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# ON INTEGRABLE FIELD THEORIES AS DIHEDRAL AFFINE GAUDIN MODELS

BENOÎT VICEDO

ABSTRACT. We introduce the notion of a *classical dihedral affine Gaudin model*, associated with an untwisted affine Kac-Moody algebra  $\tilde{\mathfrak{g}}$  equipped with an action of the dihedral group  $D_{2T}$ ,  $T \geq 1$  through (anti-)linear automorphisms. We show that a very broad family of classical integrable field theories can be recast as examples of such classical dihedral affine Gaudin models. Among these are the principal chiral model on an arbitrary real Lie group  $G_0$  and the  $\mathbb{Z}_T$ -graded coset  $\sigma$ -model on any coset of  $G_0$  defined in terms of an order  $T$  automorphism of its complexification. Most of the multi-parameter integrable deformations of these  $\sigma$ -models recently constructed in the literature provide further examples. The common feature shared by all these integrable field theories, which makes it possible to reformulate them as classical dihedral affine Gaudin models, is the fact that they are non-ultralocal. In particular, we also obtain affine Toda field theory in its lesser-known non-ultralocal formulation as another example of this construction.

We propose that the interpretation of a given classical non-ultralocal integrable field theory as a classical dihedral affine Gaudin model provides a natural setting within which to address its quantisation. At the same time, it may also furnish a general framework for understanding the massive ODE/IM correspondence since the known examples of integrable field theories for which such a correspondence has been formulated can all be viewed as dihedral affine Gaudin models.

## 1. MOTIVATION AND INTRODUCTION

The ODE/IM correspondence describes a striking and rather unexpected relation between the theory of linear Ordinary Differential Equations in the complex plane on the one hand, and that of quantum Integrable Models on the other. Concretely, the first instance of such a correspondence was formulated by V. Bazhanov, S. Lukyanov and A. Zamolodchikov for quantum KdV theory in the series of seminal papers [8] – [12], building on from the original insight of P. Dorey and R. Tateo in [25]. These works culminated in the remarkable conjecture of [12] stating that the joint spectrum of the quantum KdV Hamiltonians on the level  $L \in \mathbb{Z}_{\geq 0}$  subspace of an irreducible module over the Virasoro algebra is in bijection with the set of certain one-dimensional Schrödinger operators  $-\partial_z^2 + V_L(z)$  with ‘monster’ potentials  $V_L(z)$  of a given form. The justification for this conjecture comes from the central observation that the functional relations and analytic properties characterising the eigenvalues of the  $Q$ -operators of quantum KdV theory [9, 10] on a given joint eigenvector coincide with those satisfied by certain connection coefficients of the associated one-dimensional Schrödinger equation. These ideas were soon extended to other massless

integrable field theories associated with higher rank Lie algebras of classical type, see *e.g.* [26, 22, 5, 21] and the review [23].

Despite the variety of examples of the ODE/IM correspondence, its mathematical underpinning remained elusive for a number of years. This problem was addressed by B. Feigin and E. Frenkel in [37] where they argued that the ODE/IM correspondence for quantum  $\widehat{\mathfrak{g}}$ -KdV theory could be understood as originating from an affine analogue of the geometric Langlands correspondence. To explain this connection we make a brief digression on Gaudin models, which provide a realisation of the global geometric Langlands correspondence for rational curves over the complex numbers.

Let  $\mathfrak{g}$  be a finite-dimensional complex semisimple Lie algebra. The Gaudin model, or  $\mathfrak{g}$ -Gaudin model, is a quantum integrable spin-chain with long-range interactions [47]. If we let  $N \in \mathbb{Z}_{\geq 1}$  denote the number of sites then the algebra of observables of the model is the  $N$ -fold tensor product  $U(\mathfrak{g})^{\otimes N}$  of the universal enveloping algebra  $U(\mathfrak{g})$  of  $\mathfrak{g}$ . The quadratic Gaudin Hamiltonians are elements of  $U(\mathfrak{g})^{\otimes N}$  given by

$$H_i := \sum_{\substack{j=1 \\ j \neq i}}^N \frac{I_a^{(i)} I_a^{(j)}}{z_i - z_j} \quad (1)$$

where the  $z_i$ ,  $i = 1, \dots, N$  are arbitrary distinct complex numbers,  $\{I^a\}$  and  $\{I_a\}$  are dual bases of  $\mathfrak{g}$  with respect to a fixed non-degenerate invariant bilinear form  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{g}$ , and  $x^{(i)}$  is the element of  $U(\mathfrak{g})^{\otimes N}$  with  $x \in \mathfrak{g}$  in the  $i^{\text{th}}$  tensor factor and 1's in every other factor. The quantum integrability of the model is characterised by the existence of a large commutative subalgebra  $\mathcal{L}_z(\mathfrak{g}) \subset U(\mathfrak{g})^{\otimes N}$  with  $z := \{z_i\}_{i=1}^N \cup \{\infty\}$ , known as the *Gaudin algebra*, containing in particular the quadratic Gaudin Hamiltonians.

Let  $M_i$ ,  $i = 1, \dots, N$  be  $\mathfrak{g}$ -modules. One is interested in finding the joint spectrum of  $\mathcal{L}_z(\mathfrak{g})$  on the spin-chain  $\bigotimes_{i=1}^N M_i$ . Note that a joint eigenvalue of the Gaudin algebra defines a homomorphism  $\mathcal{L}_z(\mathfrak{g}) \rightarrow \mathbb{C}$  sending each element of  $\mathcal{L}_z(\mathfrak{g})$  to its eigenvalue. The joint spectrum can therefore be described as a subset of the maximal spectrum of the commutative algebra  $\mathcal{L}_z(\mathfrak{g})$ , *i.e.* the set of all homomorphisms  $\mathcal{L}_z(\mathfrak{g}) \rightarrow \mathbb{C}$ . It was shown by E. Frenkel in [43, Theorem 2.7(1)] that the maximal spectrum of the Gaudin algebra  $\mathcal{L}_z(\mathfrak{g})$  is isomorphic to a certain subquotient of the space of  ${}^L\mathfrak{g}$ -valued connections on  $\mathbb{P}^1$ , known as  *${}^L\mathfrak{g}$ -opers*, with regular singularities in the set  $z$ , where  ${}^L\mathfrak{g}$  denotes the Langlands dual of the Lie algebra  $\mathfrak{g}$ . In other words, each joint eigenvalue of the Gaudin algebra  $\mathcal{L}_z(\mathfrak{g})$  on the given spin-chain  $\bigotimes_{i=1}^N M_i$  will be described by such an  ${}^L\mathfrak{g}$ -oper. In fact, when all the  $\mathfrak{g}$ -modules  $M_i$  are finite-dimensional irreducibles  $V_{\lambda_i}$  of highest weights  $\lambda_i \in \mathfrak{h}^*$ , [43, Conjecture 1] states that for each integral dominant weight  $\lambda_\infty \in \mathfrak{h}^*$ , the joint spectrum of  $\mathcal{L}_z(\mathfrak{g})$  on the subspace of weight  $\lambda_\infty$  singular vectors in  $\bigotimes_{i=1}^N V_{\lambda_i}$  is in bijection with the subspace of such  ${}^L\mathfrak{g}$ -opers with residue at the points  $z_i$  and infinity given by the shifted Weyl orbits of the weights  $\lambda_i$  and  $\lambda_\infty$  respectively, and with trivial monodromy representation.

The description of the maximal spectrum of the Gaudin algebra  $\mathcal{L}_z(\mathfrak{g})$  in terms of  ${}^L\mathfrak{g}$ -opers also generalises to the case of Gaudin models with irregular singularities; see [40, 39]. Another possible generalisation of Gaudin models is given by *cyclotomic* Gaudin models, introduced in [86, 87] and more recently [88] for the case with irregular

singularities. A similar description of the corresponding cyclotomic Gaudin algebra of [86] was recently conjectured in [62] in terms of *cyclotomic*  $L\mathfrak{g}$ -opers, i.e.  $L\mathfrak{g}$ -opers equivariant under an action of the cyclic group. In fact, these descriptions of the various Gaudin algebras in terms of global  $L\mathfrak{g}$ -opers on  $\mathbb{P}^1$  follow (conjecturally in the cyclotomic case) from the ‘local’ version proved by B. Feigin and E. Frenkel in their seminal paper [36] (see also [44, 45]) which states that the space of singular vectors in the vacuum Verma module  $\mathbb{V}_0^{\text{crit}}(\mathfrak{g})$  at the critical level over the untwisted affine Kac-Moody algebra  $\widehat{\mathfrak{g}}$ , which naturally forms a commutative algebra, is isomorphic to the algebra of functions on the space of  $L\mathfrak{g}$ -opers on the formal disc.

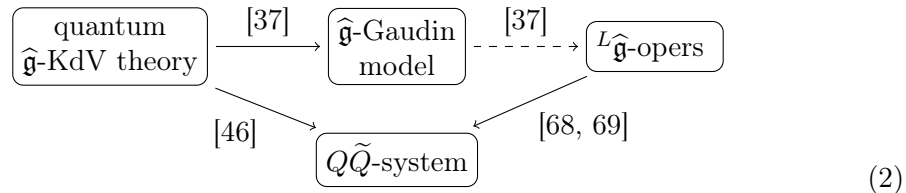
The apparent similarity between the description of the joint spectrum of the Gaudin algebra on any given spin-chain in terms of certain  $L\mathfrak{g}$ -opers and the statement of the ODE/IM correspondence for quantum KdV theory is more than just a coincidence. Indeed, as argued in [37], quantum  $\widehat{\mathfrak{g}}$ -KdV theory can be regarded as a generalised Gaudin model associated with the untwisted affine Kac-Moody algebra  $\widehat{\mathfrak{g}}$ , or  $\widehat{\mathfrak{g}}$ -Gaudin model for short, with a regular singularity at the origin and an irregular singularity of the mildest possible form at infinity. Unfortunately, much less is known at present about Gaudin models associated with general Kac-Moody algebras; see however [72, 42]. In particular, there is currently no known analogue of the Feigin-Frenkel isomorphism for describing the space of singular vectors in the suitably completed vacuum Verma module over the double affine, or toroidal, Lie algebra  $\widehat{\widehat{\mathfrak{g}}}$ . It is not even clear what the critical level should be in this setting. Nevertheless, the notion of an *affine oper*, or  $\widehat{\mathfrak{g}}$ -oper, on  $\mathbb{P}^1$  can certainly be defined [42] and so it is tempting to speculate that the description of the spectrum of the  $\mathfrak{g}$ -Gaudin Hamiltonians in terms of  $L\mathfrak{g}$ -opers persists when  $\mathfrak{g}$  is replaced by an affine Kac-Moody algebra.

In this spirit, the explicit form of the  $L\widehat{\mathfrak{g}}$ -opers which ought to describe the joint spectrum of the quantum  $\widehat{\mathfrak{g}}$ -KdV Hamiltonians on certain irreducible modules over the  $W$ -algebra associated with  $\mathfrak{g}$  was conjectured in [37], by using as a finite-dimensional analogy a certain description of the finite  $W$ -algebra for a regular nilpotent element in terms of  $L\mathfrak{g}$ -opers. Remarkably, when  $\widehat{\mathfrak{g}} = \widehat{\mathfrak{sl}}_2$  so that also  $L\widehat{\mathfrak{g}} = \widehat{\mathfrak{sl}}_2$ , these  $\widehat{\mathfrak{sl}}_2$ -opers were shown to coincide exactly, after a simple change of coordinate on  $\mathbb{P}^1$ , with the one-dimensional Schrödinger operators written down in [12]. This result not only confirms the idea that the ODE/IM correspondence can be thought of as a particular instance of the geometric Langlands correspondence but also provides strong evidence in support of the general claim that the joint spectrum of the higher Hamiltonians of an affine Gaudin model can be described in terms of affine opers for the Langlands dual affine Kac-Moody algebra.

Another approach to testing the proposed link between the joint spectrum of the quantum  $\widehat{\mathfrak{g}}$ -KdV Hamiltonians and  $L\widehat{\mathfrak{g}}$ -opers of the prescribed form is to follow the same strategy originally used to establish the ODE/IM correspondence for quantum KdV theory. Specifically, one should compare the functional relations and analytic properties of the joint eigenvalues of the  $Q$ -operators of quantum  $\widehat{\mathfrak{g}}$ -KdV theory on joint eigenvectors in the irreducibles over the  $W$ -algebra associated with  $\mathfrak{g}$ , with those satisfied by the connection coefficients of the associated  $L\widehat{\mathfrak{g}}$ -opers. This programme was initiated in [81] and was further developed very recently in [68, 69] where some

remarkable functional relations, referred to as the  $Q\tilde{Q}$ -system, were obtained for certain generalised spectral determinants of the ODE associated with the  $L\hat{\mathfrak{g}}$ -opers of [37] corresponding to highest weight states in representations of the  $W$ -algebra. Even more recently in [46], the very same  $Q\tilde{Q}$ -system was shown to arise as relations in the Grothendieck ring  $K_0(\mathcal{O})$  of the category  $\mathcal{O}$  of representations of the Borel subalgebra of the quantum affine algebra  $U_q(\hat{\mathfrak{g}})$  for an untwisted affine Kac-Moody algebra  $\hat{\mathfrak{g}}$ . Analogous relations were also conjectured to hold when  $\hat{\mathfrak{g}}$  is a twisted affine Kac-Moody algebra. Since non-local quantum  $\hat{\mathfrak{g}}$ -KdV Hamiltonians can be associated with elements of  $K_0(\mathcal{O})$  by the construction of [9, 10, 5], the joint spectrum of these Hamiltonians also satisfy the  $Q\tilde{Q}$ -system, thereby providing further evidence in favour of the ODE/IM correspondence for quantum  $\hat{\mathfrak{g}}$ -KdV theory.

The recent developments towards formulating and ultimately proving the ODE/IM correspondence for quantum  $\hat{\mathfrak{g}}$ -KdV theory, which we briefly recalled above, can be summarised in the following commutative diagram



The top line of this diagram, corresponding to the work [37], consisted of two steps. The first was to reinterpret quantum  $\hat{\mathfrak{g}}$ -KdV theory as a particular affine  $\hat{\mathfrak{g}}$ -Gaudin model. The second, which we represent by a dashed arrow to emphasise its conjectural status, was to make use of the existing description of the spectrum of  $\mathfrak{g}$ -Gaudin models in terms of  $L\mathfrak{g}$ -opers as an analogy. The big open problem here is to establish the affine counterpart of the latter statement to put the second step on a firm mathematical footing. Indeed, this would promote the top line in the above diagram to a proof of the ODE/IM correspondence for quantum  $\hat{\mathfrak{g}}$ -KdV theory. While the top line is still partly conjectural, the bottom part of the diagram provides a solid link between both sides of the ‘KdV-oper’ correspondence of [37] through the common  $Q\tilde{Q}$ -system.

Until relatively recently, the study of the ODE/IM correspondence had been limited to describing integrable structures in conformal field theories only. This left open the important question of whether similar ideas could be used to describe the spectrum of *massive* quantum integrable field theories as well. The first example of such a *massive* ODE/IM correspondence was put forward by S. Lukyanov and A. Zamolodchikov for quantum sine-Gordon and sinh-Gordon theories in their pioneering paper [64]. Specifically, they showed that the functional relations and analytic properties characterising the vacuum eigenvalues of the  $Q$ -operators of quantum sine/sinh-Gordon theory were the same as those satisfied by certain connection coefficients of the auxiliary linear problem of the classical modified sinh-Gordon equation for a suitably chosen classical solution. Subsequently, various higher rank generalisations of this massive ODE/IM correspondence for quantum affine  $\tilde{\mathfrak{g}}$ -Toda field theories were also conjectured, when  $\tilde{\mathfrak{g}}$  is of type  $A$  for rank 3 in [24] and for general rank  $n$  in [1], and more recently for

a general untwisted affine Kac-Moody algebra  $\tilde{\mathfrak{g}}$  in [52, 53] as well as examples of twisted type in [54]. Another important quantum integrable field theory for which a massive ODE/IM correspondence has been conjectured in [63], and further studied in [7, 6], is the Fateev model [35]. It can be viewed as a two-parameter deformation of the  $SU_2$  principal chiral model and as such it is equivalent [49] to the so called  $SU_2$  bi-Yang-Baxter  $\sigma$ -model [58]. Here as well the correspondence is a conjectured link between the spectrum of the Fateev model on the one hand, and solutions of the classical modified sinh-Gordon equation on the other.

A noteworthy feature of the massive ODE/IM correspondence for quantum affine  $\tilde{\mathfrak{g}}$ -Toda field theory in the non-simply-laced case is the appearance of the Langlands dual  ${}^L\tilde{\mathfrak{g}}$  of the affine Kac-Moody algebra  $\tilde{\mathfrak{g}}$  on the ODE side. This strongly suggests that the geometric Langlands correspondence may also underly the *massive* ODE/IM correspondence. One of the aims of the present paper is to make the first step towards generalising the above picture in (2) for quantum  $\hat{\mathfrak{g}}$ -KdV theory to massive quantum integrable field theories. In fact, as in the case of  $\hat{\mathfrak{g}}$ -(m)KdV theory, one typically starts from a description of the *classical* integrable field theory. Therefore, in a first instance, one is faced with the initial problem of how to quantise the given classical integrable field theory. We will argue that both problems are in fact closely related.

The most effective approach for quantising a given classical integrable field theory and establishing its quantum integrability is the quantum inverse scattering method [33, 60], whose mathematical underpinning gave rise to the theory of quantum affine algebras. In particular, it can be used to obtain functional equations such as Baxter's  $TQ$ -relation and the  $Q\tilde{Q}$ -system, all of which follow from corresponding relations in the Grothendieck ring of category  $\mathcal{O}$ . Unfortunately, the quantum inverse scattering method is well known to apply only under the restrictive assumption that the classical integrable field theory one starts with is *ultralocal*. Since the main focus of the present paper is to address the problem of quantising classical integrable field theories which violate this condition, we begin by briefly recalling why this condition is necessary in the standard quantum inverse scattering method.

The starting point of the classical inverse scattering method, as crystalised by A. Reiman and M. Semenov-Tian-Shansky in [73, 76], is to identify the phase space of the given classical integrable field theory with a coadjoint orbit in the smooth dual  $\hat{\mathfrak{G}}_1^*$  of a hyperplane  $\hat{\mathfrak{G}}_1$  in a certain central extension of the double loop algebra  $\mathfrak{G}$ . The latter consists of smooth maps from the circle  $S^1$  to the loop algebra  $\mathfrak{g}((z))$ , or possibly its twist by some finite-order automorphism of  $\mathfrak{g}$ . The smooth dual is defined relative to a certain bilinear form on  $\mathfrak{G}$  given in terms of a model dependent rational function  $\varphi(z)$ , called the *twist function*, as

$$((\mathcal{X}, \mathcal{Y}))_\varphi := \int_{S^1} d\theta \operatorname{res}_z \langle \mathcal{X}(\theta, z), \mathcal{Y}(\theta, z) \rangle \varphi(z) dz \quad (3)$$

for  $\mathcal{X}, \mathcal{Y} \in \mathfrak{G}$ . The Poisson bracket on  $\hat{\mathfrak{G}}_1^*$  is the Kostant-Kirillov  $\mathcal{R}$ -bracket associated with some solution  $\mathcal{R} \in \operatorname{End} \mathfrak{G}$  of the modified classical Yang-Baxter equation. Given any pair of differentiable functionals  $f$  and  $g$  on  $\hat{\mathfrak{G}}_1^*$ , their Poisson bracket at a generic

point  $(\mathcal{L}, 1) \in \hat{\mathfrak{G}}_1^*$ , where  $\mathcal{L} \in \mathfrak{G}$ , may be written as

$$\{f, g\}((\mathcal{L}, 1)) = \left( (df_{\mathcal{L}}, (\text{ad } \mathcal{L} \circ \mathcal{R} + \mathcal{R}^* \circ \text{ad } \mathcal{L} - (\mathcal{R} + \mathcal{R}^*)\partial_{\theta}) \cdot dg_{\mathcal{L}}) \right)_{\varphi} \quad (4)$$

where  $df_{\mathcal{L}} \in \mathfrak{G}$  denotes the Fréchet derivative of  $f$  at  $(\mathcal{L}, 1)$  and  $\mathcal{R}^*$  is the adjoint operator of  $\mathcal{R}$  with respect to (3). In this language the theory is said to be *ultralocal* if  $\mathcal{R}^* = -\mathcal{R}$ , otherwise it is *non-ultralocal*.

It is well known that the integrals of motion of the classical integrable field theory can be obtained from spectral invariants of the monodromy  $M_{\mathcal{L}}$  of the differential operator  $\partial_{\theta} + \mathcal{L}$ , which is valued in the loop group  $G((z))$ . Given any smooth functional  $\phi$  on  $G((z))$  which is central, the Fréchet derivative of the functional  $\phi^M : \mathcal{L} \mapsto \phi(M_{\mathcal{L}})$  on  $\hat{\mathfrak{G}}_1^*$  defines an element of  $\mathfrak{G}$ , and hence its Poisson bracket (4) with any other smooth functional  $f$  on  $\hat{\mathfrak{G}}_1^*$  is well defined. In particular, the involution property of the integrals of motion is established by showing that for any pair of central functionals  $\phi$  and  $\psi$  on  $G((z))$  we have  $\{\phi^M, \psi^M\} = 0$  [77]. If a functional  $\phi$  on  $G((z))$  is not central, however, then the Fréchet derivative  $d\phi^M$  will in general exhibit a jump discontinuity at the base point of  $M_{\mathcal{L}}$ . In the case of an ultralocal theory where the  $\partial_{\theta}$ -term in (4) is absent, the bracket naturally extends to such functionals  $f$  and  $g$  with discontinuous Fréchet derivatives. One can then evaluate  $\{\phi^M, \psi^M\}$  for arbitrary smooth functionals  $\phi$  and  $\psi$  [77], yielding the celebrated Sklyanin bracket on  $G((z))$ . The quantisation of the latter then serves as a starting point for the quantum inverse scattering method. By contrast, in the non-ultralocal case the bracket  $\{\phi^M, \psi^M\}$  is clearly ill-defined for arbitrary smooth functionals  $\phi$  and  $\psi$ . This issue has precluded the direct application of the quantum inverse scattering method to a wide range of important integrable field theories due to their non-ultralocal nature.

Although generalisations of the quantum inverse scattering method capable of also accommodating non-ultralocal systems do exist, see for instance [41, 48, 78], these remain applicable only to a very restricted class of non-ultralocal systems. Faced with this limitation, the common strategy for quantising a given non-ultralocal system is to attempt to ‘ultralocalise’ it by different means. These include modifying the classical field theory itself by altering its twist function, see *e.g.* [32] (and also [78, 15]), finding a suitable gauge transformation which will bring it to an ultralocal form, see *e.g.* [8, 74], or possibly by finding a dual description of the theory which would be ultralocal. Yet such attempts at ‘curing’ a classical integrable field theory of its non-ultralocality ultimately work only in a limited number of cases. Let us mention also some alternative approaches to dealing with the problem of non-ultralocality which have been put forward recently in [70] for the Alday-Arutyunov-Frolov model and very recently in [75] for the  $\lambda$ -deformation of the  $AdS_5 \times S^5$  superstring.

A possible way around the problem of dealing with the monodromy in a non-ultralocal field theory, at least for describing local integrals of motion, was first proposed in [27] on the example of the principal chiral model associated with any Lie algebra  $\mathfrak{g}$  of classical type. These ideas were subsequently developed for other models in [28, 29, 30, 31], and more recently in [61]. Specifically, it was shown in [27] that local integrals of motion in involution could be constructed directly from the Lax matrix, without needing to use the monodromy matrix. A striking property of this family

of local charges is that their degrees precisely match the pattern of exponents of the untwisted affine Kac-Moody algebra  $\tilde{\mathfrak{g}}$  associated with  $\mathfrak{g}$ . This is very reminiscent of what happens in the  $\mathfrak{g}$ -Gaudin model, where the degrees of the homogeneous integrals of motion also correspond to the pattern of exponents of  $\mathfrak{g}$ . Indeed, this parallel also served as a source of inspiration for the present work.

It is interesting to note that classical  $\widehat{\mathfrak{g}}\text{-}(m)\text{KdV}$  theory is one of those distinguished classical integrable field theories which admits both an ultralocal and a non-ultralocal formulation, related to one another through a gauge transformation. Its ultralocal description serves as the starting point in the approach of [8, 9, 10, 5] for quantising the theory using the quantum inverse scattering method. As recalled above, within this approach non-local quantum  $\widehat{\mathfrak{g}}\text{-}(m)\text{KdV}$  Hamiltonians can be associated with elements of  $K_0(\mathcal{O})$  and the  $Q\tilde{Q}$ -system is obtained from corresponding relations in  $K_0(\mathcal{O})$  established in [46]. In other words, the bottom left arrow of the diagram in (2) starts from the quantisation of  $\widehat{\mathfrak{g}}\text{-}(m)\text{KdV}$  theory in its ultralocal formulation. By contrast, treating classical  $\widehat{\mathfrak{g}}\text{-KdV}$  as a non-ultralocal theory enables one to regard it as a classical affine Gaudin model<sup>1</sup>. The proposal of [37] to then quantise  $\widehat{\mathfrak{g}}\text{-KdV}$  theory by viewing it as a classical affine Gaudin model led to the conjectural description of its quantum spectrum in terms of affine  ${}^L\widehat{\mathfrak{g}}$ -opers, corresponding to the top line of the diagram in (2). Since the ultralocal and non-ultralocal formulations of classical  $\widehat{\mathfrak{g}}\text{-KdV}$  theory are related by a gauge transformation we expect that their respective quantisations should agree. In this setting, the fact that the work [68, 69] makes the diagram in (2) commutative can be seen as evidence of this.

The goal of the present paper is to initiate a program for quantising non-ultralocal classical integrable field theories and propose a framework within which to understand the massive ODE/IM correspondence for such models. Specifically, we introduce the notion of a classical *dihedral* (or *real cyclotomic*) affine  $\tilde{\mathfrak{g}}$ -Gaudin model associated with an arbitrary untwisted affine Kac-Moody algebra  $\tilde{\mathfrak{g}}$ . We then show that classical dihedral  $\tilde{\mathfrak{g}}$ -Gaudin models describe a general class of classical non-ultralocal integrable field theories, namely those whose Poisson bracket is as in (4) with  $\mathcal{R}$ -matrix given by the standard solution of the classical Yang-Baxter equation on the (twisted) double loop algebra  $\mathfrak{G}$ . We illustrate this relation between classical dihedral  $\tilde{\mathfrak{g}}$ -Gaudin models and non-ultralocal classical integrable field theories on a wide variety of examples, listed in Table 1, including the principal chiral model on a real semisimple Lie group  $G_0$  and the  $\mathbb{Z}_T$ -graded coset  $\sigma$ -models for any  $T \in \mathbb{Z}_{\geq 2}$  as well as some of their various multi-parameter deformations introduced in recent years [57, 58, 17, 80, 50, 20]. Replacing the semisimple Lie algebra  $\mathfrak{g}$  by the Grassmann envelope of a Lie superalgebra, the present formalism also describes  $\mathbb{Z}_T$ -graded supercoset  $\sigma$ -models [89, 67, 83, 56] and various deformations recently constructed [18, 19, 51]. It is interesting to note, in particular, that the examples of integrable field theories for which a massive ODE/IM correspondence has been formulated can all be recast as classical dihedral affine Gaudin models. Our proposal is therefore that the problem of quantising non-ultralocal integrable field theories and that of formulating an ODE/IM correspondence

<sup>1</sup>Let us note here that it will also follow from §5.3, where we discuss affine  $\tilde{\mathfrak{g}}$ -Toda field theory, that  $\widehat{\mathfrak{g}}\text{-mKdV}$  theory can be regarded as a classical *cyclotomic* affine Gaudin model.



for such models can both be addressed within the context of quantisation of dihedral (affine) Gaudin models.

Non-ultralocal field theory	Divisor $\mathcal{D}$	$\sigma \in \text{Aut } \tilde{\mathfrak{g}}$
Principal chiral model (PCM)	$2 \cdot 0 + 2 \cdot \infty$	id
PCM with WZ-term	$2 \cdot k + 2 \cdot \infty, k \in \mathbb{R}^\times$	
Yang-Baxter (YB) $\sigma$ -model	$(i\eta) + 2 \cdot \infty, \eta \in \mathbb{R}_{>0}$	
YB $\sigma$ -model with WZ-term	$(k + iA) + 2 \cdot \infty, k \in \mathbb{R}^\times, A \in \mathbb{R}_{>0}$	
bi-Yang-Baxter $\sigma$ -model	$e^{i\vartheta} + e^{i(\psi+\pi)} + \infty, \vartheta, \psi \in ]0, \frac{\pi}{2}[$	
$\mathbb{Z}_T$ -graded coset $\sigma$ -model	$2 \cdot 1 + \infty$	$ \sigma  = T,$ $T \in \mathbb{Z}_{\geq 2}$
$q$ -deformation ( $q \in \mathbb{R}$ )	$e^{i\vartheta} + \infty, \vartheta \in ]0, \frac{\pi}{T}[$	
$q$ -deformation ( $ q  = 1$ )	$p + p^{-1} + \infty, p \in ]0, 1[$	
Affine Toda field theory	$2 \cdot 0 + 2 \cdot \infty$	$\sigma$ Coxeter

TABLE 1. Examples of dihedral affine Gaudin models associated with an untwisted affine Kac-Moody algebra  $\tilde{\mathfrak{g}}$ .

To end this introduction we motivate the definition of classical dihedral  $\tilde{\mathfrak{g}}$ -Gaudin models by considering the simpler case where  $\tilde{\mathfrak{g}}$  is replaced by a finite-dimensional complex semisimple Lie algebra  $\mathfrak{g}$ . The datum for a (classical)  $\mathfrak{g}$ -Gaudin model with irregular singularities can be described by a divisor  $\mathcal{D}$  on  $\mathbb{P}^1$ , *i.e.* a formal sum of a finite subset of points  $\mathbf{z} = \{z_i\}_{i=1}^N \cup \{\infty\}$  on  $\mathbb{P}^1$  weighted by positive integers  $n_x \in \mathbb{Z}_{\geq 1}$  for each  $x \in \mathbf{z}$ . We further restrict attention in this introduction to the case where  $n_x = 1$  for all  $x \in \mathbf{z}$  for simplicity. The algebra of observables of the classical  $\mathfrak{g}$ -Gaudin model is then given by the  $N$ -fold tensor product  $S(\mathfrak{g})^{\otimes N}$  of the symmetric algebra  $S(\mathfrak{g})$  on  $\mathfrak{g}$ . The classical quadratic Hamiltonians  $H_i^{cl}$ ,  $i = 1, \dots, N$  of the model are given by the same expressions as the quantum Hamiltonians  $H_i$  in (1) but regarded as elements of  $S(\mathfrak{g})^{\otimes N}$ . They can be obtained from the *Lax matrix*  $L(z)$ , defined by the expression

$$L(z)dz = \sum_{j=1}^N \frac{I_a dz}{z - z_j} \otimes I^{a(j)},$$

as the spectral invariants  $H_i^{cl} = \text{res}_{z_i} \langle L(z), L(z) \rangle dz$  where the inner product is taken over the first tensor factor, *i.e.* the auxiliary factor.

Now let  $\sigma \in \text{Aut } \mathfrak{g}$  be an automorphism of  $\mathfrak{g}$  whose order divides  $T \in \mathbb{Z}_{\geq 1}$  and pick a primitive  $T^{\text{th}}$ -root of unity  $\omega^{-1} \in \mathbb{C}^\times$ . These both induce actions of the cyclic group  $\Gamma := \mathbb{Z}_T$  on  $\mathfrak{g}$  and  $\mathbb{P}^1$ , respectively. The quadratic Gaudin Hamiltonians of a classical *cyclotomic*  $\mathfrak{g}$ -Gaudin model are similarly obtained from the same spectral invariants but using the Lax matrix defined by

$$L(z)dz = \frac{1}{T} \sum_{j=1}^N \sum_{\alpha \in \Gamma} \hat{\alpha} \left( \frac{I_a dz}{z - z_j} \right) \otimes I^{a(j)},$$

where  $\hat{\alpha}$  denotes the action of  $\alpha \in \Gamma$  on  $\mathfrak{g}$ -valued meromorphic differentials defined by combining the action on  $\mathfrak{g}$  with the pullback on differentials over  $\mathbb{P}^1$ . This Lax matrix has the  $\Gamma$ -equivariance property  $\sigma L(z) = \omega L(\omega z)$ , where  $\sigma$  acts on the auxiliary factor.

If we are also given an anti-linear automorphism  $\tau \in \overline{\text{Aut}} \mathfrak{g}$  of  $\mathfrak{g}$  which preserves the eigenspaces of  $\sigma$  then we obtain an action of the dihedral group  $\Pi := D_{2T}$  of order  $2T$  on  $\mathfrak{g}$ . Promoting also the action of  $\Gamma$  on  $\mathbb{P}^1$  to an action of  $\Pi$  by adding complex conjugation  $z \mapsto \bar{z}$ , we can define the Lax matrix of the classical *dihedral*  $\mathfrak{g}$ -Gaudin model by an expression similar to the above but replacing the sum over  $\Gamma$  by a sum over  $\Pi$ . Specifically, we should now take the tensor product over  $\mathbb{R}$  rather than  $\mathbb{C}$  and use dual basis elements of the realification of  $\mathfrak{g}$  so that we set

$$L(z)dz = \frac{1}{2T} \sum_{j=1}^N \sum_{\alpha \in \Pi} \left( \hat{\alpha} \left( \frac{I_a dz}{z - z_j} \right) \otimes I^{a(j)} + \hat{\alpha} \left( \frac{-iI_a dz}{z - z_j} \right) \otimes iI^{a(j)} \right). \quad (5)$$

By construction, this Lax matrix is  $\Pi$ -equivariant in the sense that  $\sigma L(z) = \omega L(\omega z)$  and  $\tau L(z) = L(\bar{z})$  where  $\sigma$  and  $\tau$  both act on the auxiliary factor.

In order to describe integrable field theories on the circle one should replace  $\mathfrak{g}$  in the above discussion by an untwisted affine Kac-Moody algebra  $\tilde{\mathfrak{g}}$ . Concretely this means replacing the dual bases  $\{I^a\}$  and  $\{I_a\}$  of  $\mathfrak{g}$  in the expression for the Lax matrix (5) by dual bases  $\{I^{\tilde{a}}\}$  and  $\{I_{\tilde{a}}\}$  of  $\tilde{\mathfrak{g}}$  and working in a suitable completion of the tensor product. We will demonstrate that in this affine setting the Lax matrix (5), or its generalisation to other divisors  $\mathcal{D}$ , reproduces the Lax matrices and twist functions of all the integrable field theories in Table 1.

Let us finally note that reductions of general 2-dimensional integrable field theories of Zakharov-Shabat-Mikhailov type by discrete symmetry groups, such as cyclic or dihedral groups, were introduced in the important paper [71]. The purpose of the present work is to show that many integrable field theories of interest fall, in fact, within a very restricted subclass of reduced Zakharov-Shabat-Mikhailov type field theories, namely that of affine Gaudin models with dihedral symmetry. Indeed, given an integrable field theory which can be formulated within the Zakharov-Shabat-Mikhailov scheme (often in different gauge-inequivalent ways), it is by no means guaranteed that it can also be regarded as an affine Gaudin model. In particular, an affine Gaudin formulation is typically not preserved by formal gauge transformations of the zero curvature representation. In §5 we give explicit examples of well known zero curvature representations of integrable field theories which are not of Gaudin type. Yet, as outlined above, having an affine Gaudin formulation of a given classical non-ultralocal integrable field theory puts the problem of its quantisation on a firm mathematical footing and should provide a framework for establishing an ODE/IM correspondence for it. In the present context, the reduction by a dihedral group is expected to play an important role in extensions of this correspondence to the massive case.

The plan of the article is as follows. We begin in §2 by recalling some basic results about (anti-)linear automorphisms on finite-dimensional Lie algebras and affine Kac-Moody algebras. In §3 we construct a direct sum of Takiff algebras for  $\tilde{\mathfrak{g}}$  attached to the finite subset  $\mathbf{z} \subset \mathbb{P}^1$  as a quotient of a direct sum of loop algebras of  $\tilde{\mathfrak{g}}$ , and describe its dual space in terms of certain  $\tilde{\mathfrak{g}}$ -valued meromorphic differentials on  $\mathbb{P}^1$ . The main section is §4 where we define classical dihedral  $\tilde{\mathfrak{g}}$ -Gaudin models and establish their relation to a general family of non-ultralocal integrable field theories. The Lax matrix is defined as the canonical element of the dual pair constructed in §3. Finally, §5 is

devoted to a detailed construction of important non-ultralocal integrable field theories as dihedral  $\tilde{\mathfrak{g}}$ -Gaudin models. We collect in an appendix some facts about dual pairs and our conventions on tensor index notation.

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## 2. REAL AFFINE KAC-MOODY ALGEBRAS

Let  $T \in \mathbb{Z}_{\geq 1}$ . We denote the dihedral group of order  $2T$  by  $\Pi := D_{2T} = \langle \mathfrak{s}, \mathfrak{t} \mid \mathfrak{s}^T = \mathfrak{t}^2 = (\mathfrak{s}\mathfrak{t})^2 = 1 \rangle$ , and we let  $\Gamma := \langle \mathfrak{s} \rangle \subset \Pi$  be the cyclic subgroup of order  $T$ , which is normal in  $\Pi$ . Given a complex Lie algebra  $\mathfrak{a}$ , we let  $\overline{\text{Aut}} \mathfrak{a}$  denote the group of all linear and anti-linear automorphisms of  $\mathfrak{a}$ . The subgroup  $\text{Aut} \mathfrak{a}$  of linear automorphisms is normal of index 2. We denote by  $\overline{\text{Aut}}_- \mathfrak{a}$  the subset in  $\overline{\text{Aut}} \mathfrak{a}$  of all anti-linear automorphisms of  $\mathfrak{a}$  so that  $\overline{\text{Aut}} \mathfrak{a} = \text{Aut} \mathfrak{a} \sqcup \overline{\text{Aut}}_- \mathfrak{a}$ . For any  $\chi \in \overline{\text{Aut}}_- \mathfrak{a}$  we can identify  $\overline{\text{Aut}}_- \mathfrak{a}$  with the coset  $\chi \text{Aut} \mathfrak{a}$ .

**2.1. Finite-dimensional Lie algebras.** Let  $\mathfrak{g}$  be a finite-dimensional complex Lie algebra and  $\sigma \in \text{Aut} \mathfrak{g}$  be a linear automorphism whose order divides  $T$ , *i.e.* such that  $\sigma^T = \text{id}$ . Fix a primitive  $T^{\text{th}}$ -root of unity  $\omega$  and let  $\mathfrak{g} = \bigoplus_{j=0}^{T-1} \mathfrak{g}_{(j),\mathbb{C}}$  be the decomposition of  $\mathfrak{g}$  into the eigenspaces  $\mathfrak{g}_{(j),\mathbb{C}} := \{x \in \mathfrak{g} \mid \sigma x = \omega^j x\}$  of  $\sigma$ . Let  $\tau \in \overline{\text{Aut}}_- \mathfrak{g}$  be an anti-linear involutive automorphism of  $\mathfrak{g}$ , namely such that  $\tau^2 = \text{id}$ , and let  $\mathfrak{g}_0 := \{x \in \mathfrak{g} \mid \tau x = x\}$  denote the corresponding real form of  $\mathfrak{g}$ . Its complexification  $\mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$  is naturally isomorphic to  $\mathfrak{g}$ .

We shall assume that each of the eigenspaces  $\mathfrak{g}_{(j),\mathbb{C}}$  for  $j \in \mathbb{Z}_T$  is  $\tau$ -stable, *i.e.*

$$\tau \mathfrak{g}_{(j),\mathbb{C}} = \mathfrak{g}_{(j),\mathbb{C}}. \quad (6)$$

It follows from this, and using the property  $\bar{\omega} = \omega^{-1}$ , that  $(\sigma \circ \tau)^2 = \text{id}$ . We use the property (6) to define the real subspaces  $\mathfrak{g}_{(j)} := \mathfrak{g}_{(j),\mathbb{C}} \cap \mathfrak{g}_0$  for each  $j \in \mathbb{Z}_T$  so that  $\mathfrak{g}_0 = \bigoplus_{j=0}^{T-1} \mathfrak{g}_{(j)}$ .

Note that  $\sigma^k \circ \tau \in \overline{\text{Aut}}_- \mathfrak{g}$  defines an anti-linear involutive automorphism of  $\mathfrak{g}$  for each  $k \in \mathbb{Z}_T$ . Introduce the corresponding real forms of  $\mathfrak{g}$  by  $\mathfrak{g}_k := \mathfrak{g}^{\sigma^k \circ \tau} = \{x \in \mathfrak{g} \mid \sigma^k \tau x = x\}$ . The notation reflects the fact that the case  $k = 0$  gives back the original real form  $\mathfrak{g}_0$ . We note that for each  $k \in \mathbb{Z}_T$ , the anti-linear involutive automorphism  $\sigma^k \circ \tau$  clearly also preserves the eigenspace  $\mathfrak{g}_{(j),\mathbb{C}}$  for each  $j \in \mathbb{Z}_T$ . For any  $p \in \mathbb{Z}_T$  the anti-linear map  $\omega^{-kp} \sigma^k \circ \tau$  is also an involution (but in general not an automorphism). We shall make use of the corresponding real subspaces  $\mathfrak{g}_{k,p} := \mathfrak{g}^{\omega^{-kp} \sigma^k \circ \tau} = \{x \in \mathfrak{g} \mid \sigma^k \tau x = \omega^{kp} x\}$ . In this notation we have  $\mathfrak{g}_k = \mathfrak{g}_{k,0}$ . We shall also use the notation  $\mathfrak{g}_{k,p}$  for any  $p \in \mathbb{Z}$ , which will be understood to mean  $\mathfrak{g}_{k,p \bmod T}$ .

By virtue of the relations  $\sigma^T = \tau^2 = (\sigma \circ \tau)^2 = \text{id}$  satisfied by the automorphisms  $\sigma$  and  $\tau$ , we have an action of the dihedral group  $\Pi$  on the complex Lie algebra  $\mathfrak{g}$  by linear and anti-linear automorphisms. That is, we have a group homomorphism  $r : \Pi \hookrightarrow \overline{\text{Aut}} \mathfrak{g}$ ,  $\alpha \mapsto r_\alpha$  defined by  $r_{\mathfrak{s}} := \sigma$ ,  $r_{\mathfrak{t}} := \tau$ .

Let  $\langle \cdot, \cdot \rangle : \mathfrak{g}_0 \times \mathfrak{g}_0 \rightarrow \mathbb{R}$  be a non-degenerate invariant symmetric bilinear form on  $\mathfrak{g}_0$ . We extend it to a non-degenerate symmetric invariant bilinear form on  $\mathfrak{g}$  as

$$\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{C}, \quad (x \otimes u, y \otimes v) \longmapsto \langle x, y \rangle uv, \quad (7)$$

using the canonical isomorphism between  $\mathfrak{g}$  and  $\mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$ . It has the property that  $\langle \tau x, \tau y \rangle = \langle x, y \rangle$  for any  $x, y \in \mathfrak{g}$ . We assume it is also  $\sigma$ -invariant, in other words that  $\langle \sigma x, \sigma y \rangle = \langle x, y \rangle$  for any  $x, y \in \mathfrak{g}$ .

In the following lemma we make use of the notion of dual pair recalled in §A.1.

**Lemma 2.1.** *For each  $j \in \mathbb{Z}_T$ , the triple  $(\mathfrak{g}_{(-j)}, \mathfrak{g}_{(j)}, \langle \cdot, \cdot \rangle|_{\mathfrak{g}_{(-j)} \times \mathfrak{g}_{(j)}})$  is a dual pair. In particular, the restriction of (7) to  $\mathfrak{g}_{(0)}$  is non-degenerate.*

*For any  $k, p \in \mathbb{Z}_T$ , the triple  $(\mathfrak{g}_{k,-p}, \mathfrak{g}_{k,p}, \langle \cdot, \cdot \rangle|_{\mathfrak{g}_{k,-p} \times \mathfrak{g}_{k,p}})$  is a dual pair. In particular, (7) restricts to a non-degenerate invariant symmetric bilinear form on the real form  $\mathfrak{g}_k$  for each  $k \in \mathbb{Z}_T$ .*

**2.1.1. Canonical element.** For each  $k, p \in \mathbb{Z}_T$  we shall fix a basis  $I_{k,p}^a$  for  $a = 1, \dots, \dim \mathfrak{g}$  of the real subspace  $\mathfrak{g}_{k,p}$ , and we let  $I_{a;k,-p}$  for  $a = 1, \dots, \dim \mathfrak{g}$  denote the dual basis of  $\mathfrak{g}_{k,-p}$  with respect to (7), *i.e.* such that  $\langle I_{k,p}^a, I_{b;k,-p} \rangle = \delta_b^a$  for every  $a, b = 1, \dots, \dim \mathfrak{g}$ . In the case  $p = 0$  we obtain dual bases  $I_k^a := I_{k,0}^a$  and  $I_{a;k} := I_{a;k,0}$  of  $\mathfrak{g}_k$ , for each  $k \in \mathbb{Z}_T$ . For the real form  $\mathfrak{g}_0$  we will denote these dual bases simply as  $I^a := I_0^a$  and  $I_a := I_{a,0}$ . Note that any basis of the real subspace  $\mathfrak{g}_{k,p}$  for any  $k, p \in \mathbb{Z}_T$  also forms a basis over  $\mathbb{C}$  for the complexification  $\mathfrak{g}$ . A basis for the realification  $\mathfrak{g}_{\mathbb{R}}$  is then given, for instance, by  $I^a$  and  $iI^a$  for  $a = 1, \dots, \dim \mathfrak{g}$ . We shall also fix a basis  $I^{(j,\alpha)}$  for  $\alpha = 1, \dots, \dim \mathfrak{g}_{(j),\mathbb{C}}$  of  $\mathfrak{g}_{(j)}$  and let  $I_{(-j,\alpha)}$  for  $\alpha = 1, \dots, \dim \mathfrak{g}_{(j),\mathbb{C}}$  denote the dual basis of  $\mathfrak{g}_{(-j)}$  so that  $\langle I^{(j,\alpha)}, I_{(-j,\beta)} \rangle = \delta_{\beta}^{\alpha}$  for all  $\alpha, \beta = 1, \dots, \dim \mathfrak{g}_{(j),\mathbb{C}}$ .

