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Hyper Natural Deduction for Gödel Logic — a natural deduction system for parallel reasoning[*]

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Abstract. We introduce a system of Hyper Natural Deduction for Gödel Logic as an extension of Gentzen's system of Natural Deduction. A deduction in this system consists of a finite set of derivations which uses the typical rules of Natural Deduction, plus additional rules providing means for communication between derivations. We show that our system is sound and complete for infinite-valued propositional Gödel Logic, by giving translations to and from Avron's Hypersequent Calculus. We provide conversions for normalization extending usual conversions for Natural Deduction and prove the existence of normal forms for Hyper Natural Deduction for Gödel Logic. We show that normal deductions satisfy the subformula property.

1 Introduction

Gentzen introduced his Natural Deduction system with the aim to create "*einen Formulismus [..] der dem wirklichen Schließen möglichst nahe kommt*" – a formalism which reproduces as precisely as possible the actual logical reasoning to be found in mathematical proofs [Gen35]. Natural Deduction achieves this aim. But for other purposes, in particular proving his "Hauptsatz" in the case of classical logic, Gentzen had to introduce a different, related system, his Sequent Calculus. The "Hauptsatz" shows that every proof can be transformed into normal form, and is one of the most important properties of Natural Deduction. Since their inception, both systems have been the basis for many investigations in various fields ranging from proof theory and artificial intelligence, to formal methods for system design, see [TS00,GTL89,Geu09] for further information and pointers to relevant literature.

One of the most astonishing discoveries related to Natural Deduction, which had a significant impact on proof theory and Computer Science, is the Curry-Howard isomorphism between intuitionistic Natural Deduction proofs and typed

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λ-terms. The Curry-Howard isomorphism identifies a formula with a set (or rather – type) of its proofs, and a proof of an implication $A \to B$ with a computable function that given a proof of A returns a proof of B, that is a computable function of type $A \to B$ — in the context of λ-calculus, this computation step is called β-conversion, in the context of normalisation $\to$-conversion. This correspondence is also used to design tools like theorem provers with the ability to extract programs from proofs, as done in Coq, Minlog and Nuprl [SW12]. Many approaches have been made to extend Curry-Howard correspondences to other logics [Gri89,Par92,GDQ92,Kri09] and concurrent programming [KY95,CPT12].

Following Gentzen's approach, Avron [Avr91] introduced the Hypersequent Calculus as an extension of Gentzen's Sequent Calculus [Gen35], providing a uniform framework for capturing intermediate logics, that is logics which are strictly between intuitionistic and classical logic. In particular he devised a Hypersequent Calculus which is sound and complete for one important intermediate logic called Dummett's LC [Dum59] or (infinite-valued propositional) Gödel Logic — when speaking about Hypersequent Calculus we mean this formalism in the following. Avron's approach has been very successful, and nowadays Hypersequent Calculi for a wide variety of non-classical logics as well as modal logics have been discovered [MM07,Lah13,CGT08].

A second motivation for Avron was "*to contribute towards a better understanding of the notion of logical consequence in general, and especially its possible relations with parallel computations*" [Avr91]. Avron conjectured that it may be possible to interpret his formalism with multiprocesses exchanging information, and stated "*It seems to us extremely important to determine the exact computational content of [related logics]*" [Avr91]. We claim that so far this aim has not been achieved: Previous approaches to explore this conjecture can be distinguished into either semantical (computational) or syntactic (proof theoretic) ones. Examples of the first category are game theoretic interpretation [Fer08] and extensions of the λ-calculus to work with Gödel logics [Hir12]. While these semantical approaches look tempting, they either fail to connect to the proof-theoretic side, or do not provide a computational interpretation. On the syntactical side, Baaz et al. [BCF01] introduced an extension of Natural Deduction to the hyper-level, which operates on sequences of derivations. Their system lacks a normalization procedure via conversions — normalization is shown by translation into Hypersequent Calculus, followed by cut-elimination and finally translation back into their system. As a consequence, they do not obtain a computational interpretation via a Curry-Howard correspondence. Aschieri [Asc16] extends Natural Deduction by a rule modelling Dummett's axiom [Dum59] and proves a Curry-Howard correspondence for the resulting system. However, that system resembles a different logic and has no direct relation to Avron's Hypersequent Calculus. Aschieri et. al. [ACG17] refines the previous result by providing a system which captures Gödel logic. The normalization procedure for the latter system does not extend the usual one for Natural Deduction, in the sense that $\to$-conversions can only be applied after communication rule occurrences have been permuted down, thus missing the opportunity of having an interplay between β-conversions and communication.

Our approach to Avron's conjecture is to define a new system of Hyper Natural Deduction for Gödel Logic related to Natural Deduction which is motivated by the following guiding principles: The first is that *our system should follow Gentzen's original aim to reproduce real logical reasoning,* so that our system deserves the attribute *natural* in the sense of Gentzen. We argue below that we achieve this aim. A detailed discussion with precise definitions supporting our claim is given in Sections 3.

The second principle is that *our system should honestly reflect as directly as possible Avron's original Hypersequent Calculus*, meaning that the rules of Hypersequent Calculus should correspond in a direct way to rules in our system, similar to the correspondence between Gentzen's Sequent Calculus and Natural Deduction. The translations between our system and Avron's Hypersequent Calculus as given in Sections 3.4 and 3.5 show that this aim has been achieved.

The third guiding principle is that *our formalism should admit a normalization procedure based on conversions extending the normalization procedure for Natural Deduction*, so that normal deductions enjoy similar structural properties as normal proofs in Natural Deduction do. Based on the discussion in Section 4, we provide an alternative definition of Hyper Natural Deduction for Gödel Logic in Section 6, which instead of relying on an inductive definition, provides an explicit characterization of correct deductions in our system. We then show in Section 7 that we achieve the third aim by showing that each deduction in Hyper Natural Deduction for Gödel Logic can be transformed into normal form via appropriate conversions extending usual ones, including $\rightarrow$-conversion. We also show that normal deductions satisfy the subformula property, namely that each formula occurring in a normal proof is a subformula of either an open assumption or conclusion. Our conversions on first sight look much more involved than those of Natural Deduction, but careful analysis reveals that they are a natural choice.

Future work will be dedicated to complete the Curry-Howard correspondence for Hyper Natural Deduction for Gödel Logic, and thus for Avron's Hypersequent Calculus, by defining an appropriate *parallel λ calculus* (to use an expression coined by Avron [Avr91]) by incorporating suitable elements of process calculi, in particular Milner's π calculus [MPW92], into λ calculus to extend the Curry-Howard isomorphism from Natural Deduction to Hyper Natural Deduction for Gödel Logic.

1.1 Differences to previous version

The results in this paper were first announced in [BP15]. However, there are several changes: First of all, we have switched to an inductive definition of HNGL, and we introduce the previous explicit definition as an equivalent alternative definition needed to prove normalization — in this paper we denote the alternative, explicit definition with $\mathcal{X}$. Second, we improved the mathematical presentation of $\mathcal{X}$ using notions from graph theory. Finally, we made one essential change to correct an error in the definition given in [BP15] — without that change the system defined there is not sound. In the version given in [BP15] we only demanded the existence of a total order on labels for communication and split rules, which is compatible with their occurrences on branches through derivations. We now add additional requirements for labels of contraction rules. The reason for this is that without considering contraction labels, the definition in [BP15] would not be sound for Gödel Logic. The following two prederivations form a set which is a valid deduction according to [BP15] of the formula $((A \wedge C \to B) \wedge (C \to D)) \vee ((B \wedge D \to C) \wedge (B \to A))$.

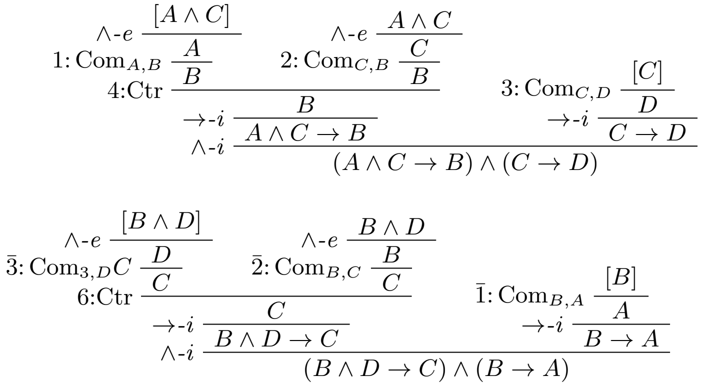


Fig. 1. A deduction given by two derivations that derives an invalid formula.

However, this formula is not valid in Gödel Logic — the counter-model is given by a valuation function v satisfying $0 < v(A) < v(B) < v(D) < v(C) < 1$, see Section 2 for a definition of valuations for Gödel Logic.

In the present paper we often consider a set of derivations as a graph, with vertices given by rule occurrences. The above set or derivations drawn as a graph has the form given in Figure 2.

A procedural reason why the deduction in Fig. 1 is not sound is that it cannot be transformed into a Hypersequent Calculus GLC derivation. In order to apply contraction 4, its premises need to be connected in the remaining graph, which involves contraction 6 and hence procedurally requires 6 before 4. But the same holds for 6 which then requires 4 before 6. So none of them can come before the other in an attempt to stepwise introducing them.

There is a cut (indicated with single slashes) which deletes edges in the graph, such that the above is a correct HNGL derivation according to [BP15]. Observe that for a different cut (indicated by double backslashes) the deduction in Fig. 1 would not be correct according to [BP15]. We conjecture that the version presented in [BP15] is correct if we demand that its conditions are satisfied for *any* possible cut operation. From a complexity perspective this gives an

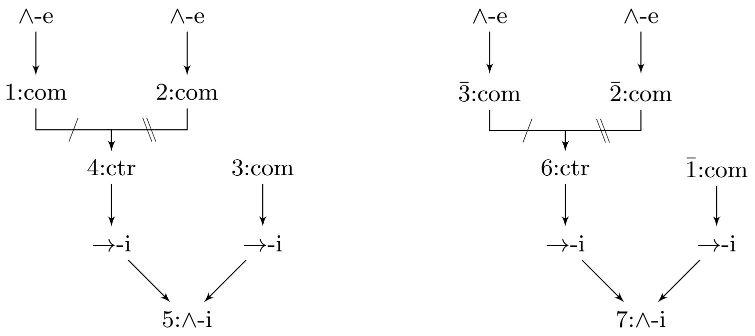


Fig. 2. The graph corresponding to the set of prederivations in Fig. 1.

exponential blow-up, thus does not present a suitable, efficient definition. In any case the above is not a correct derivation according to the new definition to be given in Section 6.

1.2 Layout of the article

We start off in Section 2 by briefly reviewing Gödel logics and Avron's Hypersequent Calculus. In Section 3 we extend Natural Deduction with new rules to form prederivations, and give an inductive definition of our system HNGL of Hyper Natural Deduction for Gödel Logic. We show that each GLC derivation can be transformed in a direct way into a HNGL deduction, and vice versa, which proves soundness and completeness of our system with respect to infinite-valued propositional Gödel logic. Section 4 discusses the reasons why the implicit definition given in the previous section is not sufficient to obtain normalization, and lays out our alternative representation. Section 5 provides definitions and basic properties of labeled graphs which are needed for an alternative, explicit definition of HNGL in the subsequent section. Section 6 introduces this alternative definition of HNGL, and proves its equivalence to the original definition. Section 7 uses the explicit representation of HNGL to provide a proof of weak normalization, and to prove that normal HNGL deductions satisfy the subformula property. A technical Appendix provides full proofs of some of the technical properties used in Sections 4–6.

2 Hypersequent Calculus

We briefly review the main results about Gödel logics. More details can be found in the handbook article on Gödel logics [BP11].

Propositional finite-valued Gödel logics were introduced by Gödel [Gö33] to show that intuitionistic logic does not have a finite characteristic matrix. They provide the first examples of intermediate logics (intermediate, that is, in strength between classical and intuitionistic logic). Dummett [Dum59] was the first to study infinite valued Gödel logics, axiomatizing the set of tautologies over infinite truth-value sets by intuitionistic logic extended by the linearity axiom $(A \to B) \vee (B \to A)$. Hence, infinite-valued propositional Gödel logic is also called Gödel-Dummett logic or Dummett's LC. In terms of Kripke semantics, the characteristic linearity axiom picks out those accessibility relations which are linear orders.

Perhaps the most surprising fact is that whereas there is only one infinite-valued propositional Gödel logic, there are infinitely many different logics at the first-order level [BLZ96,Baa96,Pre02], and in fact countably many [BGP08]. In the light of the general result of Scarpellini [Sca62] on non-axiomatizability, it is interesting that some of the infinite-valued Gödel logics belong to the limited class of recursively enumerable linearly ordered first-order logics [Hor69,TT84].

The language for propositional Gödel logics is a standard propositional language, which we fix with the following definition:

Definition 1. *The language L^0 for propositional Gödel logics consists of the propositional constant $\bot$, countably many propositional variables $(p_1, p_2, \dots)$ and the binary connectives $\wedge$, $\vee$ and $\to$. The set of well formed formulas, denoted by* $\mathrm{Frm}(L^0)$*, is defined inductively in the usual way.*

Definition 2. *Let $V \subseteq [0,1]$ be some set containing* 0 *and* 1*. We call V a* set of truth values*. A* propositional Gödel valuation v (short valuation) based on V *is a function from the set of propositional variables into V with $v(\bot) = 0$. This valuation can be extended to a function mapping formulas from* $\mathrm{Frm}(L^0)$ *into V as follows:*

$$v(A \wedge B) = \min\{v(A), v(B)\}$$
$$v(A \vee B) = \max\{v(A), v(B)\}$$
$$v(A \to B) = \begin{cases} v(B) & \text{if } v(A) > v(B) \\ 1 & \text{if } v(A) \le v(B). \end{cases}$$

A formula is called valid with respect to V *if it is mapped to* 1 *for all valuations based on V. The set of all formulas which are valid with respect to V is called* the propositional Gödel logic based on V*, and is denoted by $\mathbf{G}^0_V$.*

Infinitary propositional Gödel logic *GL is given by the propositional Gödel logic $\mathbf{G}^0_{[0,1]}$ based on the full interval* $[0,1]$.

In the following a formula is always a propositional formula. We shall use $A, B, C \dots$ to range over formulas, and $\Gamma, \Delta, \Xi \dots$ to range over finite sets of formulas.

We describe a version of Avron's Hypersequent Calculus [Avr91] following [BCF01]. As the version based on multi-conclusion sequents does not play a role for our exposition, we only define the single-conclusion version here. Thus, a *sequent* is an expression of the form $\Gamma \Rightarrow A$, where Γ is a finite set of formulas, and A is a formula. In particular, Γ being a set implies that structural rules of exchange, contraction and expansion (the converse of contraction) are build into our calculus. A *hypersequent*, in turn, is a finite multiset of sequents, which implies that the external version of exchange, but not contraction and expansion, is also build into our calculus. We shall use the usual hypersequent notation $s_1 \mid \ldots \mid s_n$ (for the multiset consisting of $s_1, \ldots, s_n$), and $\mathcal{H}, \mathcal{H}' \ldots$ to range over hypersequents. We also employ further standard notations like $\Gamma, A \Rightarrow B$ for $\Gamma \cup \{A\} \Rightarrow B$, and $\mathcal{H} \mid s$ for $\mathcal{H} \cup \{s\}$, etc.

The *Hypersequent Calculus for propositional Gödel logic*, GLC, is given by the following axioms and rules:

Axioms: $(id)\ A \Rightarrow A$ and $(\bot)\ \bot \Rightarrow A$

Cut Rule: $$cut\ \frac{\Gamma \Rightarrow A \mid \mathcal{H}_1 \qquad A, \Gamma \Rightarrow C \mid \mathcal{H}_2}{\Gamma \Rightarrow C \mid \mathcal{H}_1 \mid \mathcal{H}_2}$$

Internal Structural Rule: $$w\ \frac{\Gamma \Rightarrow C \mid \mathcal{H}}{\Gamma, B \Rightarrow C \mid \mathcal{H}}$$

External Structural Rules:

$$EW\ \frac{\Gamma \Rightarrow C \mid \mathcal{H}}{\Gamma \Rightarrow C \mid \Gamma' \Rightarrow C' \mid \mathcal{H}} \qquad EC\ \frac{\Gamma \Rightarrow C \mid \Gamma \Rightarrow C \mid \mathcal{H}}{\Gamma \Rightarrow C \mid \mathcal{H}}$$

Logical rules

$$\to,l\ \frac{\Gamma \Rightarrow A \mid \mathcal{H} \qquad \Gamma, B \Rightarrow C \mid \mathcal{H}'}{\Gamma, A \to B \Rightarrow C \mid \mathcal{H} \mid \mathcal{H}'} \qquad \to,r\ \frac{\Gamma, A \Rightarrow B \mid \mathcal{H}}{\Gamma \Rightarrow A \to B \mid \mathcal{H}}$$

$$\vee,l\ \frac{\Gamma, A \Rightarrow C \mid \mathcal{H} \qquad \Gamma, B \Rightarrow C \mid \mathcal{H}'}{\Gamma, A \vee B \Rightarrow C \mid \mathcal{H} \mid \mathcal{H}'} \qquad \vee_i,r\ \frac{\Gamma \Rightarrow A_i \mid \mathcal{H}}{\Gamma \Rightarrow A_1 \vee A_2 \mid \mathcal{H}} \quad i \in \{1,2\}$$

$$\wedge_i,l\ \frac{\Gamma, A_i \Rightarrow C \mid \mathcal{H}}{\Gamma, A_1 \wedge A_2 \Rightarrow C \mid \mathcal{H}} \quad i \in \{1,2\} \qquad \wedge,r\ \frac{\Gamma \Rightarrow A \mid \mathcal{H} \qquad \Gamma \Rightarrow B \mid \mathcal{H}'}{\Gamma \Rightarrow A \wedge B \mid \mathcal{H} \mid \mathcal{H}'}$$

Communication and Split:

$$com\ \frac{\Gamma_1 \Rightarrow A_1 \mid \mathcal{H} \qquad \Gamma_2 \Rightarrow A_2 \mid \mathcal{H}'}{\Gamma_1 \Rightarrow A_2 \mid \Gamma_2 \Rightarrow A_1 \mid \mathcal{H} \mid \mathcal{H}'} \qquad split\ \frac{\Pi, \Gamma \Rightarrow A \mid \mathcal{H}}{\Pi \Rightarrow A \mid \Gamma \Rightarrow A \mid \mathcal{H}}$$

Theorem 1 ([Avr91]). GLC *is sound and complete for infinitary propositional Gödel logic GL.*

3 Hyper Natural Deduction for Gödel Logic

The proposed system HNGL of Hyper Natural Deduction for Gödel Logic extends Gentzen's system NJ of Natural Deduction [Gen35] by two main adaptations: First, we extend NJ by four more rules (Definition 3) intended to model split, communication and external contraction in Avron's Hypersequent Calculus, plus an additional repetition rule needed for normalisation. The derivation trees obtained in this system are called 'prederivations' (Definition 5). We use the term ‘prederivations’ to stress that the derivation-like trees involving new rules in general do not derive valid assertions. Concerning terminology, we use the term ‘deduction’ to denote a well-formed hyper natural deduction. The term ‘derivation’ is also used for well-formed natural deductions elsewhere in the literature, but we avoided using it to clearly distinguish between prederivations and hyper natural deductions.

The second adaptation is that we consider sets of prederivations, which we call prehyper deductions. Not every set of prederivations provides a structure that can be interpreted as a meaningful proof in Gödel logic. We give an inductive definition of those finite sets of prederivations which are HNGL deductions in Definition 10. We then show that HNGL deductions correspond to GLC derivations by providing translations between the two formalisms (Theorems 2 and 3).

3.1 Gentzen’s system of Natural Deduction

We present NJ in the version given in [Bus98], but with an enhanced labeling of rules. As usual, a Natural Deduction style *derivation* consists of an upward rooted tree, where the nodes are formulas. Formulas at leaf nodes are called *assumptions*. All non-leaf nodes are carrying labels providing information about the rule which has been applied plus some other information (like which assumptions have been closed). We say that A is derivable from assumptions Γ and write

$$\begin{array}{c}\Gamma\\ \vdots\\ A\end{array}$$

if there is a derivation σ with root A such that the set of all open assumptions of σ is a subset of Γ. Derivations in NJ are generated inductively using the following initial, introduction and elimination rules. Rules are labeled with labels of the form

$$r : s$$

where r is a *label* from a given set of labels, and s is an *inference descriptor*. We introduce labels already now for consistency with later notation; we make it precise in Definition 4.

Any formula, viewed as a tree consisting of one node, is a derivation. Furthermore, derivations can be build using the following rules:

$$r{:}\wedge\text{-}i\ \dfrac{\begin{array}{c}\Gamma\\ \vdots\\ A\end{array}\qquad\begin{array}{c}\Delta\\ \vdots\\ B\end{array}}{A\wedge B}\qquad\qquad r{:}\wedge\text{-}e\ \dfrac{\begin{array}{c}\Gamma\\ \vdots\\ A\wedge B\end{array}}{A}\qquad \dfrac{\begin{array}{c}\Gamma\\ \vdots\\ A\wedge B\end{array}}{B}$$

$$r{:}\vee\text{-}i\ \frac{\begin{array}{c}\mathit{\Gamma}\\ \vdots\\ A\end{array}}{A\vee B}\qquad \frac{\begin{array}{c}\mathit{\Gamma}\\ \vdots\\ B\end{array}}{A\vee B}\qquad\qquad r{:}^{k}\vee\text{-}e\ \frac{\begin{array}{c}\mathit{\Gamma}\\ \vdots\\ A\vee B\end{array}\qquad \begin{array}{c}\mathit{\Delta},{}^{k}[A]\\ \vdots\\ C\end{array}\qquad \begin{array}{c}\mathit{\Pi},{}^{k}[B]\\ \vdots\\ C\end{array}}{C}$$

$$r{:}^{k}\rightarrow\text{-}i\ \frac{\begin{array}{c}\mathit{\Gamma},{}^{k}[A]\\ \vdots\\ B\end{array}}{A\rightarrow B}\qquad\qquad r{:}\rightarrow\text{-}e\ \frac{\begin{array}{c}\mathit{\Gamma}\\ \vdots\\ A\rightarrow B\end{array}\qquad \begin{array}{c}\mathit{\Delta}\\ \vdots\\ A\end{array}}{B}$$

$$r{:}\bot\text{-}i\ \frac{\begin{array}{c}\mathit{\Gamma}\\ \vdots\\ \bot\end{array}}{A}$$

The superscript k in expressions like ${}^{k}[A]$ is used to connect discharged assumptions to rule applications in the usual way; we drop it in the following if it is clear from the context.

For any rule occurrence in a derivation, we employ notions like *immediate subderivation*, *upper derivations*, *upper left (middle, right) derivation* etc. in the usual way related to the pictorial definition of the rules, see [Bus98,TS00] for more details.

3.2 Rules for HNGL

To define HNGL, we expand NJ by four new rules. Besides communication and split rules which correspond to those in GLC, we also need a contraction rule and a repetition rule to be able to effectively define all conversion rules needed for normalization. The new communication and split rules employ ideas from process algebra: They come in pairs of duals, technically realized by using labels that come in pairs, and the idea is that such pairs of dual labels form a connection called "channel" which is used during normalization to "communicate" subderivations.

With this view, communication is introduced in our system in the following way: Assume we have two HNGL deductions, one containing a derivation of A and one of B. Then we can combine these two using communication, in the spirit of Avron [Avr91], by forming one HNGL deduction:

$$\text{From}\quad \begin{array}{c}\vdots\\ A\end{array}\quad\text{and}\quad \begin{array}{c}\vdots\\ B\end{array}\quad\text{form}\quad \left\{ x{:}\,\mathrm{Com}_{A,B}\ \frac{\begin{array}{c}\vdots\\ A\end{array}}{B}\ ,\ \bar{x}{:}\,\mathrm{Com}_{B,A}\ \frac{\begin{array}{c}\vdots\\ B\end{array}}{A}\right\}$$

In this deduction, a communication channel (in the spirit of process algebra) has been establish between two derivation trees, indicated by the label x and its dual $\bar{x}$, allowing to exchange the two formulas A and B after the introduction of a pair of dually labeled communication rules.

Definition 3 (Rules of HNGL). *The set of rules of* HNGL *consists of the rules for NJ, plus the following four rules:*

$$
r{:}^{k}\,\mathrm{Spt}_{\Gamma,\Delta}\ \dfrac{\begin{matrix}{}^{k}[\Gamma],\Delta\\ \vdots\\ A\end{matrix}}{A}
\qquad
r{:}\,\mathrm{Com}_{A,B}\ \dfrac{\begin{matrix}\Gamma\\ \vdots\\ A\end{matrix}}{B}
\qquad
r{:}\,\mathrm{Ctr}\ \dfrac{\begin{matrix}\Gamma\\ \vdots\\ A\end{matrix}\qquad\begin{matrix}\Delta\\ \vdots\\ A\end{matrix}}{A}
\qquad
r{:}\,\mathrm{Rep}\ \dfrac{\begin{matrix}\Gamma\\ \vdots\\ A\end{matrix}}{A}
$$

where r is a label from a set $\mathcal{L}$ that is fixed in Definition 4.

The repetition rule is included for technical reasons, as it allows us later to express transformations of hyper natural deductions during normalisation more smoothly. We can always remove occurrences of Rep from a hyper natural deduction by contracting its premise and conclusion:

$$\mathrm{Rep}\ \dfrac{F}{F} \quad \rightsquigarrow \quad F$$

Labels play a crucial role at various places for hyper natural deductions: They are used to identify some rule occurrences, but distinguish those from others, all carrying the same inference descriptor. They also play a similar role as labels in process algebra in that they identify a communication channel. We make use of this link during normalization in Section 7.

We fix now a set of labels which stays unchanged throughout the paper. One essential property is the existence of dual labels for labels relating to communication and split. This is realised via a partial map $x \mapsto \bar{x}$ on the set of labels.

Definition 4 (Labels). *Let $\mathcal{L}$ be some fixed set. We assume the existence of some additional operations on $\mathcal{L}$ which we describe below. $\mathcal{L}$ is called the* set of labels, *elements of $\mathcal{L}$ are called* labels, *and we assume that $\mathcal{L}$ satisfies the following conditions:*

1. *There is a map from the set of labels to the set of inference descriptors. If label x is mapped to inference descriptor s, we say that x* is carrying *s, and denote this as $x{:}s$.*
2. *For each inference descriptor s, there are infinitely many labels carrying s.*
3. *There is an assignment $x \mapsto \bar{x}$ which is a partial function on $\mathcal{L}$. For a label x, if $\bar{x}$ is defined, then we say that $\bar{x}$* is the dual of *x, and we require that $\bar{\bar{x}}$ is also defined and satisfies $\bar{\bar{x}} = x$. Furthermore, if x is a label with inference descriptor $\mathrm{Spt}_{\Gamma,\Delta}$ (resp. $\mathrm{Com}_{A,B}$) then $\bar{x}$ is defined and the inference descriptor of $\bar{x}$ is $\mathrm{Spt}_{\Delta,\Gamma}$ (resp. $\mathrm{Com}_{B,A}$).*
   To express the latter condition succinctly, we say that the dual of $x\colon \mathrm{Spt}_{\Gamma,\Delta}$, denoted $\overline{x\colon \mathrm{Spt}_{\Gamma,\Delta}}$, is $\bar{x}\colon \mathrm{Spt}_{\Delta,\Gamma}$, that is $\overline{x\colon \mathrm{Spt}_{\Gamma,\Delta}} = \bar{x}\colon \mathrm{Spt}_{\Delta,\Gamma}$. Similarly for the dual of $x\colon \mathrm{Com}_{A,B}$, we have $\overline{x\colon \mathrm{Com}_{A,B}} = \bar{x}\colon \mathrm{Com}_{B,A}$.

