta) E(0) D^{-(1+\beta)}(0) t. \tag{2.11}$$

This inequality can not hold for all $t > 0$. Hence, $u(x, t)$ must blow up at some finite time t^* in the measure $D(t)$. Furthermore, we conclude from (2.11)

$$t^* \leq T = \frac{D(0)}{2\beta(1 + \beta)E(0)}.$$

If $\beta = 0$, we integrate (2.10) to get

$$D(t) \geq D(0) e^{2E(0)D^{-1}(0)t},$$

which implies $T = \infty$. □

3 Lower bound for blow-up time

In this section, we restrict $\Omega \subset \mathbb{R}^n$ ($n \geq 3$). Our goal is to determine a lower bound for blow-up time t^* . Here we impose the following constraints on data

$$\begin{aligned} \rho(s) &\geq b_1 + b_2 s^q, & b'(s) &\geq \gamma, \\ c(x)f(s(x,t)) &\leq a_1 + a_2(s(x,t))^p \left(\int_{\Omega} (s(x,t))^\alpha dx \right)^m, & s &\geq 0, \end{aligned} \quad (3.1)$$

where a_2, b_2, p, q, γ are positive constants, a_1, b_1, m are nonnegative constants, $p > 2q + 1$, $\alpha = 2r(q + 1) - 2q$, and parameter r is restricted by the condition

$$r > \max \left\{ 1, \frac{n(p - 2q - 1) + 4q}{4(q + 1)} \right\}. \quad (3.2)$$

We introduce two auxiliary functions

$$A(t) = \int_{\Omega} B(u) dx, \quad t \geq 0, \quad B(u) = \alpha \int_0^u s^{\alpha-1} b'(s) ds.$$

In this section, we also need to apply the following Sobolev inequality (see [8, Theorem 2, p. 265]) for $n \geq 3$,

$$\left(\int_{\Omega} (v^{q+1})^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{2n}} \leq C \left(\int_{\Omega} v^{2(q+1)} dx + \int_{\Omega} |\nabla v^{q+1}|^2 dx \right)^{\frac{1}{2}}, \quad (3.3)$$

where $C = C(n, \Omega)$ is an embedding constant. The main result of this section is stated as follows.

Theorem 3.1. *Let u be a classical solution of (1.1). Assume that (3.1)–(3.2) hold. If u blows up at finite time t^* in measure $A(t)$, we then conclude that blow-up time t^* is bounded from below by*

$$t^* \geq \int_{A(0)}^{\infty} \frac{d\tau}{K_1 \tau^{\frac{\alpha-1}{\alpha}} + K_2 \tau^{\frac{[4r(q+1)+2q(n-2)](1+m)-(n-2)(p-1)}{4r(q+1)+2q(n-2)-n(p-1)} + K_3 \tau^{\frac{2r(q+1)}{2r(q+1)-2q}}},$$

where

$$K_1 = a_1 \alpha |\Omega|^{\frac{1}{\alpha}} \gamma^{\frac{1-\alpha}{\alpha}}, \quad (3.4)$$

$$\begin{aligned} K_2 &= \frac{a_2 \alpha [4r(q+1) + 2q(n-2) - n(p-1)]}{4r(q+1) + 2q(n-2)} (2C^2)^{\frac{n(p-1)}{4r(q+1)+2q(n-2)-n(p-1)}} \\ &\times \left(1 + \sigma_1^{-\frac{n(p-1)}{4r(q+1)+2q(n-2)-n(p-1)}} \right) \gamma^{-\frac{[4r(q+1)+2q(n-2)](1+m)-(n-2)(p-1)}{4r(q+1)+2q(n-2)-n(p-1)}}, \end{aligned} \quad (3.5)$$

$$\begin{aligned} K_3 &= \left(\frac{b_2 q \alpha (\alpha - 1)}{r^2 (q+1)} + \frac{a_2 n \alpha (p-1)}{4r(q+1) + 2q(n-2)} (2C^2)^{\frac{n(p-1)}{4r(q+1)+2q(n-2)-n(p-1)}} \right) \\ &\times \frac{4r(q+1) - 4q}{2r(q+1) + q(n-2)} (2C^2)^{\frac{nq}{2r(q+1)-2q}} \gamma^{-\frac{2r(q+1)}{2r(q+1)-2q}} \\ &\times \left[\sigma_2^{-\frac{nq}{2r(q+1)-2q}} + \left(\frac{2r(q+1) + q(n-2)}{2nq} \right)^{-\frac{nq}{2r(q+1)-2q}} \right], \end{aligned} \quad (3.6)$$

$$\sigma_1 = \frac{b_2(\alpha - 1)[2r(q + 1) + q(n - 2)]}{a_2n(p - 1)(q + 1)r^{2(q+1)}} (2C^2)^{-\frac{n(p-1)}{4r(q+1)+2q(n-2)-n(p-1)}}, \quad (3.7)$$

$$\begin{aligned} \sigma_2 &= \frac{b_2(\alpha - 1)[2r(q + 1) + q(n - 2)]}{2nq(q + 1)} \\ &\quad \times \left[2b_2q(\alpha - 1) + \frac{a_2n(p - 1)r^{2(q+1)}}{2r(q + 1) + q(n - 2)} (2C^2)^{\frac{n(p-1)}{4r(q+1)+2q(n-2)-n(p-1)}} \right]^{-1}. \end{aligned} \quad (3.8)$$