Let  $C := I_a \otimes I^a \in \mathfrak{g} \otimes \mathfrak{g}$  denote the canonical element of  $\mathfrak{g}$ , where sums over repeated Lie algebra indices will always be implicit. Using the decomposition of  $\mathfrak{g}$  into eigenspaces of  $\sigma$ , for each  $j \in \mathbb{Z}_T$  we also have the canonical element  $C^{(j)} := I_{(-j,\alpha)} \otimes I^{(j,\alpha)} \in \mathfrak{g}_{(-j),\mathbb{C}} \otimes \mathfrak{g}_{(j),\mathbb{C}}$  of the dual pair  $(\mathfrak{g}_{(-j),\mathbb{C}}, \mathfrak{g}_{(j),\mathbb{C}}, \langle \cdot, \cdot \rangle|_{\mathfrak{g}_{(-j),\mathbb{C}} \times \mathfrak{g}_{(j),\mathbb{C}}})$ . As above, repeated indices labelling dual bases of  $\mathfrak{g}_{(-j),\mathbb{C}}$  and  $\mathfrak{g}_{(j),\mathbb{C}}$ , here  $\alpha$ , will always be implicitly summed over.

**2.2. Affine Kac-Moody algebras.** Let  $\mathcal{L}\mathfrak{g} := \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$  be the polynomial loop algebra associated with  $\mathfrak{g}$ . For any  $x \in \mathfrak{g}$  and  $n \in \mathbb{Z}$  we define  $x_n := x \otimes t^n \in \mathcal{L}\mathfrak{g}$ . The Lie bracket in  $\mathcal{L}\mathfrak{g}$  is defined by letting  $(x_m, y_n) \mapsto [x, y]_{m+n}$  and then extending to a bilinear map  $\mathcal{L}\mathfrak{g} \times \mathcal{L}\mathfrak{g} \rightarrow \mathcal{L}\mathfrak{g}$  by linearity. Similarly, define a non-degenerate invariant symmetric bilinear form on  $\mathcal{L}\mathfrak{g}$  by  $(x_m, y_n) \mapsto \langle x, y \rangle \delta_{m+n,0}$ .

The (untwisted) affine Kac-Moody algebra associated with  $\mathfrak{g}$  is defined as the vector space direct sum  $\tilde{\mathfrak{g}} := \mathcal{L}\mathfrak{g} \oplus \mathbb{C}K \oplus \mathbb{C}D$  endowed with the Lie bracket  $[\cdot, \cdot] : \tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}} \rightarrow \tilde{\mathfrak{g}}$  defined by

$$[x_m + \alpha K + xD, y_n + \beta K + yD] := [x, y]_{m+n} + nx y_n - my x_m + m\delta_{m+n,0} \langle x, y \rangle K \quad (8)$$

for any  $x, y \in \mathfrak{g}$ ,  $m, n \in \mathbb{Z}$  and  $\alpha, \beta, x, y \in \mathbb{C}$ . It is equipped with a non-degenerate invariant symmetric bilinear form  $(\cdot | \cdot) : \tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}} \rightarrow \mathbb{C}$ , defined for any  $x, y \in \mathfrak{g}$ ,  $m, n \in \mathbb{Z}$  and  $\alpha, \beta, x, y \in \mathbb{C}$  by

$$(x_m + \alpha K + xD | y_n + \beta K + yD) := \langle x, y \rangle \delta_{m+n,0} + \alpha y + \beta x. \quad (9)$$

We shall also make use of the subalgebra  $\widehat{\mathfrak{g}} := \mathcal{L}\mathfrak{g} \oplus \mathbb{C}\mathbb{K} \subset \widetilde{\mathfrak{g}}$  and the quotient  $\overline{\mathfrak{g}} := \widetilde{\mathfrak{g}}/\mathbb{C}\mathbb{K}$ . Let  $\overline{\varrho} : \widetilde{\mathfrak{g}} \rightarrow \overline{\mathfrak{g}}$  denote the canonical homomorphism with kernel  $\mathbb{C}\mathbb{K}$ . It follows from the defining relations (8) that  $\mathcal{L}\mathfrak{g}$  is isomorphic as a Lie algebra to the subquotient  $\widehat{\mathfrak{g}}/\mathbb{C}\mathbb{K}$ .

**2.2.1. Action of the dihedral group.** Consider the automorphisms  $\sigma, \tau \in \overline{\text{Aut}} \mathfrak{g}$  of  $\mathfrak{g}$  introduced in §2.1. We extend these to linear and anti-linear automorphisms of the loop algebra  $\mathcal{L}\mathfrak{g}$ , respectively, which by abuse of notation we also denote  $\sigma$  and  $\tau$ , first on homogeneous elements by letting  $\sigma(x_n) := (\sigma x)_n$ ,  $\tau(x_n) := (\tau x)_{-n}$  for  $x \in \mathfrak{g}$ ,  $n \in \mathbb{Z}$  and then to the rest of  $\mathcal{L}\mathfrak{g}$  by (anti-)linearity. Letting  $\sigma\mathbb{K} := \mathbb{K}$ ,  $\sigma\mathbb{D} := \mathbb{D}$ ,  $\tau\mathbb{K} := -\mathbb{K}$  and  $\tau\mathbb{D} := -\mathbb{D}$  further extends  $\sigma, \tau \in \overline{\text{Aut}} \mathcal{L}\mathfrak{g}$  to (anti-)linear automorphisms of the affine Kac-Moody algebra  $\widetilde{\mathfrak{g}}$ , which we shall also denote by  $\sigma, \tau \in \overline{\text{Aut}} \widetilde{\mathfrak{g}}$ .

By construction,  $\sigma, \tau \in \overline{\text{Aut}} \widetilde{\mathfrak{g}}$  satisfy  $\sigma^T = \tau^2 = (\sigma \circ \tau)^2 = \text{id}$  and thus define a representation of the dihedral group  $\Pi$  on  $\widetilde{\mathfrak{g}}$ . Specifically, we have a homomorphism

$$r : \Pi \hookrightarrow \overline{\text{Aut}} \widetilde{\mathfrak{g}}, \quad \alpha \mapsto r_\alpha \quad (10)$$

defined by  $r_s := \sigma$ ,  $r_t := \tau$ . Given any complex vector space  $V$  equipped with a real structure, namely an anti-linear involution  $\tau : V \rightarrow V$ , we extend the action of  $\Pi$  on  $\widetilde{\mathfrak{g}}$  given by (10) to the complex vector space  $\widetilde{\mathfrak{g}} \otimes V$  by defining

$$r_s(\mathbf{X} \otimes v) := r_s \mathbf{X} \otimes v, \quad r_t(\mathbf{X} \otimes v) := r_t \mathbf{X} \otimes \tau v. \quad (11)$$

In other words, if we define an action of  $\Pi$  on  $V$  by letting  $s$  act trivially and  $t$  act as  $\tau$  then (11) gives an action of  $\Pi$  on the tensor product  $\widetilde{\mathfrak{g}} \otimes V$ .

As in the finite-dimensional setting of §2.1, for each pair  $k, p \in \mathbb{Z}_T$  we consider the involutive anti-linear map  $\omega^{-kp} \sigma^k \circ \tau : \widetilde{\mathfrak{g}} \rightarrow \widetilde{\mathfrak{g}}$  which for  $p = 0$  defines an automorphism of the affine Kac-Moody algebra  $\widetilde{\mathfrak{g}}$ . We denote the corresponding real subspaces by  $\widetilde{\mathfrak{g}}_{k,p} := \widetilde{\mathfrak{g}}^{\omega^{-kp} \sigma^k \circ \tau} = \{\mathbf{X} \in \widetilde{\mathfrak{g}} \mid \sigma^k \tau \mathbf{X} = \omega^{kp} \mathbf{X}\}$ . When  $p = 0$  we obtain the real forms  $\widetilde{\mathfrak{g}}_k := \widetilde{\mathfrak{g}}_{k,0}$  of  $\widetilde{\mathfrak{g}}$  for each  $k \in \mathbb{Z}_T$ . We have the direct sum decomposition  $\widetilde{\mathfrak{g}}_{\mathbb{R}} = \widetilde{\mathfrak{g}}_{k,p} \dot{+} i \widetilde{\mathfrak{g}}_{k,p}$  of real vector spaces, where  $\widetilde{\mathfrak{g}}_{\mathbb{R}}$  denotes the realification of  $\widetilde{\mathfrak{g}}$ . Denote the corresponding projections relative to this decomposition by  $\pi_{k,p}^+ : \widetilde{\mathfrak{g}}_{\mathbb{R}} \rightarrow \widetilde{\mathfrak{g}}_{k,p}$ ,  $\mathbf{X} \mapsto \frac{1}{2}(\mathbf{X} + \omega^{-kp} \sigma^k \tau \mathbf{X})$ , and  $\pi_{k,p}^- : \widetilde{\mathfrak{g}}_{\mathbb{R}} \rightarrow i \widetilde{\mathfrak{g}}_{k,p}$ ,  $\mathbf{X} \mapsto \frac{1}{2}(\mathbf{X} - \omega^{-kp} \sigma^k \tau \mathbf{X})$ . When  $p = 0$  we denote these  $\pi_k^+ : \widetilde{\mathfrak{g}}_{\mathbb{R}} \rightarrow \widetilde{\mathfrak{g}}_k$  and  $\pi_k^- : \widetilde{\mathfrak{g}}_{\mathbb{R}} \rightarrow i \widetilde{\mathfrak{g}}_k$ , given explicitly by  $\mathbf{X} \mapsto \frac{1}{2}(\mathbf{X} \pm \sigma^k \tau \mathbf{X})$  respectively.

We denote the eigenspaces of automorphism  $\sigma \in \overline{\text{Aut}} \widetilde{\mathfrak{g}}$  by  $\widetilde{\mathfrak{g}}_{(j),\mathbb{C}} := \{\mathbf{X} \mid \sigma \mathbf{X} = \omega^j \mathbf{X}\}$  for each  $j \in \mathbb{Z}_T$ . By using the definition of  $\sigma \in \overline{\text{Aut}} \widetilde{\mathfrak{g}}$  these can be described explicitly as follows. For  $j \neq 0$ ,  $\widetilde{\mathfrak{g}}_{(j),\mathbb{C}}$  coincides with the subspace  $\mathfrak{g}_{(j),\mathbb{C}} \otimes \mathbb{C}[t, t^{-1}]$  of  $\widetilde{\mathfrak{g}}$  whereas  $\widetilde{\mathfrak{g}}_{(0),\mathbb{C}}$  can be identified with the subspace  $\mathfrak{g}_{(0),\mathbb{C}} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\mathbb{K} \oplus \mathbb{C}\mathbb{D}$ . We have the eigenspace decomposition  $\widetilde{\mathfrak{g}} = \bigoplus_{j=0}^{T-1} \widetilde{\mathfrak{g}}_{(j),\mathbb{C}}$ . It follows from the assumption (6) that  $\tau \in \overline{\text{Aut}}_- \widetilde{\mathfrak{g}}$  preserves each eigenspace. Defining corresponding real subspaces  $\widetilde{\mathfrak{g}}_{(j)} := \widetilde{\mathfrak{g}}_{(j),\mathbb{C}} \cap \widetilde{\mathfrak{g}}_0$  we have  $\widetilde{\mathfrak{g}}_0 = \bigoplus_{j=0}^{T-1} \widetilde{\mathfrak{g}}_{(j)}$ . We denote the projections relative to this decomposition by  $\pi_{(j)} : \widetilde{\mathfrak{g}}_0 \rightarrow \widetilde{\mathfrak{g}}_{(j)}$ ,  $\mathbf{X} \mapsto \frac{1}{T} \sum_{k=0}^{T-1} \omega^{-kj} \sigma^k \mathbf{X}$ . We will also need the projections  $\pi_{(j)} : \widetilde{\mathfrak{g}} \rightarrow \widetilde{\mathfrak{g}}_{(j),\mathbb{C}}$  defined by the same formulae.

The bilinear pairing (9) is both  $\sigma$ - and  $\tau$ -invariant in the sense that

$$(\sigma \mathbf{X} \mid \sigma \mathbf{Y}) = (\mathbf{X} \mid \mathbf{Y}), \quad (\tau \mathbf{X} \mid \tau \mathbf{Y}) = \overline{(\mathbf{X} \mid \mathbf{Y})}, \quad (12)$$

for any  $X, Y \in \tilde{\mathfrak{g}}$ . Recall the notion of dual pair from §A.1.

**Lemma 2.2.** *For each  $j \in \mathbb{Z}_T$ , the triple  $(\tilde{\mathfrak{g}}_{(-j)}, \tilde{\mathfrak{g}}_{(j)}, (\cdot|\cdot)|_{\tilde{\mathfrak{g}}_{(-j)} \times \tilde{\mathfrak{g}}_{(j)}}$ ) is a dual pair. In particular, the restriction of (9) to  $\tilde{\mathfrak{g}}_{(0)}$  is non-degenerate.*

*For any  $k, p \in \mathbb{Z}_T$ , the triple  $(\tilde{\mathfrak{g}}_{k,-p}, \tilde{\mathfrak{g}}_{k,p}, (\cdot|\cdot)|_{\tilde{\mathfrak{g}}_{k,-p} \times \tilde{\mathfrak{g}}_{k,p}})$  is a dual pair. In particular, (9) restricts to a non-degenerate invariant symmetric bilinear form on the real form  $\tilde{\mathfrak{g}}_k$  for each  $k \in \mathbb{Z}_T$ .*

2.2.2. *Canonical element.* Given any basis of  $\mathfrak{g}$ , such as  $I^a$ ,  $a = 1, \dots, \dim \mathfrak{g}$  which was defined initially as a basis for the real form  $\mathfrak{g}_0$ , we define a corresponding basis of  $\tilde{\mathfrak{g}}$  consisting of  $I_n^a := I^a \otimes t^n$  for  $a = 1, \dots, \dim \mathfrak{g}$  and  $n \in \mathbb{Z}$  together with the elements  $K$  and  $D$ . We denote these basis elements of  $\tilde{\mathfrak{g}}$  collectively as  $I^{\tilde{a}}$ . The dual basis of  $\tilde{\mathfrak{g}}$  with respect to (9) is then given by  $I_{a,-n} := I_a \otimes t^{-n}$  for  $a = 1, \dots, \dim \mathfrak{g}$  and  $n \in \mathbb{Z}$  together with the elements  $D$  and  $K$ . We denote the elements of this basis as  $I_{\tilde{a}}$ . A basis of the realification  $\tilde{\mathfrak{g}}_{\mathbb{R}}$  is given by  $I^{\tilde{a}}$  and  $iI^{\tilde{a}}$ .

For any anti-linear map  $\tilde{\mathfrak{g}} \rightarrow \tilde{\mathfrak{g}}$  of the form  $\chi := z \sigma^k \circ \tau$  with  $|z| = 1$  and  $k \in \mathbb{Z}_T$  we have  $\chi^2 = \text{id}$  and  $\chi x_n = (\chi x)_{-n}$  for all  $x \in \mathfrak{g}$  and  $n \in \mathbb{Z}$ . Therefore  $x_n + (\chi x)_{-n}$  is  $\chi$ -invariant. Recall the dual bases of the real subspaces  $\mathfrak{g}_{k,p}$  and  $\mathfrak{g}_{k,-p}$  for each  $k, p \in \mathbb{Z}_T$  from §2.1.1. We introduce the notation  $I_{k,p,n}^a := I_{k,p}^a \otimes t^n$  and  $I_{a;k,p,n} := I_{a;k,p} \otimes t^n$  for all  $n \in \mathbb{Z}$ . In terms of these, a basis of the real subspace  $\tilde{\mathfrak{g}}_{k,p}$  of the affine Kac-Moody algebra  $\tilde{\mathfrak{g}}$  is then given by

$$I_{k,p,0}^a, \quad \frac{1}{\sqrt{2}}(I_{k,p,n}^a + I_{k,p,-n}^a), \quad \frac{i}{\sqrt{2}}(I_{k,p,n}^a - I_{k,p,-n}^a), \quad (13a)$$

for  $n \in \mathbb{Z}_{>0}$  and  $a = 1, \dots, \dim \mathfrak{g}$  together with  $i\omega^{-kp/2}K$  and  $i\omega^{-kp/2}D$ . We denote the elements of this basis collectively as  $I_{k,p}^{\tilde{a}}$ . The dual basis of the real space  $\tilde{\mathfrak{g}}_{k,-p}$  is

$$I_{a;k,-p,0}, \quad \frac{1}{\sqrt{2}}(I_{a;k,-p,n} + I_{a;k,-p,-n}), \quad \frac{i}{\sqrt{2}}(I_{a;k,-p,n} - I_{a;k,-p,-n}), \quad (13b)$$

for  $n \in \mathbb{Z}_{>0}$  and  $a = 1, \dots, \dim \mathfrak{g}$  together with  $-i\omega^{kp/2}D$  and  $-i\omega^{kp/2}K$ . We use the notation  $I_{\tilde{a};k,-p}$  for these basis elements.

Similarly, recall the dual bases of  $\mathfrak{g}_{(j)}$  and  $\mathfrak{g}_{(-j)}$  for each  $j \in \mathbb{Z}_T$  from §2.1.1, and introduce the notation  $I_n^{(j,\alpha)} := I^{(j,\alpha)} \otimes t^n$  and  $I_{(-j,\alpha),n} := I_{(-j,\alpha)} \otimes t^n$  for all  $n \in \mathbb{Z}$ . Define dual bases of  $\tilde{\mathfrak{g}}_{(j)}$  and  $\tilde{\mathfrak{g}}_{(-j)}$ , which we denote respectively by  $I^{(j,\tilde{\alpha})}$  and  $I_{(-j,\tilde{\alpha})}$ , as follows. For  $j \neq 0$  the basis  $I^{(j,\tilde{\alpha})}$  consists of elements

$$I_0^{(j,\alpha)}, \quad \frac{1}{\sqrt{2}}(I_n^{(j,\alpha)} + I_{-n}^{(j,\alpha)}), \quad \frac{i}{\sqrt{2}}(I_n^{(j,\alpha)} - I_{-n}^{(j,\alpha)}), \quad (14a)$$

for  $n \in \mathbb{Z}_{>0}$  and  $\alpha = 1, \dots, \dim \mathfrak{g}_{(j),\mathbb{C}}$ . Its dual basis  $I_{(-j,\tilde{\alpha})}$  in  $\tilde{\mathfrak{g}}_{(-j)}$  consists of the dual elements

$$I_{(-j,\alpha),0}, \quad \frac{1}{\sqrt{2}}(I_{(-j,\alpha),n} + I_{(-j,\alpha),-n}), \quad \frac{i}{\sqrt{2}}(I_{(-j,\alpha),n} - I_{(-j,\alpha),-n}), \quad (14b)$$

for  $n \in \mathbb{Z}_{>0}$  and  $\alpha = 1, \dots, \dim \mathfrak{g}_{(j),\mathbb{C}}$ . The basis  $I^{(0,\tilde{\alpha})}$  of  $\tilde{\mathfrak{g}}_{(0)}$  comprises the same elements as in (14a) with  $j = 0$  together with  $iK$  and  $iD$ , and its dual basis  $I_{(0,\tilde{\alpha})}$  of  $\tilde{\mathfrak{g}}_{(0)}$  consists of (14b) with  $j = 0$  along with  $-iD$  and  $-iK$ .

Consider the subspaces of  $\tilde{\mathfrak{g}}$  defined by  $F_n \tilde{\mathfrak{g}} := \mathfrak{g} \otimes t^n \mathbb{C}[t]$ , for each  $n \in \mathbb{Z}_{\geq 0}$ . They define a descending  $\mathbb{Z}_{\geq 0}$ -filtration on  $\tilde{\mathfrak{g}}$ , denoted  $(F_n \tilde{\mathfrak{g}})_{n \in \mathbb{Z}_{\geq 0}}$ , in the sense that  $F_n \tilde{\mathfrak{g}} \subset F_m \tilde{\mathfrak{g}}$  for all  $n \geq m$  in  $\mathbb{Z}_{\geq 0}$ , *i.e.*  $F_0 \tilde{\mathfrak{g}} \supset F_1 \tilde{\mathfrak{g}} \supset F_2 \tilde{\mathfrak{g}} \supset F_3 \tilde{\mathfrak{g}} \supset \dots$ , and  $\bigcap_n F_n \tilde{\mathfrak{g}} = \{0\}$ .

Note that the subspaces  $F_n \tilde{\mathfrak{g}}$  define, in fact, a descending  $\mathbb{Z}_{\geq 0}$ -filtration of  $\tilde{\mathfrak{g}}$  as a Lie algebra since we have  $[F_m \tilde{\mathfrak{g}}, F_n \tilde{\mathfrak{g}}] \subset F_{m+n} \tilde{\mathfrak{g}}$  for any  $m, n \in \mathbb{Z}_{\geq 0}$ . It induces descending  $\mathbb{Z}_{\geq 0}$ -filtrations on both the realification  $\tilde{\mathfrak{g}}_{\mathbb{R}}$  and real subspaces  $\tilde{\mathfrak{g}}_{k,p}$ ,  $k, p \in \mathbb{Z}_T$  given by  $F_n \tilde{\mathfrak{g}}_{\mathbb{R}} := (F_n \tilde{\mathfrak{g}})_{\mathbb{R}}$  and  $F_n \tilde{\mathfrak{g}}_{k,p} := \pi_{k,p}^+(F_n \tilde{\mathfrak{g}})$ , respectively. We shall also make use of the ‘conjugate’ descending  $\mathbb{Z}_{\geq 0}$ -filtration of  $\tilde{\mathfrak{g}}$  as a Lie algebra defined by the subspaces  $\bar{F}_n \tilde{\mathfrak{g}} := \mathfrak{g} \otimes t^{-n} \mathbb{C}[t^{-1}]$ . Note that by definition of  $\tau \in \overline{\text{Aut}}_- \tilde{\mathfrak{g}}$  in §2.2.1 we have  $\bar{F}_n \tilde{\mathfrak{g}} = \tau(F_n \tilde{\mathfrak{g}})$ .

The subspaces  $F_n(\tilde{\mathfrak{g}} \otimes \tilde{\mathfrak{g}}) := F_n \tilde{\mathfrak{g}} \otimes \bar{F}_n \tilde{\mathfrak{g}} + \bar{F}_n \tilde{\mathfrak{g}} \otimes F_n \tilde{\mathfrak{g}}$  for  $n \in \mathbb{Z}_{\geq 0}$ , endow the tensor product  $\tilde{\mathfrak{g}} \otimes \tilde{\mathfrak{g}}$  with a descending  $\mathbb{Z}_{\geq 0}$ -filtration. Note that these subspaces are invariant under the action of  $\Pi$  on the tensor product  $\tilde{\mathfrak{g}} \otimes \tilde{\mathfrak{g}}$ . We define the completed tensor product  $\tilde{\mathfrak{g}} \hat{\otimes} \tilde{\mathfrak{g}}$  as the completion of  $\tilde{\mathfrak{g}} \otimes \tilde{\mathfrak{g}}$  with respect to the associated linear topology, in which the subspaces  $F_n(\tilde{\mathfrak{g}} \otimes \tilde{\mathfrak{g}})$  form a basis of fundamental open neighbourhoods of the origin. In other words, it is given by the corresponding inverse limit  $\tilde{\mathfrak{g}} \hat{\otimes} \tilde{\mathfrak{g}} := \varprojlim \tilde{\mathfrak{g}} \otimes \tilde{\mathfrak{g}} / F_n(\tilde{\mathfrak{g}} \otimes \tilde{\mathfrak{g}})$ .

The canonical element of  $\tilde{\mathfrak{g}}$  living in  $\tilde{\mathfrak{g}} \hat{\otimes} \tilde{\mathfrak{g}}$  is then defined as

$$\tilde{C} := I_{\tilde{a}} \otimes I^{\tilde{a}} = D \otimes K + K \otimes D + \sum_{n \in \mathbb{Z}} I_{a, -n} \otimes I_n^a \quad (15)$$

where, as in the finite dimensional case, sums over repeated Lie algebra indices, here  $\tilde{a}$ , shall always be implicit. Similarly, for each  $j \in \mathbb{Z}_T$  we let  $\tilde{C}^{(j)} := I_{(-j, \tilde{\alpha})} \otimes I^{(j, \tilde{\alpha})} = \delta_{(j)}(D \otimes K + K \otimes D) + \sum_{n \in \mathbb{Z}} I_{(-j, \alpha), -n} \otimes I_n^{(j, \alpha)}$ , where  $\delta_{(j)}$  is the periodic Kronecker delta, equal to 1 if  $j \equiv 0 \pmod{T}$  and 0 otherwise, and summation over the repeated indices  $\tilde{\alpha}$  and  $\alpha$  is implicit. As above, the infinite sum over  $n \in \mathbb{Z}$  represents an element of the subspace  $\tilde{\mathfrak{g}}_{(-j), \mathbb{C}} \hat{\otimes} \tilde{\mathfrak{g}}_{(j), \mathbb{C}}$  of the completion  $\tilde{\mathfrak{g}} \hat{\otimes} \tilde{\mathfrak{g}}$ . We have the decomposition  $\tilde{C} = \sum_{j=0}^{T-1} \tilde{C}^{(j)}$ .

The statement of the following lemma uses standard tensor index notation recalled in §A.2. Specifically, in the notation used there we take  $\mathfrak{a} = \mathfrak{b} = \tilde{\mathfrak{g}}$  and  $\mathfrak{A} = \mathbb{C}$  so that we may drop the last tensor factor in  $\mathfrak{A}$ .

**Lemma 2.3.** *For any  $X \in \tilde{\mathfrak{g}}$  we have  $[X_1 + X_2, \tilde{C}_{12}] = 0$ . Moreover, for any  $i, j \in \mathbb{Z}_T$  and  $X \in \tilde{\mathfrak{g}}_{(j), \mathbb{C}}$  we have  $[X_1, \tilde{C}_{12}^{(i+j)}] + [X_2, \tilde{C}_{12}^{(i)}] = 0$ .*

**2.2.3. Connections on  $S^1$ .** Recall the canonical map  $\bar{\varrho} : \tilde{\mathfrak{g}} \longrightarrow \bar{\mathfrak{g}}$ , whose restriction to  $\hat{\mathfrak{g}}$  is the homomorphism  $\bar{\varrho}|_{\hat{\mathfrak{g}}} : \hat{\mathfrak{g}} \rightarrow \mathcal{L}\mathfrak{g}$ , where from now on we use the isomorphism  $\hat{\mathfrak{g}}/\text{CK} \xrightarrow{\sim} \mathcal{L}\mathfrak{g}$  implicitly. In particular, we regard  $\mathcal{L}\mathfrak{g}$  as a subalgebra of  $\bar{\mathfrak{g}}$ . To make contact in §4 with classical field theories on the circle  $S^1 := \mathbb{R}/2\pi\mathbb{Z}$ , in this section we provide concrete realisations of the Lie algebras  $\mathcal{L}\mathfrak{g}$  and  $\bar{\mathfrak{g}}$  respectively in terms of  $\mathfrak{g}$ -valued trigonometric polynomials and connections on  $S^1$ .

Let  $\mathcal{T}(S^1)$  be the commutative differential  $\mathbb{C}$ -algebra of trigonometric polynomials on  $S^1$ , namely functions  $S^1 \rightarrow \mathbb{C}$  of the form  $\theta \mapsto P(e^{i\theta})$  with  $P$  a Laurent polynomial. We denote by  $\partial : \mathcal{T}(S^1) \rightarrow \mathcal{T}(S^1)$  the derivation on  $\mathcal{T}(S^1)$  sending  $\theta \mapsto P(e^{i\theta})$  to  $\theta \mapsto ie^{i\theta} P'(e^{i\theta})$ . A basis of  $\mathcal{T}(S^1)$  is  $\{e_n\}_{n \in \mathbb{Z}}$  where  $e_n : S^1 \rightarrow \mathbb{C}$ ,  $\theta \mapsto e^{in\theta}$ . Complex conjugation provides  $\mathcal{T}(S^1)$  with an anti-linear involution, which sends the basis element  $e_n$  to  $e_{-n}$ . We equip  $\mathcal{T}(S^1)$  with an action of  $\Pi$  by letting  $\mathfrak{s}$  act trivially and  $\mathfrak{t}$  act by complex conjugation.

Let  $\mathcal{T}(S^1, \mathfrak{g}) := \mathfrak{g} \otimes \mathcal{T}(S^1)$  be the space of  $\mathfrak{g}$ -valued trigonometric polynomial on  $S^1$ . We obtain an action of  $\Pi$  on  $\mathcal{T}(S^1, \mathfrak{g})$  by combining the above action on  $\mathcal{T}(S^1)$  with that on  $\mathfrak{g}$  given in §2.1. A non-degenerate symmetric bilinear form on  $\mathcal{T}(S^1, \mathfrak{g})$  is given by

$$(A|B)_{S^1} := \frac{1}{2\pi} \int_{S^1} d\theta \langle A(\theta), B(\theta) \rangle, \quad (16)$$

for any  $A, B \in \mathcal{T}(S^1, \mathfrak{g})$ , where the bilinear form (7) is extended to a map  $\langle \cdot, \cdot \rangle : \mathcal{T}(S^1, \mathfrak{g}) \times \mathcal{T}(S^1, \mathfrak{g}) \rightarrow \mathcal{T}(S^1)$  defined by  $\langle x \otimes f, y \otimes g \rangle := \langle x, y \rangle fg$  for any  $x, y \in \mathfrak{g}$  and  $f, g \in \mathcal{T}(S^1)$ .

Consider the complex vector space  $\text{Conn}_{\mathfrak{g}}(S^1)$  of  $\mathfrak{g}$ -valued connections on  $S^1$  of the form  $\ell\partial + A$  where  $\ell \in \mathbb{C}$  and  $A \in \mathcal{T}(S^1, \mathfrak{g})$ . We extend the action of  $\Pi$  on  $\mathcal{T}(S^1, \mathfrak{g})$  to  $\text{Conn}_{\mathfrak{g}}(S^1)$  by letting it act trivially on the derivative  $\partial$  and using (anti-)linearity. We refer to  $\ell\partial + A \in \text{Conn}_{\mathfrak{g}}(S^1)$  as an  $\ell$ -connection to emphasise its dependence on the coefficient  $\ell$  of the derivative term. In particular, we may regard an element of  $\mathcal{T}(S^1, \mathfrak{g})$  as defining a 0-connection in  $\text{Conn}_{\mathfrak{g}}(S^1)$ . The commutator of two connections then defines a Lie bracket on  $\text{Conn}_{\mathfrak{g}}(S^1)$ , which we denote  $[\cdot, \cdot] : \text{Conn}_{\mathfrak{g}}(S^1) \times \text{Conn}_{\mathfrak{g}}(S^1) \rightarrow \text{Conn}_{\mathfrak{g}}(S^1)$ .

**Lemma 2.4.** *We have a  $\Pi$ -equivariant isomorphism  $\bar{\mathfrak{g}} \xrightarrow{\sim} \text{Conn}_{\mathfrak{g}}(S^1)$ , under which the bilinear form (16) on  $\mathcal{T}(S^1, \mathfrak{g})$  corresponds to that on  $\mathcal{L}\mathfrak{g}$ . Its composition with  $\bar{\varrho}$  is the Lie algebra homomorphism  $\varrho : \bar{\mathfrak{g}} \rightarrow \text{Conn}_{\mathfrak{g}}(S^1)$  given by  $\varrho(D) = -i\partial$ ,  $\varrho(K) = 0$  and  $\varrho(x_n) = x \otimes e_n$ , for any  $x \in \mathfrak{g}$  and  $n \in \mathbb{Z}$ .*

The vector space  $\text{Conn}_{\mathfrak{g}}(S^1)$  is endowed with a pair of descending  $\mathbb{Z}_{\geq 0}$ -filtrations defined as the images of the subspaces  $(F_n \tilde{\mathfrak{g}})_{n \in \mathbb{Z}_{\geq 0}}$  and  $(\bar{F}_n \tilde{\mathfrak{g}})_{n \in \mathbb{Z}_{\geq 0}}$  under the linear map  $\varrho$  from Lemma 2.4. Concretely, the subspace  $F_n(\text{Conn}_{\mathfrak{g}}(S^1))$  (resp.  $\bar{F}_n(\text{Conn}_{\mathfrak{g}}(S^1))$ ) for  $n \in \mathbb{Z}_{\geq 0}$  is spanned by  $x \otimes e_m$  (resp.  $x \otimes e_{-m}$ ) with  $x \in \mathfrak{g}$  and  $m \geq n$ . Since  $\varrho \otimes \varrho$  is continuous it extends to a linear map  $\varrho \otimes \varrho : \tilde{\mathfrak{g}} \hat{\otimes} \tilde{\mathfrak{g}} \rightarrow \text{Conn}_{\mathfrak{g}}(S^1) \hat{\otimes} \text{Conn}_{\mathfrak{g}}(S^1)$ . We shall need the image of the canonical element  $\tilde{C}$  under this map. It follows from the form of  $\tilde{C}$  in (15) that its image in fact lies in the subspace  $\mathcal{T}(S^1, \mathfrak{g}) \hat{\otimes} \mathcal{T}(S^1, \mathfrak{g}) \cong_{\mathbb{C}} \mathfrak{g} \otimes \mathfrak{g} \otimes \mathcal{T}(S^1) \hat{\otimes} \mathcal{T}(S^1)$ , where the pair of conjugate descending  $\mathbb{Z}_{\geq 0}$ -filtrations on  $\mathcal{T}(S^1)$  are given by the subspaces  $F_n(\mathcal{T}(S^1))$  and  $\bar{F}_n(\mathcal{T}(S^1))$  for  $n \in \mathbb{Z}_{\geq 0}$  with bases  $\{e_{\pm m}\}_{m \in \mathbb{Z}_{\geq n}}$ , respectively. Given any element  $\kappa \in \mathcal{T}(S^1) \hat{\otimes} \mathcal{T}(S^1)$ , for each  $\theta \in S^1$  we can regard  $\kappa(\theta, \cdot)$  as the kernel of a formal distribution on  $S^1$ , in the sense that it yields a linear map  $\mathcal{T}(S^1) \rightarrow \mathbb{C}$ ,  $f \mapsto \frac{1}{2\pi} \int_{S^1} d\theta' \kappa(\theta, \theta') f(\theta')$ . In particular, the element  $(\varrho \otimes \varrho)\tilde{C} \in \mathfrak{g} \otimes \mathfrak{g} \otimes \mathcal{T}(S^1) \hat{\otimes} \mathcal{T}(S^1)$  given explicitly by

$$((\varrho \otimes \varrho)\tilde{C})(\theta, \theta') = \sum_{n \in \mathbb{Z}} I_a \otimes I^a e_{-n}(\theta) e_n(\theta') = C \delta_{\theta\theta'} \quad (17)$$

is related to the Dirac  $\delta$ -distribution  $\delta := \sum_{n \in \mathbb{Z}} e_n \otimes e_{-n} \in \mathcal{T}(S^1) \hat{\otimes} \mathcal{T}(S^1)$ . For any  $\theta \in S^1$ , the expression  $\delta(\theta, \cdot)$  is the kernel of the distribution on  $S^1$  sending the test function  $f \in \mathcal{T}(S^1)$  to  $f(\theta)$ . Note that we use the notation  $\delta_{\theta\theta'}$  instead of  $\delta(\theta, \theta')$ .



## 3. DOUBLE LOOP ALGEBRAS AND TAKIFF ALGEBRAS

Consider the Riemann sphere  $\mathbb{P}^1 := \mathbb{C} \cup \{\infty\}$  and fix a global coordinate  $z$  on  $\mathbb{C}$ . At each finite point  $x \in \mathbb{C} \subset \mathbb{P}^1$  we have the local coordinate  $\xi_x := z - x$  and at the point  $\infty \in \mathbb{P}^1$  a local coordinate is given by  $\xi_\infty := z^{-1}$ .

We denote by  $\mathbb{C}[t]$  the ring of polynomials in  $t$ , by  $\mathbb{C}\{t\}$  the ring of convergent power series in  $t$  and by  $\mathbb{C}\{\{t\}\}$  the ring of convergent Laurent series in  $t$ , *i.e.*  $f(t) \in \mathbb{C}\{\{t\}\}$  if and only if  $t^k f(t) \in \mathbb{C}\{t\}$  for some  $k \in \mathbb{Z}_{\geq 0}$ .

For each  $x \in \mathbb{P}^1$ , we let  $\mathcal{O}_x := \mathbb{C}\{\xi_x\}$  denote the local ring of germs of holomorphic functions at  $x$ , *i.e.* the ring of convergent power series in  $\xi_x$ . Denote by  $\mathfrak{m}_x := \xi_x \mathbb{C}\{\xi_x\}$  the maximal ideal of  $\mathcal{O}_x$  consisting of germs of holomorphic functions at  $x$  which vanish at  $x$ . The ring  $\mathcal{O}_x$  has a natural descending  $\mathbb{Z}_{\geq 0}$ -filtration  $\mathcal{O}_x = \mathfrak{m}_x^0 \supset \mathfrak{m}_x \supset \mathfrak{m}_x^2 \supset \mathfrak{m}_x^3 \supset \dots$  where  $\mathfrak{m}_x^n := \xi_x^n \mathbb{C}\{\xi_x\}$  for any  $n \in \mathbb{Z}_{\geq 0}$ . Denote by  $\mathcal{K}_x := \mathbb{C}\{\{\xi_x\}\}$  the field of germs of meromorphic functions at  $x$ , *i.e.* the field of convergent Laurent series in  $\xi_x$ . It has a natural descending  $\mathbb{Z}$ -filtration which by abuse of notation we also denote  $\mathfrak{m}_x^\bullet$ , namely

$$\dots \supset \mathfrak{m}_x^{-3} \supset \mathfrak{m}_x^{-2} \supset \mathfrak{m}_x^{-1} \supset \mathcal{O}_x \supset \mathfrak{m}_x \supset \mathfrak{m}_x^2 \supset \mathfrak{m}_x^3 \supset \dots$$

where we extend the notation  $\mathfrak{m}_x^n$ ,  $n \in \mathbb{Z}_{\geq 0}$  introduced above by letting  $\mathfrak{m}_x^k := \xi_x^k \mathbb{C}\{\xi_x\}$  for any  $k \in \mathbb{Z}$ . Let  $\mathfrak{p}_x := \xi_x^{-1} \mathbb{C}[\xi_x^{-1}]$  denote the set of principal parts at  $x$ , which forms a ring without identity. If we choose to also include the constant term in the principal part then we obtain the corresponding ring  $\mathfrak{p}_x^0 := \mathbb{C}[\xi_x^{-1}]$ . This ring also has a natural descending  $\mathbb{Z}_{\geq 0}$ -filtration  $\mathfrak{p}_x^0 \supset \mathfrak{p}_x \supset \mathfrak{p}_x^2 \supset \mathfrak{p}_x^3 \supset \dots$  where we use the notation  $\mathfrak{p}_x^n := \xi_x^{-n} \mathbb{C}[\xi_x^{-1}]$  defined for any  $n \in \mathbb{Z}$ .

We define an injective homomorphism

$$\mu : \Pi \hookrightarrow \overline{\text{Aut}} \mathbb{P}^1, \quad \alpha \mapsto \mu_\alpha \tag{18}$$

of  $\Pi$  into the full Möbius group of holomorphic and anti-holomorphic automorphisms of  $\mathbb{P}^1$  by letting  $\mu_s : z \mapsto \omega z$  and  $\mu_t : z \mapsto \bar{z}$ . The image of the cyclic subgroup  $\Gamma$  consists of Möbius transformations and the image of the coset  $\Gamma t$  consists of orientation-reversing Möbius transformations. Given any point  $x \in \mathbb{P}^1$  we let  $\Pi_x \subset \Pi$  denote its stabilizer under the action (18). We will refer to  $x \in \mathbb{P}^1$  as a *complex* point if  $\Pi_x$  is trivial and as a *real* point if  $\Pi_x \cap \Gamma t \neq \emptyset$ . Specifically, the set of all real points is formed of the union  $\{\infty\} \cup \bigcup_{k \in \mathbb{Z}_T} \omega^{k/2} \mathbb{R}$  since  $\Pi_x = \langle s^k t \rangle$  for any  $x \in \omega^{k/2} \mathbb{R} \setminus \{0\}$  with  $k \in \mathbb{Z}_T$  and  $\Pi_x = \Pi$  for any  $x \in \{0, \infty\}$ . Moreover, the set of complex points is the complement of the set of real points. We refer to  $\{0, \infty\}$  as the set of *fixed* points. This terminology reflects the fact that when  $T \in \mathbb{Z}_{>1}$  we have  $\Pi_x = \Pi$  if and only if  $x \in \{0, \infty\}$ . However, by convention we will still refer to 0 and  $\infty$  as fixed points even when  $T = 1$ .

Let  $N \in \mathbb{Z}_{\geq 0}$ . Pick and fix a finite set  $\mathbf{z} := \{z_1, \dots, z_N, \infty\} \subset \mathbb{P}^1$  which includes the point at infinity labelled as  $z_{N+1} = \infty$ . We will assume that the  $\Pi$ -orbits of the points in  $\mathbf{z}$  are all disjoint, in other words  $z_i \neq \mu_\alpha z_j$  for all  $\alpha \in \Pi$  and  $i \neq j$ . Denote by  $\mathbf{z}_c$  the subset of complex points in  $\mathbf{z}$  and by  $\mathbf{z}_r$  the subset of real points, so that in particular  $\infty \in \mathbf{z}_r$  and we have the disjoint union  $\mathbf{z} = \mathbf{z}_c \sqcup \mathbf{z}_r$ . We also define the

subsets

$$\mathbf{z}_f := \mathbf{z} \cap \{0, \infty\}, \quad \mathbf{z}'_r := \mathbf{z}_r \setminus \{0, \infty\}, \quad \mathbf{z}^k_r := \{x \in \mathbf{z}'_r \mid \Pi_x = \langle \mathbf{s}^k \mathbf{t} \rangle\}$$

corresponding respectively to fixed points, to real non-fixed points and to real points with stabiliser  $\langle \mathbf{s}^k \mathbf{t} \rangle$ . The disjoint union decomposition of  $\mathbf{z}$  may be refined as  $\mathbf{z} = \mathbf{z}_c \sqcup \mathbf{z}'_r \sqcup \mathbf{z}_f = \mathbf{z}_c \sqcup \bigsqcup_{k \in \mathbb{Z}_T} \mathbf{z}^k_r \sqcup \mathbf{z}_f$ . For any finite subset  $\mathbf{x} \subset \mathbb{P}^1$  we also introduce the notation  $\bar{\mathbf{x}} := \{x \in \mathbb{P}^1 \mid \bar{x} \in \mathbf{x}\}$ .

**3.1. ‘Local’ Lie algebras.** For any  $x \in \mathbb{P}^1$ , we define an action of  $\Pi_x$  on the field  $\mathcal{K}_x$  as follows. Given a germ  $[f]_x \in \mathcal{K}_x$  we choose a representative  $f : D_x \rightarrow \mathbb{C}$  on a small disc  $D_x$  around  $x$ . We set

$$\alpha.[f]_x := \begin{cases} [f \circ \mu_\alpha^{-1}]_x, & \text{if } \alpha \in \Pi_x \cap \Gamma \\ [c \circ f \circ \mu_\alpha^{-1}]_x, & \text{if } \alpha \in \Pi_x \cap \Gamma \mathbf{t}, \end{cases}$$

where  $c : \mathbb{C} \rightarrow \mathbb{C}$  denotes complex conjugation and  $\mu_\alpha : D_x \rightarrow D_x$  is the restriction of  $\mu_\alpha \in \overline{\text{Aut}} \mathbb{P}^1$  to the open disc  $D_x$ . This defines a left action of  $\Pi_x$  on  $\mathcal{K}_x$  which also preserves its  $\mathbb{Z}$ -filtration, in the sense that  $\alpha.\mathfrak{m}_x^k \subset \mathfrak{m}_x^k$  for any  $k \in \mathbb{Z}$ . By combining the actions of  $\Pi_x \subset \Pi$  on  $\tilde{\mathfrak{g}}$  and  $\mathcal{K}_x$  we obtain a natural action on the tensor product  $\tilde{\mathfrak{g}} \otimes \mathcal{K}_x$ . Explicitly, we define a homomorphism

$$\Pi_x \hookrightarrow \overline{\text{Aut}}(\tilde{\mathfrak{g}} \otimes \mathcal{K}_x), \quad \alpha \mapsto \hat{\alpha} \quad (19)$$

by letting, for any  $\alpha \in \Pi_x$ ,  $\mathbf{X} \in \tilde{\mathfrak{g}}$  and  $[f]_x \in \mathcal{K}_x$ ,  $\hat{\alpha}(\mathbf{X} \otimes [f]_x) := r_\alpha \mathbf{X} \otimes \alpha.[f]_x$ . It induces a homomorphism  $\Pi_x \hookrightarrow \text{Aut}(\tilde{\mathfrak{g}} \otimes \mathcal{K}_x)_\mathbb{R}$  into  $\mathbb{R}$ -linear automorphisms of the realification  $(\tilde{\mathfrak{g}} \otimes \mathcal{K}_x)_\mathbb{R}$ .

To every point  $x \in \mathbf{z}$  we attach the real Lie algebra  $L\tilde{\mathfrak{g}}_x := ((\tilde{\mathfrak{g}} \otimes \mathcal{K}_x)_\mathbb{R})^{\Pi_x}$ . Explicitly, to any complex point  $x \in \mathbf{z}_c$  we attach the realification  $L\tilde{\mathfrak{g}}_x = (\tilde{\mathfrak{g}} \otimes \mathcal{K}_x)_\mathbb{R}$  and to a non-fixed real point  $x \in \mathbf{z}^k_r$  we attach the real form  $L\tilde{\mathfrak{g}}_x = (\tilde{\mathfrak{g}} \otimes \mathcal{K}_x)^{\mathbf{s}^k \text{ot}}$ . If  $0 \in \mathbf{z}$  then we attach to it the  $\Pi$ -invariant subalgebra  $L\tilde{\mathfrak{g}}_0 = (\tilde{\mathfrak{g}} \otimes \mathcal{K}_0)^\Pi$ . Similarly, to the point at infinity, which by assumption always belongs to  $\mathbf{z}$ , we also attach the  $\Pi$ -invariant subalgebra  $L\tilde{\mathfrak{g}}_\infty = (\tilde{\mathfrak{g}} \otimes \mathcal{K}_\infty)^\Pi$ . Define the direct sum of real Lie algebras  $L\tilde{\mathfrak{g}}_\mathbf{z} := \bigoplus_{x \in \mathbf{z}} L\tilde{\mathfrak{g}}_x$ .

We also introduce the Lie subalgebras  $L\tilde{\mathfrak{g}}_x^+ := ((\tilde{\mathfrak{g}} \otimes \mathcal{O}_x)_\mathbb{R})^{\Pi_x} \subset L\tilde{\mathfrak{g}}_x$  at every finite point  $x \in \mathbf{z} \setminus \{\infty\}$ . In particular, if  $0 \in \mathbf{z}$  then  $L\tilde{\mathfrak{g}}_0^+ = (\tilde{\mathfrak{g}} \otimes \mathcal{O}_0)^\Pi$ . However, for reasons to be clarified in the next subsection, cf. Lemma 3.1, at infinity we consider instead the Lie subalgebra  $L\tilde{\mathfrak{g}}_\infty^+ := (\tilde{\mathfrak{g}} \otimes \mathfrak{m}_\infty)^\Pi$  of  $L\tilde{\mathfrak{g}}_\infty$ . We set  $L\tilde{\mathfrak{g}}_\mathbf{z}^+ := \bigoplus_{x \in \mathbf{z}} L\tilde{\mathfrak{g}}_x^+$ .