As each label carries exactly one inference descriptor, we assume from now on that rules are labeled by labels only, instead of the hitherto used notation $x{:}s$. We may write $x{:}s$ for convenience to the reader to succinctly express that x is carrying s.

Definition 5 (Prederivation). *A* prederivation *is a well-formed derivation tree based on the rules of* HNGL*.*

Note that "well-formed" implies adherence to all the rules of Natural Deduction, and local syntactical consistency of the additional rules. We shall use $\rho, \sigma \dots$ to range over prederivations. A label of the form $x\colon \mathrm{Spt}_{\Gamma,\Delta}$ is called *splitting label* and a rule introducing it *splitting rule* or simply *split*. One of the form $x\colon \mathrm{Com}_{A,B}$ is called *communication label* and a rule introducing it *communication rule* or simply *communication*. One of the form $x\colon \mathrm{Ctr}$ is called *contraction label* and a rule introducing it *contraction rule*, or simply *contraction*. One of the form $x\colon \mathrm{Rep}$ is called *repetition label* and a rule introducing it *repetition rule*, or simply *repetition*.

The following notations and concepts are useful:

Definition 6. *For prederivations $\rho, \rho_1, \dots, \rho_n$ we define the following:*

$\mathrm{Labels}(\rho_1, \dots, \rho_n)$ *is the set of all labels which label any rule occurring in any of the ρ_i. We single out two specific subsets of* $\mathrm{Labels}(\rho_1, \dots, \rho_n)$*, namely* $\mathrm{DLabels}(\rho_1, \dots, \rho_n)$ *for the set of labels carrying split or communication (these labels have a dual, thus the "D" in* DLabels*), and* $\mathrm{CLabels}(\rho_1, \dots, \rho_n)$ *for the set of labels carrying contraction (the "C" in* CLabels *refers to contractions). For any R, we denote with* $\mathrm{CDLabels}(R)$ *the set* $\mathrm{CLabels}(R) \cup \mathrm{DLabels}(R)$*.*

With $\mathrm{Assum}(\rho)$ *we denote the set of assumptions of ρ which are not discharged;* $\mathrm{Conc}(\rho)$ *is the final conclusion, i.e. last formula, of ρ.*

For an occurrence of a formula A in a prederivation, we define the subprederivation rooted in A *as the subtree up to and including A. For a rule occurrence r, we define the* subtree rooted in r *as the subprederivation rooted in the conclusion of r. An* immediate subprederivation of ρ *is a subprederivation rooted in one of the premises of the final rule of ρ.*

3.3 Defining the system

We define the system of Hyper Natural Deduction for Gödel Logic via an inductive definition based on hyper rules, which operates on finite sets of prederivations.

Definition 7 (Prehyper deduction). *A* prehyper deduction *is a finite set of prederivations.*

We follow [BCF01] to communicate hyper rules in a readable way. First, the symbol $\mid$ is also used to separate the elements of prehyper deductions, in the same way as it was used to separate elements of hyper sequents. We extend standard proof theoretic notation to the level of prehyper deductions: For prehyper deductions R, R' and prederivations ρ, ρ', we write $R \mid \rho$ for $R \cup \{\rho\}$, and $R \mid R' \mid \rho \mid \rho'$ for $R \cup R' \cup \{\rho, \rho'\}$, etc.

Definition 8 (Hyper rules). *A* hyper rule $\mathbf{h}$-r *of arity k is an operation which takes k prehyper deductions and produces another prehyper deduction. It is displayed in the form*

$$\mathbf{h}\text{-}r \, \frac{R_1 \qquad \cdots \qquad R_k}{R}$$

and we say that **h**-r applied to $R_1 \ldots, R_k$ yields R. *Without loss of generality we assume in this case that the sets of labels for* R_i, $\text{Labels}(R_i)$, *for* $i \leq k$, *are pairwise disjoint.*

Each NJ-rule r induces a corresponding hyper rule **h**-r. If r has k premises, then **h**-r has arity k. In addition to these, we have a hyper communication rule **h**-Com of arity 2, a hyper splitting rule **h**-Spt, a hyper contraction rule **h**-Ctr, and a hyper repetition rule **h**-Rep, all of arity 1.

Hyper rule h-r for NJ rule r: Each NJ rule induces a corresponding hyper rule in the obvious way. For example, consider the case of **h**-→-e:

$$\mathbf{h}\text{-}{\to}\text{-e}\ \dfrac{R_1 \left| \begin{array}{c}\Gamma\\ \vdots\\ A\to B\end{array}\right. \qquad\qquad R_2 \left| \begin{array}{c}\Delta\\ \vdots\\ A\end{array}\right.}{R_1 \;\Big|\; R_2 \left| \;{\to}\text{-e}\,\dfrac{\begin{array}{cc}\Gamma & \Delta\\ \vdots & \vdots\\ A\to B & A\end{array}}{B}\right.}$$

Applying the hyper rule **h**-→-e here means the following: Given two prehyper deductions — one containing a prederivation of $A \to B$ from Γ and side prehyper deduction R_1, the other one containing a prederivation of A from Δ and side prehyper deduction R_2 — we form one prehyper deduction consisting of the indicated prederivation of B from Γ, Δ, together with the union of the side prehyper deductions R_1 and R_2.

Hyper communication rule: Applying the hyper communication rule

$$\mathbf{h}\text{-Com}\ \dfrac{R_1 \left| \begin{array}{c}\Gamma\\ \vdots\\ A\end{array}\right. \qquad\qquad R_2 \left| \begin{array}{c}\Delta\\ \vdots\\ B\end{array}\right.}{R_1 \;\Big|\; R_2 \left| \; x\colon \text{Com}_{A,B}\,\dfrac{\begin{array}{c}\Gamma\\ \vdots\\ A\end{array}}{B} \;\right| \; \bar{x}\colon \text{Com}_{B,A}\,\dfrac{\begin{array}{c}\Delta\\ \vdots\\ B\end{array}}{A}}$$

means that two prehyper deductions — one containing a prederivation of A from Γ, the other containing a prederivation of B from Δ — are replaced by one prehyper deduction consisting of the two indicated components with their conclusion interchanged, together with the union of the side prehyper deductions R_1 and R_2. Here x is a fresh label carrying $\text{Com}_{A,B}$.

Hyper splitting rule: For the hyper splitting rule we have

$$\mathbf{h}\text{-}\mathrm{Spt}\;\frac{R \left| \begin{array}{c}\varGamma,\varDelta\\ \vdots\\ A\end{array}\right.}{R \left| \begin{array}{r} {}^{k}[\varGamma],\varDelta\\ \vdots \\ x{:}^{k}\,\mathrm{Spt}_{\varGamma,\varDelta}\,\dfrac{A}{A}\end{array} \right| \begin{array}{r} \varGamma,{}^{l}[\varDelta]\\ \vdots \\ \bar{x}{:}^{l}\,\mathrm{Spt}_{\varDelta,\varGamma}\,\dfrac{A}{A}\end{array}}$$

Applying this rule means that the set $\varGamma,\varDelta$ of assumptions of the indicated component of the upper prehyper deduction is split into two (not necessary disjoint) subsets $\varDelta$ and $\varGamma$. The application itself can be described in three steps: First, we fix a partitioning of the open assumption occurrences related to $\varGamma$ and $\varDelta$. Second, the indicated component of the upper prehyper deduction is duplicated while maintaining the partitioning also in the duplicated copy. Third, we discharge open assumptions related to $\varGamma$ from the first copy and add A to its root with label $x{:}^{k}\,\mathrm{Spt}_{\varGamma,\varDelta}$, and we discharge open assumptions related to $\varDelta$ from the second copy and add A to its root with label $\bar{x}{:}^{\ell}\,\mathrm{Spt}_{\varDelta,\varGamma}$, where x is a fresh label carrying $\mathrm{Spt}_{\varGamma,\varDelta}$.

Hyper contraction rule: Applying the hyper contraction rule

$$\mathbf{h}\text{-}\mathrm{Ctr}\;\frac{R \left| \begin{array}{c}\varGamma\\ \vdots\\ A\end{array}\right| \begin{array}{c}\varDelta\\ \vdots\\ A\end{array}}{R \left| \begin{array}{r} \varGamma \;\; \varDelta\\ \vdots \;\;\; \vdots \\ x{:}\,\mathrm{Ctr}\,\dfrac{A \quad A}{A}\end{array}\right.}$$

means the following. Assume that the upper prehyper deduction has two components that end in the same formula A. For the new, lower prehyper deduction we combine these two prederivations into one by appending another occurrence of A as the common root of those components, and label the new root with $x{:}\,\mathrm{Ctr}$ for some fresh label x.

Hyper repetition rule: In applying the hyper repetition rule

$$\mathbf{h}\text{-}\mathrm{Rep}\;\frac{R \left| \begin{array}{c}\varGamma\\ \vdots\\ A\end{array}\right.}{R \left| \begin{array}{r} \varGamma\\ \vdots \\ x{:}\,\mathrm{Rep}\,\dfrac{A}{A}\end{array}\right.}$$

we repeat the conclusion A of some prederivation in the upper prehyper deduction by appending another occurrence of A as the new root and label it with $x{:}\,\mathrm{Rep}$ for some fresh label x.

Definition 9 (Hyper rules for HNGL). *The* HNGL hyper rules*, also just called* hyper rules*, are given by the hyper rules corresponding to NJ rules, plus the hyper communication rule **h**-Com, the hyper splitting rule **h**-Spt, the hyper contraction rule **h**-Ctr, and the hyper repetition rule **h**-Rep.*

Definition 10 (HNGL). *We inductively define* HNGL *as the collection of prehyper deductions which contains all initial NJ deductions, viewed as singleton sets, and which is closed under applying hyper rules for* HNGL*. A prehyper deduction derived in this way is called a* deduction in Hyper Natural Deduction for Gödel Logic*, or short a* HNGL deduction*, or simply a* hyper natural deduction*.*

3.4 Translation from GLC to HNGL

In the following, we translate GLC derivations into HNGL deductions. As an illustration consider the linearity formula $C = (A \to B) \vee (B \to A)$, which has the following GLC derivation:

$$
\begin{prooftree}
\AxiomC{$A \Rightarrow A$}
\AxiomC{$B \Rightarrow B$}
\LeftLabel{com}
\BinaryInfC{$A \Rightarrow B \mid B \Rightarrow A$}
\LeftLabel{$\to,r$}
\UnaryInfC{$\Rightarrow A \to B \mid B \Rightarrow A$}
\LeftLabel{$\to,r$}
\UnaryInfC{$\Rightarrow A \to B \mid \;\Rightarrow B \to A$}
\LeftLabel{$\vee_1,r$}
\UnaryInfC{$\Rightarrow C \mid \;\Rightarrow B \to A$}
\LeftLabel{$\vee_2,r$}
\UnaryInfC{$\Rightarrow C \mid \;\Rightarrow C$}
\LeftLabel{EC}
\UnaryInfC{$\Rightarrow C$}
\end{prooftree}
$$

The corresponding HNGL deduction has the following form:

$$
\begin{prooftree}
\AxiomC{${}^1[A]$}
\LeftLabel{$x\colon \mathrm{Com}_{A,B}$}
\UnaryInfC{B}
\LeftLabel{${}^1{\to}\text{-}i$}
\UnaryInfC{$A \to B$}
\LeftLabel{$\vee\text{-}i$}
\UnaryInfC{C}
\AxiomC{${}^2[B]$}
\LeftLabel{$x\colon \mathrm{Com}_{A,B}$}
\UnaryInfC{A}
\LeftLabel{${}^2{\to}\text{-}i$}
\UnaryInfC{$B \to A$}
\LeftLabel{$\vee\text{-}i$}
\UnaryInfC{C}
\LeftLabel{$y\colon \mathrm{Ctr}$}
\BinaryInfC{C}
\end{prooftree}
$$

It can be obtained by a direct translation of the GLC derivation above using corresponding hyper rules, as shown in Figure 3. The first two lines in the figure should be read as follows: The set consisting of the initial NJ deduction given by A is a hyper natural deduction. Similar for B. To these two hyper natural deductions we apply the hyper communication rule **h**-Com to obtain the hyper natural deduction displayed below the first double line, which is a set with two prederivations. Etc.

With the next two definitions we fix the notion of a derived hypersequent of a hyper natural deduction, and how to compare hypersequents semantically.

Definition 11. *Let*

$$\mathcal{H} \quad = \quad \Gamma_1 \Rightarrow A_1 \mid \ldots \mid \Gamma_k \Rightarrow A_k$$

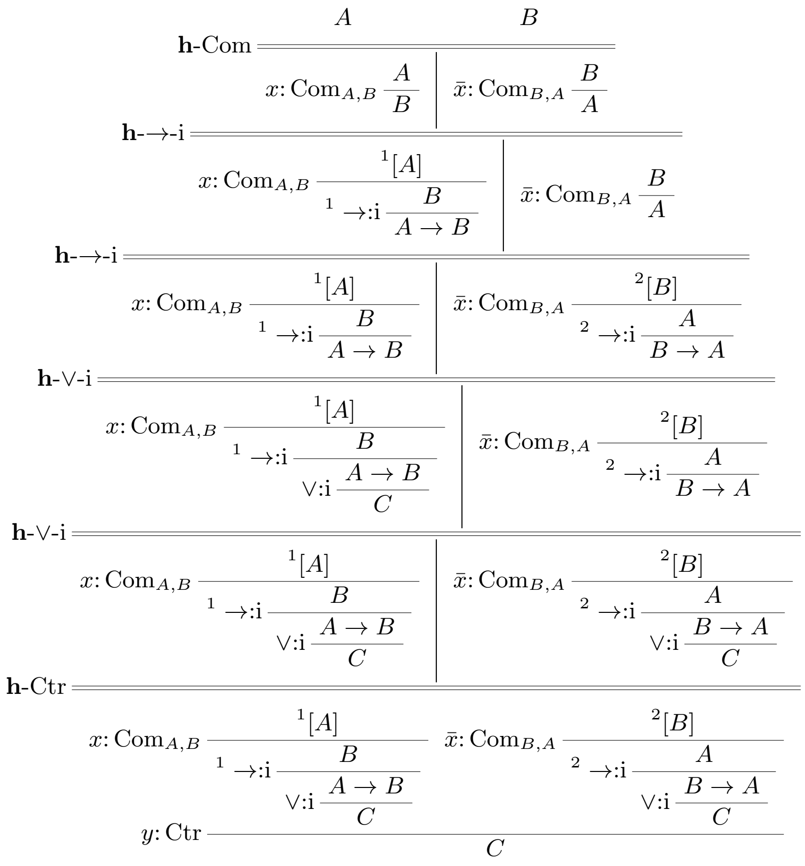


Fig. 3. HNGL deduction of linearity $\quad C \;=\; A \to B \vee B \to A$

and

$$\mathcal{H}' \quad = \quad \Delta_1 \Rightarrow B_1 \mid \ \ldots \ \mid \Delta_\ell \Rightarrow B_\ell \ \ .$$

$\mathcal{H}$ is a syntactic subhypersequent *of $\mathcal{H}'$, denoted $\mathcal{H} \sqsubseteq \mathcal{H}'$, if and only if there exists an injection $f\colon \{1,\ldots,k\} \to \{1,\ldots,\ell\}$ such that for all $i = 1,\ldots,k$,*

$$A_i = B_{f(i)} \qquad \textit{and} \qquad \Gamma_i \subseteq \Delta_{f(i)} \ \ .$$

We observe that if $\mathcal{H} \sqsubseteq \mathcal{H}'$ and $\mathcal{H}$ is valid, then also $\mathcal{H}'$ is valid, under the usual definition of validity of hypersequents which interprets

$$\Gamma_1 \Rightarrow A_1 \mid \ \ldots \ \mid \Gamma_k \Rightarrow A_k$$

as

$$(\bigwedge \Gamma_1 \to A_1) \vee \cdots \vee (\bigwedge \Gamma_k \to A_k) \ \ .$$

Definition 12. *The* derived sequent *of a prederivation ρ, denoted* $\mathrm{Seq}(\rho)$*, is given by* $\mathrm{Assum}(\rho) \Rightarrow \mathrm{Conc}(\rho)$*.*
The derived hypersequent *of a prehyper deduction $R = \{\rho_1, \ldots, \rho_k\}$, denoted* $\mathrm{HypSeq}(R)$*, is given by* $\mathrm{Seq}(\rho_1) \mid \ldots \mid \mathrm{Seq}(\rho_k)$*.*

We are now ready to translate GLC derivations into corresponding hyper natural deductions.

Theorem 2. *Assume that the hypersequent $\mathcal{H}$ has a* GLC *derivation. Then there exists a deduction R in* HNGL *such that the derived hypersequent of R is a syntactic subhypersequent of $\mathcal{H}$.*

Proof. Fix a GLC derivation D of $\mathcal{H}$. We prove the claim by induction on the length of D. If D is an axiom $A \Rightarrow A$, we form the HNGL deduction consisting just of A. If D is an axiom of the form $\bot \Rightarrow A$, then we form the HNGL deduction consisting of $\bot\text{-}i\ \dfrac{\bot}{A}$.

EC, (→,r), (∨,l), ($\vee_i$,r), (∧,r), (com), (split)
In all these cases the assertion follows by applying the corresponding hyper rule to the HNGL deductions obtained by induction hypothesis: For EC use the hyper contraction rule, for (→,r) the hyper →:i rule, for (∨,l) the hyper ∨:e rule, for ($\vee_i$,r) the hyper ∨:i rule, for (∧,r) the hyper ∧:i rule, for (com) the hyper communication rule, and for (split) the hyper splitting rule.

weakenings
Internal and external weakenings can be ignored, as we require the derived hypersequent to be a syntactic subhypersequent only.

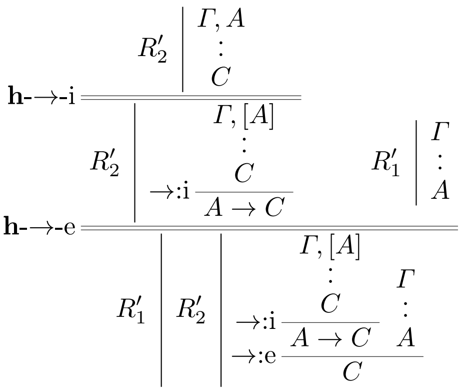
cut rule
If the last rule in D is an application of the cut rule of the form

$$\frac{\Gamma \Rightarrow A \mid \mathcal{H}_1 \qquad A, \Gamma \Rightarrow C \mid \mathcal{H}_2}{\Gamma \Rightarrow C \mid \mathcal{H}_1 \mid \mathcal{H}_2}$$

let π_1 and π_2 be the subderivations ending in the left and right, respectively, premises of the rule. By induction hypothesis we have two HNGL deductions R_1 and R_2 such that $\mathrm{HypSeq}(R_1) \sqsubseteq \Gamma \Rightarrow A \mid \mathcal{H}_1$ and $\mathrm{HypSeq}(R_2) \sqsubseteq A, \Gamma \Rightarrow C \mid \mathcal{H}_2$. Without loss of generality we assume that the sets of labels of R_1 and R_2 are disjoint.

If $\mathrm{HypSeq}(R_1) \sqsubseteq \mathcal{H}_1$ then the required HNGL deduction is given by R_1. Similar if $\mathrm{HypSeq}(R_2) \sqsubseteq \mathcal{H}_2$. Otherwise, there are prederivations ρ_1, ρ_2 such that the following holds: (i) $R_1 = R_1' \mid \rho_1$, $\mathrm{Seq}(\rho_1) \sqsubseteq \Gamma \Rightarrow A$ and $\mathrm{HypSeq}(R_1') \sqsubseteq \mathcal{H}_1$; and (ii) $R_2 = R_2' \mid \rho_2$, $\mathrm{Seq}(\rho_2) \sqsubseteq \Gamma, A \Rightarrow C$, and $\mathrm{HypSeq}(R_2') \sqsubseteq \mathcal{H}_2$. We generate

R by applying hyper rules in the following way:



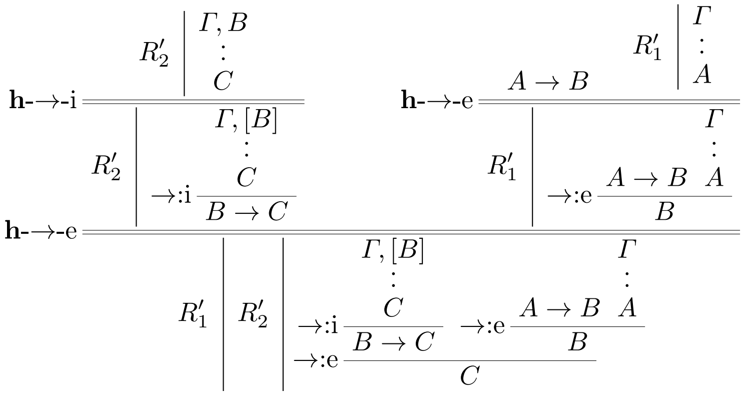
It is easy to see that

$$\mathrm{HypSeq}(R) \sqsubseteq \Gamma \Rightarrow C \mid \mathcal{H}_1 \mid \mathcal{H}_2 \ .$$

(→,l)
Assume the last rule of D is an application of (→,l) of the form

$$\to, l\ \frac{\Gamma \Rightarrow A \mid \mathcal{H}_1 \qquad \Gamma, B \Rightarrow C \mid \mathcal{H}_2}{\Gamma, A \to B \Rightarrow C \mid \mathcal{H}_1 \mid \mathcal{H}_2}$$

Similar to the case for the cut rule, we can assume using induction hypothesis that there are two hyper natural deductions R_1 and R_2 with disjoint sets of labels, such that $\mathrm{HypSeq}(R_1) \sqsubseteq \Gamma \Rightarrow A \mid \mathcal{H}_1$ and $\mathrm{HypSeq}(R_2) \sqsubseteq \Gamma, B \Rightarrow C \mid \mathcal{H}_2$. Furthermore, we can assume $R_1 = R'_1 \mid \rho_1$ with $\mathrm{HypSeq}(R'_1) \sqsubseteq \mathcal{H}_1$, and $R_2 = R'_2 \mid \rho_2$ with $\mathrm{HypSeq}(R'_2) \sqsubseteq \mathcal{H}_2$, such that $\mathrm{Seq}(\rho_1) \sqsubseteq \Gamma \Rightarrow A$ and $\mathrm{Seq}(\rho_2) \sqsubseteq \Gamma, B \Rightarrow C$. We now form the following hyper natural deduction R:



We observe that $\mathrm{HypSeq}(R) \sqsubseteq \Gamma, A \to B \Rightarrow C \mid \mathcal{H}_1 \mid \mathcal{H}_2$.

($\wedge_i$,l)
Assume the last rule in D is an application of (∧,l) of the form

$$
\wedge_i,l\,\frac{\Gamma, A_i \Rightarrow C \mid \mathcal{H}}{\Gamma, A_1 \wedge A_2 \Rightarrow C \mid \mathcal{H}}
$$

for some $i \in \{1, 2\}$. Similar to the two previous cases, we can assume using induction hypothesis that there is a hyper natural deduction R_1 such that $\mathrm{HypSeq}(R_1) \sqsubseteq \Gamma, A_i \Rightarrow C \mid \mathcal{H}$, and $R_1 = R'_1 \mid \rho_1$ with $\mathrm{HypSeq}(R'_1) \sqsubseteq \mathcal{H}$, and $\mathrm{Seq}(\rho_1) \sqsubseteq \Gamma, A_i \Rightarrow C$. We now form the following hyper natural deduction R:

$$
\mathbf{h}\text{-}{\to}\text{-e}\,\cfrac{
\mathbf{h}\text{-}{\to}\text{-i}\,\cfrac{R'_1\left|\begin{array}{c}\Gamma, A_i\\ \vdots\\ C\end{array}\right.}{R'_1\left|\begin{array}{c}\Gamma,[A_i]\\ \vdots\\ {\to}{:}\mathrm{i}\,\dfrac{C}{A_i\to C}\end{array}\right.}
\qquad
\mathbf{h}\text{-}{\wedge}\text{-e}\,\cfrac{A_1\wedge A_2}{{\wedge}\text{-e}\,\dfrac{A_1\wedge A_2}{A_i}}
}{
R'_1\left|\begin{array}{c}\Gamma,[A_i]\\ \vdots\\ {\to}{:}\mathrm{e}\,\dfrac{{\to}{:}\mathrm{i}\,\dfrac{C}{A_i\to C}\quad {\wedge}\text{-e}\,\dfrac{A_1\wedge A_2}{A_i}}{C}\end{array}\right.}
$$

We observe $\mathrm{HypSeq}(R) \sqsubseteq \Gamma, A_1 \wedge A_2 \Rightarrow C \mid \mathcal{H}$.

This concludes the proof of Theorem 2. □

A formula A is GLC derivable if the hypersequent $\Rightarrow A$ is GLC derivable. A is HNGL derivable if there is a HNGL deduction consisting of one prederivation ρ, which has no free assumptions and ends in A, that is, $\mathrm{Assum}(\rho) = \emptyset$ and $\mathrm{Conc}(\rho) = A$.

Corollary 1. *If A is* GLC *derivable, then A is also* HNGL *derivable.*

3.5 Translation from HNGL to GLC

The missing piece for soundness and completeness of HNGL is the reverse direction of Theorem 2.

Theorem 3. *Let R be a deduction in* HNGL*. Then there exists a derivation in* GLC *of* HypSeq(R)*, the derived hypersequent of R.*

Proof. The proof is by induction on the build-up of R as a hyper natural deduction.

Initial deductions
If R is an initial NJ deduction, that is a tree consisting of one node made from some formula A, then the corresponding GLC-derivation consists just of the axiom $A \Rightarrow A$.