Proof. By (3.2), we have $\alpha > 2$. It follows from Green's formula and (3.1) that

$$\begin{aligned} A'(t) &= \int_{\Omega} B'(u)u_t dx = \alpha \int_{\Omega} u^{\alpha-1}b'(u)u_t dx \\ &= \alpha \int_{\Omega} u^{\alpha-1} [\nabla \cdot (\rho(|\nabla u|^2)\nabla u) + c(x)f(u)] dx \\ &= \alpha \int_{\Omega} \nabla \cdot (u^{\alpha-1}\rho(|\nabla u|^2)\nabla u) dx - \alpha(\alpha - 1) \int_{\Omega} \rho(|\nabla u|^2)u^{\alpha-2}|\nabla u|^2 dx \\ &\quad + \alpha \int_{\Omega} u^{\alpha-1}c(x)f(u) dx \\ &\leq \alpha \int_{\partial\Omega} u^{\alpha-1}\rho(|\nabla u|^2)\frac{\partial u}{\partial\nu} dS - \alpha(\alpha - 1) \int_{\Omega} u^{\alpha-2}(b_1 + b_2|\nabla u|^{2q})|\nabla u|^2 dx \\ &\quad + a_1\alpha \int_{\Omega} u^{\alpha-1} dx + a_2\alpha \int_{\Omega} u^{\alpha+p-1} dx \left(\int_{\Omega} u^{\alpha} dx \right)^m \\ &\leq -b_2\alpha(\alpha - 1) \int_{\Omega} u^{\alpha-2}|\nabla u|^{2(q+1)} dx + a_1\alpha \int_{\Omega} u^{\alpha-1} dx \\ &\quad + a_2\alpha \int_{\Omega} u^{\alpha+p-1} dx \left(\int_{\Omega} u^{\alpha} dx \right)^m. \end{aligned} \quad (3.9)$$

We apply the Hölder inequality to get

$$\int_{\Omega} u^{\alpha-1} dx \leq |\Omega|^{\frac{1}{\alpha}} \left(\int_{\Omega} u^{\alpha} dx \right)^{\frac{\alpha-1}{\alpha}}. \quad (3.10)$$

For brevity, we denote $v = u^r$ and

$$|\nabla u^r|^{2(q+1)} = r^{2(q+1)}u^{2(r-1)(q+1)}|\nabla u|^{2(q+1)}. \quad (3.11)$$

Hence, by (3.10)–(3.11), (3.9) can be rewritten as

$$\begin{aligned} A'(t) &\leq -\frac{b_2\alpha(\alpha - 1)}{r^{2(q+1)}} \int_{\Omega} |\nabla u^r|^{2(q+1)} dx + a_1\alpha|\Omega|^{\frac{1}{\alpha}} \left(\int_{\Omega} u^{\alpha} dx \right)^{\frac{\alpha-1}{\alpha}} \\ &\quad + a_2\alpha \int_{\Omega} u^{\alpha+p-1} dx \left(\int_{\Omega} u^{\alpha} dx \right)^m \\ &= -\frac{b_2\alpha(\alpha - 1)}{r^{2(p+1)}} \int_{\Omega} |\nabla v|^{2(q+1)} dx + a_1\alpha|\Omega|^{\frac{1}{\alpha}} \left(\int_{\Omega} v^{\frac{\alpha}{r}} dx \right)^{\frac{\alpha-1}{\alpha}} \\ &\quad + a_2\alpha \int_{\Omega} v^{2(q+1)+\frac{p-2q-1}{r}} dx \left(\int_{\Omega} v^{\frac{\alpha}{r}} dx \right)^m. \end{aligned} \quad (3.12)$$

Using the Hölder inequality and the Young inequality, we have

$$\begin{aligned} \int_{\Omega} |\nabla v^{q+1}|^2 dx &= (q+1)^2 \int_{\Omega} v^{2q} |\nabla v|^2 dx \\ &\leq (q+1)^2 \left(\int_{\Omega} v^{2(q+1)} dx \right)^{\frac{q}{q+1}} \left(\int_{\Omega} |\nabla v|^{2(q+1)} dx \right)^{\frac{1}{q+1}} \\ &\leq q(q+1) \int_{\Omega} v^{2(q+1)} dx + (q+1) \int_{\Omega} |\nabla v|^{2(q+1)} dx; \end{aligned}$$

that is

$$\int_{\Omega} |\nabla v|^{2(q+1)} dx \geq \frac{1}{q+1} \int_{\Omega} |\nabla v^{q+1}|^2 dx - q \int_{\Omega} v^{2(q+1)} dx. \quad (3.13)$$