There is a natural complementary subalgebra to  $L\tilde{\mathfrak{g}}_\mathbf{z}^+$  in  $L\tilde{\mathfrak{g}}_\mathbf{z}$  defined as follows. To every finite point  $x \in \mathbf{z} \setminus \{\infty\}$  we attach the Lie subalgebra  $L\tilde{\mathfrak{g}}_x^- := ((\tilde{\mathfrak{g}} \otimes \mathfrak{p}_x)_\mathbb{R})^{\Pi_x} \subset L\tilde{\mathfrak{g}}_x$  and similarly we define  $L\tilde{\mathfrak{g}}_\infty^- := (\tilde{\mathfrak{g}} \otimes \mathfrak{p}_\infty^0)^\Pi$  for the point at infinity. Recall here that  $\mathfrak{p}_\infty^0 = \mathbb{C}[\xi_\infty^{-1}]$  includes the constant term. In particular, we then have the direct sum decomposition of linear spaces  $L\tilde{\mathfrak{g}}_x = L\tilde{\mathfrak{g}}_x^+ \dot{+} L\tilde{\mathfrak{g}}_x^-$  for any  $x \in \mathbf{z}$ . We define the direct sum of Lie subalgebras  $L\tilde{\mathfrak{g}}_\mathbf{z}^- := \bigoplus_{x \in \mathbf{z}} L\tilde{\mathfrak{g}}_x^-$ , so that we have the direct sum decomposition  $L\tilde{\mathfrak{g}}_\mathbf{z} = L\tilde{\mathfrak{g}}_\mathbf{z}^+ \dot{+} L\tilde{\mathfrak{g}}_\mathbf{z}^-$ . We will be interested in a different complement of  $L\tilde{\mathfrak{g}}_\mathbf{z}^+$  in  $L\tilde{\mathfrak{g}}_\mathbf{z}$  provided by Lemma 3.1 below.

**3.2. ‘Global’ Lie algebras.** Given a finite subset  $S \subset \mathbb{P}^1$  with  $z \subset S$ , we denote by  $R_S$  the ring of meromorphic functions on  $\mathbb{P}^1$  with poles contained in  $S$ . We let  $R_S(\tilde{\mathfrak{g}}) := \tilde{\mathfrak{g}} \otimes R_S$  be the corresponding Lie algebra of  $\tilde{\mathfrak{g}}$ -valued meromorphic functions on  $\mathbb{P}^1$  with poles in  $S$ . For each  $x \in z$  there is an injective homomorphism of rings  $\iota_x : R_S \hookrightarrow \mathcal{K}_x$  which assigns to a meromorphic function  $f \in R_S$  its germ  $[f]_x \in \mathcal{K}_x$  at  $x$ . Correspondingly, there is an embedding of Lie algebras

$$\iota_x : R_S(\tilde{\mathfrak{g}}) \hookrightarrow \bigoplus_{x \in z} \tilde{\mathfrak{g}} \otimes \mathcal{K}_x \quad (20)$$

which assigns to any meromorphic function  $X \otimes f \in R_S(\tilde{\mathfrak{g}})$ , where  $X \in \tilde{\mathfrak{g}}$  and  $f \in R_S$ , the set of its germs  $\iota_x(X \otimes f) := X \otimes [f]_x \in \tilde{\mathfrak{g}} \otimes \mathcal{K}_x$  at the points  $x \in z$ .

Consider the set  $\Pi z := \{\mu_\alpha x \mid \alpha \in \Pi, x \in z\}$ . We define an action of  $\Gamma$  on  $R_{\Pi z}$  by setting  $\alpha.f := f \circ \mu_\alpha^{-1}$  for any  $f \in R_{\Pi z}$  and  $\alpha \in \Gamma$ . This lifts to an action of  $\Pi$  by letting  $\mathfrak{t}$  act as  $\mathfrak{t}.f := c \circ f \circ \mu_{\mathfrak{t}} =: \hat{f}$  on any  $f \in R_{\Pi z}$ . We therefore obtain an action of  $\Pi$  on  $R_{\Pi z}(\tilde{\mathfrak{g}}) = \tilde{\mathfrak{g}} \otimes R_{\Pi z}$ ,

$$\Pi \hookrightarrow \overline{\text{Aut}} R_{\Pi z}(\tilde{\mathfrak{g}}), \quad \alpha \mapsto \hat{\alpha} \quad (21)$$

where the action of  $\alpha \in \Pi$  is given explicitly by  $\hat{\alpha}(X \otimes f) := r_\alpha X \otimes \alpha.f$ , for  $X \in \tilde{\mathfrak{g}}$  and  $f \in R_{\Pi z}$ . Define the real Lie algebra of  $\Pi$ -invariants  $R_z^\Pi(\tilde{\mathfrak{g}}) := R_{\Pi z}(\tilde{\mathfrak{g}})^\Pi$ .

The property of invariance under the action (21) may equivalently be rephrased as follows. Let  $\bar{R}_{\Pi z}$  be the ring of anti-meromorphic functions on  $\mathbb{P}^1$  with poles contained in  $\Pi z$ , and define the corresponding Lie algebra  $\bar{R}_{\Pi z}(\tilde{\mathfrak{g}}) := \tilde{\mathfrak{g}} \otimes \bar{R}_{\Pi z}$  of  $\tilde{\mathfrak{g}}$ -valued anti-meromorphic functions. For any  $\alpha \in \Gamma$  we extend  $r_\alpha \in \text{Aut } \tilde{\mathfrak{g}}$  to a linear map  $r_\alpha : R_{\Pi z}(\tilde{\mathfrak{g}}) \rightarrow R_{\Pi z}(\tilde{\mathfrak{g}})$ ,  $r_\alpha(X \otimes f) = r_\alpha X \otimes f$ . Similarly, for any  $\alpha \in \Gamma \mathfrak{t}$  we extend  $r_\alpha \in \overline{\text{Aut}}_- \tilde{\mathfrak{g}}$  to an anti-linear map  $r_\alpha : R_{\Pi z}(\tilde{\mathfrak{g}}) \rightarrow \bar{R}_{\Pi z}(\tilde{\mathfrak{g}})$ ,  $r_\alpha(X \otimes f) = r_\alpha X \otimes c \circ f$ . Now for any  $\alpha \in \Gamma$  we define a linear map  $\mu_\alpha^* : R_{\Pi z}(\tilde{\mathfrak{g}}) \rightarrow R_{\Pi z}(\tilde{\mathfrak{g}})$  using the pullback by  $\mu_\alpha$  on the second tensor factor. On the other hand, the pullback by  $\mu_\alpha$  for  $\alpha \in \Gamma \mathfrak{t}$  defines instead a linear map  $\mu_\alpha^* : R_{\Pi z}(\tilde{\mathfrak{g}}) \rightarrow \bar{R}_{\Pi z}(\tilde{\mathfrak{g}})$ . By combining the above, we can then describe the  $\Pi$ -invariant subalgebra  $R_z^\Pi(\tilde{\mathfrak{g}})$  equivalently as

$$R_z^\Pi(\tilde{\mathfrak{g}}) = \{F \in R_{\Pi z}(\tilde{\mathfrak{g}}) \mid r_\alpha F = \mu_\alpha^* F \text{ for all } \alpha \in \Pi\}. \quad (22)$$

By restricting the embedding (20) with  $S = \Pi z$  to the above subalgebra  $R_z^\Pi(\tilde{\mathfrak{g}})$  we obtain an embedding  $\iota_z := (\iota_{z_1}, \dots, \iota_{z_N}, \iota_\infty) : R_z^\Pi(\tilde{\mathfrak{g}}) \hookrightarrow L\tilde{\mathfrak{g}}_z$ . In what follows we shall often regard  $R_z^\Pi(\tilde{\mathfrak{g}})$  as a subalgebra of  $L\tilde{\mathfrak{g}}_z$  by identifying an element  $X \in R_z^\Pi(\tilde{\mathfrak{g}})$  with its image  $\iota_z X = (\iota_x X)_{x \in z} \in L\tilde{\mathfrak{g}}_z$ .

**Lemma 3.1.** *We have the direct sum decomposition of real vector spaces  $L\tilde{\mathfrak{g}}_z = L\tilde{\mathfrak{g}}_z^+ \dot{+} R_z^\Pi(\tilde{\mathfrak{g}})$ .*

**3.3. Dual spaces.** For any  $x \in \mathbb{P}^1$  we define an action of the stabilizer subgroup  $\Pi_x \subset \Pi$  on the space of germs of meromorphic differentials  $\mathcal{K}_x d\xi_x$ . We let  $\alpha \in \Pi_x \cap \Gamma$  act on  $[\varpi]_x \in \mathcal{K}_x d\xi_x$  via pullback by  $\mu_\alpha^{-1}$ , namely  $\alpha.[\varpi]_x := [(\mu_\alpha^{-1})^* \varpi]_x$ , whereas if  $\alpha \in \Pi_x \cap \Gamma \mathfrak{t}$  then  $\mu_\alpha$  is orientation-reversing and so we set  $\alpha.[\varpi]_x := [c \circ (\mu_\alpha^{-1})^* \varpi]_x$  instead. Define an action of  $\Pi_x$  on the tensor product  $\tilde{\mathfrak{g}} \otimes \mathcal{K}_x d\xi_x$ ,

$$\Pi_x \hookrightarrow \overline{\text{Aut}}(\tilde{\mathfrak{g}} \otimes \mathcal{K}_x d\xi_x), \quad \alpha \mapsto \hat{\alpha} \quad (23)$$

given for any  $\alpha \in \Pi_x$ ,  $X \in \tilde{\mathfrak{g}}$  and  $[\varpi]_x \in \mathcal{K}_x d\xi_x$  by  $\hat{\alpha}(X \otimes [\varpi]_x) := r_\alpha X \otimes \alpha.[\varpi]_x$ .

To any point  $x \in \mathbf{z}$  we attach the real subspace of  $\Pi_x$ -invariants in the realification  $(\tilde{\mathfrak{g}} \otimes \mathcal{H}_x d\xi_x)_{\mathbb{R}}$ , namely  $\Omega_{\tilde{\mathfrak{g}}_x} := ((\tilde{\mathfrak{g}} \otimes \mathcal{H}_x d\xi_x)_{\mathbb{R}})^{\Pi_x}$ . Introduce also the subspace  $\Omega_{\tilde{\mathfrak{g}}_x}^+ := ((\tilde{\mathfrak{g}} \otimes \mathcal{O}_x d\xi_x)_{\mathbb{R}})^{\Pi_x}$  for each  $x \in \mathbf{z} \setminus \{\infty\}$  and at infinity we set  $\Omega_{\tilde{\mathfrak{g}}_{\infty}}^+ := (\tilde{\mathfrak{g}} \otimes \mathfrak{m}_{\infty}^{-1} d\xi_{\infty})^{\Pi}$ . These have natural complementary subspaces in  $\Omega_{\tilde{\mathfrak{g}}_x}$  given respectively by  $\Omega_{\tilde{\mathfrak{g}}_x}^- := ((\tilde{\mathfrak{g}} \otimes \mathfrak{p}_x d\xi_x)_{\mathbb{R}})^{\Pi_x}$  and  $\Omega_{\tilde{\mathfrak{g}}_{\infty}}^- := (\tilde{\mathfrak{g}} \otimes \mathfrak{p}_{\infty}^2 d\xi_{\infty})^{\Pi}$ . Let  $\Omega_{\tilde{\mathfrak{g}}_{\mathbf{z}}} := \bigoplus_{x \in \mathbf{z}} \Omega_{\tilde{\mathfrak{g}}_x}$  and  $\Omega_{\tilde{\mathfrak{g}}_{\mathbf{z}}}^+ := \bigoplus_{x \in \mathbf{z}} \Omega_{\tilde{\mathfrak{g}}_x}^+$ .

We shall also need the space of globally defined  $\Pi$ -invariant  $\tilde{\mathfrak{g}}$ -valued meromorphic differentials on  $\mathbb{P}^1$  with poles contained in the set  $\Pi\mathbf{z}$ . Given a finite subset  $S \subset \mathbb{P}^1$  containing  $\mathbf{z}$ , we let  $\Omega_S$  denote the space of meromorphic differentials on  $\mathbb{P}^1$  with poles at most in  $S$ . The differential  $dz$ , where  $z$  is the global coordinate on  $\mathbb{C}$ , provides an  $R_S$ -basis for  $\Omega_S$  since any  $\varpi \in \Omega_S$  can be written as  $\varpi = f dz$  for some  $f \in R_S$ . In the case  $S = \Pi\mathbf{z}$ , the group  $\Pi$  acts on the space  $\Omega_{\Pi\mathbf{z}}$  by letting  $\alpha \in \Gamma$  act as the pullback by the inverse of the multiplication map  $\mu_{\alpha} : \mathbb{P}^1 \rightarrow \mathbb{P}^1$  defined in (18) and letting  $\mathfrak{t}$  send  $\varpi = f dz$  to  $\mathfrak{t}.\varpi = \bar{f} dz$  where  $\bar{f} = c \circ f \circ \mu_{\mathfrak{t}}$ . In particular, this allows us to define an action

$$\Pi \curvearrowright \overline{\text{Aut}} \Omega_{\Pi\mathbf{z}}(\tilde{\mathfrak{g}}), \quad \alpha \mapsto \hat{\alpha} \quad (24)$$

on the tensor product  $\Omega_{\Pi\mathbf{z}}(\tilde{\mathfrak{g}}) := \tilde{\mathfrak{g}} \otimes \Omega_{\Pi\mathbf{z}}$ , given explicitly by  $\hat{\alpha}(X \otimes \varpi) := r_{\alpha} X \otimes \alpha.\varpi$  for any  $\alpha \in \Pi$ ,  $X \in \tilde{\mathfrak{g}}$  and  $\varpi \in \Omega_{\Pi\mathbf{z}}$ . Define the real vector space of  $\Pi$ -invariants  $\Omega_{\mathbf{z}}^{\Pi}(\tilde{\mathfrak{g}}) := \Omega_{\Pi\mathbf{z}}(\tilde{\mathfrak{g}})^{\Pi}$ .

This subspace can alternatively be described in a similar fashion to the subalgebra of  $\Pi$ -invariant  $\tilde{\mathfrak{g}}$ -valued rational functions (22). For this we introduce the space  $\bar{\Omega}_{\Pi\mathbf{z}}$  of anti-meromorphic differentials with poles contained in  $\Pi\mathbf{z}$ , of the form  $\varphi = f d\bar{z}$  for some  $f \in \bar{R}_{\Pi\mathbf{z}}$ . We then also define the corresponding space  $\bar{\Omega}_{\Pi\mathbf{z}}(\tilde{\mathfrak{g}}) := \tilde{\mathfrak{g}} \otimes \bar{\Omega}_{\Pi\mathbf{z}}$ . The map  $r_{\alpha} \in \overline{\text{Aut}} \tilde{\mathfrak{g}}$  extends to a linear map  $\Omega_{\Pi\mathbf{z}}(\tilde{\mathfrak{g}}) \rightarrow \Omega_{\Pi\mathbf{z}}(\tilde{\mathfrak{g}})$  for  $\alpha \in \Gamma$  and to an anti-linear map  $\Omega_{\Pi\mathbf{z}}(\tilde{\mathfrak{g}}) \rightarrow \bar{\Omega}_{\Pi\mathbf{z}}(\tilde{\mathfrak{g}})$  for  $\alpha \in \Gamma\mathfrak{t}$  in the same way as done in §3.2 for  $R_{\Pi\mathbf{z}}(\tilde{\mathfrak{g}})$ . If we also define linear maps  $\mu_{\alpha}^* : \Omega_{\Pi\mathbf{z}}(\tilde{\mathfrak{g}}) \rightarrow \Omega_{\Pi\mathbf{z}}(\tilde{\mathfrak{g}})$  (resp.  $\mu_{\alpha}^* : \Omega_{\Pi\mathbf{z}}(\tilde{\mathfrak{g}}) \rightarrow \bar{\Omega}_{\Pi\mathbf{z}}(\tilde{\mathfrak{g}})$ ) for each  $\alpha \in \Gamma$  (resp.  $\alpha \in \Gamma\mathfrak{t}$ ), then the real vector space  $\Omega_{\mathbf{z}}^{\Pi}(\tilde{\mathfrak{g}})$  may be equivalently described as

$$\Omega_{\mathbf{z}}^{\Pi}(\tilde{\mathfrak{g}}) = \{\Phi \in \Omega_{\Pi\mathbf{z}}(\tilde{\mathfrak{g}}) \mid r_{\alpha}\Phi = \mu_{\alpha}^*\Phi \text{ for all } \alpha \in \Pi\}. \quad (25)$$

As in §3.2, we have an injective map  $\iota_{\mathbf{z}} := (\iota_{z_1}, \dots, \iota_{z_N}, \iota_{\infty}) : \Omega_{\mathbf{z}}^{\Pi}(\tilde{\mathfrak{g}}) \hookrightarrow \Omega_{\tilde{\mathfrak{g}}_{\mathbf{z}}}$  which assigns to a meromorphic differential in  $\Omega_{\mathbf{z}}^{\Pi}(\tilde{\mathfrak{g}})$  the set of its germs at the points in  $\mathbf{z}$ . In what follows we will also often implicitly identify an element  $\Phi \in \Omega_{\mathbf{z}}^{\Pi}(\tilde{\mathfrak{g}})$  with its image  $\iota_{\mathbf{z}}\Phi = (\iota_x\Phi)_{x \in \mathbf{z}} \in \Omega_{\tilde{\mathfrak{g}}_{\mathbf{z}}}$ .

**Lemma 3.2.** *We have the direct sum decomposition of real vector spaces  $\Omega_{\tilde{\mathfrak{g}}_{\mathbf{z}}} = \Omega_{\tilde{\mathfrak{g}}_{\mathbf{z}}}^+ \dot{+} \Omega_{\mathbf{z}}^{\Pi}(\tilde{\mathfrak{g}})$ .*

3.3.1. *Dual pairing.* Let  $\Re : \mathbb{C} \rightarrow \mathbb{R}$ ,  $u \mapsto \Re(u)$  and  $\Im : \mathbb{C} \rightarrow \mathbb{R}$ ,  $u \mapsto \Im(u)$  denote the maps which return the real and imaginary parts of a complex number, respectively. By combining the non-degenerate bilinear form on  $\tilde{\mathfrak{g}}$  with the residue pairing, we can define a bilinear form  $\langle\langle \cdot, \cdot \rangle\rangle : \Omega_{\tilde{\mathfrak{g}}_{\mathbf{z}}} \times L\tilde{\mathfrak{g}}_{\mathbf{z}} \rightarrow \mathbb{R}$  as

$$\langle\langle \Phi, X \rangle\rangle := \Re \left( \sum_{x \in \mathbf{z}} \frac{2T}{|\Pi_x|} \text{res}_x(\Phi_x | X_x) \right) \quad (26)$$

for any  $\Phi = (\Phi_x)_{x \in \mathbf{z}} \in \Omega_{\tilde{\mathfrak{g}}_{\mathbf{z}}}$  and  $X = (X_x)_{x \in \mathbf{z}} \in L\tilde{\mathfrak{g}}_{\mathbf{z}}$ .

In what follows we make use of standard results on dual pairs recalled in §A.1.

**Lemma 3.3.** *The triple  $(\Omega_{\mathbf{z}}^{\sim}(\tilde{\mathfrak{g}}), L\tilde{\mathfrak{g}}_{\mathbf{z}}, \langle\langle \cdot, \cdot \rangle\rangle)$  is a dual pair.*

The first part of the following proposition is a generalisation of the  $\Gamma$ -equivariant strong residue theorem on  $\mathbb{P}^1$  as in [86, Lemma A.1] to the  $\Pi$ -equivariant case. For the non-equivariant version of the strong residue theorem on an arbitrary algebraic curve see, for instance, [82, §2.3].

**Proposition 3.4.** *We have  $R_{\mathbf{z}}^{\Pi}(\tilde{\mathfrak{g}})^{\perp} = \Omega_{\mathbf{z}}^{\Pi}(\tilde{\mathfrak{g}})$  and  $\Omega_{\mathbf{z}}^{\Pi}(\tilde{\mathfrak{g}})^{\perp} = R_{\mathbf{z}}^{\Pi}(\tilde{\mathfrak{g}})$ . Also,  $(\Omega_{\mathbf{z}}^{\sim}(\tilde{\mathfrak{g}})^{\perp})^{\perp} = L\tilde{\mathfrak{g}}_{\mathbf{z}}^{\perp}$  and  $(L\tilde{\mathfrak{g}}_{\mathbf{z}}^{\perp})^{\perp} = \Omega_{\mathbf{z}}^{\sim}(\tilde{\mathfrak{g}})$ .*

*Proof.* According to Lemmas 3.1, 3.2 and A.1 it suffices to show that  $\Omega_{\mathbf{z}}^{\Pi}(\tilde{\mathfrak{g}}) \perp R_{\mathbf{z}}^{\Pi}(\tilde{\mathfrak{g}})$  and  $\Omega_{\mathbf{z}}^{\sim}(\tilde{\mathfrak{g}})^{\perp} \perp L\tilde{\mathfrak{g}}_{\mathbf{z}}^{\perp}$ . We prove each of these statements in turn.

Let  $\Phi \in \Omega_{\mathbf{z}}^{\Pi}(\tilde{\mathfrak{g}})$  and  $X \in R_{\mathbf{z}}^{\Pi}(\tilde{\mathfrak{g}})$ . It follows from (22) and (25) that these have the properties  $r_{\alpha}\Phi = \mu_{\alpha}^*\Phi$  and  $r_{\alpha}X = \mu_{\alpha}^*X$  for any  $\alpha \in \Pi$ . Then by the  $\sigma$ -invariance of the bilinear form on  $\tilde{\mathfrak{g}}$  it follows that, for any  $x \in \mathbb{P}^1$  and  $\alpha = s^n \in \Gamma$ ,

$$\begin{aligned} \operatorname{res}_x(\iota_x\Phi|\iota_xX) &= \frac{1}{2\pi i} \int_{\mathcal{C}_x} (\Phi|X) = \frac{1}{2\pi i} \int_{\mathcal{C}_x} (r_{\alpha}\Phi|r_{\alpha}X) = \frac{1}{2\pi i} \int_{\mathcal{C}_x} (\mu_{\alpha}^*\Phi|\mu_{\alpha}^*X) \\ &= \frac{1}{2\pi i} \int_{\mu_{\alpha}\mathcal{C}_x} (\Phi|X) = \frac{1}{2\pi i} \int_{\mathcal{C}_{\mu_{\alpha}x}} (\Phi|X) = \operatorname{res}_{\omega^n x}(\iota_{\omega^n x}\Phi|\iota_{\omega^n x}X), \end{aligned}$$

where  $\mathcal{C}_x$  is a sufficiently small counterclockwise contour around the point  $x$ . Likewise, by the  $\tau$ -invariance of the bilinear pairing (9), in the sense of (12), we have

$$\begin{aligned} \overline{\operatorname{res}_x(\iota_x\Phi|\iota_xX)} &= -\frac{1}{2\pi i} \int_{\mathcal{C}_x} \overline{(\Phi|X)} = -\frac{1}{2\pi i} \int_{\mathcal{C}_x} (\tau\Phi|\tau X) = -\frac{1}{2\pi i} \int_{\mathcal{C}_x} (\mu_{\tau}^*\Phi|\mu_{\tau}^*X) \\ &= -\frac{1}{2\pi i} \int_{\mu_{\tau}\mathcal{C}_x} (\Phi|X) = \frac{1}{2\pi i} \int_{\mathcal{C}_{\bar{x}}} (\Phi|X) = \operatorname{res}_{\bar{x}}(\iota_{\bar{x}}\Phi|\iota_{\bar{x}}X), \end{aligned}$$

noting that  $\mu_{\tau}\mathcal{C}_x$  is oriented clockwise so that  $\mu_{\tau}\mathcal{C}_x = -\mathcal{C}_{\bar{x}}$ . Hence

$$\begin{aligned} \langle\langle \Phi, X \rangle\rangle &= \Re \left( \sum_{x \in \mathbf{z}} \frac{2T}{|\Pi_x|} \operatorname{res}_x(\iota_x\Phi|\iota_xX) \right) = \sum_{x \in \mathbf{z}_c} T (\operatorname{res}_x(\iota_x\Phi|\iota_xX) + \operatorname{res}_{\bar{x}}(\iota_{\bar{x}}\Phi|\iota_{\bar{x}}X)) \\ &\quad + \sum_{x \in \mathbf{z}'_t} T \operatorname{res}_x(\iota_x\Phi|\iota_xX) + \sum_{x \in \mathbf{z}'_f} \operatorname{res}_x(\iota_x\Phi|\iota_xX) = \sum_{x \in \Pi\mathbf{z}} \operatorname{res}_x(\iota_x\Phi|\iota_xX) = 0, \end{aligned}$$

where the last equality is by the standard residue theorem.

Now let  $\Phi \in \Omega_{\mathbf{z}}^{\sim}(\tilde{\mathfrak{g}})^{\perp}$  and  $X \in L\tilde{\mathfrak{g}}_{\mathbf{z}}^{\perp}$ . At any point  $x \in \mathbf{z} \setminus \{\infty\}$  the germs  $\Phi_x \in \Omega_{\mathbf{z}}^{\sim}(\tilde{\mathfrak{g}})^{\perp}$  and  $X_x \in L\tilde{\mathfrak{g}}_{\mathbf{z}}^{\perp}$  are elements of the spaces  $\tilde{\mathfrak{g}} \otimes \mathcal{O}_x d\xi_x$  and  $\tilde{\mathfrak{g}} \otimes \mathcal{O}_x$ , respectively. Therefore  $(\Phi_x|X_x) \in \mathcal{O}_x d\xi_x$  so that  $\operatorname{res}_x(\Phi_x|X_x) = 0$ . On the other hand, at infinity the germs  $\Phi_{\infty}$  and  $X_{\infty}$  belong to  $\tilde{\mathfrak{g}} \otimes \mathfrak{m}_{\infty}^{-1} d\xi_{\infty}$  and  $\tilde{\mathfrak{g}} \otimes \mathfrak{m}_{\infty}$ , respectively. So in this case as well we have  $(\Phi_{\infty}|X_{\infty}) \in \mathcal{O}_{\infty} d\xi_{\infty}$ , and hence  $\operatorname{res}_{\infty}(\Phi_{\infty}|X_{\infty}) = 0$ . It therefore follows that each term in the sum of residues in (26) vanishes, and thus  $\langle\langle \Phi, X \rangle\rangle = 0$ .  $\square$

**Corollary 3.5.** *The triple  $(\Omega_{\mathbf{z}}^{\Pi}(\tilde{\mathfrak{g}}), L\tilde{\mathfrak{g}}_{\mathbf{z}}^{\perp}, \langle\langle \cdot, \cdot \rangle\rangle)$  is a dual pair.*

**3.4. Divisors and Takiff algebras.** Let  $\text{Div}(\mathbb{P}^1)$  be the free abelian group generated by the points of  $\mathbb{P}^1$ . An element  $\mathcal{D} \in \text{Div}(\mathbb{P}^1)$  is called a *divisor*, which we write as a formal sum  $\mathcal{D} = \sum_{x \in \mathbb{P}^1} n_x x$  with  $n_x \in \mathbb{Z}$  being zero for all but finitely many  $x \in \mathbb{P}^1$ . The divisor  $\mathcal{D}$  is said to be non-negative, and we write  $\mathcal{D} \geq 0$ , if  $n_x \geq 0$  for all  $x \in \mathbb{P}^1$ . For any  $\mathcal{D}, \mathcal{D}' \in \text{Div}(\mathbb{P}^1)$  we write  $\mathcal{D} \geq \mathcal{D}'$  if  $\mathcal{D} - \mathcal{D}' \geq 0$ . This defines a partial ordering on the set  $\text{Div}(\mathbb{P}^1)$ . The support of  $\mathcal{D} \in \text{Div}(\mathbb{P}^1)$  is the finite subset  $\text{supp } \mathcal{D} := \{x \in \mathbb{P}^1 \mid n_x \neq 0\}$  and its degree is defined as  $\deg \mathcal{D} := \sum_{x \in \mathbb{P}^1} n_x \in \mathbb{Z}$ . Given a finite subset  $S \subset \mathbb{P}^1$  we let  $\text{Div}_{\geq 1}(S)$  denote the subset of all non-negative divisors with support  $S$ .

Fix a divisor  $\mathcal{D} \in \text{Div}_{\geq 1}(\mathbf{z})$ . We associate to it a non-negative divisor  $\Pi \mathcal{D} := \sum_{x \in \Pi \mathbf{z}} n_x x \in \text{Div}_{\geq 1}(\Pi \mathbf{z})$  with support  $\Pi \mathbf{z}$ , where for any  $x \in \Pi \mathbf{z}$ ,  $n_x \in \mathbb{Z}_{\geq 1}$  is defined by noting that there is a unique  $y \in \mathbf{z}$  such that  $x \in \Pi y$ , *i.e.*  $x$  and  $y$  lie on the same orbit of  $\Pi$ , and we set  $n_x := n_y$ . In what follows we assume  $\mathcal{D}$  to be such that  $\deg(\Pi \mathcal{D}) \geq 2$ , *i.e.* we have  $n_\infty \geq 2$  or  $|\mathbf{z}| \geq 2$ .

Given a meromorphic differential  $\varpi \in \Omega_{\Pi \mathbf{z}}$ , its order at  $x \in \mathbb{P}^1$ , denoted  $\text{ord}_x \varpi$ , is by definition equal to  $n \in \mathbb{Z}$  if its germ at  $x$  takes the form  $[\varpi]_x = \sum_{k=n}^{\infty} a_k \xi_x^k d\xi_x$  with  $a_k \in \mathbb{C}$  and  $a_n \neq 0$ . The canonical divisor of  $\varpi$  is then defined as  $(\varpi) := \sum_{x \in \mathbb{P}^1} (\text{ord}_x \varpi) x \in \text{Div}(\mathbb{P}^1)$ . Let  $\Omega_{\Pi \mathcal{D}} := \{\varpi \in \Omega_{\Pi \mathbf{z}} \mid (\varpi) \geq -\Pi \mathcal{D}\}$ . For any complex vector space  $V$  we introduce the notation

$$\Omega_{\Pi \mathcal{D}}(V) := V \otimes \Omega_{\Pi \mathcal{D}}. \quad (27)$$

The linear space  $\Omega_{\Pi \mathcal{D}}(\tilde{\mathfrak{g}})$  admits a natural action of  $\Pi$  defined in the same way as (24). We shall be interested in the subspace of  $\Pi$ -invariants  $\Omega_{\mathcal{D}}^{\Pi}(\tilde{\mathfrak{g}}) := \Omega_{\Pi \mathcal{D}}(\tilde{\mathfrak{g}})^{\Pi}$ . This is a subspace of  $\Omega_{\mathbf{z}}^{\Pi}(\tilde{\mathfrak{g}})$  which, as usual, we will implicitly identify with its image in  $\Omega_{\tilde{\mathfrak{g}}_{\mathbf{z}}}$  under the map  $\iota_{\mathbf{z}}$  from §3.3.

For each  $x \in \mathbf{z}$  and  $n \in \mathbb{Z}_{>1}$  we define the ideal  $L\tilde{\mathfrak{g}}_x^{+n} := ((\tilde{\mathfrak{g}} \otimes \mathfrak{m}_x^n)_{\mathbb{R}})^{\Pi_x}$  of  $L\tilde{\mathfrak{g}}_x^+$ . We note in particular that  $L\tilde{\mathfrak{g}}_x^{+1}$  is a proper ideal in  $L\tilde{\mathfrak{g}}_x^+$  for  $x \neq \infty$  whereas  $L\tilde{\mathfrak{g}}_{\infty}^{+1} = L\tilde{\mathfrak{g}}_{\infty}^+$ . Set  $L\tilde{\mathfrak{g}}_{\mathcal{D}}^+ := \bigoplus_{x \in \mathbf{z}} L\tilde{\mathfrak{g}}_x^{+n_x}$ , which is an ideal in the real Lie algebra  $L\tilde{\mathfrak{g}}_{\mathbf{z}}^+$ .

**Lemma 3.6.** *We have the direct sum decompositions  $(L\tilde{\mathfrak{g}}_{\mathcal{D}}^+)^{\perp} = \Omega_{\mathcal{D}}^{\Pi}(\tilde{\mathfrak{g}}) \dot{+} \Omega_{\mathcal{D}}^{\Pi}(\tilde{\mathfrak{g}})$  and  $\Omega_{\mathcal{D}}^{\Pi}(\tilde{\mathfrak{g}})^{\perp} = R_{\mathbf{z}}^{\Pi}(\tilde{\mathfrak{g}}) \dot{+} L\tilde{\mathfrak{g}}_{\mathcal{D}}^+$  of real vector spaces.*

**Proposition 3.7.** *The triple  $(\Omega_{\mathcal{D}}^{\Pi}(\tilde{\mathfrak{g}}), L\tilde{\mathfrak{g}}_{\mathbf{z}}^+/L\tilde{\mathfrak{g}}_{\mathcal{D}}^+, \langle\langle \cdot, \cdot \rangle\rangle)$  forms a dual pair, where the bilinear form*

$$\langle\langle \cdot, \cdot \rangle\rangle : \Omega_{\mathcal{D}}^{\Pi}(\tilde{\mathfrak{g}}) \times L\tilde{\mathfrak{g}}_{\mathbf{z}}^+/L\tilde{\mathfrak{g}}_{\mathcal{D}}^+ \longrightarrow \mathbb{R} \quad (28)$$

*is induced from the restriction  $\langle\langle \cdot, \cdot \rangle\rangle|_{\Omega_{\mathcal{D}}^{\Pi}(\tilde{\mathfrak{g}}) \times L\tilde{\mathfrak{g}}_{\mathbf{z}}^+}$  of (26) to  $\Omega_{\mathcal{D}}^{\Pi}(\tilde{\mathfrak{g}}) \times L\tilde{\mathfrak{g}}_{\mathbf{z}}^+$ .*

**3.4.1. Direct sum of real Takiff algebras.** The quotient Lie algebra  $L\tilde{\mathfrak{g}}_{\mathbf{z}}^+/L\tilde{\mathfrak{g}}_{\mathcal{D}}^+$  can be described in terms of real generalised Takiff algebras for  $\tilde{\mathfrak{g}}$  as follows.

We assign to each  $x \in \mathbf{z}$  a formal variable  $\varepsilon_x$ . At every finite point  $x \in \mathbf{z} \setminus \{\infty\}$  we consider the ring of polynomials  $\mathbb{C}[\varepsilon_x]$ , whereas for the point at infinity we consider instead the ideal  $\varepsilon_{\infty} \mathbb{C}[\varepsilon_{\infty}]$  of polynomials without constant terms. There is a natural action of  $\Pi$  on  $\mathbb{C}[\varepsilon_x]$  for each  $x \in \mathbf{z} \setminus \{\infty\}$  given by  $\mathfrak{s}.f(\varepsilon_x) = f(\omega^{-1}\varepsilon_x)$  and  $\mathfrak{t}.f(\varepsilon_x) = \bar{f}(\varepsilon_x)$ , for any  $f \in \mathbb{C}[\varepsilon_x]$ . Here  $\bar{f} \in \mathbb{C}[\varepsilon_x]$  denotes the complex conjugate polynomial, defined as  $\bar{f}(\varepsilon_x) = \sum_i \bar{a}_i \varepsilon_x^i$  if  $f(\varepsilon_x) = \sum_i a_i \varepsilon_x^i$ . By contrast, we let  $\Pi$  act on  $\varepsilon_{\infty} \mathbb{C}[\varepsilon_{\infty}]$

by  $\mathbf{s}.f(\varepsilon_\infty) = f(\omega \varepsilon_\infty)$  and  $\mathbf{t}.f(\varepsilon_\infty) = \bar{f}(\varepsilon_\infty)$ , for any  $f \in \varepsilon_\infty \mathbb{C}[\varepsilon_\infty]$ , with  $\bar{f} \in \varepsilon_\infty \mathbb{C}[\varepsilon_\infty]$  denoting its complex conjugate.

Now given the above divisor  $\mathcal{D} = \sum_{x \in \mathbf{z}} n_x x \in \text{Div}_{\geq 1}(\mathbf{z})$ , we can form the quotient ring  $\mathbb{C}[\varepsilon_x]/\varepsilon_x^{n_x} \mathbb{C}[\varepsilon_x]$  of  $n_x$ -truncated polynomials for  $x \in \mathbf{z} \setminus \{\infty\}$ . At infinity we have, instead, the quotient  $\varepsilon_\infty \mathbb{C}[\varepsilon_\infty]/\varepsilon_\infty^{n_\infty} \mathbb{C}[\varepsilon_\infty]$  which forms a ring without identity. The actions of  $\Pi$  on  $\mathbb{C}[\varepsilon_x]$  and  $\varepsilon_\infty \mathbb{C}[\varepsilon_\infty]$  defined above both descend to these quotients since the respective ideals  $\varepsilon_x^{n_x} \mathbb{C}[\varepsilon_x]$  and  $\varepsilon_\infty^{n_\infty} \mathbb{C}[\varepsilon_\infty]$  are invariant. By abuse of notation we denote the action of  $\alpha \in \Pi$  on the class  $f \in \mathbb{C}[\varepsilon_x]/\varepsilon_x^{n_x} \mathbb{C}[\varepsilon_x]$  also as  $\alpha.f$  and similarly for the point at infinity.

At a finite point  $x \in \mathbf{z} \setminus \{\infty\}$  we form the Lie algebra  $\mathcal{T}_x^{n_x} \tilde{\mathfrak{g}} := \tilde{\mathfrak{g}} \otimes \mathbb{C}[\varepsilon_x]/\varepsilon_x^{n_x} \mathbb{C}[\varepsilon_x]$ , which we shall refer to as a generalised Takiff algebra for  $\tilde{\mathfrak{g}}$ . For the point at infinity we define instead the nilpotent Lie algebra  $\mathcal{T}_\infty^{n_\infty} \tilde{\mathfrak{g}} := \tilde{\mathfrak{g}} \otimes \varepsilon_\infty \mathbb{C}[\varepsilon_\infty]/\varepsilon_\infty^{n_\infty} \mathbb{C}[\varepsilon_\infty]$ . By combining the action of  $\Pi$  on  $n_x$ -truncated polynomials for each  $x \in \mathbf{z}$  introduced above with that on  $\tilde{\mathfrak{g}}$ , we obtain an action of  $\Pi$  on the Lie algebra  $\mathcal{T}_x^{n_x} \tilde{\mathfrak{g}}$  for each  $x \in \mathbf{z}$ , *i.e.* a homomorphism  $\Pi \rightarrow \overline{\text{Aut}}(\mathcal{T}_x^{n_x} \tilde{\mathfrak{g}})$ ,  $\alpha \mapsto \hat{\alpha}$  given by

$$\hat{\alpha}(\mathbf{X} \otimes f) = r_\alpha \mathbf{X} \otimes \alpha.f, \quad (29)$$

for any  $\alpha \in \Pi$ ,  $\mathbf{X} \in \tilde{\mathfrak{g}}$  and  $f \in \mathbb{C}[\varepsilon_x]/\varepsilon_x^{n_x} \mathbb{C}[\varepsilon_x]$  if  $x \in \mathbf{z} \setminus \{\infty\}$  or  $f \in \varepsilon_\infty \mathbb{C}[\varepsilon_\infty]/\varepsilon_\infty^{n_\infty} \mathbb{C}[\varepsilon_\infty]$  for the point at infinity. In what follows we will be interested only in the restriction  $\Pi_x \rightarrow \overline{\text{Aut}}(\mathcal{T}_x^{n_x} \tilde{\mathfrak{g}})$  of the above homomorphism to the stabiliser subgroup  $\Pi_x \subset \Pi$ .

To any  $x \in \mathbf{z}$  we attach the real Lie algebra of  $\Pi_x$ -invariants  $\tilde{\mathfrak{g}}^{n_x} := ((\mathcal{T}_x^{n_x} \tilde{\mathfrak{g}})_{\mathbb{R}})^{\Pi_x}$ . We refer to these as real (generalised) Takiff algebras for  $\tilde{\mathfrak{g}}$ . Consider the direct sum of real Lie algebras  $\tilde{\mathfrak{g}}^{\mathcal{D}} := \bigoplus_{x \in \mathbf{z}} \tilde{\mathfrak{g}}^{n_x}$ . Note that when  $n_\infty = 1$ , the summand corresponding to the point at infinity is absent.

**Proposition 3.8.** *We have an isomorphism of real Lie algebras  $L\tilde{\mathfrak{g}}_{\mathbf{z}}^+ / L\tilde{\mathfrak{g}}_{\mathcal{D}}^+ \cong \tilde{\mathfrak{g}}^{\mathcal{D}}$ . In particular, the triple  $(\Omega_{\mathcal{D}}^{\Pi}(\tilde{\mathfrak{g}}), \tilde{\mathfrak{g}}^{\mathcal{D}}, \langle \cdot, \cdot \rangle)$  forms a dual pair where the bilinear form  $\langle \cdot, \cdot \rangle : \Omega_{\mathcal{D}}^{\Pi}(\tilde{\mathfrak{g}}) \times \tilde{\mathfrak{g}}^{\mathcal{D}} \rightarrow \mathbb{R}$  is induced from (28).*

In the next section we shall make use of explicit bases for  $\tilde{\mathfrak{g}}^{\mathcal{D}}$  and its dual  $\Omega_{\mathcal{D}}^{\Pi}(\tilde{\mathfrak{g}})$  from Proposition 3.8, which we now describe.

Recall the dual bases  $\{I^{\tilde{a}}\}$  and  $\{I_{\tilde{a}}\}$  of  $\tilde{\mathfrak{g}}$  introduced in §2.2. In terms of these, a basis of the Takiff algebra  $\mathcal{T}_x^{n_x} \tilde{\mathfrak{g}}$  for  $x \in \mathbf{z} \setminus \{\infty\}$  is given by  $I^{\tilde{a}} \otimes \varepsilon_x^p$  for  $p = 0, \dots, n_x - 1$ . Here we denote the class  $\varepsilon_x^p + \varepsilon_x^{n_x} \mathbb{C}[\varepsilon_x] \in \mathbb{C}[\varepsilon_x]/\varepsilon_x^{n_x} \mathbb{C}[\varepsilon_x]$  simply by  $\varepsilon_x^p$ . In particular, we have  $\varepsilon_x^{n_x} = 0$  for every  $x \in \mathbf{z}$ . A basis of the realification  $(\mathcal{T}_x^{n_x} \tilde{\mathfrak{g}})_{\mathbb{R}}$  then consists of  $I^{\tilde{a}} \otimes \varepsilon_x^p$  and  $iI^{\tilde{a}} \otimes \varepsilon_x^p$  for  $p = 0, \dots, n_x - 1$ . Likewise, a basis of  $(\mathcal{T}_\infty^{n_\infty} \tilde{\mathfrak{g}})_{\mathbb{R}}$  is given by  $I^{\tilde{a}} \otimes \varepsilon_\infty^{q+1}$  and  $iI^{\tilde{a}} \otimes \varepsilon_\infty^{q+1}$  for  $q = 0, \dots, n_\infty - 2$ .

A basis of  $\tilde{\mathfrak{g}}^{\mathcal{D}}$ , whose elements we denote collectively as  $I^A$ , is given by

$$I^{\tilde{a}} \otimes \varepsilon_x^p, \quad iI^{\tilde{a}} \otimes \varepsilon_x^p, \quad (30a)$$

for each  $x \in \mathbf{z}_c$  with  $p = 0, \dots, n_x - 1$ ,

$$I_{k,p}^{\tilde{a}} \otimes \varepsilon_x^p, \quad (30b)$$

for each  $x \in \mathbf{z}_r^k$ ,  $k \in \mathbb{Z}_T$  with  $p = 0, \dots, n_x - 1$ , and

$$I^{(p,\tilde{\alpha})} \otimes \varepsilon_0^p, \quad I^{(-q-1,\tilde{\alpha})} \otimes \varepsilon_\infty^{q+1}, \quad (30c)$$

for  $p = 0, \dots, n_0 - 1$  and  $q = 0, \dots, n_\infty - 2$ . The dual basis elements  $I_A$ , with respect to the pairing from Lemma 3.8, of the real vector space  $\Omega_{\mathbb{D}}^{\Pi}(\tilde{\mathfrak{g}})$  read

$$\frac{1}{2T} \sum_{\alpha \in \Pi} \hat{\alpha} \left( \frac{I_{\tilde{a}}}{(z-x)^{p+1}} dz \right), \quad \frac{1}{2T} \sum_{\alpha \in \Pi} \hat{\alpha} \left( \frac{-iI_{\tilde{a}}}{(z-x)^{p+1}} dz \right), \quad (31a)$$

for each  $x \in \mathbf{z}_c$  with  $p = 0, \dots, n_x - 1$ ,

$$\frac{1}{T} \sum_{\alpha \in \Gamma} \hat{\alpha} \left( \frac{I_{\tilde{a};k,-p}}{(z-x)^{p+1}} dz \right), \quad (31b)$$

for each  $x \in \mathbf{z}_r^k$ ,  $k \in \mathbb{Z}_T$  with  $p = 0, \dots, n_x - 1$ , and

$$I_{(-p,\tilde{\alpha})} z^{-p-1} dz, \quad -I_{(q+1,\tilde{\alpha})} z^q dz, \quad (31c)$$

for  $p = 0, \dots, n_0 - 1$  and  $q = 0, \dots, n_\infty - 2$ . Note that the factor of  $1/2T$  (resp.  $1/T$ ) in the differentials at points  $x \in \mathbf{z}_c$  (resp.  $x \in \mathbf{z}_r^k$ ) stems from the corresponding factor of  $2T/|\Pi_x| = 2T$  (resp.  $2T/|\Pi_x| = T$ ) in the definition of the bilinear form (26).

**3.4.2. Complexification of  $\tilde{\mathfrak{g}}^{\mathbb{D}}$ .** In the next section we shall also make use of an explicit description of the complexification of  $\tilde{\mathfrak{g}}^{\mathbb{D}}$  defined in Proposition 3.10 below. First we need the following lemma.

Given any complex Lie algebra  $\mathfrak{a}$  we denote by  $\bar{\mathfrak{a}}$  the complex conjugate Lie algebra, namely the realification  $\mathfrak{a}_{\mathbb{R}}$  of  $\mathfrak{a}$  endowed with the opposite complex structure given by  $i \cdot v = -iv$  for any  $v \in \bar{\mathfrak{a}}$ .