Hyper NJ rules
If R has been formed by applying a hyper rule **h**-r for NJ rule r, then we can

simply follow the standard translation of NJ to LK. For example, consider the case of **h**-→-e:

$$\mathbf{h}\text{-}{\to}\text{-e}\ \dfrac{R_1 \left|\begin{array}{c}\Gamma\\ \vdots \\ A\to B\end{array}\right. \qquad\qquad R_2 \left|\begin{array}{c}\Delta\\ \vdots\\ A\end{array}\right.}{R_1 \left| R_2 \left| \begin{array}{c}\Gamma \qquad \Delta\\ \vdots \qquad\ \vdots \\ {\to}\text{-e}\,\dfrac{A\to B \quad A}{B}\end{array}\right.\right.}$$

Let $\mathcal{H}_i = \mathrm{HypSeq}(R_i)$. By induction hypothesis there exist GLC derivations π_1 of $\mathcal{H}_1 \mid \Gamma \Rightarrow A \to B$, and π_2 of $\mathcal{H}_2 \mid \Delta \Rightarrow A$. We form the following GLC derivation:

$$cut\ \dfrac{w\,\dfrac{\begin{array}{c}\pi_1\ \vdots\\ \Gamma \Rightarrow A\to B \mid \mathcal{H}_1\end{array}}{\Gamma,\Delta\Rightarrow A\to B\mid\mathcal{H}_1} \qquad {\to},l\,\dfrac{w\,\dfrac{\begin{array}{c}\pi_2\ \vdots\\ \Delta\Rightarrow A\mid\mathcal{H}_2\end{array}}{\Gamma,\Delta\Rightarrow A\mid\mathcal{H}_2}\qquad w\,\dfrac{B\Rightarrow B}{\Gamma,\Delta,B\Rightarrow B}}{A\to B,\Gamma,\Delta\Rightarrow B\mid\mathcal{H}_2}}{\Gamma,\Delta\Rightarrow B\mid\mathcal{H}_1\mid\mathcal{H}_2}$$

Hyper communication rule

Assume R has been formed by applying a hyper communication rule **h**-Com:

$$\mathbf{h}\text{-Com}\ \dfrac{R_1 \left|\begin{array}{c}\Gamma\\ \vdots \\ A\end{array}\right. \qquad\qquad R_2 \left|\begin{array}{c}\Delta\\ \vdots\\ B\end{array}\right.}{R_1 \left| R_2 \left| \begin{array}{r}\Gamma\\ \vdots\\ x\colon \mathrm{Com}_{A,B}\,\dfrac{A}{B}\end{array}\right| \begin{array}{r}\Delta\\ \vdots\\ \bar{x}\colon \mathrm{Com}_{B,A}\,\dfrac{B}{A}\end{array}\right.\right.}$$

Let $\mathcal{H}_i = \mathrm{HypSeq}(R_i)$. By induction hypothesis there exist GLC derivations π_1 of $\mathcal{H}_1 \mid \Gamma \Rightarrow A$, and π_2 of $\mathcal{H}_2 \mid \Delta \Rightarrow B$. Applying com yields the required GLC derivation:

$$com\ \dfrac{\begin{array}{c}\pi_1\ \vdots\\ \Gamma\Rightarrow A\mid\mathcal{H}_1\end{array}\qquad \begin{array}{c}\pi_2\ \vdots\\ \Delta\Rightarrow B\mid\mathcal{H}_2\end{array}}{\Gamma\Rightarrow B\mid\Delta\Rightarrow A\mid\mathcal{H}_1\mid\mathcal{H}_2}$$

Hyper splitting rule

Assume R has been formed by applying a hyper split rule **h**-Spt:

$$\mathbf{h}\text{-Spt}\ \dfrac{R \left|\begin{array}{c}\Gamma,\Delta\\ \vdots \\ A\end{array}\right.}{R \left| \begin{array}{r}{}^{k}[\Gamma],\Delta\\ \vdots\\ x{:}^{k}\,\mathrm{Spt}_{\Gamma,\Delta}\,\dfrac{A}{A}\end{array}\right| \begin{array}{r}\Gamma,{}^{l}[\Delta]\\ \vdots\\ \bar{x}{:}^{l}\,\mathrm{Spt}_{\Delta,\Gamma}\,\dfrac{A}{A}\end{array}}$$

Let $\mathcal{H} = \text{HypSeq}(R)$. By induction hypothesis there exist a GLC derivations π of $\mathcal{H} \mid \Gamma, \Delta \Rightarrow A$. Applying split yields the required GLC derivation:

$$\begin{array}{c} \pi \;\vdots \\ \textit{split}\;\dfrac{\Gamma, \Delta \Rightarrow A \mid \mathcal{H}}{\Gamma \Rightarrow A \mid \Delta \Rightarrow A \mid \mathcal{H}} \end{array}$$

Hyper contraction rule
Assume R has been formed by applying the hyper contraction rule **h**-Ctr:

$$\mathbf{h}\text{-Ctr}\;\dfrac{R \left| \begin{array}{c}\Gamma\\ \vdots \\ A\end{array}\right| \begin{array}{c}\Delta\\ \vdots\\ A\end{array}}{R \left| \begin{array}{c} x{:}\,\text{Ctr}\,\dfrac{\begin{array}{cc}\Gamma & \Delta\\ \vdots & \vdots \\ A & A\end{array}}{A}\end{array}\right.}$$

Let $\mathcal{H} = \text{HypSeq}(R)$. By induction hypothesis there exist a GLC derivations π of $\mathcal{H} \mid \Gamma \Rightarrow A \mid \Delta \Rightarrow A$. Applying contraction yields the required GLC derivation:

$$\begin{array}{c} \pi \;\vdots \\ \textit{contr}\;\dfrac{w\;\dfrac{w\;\dfrac{\Gamma \Rightarrow A \mid \Delta \Rightarrow A \mid \mathcal{H}}{\Gamma, \Delta \Rightarrow A \mid \Delta \Rightarrow A \mid \mathcal{H}}}{\Gamma, \Delta \Rightarrow A \mid \Gamma, \Delta \Rightarrow A \mid \mathcal{H}}}{\Gamma, \Delta \Rightarrow A \mid \mathcal{H}} \end{array}$$

Hyper repetition rule
If R has been formed by applying the hyper repetition rule **h**-Rep

$$\mathbf{h}\text{-Rep}\;\dfrac{R \left| \begin{array}{c}\Gamma\\ \vdots \\ A\end{array}\right.}{R \left| \begin{array}{c} x{:}\,\text{Rep}\,\dfrac{\begin{array}{c}\Gamma\\ \vdots \\ A\end{array}}{A}\end{array}\right.}$$

then the required derivation is given already by induction hypothesis.

This concludes the proof of Theorem 3. □

As a consequence of the previous theorem we obtain the following corollary:

Corollary 2. *If A is* HNGL *derivable, then A is also* GLC *derivable.*

Corollaries 1 and 2 together show soundness and completeness of HNGL for infinitary propositional Gödel logic, by employing soundness and completeness of GLC [Avr91]:

Corollary 3. *The system* HNGL *is sound and complete for infinitary propositional Gödel logic.*

4 Discussion

The approach laid out in the previous section is very close to the one taken by Baaz et al. [BCF01], with the following differences: (1) we keep the form of natural deduction rules close to the original, while Baaz et al. use elimination style rules similar to ∨ elimination for all connectives; (2) we enrich inference labels with more information; (3) in our system a deduction carries its generation process, while Baaz et al. need to consider deduction together with their generation.

The last point exhibits a subtle difference between standard Natural Deduction and our system: In standard Natural Deduction a derivation carries already the complete information of its generation, while in our system with prehyper deduction, this is not automatically given, but needs extra information, as we have seen in the example in the introduction.

We aim for a procedural normalization which extends usual Natural Deduction, which in particular admits →-conversions whenever corresponding redices occur. Thus we need to be able to reshuffle parts of deductions and be sure that the newly constructed deduction is still well-formed. Using the inductive, implicit definition of [BCF01], or our Definition 23 of HNGL deductions, it seems impossible to directly define such conversions. To overcome this hurdle, we take a different approach in the following, namely providing an explicit characterization that allows us to directly explore the structure of HNGL deductions and define conversions on them for normalization.

In Section 6 we identify prehyper deductions via explicit conditions on the structure of prederivations, and show that they coincide with Hyper Natural Deduction for Gödel Logic deductions as defined in Section 3. In Section 7 we use the explicit characterization to define conversions and prove normalization. Although these two sections appear to be very technical, including the related technical appendix, the actual operations used to define conversions are not that complicated and in some sense natural. The quite involved technical part is needed to show that those operations have the right properties.

5 Labeled graphs

We associate to each HNGL deduction a particular directed graph. These graphs have a special structure, namely they consist of a set of directed trees with additional bidirectional connections between and within trees. The direction within trees are from leaves to root.

We use basic concepts of graph theory without defining them, like the notion of subgraphs. We use $\subseteq$ to denote the subgraph relation on graphs.

Definition 13. *A* labeled graph *$\mathcal{G}$ is a tuple (V, E, L, f) where V and L are sets, E is a binary relation on V, i.e., $E \subseteq V \times V$, and f is a function from V to L. Elements in V are called* vertices*, in E* edges*, and in L* labels*. An edge $(r, s) \in E$ with $(s, r) \in E$ is called* symmetric*.*

We write $r \to_{\mathcal{G}} s$ if $(r,s) \in E$, and indicate with $\to^*_{\mathcal{G}}$ the reflexive and transitive closure of $\to_{\mathcal{G}}$, and with $\leftrightarrow^*_{\mathcal{G}}$ the reflexive, symmetric, and transitive closure of $\to_{\mathcal{G}}$. We say that a vertex s is reachable *from a vertex r in $\mathcal{G}$ if $r \to^*_{\mathcal{G}} s$.*

The connected components *of a graph $\mathcal{G}$ are the subgraphs of $\mathcal{G}$ induced by the equivalence classes of $\leftrightarrow^*_{\mathcal{G}}$. The connected component of a vertex v in $\mathcal{G}$, denoted $[v]_{\mathcal{G}}$, is the subgraph of $\mathcal{G}$ induced by the equivalence class $[v]_{\leftrightarrow^*_{\mathcal{G}}}$ of v w.r.t. $\leftrightarrow^*_{\mathcal{G}}$.*

If no ambiguity arises we drop the index $\mathcal{G}$ of the various relations. In the following we assume that all graphs under discussion are labeled, and just call them graphs. A trivial observation is the following proposition:

Proposition 1. *If $\mathcal{G}$ is a subgraph of $\mathcal{H}$, then the connected components of $\mathcal{G}$ are subgraphs of connected components of $\mathcal{H}$.*

We now repeat some standard graph theory notions, in the context of labeled graphs. We define paths as simple paths, that is without repetition of vertices. We have to distinguish between directed and undirected paths, and if not specified otherwise a path denotes an undirected path.

Definition 14. *An* undirected path*, also called* upath *or simply* path*, through a directed graph $\mathcal{G}$ is a (non-empty) finite sequence of edges which connects a sequence of vertices such that all vertices, except possibly the first and last, are distinct. That is, a path p is a sequence of edges $(e_1, \ldots, e_k)$ such that there is a sequence of vertices $(v_0, \ldots, v_k)$ which are pairwise distinct except for $v_0 = v_k$ being possible, such that $e_j = (v_{j-1}, v_j)$ or $e_j = (v_j, v_{j-1})$ for $j = 1, \ldots, k$.*

A directed path*, also called* dipath*, through a directed graph $\mathcal{G}$ is a undirected path in which all edges are directed in the same direction. That is, a dipath p is a sequence of edges $(e_1, \ldots, e_k)$ such that there is a sequence of pairwise distinct vertices $(v_0, \ldots, v_k)$ except for $v_0 = v_k$ being possible, such that $e_j = (v_{j-1}, v_j)$ for $j = 1, \ldots, k$.*

We indicate with $v_0 - p - v_k$ that the path p is leading from v_0 to v_k.

We have that w is reachable from v in $\mathcal{G}$, $v \to^*_{\mathcal{G}} w$, if $v = w$, or there is a dipath p from v to w: $v - p - w$.

We define the following operations on graphs:

Definition 15 (Drop). *Let $\mathcal{G} = (V, E, L, f)$ be a labeled graph, and let L' be a subset of the set of labels L. The graph obtained from $\mathcal{G}$ by dropping all vertices reachable from some vertex with label in L', as well as all edges involving one of the dropped vertices, is denoted with* $\mathrm{Drop}(\mathcal{G}, L')$.

For a subset $V' \subseteq V$ of vertices, we defined $\mathrm{Drop}(\mathcal{G}, V')$ as the graph obtained from $\mathcal{G}$ by dropping the corresponding set of labels:

$$\mathrm{Drop}(\mathcal{G}, V') = \mathrm{Drop}(\mathcal{G}, \{\ell \mid \exists v \in V' : f(v) = \ell\}).$$

Let $\mathrm{Drop}(\mathcal{G}, v) = \mathrm{Drop}(\mathcal{G}, \{v\})$ *for a vertex v.*

Observe that we use Drop with different types of arguments to simplify readability. Note that the Drop operation is quite radical, as it deletes everything which can be reached from a specific node following edges in the directed graph.

Definition 16 (Cut). *Let $\mathcal{G} = (V, E, L, f)$ be a labeled graph, and let E' be a set of edges. We denote with* $\mathrm{Cut}(\mathcal{G}, E')$ *the graph obtained from $\mathcal{G}$ by removing every edge e in $E \cap E'$.*

We continue with defining a specific class of directed graphs, called C-graphs, which are central to our investigation of hyper natural deductions.

Definition 17. *Let $\mathcal{G} = (V, E, L, f)$ be a labeled graph, and let $E^c \subseteq E$ be the set of symmetric edges, that is the set of all edges $(r, s) \in E$ with $(s, r) \in E$. If* $\mathrm{Cut}(\mathcal{G}, E^c)$ *is a disjoint union of trees, we call $\mathcal{G}$ a* C-graph *or* canopy graph. *The trees in* $\mathrm{Cut}(\mathcal{G}, E^c)$ *are called* subtrees *of $\mathcal{G}$. The edges in E^c are called* cross edges*, and edges in $E^t = E \setminus E^c$ are called* tree edges.

The set of edges of G form a binary relation which we have denoted with $\rightarrow_{\mathcal{G}}$ in Definition 13. With $\rightarrow_{\mathrm{t},\mathcal{G}}$ and $\rightarrow_{\mathrm{c},\mathcal{G}}$ we denote the restriction of $\rightarrow_{\mathcal{G}}$ to the set of tree edges E^t and cross edges E^c, respectively. For $\rightarrow_{\mathcal{G}}$, $\rightarrow_{\mathrm{t},\mathcal{G}}$ and $\rightarrow_{\mathrm{c},\mathcal{G}}$, the reflexive and transitive closure is indicated with $\rightarrow^*_{\mathcal{G}}$, $\rightarrow^*_{\mathrm{t},\mathcal{G}}$ and $\rightarrow^*_{\mathrm{c},\mathcal{G}}$, the symmetric closure is indicated with $\leftrightarrow_{\mathcal{G}}$, $\leftrightarrow_{\mathrm{t},\mathcal{G}}$ and $\leftrightarrow_{\mathrm{c},\mathcal{G}}$, and the reflexive, transitive, and symmetric closure with $\leftrightarrow^*_{\mathcal{G}}$ $\leftrightarrow^*_{\mathrm{t},\mathcal{G}}$ and $\leftrightarrow^*_{\mathrm{c},\mathcal{G}}$, respectively.

Again, we drop the index $\mathcal{G}$ if no ambiguity arises. An alternative condition for C-graphs is that every connected component of the graph with all symmetric edges removed is a tree. In the following we prove some simple facts on C-graphs.

Lemma 1. *Every subtree of a C-graph $\mathcal{G}$ is contained in exactly one connected component of $\mathcal{G}$.*

Proof. Since every two nodes in a subtree are connected, the whole subtree is contained in one connected component. □

Lemma 2. *Any subgraph of a C-graph which respects E^c, that is which always removes symmetric edges in pairs, is also a C-graph.*

Proof. Let E^c_s be the set of symmetric edges of the subgraph, and E^t_s the set of tree edges of the subgraph. Due to the requirement that the subgraph respects E^c, we obtain that $E^c_s \subseteq E^c$ and $E^t_s \subseteq E^t$. Combined with the fact that every subgraph of a tree is again a set of trees, we obtain the result. □

Lemma 3. *If $\mathcal{G} = (V, E, L, f)$ is a C-graph, E' a subset of E^t, and L' a subset of L, then* $\mathrm{Cut}(\mathcal{G}, E')$ *and* $\mathrm{Drop}(\mathcal{G}, L')$ *are again C-graphs.*

Proof. By the previous lemma and the fact that both operations respect E^c. □

While we only need the case of two or three, we define the marriage of an arbitrary number of C-graphs.

Definition 18 (Marriage of C-graphs). *Assume $\mathcal{G}_i = (V_i, E_i, L_i, f_i)$ are C-graphs, for $i = 1, \ldots, n$. Let $g_i \in \mathcal{G}_i$ be the root of some subtree in $\mathcal{G}_i$. Let $\mathcal{G}_0 = (V_0, E_0, L_0, f_0)$ be the disjoint union of the graphs $\mathcal{G}_i$ for $i = 1, \ldots, n$. We call the graph $\mathcal{G} = (V, E, L, f)$ the* marriage graph of $\mathcal{G}_1, \ldots, \mathcal{G}_n$ *if one of the following two cases holds:*

(i) $V = V_0 \cup \{r\}$ *for a new vertex* r, $E = E_0 \cup \{(g_i, r) : i = 1, \ldots, n\}$, $L = L_0 \cup \{x\}$ *for a new label* x, $f = f_0 \cup \{(r, x)\}$*. That is, all the vertices* g_i *are connected to a new root vertex* r*;*

(ii) $V = V_0 \cup \{r_i : i = 1, \ldots, n\}$ *for new vertices* r_i, $E = E_0 \cup \{(g_i, r_i) : i = 1, \ldots, n\} \cup \{(r_i, r_j) : i \neq j,\ i, j = 1, \ldots, n\}$, $L = L_0 \cup \{x_i : i = 1, \ldots, n\}$ *for new labels* x_i, $f = f_0 \cup \{(r_i, x_i) : i = 1, \ldots, n\}$*. That is each of the vertices* g_i *is connected to a new vertex* r_i *via a tree edge, and each pair of the vertices* r_i *is connected with a symmetric pair of cross edges.*

Lemma 4. *The marriage graph of C-graphs is again a C-graph.*

Proof. In the first case it is easy to see that the set of symmetric edges is $E^c = \uplus_{i=1}^{n} E_i^c$, while in the second case we have to add the pairs of edges between the new vertices r_i. In both cases, dropping all symmetric edges E^c we remain with either the union of the subtrees of the single graphs in the second case, and with the same set but with n subtrees connected to a new root r, which again is a tree. □

Lemma 5. *Assume the notions from Definition 18, and let* s *be a vertex in* $\mathcal{G}_1$ *such that* g_1 *is not reachable from* s*. Then* $\mathrm{Drop}(\mathcal{G}, s)$ *is a marriage graph of* $\mathrm{Drop}(\mathcal{G}_1, s)$ *and* $\mathcal{G}_i$ *for* $i = 2, \ldots, n$ *based on the same* $g_1, \ldots, g_n$*. Furthermore, if* $E' \subseteq \cup_{i=1}^{n} E_i^t$*, then* $\mathrm{Cut}(\mathcal{G}, E')$ *is a marriage graph of* $\mathrm{Cut}(\mathcal{G}_i, E')$ *for* $i = 1, \ldots, n$ *based on the same* $g_1, \ldots, g_n$

Proof. Obvious from the fact that the connecting point is not reachable in the original graphs from the connected component of s. □

Lemma 6. *Let* $\mathcal{G}_i$ *for* $i = 1, \ldots, n$ *be C-graphs, and* $\mathcal{H}$ *one of their marriage graphs. Assume the notions of Definition 18, and let* E' *be a subset of the tree edges of the original graphs, that is* $E' \subseteq \cup_{i=1}^{n} E_i^t$*. Let* c_j *for* $j = 1, \ldots, k$ *be pairwise distinct connected components of* $\mathrm{Cut}(\mathcal{G}_1, E')$*. Let* d_i *be the connected component in* $\mathrm{Cut}(\mathcal{H}, E')$ *containing* c_i*, respectively. Then* $c_i \neq d_i$ *for at most one* i*.*

Proof. The only additional connection is via the new edges that are added during the wedding. Only one connected component of $\mathcal{G}_1$, namely the one containing g_1, obtains a new connections. Thus, the only candidate for changes is the connected component $[g_i]_{\mathrm{Cut}(\mathcal{G}, E_1')}$ of g_1 in $\mathrm{Cut}(\mathcal{G}, E_1')$. If this did not appear in the list of c_i, non of the components change, otherwise d_i is a proper superset of d_i since at least the new vertex below g_1 is included in d_1. □

6 Explicit definition of Hyper Natural Deduction for Gödel Logic

As discussed in Section 4, the *implicit* definition of hyper natural deduction via an inductive definition in Definition 23, while being intuitive, does not lend itself directly to define conversions as needed for normalization. In the following we give an explicit characterisation of hyper natural deductions, which will overcome those difficulties, in the following way: We state a set of conditions on prehyper deduction, such that the set $\mathcal{X}$ of prehyper deductions satisfying those conditions coincides with the set of all hyper natural deductions (Corollary 4).

Using the implicit definition we see that communication rules always come in pairs, each one coming from separate HNGL deductions. The conditions we are going to define model independence of parts of HNGL deductions — in a similar way non-unary rules in GLC require that the derivations of their premises are given as independent derivations. We have to model this through our conditions on $\mathcal{X}$: For example, for each pair of dually labeled communication rules in an $\mathcal{X}$ deduction, we need to identify two parts of the deduction which can serve as independent justifications of the premises of the communication application.

As a first step towards an explicit definition, we define the labeled graph associated with a prehyper deduction.

Definition 19 (Prehyper deductions as labeled graphs). *Let $R = \{\rho_1, \dots, \rho_n\}$ be a prehyper deduction. We associate a labeled graph (V, E, L, f) with R as follows:*

- *V is given as the set of rule occurrences in R;*
- *The set of labels L is given by* Labels(R)*;*
- *The labeling function f is defined by mapping each vertex to its label.*
- *The set of edges E is defined as follows: If the conclusion of a rule occurrence r is the premise of a rule occurrence r', then add an edge (r, r') to E; Furthermore, for dually labeled vertices add symmetric edges between them to E.*

To use the lemmas obtained earlier, we observe that a set of prederivations seen as a labeled graph forms a C-graph:

Lemma 7. *The associated graph of a prehyper deduction is a C-graph.*

In the context of prederivations, we have introduced the notation $x{:}s$ to denote that label x is carrying the inference descriptor s. We extend this notation to vertices of the associated labeled graph, that is, rule occurrences. Each rule occurrence v is carrying a label x which we denote as $v{:}x$. If we also want to stress the inference descriptor carried by x, say s, we write $v{:}x{:}s$.

Using this alternative view onto prehyper deductions it is easy to describe the subprederivations rooted in a rule occurrence r: this is the subgraph induced by the set of vertices from which r can be reached via tree edges, that is $\{v \in R : v \to_{\mathrm{t}}^{*} r\}$. Furthermore, it allows us to speak about connected components of a prehyper deduction R, and use notions like $[r]_R$ for a rule occurrence r in R to indicate the connected component containing r.

Connected components are intended to characterize dependent parts of a deduction. For all non-unary logical rules, like $\wedge$-i, it is a necessary condition that their premises have been derived with independent deductions. For contraction we demand the opposite, that their premises have been derived with the same deduction. To express dependence or independence via connected components, we make use of the operations Drop and Cut on graphs as defined in Definitions 15 and 16.

When forming hyper natural deductions, many rules demand that two independently derived deductions are combined. Once this has happened, we need to remove the introduced node in order to be able to discover its premises as members of different connected components — the 'Drop' operation is doing exactly this, removing a rule in a minimal way so that the result is still a reasonable deduction-like object. Contractions on the other hand are applied within one hyper natural deduction, thus we want to demand that their premises are dependent at the time of their introduction, which implies that the connection coming from a contraction rule itself is not needed for providing dependency — the 'Cut' operation is used to reflect exactly this.

In the following we need the Cut operation not on arbitrary sets of edges, but only for edges leading to a node labeled with a contraction rule. We thus extend the definition of Cut as follows:

Definition 20. *Let R be a prehyper deduction, and c be a contraction rule occurrence in R. Let r be the left premise of c. Then we denote with* $\mathrm{Cut}(R, c)$ *the graph* $\mathrm{Cut}(R, (r, c))$*. With* $\mathrm{Cut}(R)$ *we denote the graph obtained from applying the* Cut *operation to* all *contraction rules occurring in R.*

The apparent asymmetry in choosing the left predecessor of a contraction rule occurrence is harmless: We show later in Lemma 11 that the definition of HNGL does not depend on this choice. Recall also that we have defined the Drop operation on both set of labels as well as sets of vertices, see Definition 15.

6.1 Motivation of the above concepts

To make the concepts introduced above clearer, we discuss an example before giving the definition of $\mathcal{X}$.

Example 1. Consider the following hypersequent derivation:

$$3\times \to\text{-}i\ \dfrac{contr\ \dfrac{\wedge\text{-}r\ \dfrac{com_1\ \dfrac{\dfrac{B \Rightarrow B}{C, B \Rightarrow B} \qquad A \Rightarrow A}{C, B \Rightarrow A \mid A \Rightarrow B} \qquad com_2\ \dfrac{\dfrac{C \Rightarrow C}{C, B \Rightarrow C} \qquad A \Rightarrow A}{C, B \Rightarrow A \mid A \Rightarrow C}}{C, B \Rightarrow A \mid C, B \Rightarrow A \ \mid A \Rightarrow B \wedge C}}{C, B \Rightarrow A \ \mid A \Rightarrow B \wedge C}}{\Rightarrow C \to (B \to A) \mid \ \Rightarrow A \to B \wedge C}$$

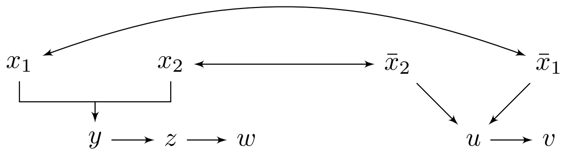
This hypersequent derivation contains two independent proofs ending in the respective communication rules, which are then merged into one derivation. We aim to translate this derivation into the following set R_{ex} consisting of two prederivations:

$$w\,\dfrac{z\,\dfrac{y{:}\,\mathrm{Ctr}\,\dfrac{x_1{:}\,\mathrm{Com}_{C,A}\,\dfrac{[C]}{A} \qquad x_2{:}\,\mathrm{Com}_{B,A}\,\dfrac{[B]}{A}}{A}}{B \to A}}{C \to (B \to A)}$$

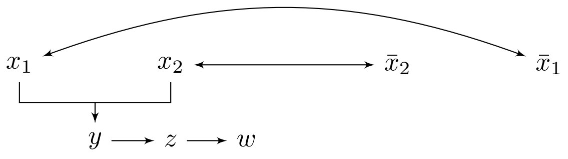
$$v\,\dfrac{u{:}\wedge\text{-}i\,\dfrac{\bar{x}_1{:}\,\mathrm{Com}_{A,C}\,\dfrac{[A]}{C} \qquad \bar{x}_2{:}\,\mathrm{Com}_{A,B}\,\dfrac{[A]}{B}}{B \wedge C}}{A \to (B \wedge C)}$$

where the communication pair $x_i/\bar{x}_i$ in the latter corresponds to com_i in the former GLC derivation.