Substituting (3.13) into (3.12), we get

$$\begin{aligned} A'(t) &\leq \frac{b_2 q \alpha (\alpha - 1)}{r^{2(q+1)}} \int_{\Omega} v^{2(q+1)} dx - \frac{b_2 \alpha (\alpha - 1)}{(q+1)r^{2(q+1)}} \int_{\Omega} |\nabla v^{q+1}|^2 dx + a_1 \alpha |\Omega|^{\frac{1}{\alpha}} \left(\int_{\Omega} v^{\frac{\alpha}{r}} dx \right)^{\frac{\alpha-1}{\alpha}} \\ &\quad + a_2 \alpha \int_{\Omega} v^{2(q+1) + \frac{p-2q-1}{r}} dx \left(\int_{\Omega} v^{\frac{\alpha}{r}} dx \right)^m. \end{aligned} \quad (3.14)$$

Now, we deal with the last term of (3.14). Applying the Hölder inequality and (3.3), we derive

$$\begin{aligned} &\int_{\Omega} v^{2(q+1) + \frac{p-2q-1}{r}} dx \left(\int_{\Omega} v^{\frac{\alpha}{r}} dx \right)^m \\ &\leq \left(\int_{\Omega} v^{\frac{\alpha}{r}} dx \right)^{\frac{4r(q+1)+2q(n-2)-(n-2)(p-1)}{4r(q+1)+2q(n-2)} + m} \left(\int_{\Omega} (v^{q+1})^{\frac{2n}{n-2}} dx \right)^{\frac{(n-2)(p-1)}{4r(q+1)+2q(n-2)}} \\ &\leq \left(\int_{\Omega} v^{\frac{\alpha}{r}} dx \right)^{\frac{4r(q+1)+2q(n-2)-(n-2)(p-1)}{4r(q+1)+2q(n-2)} + m} \left[C^{\frac{2n}{n-2}} \left(\int_{\Omega} v^{2(q+1)} dx + \int_{\Omega} |\nabla v^{q+1}|^2 dx \right)^{\frac{n}{n-2}} \right]^{\frac{(n-2)(p-1)}{4r(q+1)+2q(n-2)}} \\ &= \left(\int_{\Omega} v^{\frac{\alpha}{r}} dx \right)^{\frac{4r(q+1)+2q(n-2)-(n-2)(p-1)}{4r(q+1)+2q(n-2)} + m} \left(C^2 \int_{\Omega} v^{2(q+1)} dx + C^2 \int_{\Omega} |\nabla v^{q+1}|^2 dx \right)^{\frac{n(p-1)}{4r(q+1)+2q(n-2)}}, \end{aligned} \quad (3.15)$$

where $0 < \frac{(n-2)(p-1)}{4r(q+1)+2q(n-2)} < 1$ in view of (3.2). Using in (3.15) the basic inequality

$$(k_1 + k_2)^j \leq 2^j (k_1^j + k_2^j), \quad k_1, k_2, j > 0, \quad (3.16)$$

we have

$$\begin{aligned} &\int_{\Omega} v^{2(q+1) + \frac{p-2q-1}{r}} dx \left(\int_{\Omega} v^{\frac{\alpha}{r}} dx \right)^m \\ &\leq (2C^2)^{\frac{n(p-1)}{4r(q+1)+2q(n-2)}} \left(\int_{\Omega} v^{\frac{\alpha}{r}} dx \right)^{\frac{4r(q+1)+2q(n-2)-(n-2)(p-1)}{4r(q+1)+2q(n-2)} + m} \left(\int_{\Omega} v^{2(q+1)} dx \right)^{\frac{n(p-1)}{4r(q+1)+2q(n-2)}} \\ &\quad + (2C^2)^{\frac{n(p-1)}{4r(q+1)+2q(n-2)}} \left(\int_{\Omega} v^{\frac{\alpha}{r}} dx \right)^{\frac{4r(q+1)+2q(n-2)-(n-2)(p-1)}{4r(q+1)+2q(n-2)} + m} \\ &\quad \times \left(\int_{\Omega} |\nabla v^{q+1}|^2 dx \right)^{\frac{n(p-1)}{4r(q+1)+2q(n-2)}}. \end{aligned} \quad (3.17)$$

An application of the Young inequality to the first term of (3.17) yields

$$\begin{aligned}
& \left(\int_{\Omega} v^{\frac{\alpha}{r}} dx \right)^{\frac{4r(q+1)+2q(n-2)-(n-2)(p-1)}{4r(q+1)+2q(n-2)}+m} \left(\int_{\Omega} v^{2(q+1)} dx \right)^{\frac{n(p-1)}{4r(q+1)+2q(n-2)}} \\
&= \left[\left(\int_{\Omega} v^{\frac{\alpha}{r}} dx \right)^{\frac{[4r(q+1)+2q(n-2)](1+m)-(n-2)(p-1)}{4r(q+1)+2q(n-2)-n(p-1)}} \right]^{\frac{4r(q+1)+2q(n-2)-n(p-1)}{4r(q+1)+2q(n-2)}} \left(\int_{\Omega} v^{2(q+1)} dx \right)^{\frac{n(p-1)}{4r(q+1)+2q(n-2)}} \\
&\leq \frac{4r(q+1)+2q(n-2)-n(p-1)}{4r(q+1)+2q(n-2)} \left(\int_{\Omega} v^{\frac{\alpha}{r}} dx \right)^{\frac{[4r(q+1)+2q(n-2)](1+m)-(n-2)(p-1)}{4r(q+1)+2q(n-2)-n(p-1)}} \\
&\quad + \frac{n(p-1)}{4r(q+1)+2q(n-2)} \int_{\Omega} v^{2(q+1)} dx, \tag{3.18}
\end{aligned}$$