**Lemma 3.9.** *Let  $\mathfrak{a}$  be a complex Lie algebra,  $\tau \in \overline{\text{Aut}}_- \mathfrak{a}$  an anti-linear involutive automorphism and  $\mathfrak{a}^\tau := \{a \in \mathfrak{a} \mid \tau a = a\}$  the corresponding real form of  $\mathfrak{a}$ . We have the following isomorphisms of complex Lie algebras:*

- (i)  $\psi_c : \mathfrak{a} \oplus \bar{\mathfrak{a}} \xrightarrow{\sim} \mathfrak{a} \otimes_{\mathbb{R}} \mathbb{C}$ ,  $(a, b) \mapsto \frac{1}{2}(a \otimes 1 - ia \otimes i + b \otimes 1 + ib \otimes i)$ , under which the anti-linear involution  $\mathfrak{a} \otimes_{\mathbb{R}} \mathbb{C} \rightarrow \mathfrak{a} \otimes_{\mathbb{R}} \mathbb{C}$ ,  $a \otimes u \mapsto a \otimes \bar{u}$  corresponds to the exchange  $\mathfrak{a} \oplus \bar{\mathfrak{a}} \rightarrow \mathfrak{a} \oplus \bar{\mathfrak{a}}$ ,  $(a, b) \mapsto (b, a)$ .
- (ii)  $\psi_r : \mathfrak{a} \xrightarrow{\sim} \mathfrak{a}^\tau \otimes_{\mathbb{R}} \mathbb{C}$ ,  $a \mapsto \frac{1}{2}(a + \tau a) \otimes 1 - \frac{1}{2}i(a - \tau a) \otimes i$ , under which the anti-linear involution  $\mathfrak{a}^\tau \otimes_{\mathbb{R}} \mathbb{C} \rightarrow \mathfrak{a}^\tau \otimes_{\mathbb{R}} \mathbb{C}$ ,  $a \otimes u \mapsto a \otimes \bar{u}$  corresponds to  $\tau : \mathfrak{a} \rightarrow \mathfrak{a}$ .

*Proof.* One checks that  $\psi_c$  is  $\mathbb{C}$ -linear and that  $\phi_c : \mathfrak{a} \otimes_{\mathbb{R}} \mathbb{C} \rightarrow \mathfrak{a} \oplus \bar{\mathfrak{a}}$ ,  $a \otimes u \mapsto (ua, \bar{u}a)$  is its inverse. Moreover, the latter is seen to be a homomorphism of complex Lie algebras. It also follows from the form of the isomorphism  $\psi_c$  that complex conjugation on the second tensor factor in  $\mathfrak{a} \otimes_{\mathbb{R}} \mathbb{C}$  sends  $(a, b)$  to  $(b, a)$  in  $\mathfrak{a} \oplus \bar{\mathfrak{a}}$ . This proves (i).

Next, the map  $\psi_r$  is  $\mathbb{C}$ -linear since it is  $\mathbb{R}$ -linear and  $\psi_r(ia) = i\psi_r(a)$  for all  $a \in \mathfrak{a}$ . Its inverse is given by  $\phi_r : \mathfrak{a}^\tau \otimes_{\mathbb{R}} \mathbb{C} \rightarrow \mathfrak{a}$ ,  $a \otimes u \mapsto ua$ . The latter is clearly a homomorphism of complex Lie algebras and the remaining claim in (ii) follows from the explicit form of the isomorphism  $\psi_r$ .  $\square$

The following important proposition is a direct application of Lemma 3.9.

**Proposition 3.10.** *Let  $\tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathbb{D}} := \tilde{\mathfrak{g}}^{\mathbb{D}} \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{x \in \mathbf{z}} \tilde{\mathfrak{g}}^{n_x x} \otimes_{\mathbb{R}} \mathbb{C}$  be the complexification of  $\tilde{\mathfrak{g}}^{\mathbb{D}}$  and denote by  $c : \tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathbb{D}} \rightarrow \tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathbb{D}}$  the complex conjugation in the second tensor factor.*



We have an isomorphism of complex Lie algebras

$$\psi : \bigoplus_{x \in \mathbf{z}_f} (\mathcal{J}_x^{n_x} \tilde{\mathfrak{g}})^\Gamma \oplus \bigoplus_{x \in \mathbf{z}'_r} \mathcal{J}_x^{n_x} \tilde{\mathfrak{g}} \oplus \bigoplus_{x \in \mathbf{z}_c} (\mathcal{J}_x^{n_x} \tilde{\mathfrak{g}} \oplus \overline{\mathcal{J}_x^{n_x} \tilde{\mathfrak{g}}}) \xrightarrow{\sim} \tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}. \quad (32)$$

More precisely, we have the following isomorphisms of complex Lie algebras:

(i) For every  $x \in \mathbf{z}_c$ ,  $\mathcal{J}_x^{n_x} \tilde{\mathfrak{g}} \oplus \overline{\mathcal{J}_x^{n_x} \tilde{\mathfrak{g}}} \xrightarrow{\sim} \tilde{\mathfrak{g}}^{n_x x} \otimes_{\mathbb{R}} \mathbb{C}$  given by

$$(\mathbf{X} \otimes \varepsilon_x^p, \mathbf{Y} \otimes \varepsilon_x^q) \mapsto \mathbf{X}_{[p]}^{(x)} + \mathbf{Y}_{[q]}^{(\bar{x})} := \frac{1}{2} ((\mathbf{X} \otimes \varepsilon_x^p) \otimes 1 - (i\mathbf{X} \otimes \varepsilon_x^p) \otimes i + (\mathbf{Y} \otimes \varepsilon_x^q) \otimes 1 + (i\mathbf{Y} \otimes \varepsilon_x^q) \otimes i). \quad (33a)$$

Moreover, we have  $c(\mathbf{X}_{[p]}^{(x)}) = \mathbf{X}_{[p]}^{(\bar{x})}$  for any  $x \in \mathbf{z}_c \cup \bar{\mathbf{z}}_c$ .

(ii) For every  $x \in \mathbf{z}'_r$  with  $k \in \mathbb{Z}_T$ ,  $\mathcal{J}_x^{n_x} \tilde{\mathfrak{g}} \xrightarrow{\sim} \tilde{\mathfrak{g}}^{n_x x} \otimes_{\mathbb{R}} \mathbb{C}$  given by

$$\mathbf{X} \otimes \varepsilon_x^p \mapsto \mathbf{X}_{[p]}^{(x)} := (\pi_{k,p}^+ \mathbf{X} \otimes \varepsilon_x^p) \otimes 1 - i(\pi_{k,p}^- \mathbf{X} \otimes \varepsilon_x^p) \otimes i. \quad (33b)$$

Moreover,  $c(\mathbf{X}_{[p]}^{(x)}) = (\omega^{-kp} \sigma^k \tau \mathbf{X})_{[p]}^{(x)}$ . In particular,  $c(\mathbf{X}_{[p]}^{(x)}) = \mathbf{X}_{[p]}^{(x)}$  for  $\mathbf{X} \in \tilde{\mathfrak{g}}_{k,p}$ .

(iii) For every  $x \in \mathbf{z}_f$ ,  $(\mathcal{J}_x^{n_x} \tilde{\mathfrak{g}})^\Gamma \xrightarrow{\sim} \tilde{\mathfrak{g}}^{n_x x} \otimes_{\mathbb{R}} \mathbb{C}$  given by

$$\mathbf{X} \otimes \varepsilon_x^p \mapsto \mathbf{X}_{[p]}^{(x)} := (\pi_0^+ \mathbf{X} \otimes \varepsilon_x^p) \otimes 1 - i(\pi_0^- \mathbf{X} \otimes \varepsilon_x^p) \otimes i, \quad (33c)$$

with  $\mathbf{X} \in \tilde{\mathfrak{g}}_{(p),\mathbb{C}}$  if  $x = 0$  and  $\mathbf{X} \in \tilde{\mathfrak{g}}_{(-p),\mathbb{C}}$  if  $x = \infty$ . Moreover,  $c(\mathbf{X}_{[p]}^{(x)}) = (\tau \mathbf{X})_{[p]}^{(x)}$ . In particular,  $c(\mathbf{X}_{[p]}^{(0)}) = \mathbf{X}_{[p]}^{(0)}$  for  $\mathbf{X} \in \tilde{\mathfrak{g}}_{(p)}$  and  $c(\mathbf{X}_{[p]}^{(\infty)}) = \mathbf{X}_{[p]}^{(\infty)}$  for  $\mathbf{X} \in \tilde{\mathfrak{g}}_{(-p)}$ .

Using the notation of Proposition 3.10, a basis for  $\tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}} = \tilde{\mathfrak{g}}^{\mathcal{D}} \otimes_{\mathbb{R}} \mathbb{C}$  is given by

$$I_{[p]}^{\tilde{\alpha}(x)} := (I^{\tilde{\alpha}})_{[p]}^{(x)}, \quad I_{[p]}^{\tilde{\alpha}(\bar{x})} := (I^{\tilde{\alpha}})_{[p]}^{(\bar{x})}, \quad (34a)$$

for  $x \in \mathbf{z}_c$  with  $p = 0, \dots, n_x - 1$ ,

$$I_{k[p]}^{\tilde{\alpha}(x)} := (I_{k,p}^{\tilde{\alpha}})_{[p]}^{(x)} = (I_{k,p}^{\tilde{\alpha}} \otimes \varepsilon_x^p) \otimes 1, \quad (34b)$$

for  $x \in \mathbf{z}'_r$  with  $k \in \mathbb{Z}_T$ ,  $p = 0, \dots, n_x - 1$ , and

$$\begin{aligned} I_{[p]}^{\tilde{\alpha}(0)} &:= (I^{(p,\tilde{\alpha})})_{[p]}^{(0)} = (I^{(p,\tilde{\alpha})} \otimes \varepsilon_0^p) \otimes 1, \\ I_{[q+1]}^{\tilde{\alpha}(\infty)} &:= (I^{(-q-1,\tilde{\alpha})})_{[q+1]}^{(\infty)} = (I^{(-q-1,\tilde{\alpha})} \otimes \varepsilon_\infty^{q+1}) \otimes 1, \end{aligned} \quad (34c)$$

for  $p = 0, \dots, n_0 - 1$  and  $q = 0, \dots, n_\infty - 2$ .

We will use the notation  $I_{n,[p]}^{a(x)} := (I_n^a)_{[p]}^{(x)}$  and  $I_{a,n,[p]}^{(x)} := (I_{a,n})_{[p]}^{(x)}$  for any  $x \in \mathbf{z}_c \cup \bar{\mathbf{z}}_c \cup \mathbf{z}'_r$ ,  $p = 0, \dots, n_x - 1$ ,  $a = 1, \dots, \dim \mathfrak{g}$  and  $n \in \mathbb{Z}$ . Likewise, for the origin we write  $I_{n,[p]}^{\alpha(0)} := (I_n^{(p,\alpha)})_{[p]}^{(0)}$  and  $I_{\alpha,n,[p]}^{(0)} := (I_{(p,\alpha),n})_{[p]}^{(0)}$ , for any  $p = 0, \dots, n_0 - 1$ ,  $\alpha = 1, \dots, \dim \mathfrak{g}_{(p),\mathbb{C}}$  and  $n \in \mathbb{Z}$ , and for the point at infinity  $I_{n,[q+1]}^{\alpha(\infty)} := (I_n^{(-q-1,\alpha)})_{[q+1]}^{(\infty)}$  and  $I_{\alpha,n,[q+1]}^{(\infty)} := (I_{(-q-1,\alpha),n})_{[q+1]}^{(\infty)}$ , for  $q = 0, \dots, n_\infty - 2$ ,  $\alpha = 1, \dots, \dim \mathfrak{g}_{(-q-1),\mathbb{C}}$  and  $n \in \mathbb{Z}$ .

## 4. CLASSICAL DIHEDRAL AFFINE GAUDIN MODELS

Let  $\tilde{\mathfrak{g}}$  be an affine Kac-Moody algebra equipped with an action  $r : \Pi \rightarrow \overline{\text{Aut}} \tilde{\mathfrak{g}}$  of the dihedral group  $\Pi$  by (anti-)linear automorphisms as in §2 and let us fix a divisor  $\mathcal{D} = \sum_{x \in \mathcal{Z}} n_x x \in \text{Div}_{\geq 1}(\mathcal{Z})$ , cf. §3.4. We introduce the set of points  $\mathcal{Z} := \mathcal{z}_c \cup \bar{\mathcal{z}}_c \cup \mathcal{z}_r$ , and let  $\mathcal{Z}' := \mathcal{Z} \setminus \{0, \infty\} = \mathcal{z}_c \cup \bar{\mathcal{z}}_c \cup \mathcal{z}'_r$ . Let  $\ell$  be a tuple of complex numbers

$$\ell_0^x, \dots, \ell_{n_x-1}^x, \quad \text{for each } x \in \mathcal{Z}', \quad \ell_0^0, \dots, \ell_{n_0-1}^0, \quad \text{and} \quad \ell_1^\infty, \dots, \ell_{n_\infty-1}^\infty. \quad (35)$$

We require that  $\ell_p^0 = \ell_p^\infty = 0$  for  $p \not\equiv 0 \pmod{T}$ . Note that the latter condition on  $p$  is never satisfied when  $T = 1$ , so that in this case none of the levels at the origin and infinity are required to vanish. Throughout this section we will always assume that

$$\ell_{n_x-1}^x \neq 0 \quad (36)$$

for all  $x \in \mathcal{Z}$ . In particular, we assume that  $n_0, n_\infty \equiv 1 \pmod{T}$ . The condition (36) will be important in the discussion of §4.1.2. However, we will show in §5.3 how this assumption can be relaxed on the example of affine  $\tilde{\mathfrak{g}}$ -Toda field theory where such a condition does not hold. By convention, we set  $\ell_p^x = 0$  for every  $x \in \mathcal{Z}$  and  $p \geq n_x$ . We refer to the tuple  $\ell$  as the *levels* and require them to satisfy the following reality conditions:

$$\overline{\ell_p^x} = \ell_p^{\bar{x}} \quad \text{for } x \in \mathcal{z}_c \cup \bar{\mathcal{z}}_c, \quad (37a)$$

$$\overline{\ell_p^x} = \omega^{-kp} \ell_p^x \quad \text{for } x \in \mathcal{z}_r^k, \quad (37b)$$

$$\ell_p^x \in \mathbb{R} \quad \text{for } x \in \mathcal{z}_f. \quad (37c)$$

In this section we associate a classical dihedral affine Gaudin model to the datum  $(\tilde{\mathfrak{g}}, r, \mathcal{D}, \pi_\ell)$  where  $\pi_\ell$  is a homomorphism depending on the tuple of levels  $\ell$ , from the complexified algebra of formal observables to the complexified algebra of local observables, both introduced in §4.1 below. We will construct an explicit such homomorphism in §4.1.2 under the assumption that the condition (36) is satisfied. In §5.3 we will also give a definition of  $\pi_\ell$  when (36) fails to hold, on the example of affine  $\tilde{\mathfrak{g}}$ -Toda field theory.

**4.1. Algebra of formal observables.** Let  $S(\tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})$  denote the symmetric algebra on  $\tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}$ . The Lie bracket on  $\tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}$  uniquely extends to a Poisson bracket on  $S(\tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})$ , which we denote

$$\{\cdot, \cdot\} : S(\tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}) \times S(\tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}) \longrightarrow S(\tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}). \quad (38)$$

Explicitly, we require  $1 \in S(\tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})$  to lie in the centre of (38) and set  $\{\mathfrak{X}, \mathfrak{Y}\} := [\mathfrak{X}, \mathfrak{Y}]$  for any  $\mathfrak{X}, \mathfrak{Y} \in \tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}$ , then use linearity and the Leibniz rule to uniquely define  $\{f, g\}$  for every  $f, g \in S(\tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})$ .

The commutative Poisson algebra  $S(\tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})$  is not large enough for our purposes. For instance it does not contain the quadratic Hamiltonians constructed in §4.5 below. A suitable completion of it is defined as follows. Recalling the descending  $\mathbb{Z}_{\geq 0}$ -filtration on  $\tilde{\mathfrak{g}}$  defined in §2.2.2, to every  $n \in \mathbb{Z}_{\geq 0}$  and each finite point  $x \in \mathcal{z} \setminus \{\infty\}$  we associate the subspace  $F_n(\mathcal{J}_x^{n_x} \tilde{\mathfrak{g}}) := F_n \tilde{\mathfrak{g}} \otimes \mathbb{C}[\varepsilon_x] / \varepsilon_x^{n_x} \mathbb{C}[\varepsilon_x] \subset \mathcal{J}_x^{n_x} \tilde{\mathfrak{g}}$ , and for the point at infinity we define the subspaces  $F_n(\mathcal{J}_\infty^{n_\infty} \tilde{\mathfrak{g}}) := F_n \tilde{\mathfrak{g}} \otimes \varepsilon_\infty \mathbb{C}[\varepsilon_\infty] / \varepsilon_\infty^{n_\infty} \mathbb{C}[\varepsilon_\infty] \subset \mathcal{J}_\infty^{n_\infty} \tilde{\mathfrak{g}}$ . This defines a descending  $\mathbb{Z}_{\geq 0}$ -filtration on  $\mathcal{J}_x^{n_x} \tilde{\mathfrak{g}}$  for each  $x \in \mathcal{z}$ . At every  $x \in \mathcal{z}_c$ , let us also introduce

a ‘conjugate’ descending  $\mathbb{Z}_{\geq 0}$ -filtration on  $\mathcal{J}_x^{n_x} \tilde{\mathfrak{g}}$ , which we denote as  $(\bar{F}_n(\mathcal{J}_x^{n_x} \tilde{\mathfrak{g}}))_{n \in \mathbb{Z}_{\geq 0}}$ , defined by the subspaces,  $\bar{F}_n(\mathcal{J}_x^{n_x} \tilde{\mathfrak{g}}) := \bar{F}_n \tilde{\mathfrak{g}} \otimes \mathbb{C}[\varepsilon_x] / \varepsilon_x^{n_x} \mathbb{C}[\varepsilon_x] \subset \mathcal{J}_x^{n_x} \tilde{\mathfrak{g}}$ . For any  $x \in \mathfrak{z}_f$ , the action (29) of  $\Gamma \subset \Pi$  on  $\mathcal{J}_x^{n_x} \tilde{\mathfrak{g}}$  preserves the respective subspaces  $F_n(\mathcal{J}_x^{n_x} \tilde{\mathfrak{g}})$ ,  $n \in \mathbb{Z}_{\geq 0}$  so we may consider the  $\Gamma$ -invariant subspaces  $(F_n(\mathcal{J}_x^{n_x} \tilde{\mathfrak{g}}))^\Gamma$ . We can now define a descending  $\mathbb{Z}_{\geq 0}$ -filtration on the complex vector space  $\tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}$  introduced in Proposition 3.10 using the isomorphism (32). Specifically, we let

$$F_n(\tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}) := \psi \left( \bigoplus_{x \in \mathfrak{z}_f} (F_n(\mathcal{J}_x^{n_x} \tilde{\mathfrak{g}}))^\Gamma \oplus \bigoplus_{x \in \mathfrak{z}'_f} F_n(\mathcal{J}_x^{n_x} \tilde{\mathfrak{g}}) \oplus \bigoplus_{x \in \mathfrak{z}_c} (F_n(\mathcal{J}_x^{n_x} \tilde{\mathfrak{g}}) \oplus \overline{F_n(\mathcal{J}_x^{n_x} \tilde{\mathfrak{g}})}) \right),$$

for each  $n \in \mathbb{Z}_{\geq 0}$ . Recalling the fact that the subspaces  $F_n \tilde{\mathfrak{g}}$  define a descending  $\mathbb{Z}_{\geq 0}$ -filtration on  $\tilde{\mathfrak{g}}$  as a complex Lie algebra and that the linear map  $\psi$  from Proposition 3.10 is an isomorphism of complex Lie algebras, we have  $[F_m(\tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}), F_n(\tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})] \subset F_{m+n}(\tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})$  for every  $m, n \in \mathbb{Z}_{\geq 0}$ . It follows that  $(F_n(\tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}))_{n \in \mathbb{Z}_{\geq 0}}$  defines a descending  $\mathbb{Z}_{\geq 0}$ -filtration on  $\tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}$  as a complex Lie algebra. In turn, using this we set

$$F_n(S(\tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})) := F_n(\tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}) S(\tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}) \cap c(F_n(\tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})) S(\tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}), \quad (39)$$

where  $c : \tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}} \rightarrow \tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}$  is the anti-linear map introduced in Proposition 3.10. This defines a descending  $\mathbb{Z}_{\geq 0}$ -filtration on the commutative algebra  $S(\tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})$  by ideals. It therefore follows that the corresponding completion, which we denote by  $\hat{S}(\tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}) := \varprojlim S(\tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}) / F_n(S(\tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}))$  and call the *complexified algebra of formal observables*, is a commutative  $\mathbb{C}$ -algebra. We note that

$$c(F_n(\tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})) = \psi \left( \bigoplus_{x \in \mathfrak{z}_f} (\bar{F}_n(\mathcal{J}_x^{n_x} \tilde{\mathfrak{g}}))^\Gamma \oplus \bigoplus_{x \in \mathfrak{z}'_f} \bar{F}_n(\mathcal{J}_x^{n_x} \tilde{\mathfrak{g}}) \oplus \bigoplus_{x \in \mathfrak{z}_c} (\bar{F}_n(\mathcal{J}_x^{n_x} \tilde{\mathfrak{g}}) \oplus \overline{F_n(\mathcal{J}_x^{n_x} \tilde{\mathfrak{g}})}) \right)$$

for each  $n \in \mathbb{Z}_{\geq 0}$ , which defines the ‘conjugate’ descending  $\mathbb{Z}_{\geq 0}$ -filtration on  $\tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}$  as a complex Lie algebra.

Although the  $F_n(S(\tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}))$  are not Poisson ideals of  $S(\tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})$ , the Poisson bracket (38) is continuous at the origin with respect to the associated topology, where  $S(\tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}) \times S(\tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})$  is given the product topology, since for all  $m, n \in \mathbb{Z}_{\geq 0}$  we have  $\{F_m(S(\tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})), F_n(S(\tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}))\} \subset F_{\min(m,n)}(S(\tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}))$ . By linearity it follows that the Poisson bracket (38) is uniformly continuous and hence extends to a Poisson bracket

$$\{\cdot, \cdot\} : \hat{S}(\tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}) \times \hat{S}(\tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}) \longrightarrow \hat{S}(\tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}). \quad (40)$$

on the completion  $\hat{S}(\tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})$ , which is therefore also a Poisson algebra. We note here that the completion of the Cartesian product with respect to the product topology is the Cartesian product of the completions.

We extend the anti-linear automorphism  $c : \tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}} \rightarrow \tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}$  defined in Proposition 3.10 to an anti-linear automorphism of the Poisson algebra  $S(\tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})$ . Since it preserves each subspace  $F_n(S(\tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}))$ ,  $n \in \mathbb{Z}_{\geq 0}$  of the descending  $\mathbb{Z}_{\geq 0}$ -filtration on  $S(\tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})$  introduced in (39), by construction of the latter, it follows that the map  $c$  is continuous with respect to the associated topology. It therefore extends to an anti-linear automorphism of the

completion  $\hat{S}(\tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})$  which we still denote  $c$ . Hence we can consider the real subalgebra  $\hat{S}(\tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})^c$  of fixed points under  $c$  which we refer to as the *algebra of formal observables*.

In §4.1.1 below we use the levels (35) to define another Poisson algebra  $\hat{S}_{\ell}(\hat{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})$  and in §4.1.2 we construct a homomorphism of Poisson algebras  $\pi_{\ell} : \hat{S}(\tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}) \rightarrow \hat{S}_{\ell}(\hat{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})$ .

4.1.1. *Algebra of local observables.* Recall the Lie subalgebra  $\hat{\mathfrak{g}}$  of  $\tilde{\mathfrak{g}}$  from §2.2. Let  $\hat{\mathfrak{g}}^{\mathcal{D}}$  be the real Lie subalgebra of  $\tilde{\mathfrak{g}}^{\mathcal{D}}$  defined in the same way as  $\tilde{\mathfrak{g}}^{\mathcal{D}}$  in §3.4.1 with  $\hat{\mathfrak{g}}$  replacing  $\tilde{\mathfrak{g}}$ . Its complexification  $\hat{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}} := \hat{\mathfrak{g}}^{\mathcal{D}} \otimes_{\mathbb{R}} \mathbb{C}$  is a subalgebra of  $\tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}$ . In particular, the Poisson bracket (38) restricts to the symmetric algebra  $S(\hat{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})$  on  $\hat{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}$  which is thus a Poisson subalgebra of  $S(\tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})$ .

Let  $J_{\ell}$  denote the ideal of  $S(\hat{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})$  generated by the elements

$$K_{[p]}^{(x)} - i\ell_p^x 1, \quad K_{[q]}^{(y)} + i\ell_q^y 1, \quad K_{[r]}^{(0)} - i\ell_r^0 1, \quad K_{[s+1]}^{(\infty)} - i\ell_{s+1}^{\infty} 1$$

for every  $x \in \mathfrak{z}_c \cup \mathfrak{z}_r'$  with  $p = 0, \dots, n_x - 1$ , every  $y \in \bar{\mathfrak{z}}_c$  with  $q = 0, \dots, n_y - 1$  and  $r, s + 1 \equiv 0 \pmod{T}$ . Since these elements all lie in the centre of the Poisson bracket (38),  $J_{\ell}$  is also a Poisson ideal of  $S(\hat{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})$ . The quotient  $S_{\ell}(\hat{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}) := S(\hat{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})/J_{\ell}$  is therefore a Poisson algebra whose induced Poisson bracket we denote

$$\{\cdot, \cdot\} : S_{\ell}(\hat{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}) \times S_{\ell}(\hat{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}) \longrightarrow S_{\ell}(\hat{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}). \quad (41)$$

Let  $\hat{\mathfrak{g}}_{\mathbb{C},\ell}^{\mathcal{D}}$  be the image of the Lie subalgebra  $\hat{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}} \subset S(\hat{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})$  under the quotient map  $S(\hat{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}) \rightarrow S_{\ell}(\hat{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})$ . By a slight abuse of notation we will denote the image in  $\hat{\mathfrak{g}}_{\mathbb{C},\ell}^{\mathcal{D}}$  of an element  $X_{[p]}^{(x)} \in \hat{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}$  with  $X \in \mathcal{L}\mathfrak{g}$  by the same symbol.

The descending  $\mathbb{Z}_{>0}$ -filtration on the subalgebra  $\hat{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}$  inherited from  $\tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}$  is simply given by  $F_n(\hat{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}) = \hat{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}} \cap F_n(\tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}) = F_n(\tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})$  for each  $n \in \mathbb{Z}_{\geq 0}$ . Let  $F_n(\hat{\mathfrak{g}}_{\mathbb{C},\ell}^{\mathcal{D}})$  denote the subspaces of the induced descending  $\mathbb{Z}_{\geq 0}$ -filtration on  $\hat{\mathfrak{g}}_{\mathbb{C},\ell}^{\mathcal{D}}$ . The ideals  $F_n(S_{\ell}(\hat{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})) := F_n(\hat{\mathfrak{g}}_{\mathbb{C},\ell}^{\mathcal{D}})S_{\ell}(\hat{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}) \cap c(F_n(\hat{\mathfrak{g}}_{\mathbb{C},\ell}^{\mathcal{D}}))S_{\ell}(\hat{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})$ , for  $n \in \mathbb{Z}_{\geq 0}$ , define a descending  $\mathbb{Z}_{\geq 0}$ -filtration on the Poisson algebra  $S_{\ell}(\hat{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})$ . As in the above discussion for  $\hat{S}(\tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})$ , since the Poisson bracket (41) is uniformly continuous with respect to the associated topology, it follows that the corresponding completion  $\hat{S}_{\ell}(\hat{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}) := \varprojlim S_{\ell}(\hat{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})/F_n(S_{\ell}(\hat{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}))$ , the *complexified algebra of local observables*, is also a Poisson algebra over  $\mathbb{C}$ , whose Poisson bracket we denote

$$\{\cdot, \cdot\} : \hat{S}_{\ell}(\hat{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}) \times \hat{S}_{\ell}(\hat{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}) \longrightarrow \hat{S}_{\ell}(\hat{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}). \quad (42)$$

The restriction of  $c : \tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}} \rightarrow \tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}$  to  $\hat{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}} \subset \tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}$  extends as an anti-linear automorphism to the Poisson algebra  $S(\hat{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})$ . Using the reality conditions (37) on the tuple of levels  $\ell$  and the properties of the isomorphism  $\psi$  with regards to complex conjugation  $c$  given in Proposition 3.10, it follows that the ideal  $J_{\ell}$  is invariant under  $c$ . Hence  $c$  acts on the quotient  $S_{\ell}(\hat{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})$ . And since it preserves each of the subspaces  $F_n(S_{\ell}(\hat{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}))$ , it is continuous with respect to the associated topology and so extends to the completion  $\hat{S}_{\ell}(\hat{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})$ . We then define the *algebra of local observables* of the classical dihedral affine Gaudin model as the real subalgebra  $\hat{S}_{\ell}(\hat{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})^c$  of fixed points.

4.1.2. *Fixing the levels.* Let  $\mathcal{C}(n)$  denote the set of compositions of  $n$  if  $n \in \mathbb{Z}_{\geq 0}$  or the empty set if  $n \in \mathbb{Z}_{< 0}$ . By convention the empty composition is the only composition of 0, *i.e.*  $\mathcal{C}(0) = \{\emptyset\}$ . Given a composition  $c \in \mathcal{C}(n)$  for some  $n \in \mathbb{Z}_{\geq 0}$  we denote by  $|c|$  its length and by  $c_j \neq 0$ ,  $j = 1, \dots, |c|$  its parts so that  $c_1 + \dots + c_{|c|} = n$ .

**Lemma 4.1.** *Let  $n \in \mathbb{Z}_{\geq 1}$  and  $\ell_p \in \mathbb{C}$ ,  $p \in \mathbb{Z}_{\geq 0}$  be such that  $\ell_{n-1} \neq 0$  and  $\ell_p = 0$  for all  $p \geq n$ . Consider the system of linear equations for  $\kappa_p$ ,  $p = 0, \dots, 2n - 2$  given by  $\sum_{p \geq 0} \kappa_{p+q} \ell_{p+s} = \delta_{q,s}$  with  $q, s = 0, \dots, n - 1$ . It has the unique solution*

$$\kappa_p = \kappa_p(\ell_0, \dots, \ell_{n-1}) := \sum_{c \in \mathcal{C}(p-n+1)} (-1)^{|c|} \frac{\prod_{j=1}^{|c|} \ell_{n-1-c_j}}{(\ell_{n-1})^{|c|+1}}. \quad (43)$$

Moreover, if  $n \equiv 1 \pmod{T}$  and  $\ell_p = 0$  for  $p \not\equiv 0 \pmod{T}$  then  $\kappa_p = 0$  for  $p \not\equiv 0 \pmod{T}$ .

We define a linear map

$$\pi_{\ell} : \widetilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}} \longrightarrow \hat{S}_{\ell}(\widehat{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}) \quad (44a)$$

as follows. For any  $x \in \mathbf{z}_c \cup \mathbf{z}'_r$  and  $r = 0, \dots, n_x - 1$  we let

$$\begin{aligned} \pi_{\ell}(\mathbf{D}_{[r]}^{(x)}) &:= \frac{i}{2} \sum_{p,q \geq 0} \sum_{n \in \mathbb{Z}} \kappa_{p+q-r}^x I_{a,-n,[p]}^{(x)} I_{n,[q]}^{a(x)}, \\ \pi_{\ell}(\mathbf{K}_{[r]}^{(x)}) &:= i \ell_r^x 1, \quad \pi_{\ell}(I_{n,[r]}^{a(x)}) := I_{n,[r]}^{a(x)}, \end{aligned} \quad (44b)$$

where  $\kappa_p^x := \kappa_p(\ell_0^x, \dots, \ell_{n_x-1}^x)$  for  $p = 0, \dots, 2n_x - 2$  is given by Lemma 4.1. Note that the assumption (36) is used here for satisfying the conditions of the lemma. For any  $y \in \bar{\mathbf{z}}_c$  and  $r = 0, \dots, n_y - 1$  we let

$$\begin{aligned} \pi_{\ell}(\mathbf{D}_{[r]}^{(y)}) &:= -\frac{i}{2} \sum_{p,q \geq 0} \sum_{n \in \mathbb{Z}} \kappa_{p+q-r}^y I_{a,-n,[p]}^{(y)} I_{n,[q]}^{a(y)}, \\ \pi_{\ell}(\mathbf{K}_{[r]}^{(y)}) &:= -i \ell_r^y 1, \quad \pi_{\ell}(I_{n,[r]}^{a(y)}) := I_{n,[r]}^{a(y)}, \end{aligned} \quad (44c)$$

where  $\kappa_p^y := \kappa_p(\ell_0^y, \dots, \ell_{n_y-1}^y)$  for  $p = 0, \dots, 2n_y - 2$  is given again by Lemma 4.1. In particular, it follows from the reality conditions (37a) and the definition (43) that  $\overline{\kappa_p^x} = \kappa_p^{\bar{x}}$  for any  $x \in \mathbf{z}_c \cup \bar{\mathbf{z}}_c$ . At the origin, for any  $r = 0, \dots, n_0 - 1$  with  $r \equiv 0 \pmod{T}$  and any  $s = 0, \dots, n_0 - 1$  we let

$$\begin{aligned} \pi_{\ell}(\mathbf{D}_{[r]}^{(0)}) &:= \frac{i}{2} \sum_{p,q \geq 0} \sum_{n \in \mathbb{Z}} \kappa_{p+q-r}^0 I_{\alpha,-n,[p]}^{(0)} I_{n,[q]}^{\alpha(0)}, \\ \pi_{\ell}(\mathbf{K}_{[r]}^{(0)}) &:= i \ell_r^0 1, \quad \pi_{\ell}(I_{n,[s]}^{\alpha(0)}) := I_{n,[s]}^{\alpha(0)}, \end{aligned} \quad (44d)$$

where  $\kappa_p^0 := \kappa_p(\ell_0^0, \dots, \ell_{n_0-1}^0)$  for  $p = 0, \dots, 2n_0 - 2$  is determined by Lemma 4.1. Here we used again the assumption that  $\ell_{n_0-1}^0 \neq 0$  from (36). We note that  $\kappa_p^0 = 0$  whenever  $p \not\equiv 0 \pmod{T}$ , using the last part of Lemma 4.1, so that the double sum over  $p, q \geq 0$  restricts to  $p$  and  $q$  satisfying  $p + q \equiv 0 \pmod{T}$  and hence the implicit summation over the repeated index  $\alpha = 1, \dots, \dim \mathfrak{g}_{(p),\mathbb{C}} = \dim \mathfrak{g}_{(q),\mathbb{C}}$  in (44d) makes

sense. Finally, at infinity we set, for  $r = 0, \dots, n_\infty - 2$  with  $r + 1 \equiv 0 \pmod T$  and any  $s = 0, \dots, n_\infty - 2$ ,

$$\begin{aligned}\pi_\ell(\mathbf{D}_{[r+1]}^{(\infty)}) &:= \frac{i}{2} \sum_{p,q \geq 0} \sum_{n \in \mathbb{Z}} \kappa_{p+q-r+1}^\infty I_{\alpha, -n, [p+1]}^{(\infty)} I_{n, [q+1]}^{\alpha(\infty)}, \\ \pi_\ell(\mathbf{K}_{[r+1]}^{(\infty)}) &:= i\ell_{r+1}^\infty 1, \quad \pi_\ell(I_{n, [s+1]}^{\alpha(\infty)}) := I_{n, [s+1]}^{\alpha(\infty)},\end{aligned}\tag{44e}$$

where  $\kappa_{p+1}^\infty := \kappa_{p+1}(0, \ell_1^\infty, \dots, \ell_{n_\infty-1}^\infty)$  for each  $p = 0, \dots, 2n_\infty - 4$ . Note that  $\kappa_{p+1}(\ell, \ell_1^\infty, \dots, \ell_{n_\infty-1}^\infty)$  does not depend on  $\ell$  whenever  $p \leq 2n_\infty - 4$ , as can be seen directly from the explicit formula (43). Now from Lemma 4.1 we have  $\kappa_{p+q-r+1}^\infty = 0$  unless  $p + q - r + 1 \equiv 0 \pmod T$ , or in other words  $p + q + 2 \equiv 0 \pmod T$ , so that the implicit sum over  $\alpha = 1, \dots, \dim \mathfrak{g}_{(-p-1), \mathbb{C}} = \dim \mathfrak{g}_{(-q-1), \mathbb{C}}$  in (44e) makes sense.

*Example 4.2.* Let  $\mathcal{D} = \sum_{x \in \mathbb{P}^1} n_x x \in \text{Div}_{\geq 1}(\mathbf{z})$ . If  $x \in \mathbf{z} \setminus \{0, \infty\}$  is such that  $n_x = 1$  then

$$\pi_\ell(\mathbf{D}_{[0]}^{(x)}) = \frac{i}{2\ell_0^x} \sum_{n \in \mathbb{Z}} I_{a, -n, [0]}^{(x)} I_{n, [0]}^{a(x)}.\tag{45}$$

This coincides with the standard expression for the classical Segal-Sugawara operator  $L_0$  in terms of the Kac-Moody algebra generators  $I_{n, [0]}^{a(x)}$ . If  $x \in \mathbf{z} \setminus \{0, \infty\}$  is such that  $n_x = 2$  then instead we have

$$\begin{aligned}\pi_\ell(\mathbf{D}_{[0]}^{(x)}) &= \frac{i}{\ell_1^x} \sum_{n \in \mathbb{Z}} I_{a, -n, [0]}^{(x)} I_{n, [1]}^{a(x)} - \frac{i\ell_0^x}{2(\ell_1^x)^2} \sum_{n \in \mathbb{Z}} I_{a, -n, [1]}^{(x)} I_{n, [1]}^{a(x)}, \\ \pi_\ell(\mathbf{D}_{[1]}^{(x)}) &= \frac{i}{2\ell_1^x} \sum_{n \in \mathbb{Z}} I_{a, -n, [1]}^{(x)} I_{n, [1]}^{a(x)}.\end{aligned}\tag{46}$$

Expression (46) is the classical analogue of the generalised Segal-Sugawara operator  $L_0$  constructed using the generators  $I_{n, [0]}^{a(x)}$  and  $I_{n, [1]}^{a(x)}$  of the generalised Takiff algebra for  $\widehat{\mathfrak{g}}$ , whose quantum counterpart can be found in [3, Theorem 1].  $\triangleleft$

Recall the definition of the anti-linear automorphisms  $c$  of  $\widetilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}$  and  $\widehat{S}_\ell(\widetilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})$  defined in Proposition 3.10 and §4.1.1 respectively. The proof of the next proposition is a lengthy but straightforward case by case check from the definition of the map (44), so we omit it.

**Proposition 4.3.** *The map (44) is a homomorphism of Lie algebras which commutes with  $c$ .*

We extend the map (44) to a homomorphism  $\pi_\ell : S(\widetilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}) \rightarrow \widehat{S}_\ell(\widetilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})$  of commutative algebras. It follows from Proposition 4.3 that the latter is in fact a homomorphism of Poisson algebras. For each  $n \in \mathbb{Z}_{\geq 0}$ , the image of  $F_n(S(\widetilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}))$ , defined in (39), under this homomorphism lies in  $F_n(\widehat{S}_\ell(\widetilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})) := \ker(\widehat{S}_\ell(\widetilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}) \twoheadrightarrow S_\ell(\widetilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})/F_n(S_\ell(\widetilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})))$  so that it extends by continuity to a homomorphism

$$\pi_\ell : \widehat{S}(\widetilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}) \longrightarrow \widehat{S}_\ell(\widetilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}).\tag{47}$$

## 4.2. Fields.

4.2.1. *Formal fields.* Let  $\mathfrak{F}$  denote the completion of the tensor product of complex vector spaces  $\tilde{\mathfrak{g}} \otimes \tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}$  with respect to the descending  $\mathbb{Z}_{\geq 0}$ -filtration given by

$$F_n(\tilde{\mathfrak{g}} \otimes \tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}) := F_n \tilde{\mathfrak{g}} \otimes c(F_n \tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}) + \bar{F}_n \tilde{\mathfrak{g}} \otimes F_n(\tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}) \quad (48)$$

for all  $n \in \mathbb{Z}_{\geq 0}$ . We refer to the first tensor factor in  $\mathfrak{F}$  as the *auxiliary factor* and the second tensor factor as the *operator factor*.

We gather the basis (34) of  $\tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}$  together into a finite collection of elements of  $\mathfrak{F}$ , which we will call the *formal fields* of the dihedral affine Gaudin model, defined as

$$\mathcal{A}_{[p]}^x := I_{\tilde{\alpha}} \otimes I_{[p]}^{\tilde{\alpha}(x)}, \quad \mathcal{A}_{[p]}^{\bar{x}} := \tau I_{\tilde{\alpha}} \otimes I_{[p]}^{\tilde{\alpha}(\bar{x})}, \quad \text{for } x \in \mathbf{z}_c \quad (49a)$$

$$\mathcal{A}_{[p]}^x := I_{\tilde{\alpha}; k, -p} \otimes I_{k[p]}^{\tilde{\alpha}(x)}, \quad \text{for } x \in \mathbf{z}_r^k, k \in \mathbb{Z}_T, \quad (49b)$$

where in each case  $p = 0, \dots, n_x - 1$ , and

$$\mathcal{A}_{[p]}^0 := I_{(-p, \tilde{\alpha})} \otimes I_{[p]}^{\tilde{\alpha}(0)}, \quad \mathcal{A}_{[q+1]}^{\infty} := I_{(q+1, \tilde{\alpha})} \otimes I_{[q+1]}^{\tilde{\alpha}(\infty)}. \quad (49c)$$

with  $p = 0, \dots, n_0 - 1$  and  $q = 0, \dots, n_{\infty} - 2$ . Note in particular that for each  $x \in \mathbf{Z}$  we have  $\mathcal{A}_{[p]}^x = 0$  whenever  $p \geq n_x$ . In order to see that the formal field (49b) indeed defines an element of  $\mathfrak{F}$  it is enough to note that by a  $\mathbb{C}$ -linear change of basis, recalling the definition (13), we may rewrite it equivalently as  $\mathcal{A}_{[p]}^x = I_{\tilde{\alpha}} \otimes I_{[p]}^{\tilde{\alpha}(x)}$  making use of the notation introduced in (33b). Similar considerations apply to the formal fields (49c) at 0 and  $\infty$ .

We equip  $\tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}} = \tilde{\mathfrak{g}}^{\mathcal{D}} \otimes_{\mathbb{R}} \mathbb{C}$  with an action of  $\Pi$  by letting  $\mathfrak{s}$  act trivially and letting  $\mathfrak{t}$  act by complex conjugation on the second tensor factor, cf. Proposition 3.10. Combining this with the action (10) of  $\Pi$  on  $\tilde{\mathfrak{g}}$  we obtain an action on the tensor product  $\tilde{\mathfrak{g}} \otimes \tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}$ , cf. (11). Now each subspace  $F_n(\tilde{\mathfrak{g}} \otimes \tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})$  given by (48) is preserved by this action of  $\Pi$ . In other words,  $\Pi$  acts on  $\tilde{\mathfrak{g}} \otimes \tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}$  by continuous (anti-)linear maps. We extend this by continuity to an action of  $\Pi$  on  $\mathfrak{F}$ , which we denote by

$$r : \Pi \longrightarrow \overline{\text{Aut}} \mathfrak{F}, \quad \alpha \longmapsto r_{\alpha}. \quad (50)$$

In order to reflect the fact that the original action of  $\Pi$  on  $\tilde{\mathfrak{g}}$ , as given in (10), was defined in terms of the pair of automorphisms  $\sigma, \tau \in \overline{\text{Aut}} \tilde{\mathfrak{g}}$ , we will sometimes also write  $r_{\mathfrak{s}}$  and  $r_{\mathfrak{t}}$  simply as  $\sigma$  and  $\tau$ , respectively.

Let  $\tilde{\mathfrak{g}} \hat{\otimes} \tilde{\mathfrak{g}} \hat{\otimes} \tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}$  denote the completion of the tensor product  $\tilde{\mathfrak{g}} \otimes \tilde{\mathfrak{g}} \otimes \tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}$  with respect to the descending  $\mathbb{Z}_{\geq 0}$ -filtration whose subspace  $F_n(\tilde{\mathfrak{g}} \hat{\otimes} \tilde{\mathfrak{g}} \hat{\otimes} \tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})$ ,  $n \in \mathbb{Z}_{\geq 0}$  is defined as in §A.2. The following proposition describes identities in  $\tilde{\mathfrak{g}} \hat{\otimes} \tilde{\mathfrak{g}} \hat{\otimes} \tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}$  so that in the notation of §A.2 we have  $\mathfrak{a} = \mathfrak{b} = \tilde{\mathfrak{g}}$ ,  $\mathfrak{c} = \tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}$  and  $\mathfrak{A} = \mathbb{C}$  (hence we do not include the superfluous tensor factor in  $\mathfrak{A}$ ). Note that although the elements in the third tensor factor all belong to the Lie algebra  $\tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}} \subset \hat{S}(\tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})$ , we prefer to use the Poisson bracket notation (40) for their Lie bracket to emphasise that they ought be regarded as ‘linear’ functions in the complexified algebra of formal observables  $\hat{S}(\tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})$ . Moreover, in order to conform to the standard notation, cf. Corollary 4.5 below, we shall also drop the tensor index  $_3$  corresponding to the operator factor, in the terminology introduced in §4.2.1. For instance, instead of  $\mathcal{A}_{[p]_{13}}^x$  we shall simply write  $\mathcal{A}_{[p]_1}^x$ .

**Proposition 4.4.** *The collection of non-zero Poisson brackets between the elements of  $\mathfrak{F}$  defined in (49) reads*

$$\{\mathcal{A}_{[p]_1}^x, \mathcal{A}_{[q]_2}^x\} = -[\tilde{C}_{12}, \mathcal{A}_{[p+q]_2}^x], \quad (51a)$$

for each  $x \in \mathfrak{Z}'$  with  $p, q = 0, \dots, n_x - 1$ , and

$$\{\mathcal{A}_{[p]_1}^0, \mathcal{A}_{[q]_2}^0\} = -[\tilde{C}_{12}^{(p)}, \mathcal{A}_{[p+q]_2}^0], \quad \{\mathcal{A}_{[r+1]_1}^\infty, \mathcal{A}_{[s+1]_2}^\infty\} = -[\tilde{C}_{12}^{(-r-1)}, \mathcal{A}_{[r+s+2]_2}^\infty], \quad (51b)$$

for  $p, q = 0, \dots, n_0 - 1$  and  $r, s = 0, \dots, n_\infty - 2$ .