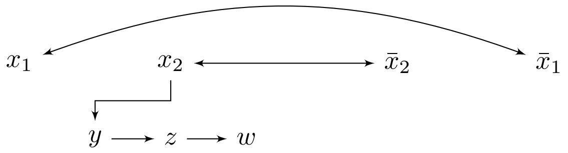
In the following we often exhibit prehyper deductions as C-graphs, but indicating vertices only by their labels. Furthermore, to make a distinction between $\wedge{:}i$ node and contraction nodes, the former ones are drawn with diagonal edges, while the latter ones are drawn via step merge. The following graph shows the associated graph of the hyper natural deduction from above (note that we bended the lower branches to save space):



Considering the connected components of this graph, we see that there is only one, the whole graph. In the following definition we give conditions on binary rules like $\wedge$-i above, and contraction rules, to express that the prederivations rooted in their premises are independent (in case of $\wedge$-i) or dependent (in case of contraction). Consider for example the conjunction introduction in the above case. We need to ensure that the components above the conjunction rule (here u) are independent, which we want to capture by having different connected components when the conjunction node is dropped:



Observe that there is still only one connected component, as the contraction rule labelled y connects $\bar{x}_1$ and $\bar{x}_2$. Applying the Cut-operation, we obtain the following graph:



There are two connected components: $\{\bar{x}_1, x_1\}$ of $\bar{x}_1$, and $\{\bar{x}_2, x_2, y, z, w\}$ of $\bar{x}_2$, reflecting that the predecessor nodes $\bar{x}_1$ and $\bar{x}_2$ of u belong to independent deductions.

6.2 Total order modulo duality

Another condition which we need for defining $\mathcal{X}$ is that for a HNGL deduction R, the CDLabels on any dipath through any prederivation in R are respecting some total order which respects duality. We make these kind of orderings precise with the following definitions.

Definition 21. *Let $\approx$ be the equivalence relation on the set $\mathcal{L}$ of labels, given by identifying dual labels; that is*

$$x \approx y \qquad \textit{iff} \qquad x = y, \ \textit{or } \bar{x} \textit{ is defined and } \bar{x} = y.$$

It follows from our assumptions on labels that $\approx$ is an equivalence relation.

Definition 22 (Total order modulo duality). *Let R be a prehyper deduction, and let L be* CDLabels(R). *Let $\approx$ be the equivalence relation on labels from the previous definition. A binary relation $\preceq$ on L is a* total order on L modulo duality *if and only if the induced relation $\precsim$ on $L/\!\approx$ given by*

$$[x]_\approx \precsim [y]_\approx \qquad \textit{iff} \qquad x \preceq y$$

is well-defined and a total order.

In the example above, CDLabels(R_{ex}) consists of $L_{\text{ex}} = \{x_1, \bar{x}_1, x_2, \bar{x}_2, y\}$, and a total order on L_{ex} modulo duality is given by

$$x_1 = \bar{x}_1 \prec x_2 = \bar{x}_2 \prec y \ \ .$$

We see by inspection that this order respects the order of occurrences of labels in L_{ex} on any branch through a prederivation in R_{ex}.

We are now in the position to define the set $\mathcal{X}$. We subsequently show that $\mathcal{X}$ gives an explicit description of Hyper Natural Deduction for Gödel Logic.

Definition 23. *Let $R = \{\rho_1, \ldots, \rho_n\}$ be a prehyper deduction, and let $\preceq$ be a total order on* CDLabels(R) *modulo duality. $(R, \preceq)$ is in $\mathcal{X}$, and called a $\mathcal{X}$* deduction*, if the following conditions are satisfied:*

1. ***(Dual labels)*** DLabels(R) *is closed under taking duals. That is, if* $l \in$ DLabels(R)*, then also* $\bar{l} \in$ DLabels(R)*. Recall the definition of dual labels:*

$$\overline{r\colon \mathrm{Spt}_{\Delta,\Gamma}} = \bar{r}\colon \mathrm{Spt}_{\Gamma,\Delta} \qquad \textit{and} \qquad \overline{r\colon \mathrm{Com}_{B,A}} = \bar{r}\colon \mathrm{Com}_{A,B}$$

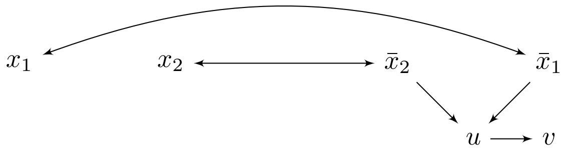
2. ***(Consistent labeling)*** *For all labels* $l \in$ Labels(R)*, all subtrees in* R *rooted in a rule labeled by* l *are different occurrences of the same prederivation.*
3. ***(Consistent splitting labeling)*** *If* $l \in$ DLabels(R) *is a splitting label, then all subprederivations of prederivations in* R *rooted in premises of rules labeled by* l *or* $\bar{l}$ *are different occurrences of the same prederivation.*
4. ***(Label ordering)*** *The label order is compatible with the tree-edge relation of* R*. That is, if* r *and* r' *are nodes with labels in* CDLabels(R)*, such that there is a path from* r *to* r' *w.r.t. tree-edges,* $r \rightarrow_{\mathrm{t}}^{*} r'$*, then* $r \preceq r'$*.*
5. ***(Independence of premises)*** *For any non-unary logical rule occurrence* r *in* R *with label* x*, that is* $r{:}x{:}\wedge$*-i,* $r{:}x{:}\vee$*-e, or* $r{:}x{:}\rightarrow$*-e, and any pair of dually labeled communication rules occurrences* r_1 *and* r_2 *in* R*, with* x *the label of* r_1 *and* $\bar{x}$ *of* r_2*, that is* $r_1{:}x$ *and* $r_2{:}\bar{x}$*, we require an independence of their premises as follows: For any two different premises* s_1 *and* s_2 *of* r*, respective premises* s_i *of* r_i*,* $i = 1, 2$*, in case of communication, we require that* s_1 *and* s_2 *are in different connected components in* Cut(Drop(R, x))*, i.e.,* $[s_1]_{\mathrm{Cut}(\mathrm{Drop}(R,x))} \neq [s_2]_{\mathrm{Cut}(\mathrm{Drop}(R,x))}$*.*
6. ***(Local dependence of contraction premises)*** *For any occurrences* r *of a contraction rule with label* c*, we stipulate a dependency of its premises as follows: Let* c' *be any contraction label such that* $c \preceq c'$*, and let* s_1 *and* s_2 *be the premises of* r*. We require that* s_1 *and* s_2 *are in the same connected component in* Cut(Drop$(R, \{c, c'\})$)*, i.e.,* $[s_1]_{\mathrm{Cut}(\mathrm{Drop}(R,\{c,c'\}))} = [s_2]_{\mathrm{Cut}(\mathrm{Drop}(R,\{c,c'\}))}$*.*
7. ***(Global dependence of prederivations)*** R *is connected.*

Note that for conditions on (in)dependence the graphs considered in the definition (like Cut(Drop(R, r))) are all C-graphs, according to Lemma 3. In the following we refer to the conditions in the previous definition as "$\mathcal{X}$-conditions", like "$\mathcal{X}$-condition 1 (dual labels)" to refer to the first, without explicitely mentioning "Definition 23".

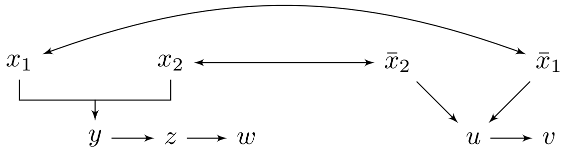
Example 2 (Cont.). We continue the example of Section 6.1 and show that the HNGL-conditions are satisfied. The conditions of dual labels and consistent communication and splitting labeling are obvious. The order has already been identified as

$$x_1 = \bar{x}_1 \prec x_2 = \bar{x}_2 \prec y \ \ .$$

Concerning independence of premises, we only have to consider the rule $\wedge$-i, and we have already discussed in Section 6.1 that the Cut operation provides the necessary separation between the connected components. Concerning the dependence of the contraction premises, we need to consider Cut(Drop(R, y)) since y is the only contraction rule:

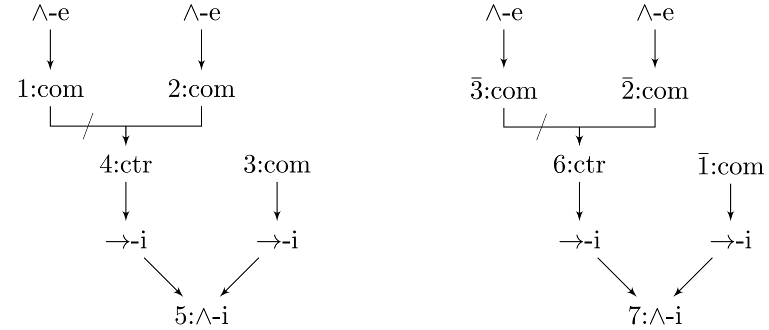


We see that there is only one connected component, and thus the requirements are fulfilled. Finally for the global dependence of prederivations, the original graph



as a whole is obviously connected. □

Example 3. Let us come back to the graph of the problematic prehyper deduction from Figure 1, reproduced here:



According to the version of HNGL as given in [BP15], the above figure would be considered a hyper natural deduction, if based on the cut as indicated with single slashes. But, as mentioned in the introduction, the derived formula of the underlying derivation is not valid in Gödel logic. The new definition of $\mathcal{X}$ is essentially unchanged to the one given in [BP15], except that $\mathcal{X}$-condition 6 (contr. premise) has changed to exclude these cases. With the new condition, assume that contraction 4 is ordered before contraction 6 ($4 \prec 6$), then the premises of contraction 4, that is 1 and 2, are not connected in $\mathrm{Cut}(\mathrm{Drop}(R, \{4, 6\}))$. Similarly in the case that contraction 6 is ordered before contraction 4, considering the premises of contraction 6 in $\mathrm{Cut}(\mathrm{Drop}(R, \{6, 4\}))$.

Observe that the previous definition induces an efficient (quadratic in number of nodes) decision procedure to determine whether a given set of prederivations forms an $\mathcal{X}$ deduction. Although testing whether two nodes are connected is known to be in logspace, and computing the Drop-operation in linear time, the test in $\mathcal{X}$-condition 6 (contr. premise) loops over all pairs of contraction nodes and thus is only quadratic. To improve efficiency one could redefine this condition to test for each contraction dependency in which all later contraction are dropped, and obtain a linear time algorithm in this way. As we are not aiming to obtain the most efficient decision procedure, we leave the $\mathcal{X}$ conditions as introduced.

We require that a total order on the labels is given as part of the definition of $\mathcal{X}$. This is not necessary as there is an efficient (quadratic) algorithm (based on topological sorting [Knu97]) which, given a prehyper deduction, determines whether such an order exists (and in this case computes it).

6.3 Properties of the set $\mathcal{X}$

In the following section we present all theorems and propositions with their proofs, and also include statements of lemmata used in their proof, but defer proofs of lemmata to a technical appendix.

A first observation is that all vertices in a tree of a premise of a rule are in the same connected component:

Lemma 8. *Let R be in $\mathcal{X}$. Let r, s and v be rule occurrences in R, such that s is a premise of rule r, that is $s \to_{\mathrm{t}} r$. If s is reachable in R by tree edges from v, then v is in the connected component of s in* $\mathrm{Cut}(\mathrm{Drop}(R, r))$*:*

$$v \to_{\mathrm{t}}^{*} s \to_{\mathrm{t}} r \qquad \Longrightarrow \qquad v \in [s]_{\mathrm{Cut}(\mathrm{Drop}(R,r))}.$$

Proof. See Appendix.

We now aim to show a more involved property that certain cycles in an $\mathcal{X}$ deductions are excluded. First we observe that for any $u \leftrightarrow^{*} v$ we can find $x_1, \ldots x_n$ and $y_1, \ldots, y_n$ such that $u = x_1$, $v = y_n$, $x_i \leftrightarrow_{\mathrm{t}}^{*} y_i$ for $i = 1, \ldots, n$, and $y_i \leftrightarrow_{\mathrm{c}} x_{i+1}$ for $i = 1, \ldots, n{-}1$, that is we have:

$$u = x_1 \leftrightarrow_{\mathrm{t}}^{*} y_1 \leftrightarrow_{\mathrm{c}} x_2 \leftrightarrow_{\mathrm{t}}^{*} y_2 \leftrightarrow_{\mathrm{c}} \ldots \leftrightarrow_{\mathrm{c}} x_n \leftrightarrow_{\mathrm{t}}^{*} y_n = v$$

We also observe that $x \leftrightarrow_{\mathrm{t}}^{*} y$ implies that there exists w such that $x \to_{\mathrm{t}}^{*} w$ and $y \to_{\mathrm{t}}^{*} w$.

Lemma 9. *Let R be a prehyper deduction, and let $\mathcal{G} = $* $\mathrm{Cut}(R)$*. Assume that there are vertices $x_1 \ldots, x_\ell, y_1, \ldots, y_\ell$ in $\mathcal{G}$ carrying dual labels, such that $x_i \leftrightarrow_{\mathrm{t}}^{*} y_i$ and $y_i \leftrightarrow_{\mathrm{c}} x_{i+1}$, for $i = 1, \ldots, \ell$, where we let $x_{\ell+1}$ denote x_1. Assume that the set of labels of the x_i, y_i contains at least three elements (that is there are at least two different labels modulo duality). Furthermore, if x_i is reachable from y_i via tree edges, and $x_i \neq y_i$, then x_i is not a splitting label; similar for the roles of y_i and x_i swapped.*

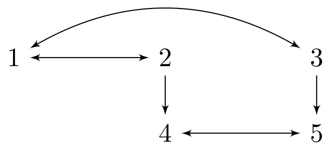
Then R is not an $\mathcal{X}$ deduction.

Proof. See Appendix.

In the above Lemma the restriction on splitting labels is essential to prove the assertion; e.g., consider the following hyper natural deduction

$$x\colon \mathrm{Com}_{B,A}\ \frac{B}{A} \qquad\qquad \begin{array}{r} \bar{x}\colon \mathrm{Com}_{A,B} \\ y\colon \mathrm{Spt}_{B,} \end{array} \begin{array}{c} [A] \\ \hline B \\ \hline B \end{array} \qquad\qquad \begin{array}{r} \bar{x}\colon \mathrm{Com}_{A,B} \\ \bar{y}\colon \mathrm{Spt}_{,B} \end{array} \begin{array}{c} [A] \\ \hline B \\ \hline B \end{array}$$

and its graph



where 1 is carrying label x, 2 and 3 are carrying label $\bar{x}$, 4 is carrying y, and 5 is carrying $\bar{y}$. With the notion from the Lemma, we can choose $x_1 = 1 = y_1$, $x_2 = 2$, $y_2 = 4$, $x_3 = 5$, $y_3 = 3$ satisfying the assumptions of the proposition, except for the last as e.g., 4 is carrying a splitting label and is reachable from 2.

Definition 24 (Regular paths). *Let R be an $\mathcal{X}$ deduction. An undirected path in* $\mathrm{Cut}(R)$ *is* irregular *if its sequence of vertices $u_1, \dots, u_k$ contains one of the following patterns: for some i,*

1. *u_i is connected to u_{i+1} via a tree edge, and $u_{i+1}, u_{i+2}, u_{i+3}$ are labeled $c, \bar{c}, c$ for some communication label c; that is*

$$u_i \rightarrow_{\mathrm{t}} u_{i+1}{:}c \leftrightarrow_{\mathrm{c}} u_{i+2}{:}\bar{c} \leftrightarrow_{\mathrm{c}} u_{i+3}{:}c$$

2. *u_{i+3} is connected to u_{i+2} via a tree edge, and u_i, u_{i+1}, u_{i+2} are labeled $c, \bar{c}, c$ for some communication label c; that is*

$$u_{i+3} \rightarrow_{\mathrm{t}} u_{i+2}{:}c \leftrightarrow_{\mathrm{c}} u_{i+1}{:}\bar{c} \leftrightarrow_{\mathrm{c}} u_i{:}c$$

3. *u_i is connected to u_{i+1} via a tree edge, and u_{i+1} and u_{i+2} carry dual splitting labels; that is, for some splitting label s,*

$$u_i \rightarrow_{\mathrm{t}} u_{i+1}{:}s \leftrightarrow_{\mathrm{c}} u_{i+2}{:}\bar{s}$$

4. *u_{i+2} is connected to u_{i+1} via a tree edge, and u_{i+1} and u_i carry dual splitting labels; that is, for some splitting label s,*

$$u_{i+2} \rightarrow_{\mathrm{t}} u_{i+1}{:}s \leftrightarrow_{\mathrm{c}} u_i{:}\bar{s}$$

5. *$u_i, u_{i+1}, u_{i+2}, u_{i+3}$ are labeled $r, \bar{r}, r, \bar{r}$ for some dual label r; that is*

$$u_i{:}r \leftrightarrow_{\mathrm{c}} u_{i+1}{:}\bar{r} \leftrightarrow_{\mathrm{c}} u_{i+2}{:}r \leftrightarrow_{\mathrm{c}} u_{i+3}{:}\bar{r}$$

   for some $r \in$ DLabels*.*

An occurrence of such a pattern on a path is called an irregular step.

An undirected path in Cut(R) *is* regular *if it is not irregular.*

Irregular steps on a path usually indicate a detour which can be avoided. Thus a reasonable conjecture is that shortest paths should be regular. We see that this is true, except for the beginning or end of a path. For example, if the path is between two copies of the same prederivation due to split they may use an unavoidable irregular step $u{:}r - v{:}\bar{r} - w{:}r$ where $u{:}r$ and $w{:}r$ refer to two different occurrences of a rule with label r.

Shortest paths are regular if we consider paths p where neither the first step is a $\rightarrow_{\mathrm{t}}$-step, nor the last a $\leftarrow_{\mathrm{t}}$-step. That is, let $u_1, u_2, \ldots, u_k$ be the sequence of vertices of p, then neither $u_1 \rightarrow_{\mathrm{t}} u_2$ nor $u_k \rightarrow_{\mathrm{t}} u_{k-1}$.

Lemma 10. *Let R be in $\mathcal{X}$. Let p be an irregular path in* Cut(R)*, such that neither the first step is a $\rightarrow_{\mathrm{t}}$-step, nor the last a $\leftarrow_{\mathrm{t}}$-step.*

Then we can find a shorter regular path p' in Cut(R) *with the same start and end vertex as p.*

Proof. See Appendix.

Definition 25 (Cycles). *A* cycle *is a path p of length ≥ 2 which connects a vertex v with itself, $v - p - v$. A cycle is* trivial *if its sequence of vertices is of the form u, v, u. A* cycle w.r.t. regular paths *is a non-trivial cycle given by a regular path.*

The next Theorem states an important property of $\mathcal{X}$, namely that cycles wrt. regular paths do not occur in $\mathcal{X}$ deductions.

Theorem 4. *Let R be an $\mathcal{X}$ deduction. Then* Cut(R) *is acyclic wrt. regular paths.*

Proof. Let R be in $\mathcal{X}$, and consider the graph $\mathcal{G} = \mathrm{Cut}(R)$. Assume that $\mathcal{G}$ contains a cycle wrt. regular paths. Let p be a regular path with sequence of vertices $u_1, u_2, \ldots, u_k, u_1$ of minimal length, that is, k is minimal with this property. Observe $k \geq 3$, because $k = 1$ is impossible for paths through $\mathcal{G}$, and for $k = 2$ we would have a trivial cycle. p must contain a tree edge, as $u, \bar{u}, u, \bar{u}$ is irregular. p must also contain a cross edge, as a path consisting only of tree edges cannot form a non-trivial cycle. Without loss of generality $u_k \leftrightarrow_{\mathrm{c}} u_1 \rightarrow_{\mathrm{t}} u_2$.

Analyzing p, we can identify indices $1 < i_1 < \cdots < i_\ell = k$ such that the u_{i_j} are carrying dual labels, and $u_{1+i_{j-1}} \leftrightarrow^*_{\mathrm{t}} u_{i_j}$ and $u_{i_j} \leftrightarrow_{\mathrm{c}} u_{1+i_j}$, for $j = 1, \ldots, \ell$ (with $i_0 = 0$ and $u_{\ell+1} = u_1$). That is, we have

$$u_1 \leftrightarrow^*_{\mathrm{t}} u_{i_1} \leftrightarrow_{\mathrm{c}} u_{1+i_1} \leftrightarrow^*_{\mathrm{t}} u_{i_2} \leftrightarrow_{\mathrm{c}} \ldots \leftrightarrow^*_{\mathrm{t}} u_{i_{\ell-1}} \leftrightarrow_{\mathrm{c}} u_{1+i_{\ell-1}} \leftrightarrow^*_{\mathrm{t}} u_k \leftrightarrow_{\mathrm{c}} u_1$$

Thus, if we set $x_j = u_{1+i_{j-1}}$ and $y_j = u_{i_j}$ for $j = 1, \ldots, \ell$, the first conditions of Lemma 9 are satisfied. As $x_1 = u_1 \rightarrow_{\mathrm{t}} u_2 \leftrightarrow^*_{\mathrm{t}} y_1$, the labels of x_1 and y_1 are different even modulo duality of labels.

Therefore, by Lemma 9, for some i, $x_i \neq y_i$, and either x_i is reachable from y_i via tree edges and x_i is a splitting rule, or y_i is reachable from x_i via tree edges and y_i is a splitting rule. Without loss of generality assume $x_i \to_{\mathrm{t}}^* y_i$ and y_i is a split rule. As $x_i \neq y_i$ there is some v such that $x_i \to_{\mathrm{t}}^* v \to_{\mathrm{t}} y_i$. Thus we have three consecutive vertices v, y_i, x_{i+1} on path p such that $v \to_{\mathrm{t}} y_i \leftrightarrow_{\mathrm{c}} x_{i+1}$, such that y_i and x_{i+1} are carrying dual splitting labels, contradicting condition 3 of Definition 24 for p. □

We now address the point made after Definition 20, that the apparent asymmetry in the definition of $\mathrm{Cut}(R, c)$ is harmless, in the sense that all $\mathcal{X}$ conditions remain valid when changing to cut arbitrary left or right premises of contractions, as long as contraction rules with the same label are cut in the same way.

Lemma 11. *The $\mathcal{X}$ conditions are preserved if the* Cut *operation cuts randomly either the left or right upper edge of a contraction rule occurrence, as long as contraction rule occurrences with the same label are cut on the same side.*

Proof. See Appendix.

We have defined the Cut operation to always delete the left edge leading to a contraction. For some conversions during normalization it is necessary to switch to alternative cuts. The following two assertions deal with this.

Lemma 12. *Let R be in $\mathcal{X}$, and r a rule occurrence in R. Let s be one of the premises of r. Then there is an alternative cut operation* Cut^* *such that the following property holds: Let C be the connected component of s in* $\mathrm{Cut}^*(\mathrm{Drop}(R, r))$*, that is $C = [s]_{\mathrm{Cut}^*(\mathrm{Drop}(R,r))}$.*
(*) For any contraction c in R with premises a and b we have $c \in C$ iff $a, b \in C$.

Proof. See Appendix.

Proposition 2. *Let R be in $\mathcal{X}$, r a rule in R, and s a premise of r. Let $C = [s]_{\mathrm{Cut}(\mathrm{Drop}(R,r))}$ be the connected component of s in* $\mathrm{Cut}(\mathrm{Drop}(R, r))$*. For any contraction occurrence c in R with premises a and b assume that $c \in C$ iff $a, b \in C$. Let R' be $R \setminus C$ in which all contraction occurrences which had one of their premises removed by C are re-label with the unary rule* Rep*. Let C' be the prehyper deduction induced by C on R.*

Then R' and C' are in $\mathcal{X}$.

Proof. The only conditions not being immediate consequences of R being in $\mathcal{X}$ are conditions 6 (contr. premise) and 7 (one class). First consider the condition on contractions, and let $c \preceq c'$ be two contractions in R' and s_1, s_2 the two premises of c. For sake of contradiction, assume that s_1 and s_2 are not connected in $\mathrm{Cut}(\mathrm{Drop}(R', \{c, c'\}))$. As they were connected in $\mathrm{Cut}(\mathrm{Drop}(R, \{c, c'\}))$ by $\mathcal{X}$-condition 6 (contr. premise) for R, there is a path p_1 connecting s_1 and s_2 in $\mathrm{Cut}(\mathrm{Drop}(R, \{c, c'\}))$, which is not present in $\mathrm{Cut}(\mathrm{Drop}(R', \{c, c'\}))$. But then p_1 must reach C, hence s_1 or s_2 are in C, contradicting that s_1 and s_2 both occur in R'.

For $\mathcal{X}$-condition 7 (one class) observe that in $\mathrm{Cut}(R)$, C is connected to rest exactly via the edge (s, r). If R' would not be connected, then there must be a path p in R connecting two different components C_1 and C_2 of R'. This path has to go through C and at least one contraction which had their premise removed by C. According to condition 6 (contr. premise) for R the premises of such contraction c are connected in $\mathrm{Cut}(\mathrm{Drop}(R, c))$, and as (s, r) is the only connection of C in $\mathrm{Cut}(R)$, c must be connected to r in R'. It follows that both C_1 and C_2 must be connected to r in R'.

Similar arguments can be given for C'. □

The next Lemma is needed to prove that any $\mathcal{X}$-deduction is a HNGL-deduction.

Lemma 13. *Consider a prehyper deduction R such that all conditions but the last $\mathcal{X}$-condition 7 (one class) are satisfied. Then the connected components of R form a partition $R = R_1 \cup \cdots \cup R_k$ such that each R_i is in $\mathcal{X}$, for $i \leq k$.*

Proof. The existence of the partition is a direct consequence of Lemma 1. Furthermore, since all the conditions but $\mathcal{X}$-condition 7 (one class) are satisfied for R, they are also satisfied for each of the R_i. In addition R_i is connected, thus each R_i is in $\mathcal{X}$. □

The next theorem establishes one direction of the equivalence between implicit and explicit definitions of HNGL.

Theorem 5. *Every* HNGL *deduction is a $\mathcal{X}$ deduction.*

Proof. Every initial NJ deductions is in $\mathcal{X}$. Thus, as HNGL is inductively defined, it suffices to show that $\mathcal{X}$ is closed under applying HNGL hyper rules, in order to show that HNGL $\subseteq \mathcal{X}$.

Let R_i be a hyper natural deduction, such that the label sets $\mathrm{Labels}(R_i)$, for $i \leq k$, are pairwise disjoint. Let $\rho_i \in R_i$ and define R_i^- as $R_i \setminus \{\rho_i\}$.

Unary hyper rules

For unary rules it is obvious that $\mathcal{X}$-conditions 1–7 are satisfied and the new figure forms a $\mathcal{X}$ deduction.

$\wedge$-i, $\vee$-e, $\rightarrow$-e, communication

For $\wedge$-i, $\vee$-e, $\rightarrow$-e, and communication, the proof works similar, we only treat $\wedge$-i: Let us assume that

$$\rho_1 = \begin{array}{c} \mathit{\Pi} \\ \vdots \\ A \end{array} \qquad\qquad \rho_2 = \begin{array}{c} \mathit{\Gamma} \\ \vdots \\ B \end{array}$$

We show that $R_\wedge = \{\rho_\wedge\} \cup R_1^- \cup R_2^-$ is in $\mathcal{X}$, where $\rho_\wedge$ is

$$\rho_\wedge := \qquad r{:}\wedge\text{-}i \; \dfrac{\begin{array}{c} \mathit{\Pi} \\ \rho_1 \vdots \\ A \end{array} \quad \begin{array}{c} \mathit{\Gamma} \\ \rho_2 \vdots \\ B \end{array}}{A \wedge B}$$

We have to show that $R_\wedge$ satisfies the $\mathcal{X}$-conditions 1–7. Combining the assumption that R_1 and R_2 are in $\mathcal{X}$, and that their sets of labels are disjoint, we see that $\mathcal{X}$-condition 1 (dual labels), $\mathcal{X}$-condition 2 (consistent labels) and 3 (consistent split) are immediately satisfied.