where we use the fact that $0 < \frac{n(p-1)}{4r(q+1)+2q(n-2)} < 1$ due to (3.2). Similarly, for the second term of (3.17), we apply the Young inequality to obtain

$$\begin{aligned}
& \left(\int_{\Omega} v^{\frac{\alpha}{r}} dx \right)^{\frac{4r(q+1)+2q(n-2)-(n-2)(p-1)}{4r(q+1)+2q(n-2)}+m} \left(\int_{\Omega} |\nabla v^{q+1}|^2 dx \right)^{\frac{n(p-1)}{4r(q+1)+2q(n-2)}} \\
&= \left[\left(\int_{\Omega} v^{\frac{\alpha}{r}} dx \right)^{\frac{[4r(q+1)+2q(n-2)](1+m)-(n-2)(p-1)}{4r(q+1)+2q(n-2)-n(p-1)}} \right]^{\frac{4r(q+1)+2q(n-2)-n(p-1)}{4r(q+1)+2q(n-2)}} \left(\int_{\Omega} |\nabla v^{q+1}|^2 dx \right)^{\frac{n(p-1)}{4r(q+1)+2q(n-2)}} \\
&\leq \frac{4r(q+1)+2q(n-2)-n(p-1)}{4r(q+1)+2q(n-2)} \sigma_1^{-\frac{n(p-1)}{4r(q+1)+2q(n-2)-n(p-1)}} \left(\int_{\Omega} v^{\frac{\alpha}{r}} dx \right)^{\frac{[4r(q+1)+2q(n-2)](1+m)-(n-2)(p-1)}{4r(q+1)+2q(n-2)-n(p-1)}} \\
&\quad + \frac{n(p-1)\sigma_1}{4r(q+1)+2q(n-2)} \int_{\Omega} |\nabla v^{q+1}|^2 dx, \tag{3.19}
\end{aligned}$$

where σ_1 is given in (3.7). Substituting (3.18)–(3.19) into (3.17), we have

$$\begin{aligned}
& \int_{\Omega} v^{2(q+1)+\frac{p-2q-1}{r}} dx \left(\int_{\Omega} v^{\frac{\alpha}{r}} dx \right)^m \\
&\leq \frac{4r(q+1)+2q(n-2)-n(p-1)}{4r(q+1)+2q(n-2)} (2C^2)^{\frac{n(p-1)}{4r(q+1)+2q(n-2)-n(p-1)}} \\
&\quad \times \left(1 + \sigma_1^{-\frac{n(p-1)}{4r(q+1)+2q(n-2)-n(p-1)}} \right) \left(\int_{\Omega} v^{\frac{\alpha}{r}} dx \right)^{\frac{[4r(q+1)+2q(n-2)](1+m)-(n-2)(p-1)}{4r(q+1)+2q(n-2)-n(p-1)}} \\
&\quad + \frac{n(p-1)}{4r(q+1)+2q(n-2)} (2C^2)^{\frac{n(p-1)}{4r(q+1)+2q(n-2)-n(p-1)}} \\
&\quad \times \left(\sigma_1 \int_{\Omega} |\nabla v^{q+1}|^2 dx + \int_{\Omega} v^{2(q+1)} dx \right). \tag{3.20}
\end{aligned}$$

Inserting (3.20) into (3.14), we deduce

$$\begin{aligned}
 A'(t) &\leq a_1 \alpha |\Omega|^{\frac{1}{\alpha}} \left(\int_{\Omega} v^{\frac{\alpha}{r}} dx \right)^{\frac{\alpha-1}{\alpha}} + \frac{a_2 \alpha [4r(q+1) + 2q(n-2) - n(p-1)]}{4r(q+1) + 2q(n-2)} (2C^2)^{\frac{n(p-1)}{4r(q+1)+2q(n-2)-n(p-1)}} \\
 &\quad \times \left(1 + \sigma_1^{-\frac{n(p-1)}{4r(q+1)+2q(n-2)-n(p-1)}} \right) \left(\int_{\Omega} v^{\frac{\alpha}{r}} dx \right)^{\frac{[4r(q+1)+2q(n-2)](1+m)-(n-2)(p-1)}{4r(q+1)+2q(n-2)-n(p-1)}} \\
 &\quad + \left(\frac{b_2 q \alpha (\alpha-1)}{r^2(q+1)} + \frac{a_2 n \alpha (p-1)}{4r(q+1) + 2q(n-2)} (2C^2)^{\frac{n(p-1)}{4r(q+1)+2q(n-2)-n(p-1)}} \right) \int_{\Omega} v^{2(q+1)} dx \\
 &\quad + \left(\frac{a_2 n \alpha (p-1) \sigma_1}{4r(q+1) + 2q(n-2)} (2C^2)^{\frac{n(p-1)}{4r(q+1)+2q(n-2)-n(p-1)}} - \frac{b_2 \alpha (\alpha-1)}{(q+1)r^2(q+1)} \right) \\
 &\quad \times \int_{\Omega} |\nabla v^{q+1}|^2 dx. \tag{3.21}
 \end{aligned}$$