*Proof.* From its definition in Proposition 3.10,  $\tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}$  is a direct sum of complexified Lie algebras  $\tilde{\mathfrak{g}}^{n_{xx}} \otimes_{\mathbb{R}} \mathbb{C}$  attached to each point  $x \in \mathfrak{z}$ . It follows from the definition of the formal fields (49) and using Proposition 3.10 that  $\{\mathcal{A}_{[p]_1}^x, \mathcal{A}_{[q]_2}^y\} = 0$  for any distinct  $x$  and  $y$  in  $\mathfrak{Z}$ . To show (51a) for  $x \in \mathfrak{z}_c$ , note that for such  $x$  we have

$$I_{\tilde{a}} \otimes I_{\tilde{b}} \otimes [I_{[p]}^{\tilde{a}(x)}, I_{[q]}^{\tilde{b}(x)}] = I_{\tilde{a}} \otimes I_{\tilde{b}} \otimes [I_{\tilde{a}}, I_{\tilde{b}}]_{[p+q]}^{(x)} = -I_{\tilde{a}} \otimes [I_{\tilde{a}}, I_{\tilde{b}}] \otimes I_{[p+q]}^{\tilde{b}(x)}, \quad (52)$$

using first the fact that  $\mathbf{X} \otimes \varepsilon_x^p \mapsto \mathbf{X}_{[p]}^{(x)}$  is a homomorphism of complex Lie algebras, since it is a composition of the isomorphism (33a) with the natural embedding  $\mathcal{T}_x^{n_x} \tilde{\mathfrak{g}} \hookrightarrow \mathcal{T}_x^{n_x} \tilde{\mathfrak{g}} \oplus \overline{\mathcal{T}_x^{n_x} \tilde{\mathfrak{g}}}$ , followed by Lemma 2.3.

The relation (51a) for  $x \in \bar{\mathfrak{z}}_c$  simply follows from applying the anti-linear involution  $\tau \otimes \tau \otimes c$  to the equality (51a) for  $x \in \mathfrak{z}_c$  and noting that  $\tau I_{\tilde{a}} \otimes \tau I_{\tilde{b}} = \tilde{C}_{12}$ , by the  $\tau$ -invariance (12) of the bilinear form on  $\tilde{\mathfrak{g}}$ .

In order to prove (51a) holds for  $x \in \mathfrak{z}'$ , recall that the formal field at such a point can be rewritten as  $\mathcal{A}_{[p]}^x = I_{\tilde{a}} \otimes I_{[p]}^{\tilde{a}(x)}$ . The result then follows by the same calculation as above in (52), using the fact that (33b) is a homomorphism of complex Lie algebras followed by Lemma 2.3.

The first relation in (51b) for the formal fields at the origin follow from a similar calculation, namely

$$\begin{aligned} I_{(-p, \tilde{\alpha})} \otimes I_{(-q, \tilde{\beta})} \otimes [I_{[p]}^{\tilde{\alpha}(0)}, I_{[q]}^{\tilde{\beta}(0)}] &= I_{(-p, \tilde{\alpha})} \otimes I_{(-q, \tilde{\beta})} \otimes [I^{(p, \tilde{\alpha})}, I^{(q, \tilde{\beta})}]_{[p+q]}^{(0)} \\ &= -I_{(-p, \tilde{\alpha})} \otimes [I^{(p, \tilde{\alpha})}, I_{(-p-q, \tilde{\beta})}] \otimes I_{[p+q]}^{\tilde{\beta}(0)} \end{aligned}$$

where in the first step we used the fact that (33c) with  $x = 0$  is a homomorphism of complex Lie algebras and in the last line we used the second part of Lemma 2.3. The proof of the second relation in (51b) is analogous.  $\square$

**4.2.2. Classical fields.** Recall the homomorphisms of Lie algebras  $\varrho$  and  $\pi_\ell$  introduced in §2.2.3 and §4.1, respectively. Consider their tensor product, namely the linear map  $\varrho \otimes \pi_\ell : \tilde{\mathfrak{g}} \otimes \tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}} \rightarrow \text{Conn}_{\mathfrak{g}}(S^1) \otimes \hat{S}_\ell(\tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})$ . Given the descending  $\mathbb{Z}_{\geq 0}$ -filtration on its domain with the subspaces (48), we endow its codomain with the image descending  $\mathbb{Z}_{\geq 0}$ -filtration whose subspaces are defined by  $(\varrho \otimes \pi_\ell)(\mathbf{F}_n(\tilde{\mathfrak{g}} \otimes \tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}))$  for each  $n \in \mathbb{Z}_{\geq 0}$ . Let  $\text{Conn}_{\mathfrak{g}}(S^1) \hat{\otimes} \hat{S}_\ell(\tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})$  denote the corresponding completion. By construction, the map  $\varrho \otimes \pi_\ell$  is then continuous at the origin and hence uniformly continuous by linearity so that it extends to the respective completions  $\varrho \otimes \pi_\ell : \mathfrak{F} \rightarrow \text{Conn}_{\mathfrak{g}}(S^1) \hat{\otimes} \hat{S}_\ell(\tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})$ .



To describe the image of the formal fields defined in (49) under the homomorphism  $\varrho \otimes \pi_\ell$ , it is convenient to introduce the following elements of the inverse limit  $\mathcal{T}(S^1) \hat{\otimes} \widehat{\mathfrak{g}}_{\mathbb{C},\ell}^{\mathbb{D}} := \varprojlim (\mathcal{T}(S^1) \otimes \widehat{\mathfrak{g}}_{\mathbb{C},\ell}^{\mathbb{D}}) / \mathbb{F}_n(\mathcal{T}(S^1) \otimes \widehat{\mathfrak{g}}_{\mathbb{C},\ell}^{\mathbb{D}})$  where  $\mathbb{F}_n(\mathcal{T}(S^1) \otimes \widehat{\mathfrak{g}}_{\mathbb{C},\ell}^{\mathbb{D}}) := \mathbb{F}_n(\mathcal{T}(S^1)) \otimes c(\mathbb{F}_n(\widehat{\mathfrak{g}}_{\mathbb{C},\ell}^{\mathbb{D}})) + \overline{\mathbb{F}}_n(\mathcal{T}(S^1)) \otimes \mathbb{F}_n(\widehat{\mathfrak{g}}_{\mathbb{C},\ell}^{\mathbb{D}})$  for all  $n \in \mathbb{Z}_{\geq 0}$ . To every point  $x \in \mathfrak{z}_c \cup \mathfrak{z}'_r$  and  $y \in \bar{\mathfrak{z}}_c$  we associate the *classical fields*  $A_{[p]}^{a,x} := \sum_{n \in \mathbb{Z}} e_{-n} \otimes I_{n,[p]}^{a(x)}$  and  $A_{[q]}^{a,y} := \sum_{n \in \mathbb{Z}} e_n \otimes I_{n,[q]}^{a(y)}$ , labelled by  $a = 1, \dots, \dim \mathfrak{g}$ ,  $p = 0, \dots, n_x - 1$  and  $q = 0, \dots, n_y - 1$ . Here we use the notation from §3.4.2. Likewise, to the origin and the point at infinity we associate the *classical fields*  $A_{[p]}^{\alpha,0} := \sum_{n \in \mathbb{Z}} e_{-n} \otimes I_{n,[p]}^{\alpha(0)}$ , and  $A_{[q+1]}^{\beta,\infty} := \sum_{n \in \mathbb{Z}} e_{-n} \otimes I_{n,[q+1]}^{\beta(\infty)}$ , labelled by  $\alpha = 1, \dots, \dim \mathfrak{g}_{(p),\mathbb{C}}$ ,  $\beta = 1, \dots, \dim \mathfrak{g}_{(-q-1),\mathbb{C}}$ ,  $p = 0, \dots, n_0 - 1$  and  $q = 0, \dots, n_\infty - 2$ . We then gather all these classical fields at each point in  $\mathfrak{Z}$  into  *$\mathfrak{g}$ -valued classical fields*, belonging to  $\mathcal{T}(S^1, \mathfrak{g}) \hat{\otimes} \widehat{\mathfrak{g}}_{\mathbb{C},\ell}^{\mathbb{D}} := \mathfrak{g} \otimes \mathcal{T}(S^1) \hat{\otimes} \widehat{\mathfrak{g}}_{\mathbb{C},\ell}^{\mathbb{D}}$  and defined by

$$A_{[p]}^x := I_a \otimes A_{[p]}^{a,x}, \quad A_{[q]}^0 := I_{(-q,\alpha)} \otimes A_{[q]}^{\alpha,0}, \quad A_{[r+1]}^\infty := I_{(r+1,\alpha)} \otimes A_{[r+1]}^{\alpha,\infty}, \quad (53)$$

for all  $x \in \mathfrak{Z}'$ ,  $p = 0, \dots, n_x - 1$ ,  $q = 0, \dots, n_0 - 1$  and  $r = 0, \dots, n_\infty - 2$ . We may now describe the image of the formal fields (49) under  $\varrho \otimes \pi_\ell$  as  $\mathfrak{g}$ -valued connections on  $S^1$  with components given by the  $\mathfrak{g}$ -valued classical fields in (53), namely

$$(\varrho \otimes \pi_\ell) \mathcal{A}_{[p]}^x = \ell_p^x \partial + A_{[p]}^x, \quad (\varrho \otimes \pi_\ell) \mathcal{A}_{[q]}^0 = \ell_q^0 \partial + A_{[q]}^0, \quad (\varrho \otimes \pi_\ell) \mathcal{A}_{[r+1]}^\infty = \ell_{r+1}^\infty \partial + A_{[r+1]}^\infty, \quad (54)$$

where we suppressed the tensor product with  $1 \in \hat{S}_\ell(\widehat{\mathfrak{g}}_{\mathbb{C}}^{\mathbb{D}})$  in the derivative terms. We recall here that  $I_a$  is by definition a basis of the real form  $\mathfrak{g}_0$ , cf. §2.1.1, so that  $\tau I_a = I_a$ .

Recall the Dirac  $\delta$ -distribution  $\delta_{\theta\theta'}$  introduced in §2.2.3. In what follows we shall also need its derivative  $\delta' := \sum_{n \in \mathbb{Z}} e'_n \otimes e_{-n} = \sum_{n \in \mathbb{Z}} i n e_n \otimes e_{-n} \in \mathcal{T}(S^1) \hat{\otimes} \mathcal{T}(S^1)$ , with the property that for any  $\theta \in S^1$ ,  $\delta'(\theta, \cdot)$  is a distribution on  $S^1$  sending a test function  $f \in \mathcal{T}(S^1)$  to  $f'(\theta)$ . We use the shorthand notation  $\delta'_{\theta\theta'}$  for  $\delta'(\theta, \theta')$ .

**Corollary 4.5.** *The non-trivial Poisson brackets between the  $\mathfrak{g}$ -valued classical fields (53) read*

$$\{A_{[p]_1}^x(\theta), A_{[q]_2}^x(\theta')\} = -[C_{12}, A_{[p+q]_2}^x(\theta)] \delta_{\theta\theta'} - \ell_{p+q}^x C_{12} \delta'_{\theta\theta'},$$

for each  $x \in \mathfrak{Z}'$  with  $p, q = 0, \dots, n_x - 1$ , and

$$\begin{aligned} \{A_{[p]_1}^0(\theta), A_{[q]_2}^0(\theta')\} &= -[C_{12}^{(p)}, A_{[p+q]_2}^0(\theta)] \delta_{\theta\theta'} - \ell_{p+q}^0 C_{12}^{(p)} \delta'_{\theta\theta'}, \\ \{A_{[r+1]_1}^\infty(\theta), A_{[s+1]_2}^\infty(\theta')\} &= -[C_{12}^{(-r-1)}, A_{[r+s+2]_2}^\infty(\theta)] \delta_{\theta\theta'} - \ell_{r+s+2}^\infty C_{12}^{(-r-1)} \delta'_{\theta\theta'}, \end{aligned}$$

for  $p, q = 0, \dots, n_0 - 1$  and  $r, s = 0, \dots, n_\infty - 2$ .

*Proof.* This follows from applying the tensor product of Lie algebra homomorphisms  $\varrho \otimes \varrho \otimes \pi_\ell$  to the identities in Proposition 4.4, and using (54) together with (17) and the analogous relation for  $\tilde{C}^{(j)}$ . We also make use of the fact that  $\partial_\theta \delta_{\theta\theta'} = -\delta'_{\theta\theta'}$ .  $\square$

**4.3. Canonical element.** Recall the dual pair  $(\Omega_{\mathcal{D}}^{\Pi}(\tilde{\mathfrak{g}}), \tilde{\mathfrak{g}}^{\mathcal{D}}, \langle\langle \cdot, \cdot \rangle\rangle)$  of Proposition 3.8. Given the dual bases  $\{I^A\}$  of  $\tilde{\mathfrak{g}}^{\mathcal{D}}$  and  $\{I_A\}$  of  $\Omega_{\mathcal{D}}^{\Pi}(\tilde{\mathfrak{g}})$  introduced in (30) and (31) respectively, we can consider the corresponding canonical element  $I_A \otimes I^A$  living in a suitable completion of the tensor product  $\Omega_{\mathcal{D}}^{\Pi}(\tilde{\mathfrak{g}}) \otimes_{\mathbb{R}} \tilde{\mathfrak{g}}^{\mathcal{D}}$  over  $\mathbb{R}$  defined below. It can be naturally rewritten as a linear combination of formal fields from §4.2.1 with coefficients given by meromorphic differentials on  $\mathbb{P}^1$ . This expression, in Proposition 4.6, will be related to the (formal) Lax matrix of the dihedral affine Gaudin model.

By regarding  $\tilde{\mathfrak{g}} \otimes_{\mathbb{R}} \tilde{\mathfrak{g}}^{\mathcal{D}}$  as a complex vector space with scalar multiplication acting in the first tensor factor, we get an isomorphism of complex vector spaces  $\tilde{\mathfrak{g}} \otimes_{\mathbb{R}} \tilde{\mathfrak{g}}^{\mathcal{D}} \cong_{\mathbb{C}} \tilde{\mathfrak{g}} \otimes \tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}$ . In turn, taking the tensor product with  $\Omega_{\Pi\mathcal{D}}$  induces a linear isomorphism

$$\zeta : \Omega_{\Pi\mathcal{D}}(\tilde{\mathfrak{g}}) \otimes_{\mathbb{R}} \tilde{\mathfrak{g}}^{\mathcal{D}} \xrightarrow{\sim} \Omega_{\Pi\mathcal{D}}(\tilde{\mathfrak{g}} \otimes \tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}), \quad (55)$$

where we made use of the notation (27). Given any  $\mathcal{X} \in \tilde{\mathfrak{g}}$ ,  $\varpi \in \Omega_{\Pi\mathcal{D}}$  and  $\mathcal{Y} \in \tilde{\mathfrak{g}}^{\mathcal{D}}$ , the linear map in (55) is given explicitly by  $\zeta(\mathcal{X} \otimes \varpi \otimes_{\mathbb{R}} \mathcal{Y}) = \mathcal{X} \otimes (\mathcal{Y} \otimes_{\mathbb{R}} 1) \otimes \varpi$ , where here we denote the tensor products over  $\mathbb{R}$  explicitly by a subscript. Recalling the action of  $\Pi$  on the subspace  $\Omega_{\Pi\mathcal{D}}(\tilde{\mathfrak{g}}) \subset \Omega_{\Pi\mathbb{z}}(\tilde{\mathfrak{g}})$  defined by (24), we extend it to an action on  $\Omega_{\Pi\mathcal{D}}(\tilde{\mathfrak{g}}) \otimes_{\mathbb{R}} \tilde{\mathfrak{g}}^{\mathcal{D}}$  by letting it act trivially on the second tensor factor. The map (55) can then be made  $\Pi$ -equivariant if we define the action of  $\Pi$  on  $\Omega_{\Pi\mathcal{D}}(\tilde{\mathfrak{g}} \otimes \tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})$  through the homomorphism  $\Pi \hookrightarrow \overline{\text{Aut}} \Omega_{\Pi\mathcal{D}}(\tilde{\mathfrak{g}} \otimes \tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})$ ,  $\alpha \mapsto \hat{\alpha}$  given explicitly by  $\hat{\alpha}(\mathcal{X} \otimes \varpi) := r_{\alpha} \mathcal{X} \otimes \alpha \cdot \varpi$  for any  $\alpha \in \Pi$ ,  $\mathcal{X} \in \tilde{\mathfrak{g}} \otimes \tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}$  and  $\varpi \in \Omega_{\Pi\mathcal{D}}$ , where the action of  $\Pi$  on  $\tilde{\mathfrak{g}} \otimes \tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}$ , denoted here  $\alpha \mapsto r_{\alpha}$ , was defined in §4.2.1.

The linear isomorphism (55) becomes continuous if we equip its codomain with the descending  $\mathbb{Z}_{\geq 0}$ -filtration given by the subspaces  $\Omega_{\Pi\mathcal{D}}(F_n(\tilde{\mathfrak{g}} \otimes \tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}))$ , cf. (48), and its domain with the induced descending  $\mathbb{Z}_{\geq 0}$ -filtration whose subspaces are

$$F_n(\Omega_{\Pi\mathcal{D}}(\tilde{\mathfrak{g}}) \otimes_{\mathbb{R}} \tilde{\mathfrak{g}}^{\mathcal{D}}) := \zeta^{-1}(\Omega_{\Pi\mathcal{D}}(F_n(\tilde{\mathfrak{g}} \otimes \tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}))), \quad (56)$$

for  $n \in \mathbb{Z}_{\geq 0}$ . Let  $\Omega_{\Pi\mathcal{D}}(\tilde{\mathfrak{g}}) \hat{\otimes}_{\mathbb{R}} \tilde{\mathfrak{g}}^{\mathcal{D}}$  be the completion of the tensor product  $\Omega_{\Pi\mathcal{D}}(\tilde{\mathfrak{g}}) \otimes_{\mathbb{R}} \tilde{\mathfrak{g}}^{\mathcal{D}}$  with respect to (56). We note that the action of  $\Pi$  on  $\Omega_{\Pi\mathcal{D}}(\tilde{\mathfrak{g}}) \otimes_{\mathbb{R}} \tilde{\mathfrak{g}}^{\mathcal{D}}$  preserves the subspaces (56) and therefore extends to the completion  $\Omega_{\Pi\mathcal{D}}(\tilde{\mathfrak{g}}) \hat{\otimes}_{\mathbb{R}} \tilde{\mathfrak{g}}^{\mathcal{D}}$ . We obtain a  $\Pi$ -equivariant linear isomorphism

$$\zeta : \Omega_{\Pi\mathcal{D}}(\tilde{\mathfrak{g}}) \hat{\otimes}_{\mathbb{R}} \tilde{\mathfrak{g}}^{\mathcal{D}} \xrightarrow{\sim} \Omega_{\Pi\mathcal{D}}(\mathfrak{F}). \quad (57)$$

Here we have used the fact that the completion of the codomain of the isomorphism (55) with respect to the descending  $\mathbb{Z}_{\geq 0}$ -filtration with subspaces  $\Omega_{\Pi\mathcal{D}}(F_n(\tilde{\mathfrak{g}} \otimes \tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}))$  is  $\Omega_{\Pi\mathcal{D}}(\mathfrak{F})$ , where we use again the notation (27). Note that the action of  $\Pi$  on the latter is given by the homomorphism  $\Pi \hookrightarrow \overline{\text{Aut}} \Omega_{\Pi\mathcal{D}}(\mathfrak{F})$ ,  $\alpha \mapsto \hat{\alpha}$  defined explicitly by  $\hat{\alpha}(\mathcal{X} \otimes \varpi) := r_{\alpha} \mathcal{X} \otimes \alpha \cdot \varpi$  for any  $\alpha \in \Pi$ ,  $\mathcal{X} \in \mathfrak{F}$  and  $\varpi \in \Omega_{\Pi\mathcal{D}}$ , where the action of  $\Pi$  on  $\mathfrak{F}$  is given in (50).

Now the  $\Pi$ -invariant subspace  $(\Omega_{\Pi\mathcal{D}}(\tilde{\mathfrak{g}}) \hat{\otimes}_{\mathbb{R}} \tilde{\mathfrak{g}}^{\mathcal{D}})^{\Pi}$  of the completion  $\Omega_{\Pi\mathcal{D}}(\tilde{\mathfrak{g}}) \hat{\otimes}_{\mathbb{R}} \tilde{\mathfrak{g}}^{\mathcal{D}}$  coincides with the completion  $\Omega_{\mathcal{D}}^{\Pi}(\tilde{\mathfrak{g}}) \hat{\otimes}_{\mathbb{R}} \tilde{\mathfrak{g}}^{\mathcal{D}}$  of the tensor product  $\Omega_{\mathcal{D}}^{\Pi}(\tilde{\mathfrak{g}}) \otimes_{\mathbb{R}} \tilde{\mathfrak{g}}^{\mathcal{D}}$  with respect to the descending  $\mathbb{Z}_{\geq 0}$ -filtration defined by the subspaces  $F_n(\Omega_{\mathcal{D}}^{\Pi}(\tilde{\mathfrak{g}}) \otimes_{\mathbb{R}} \tilde{\mathfrak{g}}^{\mathcal{D}}) := \zeta^{-1}((\Omega_{\Pi\mathcal{D}}(F_n(\tilde{\mathfrak{g}} \otimes \tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})))^{\Pi})$ . If we define the subspace of  $\Pi$ -invariants  $\Omega_{\mathcal{D}}^{\Pi}(\mathfrak{F}) := \Omega_{\Pi\mathcal{D}}(\mathfrak{F})^{\Pi}$ , then the restriction of (57) to the subspace  $\Omega_{\mathcal{D}}^{\Pi}(\tilde{\mathfrak{g}}) \hat{\otimes}_{\mathbb{R}} \tilde{\mathfrak{g}}^{\mathcal{D}}$  yields a linear

isomorphism with  $\Omega_{\mathcal{D}}^{\Pi}(\mathfrak{F})$  which we denote by the same symbol, namely

$$\zeta : \Omega_{\mathcal{D}}^{\Pi}(\tilde{\mathfrak{g}}) \hat{\otimes}_{\mathbb{R}} \tilde{\mathfrak{g}}^{\mathcal{D}} \xrightarrow{\sim} \Omega_{\mathcal{D}}^{\Pi}(\mathfrak{F}). \quad (58)$$

**Proposition 4.6.** *The canonical element  $\Phi := I_A \otimes I^A$  of the dual pair  $(\Omega_{\mathcal{D}}^{\Pi}(\tilde{\mathfrak{g}}), \tilde{\mathfrak{g}}^{\mathcal{D}}, \langle \cdot, \cdot \rangle)$ , where the tensor product is over  $\mathbb{R}$  and the infinite sum over the repeated multi-index  $A$  is implicit, defines an element of the completion  $\Omega_{\mathcal{D}}^{\Pi}(\tilde{\mathfrak{g}}) \hat{\otimes}_{\mathbb{R}} \tilde{\mathfrak{g}}^{\mathcal{D}}$ . Its image under the linear isomorphism (58) reads*

$$\zeta(\Phi) = \frac{1}{T} \sum_{k=0}^{T-1} \hat{s}^k \left[ \sum_{x \in \mathcal{Z}'} \sum_{p \geq 0} \frac{\mathcal{A}_{[p]}^x}{(z-x)^{p+1}} dz \right] + \sum_{p \geq 0} \mathcal{A}_{[p]}^0 z^{-p-1} dz - \sum_{q \geq 0} \mathcal{A}_{[q+1]}^{\infty} z^q dz. \quad (59)$$

*Proof.* The proof of both statements, namely that  $\Phi$  lives in  $\Omega_{\mathcal{D}}^{\Pi}(\tilde{\mathfrak{g}}) \hat{\otimes}_{\mathbb{R}} \tilde{\mathfrak{g}}^{\mathcal{D}}$  and that its image under (58) is of the form (59), rely on very similar computations. Since our main focus is to prove (59), we choose to begin by proving the second statement assuming that the infinite sum over  $A$  in  $\Phi$  makes sense and then comment on the proof of the first. We work separately on the dual basis elements of  $\tilde{\mathfrak{g}}^{\mathcal{D}}$  and  $\Omega_{\mathcal{D}}^{\Pi}(\tilde{\mathfrak{g}})$  associated with points in  $z_c, z'_r$  and  $z_f$ .

Consider first basis elements associated with a complex point  $x \in z_c$ . We may write the elements (31a) from the basis (31) of  $\Omega_{\mathcal{D}}^{\Pi}(\tilde{\mathfrak{g}})$  as

$$\begin{aligned} \frac{1}{2T} \sum_{\alpha \in \Pi} \hat{\alpha} \left( \frac{I_{\tilde{a}} dz}{(z-x)^{p+1}} \right) &= \frac{1}{2T} \sum_{k=0}^{T-1} \frac{\omega^{-k} \sigma^k I_{\tilde{a}} dz}{(\omega^{-k} z - x)^{p+1}} + \frac{1}{2T} \sum_{k=0}^{T-1} \frac{\omega^{-k} \sigma^k \tau I_{\tilde{a}} dz}{(\omega^{-k} z - \bar{x})^{p+1}}, \\ \frac{1}{2T} \sum_{\alpha \in \Pi} \hat{\alpha} \left( \frac{-i I_{\tilde{a}} dz}{(z-x)^{p+1}} \right) &= -\frac{1}{2T} \sum_{k=0}^{T-1} \frac{i \omega^{-k} \sigma^k I_{\tilde{a}} dz}{(\omega^{-k} z - x)^{p+1}} + \frac{1}{2T} \sum_{k=0}^{T-1} \frac{i \omega^{-k} \sigma^k \tau I_{\tilde{a}} dz}{(\omega^{-k} z - \bar{x})^{p+1}}. \end{aligned}$$

Taking their tensor product over  $\mathbb{R}$  with the corresponding dual basis elements of  $\tilde{\mathfrak{g}}^{\mathcal{D}}$  given in (30a) we obtain

$$\begin{aligned} &\left( \frac{1}{2T} \sum_{k=0}^{T-1} \frac{\omega^{-k} \sigma^k I_{\tilde{a}} dz}{(\omega^{-k} z - x)^{p+1}} + \frac{1}{2T} \sum_{k=0}^{T-1} \frac{\omega^{-k} \sigma^k \tau I_{\tilde{a}} dz}{(\omega^{-k} z - \bar{x})^{p+1}} \right) \otimes (I^{\tilde{a}} \otimes \varepsilon_x^p) \\ &+ \left( -\frac{1}{2T} \sum_{k=0}^{T-1} \frac{i \omega^{-k} \sigma^k I_{\tilde{a}} dz}{(\omega^{-k} z - x)^{p+1}} + \frac{1}{2T} \sum_{k=0}^{T-1} \frac{i \omega^{-k} \sigma^k \tau I_{\tilde{a}} dz}{(\omega^{-k} z - \bar{x})^{p+1}} \right) \otimes (i I^{\tilde{a}} \otimes \varepsilon_x^p), \quad (60) \end{aligned}$$

where the tensor product between the expressions in brackets is over  $\mathbb{R}$  and as usual the summation over repeated Lie algebra indices  $\tilde{a}$  is implicit. Applying the linear map (58) to the latter we may rewrite it as

$$\begin{aligned} &\frac{1}{T} \sum_{k=0}^{T-1} \frac{\omega^{-k} \sigma^k I_{\tilde{a}} dz}{(\omega^{-k} z - x)^{p+1}} \otimes I_{[p]}^{\tilde{a}(x)} + \frac{1}{T} \sum_{k=0}^{T-1} \frac{\omega^{-k} \sigma^k \tau I_{\tilde{a}} dz}{(\omega^{-k} z - \bar{x})^{p+1}} \otimes I_{[p]}^{\tilde{a}(\bar{x})} \\ &= \frac{1}{T} \sum_{k=0}^{T-1} \hat{s}^k \left[ \left( \frac{\mathcal{A}_{[p]}^x}{(z-x)^{p+1}} + \frac{\mathcal{A}_{[p]}^{\bar{x}}}{(z-\bar{x})^{p+1}} \right) dz \right]. \end{aligned}$$

To show that the infinite sum in (60) indeed defines an element of the completion  $\Omega_{\mathcal{D}}^{\Pi}(\tilde{\mathfrak{g}}) \hat{\otimes}_{\mathbb{R}} \tilde{\mathfrak{g}}^{\mathcal{D}}$ , consider the same expression but where instead of summing over the

index  $\tilde{a}$  we replace the dual basis elements  $I_{\tilde{a}}$  and  $I^{\tilde{a}}$  respectively by  $I_{a,-n}$  and  $I_n^a$  for some fixed  $n \in \mathbb{Z}$ . Then by applying the linear map in (55) to this we obtain an element of  $(\Omega_{\Pi\mathcal{D}}(\mathbf{F}_{|n|}(\tilde{\mathfrak{g}} \otimes \tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})))^{\Pi}$ . We therefore deduce that the original element of  $\Omega_{\mathcal{D}}^{\Pi}(\tilde{\mathfrak{g}}) \otimes_{\mathbb{R}} \tilde{\mathfrak{g}}^{\mathcal{D}}$  we started with lives in  $\mathbf{F}_{|n|}(\Omega_{\mathcal{D}}^{\Pi}(\tilde{\mathfrak{g}}) \otimes_{\mathbb{R}} \tilde{\mathfrak{g}}^{\mathcal{D}})$ , by definition of the latter. In particular, it follows that the infinite sum in (60) defines an element of the inverse limit  $\Omega_{\mathcal{D}}^{\Pi}(\tilde{\mathfrak{g}}) \hat{\otimes}_{\mathbb{R}} \tilde{\mathfrak{g}}^{\mathcal{D}}$ .

Next, for a non-fixed real point  $x \in \mathbf{z}_r^k$ , the elements (31b) from the basis (31) of  $\Omega_{\mathcal{D}}^{\Pi}(\tilde{\mathfrak{g}})$  read

$$\frac{1}{T} \sum_{\alpha \in \Gamma} \hat{\alpha} \left( \frac{I_{\tilde{a};k,-p} dz}{(z-x)^{p+1}} \right) = \frac{1}{T} \sum_{k=0}^{T-1} \frac{\omega^{-k} \sigma^k I_{\tilde{a};k,-p} dz}{(\omega^{-k} z - x)^{p+1}}.$$

Their tensor product with the elements  $I_{k,p}^{\tilde{a}} \otimes \varepsilon_x^p$  from the dual basis of  $\tilde{\mathfrak{g}}^{\mathcal{D}}$  can then be rewritten, after formally applying the linear map (58) and recalling the notation (34b), as

$$\frac{1}{T} \sum_{k=0}^{T-1} \frac{\omega^{-k} \sigma^k I_{\tilde{a};k,-p} dz}{(\omega^{-k} z - x)^{p+1}} \otimes I_{k[p]}^{\tilde{a}(x)} = \frac{1}{T} \sum_{k=0}^{T-1} \hat{s}^k \left[ \frac{\mathcal{A}_{[p]}^x}{(z-x)^{p+1}} dz \right].$$

The same argument as above applies to show that the terms in  $\Phi$  corresponding to a non-fixed real point  $x \in \mathbf{z}_r^k$  belong to the right completion. Specifically, we consider the tensor product

$$\frac{1}{T} \sum_{k=0}^{T-1} \frac{\omega^{-k} \sigma^k I_{\tilde{a};k,-p} dz}{(\omega^{-k} z - x)^{p+1}} \otimes (I_{k,p}^{\tilde{a}} \otimes \varepsilon_x^p)$$

over  $\mathbb{R}$ , where there is no implicit sum over  $\tilde{a}$  but instead  $I_{k,p}^{\tilde{a}}$  is a particular basis element from (13a) and  $I_{\tilde{a};k,-p}$  is its dual basis element from (13b). Applying the linear map (55) yields an element of  $(\Omega_{\Pi\mathcal{D}}(\mathbf{F}_n(\tilde{\mathfrak{g}} \otimes \tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})))^{\Pi}$  for some  $n \in \mathbb{Z}_{\geq 0}$ , from which it follows that the above belongs to  $\mathbf{F}_n(\Omega_{\mathcal{D}}^{\Pi}(\tilde{\mathfrak{g}}) \otimes_{\mathbb{R}} \tilde{\mathfrak{g}}^{\mathcal{D}})$  as required.

Finally, the tensor product over  $\mathbb{R}$  of the basis elements (31c) of  $\Omega_{\mathcal{D}}^{\Pi}(\tilde{\mathfrak{g}})$  with the basis elements (30c) of  $\tilde{\mathfrak{g}}^{\mathcal{D}}$  gives the last two terms in (59) after formally applying the linear map (58). The proof that we have in this case as well an element of the completion  $\Omega_{\mathcal{D}}^{\Pi}(\tilde{\mathfrak{g}}) \hat{\otimes}_{\mathbb{R}} \tilde{\mathfrak{g}}^{\mathcal{D}}$  is as above.  $\square$

#### 4.4. Classical

4.4.1. *Lax matrix algebra.* We define the *formal Lax matrix*  $\mathcal{L} \in \mathfrak{F} \otimes R_{\Pi z}$  by writing  $\zeta(\Phi) = \mathcal{L}(z)dz$ . Explicitly, from the expression (59) for  $\zeta(\Phi)$  we obtain

$$\mathcal{L}(z) = \frac{1}{T} \sum_{k=0}^{T-1} \sum_{x \in \mathfrak{Z}'} \sum_{p \geq 0} \frac{\omega^{-k} \sigma^k \mathcal{A}_{[p]}^x}{(\omega^{-k} z - x)^{p+1}} + \sum_{p \geq 0} \mathcal{A}_{[p]}^0 z^{-p-1} - \sum_{q \geq 0} \mathcal{A}_{[q+1]}^\infty z^q. \quad (62)$$

Since  $\Phi \in \Omega_{\mathcal{D}}^{\Pi}(\tilde{\mathfrak{g}}) \hat{\otimes}_{\mathbb{R}} \tilde{\mathfrak{g}}^{\mathcal{D}}$  by Proposition 4.6, its image  $\zeta(\Phi) \in \Omega_{\mathcal{D}}^{\Pi}(\mathfrak{F})$  under the isomorphism (58) is  $\Pi$ -invariant by construction. Recalling the action of  $\Pi$  on  $\Omega_{\Pi \mathcal{D}}(\mathfrak{F})$  in §4.3, it follows that the formal Lax matrix is  $\Pi$ -equivariant in the sense that  $\sigma \mathcal{L}(z) = \omega \mathcal{L}(\omega z)$  and  $\tau \mathcal{L}(z) = \mathcal{L}(\bar{z})$ .

In the following proposition we make use of the tensor index notation from §A.2 with  $\mathfrak{a} = \mathfrak{b} = \tilde{\mathfrak{g}}$ ,  $\mathfrak{c} = \tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}$  and where  $\mathfrak{A}$  is the algebra of rational functions on  $\mathbb{C}^2$ , with coordinates  $z, z'$ . Moreover, as in Proposition 4.4 we drop the index  $3$  for the operator factor and use the Poisson bracket (40) to denote its Lie bracket.

**Proposition 4.7.** *The bracket of the formal Lax matrix takes the form*

$$\{\mathcal{L}_1(z), \mathcal{L}_2(z')\} = [$$

where  $\mathfrak{A}$  denotes here the algebra of rational functions on  $\mathbb{C}^2$ , with coordinates  $z, z'$ . Applying to it the linear map  $\varrho \otimes \varrho \otimes \text{id}$  we obtain, cf. (17),

$$((\varrho \otimes \varrho \otimes \text{id}))$$

We extend (68) in the obvious way to a bilinear map  $(\cdot|\cdot) : \mathfrak{F} \otimes R_{\Pi\mathbf{z}} \times \mathfrak{F} \otimes R_{\Pi\mathbf{z}} \rightarrow \hat{S}(\widetilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}) \otimes R_{\Pi\mathbf{z}}$ , by sending the pair  $(\mathcal{X} \otimes f, \mathcal{Y} \otimes g)$  for any  $\mathcal{X}, \mathcal{Y} \in \mathfrak{F}$  and  $f, g \in R_{\Pi\mathbf{z}}$  to  $(\mathcal{X}|\mathcal{Y}) \otimes fg$ .

Recall the formal Lax matrix  $\mathcal{L} \in \mathfrak{F} \otimes R_{\Pi\mathbf{z}}$  from §4.4.1. The following proposition is to be compared with [88, Proposition 2.5] corresponding to the case of a quantum cyclotomic Gaudin model associated with a semisimple Lie algebra  $\mathfrak{g}$ .

**Proposition 4.9.** *We have*

$$\frac{1}{2}(\mathcal{L}(z)|\mathcal{L}(z)) = \frac{1}{T^2} \sum_{k=0}^{T-1} \sum_{x \in \mathfrak{Z}'} \sum_{p \geq 0} \frac{\omega^{k(p-1)} \mathcal{H}_p^x}{(z - \omega^k x)^{p+1}} + \sum_{\substack{p \geq 0 \\ p \equiv 1 \pmod{T}}} \mathcal{H}_p^0 z^{-p-1} + \sum_{\substack{q \geq 0 \\ q+2 \equiv 0 \pmod{T}}} \mathcal{H}_q^\infty z^q,$$

where the formal quadratic Gaudin Hamiltonians in  $\hat{S}(\widetilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})$  are given by

$$\begin{aligned} \mathcal{H}_p^x := & \sum_{\ell=0}^{T-1} \sum_{y \in \mathfrak{Z}' \cup \{0\}} \sum_{\substack{q, s \geq 0 \\ (\ell, y) \neq (0, x)}} \binom{q+s}{s} (-1)^s \frac{\omega^{\ell q} (\mathcal{A}_{[p+s]}^x | \sigma^\ell \mathcal{A}_{[q]}^y)}{(x - \omega^\ell y)^{q+s+1}} \\ & - T \sum_{q, s \geq 0} \binom{q}{s} x^{q-s} (\mathcal{A}_{[p+s]}^x | \mathcal{A}_{[q+1]}^\infty) + \frac{1}{2} \sum_{q=0}^{p-1} (\mathcal{A}_{[q]}^x | \mathcal{A}_{[p-q-1]}^x), \end{aligned} \quad (69a)$$

for all  $x \in \mathfrak{Z}'$  and  $p = 0, \dots, 2n_x - 1$ ,

$$\mathcal{H}_p^0 = \frac{1}{T} \sum_{x \in \mathfrak{Z}'} \sum_{q, s \geq 0} (-1)^{q+1} \binom{q+s}{s} \frac{(\mathcal{A}_{[p+s]}^0 | \mathcal{A}_{[q]}^x)}{x^{q+s+1}} - \sum_{q \geq 0} (\mathcal{A}_{[p+q]}^0 | \mathcal{A}_{[q+1]}^\infty) + \sum_{q=0}^{p-1} \frac{1}{2} (\mathcal{A}_{[p-q-1]}^0 | \mathcal{A}_{[q]}^0), \quad (69b)$$

for  $p = 0, \dots, 2n_0 - 1$  such that  $p \equiv 1 \pmod{T}$ , and

$$\mathcal{H}_q^\infty = - \sum_{x \in \mathfrak{Z}'} \sum_{p, s \geq 0} \binom{s}{p} x^{s-p} (\mathcal{A}_{[p]}^x | \mathcal{A}_{[q+s+2]}^\infty) - \sum_{p \geq 0} (\mathcal{A}_{[p]}^0 | \mathcal{A}_{[q+p+2]}^\infty) + \sum_{p=0}^q \frac{1}{2} (\mathcal{A}_{[p+1]}^\infty | \mathcal{A}_{[q-p+1]}^\infty), \quad (69c)$$

for all  $q = 0, \dots, 2n_\infty - 4$  such that  $q + 2 \equiv 0 \pmod{T}$ .

#### 4.5.1. Lax equations.

**Proposition 4.10.** *The formal quadratic Gaudin Hamiltonians from Proposition 4.9 Poisson commute, i.e.  $\{\mathcal{H}_p^x, \mathcal{H}_q^y\} = 0$ , for any pair of points  $x, y \in \mathfrak{Z}$  and  $p, q \in \mathbb{Z}_{\geq 0}$ . Moreover, we have the Lax equations  $\{\mathcal{H}_p^x, \mathcal{L}(z)\} = [\mathcal{M}_p^x(z), \mathcal{L}(z)]$  for every  $x \in \mathfrak{Z}$ , where*

$$\mathcal{M}_p^x(z) := \sum_{\ell=0}^{T-1} \sum_{s \geq 0} \frac{\sigma^\ell \mathcal{A}_{[p+s]}^x}{(\omega^{-\ell} z - x)^{s+1}}, \quad \mathcal{M}_q^0(z) := \sum_{s \geq 0} \mathcal{A}_{[q+s]}^0 z^{-s-1}, \quad \mathcal{M}_r^\infty(z) := \sum_{s \geq 0} \mathcal{A}_{[r+s+2]}^\infty z^s$$

for all  $x \in \mathfrak{Z}'$  and for  $p = 0, \dots, 2n_x - 1$ ,  $q = 0, \dots, 2n_0 - 1$  and  $r = 0, \dots, 2n_\infty - 4$  with  $q \equiv 1 \pmod{T}$  and  $r + 2 \equiv 0 \pmod{T}$ .

We define the *local quadratic Gaudin Hamiltonians* in  $\hat{S}_\ell(\widehat{\mathfrak{g}}_{\mathbb{C}}^{\mathbb{D}})$  as  $H_p^x := \pi_\ell(\mathcal{H}_p^x)$  and  $H_q^\infty := \pi_\ell(\mathcal{H}_q^\infty)$  for all  $x \in \mathfrak{Z} \setminus \{\infty\}$  with  $p = 0, \dots, 2n_x - 1$  and  $q = 0, \dots, 2n_\infty - 4$ .

Following the discussion in §4.4.2, we apply the linear map  $\varrho \otimes \pi_\ell \otimes \text{id}$  to the element  $\mathcal{M}_p^x \in \mathfrak{F} \otimes R_{\Pi\mathfrak{z}}$  from Proposition 4.10. It follows from the assumption (36) made at the start of this section and the explicit form of  $\mathcal{M}_p^x$  in Proposition 4.10 that the result takes the form, cf. (64),

$$(\varrho \otimes \pi_\ell \otimes \text{id})\mathcal{M}_p^x(z) = \varphi_p^x(z)(\partial + \mathcal{M}_p^x(z)) \in \text{Conn}_{\mathfrak{g}}(S^1) \hat{\otimes} \widehat{\mathfrak{g}}_{\mathbb{C},\ell}^{\mathbb{D}} \otimes R_{\Pi\mathfrak{z}},$$

where  $\varphi_p^x(z) \in R_{\Pi\mathfrak{z}}$  is not identically zero provided  $p \leq n_x - 1$  when  $x \in \mathfrak{Z} \setminus \{\infty\}$  and  $p \leq n_\infty - 3$  for the point at infinity. Here  $\varphi_p^x(z)\mathcal{M}_p^x(z) \in \mathcal{T}(S^1, \mathfrak{g}) \hat{\otimes} \widehat{\mathfrak{g}}_{\mathbb{C},\ell}^{\mathbb{D}} \otimes R_{\Pi\mathfrak{z}}$  is a linear combination of the  $\mathfrak{g}$ -valued classical fields (53) with coefficients given by rational functions in  $z$  with poles in  $\Pi\mathfrak{z}$ . In §5.3.3 below we will consider also the case where the assumption (36) fails in the context of affine Toda field theory.

**Corollary 4.11.** *The local quadratic Gaudin Hamiltonians Poisson commute. Moreover, we have zero curvature equations  $\{H_p^x, \mathcal{L}(z)\} = \partial \mathcal{N}_p^x(z) + [\mathcal{L}(z), \mathcal{N}_p^x(z)]$ , where  $\mathcal{N}_p^x(z) := \varphi_p^x(z)(\mathcal{L}(z) - \mathcal{M}_p^x(z))$ .*

4.5.2. *Hamiltonian and momentum.* In all the examples of integrable field theories discussed in §5 below, the Hamiltonian of the model will be related to a specific linear combination of the formal quadratic Gaudin Hamiltonians introduced in Proposition 4.9. Recall that the latter are associated with the set  $\Pi\mathfrak{z}$  of poles of the twist function and appear as coefficients in the partial fraction decomposition of  $\frac{1}{2}(\mathcal{L}(z)|\mathcal{L}(z))$ . The Hamiltonian of a specific model will instead be naturally associated with *zeroes* of the twist function. Specifically, to each zero  $x \in \mathbb{P}^1$  of the twist function, *i.e.* such that  $\varphi(x) = 0$ , we associate a formal quadratic Hamiltonian

$$\mathcal{H}_x := \text{res}_x \frac{1}{2}(\mathcal{L}(z)|\mathcal{L}(z))\varphi(z)^{-1}dz. \quad (70)$$

It is given by a linear combination of the formal quadratic Gaudin Hamiltonians from Proposition 4.9. We can then associate with (70) a local Hamiltonian by applying to it the homomorphism  $\pi_\ell$  from §4.1.2, namely  $H_x := \pi_\ell(\mathcal{H}_x) \in \hat{S}_\ell(\widehat{\mathfrak{g}}_{\mathbb{C}}^{\mathbb{D}})$ .

We shall also need the local momentum of the integrable field theory, which is an element  $P \in \hat{S}_\ell(\widehat{\mathfrak{g}}_{\mathbb{C}}^{\mathbb{D}})$  whose Poisson bracket with any classical field, cf. §4.2.2, is equal to its derivative, *i.e.*  $\{P, A_{[p]}^x\} = \partial A_{[p]}^x$  and  $\{P, A_{[q+1]}^\infty\} = \partial A_{[q+1]}^\infty$  for all  $x \in \mathfrak{Z} \setminus \{\infty\}$  with  $p = 0, \dots, n_x - 1$  and  $q = 0, \dots, n_\infty - 2$ . For every example of dihedral affine Gaudin model considered in §5 we always have  $n_\infty \leq 2$ , cf. §3.4. If  $n_\infty = 1$  then there is no classical field attached to infinity, as will be the case in §5.2. When  $n_\infty = 2$  the only  $\mathfrak{g}$ -valued classical field at infinity is  $A_{[1]}^\infty$  which turns out to be a Casimir of the Poisson bracket (42), see Corollary 4.5. By a suitable modification of the homomorphism  $\pi_\ell$  from §4.1.2 we will set this Casimir to a constant in both §5.1 and §5.3. In each of these cases the local momentum can be defined as  $\mathcal{P} := \pi_\ell(P)$  with the *formal momentum* given by

$$\mathcal{P} := -i \sum_{x \in \mathfrak{z} \setminus \{\infty\}} D_{[0]}^{(x)} + i \sum_{x \in \mathfrak{z}_c} D_{[0]}^{(\bar{x})}. \quad (71)$$



By inspection, cf. also the proof of Proposition 4.13 in §4.5.3 below, we see it Poisson commutes with the formal quadratic Gaudin Hamiltonians defined in Proposition 4.9, *i.e.* we have  $\{\mathcal{P}, \mathcal{H}_p^x\} = 0$  for all  $x \in \mathfrak{Z}$  and  $p \geq 0$ .