Consider now $\mathcal{X}$-condition 4 (label order): Since the label sets are disjoint, we can combine the two orders of R_1 and R_2 in any way (even mixing), as long as the combination is order preserving. In particular, we can choose $\text{Labels}(R_1) \prec \text{Labels}(R_2)$. For the case of the communication rule we stipulate that the communication label is placed after any other label in the new order.

Before turning our attention to the critical condition on binary rules, let us first treat the remaining two simpler conditions: $\mathcal{X}$-condition 6 (contr. premise) and $\mathcal{X}$-condition 7 (one class) are trivially satisfied, as we are creating new connections and the connected components can only become bigger.

Finally, let us turn to the critical $\mathcal{X}$-condition 5 (indep. premises), and consider another binary, ternary, or communication rule occurring in $R_\wedge$. We only consider the case of $\vee$-e, all other cases can be dealt with in a similar way. Thus, we assume that we are in the following situation:

$$
\begin{array}{c}
\begin{array}{cc} \mathit{\Pi} & \mathit{\Pi} \\ \rho_1 \vdots & \rho_2 \vdots \end{array}\\
r{:}\wedge\text{-}i \ \dfrac{A \qquad B}{A \wedge B}\\
\rho_\wedge
\end{array}
\qquad \cdots \qquad
\begin{array}{c}
\begin{array}{ccc} \mathit{\Delta} & \mathit{\Delta}, {}^k[C] & \mathit{\Pi}, {}^k[D] \\ \sigma_0 \vdots & \sigma_1 \vdots & \sigma_2 \vdots \end{array}\\
s{:}^k\vee\text{-}e \ \dfrac{C \vee D \qquad F \qquad F}{F}\\
\vdots\\
\rho'
\end{array}
$$

We assume without loss of generality that s:$\vee$-e is occurring in R_1.

First assume that the vertex r:$\wedge$-i is reachable from the vertex s:${}^k\vee$-e in $R_\wedge$. In this case, the operation $\text{Drop}(R_\wedge, s)$ also drops the connection (or *marriage*) node between the two original graphs, and thus the connected components of the premises of s computed in $\text{Cut}(\text{Drop}(R_\wedge, s))$ are the same as computed in $\text{Cut}(\text{Drop}(R_1, s))$. As a consequence, $\mathcal{X}$-condition 5 (indep. premises) continues to hold in this case, as R_1 is an $\mathcal{X}$ deduction.

It remains the case where r is not reachable from s. By assumption we have $\rho' \in R_1$. Define $G_1 = \text{Cut}(\text{Drop}(R_1, s))$, $G_2 = \text{Cut}(R_2)$, and $G = \text{Cut}(\text{Drop}(R, s))$. By Lemma 5 we see that G and the G_i satisfy the conditions of Lemma 6. Let c_i be the connected component of $\text{Conc}(\sigma_i)$ for $i = 0, 1, 2$ in G_1, and d_i be those in G. By the assumption that R_1 is in $\mathcal{X}$, $\mathcal{X}$-condition 5 (indep. premises) implies that the c_i are pairwise different. From Lemma 1 and the fact that the G_i are sub graphs of G, we obtain that $c_i \subseteq d_i$. Finally from Lemma 6 we obtain that for at most one $i = 1, 2, 3$, $c_i \neq d_i$.

Now assume for the sake of contradiction that the condition for rule s in $R_\wedge$ does not hold, thus, at least two connected components d_a and d_b ($a \neq b$, $a, b \in \{1, 2, 3\}$) coincide, $d_a = d_b$. Combined with the above that at most one $c_i \neq d_i$ we obtain that either $c_a = d_a$ or $c_b = d_b$. Assume without loss of generality that $c_a = d_a$. Together with $d_a = d_b$ we obtain that $c_a = d_b$, and $c_b \subseteq d_b = c_a$, contradicting that c_a and c_b are different connected components.

contraction
In the case of a contraction rule, we are starting from slightly different assumptions, namely that two prederivations in R_1 have been combined with a contraction rule to form a new set R'. Again, we have to show that R' is in $\mathcal{X}$. As in the previous case, it is easy to see that all conditions but $\mathcal{X}$-condition 5 (indep. premises) are trivially satisfied. For condition $\mathcal{X}$-condition 5 for R', we consider another binary rule r in R' for which we have to show independence of its premises. Since we compute connected components in $\mathrm{Cut}(\mathrm{Drop}(R', r))$, and the Cut in particular removes one edge of the new contraction node, we observe that the connected components in $\mathrm{Cut}(\mathrm{Drop}(R', r))$ and in $\mathrm{Cut}(\mathrm{Drop}(R_1, r))$ are the same (modulo one new vertex in $\mathrm{Cut}(\mathrm{Drop}(R', r))$ coming from the new contraction node). In particular the property that the premises of r are not connected in $\mathrm{Cut}(\mathrm{Drop}(R_1, r))$, as R_1 is a hyper natural deduction, is preserved in $\mathrm{Cut}(\mathrm{Drop}(R', r))$.

split rule
Next, we consider the splitting rule: Assume that we want to extend $\rho_1 \in R_1$ with a splitting rule, where ρ_1 is a prederivation of A from assumptions $\mathit{\Gamma}, \mathit{\Delta}$. Let ρ_1' be a copy of ρ_1 (without any renaming of labels) and define R_S as $\{\rho_a, \rho_b\} \cup R_1^-$, where

$$\rho_a = \begin{array}{r} [\mathit{\Gamma}]^k, \mathit{\Delta} \\ \rho_1 \vdots \\ x{:}^k\, \mathrm{Spt}_{\mathit{\Gamma},\mathit{\Delta}}\, \dfrac{A}{A} \end{array} \qquad \text{and} \qquad \rho_b = \begin{array}{r} \mathit{\Gamma}, [\mathit{\Delta}]^k \\ \rho_1' \vdots \\ \bar{x}{:}^k\, \mathrm{Spt}_{\mathit{\Delta},\mathit{\Gamma}}\, \dfrac{A}{A} \end{array}$$

Checking the $\mathcal{X}$-conditions, we see that due to R_1 being in $\mathcal{X}$, $\mathcal{X}$-conditions 1 (dual labels)–2 (consistent labels) are trivially satisfied. $\mathcal{X}$-condition 3 (consistent split) is satisfied by construction of R_S. Considering $\mathcal{X}$-condition 4 (label order), the new order is defined by adding the new splitting rule at the end (largest element) of the order for R_1.

Consider now $\mathcal{X}$-condition 5 (indep. premises): As in the case of logical rules above, we consider another binary or ternary logical rule, or communication rule r occurring in R_S, and want to check independence of its premises. Assume for sake of contradiction that two premises of r, a and b, are connected in $\mathrm{Cut}(\mathrm{Drop}(R_S, r))$ — they are not connected in $\mathrm{Cut}(\mathrm{Drop}(R_1, r))$ as R_1 is in $\mathcal{X}$. Let p be a path from a to b in $\mathrm{Cut}(\mathrm{Drop}(R_S, r))$. Denote with p' the result of changing each vertex on path p, which also occurs in ρ_1', into the corresponding vertex in ρ_1. It is not difficult to see that p' is a path in $\mathrm{Cut}(\mathrm{Drop}(R_1, r))$ which connects a and b, contradicting that R_1 is in $\mathcal{X}$.

$\mathcal{X}$-condition 7 (one class), requiring that the graph R_S is connected, is satisfied using that ρ_1 and ρ_1' are connected via the new splitting nodes.

This concludes the proof of the lemma. □

The remaining part of this section deals with the reverse direction of Theorem 5, that all $\mathcal{X}$ deductions are HNGL deductions. To this end we need to identify a potential hyper rule which can be used to generate a given $\mathcal{X}$ deduction.

Definition 26 (Reducible deduction). *Let R be an $\mathcal{X}$ deduction, and r the root of a prederivation in R. We define that r* is reducible in *R as follows: Unary logical rules and contraction rules are always reducible. Binary and ternary rules r are reducible if any two premises of r are* not *connected in* $\mathrm{Drop}(R, r)$*. A communication rule r is reducible if the premises of r and $\bar{r}$ are again not connected in* $\mathrm{Drop}(R, r)$*, and in addition all occurrences of r and $\bar{r}$ are as roots of prederivations in R. Split rules r are reducible if all occurrences of r and $\bar{r}$ are as roots of prederivations in R.*

Lemma 14 (Retraction). *If a rule r of $R \in \mathcal{X}$ is reducible, then the prederivations associated to the connected components of the premises of r again form $\mathcal{X}$ deductions.*

Proof. Let R' be R with the rule occurrence r removed. R' clearly satisfies the assumptions of Lemma 13. The assertion follows from that Lemma. □

Definition 27. *Let R be in $\mathcal{X}$. A path passes* horizontally *through a contraction node c with premises s_1 and s_2, if either (s_1, c) and (c, s_2), or (s_2, c) and (c, s_1) are part of the path.*

The central proposition of our proof to show that every $\mathcal{X}$ deduction is also a HNGL deduction is the property that we can always find a reducible rule. An $\mathcal{X}$ deduction is called *trivial* if it consists of exactly one prederivation formed by exactly one formula.

Proposition 3. *Any $\mathcal{X}$ deduction is either trivial or has a reducible rule.*

We first give a general overview of the layout for the proof of this proposition: Assume some $\mathcal{X}$ deduction does not contain a reducible rule. First we show that if all roots of prederivations are communication or split rules, then we can obtain a contradiction to the order on dual labels for this $\mathcal{X}$ deduction. Thus, there must be some non-reducible logical rule at a root position. Starting from such a rule we construct a sequence of dual labels which forms a cycle, contradicting Lemma 9.

Proof. Assume that $(R, \preceq)$ is a non-trivial $\mathcal{X}$ deduction without a reducible rule. Contraction and unary rules occurring as roots of prederivations are always reducible. Thus the rules occurring as roots of prederivations of R are either non-unary logical rules, communication, or split.

First asume that no logical rule occurs as a root of a prederivation in R. Let $r_1, \ldots, r_n$ be the roots of prederivations in R, and $m_1, \ldots, m_n$ their respective labels. Without loss of generality assume that m_1 is maximal amongst the labels ($\bar{m}_1$ could be another maximal element). Since r_1 is not reducible, at least one rule labeled with m_1 or $\bar{m}_1$ must occur within another prederivation, without loss of generality above r_2. But then $m_1 \prec m_2$ contradicting maximality of m_1.

Thus, we can assume that there is at least one non-reducible logical rule occurring at a root. We first proof the following fact:

Fact 1: Let r be a non reducible logical rule occurring as a root in R. Then there is a path in $\mathrm{Drop}(R, r)$ from one premise of r to another premise of r, which passes through exactly one contraction node horizontally. With the exception of the horizontal pass through the contraction, this path is a path in $\mathrm{Cut}(\mathrm{Drop}(R, r))$.

To prove this fact, assume first that for all contractions c in R with premises a and b, a and b are connected in $\mathrm{Cut}(\mathrm{Drop}(R, \{c, r\}))$. Let p be some path in $\mathrm{Drop}(R, r)$ between two distinct premises of r, say s_1 and s_2 — such a path must exist as we assumed that r is non-reducible. Consider p as a path in $\mathrm{Cut}(\mathrm{Drop}(R, r))$: for any edge in p removed by Cut, it must have been of the form (a, c) or (c, a) for c a contraction and a one of its premises. Let b be the second premise of c. By assumption, there is a path in $\mathrm{Cut}(\mathrm{Drop}(R, \{c, r\}))$ which connects a with b — use this path to repair the cut connection between a and c in p. This can be done for any cut connection of p in $\mathrm{Cut}(\mathrm{Drop}(R, r))$, which yields that s_1 and s_2 are connected in $\mathrm{Cut}(\mathrm{Drop}(R, r))$. But this contradicts $\mathcal{X}$-condition 5 (indep. premises) for r, since R is in $\mathcal{X}$.

So we must have that there is a contraction c occurring in R with premises a and b, such that a and b are not connected in $\mathrm{Cut}(\mathrm{Drop}(R, \{c, r\}))$. By $\mathcal{X}$-condition 6 (contr. premise) a and b are connected in $\mathrm{Cut}(\mathrm{Drop}(R, c))$, hence there must be a path in $\mathrm{Cut}(\mathrm{Drop}(R, c))$ of the form $b - p_2 - s_2 - r - s_1 - p_1 - a$ where s_1 and s_2 are premises of r. Consider the path $s_1 - p_1 - a - c - b - p_2 - s_2$: this is obviously a path in $\mathrm{Drop}(R, r)$ which passes though exactly one contraction node (namely c) horizontally. (End of proof of Fact 1.)

Fact 2: For any non-reducible logical rule r occurring as a root in R, there is a contraction s occurring in R, and another non-reducible logical rule r' occurring as a root in R, such that

(a) There is a path in $\mathrm{Drop}(R, r)$ from two different premises of r which passes horizontally through exactly one contraction node, namely s. With the exception of the horizontal pass through s, this path is a path in $\mathrm{Cut}(\mathrm{Drop}(R, r))$

(b) $s \rightarrow^*_{\mathrm{tc},R} r'$

To prove this fact, consider a non-reducible logical root r in R. By Fact 1 we can find such a contraction s satisfying (a). If we would have $s \rightarrow^*_{\mathrm{t},R} r$, that is contraction s occurs in the tree above r, we obtain from $\mathcal{X}$-condition 6 (contr. premise) that there is a path in $\mathrm{Cut}(\mathrm{Drop}(R, s))$ connecting the premises of s; this path then also exists in $\mathrm{Cut}(\mathrm{Drop}(R, r))$ and can be used to connect the premises of r, contradicting $\mathcal{X}$-condition 5 (indep. premises). To find another root r' satisfying (c), consider the following process: Let u_1 be the root of the tree in which s is occurring. If u_1 is a logical rule we can choose $r' = u_1$. Otherwise, u_1 must be communication or split. As u_1 is non-reducible, there must be a rule occurrence u'_1 which is not a root in R, and which is labeled with the same or dual label of u_1. Continue with the same process, with u_1 in place of s, to construct roots u_2, u_3 etc. As R is finite, this process must stop in some logical root r', satisfying (b). r' must be different from r as otherwise the last u_i would have been a communication rule with dependent premises, contradicting $\mathcal{X}$-condition 5 (indep. premises). (End of proof of Fact 2.)

By repeatedly applying Fact 2, construct a sequence r_1, s_1, r_2, s_2... of non-reducible roots r_i and contractions s_i, such that there exists a path in $\mathrm{Drop}(R, r_i)$ passing horizontally exactly through s_i, and $s_i \to^*_{\mathrm{tc},R} r_{i+1}$. As R is finite, some r_i's must be the same. Let r_1, s_1, r_2, $\dots$, s_k, $r_{k+1} = r_1$ be a sequence of this form of minimal length. By construction (Fact 2) $r_1 \neq r_2$, hence $k \geq 2$.

If $s_1 \succeq s_2$, then $\mathcal{X}$-condition 6 (contr. premise) yields that the premises of s_2 are connected in $\mathrm{Cut}(\mathrm{Drop}(R, \{s_1, s_2\}))$. As $s_1 \to^*_{\mathrm{tc},R} r_2$ we obtain that they are also connected in $\mathrm{Cut}(\mathrm{Drop}(R, r_2))$. Together with (a) we obtain that the premises of r_2 would be connected in $\mathrm{Cut}(\mathrm{Drop}(R, r_2))$, contradicting $\mathcal{X}$-condition 5 (indep. premises) for r_2. Hence $s_1 \prec s_2$. The same is true for all s_i, thus $s_i \prec s_{i+1}$ for $i = 1, \dots, k$ (where $s_{k+1} = s_1$).

Thus we have the following chain

$$s_1 \prec s_2 \prec s_3 \cdots \prec s_\ell \prec s_1$$

contradicting the totality of $\prec$. □

Theorem 6. *Every $\mathcal{X}$ deduction is a* HNGL *deduction.*

Proof. The proof is by induction on the total number of nodes of $R \in \mathcal{X}$. If R is trivial, i.e. just consists of an initial NJ derivation, then clearly $R \in$ HNGL.

Otherwise, we use Proposition 3 to find a reducible rule r in R. Let R_i for $i = 1, \dots, k$ ($k \leq 3$), be the connected components in $\mathrm{Drop}(R, r)$ of the premises of r. We observe that applying the hyper rule corresponding to r to $R_1, \dots, R_k$ again yields R. Thus it suffices to show that the R_i's are HNGL deductions in order to finish the proof of Theorem 6.

Using the Retraction Lemma 14 we obtain that $R_i \in \mathcal{X}$. As R_i has fewer nodes than R we can apply the induction hypothesis to obtain that $R_i \in$ HNGL. □

Theorems 5 and 6 together show that $\mathcal{X}$ and HNGL are the same.

Corollary 4. *$\mathcal{X}$ deductions and* HNGL *deductions are the same.*

In the following we identify $\mathcal{X}$ and HNGL, and treat $\mathcal{X}$ as an alternative description for hyper natural deductions.

7 Normalization

The system NJ of Natural Deduction possesses the important property that every derivation in NJ can be transformed into normal form. The aim of this section is to describe a similar procedure for HNGL deductions. We follow [TS00, Chapter 6] for providing a normalization strategy for HNGL via a specific set of conversions. Our approach extends [TS00] in the sense that when restricted to NJ, it coincides with the latter.

Staying close to the usual normalization procedure for Natural Deduction implies that →-conversions can always be performed wherever corresponding redices occur. This is important for our long-term aim to establish a Curry-Howard correspondence of Hyper Natural Deduction for Gödel Logic to some kind of parallel lambda calculus. A $\to$-conversion in HNGL will correspond to a β-step for any kind of parallel lambda calculus. And as β-steps in lambda calculus are the most important computational steps, there should be no restriction on their execution. Without that freedom we do not think to be able to achieve a *full* answer to the question of characterising the parallel computation content of Hypersequent Calculus or Hyper Natural Deduction for Gödel Logic.

Thus, we want to be able to apply transformations like $\to$-conversions as liberally as possible. To be able to effectively describe the parts that need to be reshuffled during such a conversion, we have to make use of the explicit description of HNGL-deductions as $\mathcal{X}$-deductions.

The notion of normal deduction depends on notions like segment and cuts, which in turn need the notions minor and major premise of rules. We thus start by defining the latter for communication, split and contraction rules. As communication and split are linking different prederivations, we cannot define the minor premise of such a rule locally, i.e. dependent on just this rule, anymore, but have to define them in the context of a HNGL deduction.

Definition 28 (Minor premises for communication, split and contraction rules). *Let R be a prehyper deduction.*

A minor premise w.r.t. *R of a communication rule of the form*

$$x\colon \mathrm{Com}_{A,B}\ \frac{A}{B}$$

is any occurrence of B in R as the premise of a rule labeled by $\overline{x\colon \mathrm{Com}_{A,B}}$.

The premise of a splitting rule is the minor premise *of that splitting occurrence.*

Both premises of a contraction rule are minor premises *of that contraction occurrence.*

During normalization, elimination rules are permuted over minor premises of other disjunction elimination like rules (del-rules, see below) until they reach an introduction rule. For communication, split and contraction the reason for calling premises "minor" is the same as above, during normalization elimination rules are just "permuted". But due to the non-local nature of minor premises for communication and split the situation now is much more involved.

We adapt the notion of segment, cut, cut-rank, and critical cut from [TS00, Def. 6.1.2] to take the additional minor premises into account, which can be conveniently done by just defining the "del-rules of HNGL" (see [TS00, Def. 6.1.1]). For the benefit of the reader we restate the definition of the former notions from [TS00, Def. 6.1.2] as well. Let *I-rules* denote the introduction rules for logical connectives, and *E-rules* denote the elimination rules for logical connectives. In E-rule applications, the premise containing the occurrence of the logical operator being eliminated is called *major*, the others *minor*. With $|A|$ we denote the length of formula A, given by the number of occurrences of logical connectives in A.

Definition 29. *The* del-rules *of* HNGL *are* $\vee$*-e, contr, com and split.*

We adapt the notion of segment from [TS00, Def. 6.1.2] to one which allows segments to extend via communication rules from one prederivation to another.

Definition 30 (Segment). *A* segment *(of* length k*) in a* HNGL *deduction* R *is a sequence* $A_1, \dots, A_k$ *of formula occurrences of a formula* A *in* R *such that the following conditions are satisfied:*

– *for* $1 < i \leq k$*, there are rule occurrences* r_i *in* R*, such that* r_i *is a del-rule,* A_{i-1} *is minor premise of* r_i*, and* A_i *is the conclusion of* r_i *— observe that* r_i *may be a communication rule, in which case the two formula occurrences may be in different prederivations in* R*;*

– A_k *is not a minor premise of a del-rule application,*

– A_1 *is not the conclusions of a del-rule application.*

A segment is maximal*, or a* cut (segment) *if* A_k *is the major premise of an E-rule, and either* $k > 1$*, or* $k = 1$ *and* A_1 *is the conclusion of an I-rule. The* cut-rank $\textsc{cr}(s)$ *of a maximal segment* s *is* $|A|$*. The* cut-rank $\textsc{cr}(R)$ *of a* HNGL *deduction* R *is the maximum of the cut-ranks of cuts of* R*. If there is no cut, the cut-rank of* R *is zero. A* critical *cut of* R *is a cut of maximal cut-rank among all cuts in* R*. We shall use* s, s' *for segments.*

We shall say that s *is a* subformula *of* s' *if the formula* A *in* s *is a subformula of* B *in* s'*. A deduction without critical cuts is said to be* normal.

Observe that this definition coincides with [TS00, Def. 6.1.2] if restricted to Natural Deduction NJ.

7.1 Conversions

We extend the conversions defined in [TS00, Chap. 6.1] to deal with our additional cases involving communication, splitting and contraction rules. We remark already at this point, that due to $\mathcal{X}$-condition 2 (consistent labels), conversions need to be applied to all occurrences of redices carrying the same labels. Thus, in the following, when converting a redex, we implicitly assume that the conversion is applied to all such redex occurrences carrying the same labels.

Some conversions, like $\rightarrow$-conversion, copy subderivations above other subderivations. To certify that the resulting prehyper deduction again form a valid deduction, we need to change the total order on labels to be consistent with the newly created prederivations. The following lemma addresses this point by showing that certain rearrangements of the order are acceptable.

Lemma 15. *Let* $(R, \preceq)$ *be a deduction in* HNGL*. Let* r *be an occurrence of a non-unary logical rule, and let* s *be a premise of* r*. Let* $C = [s]_{\mathrm{Cut}(\mathrm{Drop}(R,r))}$ *be the connected component of* s *in* $\mathrm{Cut}(\mathrm{Drop}(R, r))$*. For any contraction occurrence* c *in* R *with premises* a *and* b *assume that* $c \in C$ *iff* $a, b \in C$*.*

Let L_1 *be the set of labels in* $\mathrm{CDLabels}(R)$ *which occur in* C*, and* L_0 *the remaining ones, i.e.* $L_0 = \mathrm{CDLabels}(R) \setminus L_1$*. Let* $\preceq'$ *be a total order modulo* *duality on* CDLabels(R)*, which coincides with* $\preceq$ *on* L_i*, for* $i = 0, 1$*, and puts* L_1 *before* L_0*, i.e.* $y \prec' x$ *for any* $x \in L_0$ *and* $y \in L_1$*.*

Then $(R, \preceq')$ *is also a deduction in* HNGL*.*

Proof. The only conditions which are affected from changing the order, are $\mathcal{X}$-conditions 4 (label order) and 6 (contr. premise).

For $\mathcal{X}$-condition 4 (label order) let r and r' be nodes with labels in CDLabels(R), such that there is a dipath from r to r' w.r.t. tree-edges, $r \rightarrow_{\mathrm{t}}^{*} r'$. If both r and r' are in the same L_i, $i = 0, 1$, then $r \preceq r'$ as $\preceq$ and $\preceq'$ coincide by construction. So the critical case is when r and r' are in different L_i's. But then we must have $r \in L_1$ and $r' \in L_0$ by construction, hence $r \preceq' r'$ as $\preceq'$ puts L_1 before L_0.

For $\mathcal{X}$-condition 6 (contr. premise), assume for sake of contradiction that it is violated under $\preceq'$. That is, for some pair of contraction rules c and c', with premises v and t of c, we have that $c \preceq' c'$ but v and t are not connected in $\mathrm{Cut}(\mathrm{Drop}(R, \{c, c'\}))$. As this condition is not violated under $\preceq$, we must have that $c' \prec c$, hence $c \in L_1$ and $c' \in L_0$ by construction. Without loss of generality assume that the edge (v, c) is the one which is removed in $\mathrm{Cut}(R)$. Let p_0 be a path connecting s and c in $\mathrm{Cut}(\mathrm{Drop}(R, r))$, which exists as $c \in L_1$.

Using $\mathcal{X}$-condition 6 (contr. premise) for $(R, \preceq)$, there is a path p from t to v in $\mathrm{Cut}(\mathrm{Drop}(R, c))$. As p is destroyed in $\mathrm{Cut}(\mathrm{Drop}(R, \{c, c'\}))$, there must be an edge (f, g) such that p is of the form $t - p_1 - f \rightarrow_{\mathrm{t}} g - p_2 - v$, with p_1 a path in $\mathrm{Cut}(\mathrm{Drop}(R, \{c, c'\}))$ and $c' \rightarrow_{\mathrm{t}}^{*} g$ via a path p_3 in $\mathrm{Cut}(R)$.

Now consider the path p_4 in $\mathrm{Cut}(R)$ given by $c \leftarrow_{\mathrm{t}} t - p_1 - f \rightarrow_{\mathrm{t}} g - p_3 - c'$. As this path is destroyed in $\mathrm{Cut}(\mathrm{Drop}(R, r))$, there must be an edge (a, b) such that p_4 is of the form $c \leftarrow_{\mathrm{t}} t - p_5 - a \rightarrow_{\mathrm{t}} b - p_6 - c'$, with p_5 a path in $\mathrm{Cut}(\mathrm{Drop}(R, r))$ and $r \rightarrow_{\mathrm{t}}^{*} b$ via a path p_7 in $\mathrm{Cut}(R)$.