Next, we pay our attention to the third term of (3.21). By the Hölder inequality and (3.3), we obtain

$$\begin{aligned}
 &\int_{\Omega} v^{2(q+1)} dx \\
 &\leq \left(\int_{\Omega} v^{\frac{\alpha}{r}} dx \right)^{\frac{2r(q+1)}{2r(q+1)+q(n-2)}} \left(\int_{\Omega} (v^{q+1})^{\frac{2n}{n-2}} dx \right)^{\frac{q(n-2)}{2r(q+1)+q(n-2)}} \\
 &\leq \left(\int_{\Omega} v^{\frac{\alpha}{r}} dx \right)^{\frac{2r(q+1)}{2r(q+1)+q(n-2)}} \left[C^{\frac{2n}{n-2}} \left(\int_{\Omega} v^{2(q+1)} dx + \int_{\Omega} |\nabla v^{q+1}|^2 dx \right)^{\frac{n}{n-2}} \right]^{\frac{q(n-2)}{2r(q+1)+q(n-2)}} \\
 &= \left(\int_{\Omega} v^{\frac{\alpha}{r}} dx \right)^{\frac{2r(q+1)}{2r(q+1)+q(n-2)}} \left(C^2 \int_{\Omega} v^{2(q+1)} dx + C^2 \int_{\Omega} |\nabla v^{q+1}|^2 dx \right)^{\frac{nq}{2r(q+1)+q(n-2)}}, \tag{3.22}
 \end{aligned}$$

where $0 < \frac{q(n-2)}{2r(q+1)+q(n-2)} < 1$ in view of (3.2). Using (3.16) in (3.22), we have

$$\begin{aligned}
 &\int_{\Omega} v^{2(q+1)} dx \\
 &\leq (2C^2)^{\frac{nq}{2r(q+1)+q(n-2)}} \left(\int_{\Omega} v^{\frac{\alpha}{r}} dx \right)^{\frac{2r(q+1)}{2r(q+1)+q(n-2)}} \left(\int_{\Omega} v^{2(q+1)} dx \right)^{\frac{nq}{2r(q+1)+q(n-2)}} \\
 &\quad + (2C^2)^{\frac{nq}{2r(q+1)+q(n-2)}} \left(\int_{\Omega} v^{\frac{\alpha}{r}} dx \right)^{\frac{2r(q+1)}{2r(q+1)+q(n-2)}} \left(\int_{\Omega} |\nabla v^{q+1}|^2 dx \right)^{\frac{nq}{2r(q+1)+q(n-2)}}. \tag{3.23}
 \end{aligned}$$

For the first term of (3.23), we use the Young inequality to get

$$\begin{aligned}
& (2C^2)^{\frac{nq}{2r(q+1)+q(n-2)}} \left(\int_{\Omega} v^{\frac{\alpha}{r}} dx \right)^{\frac{2r(q+1)}{2r(q+1)+q(n-2)}} \left(\int_{\Omega} v^{2(q+1)} dx \right)^{\frac{nq}{2r(q+1)+q(n-2)}} \\
&= \left[(2C^2)^{\frac{nq}{2r(q+1)-2q}} \left(\frac{2r(q+1)+q(n-2)}{2nq} \right)^{-\frac{nq}{2r(q+1)-2q}} \left(\int_{\Omega} v^{\frac{\alpha}{r}} dx \right)^{\frac{2r(q+1)}{2r(q+1)-2q}} \right]^{\frac{2r(q+1)-2q}{2r(q+1)+q(n-2)}} \\
&\quad \times \left(\frac{2r(q+1)+q(n-2)}{2nq} \int_{\Omega} v^{2(q+1)} dx \right)^{\frac{nq}{2r(q+1)+q(n-2)}} \\
&\leq \frac{2r(q+1)-2q}{2r(q+1)+q(n-2)} (2C^2)^{\frac{nq}{2r(q+1)-2q}} \left(\frac{2r(q+1)+q(n-2)}{2nq} \right)^{-\frac{nq}{2r(q+1)-2q}} \left(\int_{\Omega} v^{\frac{\alpha}{r}} dx \right)^{\frac{2r(q+1)}{2r(q+1)-2q}} \\
&\quad + \frac{1}{2} \int_{\Omega} v^{2(q+1)} dx, \tag{3.24}
\end{aligned}$$

where $0 < \frac{nq}{2r(q+1)+q(n-2)} < 1$ in consideration of (3.2). We again use the Young inequality to second term of (3.23) to obtain