**4.5.3. Constraints and Hamiltonian reduction.** The integrable field theories discussed in §5.2 below all possess Hamiltonian constraints. To account for such constraints, in this section we introduce a set of first class constraints in the dihedral affine Gaudin model and describe the associated Hamiltonian reduction of the complexified Poisson algebra of formal observables  $\hat{S}(\widehat{\mathfrak{g}}_{\mathbb{C}}^{\mathbb{D}})$ . The reduction remains non-trivial when passing to the complexified Poisson algebra of local observables  $\hat{S}_{\ell}(\widehat{\mathfrak{g}}_{\mathbb{C}}^{\mathbb{D}})$  provided the tuple of levels  $\ell$  satisfies an extra condition.

We introduce the *formal constraint*  $\mathcal{C} := \text{res}_{\infty} \zeta(\Phi) = \sum_{x \in \mathfrak{Z} \setminus \{\infty\}} \pi_{(0)} \mathcal{A}_{[0]}^x \in \mathfrak{F}$ , where the second equality follows from the explicit form of the formal Lax matrix in (62). This can be rewritten explicitly as

$$\mathcal{C} = \mathbf{D} \otimes i\mathcal{K} + \mathbf{K} \otimes i\mathcal{P} + \sum_{n \in \mathbb{Z}} I_{(0, \alpha), -n} \otimes \mathcal{C}_n^{\alpha} \quad (72)$$

where  $\mathcal{P}$  is the formal momentum given by (71), we introduced  $\mathcal{K} := -i \sum_{x \in \mathfrak{Z} \setminus \{\infty\}} \mathbf{K}_{[0]}^{(x)} + i \sum_{x \in \mathfrak{z}_c} \mathbf{K}_{[0]}^{(\bar{x})}$  and the modes  $\mathcal{C}_n^{\alpha} := \sum_{x \in \mathfrak{Z} \setminus \{\infty\}} (I_n^{(0, \alpha)})_{[0]}^{(x)} + \sum_{x \in \mathfrak{z}_c} (I_{-n}^{(0, \alpha)})_{[0]}^{(\bar{x})}$  with  $\alpha = 1, \dots, \dim \mathfrak{g}_{(0), \mathbb{C}}$  and  $n \in \mathbb{Z}$  of the formal constraint  $\mathcal{C}$ . Note that  $\mathcal{K}$  lies in the Poisson centre of  $\hat{S}(\widehat{\mathfrak{g}}_{\mathbb{C}}^{\mathbb{D}})$  and that  $\mathcal{C}_n^{\alpha} \in \mathbf{F}_n(\widehat{\mathfrak{g}}_{\mathbb{C}}^{\mathbb{D}})$  and  $\mathcal{C}_{-n}^{\alpha} \in c(\mathbf{F}_n(\widehat{\mathfrak{g}}_{\mathbb{C}}^{\mathbb{D}}))$  for all  $\alpha = 1, \dots, \dim \mathfrak{g}_{(0), \mathbb{C}}$  and  $n \in \mathbb{Z}_{\geq 0}$ , recalling the definition of the pair of conjugate descending  $\mathbb{Z}_{\geq 0}$ -filtrations on  $\widehat{\mathfrak{g}}_{\mathbb{C}}^{\mathbb{D}}$  introduced in §4.1.

Let  $J_{\mathcal{C}}$  denote the ideal of the commutative algebra  $S(\widehat{\mathfrak{g}}_{\mathbb{C}}^{\mathbb{D}})$  generated by  $\mathcal{K}$  and  $\mathcal{C}_n^{\alpha}$  for all  $\alpha = 1, \dots, \dim \mathfrak{g}_{(0), \mathbb{C}}$  and every  $n \in \mathbb{Z}$ . Let  $\hat{J}_{\mathcal{C}} := \varprojlim J_{\mathcal{C}} / (J_{\mathcal{C}} \cap \mathbf{F}_n(S(\widehat{\mathfrak{g}}_{\mathbb{C}}^{\mathbb{D}})))$  be the corresponding ideal of the completion  $\hat{S}(\widehat{\mathfrak{g}}_{\mathbb{C}}^{\mathbb{D}})$ . We would like to set  $\mathcal{K}$  and every  $\mathcal{C}_n^{\alpha}$  for  $\alpha = 1, \dots, \dim \mathfrak{g}_{(0), \mathbb{C}}$  and  $n \in \mathbb{Z}$  to zero. In other words, we want to impose the set of constraints

$$\mathcal{K} \approx 0, \quad \mathcal{C}_n^{\alpha} \approx 0 \quad (73)$$

for  $\alpha = 1, \dots, \dim \mathfrak{g}_{(0), \mathbb{C}}$  and  $n \in \mathbb{Z}$ . However, simply working in the quotient algebra  $\hat{S}(\widehat{\mathfrak{g}}_{\mathbb{C}}^{\mathbb{D}}) / \hat{J}_{\mathcal{C}}$  is not enough since the latter is not a Poisson algebra. Indeed, although  $\hat{J}_{\mathcal{C}}$  is an ideal of  $\hat{S}(\widehat{\mathfrak{g}}_{\mathbb{C}}^{\mathbb{D}})$  it is not a *Poisson* ideal, *i.e.* we do not have  $\{\hat{S}(\widehat{\mathfrak{g}}_{\mathbb{C}}^{\mathbb{D}}), \hat{J}_{\mathcal{C}}\} \subset \hat{J}_{\mathcal{C}}$ .

**Proposition 4.12.** *The ideal  $\hat{J}_{\mathcal{C}} \subset \hat{S}(\widehat{\mathfrak{g}}_{\mathbb{C}}^{\mathbb{D}})$  is a Poisson subalgebra, *i.e.*  $\{\hat{J}_{\mathcal{C}}, \hat{J}_{\mathcal{C}}\} \subset \hat{J}_{\mathcal{C}}$ . In other words, the set of constraints in (73) is first class.*

Consider the normaliser of  $\hat{J}_{\mathcal{C}}$  in  $\hat{S}(\widehat{\mathfrak{g}}_{\mathbb{C}}^{\mathbb{D}})$  defined by  $N(\hat{J}_{\mathcal{C}}) := \{\mathcal{X} \in \hat{S}(\widehat{\mathfrak{g}}_{\mathbb{C}}^{\mathbb{D}}) \mid \{\mathcal{X}, \hat{J}_{\mathcal{C}}\} \subset \hat{J}_{\mathcal{C}}\}$ . This is the subalgebra of *first class* elements in  $\hat{S}(\widehat{\mathfrak{g}}_{\mathbb{C}}^{\mathbb{D}})$ . By Proposition 4.12 we have that  $\hat{J}_{\mathcal{C}}$  is contained in  $N(\hat{J}_{\mathcal{C}})$ , but then by definition of  $N(\hat{J}_{\mathcal{C}})$  it is in fact a Poisson ideal. The subquotient  $\hat{S}(\widehat{\mathfrak{g}}_{\mathbb{C}}^{\mathbb{D}})_{\text{red}} := N(\hat{J}_{\mathcal{C}}) / \hat{J}_{\mathcal{C}}$  is therefore a Poisson algebra, called the *Hamiltonian reduction* of  $\hat{S}(\widehat{\mathfrak{g}}_{\mathbb{C}}^{\mathbb{D}})$  with respect to the set of first class constraints given in (73). It consists of equivalence classes of first class observables

in  $\hat{S}(\widehat{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})$ , where two such observables are considered equivalent if they differ by an element of  $\hat{J}_{\mathcal{C}}$ , *i.e.* a term proportional to the constraints.

**Proposition 4.13.** *Let  $n_{\infty} = 1$  in the notation of §3.4. Then the formal momentum  $\mathcal{P}$  and the collection of formal quadratic Gaudin Hamiltonians  $\mathcal{H}_p^x$  for every  $x \in \mathcal{Z}$  and  $p \geq 0$  all belong to  $N(\hat{J}_{\mathcal{C}})$ . Hence they descend to the Hamiltonian reduction  $\hat{S}(\widehat{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})_{\text{red}}$ .*

The set of elements  $\mathcal{K}$  and  $\mathcal{C}_n^{\alpha}$ ,  $\alpha = 1, \dots, \dim \mathfrak{g}_{(0),\mathbb{C}}$ ,  $n \in \mathbb{Z}$  is seen to be invariant under the anti-linear automorphism  $c : \widehat{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}} \rightarrow \widehat{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}$  using Proposition 3.10. Specifically,  $c(\mathcal{K}) = \mathcal{K}$  and  $c(\mathcal{C}_n^{\alpha}) = \mathcal{C}_{-n}^{\alpha}$  for all  $\alpha = 1, \dots, \dim \mathfrak{g}_{(0),\mathbb{C}}$  and  $n \in \mathbb{Z}$ . It follows that the ideal  $\hat{J}_{\mathcal{C}}$  is stable under the action of the extension of the anti-linear automorphism  $c$  to the completion  $\hat{S}(\widehat{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})$  defined in §4.1. In particular, its normaliser  $N(\hat{J}_{\mathcal{C}})$  is also stable since the anti-linear map  $c$  is an automorphism of the Poisson algebra  $\hat{S}(\widehat{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})$ . We may therefore consider the real subalgebra  $\hat{S}(\widehat{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})_{\text{red}}^c$  of fixed points under  $c$  in the Hamiltonian reduction  $\hat{S}(\widehat{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})_{\text{red}}$ .

Since the linear map  $\pi_{\ell}$  in (47) is a homomorphism of Poisson algebras, it sends the ideal and Poisson subalgebra  $\hat{J}_{\mathcal{C}}$  of  $\hat{S}(\widehat{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})$  from Proposition 4.12 to an ideal and Poisson subalgebra  $\pi_{\ell}(\hat{J}_{\mathcal{C}})$  of  $\hat{S}_{\ell}(\widehat{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})$ . Let us describe the latter more explicitly.

Recall from §4.1.1 that, given any element  $\mathbf{X}_{[p]}^{(x)} \in \widehat{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}$  with  $x \in \mathcal{Z} \setminus \{\infty\}$ ,  $\mathbf{X} \in \mathcal{L}\mathfrak{g}$  and  $p = 0, \dots, n_x - 1$ , we denote its image in  $\widehat{\mathfrak{g}}_{\mathbb{C},\ell}^{\mathcal{D}}$  by the same symbol. In particular, we will also keep the same notation for the image in  $\widehat{\mathfrak{g}}_{\mathbb{C},\ell}^{\mathcal{D}}$  of the modes  $\mathcal{C}_n^{\alpha}$  of the formal constraint, so that  $\pi_{\ell}(\mathcal{C}_n^{\alpha}) = \mathcal{C}_n^{\alpha}$  for every  $\alpha = 1, \dots, \dim \mathfrak{g}_{(0),\mathbb{C}}$  and  $n \in \mathbb{Z}$ , cf. the definition (44). On the other hand, the central element  $\mathcal{K}$  is sent to  $\pi_{\ell}(\mathcal{K}) = (\sum_{x \in \mathcal{Z} \setminus \{\infty\}} \ell_0^x) 1 \in \hat{S}_{\ell}(\widehat{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})$ . In other words,  $\pi_{\ell}(\hat{J}_{\mathcal{C}})$  is a proper ideal of  $\hat{S}_{\ell}(\widehat{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})$  if and only if the tuple of levels  $\ell$  satisfies the condition

$$\text{res}_{\infty} \varphi(z) dz = \sum_{x \in \mathcal{Z} \setminus \{\infty\}} \ell_0^x = 0. \quad (74)$$

In the remainder of this section we will assume this condition to hold. Then applying the homomorphism  $\varrho \otimes \pi_{\ell}$  to the formal constraint (72) we obtain the  $\mathfrak{g}_{(0)}$ -valued classical field  $(\varrho \otimes \pi_{\ell})\mathcal{C} = \sum_{n \in \mathbb{Z}} (I_{(0,\alpha)} \otimes e_{-n}) \otimes \mathcal{C}_n^{\alpha}$  in  $\mathcal{T}(S^1, \mathfrak{g}) \hat{\otimes} \widehat{\mathfrak{g}}_{\mathbb{C},\ell}^{\mathcal{D}}$ , where we note that the derivative term disappeared by virtue of the conditions (74).

Since the linear map  $\pi_{\ell}$  in (47) is a homomorphism of Poisson algebras it maps elements of  $N(\hat{J}_{\mathcal{C}})$  into the normaliser of  $\pi_{\ell}(\hat{J}_{\mathcal{C}})$  in  $\hat{S}_{\ell}(\widehat{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})$ , namely  $N(\pi_{\ell}(\hat{J}_{\mathcal{C}})) := \{\mathcal{X} \in \hat{S}_{\ell}(\widehat{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}) \mid \{\mathcal{X}, \pi_{\ell}(\hat{J}_{\mathcal{C}})\} \subset \pi_{\ell}(\hat{J}_{\mathcal{C}})\}$ . In other words, by restricting (47) to the normaliser  $N(\hat{J}_{\mathcal{C}}) \subset \hat{S}(\widehat{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})$  we obtain a homomorphism of Poisson algebras  $\pi_{\ell} : N(\hat{J}_{\mathcal{C}}) \rightarrow N(\pi_{\ell}(\hat{J}_{\mathcal{C}}))$ , which sends the subalgebra of first class formal observables in  $\hat{S}(\widehat{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})$  to the subalgebra of first class local observables in  $\hat{S}_{\ell}(\widehat{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})$ .

As in the formal setting, the ideal  $\pi_{\ell}(\hat{J}_{\mathcal{C}})$  of  $\hat{S}_{\ell}(\widehat{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})$  is contained in  $N(\pi_{\ell}(\hat{J}_{\mathcal{C}}))$ , and by definition of the latter it follows that  $\pi_{\ell}(\hat{J}_{\mathcal{C}})$  is in fact a Poisson ideal of  $N(\pi_{\ell}(\hat{J}_{\mathcal{C}}))$ . The corresponding subquotient

$$\hat{S}_{\ell}(\widehat{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})_{\text{red}} := N(\pi_{\ell}(\hat{J}_{\mathcal{C}})) / \pi_{\ell}(\hat{J}_{\mathcal{C}})$$

is therefore a Poisson algebra. It is the *Hamiltonian reduction* of  $\hat{S}_\ell(\widehat{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})$  with respect to the set of first class constraints  $\mathcal{C}_n^\alpha \approx 0$  for  $\alpha = 1, \dots, \dim \mathfrak{g}_{(0),\mathbb{C}}$  and  $n \in \mathbb{Z}$ , *i.e.*  $(\varrho \otimes \pi_\ell)\mathcal{C} \approx 0$ . Finally, the  $c$ -equivariance of  $\pi_\ell$ , cf. Proposition 4.3, ensures that the ideal  $\pi_\ell(\hat{J}_c)$  of  $\hat{S}_\ell(\widehat{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})$  and its normaliser  $N(\pi_\ell(\hat{J}_c))$  are both stable under the action of  $c$  so that we can consider the real subalgebra  $\hat{S}_\ell(\widehat{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})_c^{\text{red}}$  of fixed points under  $c$  in  $\hat{S}_\ell(\widehat{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})_{\text{red}}$ .

## 5. EXAMPLES OF NON-ULTRALOCAL FIELD THEORIES

In this section we explicitly show how all the classical integrable field theories listed in Table 1 of the introduction can be formulated as dihedral affine Gaudin models.

In the case of the principal chiral model and its various multi-parameter integrable deformations, discussed in §5.1, the affine Gaudin model formulation exactly matches the conventional description of these models for an arbitrary real finite-dimensional Lie group. We relate the standard notation for the Lie algebra valued fields and Lax matrix of these models to the notation introduced in §4.2.2 and §4.4.2, respectively, in the context of a general classical dihedral affine Gaudin model.

In §5.2 we turn to the description of  $\mathbb{Z}_T$ -graded coset  $\sigma$ -models and their integrable deformations. Here as well, the affine Gaudin model formulation of these models exactly matches the standard one. More precisely, for the symmetric space  $\sigma$ -model, corresponding to the case  $T = 2$ , two different formulations exist in the literature. We show that only one of these formulations admits an affine Gaudin model description and argue, at the end of §5.2.1, that the other one does not. The bi-Yang-Baxter  $\sigma$ -model is also formulated as an affine Gaudin model in §5.2.3. More precisely, we match its description as a two-parameter deformation of a symmetric space  $\sigma$ -model established in [14] within the Hamiltonian formalism. Its more standard description as a deformation of the principal chiral model, originally given in [58], was only shown to be integrable within the Lagrangian formalism [59]. This alternative formulation therefore does not lend itself to an affine Gaudin model interpretation.

We give a general treatment of affine Toda field theory in §5.3. Whereas the Lie algebra homomorphism  $\pi_\ell$  constructed in §4.1.2 could be used in all previous examples, here this is no longer possible because the basic assumption (36) on the levels does not hold in the present case. We thus introduce an alternative homomorphism  $\tilde{\pi}_\ell$  which can be seen as realising the formal fields of the affine Gaudin model in terms of classical Toda fields. In the case of affine Toda field theories there is another substantial difference to the previous cases. The standard formulation of affine Toda field theory is ultralocal, whereas the affine Gaudin model formulation is inherently non-ultralocal, cf. Corollary 4.8. We show that the Lax matrices in both formulations are related by a gauge transformation. We illustrate the general discussion of affine Toda field theories in detail on the example of the sinh-Gordon model, corresponding to the  $\mathfrak{sl}_2$  case, in §5.3.4.

**5.1. Principal chiral model and deformations.** Throughout this subsection we set  $T = 1$  so that  $\Pi = D_2 = \langle \mathfrak{t} \mid \mathfrak{t}^2 = 1 \rangle \simeq \mathbb{Z}_2$ . In particular, the automorphism  $\sigma$  of  $\mathfrak{g}$  is the identity. We pick and fix an anti-linear automorphism  $\tau \in \overline{\text{Aut}}_- \mathfrak{g}$  and denote by  $\mathfrak{g}_0$  the corresponding real form.

5.1.1. *Principal chiral model.* Consider the divisor

$$\mathcal{D} = 2 \cdot 0 + 2 \cdot \infty. \quad (75)$$

In the notation of §4 we have  $\mathfrak{Z} = \{0, \infty\}$  and  $\mathfrak{Z}' = \emptyset$ . There are three formal fields  $\mathcal{A}_{[0]}^0$ ,  $\mathcal{A}_{[1]}^0$  and  $\mathcal{A}_{[1]}^\infty$ . Since 0 and  $\infty$  are both real points and we are in the non-cyclotomic setting, these fields are defined by (49b). In terms of them, the formal Lax matrix is given by  $\mathcal{L}(z) = \mathcal{A}_{[1]}^0 z^{-2} + \mathcal{A}_{[0]}^0 z^{-1} - \mathcal{A}_{[1]}^\infty$ , cf. Proposition 4.6.

We fix the levels  $\ell_0^0 = 0$ ,  $\ell_1^0 = 1$  and  $\ell_1^\infty = 1$ . Consider the  $\mathfrak{g}_0$ -valued classical fields  $j_p := A_{[p]}^0$  for  $p = 0, 1$  associated with the origin and  $\xi := A_{[1]}^\infty$  associated with infinity, defined by (54). By Corollary 4.5 these satisfy the Poisson brackets

$$\{j_{01}(\theta), j_{02}(\theta')\} = -[C_{12}, j_{02}(\theta)] \delta_{\theta\theta'}, \quad (76a)$$

$$\{j_{01}(\theta), j_{12}(\theta')\} = -[C_{12}, j_{12}(\theta)] \delta_{\theta\theta'} - C_{12} \delta'_{\theta\theta'}, \quad (76b)$$

$$\{j_{11}(\theta), j_{12}(\theta')\} = 0, \quad (76c)$$

and  $\{\xi_1(\theta), \xi_2(\theta')\} = 0$ , with all other brackets being zero. In particular, the  $\mathfrak{g}_0$ -valued classical field  $\xi$  is a Casimir of the Poisson bracket. We therefore choose to set it to zero from now on. This is formally achieved by altering slightly the definition of the homomorphism  $\pi_\ell$  in (44). Specifically, we replace the definitions (44e) of  $\pi_\ell$  on the elements of  $\tilde{\mathfrak{g}}^{n_\infty} \otimes_{\mathbb{R}} \mathbb{C}$  attached to infinity by

$$\pi_\ell(\mathcal{D}_{[1]}^{(\infty)}) := 0, \quad \pi_\ell(\mathcal{K}_{[1]}^{(\infty)}) := i, \quad \pi_\ell(I_{n, [1]}^{\alpha(\infty)}) := 0. \quad (77)$$

The resulting linear map  $\pi_\ell : \tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}} \rightarrow \hat{S}_\ell(\hat{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}})$  is still seen to be a homomorphism of Lie algebras, cf. Proposition 4.3, using the fact that  $n_\infty = 2$ . In what follows we use this altered definition of  $\pi_\ell$ . We recognise (76) as the Poisson brackets of the principal chiral model written in terms of the components of the current 1-form  $j = -dgg^{-1}$ , where  $g$  denotes the principal chiral field, see for instance [34, §I.5 of Part 2].

Applying the representation  $\varrho \otimes \pi_\ell \otimes \text{id}$  to the formal Lax matrix  $\mathcal{L}(z)$ , as in §4.4.2, and comparing with the general expression (64) we read off the Lax matrix and twist function to be

$$\mathcal{L}(z) = \frac{1}{1-z^2}(j_1 + zj_0), \quad \varphi(z) = \frac{1}{z^2} - 1, \quad (78)$$

which coincide with those of the principal chiral model. See *e.g.* [34, §I.3 of Part 2] for the Lax matrix and [79, 15] for the twist function.

It remains to be checked that the Hamiltonian of the principal chiral model can be obtained from the formal quadratic Gaudin Hamiltonians of Proposition 4.9. The twist function defined in (78) has a pair of simple zeroes located at  $\pm 1$ . We then find that the associated formal quadratic Hamiltonians (70) read  $\mathcal{H}_{\pm 1} = \mp \frac{1}{4}(\mathcal{A}_{[0]}^0 \pm \mathcal{A}_{[1]}^0 \mp \mathcal{A}_{[1]}^\infty | \mathcal{A}_{[0]}^0 \pm \mathcal{A}_{[1]}^0 \mp \mathcal{A}_{[1]}^\infty)$ . The terms involving  $\mathcal{D}$  and  $\mathcal{K}$  on the right hand side take the form  $\mp \frac{1}{2}(\mathcal{D}_{[0]}^{(0)} \pm \mathcal{D}_{[1]}^{(0)} \mp \mathcal{D}_{[1]}^{(\infty)}) (\mathcal{K}_{[0]}^{(0)} \pm \mathcal{K}_{[1]}^{(0)} \mp \mathcal{K}_{[1]}^{(\infty)})$ . However, these disappear upon applying the homomorphism  $\pi_\ell$  since  $\ell_1^0 \pm \ell_0^0 - \ell_1^\infty = 0$ , which is related to the fact that  $\pm 1$  are both simple zeroes of the twist function  $\varphi(z)$  in (78). Recalling that we have set  $\xi$  to zero, cf. (77), we therefore find  $H_{\pm 1} = \pi_\ell(\mathcal{H}_{\pm 1}) = \mp \frac{1}{4}(j_0 \pm j_1 | j_0 \pm j_1)$

where in the last equality we used Lemma 2.4. The difference of these local quadratic Hamiltonians is

$$H := H_{-1} - H_1 = \frac{1}{4\pi} \int_{S^1} d\theta (\langle j_0(\theta), j_0(\theta) \rangle + \langle j_1(\theta), j_1(\theta) \rangle), \quad (79)$$

which coincides, up to an overall factor, with the Hamiltonian of the principal chiral model, see [34, §I.5 of Part 2]. Using (46) from example 4.2 we find the momentum  $P = \pi_\ell(\mathcal{P})$  to be  $P = \frac{1}{2\pi} \int_{S^1} d\theta \langle j_0(\theta), j_1(\theta) \rangle$ . Note that  $\mathcal{H}_{-1} - \mathcal{H}_1 = \mathcal{H}_1^0 + \mathcal{H}_3^0 + \mathcal{H}_0^\infty$  in the notation of Proposition 4.9. Moreover, in the present case we have  $\pi_\ell(\mathcal{H}_3^0) = \pi_\ell(\mathcal{H}_0^\infty) = 0$  so that the local quadratic Hamiltonian can also be obtained as  $H = \pi_\ell(\mathcal{H}_1^0)$ . In particular, applying Corollary 4.11 we obtain the zero curvature equation  $\{H, \mathcal{L}(z)\} = \partial \mathcal{M}(z) + [\mathcal{L}(z), \mathcal{M}(z)]$ , where, noting that  $\varphi_1^0(z) = z^{-1}$  and  $\mathcal{M}_1^0(z) = j_1$ , we have

$$\mathcal{M}(z) := \mathcal{N}_1^0(z) = \frac{1}{1 - z^2} (j_0 + z j_1).$$

We deduce from the above analysis that the classical dihedral affine Gaudin model associated with the divisor (75) and the corresponding choice of levels as above coincides with the principal chiral model described in terms of the current 1-form  $j = -dgg^{-1}$ . Recall, however, that the actual phase space of the principal chiral model is given by the cotangent bundle  $T^*\mathcal{L}G_0$  of the loop group  $\mathcal{L}G_0$  of the real Lie group  $G_0$  with Lie algebra  $\mathfrak{g}_0$ . The above formulation in terms of the 1-form current  $j$  therefore only describes the principal chiral model dynamics on the quotient  $(T^*\mathcal{L}G_0)/G_0$  of the cotangent bundle by the right action of the subgroup of constant loops. Obtaining a complete description of the principal chiral model requires introducing a  $G_0$ -valued field  $g$  satisfying  $j_1 = -\partial gg^{-1}$  so that together with the  $\mathfrak{g}$ -valued field  $X := -g^{-1}j_0g$  they parametrise the global (left) trivialisation of  $T^*\mathcal{L}G_0$ . We shall not discuss this issue further here, and refer to [85] for further details.

5.1.2. *Two-parameter deformations.* Integrable deformations of the principal chiral model may be constructed by altering the data of the affine Gaudin model of §5.1.1 in various ways. In what follows we will only be concerned with deformations which alter the divisor (75) itself. More precisely, we shall deform the divisor  $\Pi\mathcal{D}$ , cf. §3.4, while preserving its  $\Pi$ -invariance. One possible way of doing so is to split the double pole at the origin into a pair of simple poles, either both real or complex conjugate of one another. We refer to these as the real and complex branches respectively [85].

To describe the deformation in the complex branch we use the divisor  $\mathcal{D} = x_+ + 2 \cdot \infty$  for arbitrary  $x_+ \in \mathbb{C}$  such that  $\Im x_+ > 0$ . Writing  $x_+ = k + iA$  we can regard  $k \in \mathbb{R}$  and  $A \in \mathbb{R}_{>0}$  as two *real* deformation parameters. We shall treat the limiting case when  $A \rightarrow 0$  with  $k \neq 0$  in §5.1.3 below. Similarly, to construct the deformation in the real branch we would start from the divisor

$$\mathcal{D} = x_+ + x_- + 2 \cdot \infty, \quad (80)$$

with distinct  $x_\pm \in \mathbb{R}$  being the two real deformation parameters. To treat the two branches in a unified way we note that in both cases the divisor  $\Pi\mathcal{D}$  takes the form

$$\Pi\mathcal{D} = x_+ + x_- + 2 \cdot \infty,$$

with  $x_+ \neq x_-$ , where  $x_{\pm} \in \mathbb{R}$  in the real branch and  $x_- = \bar{x}_+$  in the complex branch. In particular, we then have  $\mathfrak{Z} = \{x_+, x_-, \infty\}$  in the notation of §4.

There are three formal fields  $\mathcal{A}_{[0]}^{x_+}$ ,  $\mathcal{A}_{[0]}^{x_-}$  and  $\mathcal{A}_{[1]}^{\infty}$ , which are respectively attached to the points  $x_+$ ,  $x_-$  and infinity. By Proposition 4.6, the formal Lax matrix is expressed in terms of these as

$$\mathcal{L}(z) = \frac{\mathcal{A}_{[0]}^{x_+}}{z - x_+} + \frac{\mathcal{A}_{[0]}^{x_-}}{z - x_-} - \mathcal{A}_{[1]}^{\infty}. \quad (81)$$

To fix the levels, recall that the twist function (78) of the principal chiral model has a double pole at the origin. The requirement that this double pole is recovered in the limit  $x_{\pm} \rightarrow 0$  uniquely fixes the singular behaviour of the levels associated with the points  $x_+$  and  $x_-$ . We shall also require that the zeroes of the twist function remain fixed at  $\pm 1$  under the deformation, which leads us to set

$$\ell_0^{x_{\pm}} = \pm \frac{1 - x_{\pm}^2}{x_+ - x_-}, \quad \ell_1^{\infty} = 1. \quad (82)$$

We introduce the  $\mathfrak{g}$ -valued classical fields  $\mathcal{J}_{\pm} := A_{[0]}^{x_{\pm}}$  and  $\xi := A_{[1]}^{\infty}$ , which according to Corollary 4.5 satisfy the Poisson brackets  $\{\mathcal{J}_{\pm 1}(\theta), \mathcal{J}_{\pm 2}(\theta')\} = -[C_{12}, \mathcal{J}_{\pm 2}(\theta)]\delta_{\theta\theta'} - \ell_0^{x_{\pm}} C_{12} \delta'_{\theta\theta'}$ , with all other brackets being zero. As in §5.1.1 we observe that  $\xi$  is a Casimir so in what follows we will set it to zero by suitably modifying the definition of  $\pi_{\ell}$  on the elements of  $\tilde{\mathfrak{g}}^{n\infty} \otimes_{\mathbb{R}} \mathbb{C}$  given in (44e), replacing it by (77).

To obtain the Lax matrix and twist function we apply the linear map  $\varrho \otimes \pi_{\ell} \otimes \text{id}$  to the formal Lax matrix in (81). We find

$$\begin{aligned} (\varrho \otimes \pi_{\ell} \otimes \text{id})\mathcal{L}(z) &= \frac{\ell_0^{x_+} \partial + \mathcal{J}_+}{z - x_+} + \frac{\ell_0^{x_-} \partial + \mathcal{J}_-}{z - x_-} - \partial \\ &= \frac{1 - z^2}{(z - x_+)(z - x_-)} \left( \partial + \frac{1}{1 - z^2} (j_1 + zj_0) \right), \end{aligned}$$

where in the second line we introduced the linear combinations  $j_1 := -x_- \mathcal{J}_+ - x_+ \mathcal{J}_-$  and  $j_0 := \mathcal{J}_+ + \mathcal{J}_-$  of the  $\mathfrak{g}$ -valued classical fields. In terms of these, the Lax matrix takes the same form as for the principal chiral model in (78). However, these fields satisfy a two parameter deformation of the Poisson bracket (76) which reads

$$\{j_{01}(\theta), j_{02}(\theta')\} = -[C_{12}, j_{02}(\theta)]\delta_{\theta\theta'} + (x_+ + x_-)C_{12}\delta'_{\theta\theta'}, \quad (83a)$$

$$\{j_{01}(\theta), j_{12}(\theta')\} = -[C_{12}, j_{12}(\theta)]\delta_{\theta\theta'} - (1 + x_+x_-)C_{12}\delta'_{\theta\theta'}, \quad (83b)$$

$$\{j_{11}(\theta), j_{12}(\theta')\} = [C_{12}, (x_+ + x_-)j_{12} + x_+x_-j_{02}(\theta)]\delta_{\theta\theta'} + (x_+ + x_-)C_{12}\delta'_{\theta\theta'}. \quad (83c)$$

Such a deformation of the Poisson brackets of the principal chiral model was considered in [4] for  $\mathfrak{g} = \mathfrak{su}_2$ , but the same brackets immediately extend to any Lie algebra (see *e.g.* [55, 20]). By construction, the Poisson brackets (83) for the  $\mathfrak{g}$ -valued classical fields  $j_0$  and  $j_1$  are equivalent to the Lax matrix (78) satisfying the Poisson bracket from Corollary 4.8 with the twist function

$$\varphi(z) = \frac{1 - z^2}{(z - x_+)(z - x_-)}. \quad (84)$$

Consider now the collection of formal quadratic Gaudin Hamiltonians introduced in Proposition 4.9. Since the twist function (84) still has simple zeroes at the points  $\pm 1$ , the specific linear combinations (70) of the above quadratic Gaudin Hamiltonians read  $\mathcal{H}_{\pm 1} = \mp \frac{1}{4}(1 \mp x_+)(1 \mp x_-)(\mathcal{L}(\pm 1)|\mathcal{L}(\pm 1))$ . Just as in §5.1.1, the terms on the right hand side involving  $\mathbb{D}$  and  $\mathbb{K}$  disappear once we apply the homomorphism  $\pi_{\ell}$  using the fact that  $\varphi(\pm 1) = 0$ . We then find

$$H := H_{-1} - H_1 = \int_{S^1} d\theta \frac{(1 + x_+ x_-)(\langle j_0(\theta), j_0(\theta) \rangle + \langle j_1(\theta), j_1(\theta) \rangle) + 2(x_+ + x_-)\langle j_0(\theta), j_1(\theta) \rangle}{4\pi(1 - x_+^2)(1 - x_-^2)},$$

which coincides, in the complex branch, with the Hamiltonian for the two-parameter integrable deformation of the principal chiral model constructed in [17]. The latter model has two interesting limits. Writing  $x_+ = k + iA$ , the limit  $k \rightarrow 0$  corresponds to the Yang-Baxter  $\sigma$ -model first introduced by Klimcik in [57]. Alternatively, taking the limit  $A \rightarrow 0$  yields the principal chiral model with a Wess-Zumino term which we describe in more detail in §5.1.3 below. Note also that the momentum  $P = \pi_{\ell}(\mathcal{P})$  is found, using (45) from example 4.2, to be

$$P = \int_{S^1} d\theta \frac{(x_+ + x_-)(\langle j_0(\theta), j_0(\theta) \rangle + \langle j_1(\theta), j_1(\theta) \rangle) + 2(1 + x_+ x_-)\langle j_0(\theta), j_1(\theta) \rangle}{4\pi(1 - x_+^2)(1 - x_-^2)}.$$

The classical integrable structure of the real branch was studied in [55] but the action of the corresponding model has not yet been identified. The above analysis shows that at the Hamiltonian level this model is described by a dihedral affine Gaudin model associated with the divisor (80), where  $x_{\pm} \in \mathbb{R}$ , together with the choice of levels (82). When  $x_+ = -x_-$  we obtain a one-parameter deformation described by an integrable gauged WZW-type theory introduced by Sfetsos [80], which interpolates between the WZW model and the non-abelian  $T$ -dual of the principal chiral model.

The remark made about the principal chiral field at the end of §5.1.1 also applies to its two-parameter deformation. Consider for simplicity the one-parameter deformation where  $x_+ = -x_-$ . In this case, defining a pair of fields parametrising the global (left) trivialisation of the cotangent bundle  $T^*\mathcal{L}G_0$  of the loop group of the real Lie group  $G_0$  requires the introduction of a solution  $R \in \text{End } \mathfrak{g}_0$  of the modified classical Yang-Baxter equation on  $\mathfrak{g}_0$ . We refer to [85] for more details.

**5.1.3. Principal chiral model with WZ-term.** Another way to deform the divisor  $\Pi\mathcal{D}$ , where  $\mathcal{D}$  is given by (75), besides splitting its double pole at the origin into a pair of simple poles as in §5.1.2, is to simply shift this double pole along the real axis. That is, we define the divisor

$$\mathcal{D} = 2 \cdot k + 2 \cdot \infty, \tag{85}$$

where  $k \in \mathbb{R} \setminus \{0\}$  plays the role of the deformation parameter. For a suitable choice of levels, see below, this can also be seen as a limit of the setup in §5.1.2 where the pair of simple poles  $x_{\pm}$  collide at  $k$ .

As in the principal chiral model case we have three formal fields  $\mathcal{A}_{[0]}^k$ ,  $\mathcal{A}_{[1]}^k$  and  $\mathcal{A}_{[1]}^{\infty}$ . By Proposition 4.6, the corresponding formal Lax matrix reads  $\mathcal{L}(z) = \mathcal{A}_{[1]}^k(z - k)^{-2} + \mathcal{A}_{[0]}^k(z - k)^{-1} - \mathcal{A}_{[1]}^{\infty}$ . We require the deformation not to alter the position of the zeroes of the twist function at  $\pm 1$ , as in §5.1.2. This will ensure that the Lax

matrix takes the same form as in the principal chiral model (78) for suitably defined linear combinations  $j_0$  and  $j_1$  of the classical fields. We therefore fix the levels to be  $\ell_0^k = -2k$ ,  $\ell_1^k = 1 - k^2$  and  $\ell_1^\infty = 1$ .

If we define the  $\mathfrak{g}$ -valued classical fields  $\tilde{j}_p := A_{[p]}^k$  for  $p = 0, 1$  and  $\xi := A_{[1]}^\infty$ , then according to Corollary 4.5 their Poisson brackets read

$$\{\tilde{j}_{01}(\theta), \tilde{j}_{02}(\theta')\} = -[C_{12}, \tilde{j}_{01}(\theta)]\delta_{\theta\theta'} + 2kC_{12}\delta'_{\theta\theta'}, \quad (86a)$$

$$\{\tilde{j}_{01}(\theta), \tilde{j}_{12}(\theta')\} = -[C_{12}, \tilde{j}_{12}(\theta)]\delta_{\theta\theta'} - (1 - k^2)C_{12}\delta'_{\theta\theta'}, \quad (86b)$$

$$\{\tilde{j}_{11}(\theta), \tilde{j}_{12}(\theta')\} = 0, \quad (86c)$$

and  $\{\xi_1(\theta), \xi_2(\theta')\} = 0$ , the remaining brackets all being zero. As usual we shall set  $\xi$  to zero since it is a Casimir, cf. §5.1.1 and §5.1.2.

The Lax matrix and twist function are then constructed by applying the linear map  $\varrho \otimes \pi_\ell \otimes \text{id}$  to the formal Lax matrix  $\mathcal{L}(z)$ . We have

$$(\varrho \otimes \pi_\ell \otimes \text{id})\mathcal{L}(z) = \frac{(1 - k^2)\partial + \tilde{j}_1}{(z - k)^2} + \frac{-2k\partial + \tilde{j}_0}{z - k} - \partial = \frac{1 - z^2}{(z - k)^2} \left( \partial + \frac{1}{1 - z^2} (j_1 + zj_0) \right),$$

where we introduced the linear combinations  $j_1 := \tilde{j}_1 - k\tilde{j}_0$  and  $j_0 := \tilde{j}_0$ . In terms of these, the expression for the Lax matrix is exactly the same as in the principal chiral model (78). However, the Poisson brackets between the classical fields  $j_0$  and  $j_1$  are a deformation of (76). They follow from (86) and take the form

$$\{j_{01}(\theta), j_{02}(\theta')\} = -[C_{12}, j_{01}(\theta)]\delta_{\theta\theta'} + 2kC_{12}\delta'_{\theta\theta'}, \quad (87a)$$

$$\{j_{01}(\theta), j_{12}(\theta')\} = -[C_{12}, j_{12}(\theta)]\delta_{\theta\theta'} - (1 + k^2)C_{12}\delta'_{\theta\theta'}, \quad (87b)$$

$$\{j_{11}(\theta), j_{12}(\theta')\} = [C_{12}, 2k j_{12}(\theta) + k^2 j_{02}(\theta)]\delta_{\theta\theta'} + 2kC_{12}\delta'_{\theta\theta'}. \quad (87c)$$

This corresponds precisely to the coinciding point limit  $x_\pm \rightarrow k$  of the Poisson brackets (83) of §5.1.2. We note that the twist function

$$\varphi(z) = \frac{1 - z^2}{(z - k)^2} \quad (88)$$

can also be obtained from the two-parameter twist function (84) by taking the same coinciding point limit.

Since the twist function (88) still has a pair of simple zeroes at  $\pm 1$  by construction, we may associate to these formal quadratic Hamiltonians (70) which read  $\mathcal{H}_{\pm 1} = \mp \frac{1}{4}(1 \mp k)^2 (\mathcal{L}(\pm 1)|\mathcal{L}(\pm 1))$ . Applying the homomorphism  $\pi_\ell$  we find that the terms involving  $\mathbf{D}$  and  $\mathbf{K}$  drop out because  $\varphi(\pm 1) = 0$ . We therefore obtain

$$H := H_{-1} - H_1 = \int_{S^1} d\theta \frac{(1 + k^2)(\langle j_0(\theta), j_0(\theta) \rangle + \langle j_1(\theta), j_1(\theta) \rangle) + 4k\langle j_0(\theta), j_1(\theta) \rangle}{4\pi(1 - k^2)^2}.$$

This Hamiltonian and the Poisson brackets (87) agree with those resulting from a Hamiltonian analysis of the principal chiral model action with a Wess-Zumino term added, see *e.g.* [20] and setting  $\eta = 0$  in the notation used there.



**5.2.  $\mathbb{Z}_T$ -graded coset  $\sigma$ -model and deformations.** Let  $\mathfrak{g}$  be a finite-dimensional complex Lie algebra and  $\sigma \in \text{Aut } \mathfrak{g}$  an automorphism of order  $T \in \mathbb{Z}_{\geq 2}$ . We also fix an anti-linear involution  $\tau \in \overline{\text{Aut}}_- \mathfrak{g}$  and denote by  $\mathfrak{g}_0$  the corresponding real form of  $\mathfrak{g}$ . In the present case we have  $\Pi = D_{2T} \simeq \mathbb{Z}_T \rtimes \mathbb{Z}_2$ . Throughout this section we will impose the set of first class constraints (73) as described in §4.5.3.

5.2.1.  *$\mathbb{Z}_T$ -graded coset  $\sigma$ -model.* Consider the divisor

$$\mathcal{D} = 2 \cdot 1 + \infty. \quad (89)$$

In the notation of §4 we have  $\mathfrak{Z} = \{1, \infty\}$  and  $\mathfrak{Z}' = \{1\}$ . There are two formal fields  $\mathcal{A}_{[p]}^1$ ,  $p = 0, 1$  associated with the point 1. However, since  $n_\infty = 1$  there is no formal field associated with the point at infinity. It follows from Proposition 4.6 that the formal Lax matrix is given by

$$\mathcal{L}(z) = \frac{1}{T} \sum_{k=0}^{T-1} \left( \frac{\omega^k \sigma^k \mathcal{A}_{[1]}^1}{(z - \omega^k)^2} + \frac{\sigma^k \mathcal{A}_{[0]}^1}{z - \omega^k} \right), \quad (90)$$

where  $\omega$  is a primitive  $T^{\text{th}}$ -root of unity. We shall fix the levels to be  $\ell_0^1 = 0$  and  $\ell_1^1 = 1$ , and define the  $\mathfrak{g}$ -valued classical fields  $A := \mathcal{A}_{[1]}^1$ ,  $\Pi := \mathcal{A}_{[0]}^1$ . Using Corollary 4.5 we find their Poisson brackets to be

$$\{A_1(\theta), A_2(\theta')\} = 0, \quad (91a)$$

$$\{A_1(\theta), \Pi_2(\theta')\} = -[C_{12}, A_2(\theta)] \delta_{\theta\theta'} - C_{12} \delta'_{\theta\theta'}, \quad (91b)$$

$$\{\Pi_1(\theta), \Pi_2(\theta')\} = -[C_{12}, \Pi_2(\theta)] \delta_{\theta\theta'}. \quad (91c)$$

These coincide with the Poisson brackets of the  $\mathbb{Z}_2$ -graded coset  $\sigma$ -model (for which  $T = 2$ ), also known as the symmetric space  $\sigma$ -model, see *e.g.* [15]. If  $\mathfrak{g}$  is the Grassmann envelope of a Lie superalgebra then these Poisson brackets are also those of the  $\mathbb{Z}_4$ -graded supercoset  $\sigma$ -model, *i.e.* semi-symmetric space  $\sigma$ -model, for which  $\sigma \in \text{Aut } \mathfrak{g}$  is of order  $T = 4$  [67, 83], or more generally those of the  $\mathbb{Z}_{2n}$ -graded supercoset  $\sigma$ -model for which  $\sigma \in \text{Aut } \mathfrak{g}$  is of order  $T = 2n$  with  $n \in \mathbb{Z}_{\geq 2}$  [56].

To compute the Lax matrix and twist function we apply the linear map  $\varrho \otimes \pi_\ell \otimes \text{id}$  to the formal Lax matrix (90) which gives

$$\begin{aligned} (\varrho \otimes \pi_\ell \otimes \text{id})\mathcal{L}(z) &= \frac{1}{T} \sum_{k=0}^{T-1} \left( \frac{\omega^k \partial + \omega^k \sigma^k A}{(z - \omega^k)^2} + \frac{\sigma^k \Pi}{z - \omega^k} \right) \\ &= \frac{Tz^{T-1}}{(1 - z^T)^2} \left( \partial + \sum_{j=1}^T \frac{T - j + jz^{-T}}{T} z^j A^{(j)} + \sum_{j=1}^T \frac{1 - z^{-T}}{T} z^j \Pi^{(j)} \right). \end{aligned} \quad (92)$$

Here we have introduced the projections of the  $\mathfrak{g}_0$ -valued classical fields  $A$  and  $\Pi$  onto the graded subspaces  $\mathfrak{g}_{(j)}$  for  $j \in \mathbb{Z}_T$  relative to the direct sum decomposition of  $\mathfrak{g}_0$  given in §2.1. Specifically, we set  $A^{(j)} := \pi_{(j)} A$ ,  $\Pi^{(j)} := \pi_{(j)} \Pi$ . Comparing the above

expression with the general form in (64) we read off the Lax matrix to be

$$\mathcal{L}(z) = \sum_{j=1}^T \frac{T-j+jz^{-T}}{T} z^j A^{(j)} + \sum_{j=1}^T \frac{1-z^{-T}}{T} z^j \Pi^{(j)}, \quad (93)$$

and the corresponding twist function

$$\varphi(z) = \frac{Tz^{T-1}}{(1-z^T)^2}. \quad (94)$$

When  $T = 2$  these are precisely the expressions for the Lax matrix and corresponding twist function of the  $\mathbb{Z}_2$ -graded coset  $\sigma$ -model, see *e.g.* [15]. If  $\mathfrak{g}$  is taken to be the Grassmann envelope of a Lie superalgebra and  $T = 4$  then (93) is the Hamiltonian Lax matrix of a semi-symmetric space  $\sigma$ -model, *i.e.* a  $\mathbb{Z}_4$ -graded supercoset  $\sigma$ -model [67, 84], the Green-Schwarz superstring on  $AdS_5 \times S^5$  being a particular example of the latter. More generally, when  $\mathfrak{g}$  is the Grassmann envelope of a Lie superalgebra and  $T = 2n$  with  $n \in \mathbb{Z}_{\geq 2}$ , the expressions for the Lax matrix (93) and the twist function (94) agree up to differences in conventions with those obtained from the Hamiltonian analysis of the  $\mathbb{Z}_{2n}$ -graded supercoset  $\sigma$ -model [89] performed in [56]. Moreover, for a semisimple Lie algebra  $\mathfrak{g}$  and general  $T \in \mathbb{Z}_{\geq 2}$ , the above expressions (93) and (94) reproduce, again up to conventions, those of the bosonic truncation of the  $\mathbb{Z}_{2T}$ -graded supercoset  $\sigma$ -model from [56].