In this case, b must be either a non-unary logical rule, or a communication, thus we have a second premise a' such that p_7 is of the form $r - p_8 - a' \rightarrow_{\mathrm{t}} b$ or $r - p_8 - a' \rightarrow_{\mathrm{t}} \bar{b} \leftrightarrow_{\mathrm{t}} b$. In both cases, b has to satisfy $\mathcal{X}$-condition 5 (indep. premises), which is violated because $a - p_5 - t \rightarrow_{\mathrm{t}} c - p_0 - s \rightarrow_{\mathrm{t}} r - p_8 - a'$ is a path in $\mathrm{Cut}(\mathrm{Drop}(R, b))$. □

The detour conversions $\wedge$-*conversion,* $\vee$-*conversion* and $\rightarrow$-*conversion*, and the *simplification conversions* from [TS00, Chap. 6.1] stay in principle the same. The additional point to take care of is that whole connected components need to be removed if parts of prederivations get discarded. We make this precise in the case of $\wedge$-conversion, other cases are similar. Consider an $\wedge$-conversion redex occurring in a hyper natural deduction R of the form

$$x{:}\wedge\text{-}e\,\dfrac{r{:}\wedge\text{-}i\,\dfrac{\begin{matrix}\sigma_1 \vdots \\ s\,\dfrac{}{A_1}\end{matrix} \qquad \begin{matrix}\sigma_2 \vdots \\ t\,\dfrac{}{A_2}\end{matrix}}{A_1 \wedge A_2}}{A_1}$$

By Lemma 12 we can find a cut operation Cut such that for C the connected component of t in $\mathrm{Cut}(\mathrm{Drop}(R, r))$, $C = [t]_{\mathrm{Cut}(\mathrm{Drop}(R,r))}$, for all contraction occurrences c in R with premises a and b, $c \in C$ iff $a, b \in C$. Applying Proposition 2, we obtain for $R' := R \setminus C$, in which all contraction occurrences which had one of their premises removed by C with the unary rule Rep, that this R' is a hyper natural deduction.

In R', any occurrence of the above redex has been transformed to

$$x{:}\wedge\text{-}e\;\dfrac{r{:}\wedge\text{-}i\;\dfrac{s\;\dfrac{\sigma_1\,\vdots}{A_1}\qquad A_2}{A_1 \wedge A_2}}{A_1}$$

We now convert such trees into

$$s\;\dfrac{\sigma_1\,\vdots}{A_1}$$

The resulting prehyper deduction again forms a hyper natural deduction, and has the property that its derived hypersequent is a syntactic subhypersequent of the derived hypersequent of R.

We also consider the case of $\to$-conversion in detail. Consider an $\to$-conversion redex occurring in a hyper natural deduction $(R, \preceq)$ of the form

$$y{:}\,\to\text{-}e\;\dfrac{{}^1x{:}\,\to\text{-}i\;\dfrac{t\;\dfrac{\begin{array}{c}{}^1[A]\\ \sigma_1\,\vdots\end{array}}{B}}{A \to B}\qquad s\;\dfrac{\sigma_2\,\vdots}{A}}{B}$$

If no assumption of σ_1 is discharged at $\to$-i, then this redex converts to

$$\begin{array}{c}\sigma_1\,\vdots\\ B\end{array}$$

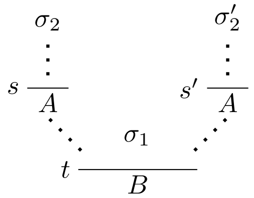
and we remove the connected component of s in Cut(Drop(R, y)) in a similar way as discussed above for $\wedge$-conversion.

Now consider the case that at least one assumption of σ_1 is discharged at x: $\to$-i. We only treat the case of two discharged assumptions, the other cases follow a similar pattern. We cannot simply copy σ_2 above these discharged assumptions, as we have to ensure that the resulting figure again forms a valid hyper natural deduction. In particular, we need to make sure that the $\mathcal{X}$-condition 5 (indep. premises) is satisfied.

Let CC_i be the connected component of σ_i in Cut(Drop(R, y)). Let L_i be the set of labels in CDLabels(R) which occur in CC_i, and $L_0 = \text{CDLabels}(R) \setminus (L_1 \cup L_2)$. Using Lemma 15 we can assume without loss of generality that $\preceq$ puts L_2 before L_1, and L_1 before L_0.

CC_2 induces a prehyper deduction in R, by choosing those subtrees of prederivations in R which are given by nodes in CC_2. The induced component can be written as $R_2 := \{\sigma_2, \rho_1, \ldots, \rho_k\}$. Create an independent copy R_2' of R_2 of the form $R_2' := \{\sigma_2', \rho_1', \ldots, \rho_k'\}$ by choosing fresh labels for all rules. Using fresh contraction labels $c_1, \ldots, c_k$ form prederivations $\tilde{\rho}_i$ by combining ρ'_i and ρ_i with a contraction labeled c_i.

Also form the following prederivation, which we abbreviate with δ:



Let L_3 be the set $\mathrm{CDLabels}(R'_2)$, and define a new order $\preceq'$ on $L_0 \cup L_1 \cup L_2 \cup L_3 \cup \{c_1, \ldots, c_k\}$, which satisfies

– on L_i, $\preceq'$ coincides with $\preceq$, for $i = 0, 1, 2, 3$;

– $\preceq'$ orders $\{c_1, \ldots, c_k\}$ arbitrarily;

– $\preceq'$ orders L_3 before L_2, L_2 before L_1, L_1 before $\{c_1, \ldots, c_k\}$, and the latter before L_0:

$$L_3 \preceq' L_2 \preceq' L_1 \preceq' \{c_1, \ldots, c_k\} \preceq' L_0$$

Now convert R in the following way:

– replace all subtrees rooted in y with δ;

– replace ρ_i with $\tilde{\rho}_i$.

We argue that the resulting prehyper deduction R' forms a hyper natural deduction. By construction it is obvious that its derived hypersequent is a syntactic subhypersequent of the derived hypersequent of R.

With the assumption that at least one discharged assumption $[A]$ is occurring in σ_1, we obtain that all conditions except $\mathcal{X}$-conditions 5 (indep. premises) and 6 (contr. premise) are immediately satisfied.

We observe that, as R_2 relates to a connected component in $\mathrm{Cut}(\mathrm{Drop}(R, y))$, the only connection between $\mathrm{Cut}(R_2)$ to the rest in $\mathrm{Cut}(R')$ is via s. Similarly for R'_2: let s' be the copy of s in R'_2, then the only connection between $\mathrm{Cut}(R'_2)$ to the rest of $\mathrm{Cut}(R')$ is via s'.

Consider $\mathcal{X}$-condition 5 (indep. premises) for R'. Let r be a non-unary logical rule occurring in R', and a and b two of its premises. For sake of contradiction, assume that there is a path p from a to b in $\mathrm{Cut}(\mathrm{Drop}(R', r))$. This path cannot go through s, as it would have to go via s a second time to be able to go from a to b, as s is the only connection between $\mathrm{Cut}(R_2)$ to the rest of $\mathrm{Cut}(R')$. Similarly it cannot go through s'. Thus the path is either completely in R_2, or completely in R'_2, or completely outside of R_2 and R'_2 in R'. In any case, p gives rise to a similar path in $\mathrm{Cut}(\mathrm{Drop}(R, r))$, contradicting $\mathcal{X}$-condition 5 (indep. premises) for R.

Consider $\mathcal{X}$-condition 6 (contr. premise) for R'. Let c and c' be two contraction rules occurring in R', with $c \preceq' c'$. Let a and b be the premises of c. Assume c, c' are both in L_3, L_2, or $L_1 \cup L_0$, and p a path in $\mathrm{Cut}(\mathrm{Drop}(R, \{c, c'\}))$ connecting a and b (or corresponding a, b in case $i = 3$). As before, p cannot go through s or s', thus has to stay completely in R_2, or completely in R'_2, or completely outside of R_2 and R'_2 in R'. In any case, p gives rise to a similar path in

Cut(Drop(R', $\{c, c'\}$)). Now assume that c is in L_i for some $i = 1, 2, 3$, but c' not. Consider p a path in Cut(Drop(R, $\{c\}$)) connecting a and b (or corresponding a, b in case $i = 3$). Again, p cannot go through s or s', hence p gives rise to a similar path in Cut(Drop(R', $\{c, c'\}$)). Finally, consider $c = c_i$ for some $i = 1, \dots k$, and $c \preceq' c'$. Then a is the root of $\rho_i \in R_2$, and b of $\rho'_i \in R'_2$. Thus, using Lemma 8, we can find the following path in Cut(Drop(R', $\{c, c'\}$)): $a - s - t - s' - b$.

The permutation conversion $\vee$*-perm conversion for E-rules* from [TS00, Chap. 6.1] in principle also stays the same. As parts of the prederivation get duplicated, we have to use a similar construction as in the previous case.

Consider an $\vee$-perm conversion redex occurring in a hyper natural deduction $(R, \preceq)$ of the form

$$y{:}\textit{E-rule}\;\dfrac{x{:}\vee\text{-}e\;\dfrac{\overset{\sigma_0\,\vdots}{A \vee B}\qquad \overset{\sigma_1\,\vdots}{C}\qquad \overset{\sigma_2\,\vdots}{C}}{C}\qquad\qquad \sigma_3}{D}$$

Let CC be the connected component of σ_3 in Cut(Drop(R, y)). CC induces a prehyper deduction in R, by choosing those subtrees of prederivations in R which are given by nodes in CC. The induced component can be written as $R_1 := \{\sigma_3, \rho_1, \dots, \rho_k\}$. Create an independent copy of R_1 of the form $R'_1 := \{\sigma'_3, \rho'_1, \dots, \rho'_k\}$ by choosing fresh labels for all rules. Using fresh contraction labels $c_1, \dots, c_k$ form prederivations $\tilde{\rho}_i$ by combining ρ_i and ρ'_i with a contraction labeled c_i. Also form the following prederivation, which we abbreviate with δ:

$$\vee\text{-}e\;\dfrac{\overset{\sigma_0\,\vdots}{A \vee B}\qquad \textit{E-rule}\;\dfrac{\overset{\sigma_1\,\vdots}{C}\qquad \sigma_3}{D}\qquad \textit{E-rule}\;\dfrac{\overset{\sigma_2\,\vdots}{C}\qquad \sigma'_3}{D}}{D}$$

Define a new order $\preceq'$ on the new set of labels similar to the case of $\rightarrow$-conversion. Now convert R in the following way:

- replace all subtrees rooted in y with δ;
- replace ρ_i with $\tilde{\rho}_i$.

The resulting $(R', \preceq')$ is again a hyper natural deduction deriving the same hypersequent as R.

In summary, we obtain that the conversions from [TS00, Chap. 6.1] transform hyper natural deductions into hyper natural deductions.

Proposition 4. *The adapted detour conversions ($\wedge$-conversion, $\vee$-conversion and $\rightarrow$-conversion), the simplification conversions and the permutation conversion ($\vee$-perm conversion for E-rules) from [TS00, Chap. 6.1] convert* HNGL *deductions into* HNGL *deductions, with the additional property that the converted* HNGL *deduction derives a structural subhypersequent of the original* HNGL *deduction.*

We now turn our attention to three more permutation conversions which deal with permuting an E-rule over a contraction, communication and split rule. For the following, let $(R, \preceq)$ be a hyper natural deduction.

contr-perm conversion:

$$\textit{E-rule}\ \dfrac{\textit{contr}\ \dfrac{\overset{\sigma_1}{A} \qquad \overset{\sigma_2}{A}}{A} \qquad \sigma}{B}$$

converts to

$$\textit{contr}\ \dfrac{\textit{E-rule}\ \dfrac{\overset{\sigma_1}{A} \qquad \sigma}{B} \qquad \textit{E-rule}\ \dfrac{\overset{\sigma_2}{A} \qquad \sigma'}{B}}{B}$$

where σ' is obtained as an independent copy from σ similar to the construction for $\vee$-perm conversion for E-rules. With the same argumentation as before we obtain that the resulting set of prederivations form again a hyper natural deduction deriving the same hyper sequent.

In the following we describe *com-perm conversion* and *split-perm conversion* only for redices involving $\to$-e, the cases for other E-rules are similar.

com-perm conversion: Consider a redex involving communication of the following form, where $\ell = x\colon \mathrm{Com}_{C,A\to B}$.

$$y\colon \to\text{-}e\ \dfrac{x\colon \mathrm{Com}_{C,A\to B}\ \dfrac{\begin{matrix}\Delta\\ \sigma_1\,\vdots\\ C\end{matrix}}{A\to B} \qquad \begin{matrix}\Pi\\ \sigma_2\,\vdots\\ A\end{matrix}}{B}$$

By $\mathcal{X}$-condition 1 (dual labels), also $\bar{\ell} = \bar{x}\colon \mathrm{Com}_{A\to B,C}$ must occur in R:

$$\bar{x}\colon \mathrm{Com}_{A\to B,C}\ \dfrac{\begin{matrix}\Gamma\\ \sigma_0\,\vdots\\ A\to B\end{matrix}}{C}$$

Define the following prederivations:

$$\mu\colon \qquad \to\text{-}e\ \dfrac{\begin{matrix}\Gamma\\ \sigma_0\,\vdots\\ A\to B\end{matrix} \qquad \begin{matrix}\Pi\\ \sigma_2\,\vdots\\ A\end{matrix}}{B}$$

$$\delta_1\colon \quad \bar{x}\colon \mathrm{Com}_{B,C}\ \dfrac{z\colon^1 \mathrm{Spt}_{\Gamma,\Pi}\ \dfrac{\begin{matrix}\Gamma, {}^1[\Pi]\\ \mu\,\vdots\\ B\end{matrix}}{B}}{C} \qquad\qquad \delta_3\colon \quad \to\text{-}i\ \dfrac{x\colon \mathrm{Com}_{C,B}\ \dfrac{\begin{matrix}\Delta\\ \sigma_1\,\vdots\\ C\end{matrix}}{B}}{A\to B}$$

$$\delta_2:\qquad \dfrac{\bar{z}{:}^1\,\mathrm{Spt}_{\Pi,\Gamma}\ \dfrac{\begin{matrix}{}^1[\Gamma],\Pi\\ \mu\,\vdots\\ B\end{matrix}}{B} \qquad x{:}\,\mathrm{Com}_{C,B}\ \dfrac{\begin{matrix}\Delta\\ \sigma_1\,\vdots\\ C\end{matrix}}{B}}{B}\ \textit{contr}$$

Then w.r.t. the previously defined com-perm conversion redex, R converts to R' which is obtained by applying the following three steps to R:

- replace all subtrees rooted in $\bar{x}{:}\,\mathrm{Com}_{A\to B,C}$ with δ_1;
- replace all subtrees rooted in $y{:}\ \to$-e with δ_2;
- for *all remaining* occurrences of $x{:}\,\mathrm{Com}_{C,A\to B}$, that is those not occurring as major premise of $y{:}\ \to$-e, replace the subtrees rooted in them by δ_3.

As the subtrees and their replacements derive the same hypersequent, it is obvious that also R' derives the same hypersequent as R.

To see that R' is also a hyper natural deduction, we define a new label order $\preceq'$ and consider all conditions in turn. Using Lemma 15 we can assume that $\preceq$ puts the CDLabels of the connected component of σ_2 in $\mathrm{Cut}(\mathrm{Drop}(R,y))$ before the remaining labels. From this we obtain $\preceq'$ by replacing $x{:}\,\mathrm{Com}_{A,B}$ in $\preceq$ with $x{:}\,\mathrm{Com}_{B,C}$, and inserting directly before it the new label $z{:}\,\mathrm{Spt}_{\Gamma,\Pi}$.

$\mathcal{X}$-condition 1 (dual labels): As all parts of the original deduction have been reused, this condition is obviously satisfied.

$\mathcal{X}$-conditions 2 (consistent labels) and 3 (consistent split) are obviously satisfied by construction, using that R is a hyper natural deduction.

$\mathcal{X}$-condition 4 (label order): Since we are introducing the new labels consistently with old positions, this condition is easily satisfied.

$\mathcal{X}$-condition 7 (one class) requires that all prederivations are in one connected component in R'. This is satisfied as the replacement trees, like the original ones, are all connected.

$\mathcal{X}$-condition 5 (indep. premises) needs careful consideration. We first observe that in $\mathrm{Cut}(\mathrm{Drop}(R',\{x,y\}))$ the connected components of σ_0, σ_1 and σ_2 are pairwise disjoint, which we can easily deduce from a similar situation in R. Thus we immediately see that $\mathcal{X}$-condition 5 in R' for $y{:}\ \to$-e and $x{:}\,\mathrm{Com}_{B,C}$ are both satisfied.

Now consider a non-unary logical rule r (communication behaves similar) other than the ones considered above. If there is no path in $\mathrm{Cut}(R)$ from r to the root of σ_0, σ_1 or σ_2, $\mathcal{X}$-condition 5 is satisfied as σ_0, σ_1 and σ_2 are all in the same connected component in $\mathrm{Cut}(R)$ and $\mathrm{Cut}(R')$. If the root of σ_0 can be reached from r in $\mathrm{Cut}(R)$, then the connected components of σ_0, σ_1 and σ_2 are pairwise disjoint, whether computed in $G := \mathrm{Cut}(\mathrm{Drop}(R,r))$ or $G' := \mathrm{Cut}(\mathrm{Drop}(R',r))$. If the root of σ_2 can be reached from r in $\mathrm{Cut}(R)$, then the connected components of σ_0, σ_1 and σ_2 are pairwise disjoint in G', where in G those of σ_0 and σ_1 fall together, and stays disjoint from σ_2. In both cases it is obvious that in G' we are not creating new dependencies between premises of r.

Now assume that the root of σ_1 can be reached from r in $\mathrm{Cut}(R)$. Then the connected components of σ_0, σ_1 and σ_2 are pairwise disjoint in $G := \mathrm{Cut}(\mathrm{Drop}(R,r))$, but in $G' := \mathrm{Cut}(\mathrm{Drop}(R',r))$ the connected components of σ_0 and σ_2 coincide, and are disjoint from the one of σ_1. Assume for sake of contradiction, that the premises of r are connected in G'. In particular, there must be a path from one of the premises of r to the root of σ_2 in G' and in G. But then the premises of $y\colon \to$-e are dependent in $\mathrm{Cut}(\mathrm{Drop}(R, y))$ via latter path and the path from r to the root of σ_1.

$\mathcal{X}$-condition 6 (contr. premise) also needs careful consideration. The premises of the newly introduced contraction rule at the root of δ_2 are connected using $x\colon \mathrm{Com}_{C,B}$ and $z\colon \mathrm{Spt}_{\Gamma,\Pi}$. Now consider any other contraction c in R' and a second one c' such that $c \preceq' c'$. With a similar analysis as in the previous case, we see that if there is no path in $\mathrm{Cut}(R)$ from c or c' to the root of σ_2, then it is obvious that in $G' := \mathrm{Cut}(\mathrm{Drop}(R', \{c, c'\}))$ we are not destroying dependencies between premises of c, simply because connected components only are increased in these cases.

So assume that the root of σ_2 can be reached from c or c' in $\mathrm{Cut}(R)$. Then the connected components of σ_0, σ_1 and σ_2 are pairwise disjoint in G', where in $G := \mathrm{Cut}(\mathrm{Drop}(R, \{c, c'\}))$ those of σ_0 and σ_1 fall together, and stay disjoint from σ_2. Assume for sake of contradiction, that the premises of c are disconnected in G'. In particular, there must be a path from the premises of c to the roots of σ_0 and σ_1 in G' and in G. If the root of σ_2 can be reached from c', then c' is in the connected component of σ_2, and as those CDLabels are put before others and $c \preceq' c'$, we must also have that c is also in the connected component of σ_2, all relative to $\mathrm{Cut}(\mathrm{Drop}(R, y))$. But then the premises of $y\colon \to$-e are dependent in $\mathrm{Cut}(\mathrm{Drop}(R, y))$ via c being in the connected component of σ_2, and both premises of c being connected to the left premise of $y\colon \to$-e, in $\mathrm{Cut}(\mathrm{Drop}(R, y))$.

split-perm conversion: Consider a redex involving split of the following form, where $\ell = x\colon \mathrm{Spt}_{\Delta,\Gamma}$.

$$y\colon \to\text{-}e\;\dfrac{{}^{1}\ell\;\dfrac{\begin{array}{c}\Gamma, {}^{1}[\Delta]\\ \sigma_0\,\vdots\\ A \to B\end{array}}{A \to B} \qquad \begin{array}{c}\Pi\\ \sigma_1\,\vdots\\ A\end{array}}{B}$$

By $\mathcal{X}$-condition 1 (dual labels), also $\bar{\ell}$ must occur in R in the form

$$ {}^{2}\bar{\ell}\;\dfrac{\begin{array}{c}{}^{2}[\Gamma], \Delta\\ \sigma_0\,\vdots\\ A \to B\end{array}}{A \to B}$$

We abbreviate subprederivations occurring above as

$$\sigma_2\colon\quad {}^{2}\bar{\ell}\;\dfrac{\begin{array}{c}{}^{2}[\Gamma], \Delta\\ \sigma_0\,\vdots\\ A \to B\end{array}}{A \to B} \qquad\qquad \sigma_3\colon\quad {}^{1}\ell\;\dfrac{\begin{array}{c}\Gamma, {}^{1}[\Delta]\\ \sigma_0\,\vdots\\ A \to B\end{array}}{A \to B}$$

$$\sigma_4: \qquad y\colon \to\text{-}e\ \dfrac{\begin{matrix}\Gamma\\ \sigma_3\,\vdots\\ A\to B\end{matrix} \qquad \begin{matrix}\Pi\\ \sigma_1\,\vdots\\ A\end{matrix}}{B}$$

Let μ be the following prederivation:

$$\mu: \qquad \to\text{-}e\ \dfrac{\begin{matrix}\Gamma,\Delta\\ \sigma_0\,\vdots\\ A\to B\end{matrix} \qquad \begin{matrix}\Pi\\ \sigma_1\,\vdots\\ A\end{matrix}}{B}$$

We consider cases depending on whether there are occurrences of $x\colon \mathrm{Spt}_{\Delta,\Gamma}$ other than those in occurrences of σ_4.

Case I) $\ell = x\colon \mathrm{Spt}_{\Delta,\Gamma}$ only occurs within σ_4 in R. Define the following two prederivations

$$\delta_1: \quad \to\text{-}i\ \dfrac{x{:}^1\,\mathrm{Spt}_{(\Gamma,\Pi),\Delta}\ \dfrac{\begin{matrix}{}^1[\Gamma],\Delta,{}^1[\Pi]\\ \mu\,\vdots\\ B\end{matrix}}{B}}{A\to B} \qquad\qquad \delta_2: \quad \bar{x}{:}^2\,\mathrm{Spt}_{\Delta,(\Gamma,\Pi)}\ \dfrac{\begin{matrix}\Gamma,{}^2[\Delta],\Pi\\ \mu\,\vdots\\ B\end{matrix}}{B}$$

W.r.t. the redex defined above, R converts to R' which is obtained by applying the following two steps to R:

– replace all occurrences of σ_2 with δ_1,

– replace all occurrences of σ_4 with δ_2.

Case II) $\ell = x\colon \mathrm{Spt}_{\Delta,\Gamma}$ occurs outside of occurrences of σ_4. Define the following three prederivations

$$\delta_0: \quad x{:}^1\,\mathrm{Spt}_{\Pi,(\Gamma,\Delta)}\ \dfrac{\begin{matrix}\Gamma,\Delta,{}^1[\Pi]\\ \mu\,\vdots\\ B\end{matrix}}{B} \qquad\qquad \delta_2: \quad \bar{x}{:}^2\,\mathrm{Spt}_{(\Gamma,\Delta),\Pi}\ \dfrac{\begin{matrix}{}^2[\Gamma,\Delta],\Pi\\ \mu\,\vdots\\ B\end{matrix}}{B}$$

$$\delta_1: \quad \to\text{-}i\ \dfrac{z{:}^3\,\mathrm{Spt}_{\Gamma,\Delta}\ \dfrac{\begin{matrix}{}^3[\Gamma],\Delta\\ \delta_0\,\vdots\\ B\end{matrix}}{B}}{A\to B} \qquad\qquad \delta_3: \quad \to\text{-}i\ \dfrac{\bar{z}{:}^4\,\mathrm{Spt}_{\Delta,\Gamma}\ \dfrac{\begin{matrix}\Gamma,{}^4[\Delta]\\ \delta_0\,\vdots\\ B\end{matrix}}{B}}{A\to B}$$

W.r.t. the redex defined above, R converts to R' which is obtained by applying the following three steps to R:

– replace all occurrences of σ_2 with δ_1;

– replace all occurrences of σ_4 with δ_2.

– replace *all remaining* occurrences of σ_3, that is those not occurring as major premises of y: →-e, with δ_3.

To see that in both cases R' is also a hyper natural deduction, we define a new label order $\preceq'$ and consider again all conditions. Using Lemma 15 we can assume that $\preceq$ puts the CDLabels of the connected component of σ_1 in $\mathrm{Cut}(\mathrm{Drop}(R, y))$ before the remaining labels. From this we obtain $\preceq'$ by replacing $x\colon \mathrm{Spt}_{\Delta,\Gamma}$ in $\preceq$ with $x\colon \mathrm{Spt}_{(\Gamma,\Pi),\Delta}$ for Case I), and for Case II) with $x\colon \mathrm{Spt}_{\Pi,(\Gamma,\Delta)}$ and inserting directly after it the new label $z\colon \mathrm{Spt}_{\Gamma,\Delta}$.

$\mathcal{X}$-conditions 1 (dual labels), 2 (consistent labels), 3 (consistent split), 4 (label order) and 7 (one class) follow similar as in the case for com-perm conversion. $\mathcal{X}$-condition 5 (indep. premises) also follows immediately, because connected components when computed over R' at most can get smaller compared to being computer over R.

So let turn to $\mathcal{X}$-condition 6 (contr. premise), and consider contractions c and c' in R' such that $c \preceq' c'$. With a similar analysis as in the case for com-perm conversion, we see that if there is no path in $\mathrm{Cut}(R)$ from c or c' to the root of σ_1, then it is obvious that in $G' := \mathrm{Cut}(\mathrm{Drop}(R', \{c, c'\}))$ we are not destroying dependencies between premises of c, because connected components stay the same.

So assume that the root of σ_1 can be reached from c or c' in $\mathrm{Cut}(R)$. Then the connected component of σ_0 in G' is smaller than computed in $G := \mathrm{Cut}(\mathrm{Drop}(R, \{c, c'\}))$. Assume for sake of contradiction, that the premises of c are disconnected in G'. Then there must be a path in G' from the premises of c which passes through the root of σ_0 in δ_1 or δ_3, and corresponds to a path p in G which passes through the root of σ_0 in σ_2 respectively σ_3.

If c would not be in the connected component of σ_1, we obtain that c' is in the connected component of σ_1, by assumption that the root of σ_1 can be reached from c or c'. As those CDLabels in the connected component of σ_1 are put before others and $c \preceq' c'$, we must then have that c is also in the connected component of σ_1, a contradiction. Thus c is in the connected component of σ_1 w.r.t. $\mathrm{Cut}(\mathrm{Drop}(R, y))$.

So the latter together with that both premises of c are connected to the root of σ_3 in G, we obtain that either the premises of $y\colon \to$-e are dependent in $\mathrm{Cut}(\mathrm{Drop}(R, y))$, or we have created a regular cycle in $Cut(R)$, contradicting our assumption that $\mathcal{X}$-condition 6 (contr. premise) fails in G'.

In summary, we obtain that the new conversions transform hyper natural deductions into hyper natural deductions.

Proposition 5. *Contr-perm, com-perm and split-perm conversions convert* HNGL *deductions into* HNGL *deductions, with the property that the converted hyper natural deduction derives a structural subhypersequent of the original hyper natural deduction.*

7.2 Normalization

Before we turn to the central theorem in this section on normalization, we need to develop an appropriate notion of branch for hyper natural deductions, taking into account that minor premises of communications can jump between subtrees.