$$\begin{aligned}
& (2C^2)^{\frac{nq}{2r(q+1)+q(n-2)}} \left(\int_{\Omega} v^{\frac{\alpha}{r}} dx \right)^{\frac{2r(q+1)}{2r(q+1)+q(n-2)}} \left(\int_{\Omega} |\nabla v^{q+1}|^2 dx \right)^{\frac{nq}{2r(q+1)+q(n-2)}} \\
&\leq \left[(2C^2)^{\frac{nq}{2r(q+1)-2q}} \left(\int_{\Omega} v^{\frac{\alpha}{r}} dx \right)^{\frac{2r(q+1)}{2r(q+1)-2q}} \sigma_2^{-\frac{nq}{2r(q+1)-2q}} \right]^{\frac{2r(q+1)-2q}{2r(q+1)+q(n-2)}} \left(\sigma_2 \int_{\Omega} |\nabla v^{q+1}|^2 dx \right)^{\frac{nq}{2r(q+1)+q(n-2)}} \\
&\leq \frac{2r(q+1)-2q}{2r(q+1)+q(n-2)} (2C^2)^{\frac{nq}{2r(q+1)-2q}} \sigma_2^{-\frac{nq}{2r(q+1)-2q}} \left(\int_{\Omega} v^{\frac{\alpha}{r}} dx \right)^{\frac{2r(q+1)}{2r(q+1)-2q}} \\
&\quad + \frac{nq}{2r(q+1)+q(n-2)} \sigma_2 \int_{\Omega} |\nabla v^{q+1}|^2 dx, \tag{3.25}
\end{aligned}$$

where σ_2 is defined in (3.8). Substituting (3.24)–(3.25) into (3.23), we get

$$\begin{aligned}
& \int_{\Omega} v^{2(q+1)} dx \\
&\leq \frac{4r(q+1)-4q}{2r(q+1)+q(n-2)} (2C^2)^{\frac{nq}{2r(q+1)-2q}} \left[\left(\frac{2r(q+1)+q(n-2)}{2nq} \right)^{-\frac{nq}{2r(q+1)-2q}} + \sigma_2^{-\frac{nq}{2r(q+1)-2q}} \right] \\
&\quad \times \left(\int_{\Omega} v^{\frac{\alpha}{r}} dx \right)^{\frac{2r(q+1)}{2r(q+1)-2q}} + \frac{2nq\sigma_2}{2r(q+1)+q(n-2)} \int_{\Omega} |\nabla v^{q+1}|^2 dx. \tag{3.26}
\end{aligned}$$

Inserting (3.26) into (3.21), we have

$$\begin{aligned}
A'(t) &\leq a_1 \alpha |\Omega|^{\frac{1}{\alpha}} \left(\int_{\Omega} v^{\frac{\alpha}{r}} dx \right)^{\frac{\alpha-1}{\alpha}} + J_1 \left(\int_{\Omega} v^{\frac{\alpha}{r}} dx \right)^{\frac{[4r(q+1)+2q(n-2)](1+m)-(n-2)(p-1)}{4r(q+1)+2q(n-2)-n(p-1)}} \\
&\quad + J_2 \left(\int_{\Omega} v^{\frac{\alpha}{r}} dx \right)^{\frac{2r(q+1)}{2r(q+1)-2q}}, \tag{3.27}
\end{aligned}$$

where

$$J_1 = \frac{a_2\alpha[4r(q+1) + 2q(n-2) - n(p-1)]}{4r(q+1) + 2q(n-2)} (2C^2)^{\frac{n(p-1)}{4r(q+1)+2q(n-2)-n(p-1)}} \left(1 + \sigma_1^{-\frac{n(p-1)}{4r(q+1)+2q(n-2)-n(p-1)}}\right),$$

$$J_2 = \left(\frac{b_2q\alpha(\alpha-1)}{r^2(q+1)} + \frac{a_2n\alpha(p-1)}{4r(q+1) + 2q(n-2)} (2C^2)^{\frac{n(p-1)}{4r(q+1)+2q(n-2)-n(p-1)}}\right)$$

$$\times \frac{4r(q+1) - 4q}{2r(q+1) + q(n-2)} (2C^2)^{\frac{nq}{2r(q+1)-2q}} \left[\left(\frac{2r(q+1) + q(n-2)}{2nq}\right)^{-\frac{nq}{2r(q+1)-2q}} + \sigma_2^{-\frac{nq}{2r(q+1)-2q}}\right].$$

From (3.1), it follows that

$$B(u) = \alpha \int_0^u b'(s)s^{\alpha-1} ds \geq \alpha\gamma \int_0^u s^{\alpha-1} ds = \gamma u^\alpha;$$

that is

$$v^{\frac{\alpha}{r}} = u^\alpha \leq \frac{1}{\gamma} B(u). \quad (3.28)$$

Combining (3.27) and (3.28), we get

$$A'(t) \leq a_1\alpha|\Omega|^{\frac{1}{\alpha}} \gamma^{\frac{\alpha-1}{\alpha}} \left(\int_{\Omega} B(u) dx\right)^{\frac{\alpha-1}{\alpha}} + J_1\gamma^{-\frac{[4r(q+1)+2q(n-2)](1+m)-(n-2)(p-1)}{4r(q+1)+2q(n-2)-n(p-1)}}$$

$$\times \left(\int_{\Omega} B(u) dx\right)^{\frac{[4r(q+1)+2q(n-2)](1+m)-(n-2)(p-1)}{4r(q+1)+2q(n-2)-n(p-1)}} + J_2\gamma^{-\frac{2r(q+1)}{2r(q+1)-2q}} \left(\int_{\Omega} B(u) dx\right)^{\frac{2r(q+1)}{2r(q+1)-2q}}$$