The formal constraint defined in §4.5.3 is given by  $\mathcal{C} = \pi_{(0)} \mathcal{A}_{[0]}^1$ . We note that the above choice of levels trivially satisfy the condition (74) of §4.5.3. Applying to it the linear map  $\varrho \otimes \pi_\ell$  and using the  $\Pi$ -equivariance of  $\varrho$  we find  $(\varrho \otimes \pi_\ell) \mathcal{C} = \Pi^{(0)}$ . This coincides with the constraint of the  $\mathbb{Z}_2$ -graded coset  $\sigma$ -model, see *e.g.* [15], or more generally with that of the  $\mathbb{Z}_T$ -graded coset  $\sigma$ -model for  $T \in \mathbb{Z}_{\geq 2}$  [56].

Finally, we turn to the definition of the Hamiltonian. The twist function (94) has zeroes at the origin and infinity. Consider the associated quadratic Hamiltonians defined as in (70) for  $x \in \{0, \infty\}$ .

**Lemma 5.1.** *The quadratic Hamiltonians (70) are given by*

$$\begin{aligned} \mathcal{H}_0 &= \frac{1}{2T} \sum_{j=1}^{T-1} \left( j \mathcal{A}_{[1]}^{(j)} - \mathcal{A}_{[0]}^{(j)} \middle| (T-j) \mathcal{A}_{[1]}^{(-j)} - \mathcal{A}_{[0]}^{(-j)} \right), \\ \mathcal{H}_\infty &= -\frac{1}{2T} \sum_{j=1}^{T-1} \left( (T-j) \mathcal{A}_{[1]}^{(j)} + \mathcal{A}_{[0]}^{(j)} \middle| j \mathcal{A}_{[1]}^{(-j)} + \mathcal{A}_{[0]}^{(-j)} \right) - \left( \mathcal{A}_{[1]}^{(0)} \middle| \mathcal{A}_{[0]}^{(0)} \right), \end{aligned}$$

where  $\mathcal{A}_{[p]}^{(j)} := \pi_{(j)} \mathcal{A}_{[p]}^1$  for  $p = 0, 1$ .

Note that the expression for  $\mathcal{H}_0$  in Lemma 5.1 does not involve  $D$  and  $K$  since these are in the grade 0 component  $\tilde{\mathfrak{g}}_{(0)}$  of  $\tilde{\mathfrak{g}}$ , cf. §2.2.1. Thus  $H_0 := \pi_\ell(\mathcal{H}_0) = \frac{1}{2T} \sum_{j=1}^{T-1} (j A^{(j)} - \Pi^{(j)} | (T-j) A^{(-j)} - \Pi^{(-j)})$ . On the other hand, we also have  $P := \pi_\ell(\mathcal{P}) = -i\pi_\ell(D_{[0]}^{(1)}) = (A | \Pi) = \sum_{j=0}^{T-1} (A^{(j)} | \Pi^{(-j)})$ , cf. example 4.2. We define

the Hamiltonian as  $H := 2H_0 + P$ , which reads

$$H = \frac{1}{2\pi T} \int_{S^1} d\theta \left( \sum_{j=0}^{T-1} \left( j(T-j) \langle A^{(j)}(\theta), A^{(-j)}(\theta) \rangle + (T-2j) \langle A^{(j)}(\theta), \Pi^{(-j)}(\theta) \rangle \right) + \sum_{j=1}^{T-1} \langle \Pi^{(j)}(\theta), \Pi^{(-j)}(\theta) \rangle \right).$$

As discussed in §4.5.3, in the reduced theory the Hamiltonian is given by this same expression but up to the addition of a term proportional to the constraint  $\Pi^{(0)}$ . In the simplest case when  $T = 2$  the above expression for  $H$  coincides with the Hamiltonian of the symmetric space  $\sigma$ -model, see *e.g.* [85]. If, however, we take  $\mathfrak{g}$  to be the Grassmann envelope of a Lie superalgebra and  $T = 2n$  then  $H$  coincides, up to conventions, with the Hamiltonian of the  $\mathbb{Z}_{2n}$ -graded supercoset  $\sigma$ -model [56]. Moreover, for semisimple  $\mathfrak{g}$  and  $T \in \mathbb{Z}_{\geq 2}$  arbitrary, the Hamiltonian  $H$  reproduces that of the bosonic truncation of the  $\mathbb{Z}_{2T}$ -graded supercoset  $\sigma$ -model [56].

The twist function (94) in the case  $T = 2$ , namely

$$\varphi(z) = \frac{2z}{(1-z^2)^2}, \quad (95)$$

coincides with the twist function of the symmetric space  $\sigma$ -model in both the formalism of [79] and [85]. More precisely, due to a difference in conventions, the twist function in [79] which is denoted as  $\phi(z^2)$  there, is related to the twist function (95) as  $\phi(z^2) = -2z\varphi(z)$ . However, it is worth pointing out here that the Lax matrix (93), which coincides with that given in [85, (3.12)] for the case  $T = 2$ , namely

$$\mathcal{L}(z) = A^{(0)} + \frac{1}{2}(1-z^{-2})\Pi^{(0)} + \frac{1}{2}(z+z^{-1})A^{(1)} + \frac{1}{2}(z-z^{-1})\Pi^{(1)},$$

differs from the expression in [79, (61)] for the same model, which in the present notation reads

$$\widetilde{\mathcal{L}}(z) = -A^{(0)} - \frac{1}{2}(z+z^{-1})A^{(1)} - \frac{1}{2}(z-z^{-1})\Pi^{(1)}.$$

Aside from a trivial overall sign, we see that the difference between these two Lax matrices is a term proportional to the Hamiltonian constraint  $\Pi^{(0)}$ . However, the presence of the latter is crucial. Indeed, whereas the Lax matrix of [85, (3.12)] admits an affine Gaudin model description, as shown above, the Lax matrix of [79, (61)] does not. This can be seen explicitly from the form of the  $r$ - and  $s$ -matrices in [79, Theorem 3.1]. Indeed, the  $r$ - and  $s$ -matrices of [79] which we denote here by  $\widetilde{r}_{12}(z, z')$  and  $\widetilde{s}_{12}(z, z')$  respectively, take the form

$$\widetilde{r}_{12}(z, z') = r_{12}(z, z') + \frac{z^2 - z'^2}{4} C_{12}^{(0)}, \quad \widetilde{s}_{12}(z, z') = s_{12}(z, z') - \frac{2 - z^2 - z'^2}{4} C_{12}^{(0)}, \quad (96)$$

in terms of the  $r$ - and  $s$ -matrices in (66) which in the present case with  $T = 2$  explicitly read

$$\begin{aligned} r_{12}(z, z') &= \frac{2 - 2z^2 - 2z'^2 + z^4 + z'^4}{4(z^2 - z'^2)} C_{12}^{(0)} + \frac{z^2 + z'^2 - 4z^2 z'^2 + z^4 z'^2 + z^2 z'^4}{4zz'(z^2 - z'^2)} C_{12}^{(1)}, \\ s_{12}(z, z') &= \frac{2 - z^2 - z'^2}{4} C_{12}^{(0)} + \frac{(1 - z^2 z'^2)}{4zz'} C_{12}^{(1)}. \end{aligned}$$

We refer the reader to [83, §3] for a more detailed discussion of such a difference in the  $r$ - and  $s$ -matrices coming from the omission of the Hamiltonian constraint  $\Pi^{(0)}$  in the Lax matrix, in the context of semi-symmetric space  $\sigma$ -models, *i.e.* the case  $T = 4$ . Since the  $r$ - and  $s$ -matrices (96) of [79] differ from those of a general affine Gaudin model derived in Corollary 4.8, it is therefore clear that the formulation of the symmetric space  $\sigma$ -model given in [79] cannot admit a Gaudin model interpretation.

**5.2.2. One-parameter deformations.** To construct integrable deformations of the  $\mathbb{Z}_T$ -graded coset  $\sigma$ -model we proceed as we did in §5.1.2 for deforming the principal chiral model. Namely we will deform the divisor  $\langle t \rangle \mathcal{D}$ , with  $\mathcal{D}$  the divisor (89) underlying the affine Gaudin model from §5.2.1, while preserving its  $\langle t \rangle$ -invariance. There are two ways of splitting the double pole at 1, either into two real points or into a pair of complex conjugate points, which we will also refer to as the real and complex branches respectively. We note, however, that moving the double pole in the divisor (89) from 1 to  $r > 0$  amounts to a trivial rescaling  $z \mapsto z' := r^{-1}z$  of the coordinate on  $\mathbb{P}^1$  along with a rescaling of the formal field  $\mathcal{A}_{[1]}^1$  appearing in (90) by  $r$ , namely  $\mathcal{A}_{[1]}^r = r\mathcal{A}_{[1]}^1$ , cf. [19]. It follows that the Poisson brackets (91) of the  $\mathfrak{g}$ -valued classical fields remain undeformed except for a possible change in the level in front of the  $\delta'_{\theta\theta'}$ -term in (91b). We shall therefore only consider one-parameter deformations. To build the deformation in the complex branch we start from the divisor

$$\mathcal{D} = e^{i\vartheta} + \infty \quad (97a)$$

where  $\vartheta \in ]0, \frac{\pi}{T}[$  plays the role of the real deformation parameter. Similarly, in the real branch the deformation is constructed using the divisor

$$\mathcal{D} = \lambda^{1/2} + \lambda^{-1/2} + \infty \quad (97b)$$

where  $\lambda \in \mathbb{R}_{>0}$  is the deformation parameter. In order to treat both branches at once we note that in both cases the divisor  $\langle t \rangle \mathcal{D}$  is of the form  $\langle t \rangle \mathcal{D} = x_+ + x_- + \infty$  where  $x_- := x_+^{-1}$ . When  $x_+ \in \mathbb{R}$  this corresponds to the real branch whereas  $|x_+| = 1$  corresponds to the complex branch. In the notation of §4 we have  $\mathfrak{Z} = \{x_+, x_-, \infty\}$  and  $\mathfrak{Z}' = \{x_+, x_-\}$ .

By virtue of Proposition 4.6, the formal Lax matrix can be written in terms of the pair of formal fields  $\mathcal{A}_{[0]}^{x_{\pm}}$  associated with the pair of points  $x_{\pm}$  as

$$\mathcal{L}(z) = \frac{1}{T} \sum_{k=0}^{T-1} \left( \frac{\sigma^k \mathcal{A}_{[0]}^{x_+}}{z - \omega^k x_+} + \frac{\sigma^k \mathcal{A}_{[0]}^{x_-}}{z - \omega^k x_-} \right). \quad (98)$$

We fix the levels to be

$$\ell_0^{x^\pm} = \pm \frac{T}{x_+^T - x_-^T}. \quad (99)$$

Define the pair of  $\mathfrak{g}$ -valued classical fields  $\mathcal{J}_\pm := A_{[0]}^{x^\pm}$ . If the deformation is in the real branch then these are in fact both  $\mathfrak{g}_0$ -valued and moreover  $\ell^{x^\pm} \in \mathbb{R}$ . By contrast, in the complex branch we have  $\tau\mathcal{J}_+ = \mathcal{J}_-$ , recalling the action of  $\Pi$  on  $\mathfrak{F}$  from §4.2.1, cf. also (54). Moreover, the levels (99) satisfy  $\ell^{x_+} = \ell^{x_-}$ . It follows from Corollary 4.5 that the non-trivial Poisson brackets of the classical fields are given by

$$\{\mathcal{J}_{\pm 1}(\theta), \mathcal{J}_{\pm 2}(\theta')\} = -[C_{12}, \mathcal{J}_{\pm 2}(\theta)]\delta_{\theta\theta'} - \ell_0^{x^\pm} C_{12} \delta'_{\theta\theta'}. \quad (100)$$

The Lax matrix and twist function are then obtained from the formal Lax matrix (98) by the procedure described in §4.4.2. We find

$$\begin{aligned} (\varrho \otimes \pi_\ell \otimes \text{id})\mathcal{L}(z) &= \frac{1}{T} \sum_{k=0}^{T-1} \left( \frac{\ell_0^{x_+} \partial + \sigma^k \mathcal{J}_+}{z - \omega^k x_+} + \frac{\ell_0^{x_-} \partial + \sigma^k \mathcal{J}_-}{z - \omega^k x_-} \right) \\ &= \frac{Tz^{T-1}}{(z^T - x_+^T)(z^T - x_-^T)} \left( \partial + \sum_{j=1}^T \frac{T-j + jz^{-T}}{T} z^j A^{(j)} + \sum_{j=1}^T \frac{1-z^{-T}}{T} z^j \Pi^{(j)} \right). \end{aligned} \quad (101)$$

Here we defined the following linear combination of the  $\mathfrak{g}$ -valued classical fields,

$$A^{(j)} := \frac{1-x_-^T}{T} x_+^{T-j} \mathcal{J}_+^{(j)} + \frac{1-x_+^T}{T} x_-^{T-j} \mathcal{J}_-^{(j)}, \quad (102a)$$

$$\Pi^{(j)} := \frac{j+(T-j)x_-^T}{T} x_+^{T-j} \mathcal{J}_+^{(j)} + \frac{j+(T-j)x_+^T}{T} x_-^{T-j} \mathcal{J}_-^{(j)}, \quad (102b)$$

for  $j = 1, \dots, T$ , and where  $\mathcal{J}_\pm^{(j)} := \pi_{(j)} \mathcal{J}_\pm$ . By comparing (101) with the general form in (64) we deduce that, when written in terms of the classical fields (102), the Lax matrix is of exactly the same form as it was in §5.2.1 for the undeformed case, cf. (93). On the other hand, the twist function (94) gets deformed to

$$\varphi(z) = \frac{Tz^{T-1}}{(z^T - x_+^T)(z^T - x_-^T)}. \quad (103)$$

As expected, we recover the original twist function (94) in the limit  $x_\pm \rightarrow 1$ . Now by Corollary 4.8, the twist function being deformed corresponds to the fact that the fields (102) satisfy a deformation of the Poisson brackets (91). These can be derived using the expressions (102) and the Poisson brackets (100). However, since they are equivalent to (100) we will not give their explicit form. See [17, Appendix D] for explicit expressions in the case  $T = 2$ .

In the present case, the formal constraint defined in §4.5.3 reads  $\mathcal{C} = \pi_{(0)} \mathcal{A}_{[0]}^{x_+} + \pi_{(0)} \mathcal{A}_{[0]}^{x_-}$ . We note that the levels (99) still satisfy the condition (74) of §4.5.3. Upon applying the linear map  $\varrho \otimes \pi_\ell$  we find

$$(\varrho \otimes \pi_\ell)\mathcal{C} = \mathcal{J}_+^{(0)} + \mathcal{J}_-^{(0)} = \Pi^{(0)}, \quad (104)$$

where the first equality uses the  $\Pi$ -equivariance of  $\varrho$  and the second equality follows from the definition (102b) with  $j = T$ . When  $T = 2$  this corresponds to the constraint of the deformed  $\mathbb{Z}_2$ -graded coset  $\sigma$ -model [17].

Consider, finally, the quadratic Hamiltonians defined as in (70), namely  $\hat{\mathcal{H}}_x := \text{res}_x \frac{1}{2} (\mathcal{L}(z) | \mathcal{L}(z)) \varphi(z)^{-1} dz$ , for  $x \in \{0, \infty\}$ , where  $\varphi(z)$  is the twist function given in (103) whose zeroes are at the origin and infinity. By comparing the expression for the formal Lax matrix (98) to the right hand side of the first line in (101), it follows by the same reasoning leading to the second line in (101) that we can write, cf. (93) in the undeformed case,

$$\tilde{\mathcal{L}}(z) := \varphi(z)^{-1} \mathcal{L}(z) = \sum_{j=1}^T \frac{T-j+jz^{-T}}{T} z^j \mathcal{A}_{[1]}^{(j)} + \sum_{j=1}^T \frac{1-z^{-T}}{T} z^j \mathcal{A}_{[0]}^{(j)}, \quad (105)$$

where here we defined the formal fields  $\mathcal{A}_{[p]}^{(j)}$  for  $j \in \mathbb{Z}_T$  and  $p = 0, 1$ , by analogy with the definition of the  $\mathfrak{g}$ -valued classical fields (102), as

$$\mathcal{A}_{[1]}^{(j)} := \frac{1-x_+^T}{T} x_+^{T-j} \pi_{(j)} \mathcal{A}_{[0]}^{x_+} + \frac{1-x_+^T}{T} x_-^{T-j} \pi_{(j)} \mathcal{A}_{[0]}^{x_-}, \quad (106a)$$

$$\mathcal{A}_{[0]}^{(j)} := \frac{j+(T-j)x_-^T}{T} x_+^{T-j} \pi_{(j)} \mathcal{A}_{[0]}^{x_+} + \frac{j+(T-j)x_+^T}{T} x_-^{T-j} \pi_{(j)} \mathcal{A}_{[0]}^{x_-}, \quad (106b)$$

for  $j = 1, \dots, T$ .

**Lemma 5.2.** *The expression for the quadratic Hamiltonians in terms of (106) are related to those of the quadratic Hamiltonians from Lemma 5.1 as*

$$\hat{\mathcal{H}}_0 = \mathcal{H}_0, \quad \hat{\mathcal{H}}_\infty = \mathcal{H}_\infty + \frac{2-x_+^T-x_-^T}{2T} \left( \mathcal{A}_{[0]}^{(0)} | \mathcal{A}_{[0]}^{(0)} \right).$$

Applying  $\pi_\ell$  to  $\hat{\mathcal{H}}_0$  we obtain  $\hat{H}_0 := \pi_\ell(\hat{\mathcal{H}}_0) = \pi_\ell(\mathcal{H}_0) = H_0$  using Lemma 5.2, where the latter was defined in §5.2.1. Likewise, the momentum reads  $\hat{P} := \pi_\ell(\mathcal{P}) = P - \frac{2-x_+^T-x_-^T}{2T} (\Pi^{(0)} | \Pi^{(0)})$ . We define the Hamiltonian of the deformed  $\mathbb{Z}_T$ -graded coset  $\sigma$ -model as

$$\hat{H} := 2\hat{H}_0 + \hat{P} = H - \frac{2-x_+^T-x_-^T}{4\pi T} \int_{S^1} d\theta \langle \Pi^{(0)}(\theta), \Pi^{(0)}(\theta) \rangle. \quad (107)$$

Note that since  $H$  and  $\hat{H}$  differ by a term proportional to the constraint (104), they define equivalent Hamiltonians in the reduced theory, cf. §4.5.3. When  $T = 2$  the expression (107) coincides with the Hamiltonian of the deformed symmetric space  $\sigma$ -model in [85, §4.3] for the complex branch and [50, (5.5)] for the real branch. For  $T = 4$  and  $\mathfrak{g}$  the Grassmann envelope of a Lie superalgebra, the expression (107) corresponds, in the complex branch, to the Hamiltonian of the so called  $\eta$ -deformed semi-symmetric space  $\sigma$ -model [18, 19] where  $\eta = \tan \vartheta$  with  $x_+ = e^{i\vartheta}$ . Likewise, in the real branch with  $x_+ = \lambda^{1/2} \in \mathbb{R}$  it corresponds to the Hamiltonian of the so called  $\lambda$ -deformation [51]. More precisely, the  $\eta$ - and  $\lambda$ -deformations can both be defined as certain ‘lifts’ to the cotangent bundle  $T^* \mathcal{L}G_0$  of the dihedral affine Gaudin model with divisor (97a) and (97b) respectively, and levels (99), cf. discussions at the end of §5.1.1 and §5.1.2.

The maximal deformation limit in the complex branch is  $\vartheta \rightarrow \frac{\pi}{T}$ . It corresponds to the point  $x_+$  coinciding with  $\omega x_-$ . In this limit the divisor becomes  $\mathcal{D} = 2 \cdot x_+ + \infty$  with  $x_+ = \omega^{1/2} \in \mathcal{Z}_r^1$  in the notation of §3. In particular, the  $\mathfrak{g}$ -valued fields  $\omega^{-p/2} A_{[p]}^{x_+}$  for  $p = 0, 1$ , associated with the double point at  $x_+$ , take value in the real form  $\mathfrak{g}_1$ . This is by contrast with the  $\mathfrak{g}$ -valued classical fields  $A_{[p]}^1$  associated with 1 in the undeformed model of §5.2.1 which take value in the real form  $\mathfrak{g}_0$ .

**5.2.3. bi-Yang-Baxter  $\sigma$ -model.** The bi-Yang-Baxter  $\sigma$ -model [58] is yet another two-parameter deformation of the principal chiral model different from the one presented in §5.1.2. Its integrability within the Hamiltonian formalism was studied in [14]. It was found to correspond to a non-cyclotomic one-parameter deformation of the deformed  $\mathbb{Z}_2$ -graded coset  $\sigma$ -model discussed in §5.2.2. In other words, the  $\Pi = D_4$  dihedral symmetry of the latter is broken down to a  $\langle \mathfrak{t} \rangle = D_2 = \mathbb{Z}_2$  symmetry. This is achieved by deforming the divisor  $\Pi\mathcal{D} = e^{i\vartheta} + e^{-i\vartheta} + e^{i(\vartheta+\pi)} + e^{-i(\vartheta+\pi)} + \infty$ , with  $\mathcal{D}$  given in (97a), to the new divisor  $\langle \mathfrak{t} \rangle \mathcal{D}'$  where  $\mathcal{D}' := e^{i\vartheta} + e^{i(\psi+\pi)} + \infty$  and  $\vartheta, \psi \in ]0, \frac{\pi}{2}[$ . In the notation of §4 we have  $\mathcal{Z} = \{x_+, x_-, y_+, y_-, \infty\}$ , where we defined  $x_{\pm} := e^{\pm i\vartheta}$  and  $y_{\pm} := e^{\pm i(\psi+\pi)}$ , and  $\mathcal{Z}' = \{x_+, x_-, y_+, y_-\}$ .

There are four formal fields  $\mathcal{A}_{[0]}^{x_{\pm}}$  and  $\mathcal{A}_{[0]}^{y_{\pm}}$  associated to the points of  $\mathcal{Z} \setminus \{\infty\}$ . The levels are chosen as

$$\ell_0^{x_{\pm}} = \pm \frac{2}{(x_+ + x_- - y_+ - y_-)(x_+ - x_-)}, \quad \ell_0^{y_{\pm}} = \mp \frac{2}{(x_+ + x_- - y_+ - y_-)(y_+ - y_-)}.$$

They satisfy the condition (74) from §4.5.3. Letting  $\mathcal{J}_{\pm} := \mathcal{A}_{[0]}^{x_{\pm}}$  and  $\tilde{\mathcal{J}}_{\pm} := \mathcal{A}_{[0]}^{y_{\pm}}$  denote the four  $\mathfrak{g}$ -valued classical fields, their non-trivial Poisson brackets read

$$\{\mathcal{J}_{\pm 1}(\theta), \mathcal{J}_{\pm 2}(\theta')\} = -[C_{12}, \mathcal{J}_{\pm 2}(\theta)] \delta_{\theta\theta'} - \ell_0^{x_{\pm}} C_{12} \delta'_{\theta\theta'}, \quad (108a)$$

$$\{\tilde{\mathcal{J}}_{\pm 1}(\theta), \tilde{\mathcal{J}}_{\pm 2}(\theta')\} = -[C_{12}, \tilde{\mathcal{J}}_{\pm 2}(\theta)] \delta_{\theta\theta'} - \ell_0^{y_{\pm}} C_{12} \delta'_{\theta\theta'}, \quad (108b)$$

from Corollary 4.5. These coincide with the Poisson brackets from [14, §5.1].

As usual, the Lax matrix and twist function of the model are obtained by applying the linear map  $\varrho \otimes \pi_{\ell} \otimes \text{id}$  to the formal Lax matrix given by Proposition 4.6, as in §4.4.2. Up to an overall normalisation, we find the Lax matrix and twist function of the bi-Yang-Baxter  $\sigma$ -model in the Hamiltonian formalism obtained in [14]. We also note that in the limit when  $\psi = \vartheta$ , *i.e.*  $x_{\pm} = -y_{\pm}$ , we recover the twist function of the deformed  $\mathbb{Z}_2$ -graded coset  $\sigma$ -model, namely (103) with  $T = 2$ .

The formal constraint defined in §4.5.3 is given by the sum of all formal fields  $\mathcal{C} = \mathcal{A}_{[0]}^{x_+} + \mathcal{A}_{[0]}^{x_-} + \mathcal{A}_{[0]}^{y_+} + \mathcal{A}_{[0]}^{y_-}$ . Note that  $\pi_{(0)} = \text{id}$  since in the present case  $T = 1$ . Applying the linear map  $\varrho \otimes \pi_{\ell}$  we find that the  $\mathfrak{g}$ -valued classical fields attached to the four points in  $\mathcal{Z} \setminus \{\infty\}$  satisfy the constraint  $(\varrho \otimes \pi_{\ell})\mathcal{C} = \mathcal{J}_+ + \mathcal{J}_- + \tilde{\mathcal{J}}_+ + \tilde{\mathcal{J}}_- \approx 0$ . The latter is equivalent to the Hamiltonian constraint  $X + \tilde{X} \approx 0$  of [14], if in the notation of that paper, *cf.* equations (5.1) there, we replace  $X$  in (5.1b) by  $\tilde{X}$  to get the expressions for  $\tilde{\mathcal{J}}_{\pm}$  before imposing the constraint.

**5.3. Affine Toda field theory.** Let  $\mathfrak{g}$  be a finite-dimensional simple Lie algebra over  $\mathbb{C}$  of rank  $\ell := \text{rk } \mathfrak{g}$ . Let  $\mathfrak{h}$  be a Cartan subalgebra and denote by  $\Delta \subset \mathfrak{h}^*$  the root system of  $(\mathfrak{g}, \mathfrak{h})$ . We have the root space decomposition  $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathbb{C} E_{\alpha}$ . Fix a

basis of simple roots  $\{\alpha_i\}_{i=1}^\ell$  and let  $\theta \in \Delta$  be the corresponding maximal root. The Coxeter number of  $\mathfrak{g}$  is defined as  $h := \text{ht } \theta + 1$ , where  $\text{ht } \alpha$  denotes the height of a root  $\alpha \in \Delta$  relative to the system of simple roots  $\{\alpha_i\}_{i=1}^\ell$ . The extended system of simple roots is defined by adjoining  $\alpha_0 := -\theta$  to  $\{\alpha_i\}_{i=1}^\ell$ . We will make use of the shorthands  $E_i := E_{\alpha_i}$  and  $F_i := E_{-\alpha_i}$  for  $i = 0, \dots, \ell$ .

We fix a non-degenerate invariant bilinear form  $\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C}$ . To any  $\alpha \in \Delta$  we associate the Cartan element  $H_\alpha \in \mathfrak{h}$  defined by  $\langle H_\alpha, H \rangle = \alpha(H)$  for all  $H \in \mathfrak{h}$ . We will use the shorthand  $H_i := H_{\alpha_i}$  for  $i = 0, \dots, \ell$ . Restricting  $\langle \cdot, \cdot \rangle$  to the Cartan subalgebra  $\mathfrak{h}$ , the bijection  $\mathfrak{h}^* \xrightarrow{\sim} \mathfrak{h}$ ,  $\alpha \mapsto H_\alpha$  induces a bilinear form on  $\mathfrak{h}^*$  which we also denote  $\langle \cdot, \cdot \rangle$ . We normalise the bilinear form on  $\mathfrak{g}$  such that  $\langle \theta, \theta \rangle = 2$ . Introduce the coroot  $\check{\alpha} := \frac{2}{\langle \alpha, \alpha \rangle} H_\alpha \in \mathfrak{h}$  for each  $\alpha \in \Delta$ . Letting  $\epsilon_i := \frac{2}{\langle \alpha_i, \alpha_i \rangle}$  for all  $i = 0, \dots, \ell$  we then have  $\check{\alpha}_i = \epsilon_i H_i$ . Note that  $\epsilon_0 = 1$  from the choice of normalisation of the bilinear form  $\langle \cdot, \cdot \rangle$ . The root vectors  $E_{-\alpha}$ ,  $\alpha \in \Delta^+$  can be normalised such that  $[E_\alpha, E_{-\alpha}] = \check{\alpha}$ . We then have

$$[\check{\alpha}_i, E_j] = a_{ij} E_j, \quad [\check{\alpha}_i, F_j] = -a_{ij} F_j, \quad [E_i, F_j] = \delta_{ij} \check{\alpha}_i,$$

for  $i, j = 0, \dots, \ell$ , where  $a_{ij} := \alpha_j(\check{\alpha}_i)$  are the components of the Cartan matrix of the untwisted affine Kac-Moody algebra  $\tilde{\mathfrak{g}}$  associated with  $\mathfrak{g}$ . Writing the maximal root  $\theta$  in the basis of simple roots as  $\theta = \sum_{i=1}^\ell a_i \alpha_i \in \Delta$  with  $a_i \in \mathbb{Z}_{>0}$  and defining  $a_0 := 1$ , we have  $\sum_{j=1}^\ell a_{ij} a_j = 0$ . It then also follows that  $\check{\theta} = \sum_{i=1}^\ell \check{a}_i \check{\alpha}_i$  where  $\check{a}_i := a_i \epsilon_i^{-1}$ . And since  $\check{a}_0 = 1$  we deduce that  $\sum_{j=1}^\ell a_{ij} \check{a}_i = 0$ . We note that  $a_{ij} \epsilon_j = \langle \check{\alpha}_j, \check{\alpha}_i \rangle =: c_{ij}$  and  $a_{ij} \epsilon_i^{-1} = \langle \alpha_j, \alpha_i \rangle =: b_{ij}$  are both symmetric. Since  $\langle E_i, F_i \rangle = \epsilon_i$ , it is convenient to define  $E^i := \epsilon_i^{-1} E_i$  and  $F^i := \epsilon_i^{-1} F_i$  for all  $i = 0, \dots, \ell$ . We also let  $\{H^i\}_{i=1}^\ell$  be a dual basis of  $\{H_i\}_{i=1}^\ell$  in  $\mathfrak{h}$ , *i.e.*  $\langle H^i, H_j \rangle = \delta_j^i$  for every  $i, j = 1, \dots, \ell$ . Note that for any  $j = 0, \dots, \ell$  we have the relation  $\sum_{k=1}^\ell \alpha_j(H_k) H^k = H_j$ .

Let  $\omega := e^{2\pi i/h}$  and define an automorphism  $\sigma \in \text{Aut } \mathfrak{g}$  by  $\sigma(E_i) = \omega E_i$ ,  $\sigma(\check{\alpha}_i) = \check{\alpha}_i$  and  $\sigma(F_i) = \omega^{-1} F_i$ , for  $i = 1, \dots, \ell$ . This is a Coxeter automorphism in the outer automorphism class of the identity, *i.e.*  $[\sigma] = [\text{id}] \in \text{Aut } \mathfrak{g}/\text{Inn } \mathfrak{g}$ . That is, the invariant subalgebra  $\mathfrak{g}_{(0), \mathbb{C}} = \mathfrak{h}$  is abelian and  $\sigma$  has minimum order among all automorphisms  $v \in \text{Aut } \mathfrak{g}$  such that  $[v] = [\text{id}] \in \text{Aut } \mathfrak{g}/\text{Inn } \mathfrak{g}$  and  $\mathfrak{g}^v$  is abelian, see *e.g.* [13]. Because  $\sigma$  lies in the outer automorphism class of the identity, its action on any root vector  $E_\alpha$ ,  $\alpha \in \Delta$  reads

$$\sigma(E_\alpha) = \omega^{\text{ht } \alpha} E_\alpha. \quad (109)$$

We consider the anti-linear automorphism  $\tau \in \overline{\text{Aut } \mathfrak{g}}$  defined by

$$\tau(\check{\alpha}_i) = \check{\alpha}_i, \quad \tau(E_\alpha) = E_\alpha, \quad (110)$$

for all  $i = 1, \dots, \ell$  and  $\alpha \in \Delta$ . The corresponding real Lie algebra  $\mathfrak{g}_0$  is the split real form of  $\mathfrak{g}$ . Recall the subspaces  $\mathfrak{g}_{(j)} \subset \mathfrak{g}_0$  for  $j \in \mathbb{Z}_h$  defined in §2.1. We have  $\mathfrak{g}_{(0)} = \mathfrak{h}_0 := \text{span}_{\mathbb{R}}\{H_i\}_{i=1}^\ell$ . Moreover, it follows from (109) that a basis of the subspace  $\mathfrak{g}_{(1)}$  (resp.  $\mathfrak{g}_{(-1)}$ ) is given by  $E_i$  (resp.  $F_i$ ) for  $i = 0, \dots, \ell$ . In other words,  $\mathfrak{g}_{(1)} = \text{span}_{\mathbb{R}}\{E_i\}_{i=0}^\ell$  and  $\mathfrak{g}_{(-1)} = \text{span}_{\mathbb{R}}\{F_i\}_{i=0}^\ell$ .



5.3.1. *Divisor and levels.* Consider the divisor

$$\mathcal{D} = 2 \cdot 0 + 2 \cdot \infty. \quad (111)$$

We have  $\mathfrak{Z} = \{0, \infty\}$  so  $\mathfrak{Z}' = \emptyset$ . By Proposition 4.6 the formal Lax matrix is  $\mathcal{L}(z) = \mathcal{A}_{[0]}^0 z^{-1} + \mathcal{A}_{[1]}^0 z^{-2} - \mathcal{A}_{[1]}^\infty$ . The formal fields  $\mathcal{A}_{[p]}^0$  for  $p = 0, 1$  and  $\mathcal{A}_{[1]}^\infty$  are defined in (49c). Recalling the notation (34c), they read

$$\mathcal{A}_{[0]}^0 = \mathsf{D} \otimes \mathsf{K}_{[0]}^{(0)} + \mathsf{K} \otimes \mathsf{D}_{[0]}^{(0)} + \sum_{n \in \mathbb{Z}} \sum_{j=1}^{\ell} H_{j,-n} \otimes H_{n,[0]}^{j(0)}, \quad (112a)$$

$$\mathcal{A}_{[1]}^0 = \sum_{n \in \mathbb{Z}} \sum_{j=0}^{\ell} F_{j,-n} \otimes E_{n,[1]}^{j(0)}, \quad \mathcal{A}_{[1]}^\infty = \sum_{n \in \mathbb{Z}} \sum_{j=0}^{\ell} E_{j,-n} \otimes F_{n,[1]}^{j(\infty)}. \quad (112b)$$

It follows from the definition of the action (50) of  $\mathfrak{t} \in \Pi$  on formal fields in  $\widetilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}$ , the reality conditions (110) and those on the generators  $\mathsf{K}, \mathsf{D}$  defined in §2.2.1 that the formal fields in (112) are all real in the sense that  $r_{\mathfrak{t}} \mathcal{A}_{[p]}^0 = \mathcal{A}_{[p]}^0$  for  $p = 0, 1$  and  $r_{\mathfrak{t}} \mathcal{A}_{[1]}^\infty = \mathcal{A}_{[1]}^\infty$ , cf. (114) below. The Poisson brackets among the formal fields (112) are determined from Proposition 4.4 to be

$$\{\mathcal{A}_{[0]1}^0, \mathcal{A}_{[0]2}^0\} = -[\widetilde{C}_{12}^{(0)}, \mathcal{A}_{[0]2}^0], \quad \{\mathcal{A}_{[0]1}^0, \mathcal{A}_{[1]2}^0\} = -[\widetilde{C}_{12}^{(0)}, \mathcal{A}_{[1]2}^0],$$

with all other brackets vanishing. Recall from the proof of Proposition 4.4 that these Poisson brackets are equivalent to the Lie brackets on  $\widetilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}$ . The latter is spanned by  $\mathsf{D}_{[0]}^{(0)}, \mathsf{K}_{[0]}^{(0)}$  together with  $H_{n,[0]}^{j(0)}$  for  $j = 1, \dots, \ell, n \in \mathbb{Z}$  and  $E_{n,[1]}^{j(0)}, F_{n,[1]}^{j(\infty)}$  for  $j = 0, \dots, \ell, n \in \mathbb{Z}$ , in terms of which its non-trivial Lie brackets read

$$[H_{m,[0]}^{i(0)}, H_{n,[0]}^{j(0)}] = m \delta_{m+n,0} \langle H^i, H^j \rangle \mathsf{K}_{[0]}^{(0)}, \quad [\mathsf{D}_{[0]}^{(0)}, H_{n,[0]}^{j(0)}] = n H_{n,[0]}^{j(0)}, \quad (113a)$$

$$[H_{m,[0]}^{i(0)}, E_{n,[1]}^{j(0)}] = \alpha_j \langle H^i, E_{m+n,[1]}^{j(0)} \rangle, \quad [\mathsf{D}_{[0]}^{(0)}, E_{n,[1]}^{j(0)}] = n E_{n,[1]}^{j(0)}. \quad (113b)$$

It follows from Proposition 3.10(iii) that under the anti-linear map  $c : \widetilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}} \rightarrow \widetilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}$  we have

$$c(H_{n,[0]}^{i(0)}) = H_{-n,[0]}^{i(0)}, \quad c(E_{n,[1]}^{j(0)}) = E_{-n,[1]}^{j(0)}, \quad c(F_{n,[1]}^{j(\infty)}) = F_{-n,[1]}^{j(\infty)}, \quad (114a)$$

$$c(\mathsf{K}_{[0]}^{(0)}) = -\mathsf{K}_{[0]}^{(0)}, \quad c(\mathsf{D}_{[0]}^{(0)}) = -\mathsf{D}_{[0]}^{(0)} \quad (114b)$$

for any  $i = 1, \dots, \ell, j = 0, \dots, \ell$  and  $n \in \mathbb{Z}$ .

We fix the levels to be  $\ell_0^0 = 1, \ell_1^0 = 0$  and  $\ell_1^\infty = 0$ . Note that by contrast with the examples discussed in §5.1 and §5.2, since  $\ell_1^0 = \ell_1^\infty = 0$  the assumption (36) made throughout §4 fails to hold. In particular, we cannot use the homomorphism  $\pi_\ell$  as defined in (44). Instead we will define a homomorphism  $\widetilde{\pi}_\ell$  from the complex Lie algebra  $\widetilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}$  of Proposition 3.10 to a completion of the symmetric algebra on a certain Heisenberg-type algebra.

5.3.2. *Toda fields.* Let  $\mathcal{L}\mathfrak{h} := \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}]$  and  $\mathcal{L}\mathfrak{g}_{(1), \mathbb{C}} := \mathfrak{g}_{(1), \mathbb{C}} \otimes \mathbb{C}[t, t^{-1}]$ . We consider the subspaces  $\mathcal{H}^{(+)} := \mathcal{L}\mathfrak{h} \dot{+} \mathcal{L}\mathfrak{g}_{(1), \mathbb{C}}$  and  $\mathcal{H}^{(-)} := \mathcal{L}\mathfrak{h}$  of the loop algebra  $\mathcal{L}\mathfrak{g}$ , cf. §2.2. Let  $(\cdot | \cdot) : \mathcal{H}^{(+)} \times \mathcal{H}^{(-)} \rightarrow \mathbb{C}$  denote the restriction of the bilinear form on  $\mathcal{L}\mathfrak{g}$  to this pair of subspaces. We endow  $\mathcal{H}^{(-)}$  with its natural abelian Lie algebra structure coming from  $\mathcal{L}\mathfrak{g}$ . The Lie algebra structure on  $\mathcal{H}^{(+)}$  is defined by letting  $\mathcal{L}\mathfrak{h}$  and  $\mathcal{L}\mathfrak{g}_{(1), \mathbb{C}}$  be abelian subalgebras and letting  $[\cdot, \cdot] : \mathcal{L}\mathfrak{h} \times \mathcal{L}\mathfrak{g}_{(1), \mathbb{C}} \rightarrow \mathcal{L}\mathfrak{g}_{(1), \mathbb{C}}$  be given by the restriction of the Lie bracket on  $\mathcal{L}\mathfrak{g}$ . Consider the direct sum of complex vector spaces  $\mathcal{H} := \mathcal{H}^{(+)} \oplus \mathcal{H}^{(-)} \oplus \mathbb{C}\mathbf{1}$ . For any  $\mathbf{X} \in \mathcal{H}^{(+)}$  we use the notation  $\mathbf{X}^{(+)} := (\mathbf{X}, 0, 0) \in \mathcal{H}$  to represent  $\mathbf{X}$  regarded as an element of  $\mathcal{H}$ , and similarly  $\mathbf{Y}^{(-)} := (0, \mathbf{Y}, 0) \in \mathcal{H}$  for any  $\mathbf{Y} \in \mathcal{H}^{(-)}$ . We define a Lie bracket on  $\mathcal{H}$  by setting

$$[\mathbf{X}^{(+)} + \mathbf{X}'^{(-)} + \alpha\mathbf{1}, \mathbf{Y}^{(+)} + \mathbf{Y}'^{(-)} + \beta\mathbf{1}] := [\mathbf{X}, \mathbf{Y}]^{(+)} + ((\mathbf{X}|\mathbf{Y}') - (\mathbf{Y}|\mathbf{X}'))\mathbf{1}.$$

for any  $\mathbf{X}, \mathbf{Y} \in \mathcal{H}^{(+)}$ ,  $\mathbf{X}', \mathbf{Y}' \in \mathcal{H}^{(-)}$  and  $\alpha, \beta \in \mathbb{C}$ . A basis of  $\mathcal{H}$  is given by  $E_n^{i(+)}$  and  $H_n^{j(\pm)}$  for  $i = 0, \dots, \ell$ ,  $j = 1, \dots, \ell$  and  $n \in \mathbb{Z}$ , together with  $\mathbf{1}$ . In terms of these the non-trivial Lie brackets on  $\mathcal{H}$  read

$$[H_m^{i(+)}, H_n^{j(-)}] = \delta_{m+n, 0} \langle H^i, H^j \rangle \mathbf{1}, \quad [H_m^{i(+)}, E_n^{j(+)}] = \alpha_j (H^i) E_{m+n}^{j(+)}. \quad (115)$$

We introduce an anti-linear automorphism  $c : \mathcal{H} \rightarrow \mathcal{H}$  simply by letting

$$c(H_n^{i(\pm)}) = H_{-n}^{i(\pm)}, \quad c(E_n^{j(+)} ) = E_{-n}^{j(+)}, \quad c(\mathbf{1}) = \mathbf{1}, \quad (116)$$

for all  $i = 1, \dots, \ell$ ,  $j = 0, \dots, \ell$  and  $n \in \mathbb{Z}$ .

The Lie bracket on  $\mathcal{H}$  extends by the Leibniz rule and linearity to a Poisson bracket on the symmetric algebra  $S(\mathcal{H})$ , cf. §4.1, which we denote  $\{\cdot, \cdot\} : S(\mathcal{H}) \times S(\mathcal{H}) \rightarrow S(\mathcal{H})$ . Recall the descending  $\mathbb{Z}_{\geq 0}$ -filtrations  $F_n \tilde{\mathfrak{g}}$  and  $\bar{F}_n \tilde{\mathfrak{g}}$  on  $\tilde{\mathfrak{g}}$ , or equivalently on  $\mathcal{L}\mathfrak{g}$ , defined in §2.2.2. We similarly define a pair of descending  $\mathbb{Z}_{\geq 0}$ -filtrations  $(F_n \mathcal{H})_{n \in \mathbb{Z}_{\geq 0}}$  and  $(\bar{F}_n \mathcal{H})_{n \in \mathbb{Z}_{\geq 0}}$  on the vector space  $\mathcal{H}$  as

$$F_n \mathcal{H} := (\mathfrak{h} \dot{+} \mathfrak{g}_{(1), \mathbb{C}}) \otimes t^n \mathbb{C}[t] \oplus \mathfrak{h} \otimes t^n \mathbb{C}[t], \quad (117a)$$

$$\bar{F}_n \mathcal{H} := (\mathfrak{h} \dot{+} \mathfrak{g}_{(1), \mathbb{C}}) \otimes t^{-n} \mathbb{C}[t^{-1}] \oplus \mathfrak{h} \otimes t^{-n} \mathbb{C}[t^{-1}], \quad (117b)$$

for  $n \in \mathbb{Z}_{\geq 0}$ . In turn, we use this to define a descending  $\mathbb{Z}_{\geq 0}$ -filtration on the commutative algebra  $S(\mathcal{H})$  by ideals  $F_n(S(\mathcal{H})) := (F_n \mathcal{H})S(\mathcal{H}) \cap (\bar{F}_n \mathcal{H})S(\mathcal{H})$ , cf. (39). The corresponding completion is a commutative algebra over  $\mathbb{C}$  which we denote by  $\hat{S}(\mathcal{H}) := \varprojlim S(\mathcal{H})/F_n(S(\mathcal{H}))$ . In the same way as in §4.1, the Poisson bracket on  $S(\mathcal{H})$  extends to one on the completion which we also denote

$$\{\cdot, \cdot\} : \hat{S}(\mathcal{H}) \times \hat{S}(\mathcal{H}) \longrightarrow \hat{S}(\mathcal{H}).$$

**Lemma 5.3.** *The set of elements  $\hat{E}_n^{j(+)} := \sum_{p \in \mathbb{Z}} p E_{n-p}^{j(+)} H_{j,p}^{(-)} - n E_n^{j(+)}$  for all  $n \in \mathbb{Z}$ ,  $j = 0, \dots, \ell$  and  $\mathbf{1} - 1$ , generate a Poisson ideal of  $\hat{S}(\mathcal{H})$ . We denote the corresponding quotient by  $\hat{S}_\ell(\mathcal{H})$ .*

The map  $c : \mathcal{H} \rightarrow \mathcal{H}$  extends as an anti-linear automorphism to  $S(\mathcal{H})$ . And since it preserves the subspaces  $F_n(S(\mathcal{H}))$  for each  $n \in \mathbb{Z}_{\geq 0}$  by sending the pair of subspaces in (117) to one another, it extends by continuity to an anti-linear automorphism of the completion  $\hat{S}(\mathcal{H})$ . Moreover, since  $c(\hat{E}_n^{j(+)} ) = -\hat{E}_{-n}^{j(+)}$  and  $c(\mathbf{1} - 1) = \mathbf{1} - 1$ , the

corresponding Poisson ideal of  $\hat{S}(\mathcal{H})$  from Lemma 5.3 is invariant under it. Thus we obtain an anti-linear automorphism

$$c : \hat{S}_\ell(\mathcal{H}) \longrightarrow \hat{S}_\ell(\mathcal{H}) \quad (118)$$

of the quotient defined in Lemma 5.3.