Definition 31. *Let R be a hyper natural deduction, and $\mathcal{G} = (V, E, L, f)$ its associated graph, with tree edges $\to_{\mathrm{t}}$ and cross edges $\to_{\mathrm{c}}$. The* branch relation *$\to_{\mathrm{b}}$ for R is defined as a binary relation on V in the following way: $u \to_{\mathrm{b}} v$ iff either v is not a communication and $u \to_{\mathrm{t}} v$, or v is a communication and for some $w \in V$, $u \to_{\mathrm{t}} w \to_{\mathrm{c}} v$.*

Lemma 16. *Let $(R, \preceq)$ be a hyper natural deduction, and $\to_{\mathrm{b}}$ its branch relation. Then $(R, \to_{\mathrm{b}})$ is a directed acyclic graph.*

Proof. Assume there would be a cycle in $(R, \to_{\mathrm{b}})$. This cycle must contain at least two nodes carrying distinct dual labels, which cannot be ordered wrt. $\preceq$ as the latter has to respect tree edges. □

Definition 32. *Let $(R, \to_{\mathrm{b}})$ be the branch relation for a hyper natural deduction R. The* branches through R *are the dipaths wrt. $\to_{\mathrm{b}}$ of maximal length.*

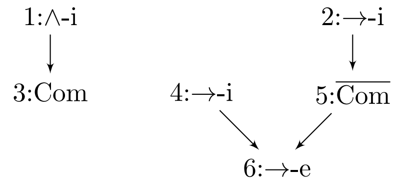
Let b and b' be two branches, then we say that b is left of *b', b* lof *b', iff there exists a node v on b with s and t two of v's premises such that v is not a contraction rule, s is left of t (as premises of v), s occurs in b, and b' has a common node with $[t]_{\mathrm{Cut}(\mathrm{Drop}(R,v))}$.*

A branch b is a rightmost branch of a set of branches B iff it is not left of any other branch in B.

To illustrate the definition of branches, consider the following hyper natural deduction R:

$$3{:}\mathrm{Com}\ \dfrac{1{:}\wedge\text{-}i\ \dfrac{A \qquad B}{A \wedge B}}{C \to C} \qquad\qquad 6{:}\to\text{-}e\ \dfrac{4{:}\to\text{-}i\ \dfrac{D}{A \wedge B \to D} \qquad 5{:}\overline{\mathrm{Com}}\ \dfrac{2{:}\to\text{-}i\ \dfrac{[C]}{C \to C}}{A \wedge B}}{D}$$

The associated graph is given by:



The branches through R are $b_1 := 1 \to_{\mathrm{b}} 5 \to_{\mathrm{b}} 6$, $b_2 := 2 \to_{\mathrm{b}} 3$ and $b_3 := 4 \to_{\mathrm{b}} 6$. b_3 is left of b_1 and b_2, and b_1 and b_2 are incompatible w.r.t. "left of". Thus b_1 and b_2 are rightmost branches in $\{b_1, b_2, b_3\}$.

Lemma 17. *Let $(R, \to_{\mathrm{b}})$ be the branch relation for a hyper natural deduction R. Any non-empty set of branches has a rightmost element.*

Proof. Assume a non-empty finite set of branches B of a hyper natural deduction R does not contains a rightmost element. Then we have a sequence of branches $b_1, \ldots, b_{k+1} = b_1$ in B with b_i lof b_{i+1}. W.l.o.g. assume that the cut operation does not cut any edge on the first branch, using Lemma 11. For each i, let v_i be the node in b_i which certifies $b_i \operatorname{lof} b_{i+1}$ in the definition of "left of", and t_i be the premise of v_i such that b_{i+1} has a common node with $[t_i]_{\mathrm{Cut}(\mathrm{Drop}(R,v_i))}$. Hence there is a path in $\mathrm{Cut}(\mathrm{Drop}(R,v_i))$ from t_i to some node in b_{i+1} — let w_{i+1} be the first node on this path which is in b_{i+1}. Let p_i be the path from $v_i - t_i$ to w_{i+1} following the latter described path.

Let q_i be a path in $\mathrm{Cut}(R)$ which goes from w_i to v_i — as premises of contractions are connected in $\mathrm{Cut}(R)$, this can be done even in the case that edges on b_i are removed by the cut.

If w_1 is above v_1 on b_1, choose $\bar{q}_1$ as the path from w_1 to s_1 along b_1. Then we can form the following cycle in $\mathrm{Cut}(R)$:

$$v_1 \leftarrow_{\mathrm{t}} t_1 - p_1 - w_2 - \cdots - w_1 - \bar{q}_1 - s_1 \rightarrow_{\mathrm{t}} v_1$$

Otherwise, if w_1 is below v_1 on b_1, choose $\bar{w}_1$ as the penultimate node on p_k, and $\bar{p}_k$ as its first part: $v_k - \bar{p}_k - \bar{w}_1$ — we have $\bar{w}_1 \rightarrow_{\mathrm{t}} w_1$ or $\bar{w}_1 \rightarrow_{\mathrm{c}} w_1$ as $\bar{w}_1$ is not on b_1. Also let w_1^+ be the predecessor of w_1 on b_1, i.e. $w_1^+ \rightarrow_{\mathrm{t}} w_1$, and choose $\bar{q}_1$ as the path from w_1^+ to v_1 along b_1. Then we can form the following cycle in $\mathrm{Cut}(R)$:

$$w_1 \leftarrow_{\mathrm{t}} w_1^+ - \bar{q}_1 - v_1 - p_1 - \cdots - \bar{p}_k - \bar{w}_1 \rightarrow_{\mathrm{tc}} w_1$$

In both cases we can use Lemma 10 to find a regular cycle in $\mathrm{Cut}(R)$, contradicting Theorem 4. □

When applying conversions, several copies of the same subprederivation may be produced, like in the case of com-perm-conversion. Segments contained in them will have identical label sequences. Thus, for the proof of normalization those segments are counted only once, whose sequence of labels of rules inferring consecutive elements are identical. To capture this, we define the notion of the label sequence of a segment.

Definition 33. *Let R be a hyper natural deduction, and $s = A_1, \dots, A_k$ a segment in R of formula A. With s we associate a sequence of rule occurrences $r_2, \dots, r_k$ in R such that A_{i-1} is a minor premise of r_i, and A_i the conclusion of r_i. Let n_i be the label of r_i, for $i = 2, \dots, k$. Then we associate with s the* sequence of labels *$n_2, \dots, n_k$, which is the sequence of labels of rules associated with s. The* length *of this sequence of labels is k.*

A critical label sequence *in R is the sequence of labels of a critical cut in R.*

We are now ready to proof our main theorem on normalization.

Theorem 7 (Normalization). *Each* HNGL *deduction R reduces to a normal* HNGL *deduction.*

Proof. We adapt the proof of normalization [TS00, Theorem 6.1.8] to our setting. It is an extension of [TS00] in the sense that they are the same when restricted to NJ.

We use main induction on the cut-rank n of R, with side-induction on m, the sum of lengths of all critical label sequences in R.

By a suitable choice of the critical cut to which we apply a conversion we can achieve that either n decreases, or that n remains constant but m decreases. Let us call s a *t.c.c.* (top critical cut) in R if no critical cut occurs in a branch above s. Choose a rightmost t.c.c. s. Applying a conversion to s, the resulting R' has a lower cut-rank (if s has length 1, and it is the only maximal segment in R), or has the same cut-rank but a lower value for m.

We discuss two cases in detail, which contain typical construction steps occurring in conversions during normalisation. These cases are →- and com-perm conversion.

For the case of a →-conversion redex of the form

$$y\colon \to\text{-}e\;\dfrac{{}^{1}x\colon \to\text{-}i\;\dfrac{\begin{matrix}[A]^1\\ \sigma_1\,\vdots\\ B\end{matrix}}{A\to B}\qquad s\;\dfrac{\sigma_2\,\vdots}{A}}{B}$$

we observe that by definition of "rightmost" σ_2 and its connected component in $\mathrm{Cut}(\mathrm{Drop}(R,y))$ has no common node with any other critical segment. Also the additional contractions used to reconnect copies of connected components of σ_2 are not impacting any other critical segment for the same reason. Thus reducing this redex does not impact the length of any other critical segments, but eliminates the current one.

Consider the case of a communication redex of the form

$$y\colon \to\text{-}e\;\dfrac{x\colon \mathrm{Com}_{C,A\to B}\;\dfrac{\begin{matrix}\varDelta\\ \sigma_1\,\vdots\\ C\end{matrix}}{A\to B}\qquad \begin{matrix}\varPi\\ \sigma_2\,\vdots\\ A\end{matrix}}{B}$$

where also

$$\bar{x}\colon \mathrm{Com}_{A\to B,C}\;\dfrac{\begin{matrix}\varGamma\\ \sigma_0\,\vdots\\ A\to B\end{matrix}}{C}$$

is occurring in R. The conversions are defined via the following prederivations:

$$\mu\colon\qquad \to\text{-}e\;\dfrac{\begin{matrix}\varGamma\\ \sigma_0\,\vdots\\ A\to B\end{matrix}\qquad \begin{matrix}\varPi\\ \sigma_2\,\vdots\\ A\end{matrix}}{B}$$

$$\delta_1: \quad \bar{x}{:}\,\mathrm{Com}_{B,C}\ \dfrac{z{:}^1\,\mathrm{Spt}_{\Gamma,\Pi}\ \dfrac{\begin{array}{c}\Gamma, {}^1[\Pi]\\ \mu\,\vdots\\ B\end{array}}{B}}{C} \qquad\qquad \delta_3: \quad \to\text{-}i\ \dfrac{x{:}\,\mathrm{Com}_{C,B}\ \dfrac{\begin{array}{c}\Delta\\ \sigma_1\,\vdots\\ C\end{array}}{B}}{A \to B}$$

$$\delta_2: \quad contr\ \dfrac{\bar{z}{:}^1\,\mathrm{Spt}_{\Pi,\Gamma}\ \dfrac{\begin{array}{c}{}^1[\Gamma], \Pi\\ \mu\,\vdots\\ B\end{array}}{B} \qquad x{:}\,\mathrm{Com}_{C,B}\ \dfrac{\begin{array}{c}\Delta\\ \sigma_1\,\vdots\\ C\end{array}}{B}}{B}$$

The critical segment under consideration involves $A \to B$, thus the repetition of B in δ_1 and δ_2 has no impact on the length of a critical segment. C may occur in another critical segment containing the communication. This is harmless as the length of those critical segments does not change. Furthermore, the duplication of derivations like μ does not effect the length measure as it is defined over label sequences, which do not change by performing this conversion. □

7.3 Subformula property

We adapt the notion of track from [TS00, Def. 6.2.2], so that tracks for hyper natural deductions may span communication rules, thus may jump from one prederivation to another. For two formula occurrences A and B, we say that A *is premise of* B if there is a rule occurrence r such that A is a (major or minor) premise of r, and B is the conclusion of r. In this case we say that B *is a successor of* A. Observe that in case of A being the premise of a communication, the successor of A is the conclusion of a dual communication, thus jumps from one branch containing A to another.

Definition 34 ([TS00, Def. 6.2.2]). *A track of a hyper natural deduction R is a sequence of formula occurrences $A_0, \ldots, A_k$ such that*

1. *A_0 is a top formula occurrence in R not discharged by an application of del-rule;*
2. *A_i for $i < k$ is not the minor premise of an instance of →-e, and either*
   (a) *A_i is not the major premise of an instance of a del-rule, and A_{i+1} is successor of A_i, or*
   (b) *A_i is the major premise of an instance u of a del-rule, and A_{i+1} is an assumption discharged by u;*
3. *A_k is either*
   (a) *the minor premise of an instance of →-e, or*
   (b) *the conclusion of a root node in R, or*
   (c) *the major premise of an instance u of a del-rule in case there are no assumptions discharged by u.*

We repeat the definitions of *strictly positive subformula* and *strictly positive part* from [TS00]. A formula A is a strictly positive subformula of itself. If $B \wedge C$ or $B \vee C$ is a strictly positive subformula of A, then so are B and C. If $B \to C$ is a strictly positive subformula of A, then so is C. A strictly positive subformula of A is also called a *strictly positive part (s.p.p.)* of A.

The following proposition is literally the same as in [TS00, Prop. 6.2.4]. Again observe that tracks may span communication rules, thus may jump from one prederivation to another. A track below is written as $\tau = s_0, \ldots, s_k$ using segments instead of formula occurrences in the obvious way.

Proposition 6 ([TS00, Prop. 6.2.4]). *Let R be a normal hyper natural deduction, and let $\tau = s_0, \ldots, s_k$ be a track in R. Then there is a segment s_i in τ, the* minimum *segment or* minimum *part of the track, which separates two (possibly empty) parts of τ, called the* E-part (elimination part) *and the* I-part (introduction part) *of τ such that:*

1. *for each s_j in the E-part we have $j < i$, s_j is a major premise of an E-rule, and s_{j+1} is a s.p.p. of s_j, and therefore each s_j is an s.p.p. of s_0;*
2. *for each s_j in the I-part we have $j > i$, and if $j \neq k$, then s_j is a premise of an I-rule and a s.p.p. of s_{j+1}, so each s_j is a s.p.p. of s_k;*
3. *if $i \neq k$, then s_i is a premise of an I-rule or a premise of $\perp_i$ and is an s.p.p. of s_0;*

Proposition 7. *In a normal* HNGL *deduction, each formula occurrence belongs to some track.*

Proof. By induction on the height of normal deductions. □

Before being able to prove the subformula property, we have to choose a nice order on labels which orders also all labels on tracks.

Proposition 8. *Let R be a hyper natural deduction. We can choose a total order $\preceq$ on* CDLabels(R) *modulo duality such that $\mathcal{X}$-condition 4 (label order) extends to tracks: $(R, \preceq)$ is a hyper natural deduction, and for any track in R, the labels in* CDLabels(R) *of rules on this track are ordered according to $\preceq$.*

Proof. Using Lemma 15, we can assume that for any del-rule, the labels of the connected component of its major premise are ordered before all other labels, w.r.t. $\preceq$. To see this, let r be an occurrence of $\vee$-e, with major premises u. According to Lemma 15, we can choose $\preceq$ such that the labels of $[u]_{\mathrm{Cut}(\mathrm{Drop}(R,r))}$ come before the rest. Then it is easy to see that tracks respect this ordering: if the track follows the tree order then this follows from $\mathcal{X}$-condition 4 (label order); if it jumps to another prederivation then this happens due to a communication rule, which extends the tree order; and if it jumps within a prederivation, this is from a major premise of some del-rule r to one of the discharged assumptions of r, but in this case the choice of $\preceq$ enforces that the order is respected. □

Each track decomposes into *track-parts* which do not span communications. That is, for a track $A_0, \ldots, A_k$ in a hyper natural deduction R we can find $i_0 = -1 < i_1 < \cdots < i_\ell < i_{\ell+1} = k$ such that within each part $A_{i_j+1}, \ldots, A_{i_{j+1}}$, $j = 0, \ldots, \ell$, consecutive formulas are not minor premise and conclusion of communication, but A_{i_j} is minor premise of some communication, and A_{i_j+1} its conclusion, $j = 1, \ldots, \ell$. Thus track-parts may start at top of R, or at conclusions of communications, and end in conclusions of roots of R, or minor premises of $\rightarrow$-e, or minor premises of communications.

Definition 35. *Let $(R, \preceq)$ be a hyper natural deduction, and let $c_1, \ldots, c_k$ be the communication labels in R. Without loss of generality assume that they form a decreasing sequence w.r.t. $\preceq$, that is $c_1 \succ c_2 \succ \cdots \succ c_k$.*

A track-part of order $(c_1, 0)$ is a track-part ending in a conclusion of a root in R, and starting at the top of R or at a conclusion of a rule labeled c_1 or $\bar{c}_1$.

A track-part of order $(c_i, 0)$, for $i > 1$, is a track-part ending in a conclusion of a root in R and starting at a conclusion of a rule labeled c_i or $\bar{c}_i$, or a track-part ending in a premise of some rule labeled c_j or $\bar{c}_j$ for $j < i$, and starting at the top of R or at a conclusion of a rule labeled c_i or $\bar{c}_i$.

A track-part of order $(c_i, n+1)$, for $i \geq 1$, is a track-part ending in a minor premise of an $\rightarrow$-e application with major premise belonging to a track-part of order (c_i, n), and starting at the top of R or at a conclusion of a rule labeled c_i or $\bar{c}_i$.

Theorem 8 (Subformula Property). *Let R be a normal hyper natural deduction with derived hypersequent $\mathcal{H}$. Then each formula in R is a subformula of a formula in $\mathcal{H}$.*

Proof. Let $(R, \preceq)$ be a normal hyper natural deduction, and let $c_1, \ldots, c_k$ be the communication labels in R, ordered as $c_1 \succ c_2 \succ \cdots \succ c_k$. The conclusion of each communication rule in R occurs either in an E-part or an I-part of some track in R (for this we count the occurrences in the minimal part to the E-part.) Let us call those occurring in E-part *E-com formulas*, and those in I-parts *I-com formulas*. Let $\mathcal{E}$ be the set of all E-com formulas occurring in R.

Claim 1: Each formula in R is a subformula of a formula in $\mathcal{H}$ or $\mathcal{E}$.

We prove this claim for track-parts of order (c_i, n), by main induction on i and side induction on n.

For example, let $A_0, \ldots, A_k$ be a track-part of order $(c_1, 0)$, and assume that A_0 is the conclusion of c_1 and A_k at a root of R. Then A_k is occurring as a formula in $\mathcal{H}$. If A_0 is a I-com formula, then all formulas occurring on this track-part are subformulas of A_k, hence of $\mathcal{H}$. If A_0 is an E-com formula, then all occurring on this track-part are subformulas of either A_0, hence of $\mathcal{E}$, or of A_k, hence of $\mathcal{H}$.

As another example consider a track-part $A_0, \ldots, A_k$ of order $(c_i, 0)$, $i > 1$, and assume that A_0 is at the top of R and A_k a premise of c_j for $j < i$. Hence all formulas on this track are either subformulas of A_0, or subformulas of A_k. The formula A_k also occurs as conclusion of $\bar{c}_j$. If this occurrence is a I-com formula, then it is subformula of $\mathcal{H} \cup \mathcal{E}$ by i.h. If it is an E-com formula, then it clearly is in $\mathcal{E}$. If A_0 is not discharged, then it is in $\mathcal{H}$. If it is discharged, then this must have happened through an I-rule in a way that A_0 is part of its conclusion, where the I-rule occurred within the track, or below it. If within the track, then A_0 is a subformula of A_k. If below the track, then it has already been shown to be subformula of $\mathcal{H} \cup \mathcal{E}$ by i.h.

Claim 2: Each formula in $\mathcal{E}$ is a subformula of a formula in $\mathcal{H}$.

Let $B_1, \dots, B_m$ be the formulas in $\mathcal{E}$, and n_j the communication label of which B_j is the conclusion. Without loss of generality assume $n_j \prec n_{j+1}$. We prove that B_j is a subformula of a formula in $\mathcal{H}$ by induction on j. Consider a track-part ending in B_j. As B_j is a E-com formula, it is a subformula of the first formula in the track-part. If the first element in the track-part is on top of R then we already know that it is a subformula of a formula in $\mathcal{H}$ by Claim 1. If it is a $B_{j'}$ for $j' < j$, then we obtain this by i.h. □

8 Conclusion

While we have achieved our main aim of giving a system of Hyper Natural Deduction for Gödel Logic which admits a normalisation procedure extending the usual one for Natural Deduction, many questions remain open: First of all our long term aim to provide a Curry-Howard style connection to some form of parallel λ-calculus is still outstanding. Procedural normalization as achieved in this paper is a main step in the direction of a suitable Curry-Howard correspondence.

Another open area is to study a normalisation procedure on the hyper-rule level based on the definition of HNGL in Section 3, and to describe its relationship to cut-elimination for the Hypersequent Calculus GLC. A related question is whether the normalization procedure described in this paper could be resembled on the hyper-rule level.

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A Technical Appendix

This technical appendix collects the following items: Lemmata only stated in Section 6 are restated and proven here. The restated lemmata are numbered as in the main part. Further lemmata and definitions needed for their proofs are also included, and are numbered within the appendix.

Lemma A.1. *Let R be an $\mathcal{X}$ deduction, and v and v' be two occurrences of rules in R, such that v' is reachable by tree edges from v, that is, $v \to^*_{\mathrm{t},R} v'$. Then* $\mathrm{Drop}(R, v)$ *is a subgraph of* $\mathrm{Drop}(R, v')$*,* $\mathrm{Drop}(R, v) \subseteq \mathrm{Drop}(R, v')$*.*

Proof. Let n be label of v and m the label of v'. The only complication that might occur is that the Drop operation removes all occurrences of rules labeled with n and m, respectively. But by $\mathcal{X}$-condition 2 (consistent labels) we see that every occurrence of a rule labeled with m carries an occurrence of a rule labeled with n in the subprederivation leading up to it. The Drop operation removes at least all vertices reachable via tree edges, so we see that the set of vertices removed for $\mathrm{Drop}(R, m)$ is a subset of $\mathrm{Drop}(R, n)$, which proves the lemma. □

Lemma A.2. *Let R be an $\mathcal{X}$ deduction. Any dipath in R is acyclic.*

Proof. Observe that by $\mathcal{X}$-condition 4 (label order), if $r \rightarrow_t^* r'$, then $r \preceq r'$. Furthermore, since $\preceq$ is an order modulo duality, $r \rightarrow_{tc}^* r'$ also implies that $r \preceq r'$. Assume that there is an cyclic dipath p of the form $v \rightarrow_{tc}^* v$ in R. So there exists u and w which carry labels from DLabels(R) such that the cycle given by p contains

$$u \rightarrow_t^* v \rightarrow_t^* w.$$

Thus, $u \rightarrow_t^* w$ and we obtain $u \preceq w$. We also obtain $w \rightarrow_{tc}^* u$ as part of dipath p, thus $w \preceq u$, contradiction as u and w are distinct. □

Next we turn our attention to connected components: When testing for $\mathcal{X}$-conditions 5 (indep. premises) and 6 (contr. premise), we only consider the connected components of the premises of a non-unary logical rule. We want to show that in fact all the vertices in the subprederivation rooted in a rule occurrence r are in the same connected component. These connected components are computed in Cut(Drop(R, r)), though, which means we have to be careful that neither the Cut nor the Drop-operation destroys this aim.

Lemma A.3. *Let R be in $\mathcal{X}$. Let r, s and v be rule occurrences in R, such that s is a premise of rule r, that is $s \rightarrow_{\mathrm{t},R} r$. If s is reachable in R by tree edges from v, then it is also reachable in* Drop(R, r) *by tree edges from v, i.e.,*

$$v \rightarrow_{\mathrm{t},R}^* s \rightarrow_{\mathrm{t},R} r \quad \Rightarrow \quad v \rightarrow_{\mathrm{t},\mathrm{Drop}(R,r)}^* s$$

In other words, the Drop operation does not delete tree edges above the rule occurrence that is dropped. This is a first step into showing later that certain cycles cannot happen in $\mathcal{X}$ deductions.

Proof. Assuming that $v \rightarrow_{\mathrm{t},R}^* s \rightarrow_{\mathrm{t},R} r$, let us write the dipath from v to r in more details:

$$v = v_0 \rightarrow_{\mathrm{t},R} v_1 \rightarrow_{\mathrm{t},R} \dots \rightarrow_{\mathrm{t},R} v_j \rightarrow_{\mathrm{t},R} \dots \rightarrow_{\mathrm{t},R} v_k = s \rightarrow_{\mathrm{t},R} r$$

If $v \rightarrow_{\mathrm{t},\mathrm{Drop}(R,r)}^* s$ does not hold, some v_j must be reachable from r by a dipath in R. Thus we obtain a proper loop in R via $v_j \rightarrow_{\mathrm{t},R}^* s \rightarrow_{\mathrm{t},R} r \rightarrow_R^* v_j$, contradicting the previous Lemma A.2. □

Lemma 8. *Let R be in $\mathcal{X}$. Let r, s and v be rule occurrences in R, such that s is a premise of rule r, that is $s \rightarrow_{\mathrm{t}} r$. If s is reachable in R by tree edges from v, then v is in the connected component of s in* Cut(Drop(R, r))*:*

$$v \rightarrow_{\mathrm{t}}^* s \rightarrow_{\mathrm{t}} r \quad \Longrightarrow \quad v \in [s]_{\mathrm{Cut}(\mathrm{Drop}(R,r))}.$$

In other words, everything that is above (wrt tree edges) the premise of a rule falls into the same connected component in Cut(Drop(R, r)).

Proof. By the previous lemma we obtain that $v \rightarrow_{\mathrm{t},\mathrm{Drop}(R,r)}^* s$. What remains to show is that even if we remove some of the tree edges leading to contractions, we still can reach s from v via $\leftrightarrow_{\mathrm{Cut}(\mathrm{Drop}(R,r))}$, which is equivalent to $v \in [s]_{\mathrm{Cut}(\mathrm{Drop}(R,r))}$. Since $v \rightarrow_{\mathrm{t},\mathrm{Drop}(R,r)}^* s$ there exists a dipath

$$v = v_0 \rightarrow_{\mathrm{t},\mathrm{Drop}(R,r)} v_1 \rightarrow_{\mathrm{t},\mathrm{Drop}(R,r)} \dots \rightarrow_{\mathrm{t},\mathrm{Drop}(R,r)} v_k = s$$

We now show by induction on the length of the chain that all the v_j are in the connected component of s in Cut(Drop(R, r)). If $v = v_0 = v_k = s$ the fact is obvious. Otherwise assume that we already know that $v_{j+1} \in [s]_{\mathrm{Cut}(\mathrm{Drop}(R,r))}$ and we want to show that the same holds also for v_j.

If v_{j+1} is *not* a contraction rule, then Cut does not remove edges leading to v_{j+1}, hence $v_j \in [s]_{\mathrm{Cut}(\mathrm{Drop}(R,r))}$ follows by induction hypothesis.

If v_{j+1} is a contraction we know that there are two predecessors: v_j and v'. The cut operations only cuts one of the two edges (v', v_{j+1}) or (v_j, v_{j+1}). In the former case we already have $v_j \to_{\mathrm{t},\mathrm{Cut}(\mathrm{Drop}(R,r))} v_{j+1}$ and by induction hypothesis we obtain that $v_j \in [s]_{\mathrm{Cut}(\mathrm{Drop}(R,r))}$.

Consider now the latter case: According to $\mathcal{X}$-condition 6 (contr. premise) we know that

$$[v_j]_{\mathrm{Cut}(\mathrm{Drop}(R,v_{j+1}))} = [v']_{\mathrm{Cut}(\mathrm{Drop}(R,v_{j+1}))},$$

that is there is a path p in Cut(Drop(R, v_{j+1})) from v_j to v'. Note that we are applying Drop to v_{j+1} and not r. Using Lemma A.1 we see that Drop(R, v_{j+1}) $\subseteq$ Drop(R, r) and thus also

$$\mathrm{Cut}(\mathrm{Drop}(R, v_{j+1})) \subseteq \mathrm{Cut}(\mathrm{Drop}(R, r)).$$

As a consequence, the path p is also in Cut(Drop(R, r)).