$$= K_1 A(t)^{\frac{\alpha-1}{\alpha}} + K_2 A(t)^{\frac{[4r(q+1)+2q(n-2)](1+m)-(n-2)(p-1)}{4r(q+1)+2q(n-2)-n(p-1)}} + K_3 A(t)^{\frac{2r(q+1)}{2r(q+1)-2q}}, \quad (3.29)$$

where K_1, K_2 and K_3 are defined in (3.4), (3.5) and (3.6), respectively. We integrate (3.29) from 0 to t to obtain

$$\int_{A(0)}^{A(t)} \frac{d\tau}{K_1 \tau^{\frac{\alpha-1}{\alpha}} + K_2 \tau^{\frac{[4r(q+1)+2q(n-2)](1+m)-(n-2)(p-1)}{4r(q+1)+2q(n-2)-n(p-1)}} + K_3 \tau^{\frac{2r(q+1)}{2r(q+1)-2q}}} \leq t.$$

Letting $t \rightarrow t^*$, a lower bound for t^* is given by

$$t^* \geq \int_{A(0)}^{\infty} \frac{d\tau}{K_1 \tau^{\frac{\alpha-1}{\alpha}} + K_2 \tau^{\frac{[4r(q+1)+2q(n-2)](1+m)-(n-2)(p-1)}{4r(q+1)+2q(n-2)-n(p-1)}} + K_3 \tau^{\frac{2r(q+1)}{2r(q+1)-2q}}}. \quad \square$$

4 Application

In this section, an example is presented to illustrate the applications of Theorems 2.1 and 3.1.

Example 4.1. Let u be a classical solution of the following problem:

$$\begin{cases} (u + \ln(1+u))_t = \nabla \cdot \left(\frac{1}{10} (1 + |\nabla u|) \nabla u\right) + (3 + |x|^2) u^{\frac{5}{2}} \left(\int_{\Omega} u^3 dx\right)^{\frac{1}{4}} & \text{in } \Omega \times (0, t^*), \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, t^*), \\ u(x, 0) = 1 + (1 - |x|^2)^2 & \text{in } \bar{\Omega}, \end{cases}$$

where $\Omega = \{x = (x_1, x_2, x_3) \mid |x|^2 = x_1^2 + x_2^2 + x_3^2 < 1\}$ is a ball of \mathbb{R}^3 . We then have

$$b(u) = u + \ln(1 + u), \quad \rho(|\nabla u|^2) = \frac{1}{10}(1 + |\nabla u|), \quad c(x) = 3 + |x|^2, \quad (4.1)$$

$$f(u) = u^{\frac{5}{2}} \left(\int_{\Omega} u^3 dx \right)^{\frac{1}{4}}, \quad u_0(x) = 1 + (1 - |x|^2)^2. \quad (4.2)$$

It follows from (2.1)–(2.2) and (4.1)–(4.2) that

$$\begin{aligned} F(u) &= \int_0^u f(s) ds = \int_0^u s^{\frac{5}{2}} \left(\int_{\Omega} s^3 dx \right)^{\frac{1}{4}} ds, \\ G(u) &= 2 \int_0^u sb'(s) ds = 2 \int_0^u s \left(1 + \frac{1}{1+s} \right) ds = u^2 + 2u - 2 \ln(1 + u), \\ P(|\nabla u|^2) &= \int_0^{|\nabla u|^2} \rho(s) ds = \frac{1}{10} \int_0^{|\nabla u|^2} \left(1 + s^{\frac{1}{2}} \right) ds = \frac{1}{10} |\nabla u|^2 + \frac{1}{15} |\nabla u|^3, \\ D(t) &= \int_{\Omega} G(u) dx = \int_{\Omega} (u^2 + 2u - 2 \ln(1 + u)) dx, \\ E(t) &= - \int_{\Omega} P(|\nabla u|^2) dx + 2 \int_{\Omega} c(x) F(u) dx \\ &= \int_{\Omega} \left(-\frac{1}{10} |\nabla u|^2 - \frac{1}{15} |\nabla u|^3 \right) dx + 2 \int_{\Omega} (3 + |x|^2) \left[\int_0^u s^{\frac{5}{2}} \left(\int_{\Omega} s^3 dx \right)^{\frac{1}{4}} ds \right] dx. \end{aligned}$$

Selecting $\beta = \frac{1}{2}$, it is easy to check that (2.3)–(2.4) hold. Moreover, we compute

$$\begin{aligned} D(0) &= \int_{\Omega} (u_0^2 + 2u_0 - 2 \ln(1 + u_0)) dx \\ &= \int_{\Omega} \left((|x|^4 - 2|x|^2 + 2)^2 + 2(|x|^4 - 2|x|^2 + 2) - 2 \ln(|x|^4 - 2|x|^2 + 3) \right) dx \\ &= 10.1931. \end{aligned}$$