We are now in a position to define an analogue of the homomorphism  $\pi_\ell$  in the present case, replacing the definition (44). Pick real parameters  $\mu_i, \nu_i \in \mathbb{R}$  for each  $i = 0, \dots, \ell$  such that  $\mu_i \nu_i \geq 0$ . Consider the linear map

$$\tilde{\pi}_\ell : \tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}} \longrightarrow \hat{S}_\ell(\mathcal{H}), \quad (119a)$$

defined on the basis elements of  $\tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}$  as

$$\tilde{\pi}_\ell(E_{n,[1]}^{i(0)}) := \frac{\mu_i}{\sqrt{2}} E_n^{i(+)}, \quad \tilde{\pi}_\ell(F_{n,[1]}^{i(\infty)}) := -\frac{\nu_i}{\sqrt{2}} \delta_{n,0}, \quad \tilde{\pi}_\ell(\mathbf{K}_{[0]}^{(0)}) := i, \quad (119b)$$

$$\tilde{\pi}_\ell(H_{n,[0]}^{j(0)}) := H_n^{j(+)} - \frac{in}{2} H_n^{j(-)}, \quad \tilde{\pi}_\ell(\mathbf{D}_{[0]}^{(0)}) := \sum_{n \in \mathbb{Z}} \sum_{k=1}^{\ell} n H_{k,-n}^{(+)} H_n^{k(-)}, \quad (119c)$$

for  $i = 0, \dots, \ell$ ,  $j = 1, \dots, \ell$  and  $n \in \mathbb{Z}$ . This map is seen to be equivariant with respect to the pair of anti-linear automorphisms  $c$  on  $\tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}$  and  $\hat{S}_\ell(\mathcal{H})$  defined respectively in Proposition 3.10 and (118), using (114) and (116).

**Lemma 5.4.** *The map (119) is a homomorphism of Lie algebras.*

We define the *formal Toda fields* as

$$\mathcal{A}^{(\pm)} := \sum_{n \in \mathbb{Z}} \sum_{j=1}^{\ell} H_{j,-n} \otimes H_n^{j(\pm)}, \quad \mathcal{E} = \sum_{n \in \mathbb{Z}} \sum_{j=0}^{\ell} F_{j,-n} \otimes E_n^{j(+)}.$$

These are elements of the completed tensor product  $\mathcal{L}(\mathfrak{h} \dot{+} \mathfrak{g}_{(-1),\mathbb{C}}) \hat{\otimes} \hat{S}_\ell(\mathcal{H})$ , and both are real in the sense that they are invariant under the action of  $\tau \otimes c$ .

**Lemma 5.5.** *The collection of non-trivial Poisson brackets between the formal Toda fields read*

$$\{\mathcal{A}_1^{(+)}, \mathcal{A}_2^{(-)}\} = \hat{C}_{12}^{(0)}, \quad \{\mathcal{A}_1^{(+)}, \mathcal{E}_2\} = -[\hat{C}_{12}^{(0)}, \mathcal{E}_2],$$

where  $\hat{C}_{12}^{(0)} := \sum_{n \in \mathbb{Z}} \sum_{j=1}^{\ell} H_{j,-n} \otimes H_n^j$ .

The *Toda field*  $\Phi$  and its conjugate momentum  $\Pi$  are the  $\mathfrak{h}_0$ -valued classical fields defined as

$$\Phi := (\varrho \otimes \text{id})\mathcal{A}^{(-)} = \sum_{j=1}^{\ell} H_j \otimes \sum_{n \in \mathbb{Z}} e_{-n} \otimes H_n^{j(-)}, \quad (120a)$$

$$\Pi := (\varrho \otimes \text{id})\mathcal{A}^{(+)} = \sum_{j=1}^{\ell} H_j \otimes \sum_{n \in \mathbb{Z}} e_{-n} \otimes H_n^{j(+)}. \quad (120b)$$

To see that  $\Pi$  is the conjugate momentum we apply  $\varrho \otimes \varrho \otimes \text{id}$  to the first relation in Lemma 5.5. This yields the canonical Poisson brackets  $\{\Pi_1(\theta), \Phi_2(\theta')\} = C_{12}^{(0)} \delta_{\theta\theta'}$ .

Applying the root  $\alpha_j$  to the Toda field  $\Phi$  we obtain the classical field

$$\alpha_j(\Phi) = \sum_{n \in \mathbb{Z}} e_{-n} \otimes \sum_{k=1}^{\ell} \alpha_j(H_k) H_n^{k(-)} = \sum_{n \in \mathbb{Z}} e_{-n} \otimes \sum_{k=1}^{\ell} H_{j,n}^{(-)}. \quad (121)$$

We also define the exponential of  $\alpha_j(\Phi)$  as the classical field

$$e^{\alpha_j(\Phi)} := \sum_{n \in \mathbb{Z}} e_{-n} \otimes E_n^{j(+)}. \quad (122)$$

The justification for this definition comes from Lemma 5.6 below. First we note that it is related to the formal field  $\mathcal{E}$  defined above as  $(\varrho \otimes \text{id})\mathcal{E} = \sum_{j=0}^{\ell} F_j \otimes e^{\alpha_j(\Phi)}$ . Then applying the linear map  $\varrho \otimes \varrho \otimes \text{id}$  to the second relation in Lemma 5.5 gives

$$\begin{aligned} \sum_{j=0}^{\ell} \{ \Pi_1(\theta), (F_j \otimes e^{\alpha_j(\Phi)})_2(\theta') \} &= - \sum_{j=0}^{\ell} \sum_{i=1}^{\ell} H_i \otimes [H^i, F_j] \otimes e^{\alpha_j(\Phi(\theta))} \delta_{\theta\theta'} \\ &= \sum_{j=0}^{\ell} \sum_{i=1}^{\ell} \alpha_j(H^i) H_i \otimes F_j \otimes e^{\alpha_j(\Phi(\theta))} \delta_{\theta\theta'} = \sum_{j=0}^{\ell} H_j \otimes F_j \otimes e^{\alpha_j(\Phi(\theta))} \delta_{\theta\theta'}, \end{aligned}$$

where we introduced the notation  $e^{\alpha_j(\Phi(\theta))} := e^{\alpha_j(\Phi)}(\theta)$  for  $j = 0, \dots, \ell$ . Equivalently, by writing the conjugate momentum  $\Pi$  in components as  $\Pi = \sum_{i=1}^{\ell} H_i \otimes \Pi^i$ , that is  $\Pi^i := \sum_{n \in \mathbb{Z}} e_{-n} \otimes H_n^{i(+)}$ , we can express the above relation in components as  $\{ \Pi^i(\theta), e^{\alpha_j(\Phi(\theta'))} \} = e^{\alpha_j(\Phi(\theta))} \delta_{\theta\theta'} \delta_{ij}$  and  $\{ \Pi^i(\theta), e^{\alpha_0(\Phi(\theta'))} \} = -a_i e^{\alpha_0(\Phi(\theta))} \delta_{\theta\theta'}$  for all  $i, j = 1, \dots, \ell$ . Here we have used the fact that  $H_0 = -\sum_{i=1}^{\ell} a_i H_i$ .

**Lemma 5.6.** *For all  $j = 0, \dots, \ell$ , we have  $\partial(e^{\alpha_j(\Phi)}) = \alpha_j(\partial\Phi)e^{\alpha_j(\Phi)}$ .*

*Proof.* Taking the derivative of the field (122) we find  $\partial(e^{\alpha_j(\Phi)}) = -i \sum_{n \in \mathbb{Z}} e_{-n} \otimes n E_n^{j(+)}$ . On the other hand, the derivative  $\partial\alpha_j(\Phi) = \alpha_j(\partial\Phi)$  of the classical field (121) takes the form  $\alpha_j(\partial\Phi) = -i \sum_{p \in \mathbb{Z}} e_{-p} \otimes p H_{j,p}^{(-)}$ . Multiplying it by (122) we obtain

$$\alpha_j(\partial\Phi)e^{\alpha_j(\Phi)} = -i \sum_{m,p \in \mathbb{Z}} e_{-m-p} \otimes p E_m^{j(+)} H_{j,p}^{(-)} = -i \sum_{n \in \mathbb{Z}} e_{-n} \otimes \sum_{p \in \mathbb{Z}} p E_{n-p}^{j(+)} H_{j,p}^{(-)},$$

where in the first equality we used the relation  $e_{-m}e_{-p} = e_{-m-p}$ , cf. §2.2.3, and in the last step we performed the change of variables  $m \rightarrow n-p$ . The result now follows from Lemma 5.3.  $\square$

**5.3.3. Lax matrix and Hamiltonian.** We define the Lax matrix and twist function as in §4.4.2, namely  $(\varrho \otimes \tilde{\pi}_{\ell} \otimes \text{id})\mathcal{L}(z) = \varphi(z)(\partial + \mathcal{L}(z))$ , cf. (64). To compute these we begin by applying the linear map  $\varrho \otimes \tilde{\pi}_{\ell}$  to the expressions (112) for the formal fields, with  $\varrho$  defined as in §2.2.3 and  $\tilde{\pi}_{\ell}$  in (119). We find  $(\varrho \otimes \tilde{\pi}_{\ell})\mathcal{A}_{[0]}^0 = \partial + \Pi + \frac{1}{2}\partial\Phi$ ,  $(\varrho \otimes \tilde{\pi}_{\ell})\mathcal{A}_{[1]}^0 = \frac{1}{\sqrt{2}} \sum_{j=0}^{\ell} \mu_j F_j \otimes e^{\alpha_j(\Phi)}$  and  $(\varrho \otimes \tilde{\pi}_{\ell})\mathcal{A}_{[1]}^{\infty} = -\frac{1}{\sqrt{2}} \sum_{j=0}^{\ell} \nu_j E_j$ . Applying

the linear map  $\varrho \otimes \tilde{\pi}_\ell \otimes \text{id}$  to the formal Lax matrix and using the above we find the twist function  $\varphi(z) = z^{-1}$  and the Lax matrix

$$\mathcal{L}(z) = \Pi + \frac{1}{2}\partial\Phi + \frac{1}{\sqrt{2}} \sum_{j=0}^{\ell} \left( e^{\alpha_j(\Phi)} z^{-1} \mu_j F_j + z \nu_j E_j \right), \quad (123)$$

where we drop some tensor products to conform to standard notations. By Corollary 4.8 it satisfies the non-ultralocal Poisson algebra (65) with  $r$ - and  $s$ -matrices given by

$$r_{12}(z, w) = -\frac{1}{2}C_{12}^{(0)} + \frac{1}{w^h - z^h} \sum_{r=0}^{h-1} z^r w^{h-r} C_{12}^{(-r)}, \quad s_{12}(z, w) = \frac{1}{2}C_{12}^{(0)}. \quad (124)$$

The above expression (123) coincides with the Lax matrix of affine Toda field theory in its non-ultralocal formulation. In order to see this, note that if we formally apply a gauge transformation by  $e^{\frac{1}{2}\Phi}$  to (123) then we obtain

$$\mathcal{L}'(z) := -\frac{1}{2}\partial\Phi + e^{\frac{1}{2}\text{ad}\Phi} \mathcal{L}(z) = \Pi + \frac{1}{\sqrt{2}} \sum_{j=0}^{\ell} e^{\frac{1}{2}\alpha_j(\Phi)} (z^{-1} \mu_j F_j + z \nu_j E_j). \quad (125)$$

This is the usual Lax matrix used in the standard treatment of affine Toda field theory, see *e.g.* [2, ch. 12]. It satisfies the *ultralocal* Poisson bracket  $\{\mathcal{L}'_1(z), \mathcal{L}'_2(w)\} = [r_{12}(z, w), \mathcal{L}'_1(z) + \mathcal{L}'_2(w)]\delta_{\theta\theta'}$ .

Finally, we show that the Hamiltonian of affine Toda field theory can be obtained from the local quadratic Gaudin Hamiltonians. It follows from Proposition 4.9 that the only non-trivial formal quadratic Gaudin Hamiltonian is  $\mathcal{H}_1^0 = -(\mathcal{A}_{[1]}^0 | \mathcal{A}_{[1]}^\infty) + \frac{1}{2}(\mathcal{A}_{[0]}^0 | \mathcal{A}_{[0]}^0)$ . Note that this can also be obtained from the formal quadratic Hamiltonian (70) associated with infinity, the only zero of the twist function  $\varphi(z) = z^{-1}$ , since this reads  $\mathcal{H}_\infty = \text{res}_\infty \frac{1}{2}(\mathcal{L}(z) | \mathcal{L}(z))\varphi(z)^{-1} dz = -\mathcal{H}_1^0$ . When applying the homomorphism  $\tilde{\pi}_\ell$  defined in §5.3.2, the terms not involving  $D$  and  $K$  give

$$\sum_{i=0}^{\ell} \frac{m_i^2}{4\pi} \epsilon_i \int_{S^1} d\theta e^{\alpha_i(\Phi(\theta))} + \frac{1}{2}(\Pi|\Pi) + \frac{1}{8}(\partial\Phi|\partial\Phi) + \frac{1}{2}(\Pi|\partial\Phi). \quad (126)$$

Here we introduced the Toda masses  $m_i \in \mathbb{R}_{\geq 0}$  by  $m_i^2 := \mu_i \nu_i$  and used the relation  $\langle E_i, F_j \rangle = \delta_{ij} \epsilon_i$  for all  $i, j = 0, \dots, \ell$ . The first three terms on the right hand side of (126) are exactly (half) the Hamiltonian of affine Toda field theory

$$H := (\Pi|\Pi) + \frac{1}{4}(\partial\Phi|\partial\Phi) + \sum_{i=0}^{\ell} \frac{m_i^2}{2\pi} \epsilon_i \int_{S^1} d\theta e^{\alpha_i(\Phi(\theta))}. \quad (127)$$

Moreover, the last term on the right hand side of (126) is (half) the momentum since

$$P := \tilde{\pi}_\ell(\mathcal{P}) = -i\tilde{\pi}_\ell(D_{[0]}^{(0)}) = -i \sum_{n \in \mathbb{Z}} \sum_{k=1}^{\ell} n H_{k,-n}^{(+)} H_n^{k(-)} = (\Pi|\partial\Phi)$$

where the second equality is from the definition (71) of the formal momentum, the third equality from (119c) and the last equality uses the definition (120) of the Toda field and its conjugate momentum.

Consider now the terms in  $\frac{1}{2}(\mathcal{A}_{[0]}^0 | \mathcal{A}_{[0]}^0)$  containing D and K. They are of the form

$$\frac{1}{2} \left( \mathbf{D}_{[0]}^{(0)} \mathbf{K}_{[0]}^{(0)} + \mathbf{K}_{[0]}^{(0)} \mathbf{D}_{[0]}^{(0)} \right) = \mathbf{D}_{[0]}^{(0)} \mathbf{K}_{[0]}^{(0)}. \quad (128)$$

We have  $\tilde{\pi}_\ell(\mathbf{K}_{[0]}^{(0)}) = i$  and  $\tilde{\pi}_\ell(\mathbf{D}_{[0]}^{(0)}) = i(\Pi | \partial\Phi)$ , hence combining the above we find

$$H_0 := \tilde{\pi}_\ell(\mathcal{H}_1^0) = \frac{1}{2}H - \frac{1}{2}(\Pi | \partial\Phi) = \frac{1}{2}(H - P).$$

In particular, as was the case for the  $\mathbb{Z}_T$ -graded coset  $\sigma$ -model and its deformation in §5.2, the Hamiltonian (127) of affine Toda field theory is given by  $H = 2H_0 + P$ .

Applying the homomorphism  $\varrho \otimes \tilde{\pi}_\ell$  to the Lax equation  $\{\mathcal{H}_1^0, \mathcal{L}(z)\} = [\mathcal{M}_1^0(z), \mathcal{L}(z)]$  from Proposition 4.10 and dividing through by  $\varphi(z)$  we find  $\{H_0, \mathcal{L}(z)\} = z^{-1}[(\varrho \otimes \tilde{\pi}_\ell)\mathcal{A}_{[1]}^0, \partial + \mathcal{L}(z)]$ . Combining this with the fact that  $\{P, \mathcal{L}(z)\} = \partial\mathcal{L}(z)$  and using the definition of  $H$  we find the zero curvature equation  $\{H, \mathcal{L}(z)\} = \partial\mathcal{M}(z) + [\mathcal{L}(z), \mathcal{M}(z)]$ , where we defined  $\mathcal{M}(z) := \mathcal{L}(z) - 2(\varrho \otimes \tilde{\pi}_\ell)\mathcal{A}_{[1]}^0$ . The explicit form of the latter reads  $\mathcal{M}(z) = \Pi + \frac{1}{2}\partial\Phi + \frac{1}{\sqrt{2}}\sum_{j=0}^{\ell} (-e^{\alpha_j(\Phi)}z^{-1}\mu_j F_j + z\nu_j E_j)$ . In terms of the gauge transformed Lax matrix (125) we have  $\{H, \mathcal{L}'(z)\} = \partial\mathcal{M}'(z) + [\mathcal{L}'(z), \mathcal{M}'(z)]$ , where  $\mathcal{M}'(z) := -\Pi + e^{\frac{1}{2}\text{ad}\Phi}\mathcal{M}(z)$  is given explicitly by

$$\mathcal{M}'(z) = \frac{1}{2}\partial\Phi + \frac{1}{\sqrt{2}}\sum_{j=0}^{\ell} e^{\frac{1}{2}\alpha_j(\Phi)}(-z^{-1}\mu_j F_j + z\nu_j E_j).$$

Up to differences in conventions, this agrees with the temporal component of the Lax connection in the standard treatment of affine Toda field theory [2, ch. 12].

5.3.4. *The  $\mathfrak{g} = \mathfrak{sl}_2$  case: sinh-Gordon model.* It is helpful to illustrate the above general procedure for obtaining the Lax matrix (123) of affine Toda field theory from the formal Lax matrix

$$\mathcal{L}(z) = \mathcal{A}_{[0]}^0 z^{-1} + \mathcal{A}_{[1]}^0 z^{-2} - \mathcal{A}_{[1]}^\infty \quad (129)$$

of the dihedral affine Gaudin model explicitly in the case  $\mathfrak{g} = \mathfrak{sl}_2$ , which corresponds to the sinh-Gordon model.

Let  $\{E, H, F\}$  be the standard basis of  $\mathfrak{g} = \mathfrak{sl}_2$ . A basis of the corresponding untwisted affine Kac-Moody algebra  $\tilde{\mathfrak{g}} = \tilde{\mathfrak{sl}}_2 = \mathfrak{sl}_2[t, t^{-1}] \oplus \mathbb{C}\mathbf{K} \oplus \mathbb{C}\mathbf{D}$  then consists of  $E_n = E \otimes t^n$ ,  $H_n = H \otimes t^n$  and  $F_n = F \otimes t^n$  for all  $n \in \mathbb{Z}$  as well as  $\mathbf{K}$  and  $\mathbf{D}$ . Its dual basis, with respect to the standard bilinear form on  $\mathfrak{sl}_2$ , cf. (9), induced from the non-degenerate invariant bilinear form  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{sl}_2$  normalised such that  $\langle H, H \rangle = 2$ , then consists of the elements  $F_{-n}$ ,  $\frac{1}{2}H_{-n}$  and  $E_{-n}$  for all  $n \in \mathbb{Z}$  as well as  $\mathbf{D}$  and  $\mathbf{K}$ . In the notation introduced at the start of §5.3 we have  $E_1 = E^1 = F_0 = F^0 = E$ ,  $F_1 = F^1 = E_0 = E^0 = F$ ,  $H^1 = H$  and  $H_1 = \frac{1}{2}H$ . As we shall not use this notation here it should not lead to confusion with the above notation  $E_n$ ,  $F_n$  and  $H_n$  for  $n \in \mathbb{Z}$ .

We extend the Coxeter automorphism  $\sigma \in \text{Aut } \mathfrak{sl}_2$  defined by  $\sigma(E) = -E$ ,  $\sigma(H) = H$  and  $\sigma(F) = -F$  to the affine Kac-Moody algebra, cf. §2.2.1, by letting  $\sigma(E_n) = -E_n$ ,  $\sigma(H_n) = H_n$  and  $\sigma(F_n) = -F_n$  for any  $n \in \mathbb{Z}$  as well as  $\sigma(\mathbf{K}) = \mathbf{K}$  and  $\sigma(\mathbf{D}) = \mathbf{D}$ . A basis for the 1-eigenspace  $\tilde{\mathfrak{g}}_{(0), \mathbb{C}} \subset \tilde{\mathfrak{g}}$  of  $\sigma \in \text{Aut } \tilde{\mathfrak{sl}}_2$  consists of  $H_n$ ,  $n \in \mathbb{Z}$  together

with  $\mathbf{K}$  and  $\mathbf{D}$ , while the  $(-1)$ -eigenspace  $\widetilde{\mathfrak{g}}_{(1),\mathbb{C}} \subset \widetilde{\mathfrak{g}}$  has basis elements  $E_n$  and  $F_n$  for  $n \in \mathbb{Z}$ .

The affine Gaudin model we are considering has divisor  $\mathcal{D} = 2 \cdot 0 + 2 \cdot \infty$ . This means that to the sites at the origin and infinity we attach certain copies of the basis elements of the affine Kac-Moody algebra  $\widetilde{\mathfrak{sl}}_2$ , namely

$$\begin{aligned} \text{origin :} & \quad H_{n,[0]}^{(0)}, E_{n,[1]}^{(0)} \text{ and } F_{n,[1]}^{(0)} \text{ for each } n \in \mathbb{Z}, \text{ and } \mathbf{K}_{[0]}^{(0)}, \mathbf{D}_{[0]}^{(0)}, \\ \text{infinity :} & \quad E_{n,[1]}^{(\infty)} \text{ and } F_{n,[1]}^{(\infty)} \text{ for each } n \in \mathbb{Z}. \end{aligned}$$

In order to avoid overburdening the present section with technicalities, we shall not discuss the precise definitions of these objects further here. We refer to §3.4.1 for the definition of the Lie algebra  $\widetilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}$  to which they belong. We shall simply note that their Lie brackets read, cf. (113),

$$[H_{n,[0]}^{(0)}, H_{-n,[0]}^{(0)}] = 2n\mathbf{K}_{[0]}^{(0)}, \quad [\mathbf{D}_{[0]}^{(0)}, H_{n,[0]}^{(0)}] = nH_{n,[0]}^{(0)}, \quad (130a)$$

$$[H_{m,[0]}^{(0)}, E_{n,[1]}^{(0)}] = E_{m+n,[1]}^{(0)}, \quad [\mathbf{D}_{[0]}^{(0)}, E_{n,[1]}^{(0)}] = nE_{n,[1]}^{(0)}, \quad (130b)$$

for all  $m, n \in \mathbb{Z}$ , with all other Lie brackets being trivial.

The three formal fields  $\mathcal{A}_{[0]}^0$ ,  $\mathcal{A}_{[1]}^0$  and  $\mathcal{A}_{[1]}^\infty$  of the dihedral affine Gaudin model under consideration may now be written in the present  $\mathfrak{sl}_2$  case as, cf. (112),

$$\mathcal{A}_{[0]}^0 = \mathbf{D} \otimes \mathbf{K}_{[0]}^{(0)} + \mathbf{K} \otimes \mathbf{D}_{[0]}^{(0)} + \sum_{n \in \mathbb{Z}} \frac{1}{2} H_{-n} \otimes H_{n,[0]}^{(0)}, \quad (131a)$$

$$\mathcal{A}_{[1]}^0 = \sum_{n \in \mathbb{Z}} (E_{-n} \otimes F_{n,[1]}^{(0)} + F_{-n} \otimes E_{n,[1]}^{(0)}), \quad \mathcal{A}_{[1]}^\infty = \sum_{n \in \mathbb{Z}} (E_{-n} \otimes F_{n,[1]}^{(\infty)} + F_{-n} \otimes E_{n,[1]}^{(\infty)}). \quad (131b)$$

Following the nomenclature introduced at the start of §4.2.1, we refer to the first tensor factor in each of these expressions as the *auxiliary factor*. In order to obtain more familiar looking  $\mathfrak{sl}_2$ -valued fields on the circle  $S^1$  from the above formal fields, we should represent the affine Kac-Moody algebra elements in the auxiliary factor in terms of  $\mathfrak{sl}_2$ -valued connections on  $S^1$  with trigonometric polynomial coefficients, cf. §4.2.2. Specifically, we apply the representation  $\varrho$  defined in Lemma 2.4, which in the present case explicitly reads

$$\varrho(\mathbf{D}) = -i\partial_\theta, \quad \varrho(\mathbf{K}) = 0, \quad \varrho(E_n) = Ee^{in\theta}, \quad \varrho(H_n) = He^{in\theta}, \quad \varrho(F_n) = Fe^{in\theta},$$

in terms of the coordinate  $\theta \in \mathbb{R}/2\pi\mathbb{Z} = S^1$  on the circle. To be even more concrete, we shall modify the above representation so that the basis elements of  $\mathfrak{sl}_2$  are further represented by  $2 \times 2$  matrices. That is, instead of applying  $\varrho$  to the auxiliary factor of the formal fields and the formal Lax matrix, we shall use a representation  $\widetilde{\varrho}$  of  $\widetilde{\mathfrak{g}}$  into  $2 \times 2$  matrix connections on the circle  $S^1$ , defined by

$$\begin{aligned} \widetilde{\varrho}(\mathbf{D}) &= -i\partial_\theta, & \widetilde{\varrho}(\mathbf{K}) &= 0, & \widetilde{\varrho}(E_n) &= \begin{pmatrix} 0 & e^{in\theta} \\ 0 & 0 \end{pmatrix}, \\ \widetilde{\varrho}(H_n) &= \begin{pmatrix} e^{in\theta} & 0 \\ 0 & -e^{in\theta} \end{pmatrix}, & \widetilde{\varrho}(F_n) &= \begin{pmatrix} 0 & 0 \\ e^{in\theta} & 0 \end{pmatrix}. \end{aligned}$$

For instance, in the case of the formal field  $\mathcal{A}_{[1]}^0$  defined in (131b) we obtain

$$(\tilde{\varrho} \otimes \text{id})\mathcal{A}_{[1]}^0 = \begin{pmatrix} 0 & \sum_{n \in \mathbb{Z}} e^{-in\theta} F_{n,[1]}^{(0)} \\ \sum_{n \in \mathbb{Z}} e^{-in\theta} E_{n,[1]}^{(0)} & 0 \end{pmatrix}.$$

The two off-diagonal components of the matrix on the right hand side are formal distributions on  $S^1$  with Fourier coefficients  $F_{n,[1]}^{(0)}$  and  $E_{n,[1]}^{(0)}$ , respectively. That is, their integral pairing with any trigonometric polynomial yields an element of the Lie algebra  $\tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathfrak{D}}$ .

To formulate the sinh-Gordon model as an affine Gaudin model we should realise all the fields of the latter in terms of sinh-Gordon fields. This is the purpose of the homomorphism (119) defined in §5.3.2, expressing the Fourier modes of the affine Gaudin model fields in terms of those of the classical Toda fields. Let us spell out its definition in the present  $\mathfrak{sl}_2$  case. Let  $\mathcal{H}$  denote the Lie algebra spanned by elements  $E_n^{(+)}$ ,  $F_n^{(+)}$  and  $H_n^{(\pm)}$  for all  $n \in \mathbb{Z}$  subject to the relations, cf. (115),

$$[H_n^{(+)}, H_{-n}^{(-)}] = 2, \quad [H_m^{(+)}, E_n^{(+)}] = E_{m+n}^{(+)}, \quad [H_m^{(+)}, F_n^{(+)}] = F_{m+n}^{(+)}, \quad (132)$$

with all other Lie brackets being trivial. In terms of these, let us introduce the sinh-Gordon field and its conjugate momentum as

$$\phi(\theta) := \frac{1}{2} \sum_{n \in \mathbb{Z}} e^{-in\theta} H_n^{(-)}, \quad \pi(\theta) := \frac{1}{2} \sum_{n \in \mathbb{Z}} e^{-in\theta} H_n^{(+)}$$

Note that their Poisson bracket is normalised as  $\{\pi(\theta), \phi(\theta')\} = \frac{1}{2} \delta_{\theta\theta'}$ . This can be seen directly from the first relation in (132) or, alternatively, from the Poisson bracket  $\{\Pi_1(\theta), \Phi_2(\theta')\} = C_{12}^{(0)} \delta_{\theta\theta'}$  by noting that we have  $C^{(0)} = \frac{1}{2} H \otimes H$  and observing that the scalar fields  $\phi$  and  $\pi$  are related to the  $\mathfrak{h}_0$ -valued fields  $\Phi$  and  $\Pi$  defined in (120) simply as  $\Phi = \phi H$  and  $\Pi = \pi H$ . Moreover, in the notation (122), which was motivated by Lemma 5.6, the exponential of (twice) the sinh-Gordon field  $\phi$  is then given by  $e^{2\phi(\theta)} = \sum_{n \in \mathbb{Z}} e^{-in\theta} E_n^{(+)}$  where we used the fact that  $\alpha_1(H) = 2$  and also  $e^{-2\phi(\theta)} = \sum_{n \in \mathbb{Z}} e^{-in\theta} F_n^{(+)}$  since we have  $\alpha_0(H) = -\alpha_1(H) = -2$ .

The desired homomorphism (119) in the present case takes the form

$$\tilde{\pi}_{\ell}(E_{n,[1]}^{(0)}) = \frac{m}{\sqrt{2}} E_n^{(+)}, \quad \tilde{\pi}_{\ell}(F_{n,[1]}^{(0)}) = \frac{m}{\sqrt{2}} F_n^{(+)}, \quad \tilde{\pi}_{\ell}(E_{n,[1]}^{(\infty)}) = \tilde{\pi}_{\ell}(F_{n,[1]}^{(\infty)}) = -\frac{m}{\sqrt{2}} \delta_{n,0}, \quad (133a)$$

$$\tilde{\pi}_{\ell}(K_{[0]}^{(0)}) = i, \quad \tilde{\pi}_{\ell}(H_{n,[0]}^{(0)}) = H_n^{(+)} - \frac{in}{2} H_n^{(-)}, \quad \tilde{\pi}_{\ell}(D_{[0]}^{(0)}) = \sum_{n \in \mathbb{Z}} \sum_{k=1}^{\ell} n H_{-n}^{(+)} H_n^{(-)}, \quad (133b)$$

for all  $n \in \mathbb{Z}$ . Here we have fixed all the parameters  $\mu_0 = \mu_1 = \nu_0 = \nu_1 = m$  in terms of a single real parameter  $m \in \mathbb{R}_{\geq 0}$  which will turn out to be the mass of the sinh-Gordon field. Note that to make sense of the infinite sum on the right hand side of the last expression in (133b), the map  $\tilde{\pi}_{\ell}$  takes value in a suitable completion of the symmetric algebra  $S(\mathcal{H})$ , see §5.3.2 for details. More precisely, we work in its quotient by the relation  $\sum_{p \in \mathbb{Z}} p E_{n-p}^{(+)} H_p^{(-)} = n E_n^{(+)}$ , cf. Lemma 5.3. According to



Lemma 5.4, the map  $\tilde{\pi}_\ell$  defined by (133) then provides a realisation of the Lie algebra  $\tilde{\mathfrak{g}}_{\mathbb{C}}^{\mathcal{D}}$  with relations (130).

We may now proceed to construct the Lax matrix of the sinh-Gordon model from the formal Lax matrix (129) of the affine Gaudin model. Substituting the expressions (131) for the formal fields of the affine Gaudin model into  $\mathcal{L}(z)$ , it explicitly reads

$$\begin{aligned} \mathcal{L}(z) = & \left( D \otimes K_{[0]}^{(0)} + K \otimes D_{[0]}^{(0)} + \sum_{n \in \mathbb{Z}} \frac{1}{2} H_{-n} \otimes H_{n,[0]}^{(0)} \right) z^{-1} \\ & + \sum_{n \in \mathbb{Z}} (E_{-n} \otimes F_{n,[1]}^{(0)} + F_{-n} \otimes E_{n,[1]}^{(0)}) z^{-2} - \sum_{n \in \mathbb{Z}} (E_{-n} \otimes F_{n,[1]}^{(\infty)} + F_{-n} \otimes E_{n,[1]}^{(\infty)}). \end{aligned}$$

To begin with, applying to it just the representation  $\tilde{\varrho} \otimes \text{id} \otimes \text{id}$ , *i.e.* applying  $\tilde{\varrho}$  defined above to the auxiliary factor, we find

$$\begin{aligned} (\tilde{\varrho} \otimes \text{id} \otimes \text{id})\mathcal{L}(z) = & -K_{[0]}^{(0)} z^{-1} i \partial_\theta + \sum_{n \in \mathbb{Z}} \begin{pmatrix} \frac{1}{2} e^{-in\theta} H_{n,[0]}^{(0)} & 0 \\ 0 & -\frac{1}{2} e^{-in\theta} H_{n,[0]}^{(0)} \end{pmatrix} z^{-1} \\ & + \sum_{n \in \mathbb{Z}} \begin{pmatrix} 0 & e^{-in\theta} F_{n,[1]}^{(0)} \\ e^{-in\theta} E_{n,[1]}^{(0)} & 0 \end{pmatrix} z^{-2} - \sum_{n \in \mathbb{Z}} \begin{pmatrix} 0 & e^{-in\theta} F_{n,[1]}^{(\infty)} \\ e^{-in\theta} E_{n,[1]}^{(\infty)} & 0 \end{pmatrix}. \end{aligned}$$

Next, applying the further representation  $\text{id} \otimes \tilde{\pi}_\ell \otimes \text{id}$ , with  $\tilde{\pi}_\ell$  given in (133), to the above expression we find

$$\begin{aligned} (\tilde{\varrho} \otimes \tilde{\pi}_\ell \otimes \text{id})\mathcal{L}(z) = & \frac{\partial_\theta}{z} + \frac{1}{z} \begin{pmatrix} \pi(\theta) + \frac{1}{2} \phi'(\theta) & 0 \\ 0 & -\pi(\theta) - \frac{1}{2} \phi'(\theta) \end{pmatrix} \\ & + \frac{m}{\sqrt{2} z^2} \begin{pmatrix} 0 & e^{-2\phi(\theta)} \\ e^{2\phi(\theta)} & 0 \end{pmatrix} + \frac{m}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

Comparing this with the general expression (64), we read off the twist function as the coefficient of the  $\partial_\theta$ -term to be  $\varphi(z) = z^{-1}$  and then the Lax matrix is

$$\mathcal{L}(z) = \begin{pmatrix} \pi(\theta) + \frac{1}{2} \phi'(\theta) & \frac{m}{\sqrt{2}} (z^{-1} e^{-2\phi(\theta)} + z) \\ \frac{m}{\sqrt{2}} (z^{-1} e^{2\phi(\theta)} + z) & -\pi(\theta) - \frac{1}{2} \phi'(\theta) \end{pmatrix}. \quad (134)$$

This expression can equally be obtained directly from the general result (123), specialised to the case  $\mathfrak{g} = \mathfrak{sl}_2$ , after representing the auxiliary factor of this  $\mathfrak{sl}_2$ -valued Lax matrix in terms of  $2 \times 2$  matrices.

The Lax matrix (134) is a non-ultralocal version of the Lax matrix of the sinh-Gordon model. But it is gauge equivalent to the standard ultralocal Lax matrix. Indeed, if we perform a gauge transformation by  $\text{diag}(e^{\frac{1}{2}\phi(\theta)}, e^{-\frac{1}{2}\phi(\theta)})$  on it, or even more directly if we represent the auxiliary factor of the Lax matrix (125) in the present case by  $2 \times 2$  matrices, then we arrive at

$$\mathcal{L}'(z) = \begin{pmatrix} \pi(\theta) & \frac{m}{\sqrt{2}} (z^{-1} e^{-\phi(\theta)} + z e^{\phi(\theta)}) \\ \frac{m}{\sqrt{2}} (z^{-1} e^{\phi(\theta)} + z e^{-\phi(\theta)}) & -\pi(\theta) \end{pmatrix}. \quad (135)$$

This is the standard ultralocal Lax matrix of the sinh-Gordon model in the Hamiltonian formalism, see for instance [2, §12.7] where the momentum is written as  $\pi = \frac{1}{2}\partial_t\phi$  so that the canonical Poisson bracket reads  $\{\partial_t\phi(\theta), \phi(\theta')\} = \delta_{\theta\theta'}$ .

Unlike its non-ultralocal counterpart (134), however, and this is the crucial point, the Lax matrix (135) cannot be obtained as a representation of the formal Lax matrix of an affine Gaudin model in any natural way. One can see this directly from the form of the Hamiltonian (127) of the affine Gaudin model obtained in §5.3.3. Indeed, in the present case with  $\mathfrak{g} = \mathfrak{sl}_2$ , the Hamiltonian and momentum take on the simple form

$$H = \frac{1}{2\pi} \int_{S^1} d\theta (2\pi(\theta)^2 + \frac{1}{2}\phi'(\theta)^2 + m^2 e^{2\phi(\theta)} + m^2 e^{-2\phi(\theta)}), \quad P = \frac{1}{\pi} \int_{S^1} d\theta \pi(\theta)\phi'(\theta).$$

Here we have used the identification  $\Phi = \phi H$  and  $\Pi = \pi H$  made above, together with the fact that  $\langle H, H \rangle = 2$ . Recall here also the definition (16) which according to Lemma 2.4 is identified with the bilinear form  $(\cdot|\cdot)$  on  $\mathcal{L}\mathfrak{g}$  defined in (2.2). Since the ultralocal Lax matrix (135) does not explicitly depend on the derivative  $\phi'(\theta)$  of the sinh-Gordon field, the term  $\phi'(\theta)^2$  in the density of the Hamiltonian cannot arise from a bilinear expression in the Lax matrix  $\mathcal{L}(z)$ , as it should do if the ultralocal formulation of sinh-Gordon could be given an affine Gaudin model interpretation. One of the effects of the gauge transformation from (135) to (134) is precisely to introduce such a dependence on  $\phi'(\theta)$  in the Lax matrix.

## APPENDIX A. NOTATIONS AND SOME USEFUL LEMMAS

**A.1. Dual pairs.** Let  $V$  and  $W$  be a pair of real vector spaces and  $\langle \cdot, \cdot \rangle : V \times W \rightarrow \mathbb{R}$  be a bilinear form. For any two vectors  $x \in V$  and  $y \in W$  satisfying  $\langle x, y \rangle = 0$  we will write  $x \perp y$ . More generally, for any subspace  $E \subset V$  and  $y \in W$  we write  $E \perp y$  if  $\langle x, y \rangle = 0$  for all  $x \in E$ . Similarly, we write  $x \perp F$  if  $\langle x, y \rangle = 0$  for any  $y \in F$ . Also, for any two subspaces  $E \subset V$  and  $F \subset W$ , we write  $E \perp F$  if  $\langle x, y \rangle = 0$  for all  $x \in E$  and  $y \in F$ . We define the orthogonal complement in  $W$  of a subspace  $E \subset V$  by  $E^\perp := \{y \in W \mid E \perp y\}$ . Likewise, for any subspace  $F \subset W$  we define its orthogonal complement in  $V$  by  $F^\perp := \{x \in V \mid x \perp F\}$ . We will call the triple  $(V, W, \langle \cdot, \cdot \rangle)$  a *dual pair* if the bilinear form  $\langle \cdot, \cdot \rangle$  is non-degenerate both on the left, *i.e.*  $W^\perp = \{0\}$ , and on the right, *i.e.*  $V^\perp = \{0\}$ . In other words, we say  $(V, W, \langle \cdot, \cdot \rangle)$  is a dual pair if  $x \perp W$  implies  $x = 0$  and  $V \perp y$  implies  $y = 0$ .

**Lemma A.1.** *Let  $(V, W, \langle \cdot, \cdot \rangle)$  be a dual pair of real vector spaces. Let  $V = V_+ \dot{+} V_-$  and  $W = W_+ \dot{+} W_-$  be direct sum decompositions such that  $V_\pm \perp W_\pm$ . Then  $V_\pm^\perp = W_\pm$  and  $W_\pm^\perp = V_\pm$ . In particular,  $(V_\pm, W_\mp, \langle \cdot, \cdot \rangle|_{V_\pm \times W_\mp})$  are dual pairs.*

Given a dual pair  $(V, W, \langle \cdot, \cdot \rangle)$ , we endow any pair of subspaces  $E \subset V$  and  $F \subset W$  with the restricted bilinear form  $\langle \cdot, \cdot \rangle|_{E \times F} : E \times F \rightarrow \mathbb{R}$ . In general this will be degenerate, with left kernel  $E \cap F^\perp = \{x \in E \mid x \perp F\}$  and right kernel  $F \cap E^\perp = \{y \in F \mid E \perp y\}$ .

**Lemma A.2.** *The triple  $(E/(E \cap F^\perp), F/(F \cap E^\perp), \langle \cdot, \cdot \rangle)$ , where  $\langle \cdot, \cdot \rangle : E/(E \cap F^\perp) \times F/(F \cap E^\perp) \rightarrow \mathbb{R}$  is induced from the restriction  $\langle \cdot, \cdot \rangle|_{E \times F} : E \times F \rightarrow \mathbb{R}$ , is a dual pair.*

**A.2. Tensor index notation.** Let  $\mathfrak{a}$ ,  $\mathfrak{b}$  and  $\mathfrak{c}$  be complex Lie algebras, each equipped with an anti-linear involutive automorphism which we commonly denote by  $\tau$ . Let us suppose further that these are equipped with descending  $\mathbb{Z}_{\geq 0}$ -filtrations  $(\mathbf{F}_n \mathfrak{a})_{n \in \mathbb{Z}_{\geq 0}}$ ,  $(\mathbf{F}_n \mathfrak{b})_{n \in \mathbb{Z}_{\geq 0}}$  and  $(\mathbf{F}_n \mathfrak{c})_{n \in \mathbb{Z}_{\geq 0}}$  as complex Lie algebras, respectively. So in particular, for every  $m, n \in \mathbb{Z}_{\geq 0}$  we have  $[\mathbf{F}_m \mathfrak{a}, \mathbf{F}_n \mathfrak{a}] \subset \mathbf{F}_{m+n} \mathfrak{a}$  and similarly for  $\mathfrak{b}$  and  $\mathfrak{c}$ . Let  $\mathfrak{A}$  be a commutative algebra over  $\mathbb{C}$ . Consider elements

$$\begin{aligned} x &= \sum_i x_i^1 \otimes x_i^2 \otimes f_i \in \mathfrak{a} \hat{\otimes} \mathfrak{b} \otimes \mathfrak{A}, & y &= \sum_i y_i^1 \otimes y_i^2 \otimes g_i \in \mathfrak{a} \hat{\otimes} \mathfrak{c} \otimes \mathfrak{A}, \\ z &= \sum_i z_i^1 \otimes z_i^2 \otimes h_i \in \mathfrak{b} \hat{\otimes} \mathfrak{c} \otimes \mathfrak{A}, \end{aligned}$$

where  $\mathfrak{a} \hat{\otimes} \mathfrak{b}$  is the completion of the tensor product  $\mathfrak{a} \otimes \mathfrak{b}$  with respect to the descending  $\mathbb{Z}_{\geq 0}$ -filtration with subspaces  $\mathbf{F}_n(\mathfrak{a} \otimes \mathfrak{b}) = \mathbf{F}_n \mathfrak{a} \otimes \tau(\mathbf{F}_n \mathfrak{b}) + \tau(\mathbf{F}_n \mathfrak{a}) \otimes \mathbf{F}_n \mathfrak{b}$  for each  $n \in \mathbb{Z}_{\geq 0}$  and likewise for  $\mathfrak{a} \hat{\otimes} \mathfrak{c}$  and  $\mathfrak{b} \hat{\otimes} \mathfrak{c}$ . So in particular, the sums over  $i$  in the elements  $x$ ,  $y$  and  $z$  above may be infinite. Given any  $u \in \mathfrak{a}$  and  $v \in \mathfrak{b}$  we use the tensor index notation

$$[u_1, x_{12}] := \sum_i [u, x_i^1] \otimes x_i^2 \otimes f_i, \quad [v_2, x_{12}] := \sum_i x_i^1 \otimes [v, x_i^2] \otimes f_i.$$

Similarly, following standard conventions we define the elements

$$\begin{aligned} [x_{12}, y_{13}] &:= \sum_{i,j} [x_i^1, y_j^1] \otimes x_i^2 \otimes y_j^2 \otimes f_i g_j, & [y_{13}, z_{23}] &:= \sum_{i,j} y_i^1 \otimes z_j^1 \otimes [y_i^2, z_j^2] \otimes g_i h_j, \\ [x_{12}, z_{23}] &:= \sum_{i,j} x_i^1 \otimes [x_i^2, z_j^1] \otimes z_j^2 \otimes f_i h_j, \end{aligned}$$

of  $\mathfrak{a} \hat{\otimes} \mathfrak{b} \hat{\otimes} \mathfrak{c} \otimes \mathfrak{A}$  where  $\mathfrak{a} \hat{\otimes} \mathfrak{b} \hat{\otimes} \mathfrak{c}$  denotes the completion of  $\mathfrak{a} \otimes \mathfrak{b} \otimes \mathfrak{c}$  with respect to the descending  $\mathbb{Z}_{\geq 0}$ -filtration defined by

$$\begin{aligned} \mathbf{F}_n(\mathfrak{a} \otimes \mathfrak{b} \otimes \mathfrak{c}) &= \mathbf{F}_n \mathfrak{a} \otimes \tau(\mathbf{F}_n \mathfrak{b}) \otimes \mathfrak{c} + \mathbf{F}_n \mathfrak{a} \otimes \mathfrak{b} \otimes \tau(\mathbf{F}_n \mathfrak{c}) + \mathfrak{a} \otimes \mathbf{F}_n \mathfrak{b} \otimes \tau(\mathbf{F}_n \mathfrak{c}) \\ &\quad + \tau(\mathbf{F}_n \mathfrak{a}) \otimes \mathbf{F}_n \mathfrak{b} \otimes \mathfrak{c} + \tau(\mathbf{F}_n \mathfrak{a}) \otimes \mathfrak{b} \otimes \mathbf{F}_n \mathfrak{c} + \mathfrak{a} \otimes \tau(\mathbf{F}_n \mathfrak{b}) \otimes \mathbf{F}_n \mathfrak{c} \end{aligned}$$

for all  $n \in \mathbb{Z}_{\geq 0}$ . If tensor indices appear in decreasing order then one should permute the tensor factors. For instance, if  $w = \sum_i w_i^1 \otimes w_i^2 \otimes k_i \in \mathfrak{b} \hat{\otimes} \mathfrak{a} \otimes \mathfrak{A}$  then we let

$$[w_{21}, y_{13}] := \sum_{i,j} [w_i^2, y_j^1] \otimes w_i^1 \otimes y_j^2 \otimes k_i g_j,$$

and similarly for other possible permutations of the tensor indices.

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