Since $1 \leq u_0 \leq 2$, we have

$$\begin{aligned} F(u_0) &= \int_0^{u_0} s^{\frac{5}{2}} \left(\int_{\Omega} s^3 dx \right)^{\frac{1}{4}} ds \geq \int_{\frac{1}{2}}^{u_0} s^{\frac{5}{2}} \left(\int_{\Omega} s^3 dx \right)^{\frac{1}{4}} ds \geq \int_{\frac{1}{2}}^{u_0} s^{\frac{5}{2}} \left(\int_{\Omega} \left(\frac{1}{2} \right)^3 dx \right)^{\frac{1}{4}} ds \\ &= \frac{2}{7} \times \left(\frac{1}{2} \right)^{\frac{3}{4}} |\Omega|^{\frac{1}{4}} u_0^{\frac{7}{2}} - \frac{2}{7} \times \left(\frac{1}{2} \right)^{\frac{17}{4}} |\Omega|^{\frac{1}{4}} = 0.2430 u_0^{\frac{7}{2}} - 0.0215. \end{aligned} \quad (4.3)$$

In view of $E(t)$ and (4.3), we have

$$\begin{aligned} E(0) &= - \int_{\Omega} P(|\nabla u_0|^2) dx + 2 \int_{\Omega} c(x) F(u_0) dx \\ &\geq \int_{\Omega} \left(-\frac{1}{10} |\nabla u_0|^2 - \frac{1}{15} |\nabla u_0|^3 \right) dx + 2 \int_{\Omega} (3 + |x|^2) \left(0.2430 u_0^{\frac{7}{2}} - 0.0215 \right) dx \\ &= \int_{\Omega} \left(-1.6 (1 - |x|^2)^2 |x|^2 - \frac{64}{15} (1 - |x|^2)^3 |x|^3 \right. \\ &\quad \left. + (6 + 2|x|^2)[0.2430(|x|^4 - 2|x|^2 + 2)^{\frac{7}{2}} - 0.0215] \right) dx \\ &= 15.3826. \end{aligned}$$

Consequently, by Theorem 2.1, we know that the solution u blows up at $t^* \leq T$ in the measure $D(t)$ and

$$T = \frac{D(0)}{2\beta(1+\beta)E(0)} \leq 0.4418. \quad (4.4)$$

Next, we apply Theorem 3.1 to obtain a lower bound for t^* . Here we have $n = 3$ and $|\Omega| = \frac{4}{3}\pi$. Choosing $a_1 = 0$, $a_2 = 4$, $b_1 = \frac{1}{10}$, $b_2 = \frac{1}{10}$, $\gamma = 1$, $m = \frac{1}{4}$, $p = \frac{5}{2}$, $q = \frac{1}{2}$, $r = \frac{4}{3}$, and $\alpha = 3$, we can check that (3.1)–(3.2) hold. The Sobolev embedding constant $C = 4^{\frac{1}{3}}3^{-\frac{1}{2}}\pi^{-\frac{2}{3}}$ is given in [11]. Inserting above constants into (3.4)–(3.8), we obtain $\sigma_1 = 0.0385$, $\sigma_2 = 0.0546$, $K_1 = 0$, $K_2 = 59.1007$, $K_3 = 9.5161$. Moreover, we compute

$$B(u) = \alpha \int_0^u b'(s)s^{\alpha-1}ds = 3 \int_0^u s^2 \left(1 + \frac{1}{1+s}\right) ds = u^3 + \frac{3}{2}u^2 + 3\ln(u+1) - 3u,$$

$$A(t) = \int_{\Omega} B(u)dx = \int_{\Omega} \left(u^3 + \frac{3}{2}u^2 + 3\ln(u+1) - 3u\right) dx,$$

and

$$\begin{aligned} A(0) &= \int_{\Omega} \left(u_0^3 + \frac{3}{2}u_0^2 + 3\ln(u_0+1) - 3u_0\right) dx \\ &= \int_{\Omega} \left(\left(|x|^4 - 2|x|^2 + 2\right)^3 + \frac{3}{2}\left(|x|^4 - 2|x|^2 + 2\right)^2\right. \\ &\quad \left.+ 3\ln\left(|x|^4 - 2|x|^2 + 2\right) - 3\left(|x|^4 - 2|x|^2 + 2\right)\right) dx \\ &= 13.1535. \end{aligned}$$

Since u blows up at t^* in measure $D(t)$, we know that u blows up at a finite time t^* . Therefore, u blows up at t^* in measure $A(t)$. By Theorem 3.1, we obtain a lower bound for the blow-up time

$$\begin{aligned} t^* &\geq \int_{A(0)}^{\infty} \frac{d\tau}{K_1\tau^{\frac{\alpha-1}{\alpha}} + K_1\tau^{\frac{[4r(q+1)+2q(n-2)](1+m)-(n-2)(p-1)}{4r(q+1)+2q(n-2)-n(p-1)}} + K_2\tau^{\frac{2r(q+1)}{2r(q+1)-2q}}} \\ &= \int_{13.1535}^{\infty} \frac{d\tau}{59.1007\tau^{\frac{13}{6}} + 9.5161\tau^{\frac{4}{3}}} = 7.0988 \times 10^{-4}. \end{aligned} \quad (4.5)$$

It follows from (4.4)–(4.5) that

$$7.0988 \times 10^{-4} \leq t^* \leq 0.4418.$$

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