We consider the following path in Cut(Drop(R, r)): $v_j{-}p{-}v' \to_{\mathrm{t},\mathrm{Cut}(\mathrm{Drop}(R,r))} v_{j+1}$, and obtain again by the induction hypothesis that $v_j \in [s]_{\mathrm{Cut}(\mathrm{Drop}(R,r))}$, which completes the proof. □

We will show now that some information on the order of rules other than communication, split and contraction rules can be derived. For this we define an extension of the total order on CDLabels(R).

Definition A.1. *Let $(R, \preceq)$ be in $\mathcal{X}$. We extend the total order $\preceq$ on* CDLabels(R) *with two elements $-\infty$ and $+\infty$ in the natural way as a new minimum and maximum, respectively, and define* $\min \emptyset = +\infty$ *and* $\max \emptyset = -\infty$.

For each label $n \in$ Labels(R) *we define two subsets $n^\uparrow$ and $n^\downarrow$ of* DLabels(R) *as follows:*

$$n^\uparrow = \{d \in \mathrm{DLabels}(R) \mid \exists u, v \in R,\ v{:}n \to^*_{\mathrm{t},R} u{:}d\}$$
$$n^\downarrow = \{d \in \mathrm{DLabels}(R) \mid \exists u, v \in R,\ u{:}d \to^*_{\mathrm{t},R} v{:}n\}$$

That is we collect in $n^\uparrow$ all dual labels that are reachable with tree edges from any occurrence of a rule labeled with n, and conversely for $n^\downarrow$.

Furthermore, we define labels n^+ and $n^- \in$ DLabels(R) *as follows:*

$$n^+ = \min n^\uparrow \qquad\qquad n^- = \max n^\downarrow$$

Lemma A.4. *Let $(R, \preceq)$ be in $\mathcal{X}$ and v a rule occurrence in R with n label of v. If $n \in$* DLabels(R) *then $n^- = n^+ = n$. If $n \notin$* DLabels(R) *then $n^- \prec n^+$.*

Proof. If $n \in \mathrm{DLabels}(R)$, we obtain using $\mathcal{X}$-condition 4 (label order) that $n^+ = \min n^\uparrow = n$ and $n^- = \max n^\downarrow = n$.

In case $n \notin \mathrm{DLabels}(R)$ we obtain, since $n^+ = \min n^\uparrow$, that there are rule occurrences $v{:}n$ and $v'{:}n^+$ in R such that $v \to^*_{\mathrm{t},R} v'$. Similarly, there are rule occurrences $u{:}n$ and $u'{:}n^-$ in R such that $u' \to^*_{\mathrm{t},R} u$. Due to $\mathcal{X}$-condition 2 (consistent labels), there has to be another rule occurrence v'' that is labeled with n^- such that $v'' \to^*_{\mathrm{t},R} v$. Thus we obtain

$$v'' \to^*_{\mathrm{t},R} v \to^*_{\mathrm{t},R} v'$$

and because v is not a dual label, there has to be at least one tree edge in the above chain, from which we obtain that $n^- \prec n^+$ using $\mathcal{X}$-condition 4 (label order).

We assumed above that both $n^\uparrow$ and $n^\downarrow$ are non-empty; the case that some are empty follows immediately from the definitions of n^+ and n^+. □

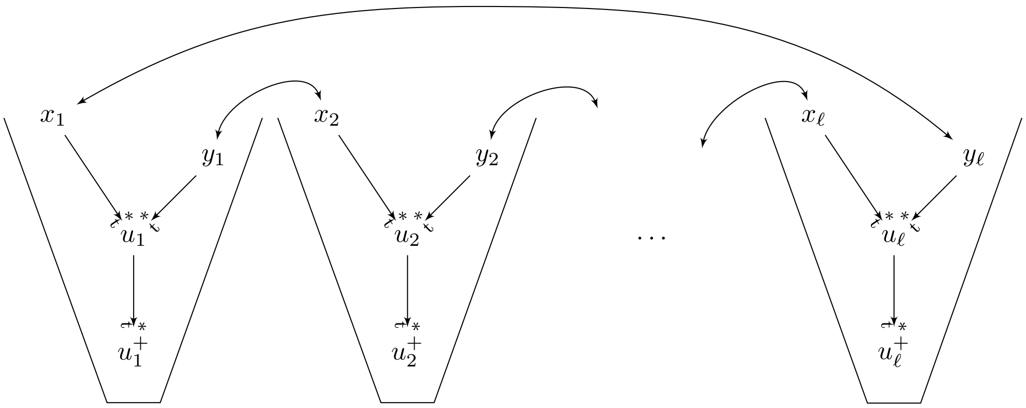
Lemma 9. *Let R be a prehyper deduction, and let $\mathcal{G} = \mathrm{Cut}(R)$. Assume that there are vertices $x_1 \ldots, x_\ell, y_1, \ldots, y_\ell$ in $\mathcal{G}$ carrying dual labels, such that $x_i \leftrightarrow^*_{\mathrm{t}} y_i$ and $y_i \leftrightarrow_{\mathrm{c}} x_{i+1}$, for $i = 1, \ldots, \ell$, where we let $x_{\ell+1}$ denote x_1. Assume that the set of labels of the x_i, y_i contains at least three elements (that is there are at least two different labels modulo duality). Furthermore, if x_i is reachable from y_i via tree edges, and $x_i \neq y_i$, then x_i is not a splitting label; similar for the roles of y_i and x_i swapped.*

Then R is not an $\mathcal{X}$ deduction.

Proof. The reachability relations (like $\to^*_{\mathrm{t}}$) are always over $\mathcal{G}$ unless otherwise indicated. We show the theorem by induction on ℓ. For sake of contradiction we assume that R is in $\mathcal{X}$.

Our assumption, that the set of labels of the x_i, y_i contains at least three elements, implies that $\ell > 1$. Let u_i be the first with respect to $\to_{\mathrm{t}}$ such that $x_i \to^*_{\mathrm{t}} u_i$ and $y_i \to^*_{\mathrm{t}} u_i$, for $i = 1, \ldots, \ell$. Let r_i be the label of u_i. Recall that r^+ for non-dual labels indicates the minimal dual label with respect to $\to_{\mathrm{t}}$ after any occurrence of a rule labeled r, see Definition A.1. Also recall that $d^+ = d$ for $d \in \mathrm{DLabels}(R)$.

Consider r_i^+: If $r_i^+ \neq +\infty$, then by definition we only know that there are occurrence $v{:}r_i$ and $u_i^+{:}r_i^+$ such that $v \to^*_{\mathrm{t}} u_i^+$. By $\mathcal{X}$-condition 2 (consistent labels) there are rules x'_i and y'_i with the same labels as x_i and y_i, respectively, occurring in the tree above v. We can obtain a chain of the same length via the tree containing x'_i, y'_i, v, and u_i^+. Thus, we assume in the following that $u_i \to^*_{\mathrm{t}} u_i^+$, that is the vertex that is labeled with r_i^+ is reachable from u_i with tree edges. We are in the following situation:



If two of the r_i^+ different from $+\infty$ are the same, then we can obtain a shorter cycle by omitting the part between two occurrences and apply the induction hypothesis. If two of the r_i^+ equal $+\infty$, then we obtain a contradiction to $\mathcal{X}$-condition 5 (indep. premises) for either of the corresponding u_i^+'s. Thus, we can assume that the r_i^+ are pairwise distinct.

Let m be maximal amongst all r_i^+ (a second maximal element may be $\bar{m}$). By the previous paragraph, there is exactly one r_i^+ with $m = r_i^+$. Without loss of generality assume that $m = r_1^+$.

Case 1) If m is the label of y_1, then $y_1 = u_1 = u_1^+$ since m is the label of u_1^+. Furthermore, due to the assumption of the Theorem, $\bar{m}$ is the label of x_2. If m is maximal, so is $\bar{m}$, thus $x_2 = u_2 = u_2^+$.

If $x_1 = y_1$ then $\bar{m}$ would have to occur twice, which we have excluded above. Similarly if $x_2 = y_2$. Thus we can choose v and w as premises of y_1 and x_2 such that $x_1 \to_{\mathrm{t}}^* v \to_{\mathrm{t}} y_1$ and $y_2 \to_{\mathrm{t}}^* w \to_{\mathrm{t}} x_2$, obtaining the following situation:

$$\begin{tikzcd}
x_1 \arrow[d, "{\mathrm{t}}^*"] & \bar{y}_2 \arrow[d, "{\mathrm{t}}^*"] \\
v \arrow[d, "\mathrm{t}"] & w \arrow[d, "\mathrm{t}"] \\
y_1{:}m & x_2{:}\bar{m}
\end{tikzcd}$$

By assumption, m cannot be a splitting label, so it has to be a communication label.

We first consider $\ell = 2$: In this case there are two pairs of dual labels (x_1, y_2) and (x_2, y_1). But then v and w are in the same connected component in $\mathrm{Cut}(\mathrm{Drop}(R, m))$ due to x_1 being dual to y_2, contradicting $\mathcal{X}$-condition 5 (indep. premises).

If $\ell \geq 3$ we show that $u_3, \dots, u_\ell$ are in $\mathrm{Cut}(\mathrm{Drop}(R, m))$, hence the connected components of v and w in $\mathrm{Cut}(\mathrm{Drop}(R, m))$ are the same, contradicting again $\mathcal{X}$-condition 5 (indep. premises).

To show the above, assume without loss of generality that $u_3 \notin \mathrm{Cut}(\mathrm{Drop}(R, m))$. Thus there is another rule occurrence $x{:}m$ or $x{:}\bar{m}$ such that $x \to_R^* u_3 \to_{\mathrm{t},R}^* r_3^+{:}u_3^+$. Thus we have that $m \leq r_3^+$, which is in contradiction to the maximality and uniqueness of m.

Case 2) If m is the label of x_1, then $\bar{m}$ is the label of y_ℓ and a similar proof as in Case 1) applies.

Case 3) If m is neither the label of x_1 nor y_1, then we are in the case that u_1 is a non-unary logical rule, and m is the label of u_1^+. Let v and w be two premises of u_1 such that $x_1 \to_{\mathrm{t}}^* v \to_{\mathrm{t}} u_1$ and $y_1 \to_{\mathrm{t}}^* w \to_{\mathrm{t}} u_1$. Note that u_1 could be a ternary logical rule and we have to choose the correct premises.

$$\begin{tikzcd}
x_1 \arrow[dr, "*", "\mathrm{t}"'] & & & & y_1 \arrow[dl, "*"', "\mathrm{t}"] \\
& v \arrow[dr, "\mathrm{t}"'] & & w \arrow[dl, "\mathrm{t}"] & \\
& & u_1 \arrow[d, "*", "\mathrm{t}"'] & & \\
& & u_1^+ & &
\end{tikzcd}$$

We show u_2,...,u_ℓ are in $\mathrm{Cut}(\mathrm{Drop}(R, m))$, hence the connected components of v and w in $\mathrm{Cut}(\mathrm{Drop}(R, m))$ are the same, contradicting again $\mathcal{X}$-condition 5 (indep. premises).

To show the above, assume without loss of generality that $u_2 \notin \mathrm{Cut}(\mathrm{Drop}(R, m))$. Thus there is another rule occurrence $x{:}m$ such that $x \to_R^* u_2 \to_{\mathrm{t},R}^* u_2^+{:}r_2^+$. Thus we have that $m \leq r_2^+$, which is in contradiction to maximality and uniqueness of m.

This completes the proof. □

Lemma 10. *Let R be in $\mathcal{X}$. Let p be an irregular path in* Cut(R)*, such that neither the first step is a $\to_{\mathrm{t}}$-step, nor the last a $\leftarrow_{\mathrm{t}}$-step.*

Then we can find a shorter regular path p' in Cut(R) *with the same start and end vertex as p.*

Proof. The proof is by induction on the number of irregular steps within an undirected path. In the critical case, assume that the first occurrence of an irregular step on some path p is of form 3 in Definition 24:

$$u_1 - p_1 - u_i \to_{\mathrm{t}} u_{i+1}{:}s \leftrightarrow_{\mathrm{c}} u_{i+2}{:}\bar{s} - p_2 - u_k$$

for some i and splitting label s. As the first step in p is not a $\to_{\mathrm{t}}$-step, u_1 and u_i are in different trees in $\mathrm{Cut}(R)$. Hence there must be some $j < i$ such that p_1 has the form

$$u_1 - p_1' - u_j \leftrightarrow_{\mathrm{c}} u_{j+1} \to_{\mathrm{t}}^* u_i \to_{\mathrm{t}} u_{i+1}$$

By $\mathcal{X}$-condition 3 (consistent split), the derivations leading to u_{i+1} and u_{i+2} are the same. Hence we can find

$$u_{j+1}' \to_{\mathrm{t}}^* u_i' \to_{\mathrm{t}} u_{i+2}$$

such that u_ν and u'_ν carry the same label, for $\nu = j{+}1, \dots, i$. Thus

$$u_1 - p'_1 - u_j \leftrightarrow_{\mathrm{c}} u'_{j+1} \rightarrow^*_{\mathrm{t}} u'_i \rightarrow_{\mathrm{t}} u_{i+2}$$

is also a path in $\mathrm{Cut}(R)$, containing one less irregular step than p. □

Lemma A.5. *Let R be in $\mathcal{X}$. Let u and r be rule occurrences in R, such that there is a undirected path connecting u and r in* $\mathrm{Cut}(R)$.

Then there is a rule occurrence r' in R (r' may be the same as r) carrying the same label as r, such that u and r' are connected in $\mathrm{Cut}(R)$ *with a regular path.*

Proof. Let p be the path in $\mathrm{Cut}(R)$ connecting u and r: $r - p - u$. We can prove by induction on the number of irregular steps within p that we can find another rule occurrence r' in R carrying the same label as r, such that u is reachable from r' in $\mathrm{Cut}(R)$ with a regular path.

In the critical case, assume that the first occurrence of an irregular step on p happens in the same tree in $\mathrm{Cut}(R)$ in which r occurs – the case that it is occurring in a different tree is dealt with a proof similar to the previous one. That is, p is of the form

$$r \rightarrow^*_{\mathrm{t}} u_1 \rightarrow_{\mathrm{t}} u_2{:}s \leftrightarrow_{\mathrm{c}} u_3{:}\bar{s} - p' - u$$

for some splitting label s (the case that the first irregular step is based on communication is done similarly).

By $\mathcal{X}$-condition 3 (consistent split), the derivations leading to u_2 and u_3 are the same. Hence we can find

$$r' \rightarrow^*_{\mathrm{t}} u'_1 \rightarrow_{\mathrm{t}} u_3$$

such that r and r', and u_1 and u'_1, respectively, carry the same label. Thus

$$r' \rightarrow^*_{\mathrm{t}} u'_1 \rightarrow_{\mathrm{t}} u_3 - p' - u$$

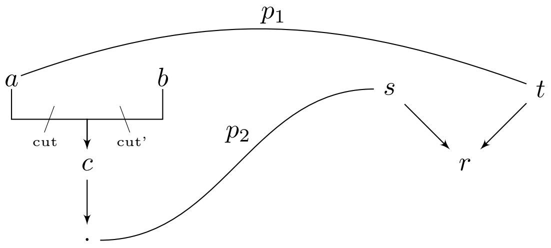
is also a path in $\mathrm{Cut}(R)$, containing one less irregular step than p. □

Lemma 11. *The $\mathcal{X}$ conditions are preserved if the* Cut *operation cuts randomly either the left or right upper edge of a contraction rule occurrence, as long as contraction rule occurrences with the same label are cut on the same side.*

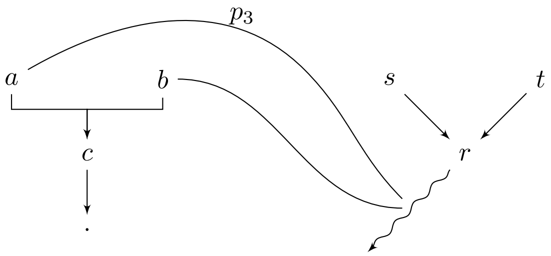
Proof. We show that if we switch the cutting position of all occurrences of contractions with the same label c to a different side, the property of being in $\mathcal{X}$ is preserved. The only two properties we need to check are those where the Cut operation is employed, that is conditions 5 (indep. premises) and 6 (contr. premise).

Assume that the contraction has premises a and b and that the cut operation drops the edge (a, c). In the following we will indicate with Cut' the graph where we drop (b, c) instead of (a, c) for all occurrences of c.

For $\mathcal{X}$-condition 5 (indep. premises), we consider a non-unary logical rule r with premises s and t, and assume that s and t are connected in $\mathrm{Cut}'(\mathrm{Drop}(R,r))$, but not connected in $\mathrm{Cut}(\mathrm{Drop}(R,r))$. Thus, there is a path p_1 in $\mathrm{Cut}'(\mathrm{Drop}(R,r))$ from t to a: $t - p_1 - a$, and another path p_2 in $\mathrm{Cut}'(\mathrm{Drop}(R,r))$ from c to s: $c - p_2 - s$ such that their combination exhibits the dependency of s and t in $\mathrm{Cut}'(\mathrm{Drop}(R,r))$ via $t - p_1 - a \to_{\mathrm{t}} c - p_2 - s$.

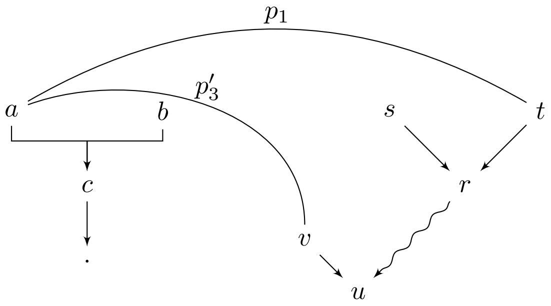


Furthermore, due to $\mathcal{X}$-condition 6 (contr. premise), there is a path p_3 in $\mathrm{Cut}(\mathrm{Drop}(R,c))$ from a to b: $a - p_3 - b$.



First assume that p_3 is also a path in $\mathrm{Cut}(\mathrm{Drop}(R,r))$. Then we can combine the paths to $t - p_1 - a - p_3 - b \to_{\mathrm{t}} c - p_2 - s$ which is a path in $\mathrm{Cut}(\mathrm{Drop}(R,r))$, contradicting the assumption that s and t are not connected in $\mathrm{Cut}(\mathrm{Drop}(R,r))$.

If this is not the case, there is a rule occurrence u in p_3 such that $r \to^*_{\mathrm{Cut}(R)} u$, as this is the part that is dropped to obtain $\mathrm{Drop}(R,r)$. Choose u and v such that p_3 is of the form $a - p_3' - v \to_{\mathrm{t}} u - p_3'' - b$ or $b - p_3' - v \to_{\mathrm{t}} u - p_3'' - a$, and p_3' is a path in $\mathrm{Cut}(\mathrm{Drop}(R,r))$, which is possible since a, b, and c are in $\mathrm{Cut}(\mathrm{Drop}(R,r))$.



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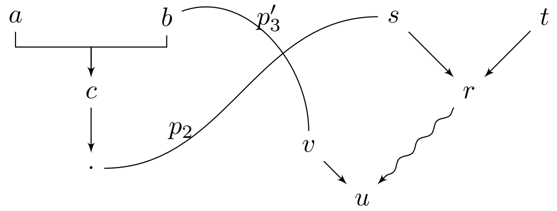
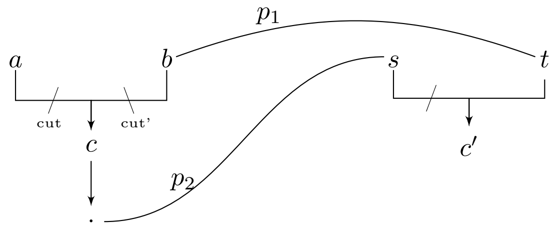


By Lemma A.5 we can assume that the path from r to u in $\mathrm{Cut}(R)$ is regular, because if not then we can find some r' carrying the same label as r, such that u can be reached from r' via a regular path. Then consider the same situation with the premises of r' in place of s and t.

As v is not reachable from r in $\mathrm{Cut}(R)$, u must be a non-unary logical rule, or communication. Let w be the other premise of u on path from p_3''. Then v and w are in the same connected component in $\mathrm{Cut}(\mathrm{Drop}(R, u))$ contradicting $\mathcal{X}$-condition 5 (indep. premises), either via $w - r - t - p_1 - a - p_3' - v$ or $w - r - s - p_2 - c - b - p_3' - v$.

Consider now $\mathcal{X}$-condition 6 (contr. premise), and let c_1 and c_2 be further contraction occurrences with $c_1 \preceq c_2$, s and t the premises of c_1 such that s and t are connected in $\mathrm{Cut}(\mathrm{Drop}(R, \{c_1, c_2\}))$ but not in $\mathrm{Cut}'(\mathrm{Drop}(R, \{c_1, c_2\}))$. Thus, there are paths p_1 and p_2 in $\mathrm{Cut}'(\mathrm{Drop}(R, r))$ connecting s and t to b and c such that their combination exhibits the dependency of s and t in $\mathrm{Cut}(\mathrm{Drop}(R, r))$ via either $t - p_1 - b \rightarrow_{\mathrm{t}} c - p_2 - s$ or $s - p_1 - b \rightarrow_{\mathrm{t}} c - p_2 - t$.



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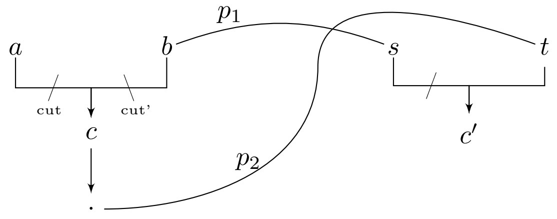
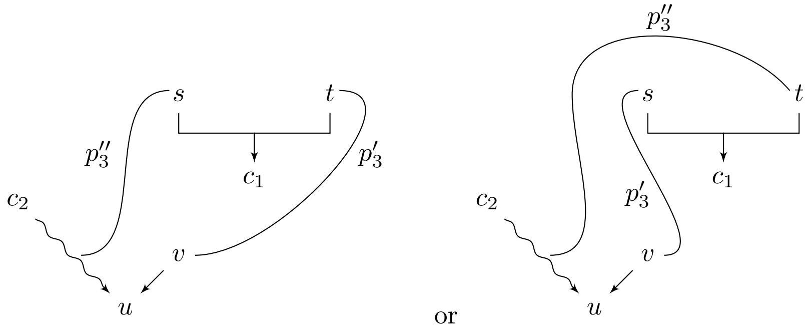


If $c \preceq c_2$, $\mathcal{X}$-condition 6 (contr. premise) yields a path p_3 in $\mathrm{Cut}(\mathrm{Drop}(R, \{c, c_2\}))$ from a to b: $a - p_3 - b$. Similar to the previous case we obtain a contradiction as

p_3 is not a path in $\mathrm{Cut}(\mathrm{Drop}(R, \{c_1, c_2\}))$, and hence there is some u in p_3 such that $c_1 \rightarrow^*_{\mathrm{Cut}(R)} u$ — c_1 now plays the role of r in the previous case.

Otherwise, $c_2 \prec c$, and we can use $\mathcal{X}$-condition 6 (contr. premise) again to find a path p_3 from s to t in $\mathrm{Cut}(\mathrm{Drop}(R, \{c_1, c\}))$. Obviously p_3 is also a path in $\mathrm{Cut}'(\mathrm{Drop}(R, \{c_1, c\}))$ as c is dropped. p_3 is not a path in $\mathrm{Cut}'(\mathrm{Drop}(R, \{c_1, c_2\}))$ as s and t are not connected in this graph by assumption. Similar to the previous cases, we can find u and v on p_3 such that p_3 is of the form $s - p'_3 - v - u - p''_3 - t$ or $t - p'_3 - v - u - p''_3 - s$, p'_3 is a path in $\mathrm{Cut}(\mathrm{Drop}(R, \{c, c_1, c_2\}))$ and p''_3 is a path in $\mathrm{Cut}(\mathrm{Drop}(R, u))$.



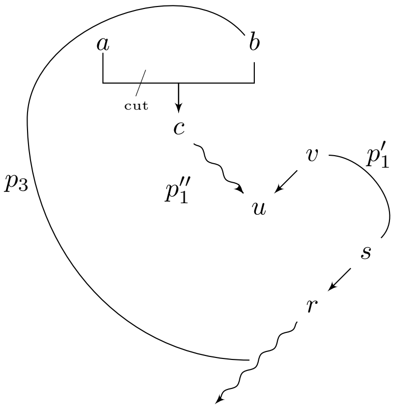
As before, we can assume that u is a non-unary logical rule, or communication, and u's premises are in the same connected component in $\mathrm{Cut}(\mathrm{Drop}(R, u))$ contradicting $\mathcal{X}$-condition 5 (indep. premises), either via $v - p'_3 - s - p_2 - p_1 - t - p''_3 - u$ or $v - p'_3 - t - p_1 - p_2 - s - p''_3 - u$. □

Lemma A.6. *Let R be in $\mathcal{X}$, and r be any rule occurrence, and s one of r's premises. Let C be the connected component of s in $\mathrm{Cut}(\mathrm{Drop}(R, r))$, $C = [s]_{\mathrm{Cut}(\mathrm{Drop}(R,r))}$. Let $c \in C$ be a contraction occurring in C. Denote the premises of c with a and b, and assume that $b \in C$ but $a \notin C$. This in particular implies that edge (b, c) is present in $\mathrm{Cut}(R)$, but (a, c) is removed.*

Any path from s to c in $\mathrm{Cut}(\mathrm{Drop}(R, r))$ contains the edge (b, c).

Proof. Assume otherwise. Then there is a path p_1 from s to c in $\mathrm{Cut}(\mathrm{Drop}(R, r))$ which does not contain the edge (b, c). By $\mathcal{X}$-condition 6 (contr. premise) there is a path p_2 in $\mathrm{Cut}(\mathrm{Drop}(R, c))$ connecting a and b. p_2 is not a path in $\mathrm{Cut}(\mathrm{Drop}(R, r))$ as $a \notin C$. Furthermore, s is not reachable from c as otherwise $\mathrm{Cut}(\mathrm{Drop}(R, c))$ would be a subgraph of $\mathrm{Cut}(\mathrm{Drop}(R, r))$, contradicting that p_2 is a path in $\mathrm{Cut}(\mathrm{Drop}(R, c))$ but not in $\mathrm{Cut}(\mathrm{Drop}(R, r))$.

Thus, we obtain a path p_3 connecting b and s in $\mathrm{Cut}(\mathrm{Drop}(R, c))$. Furthermore, there are vertices u and v such that p_1 is of the form $s - p'_1 - v \rightarrow_{\mathrm{t}} u - p''_1 - c$, u is reachable from c in $\mathrm{Cut}(R)$ and p'_1 is a path in $\mathrm{Cut}(\mathrm{Drop}(R, c))$.



As in the proof of the previous theorem, u can be chosen to be a non-unary logical rule, or communication. Let w be another premise of u besides v, which is reachable from c. Then v and w are in the same connected component in $\mathrm{Cut}(\mathrm{Drop}(R, u))$ contradicting $\mathcal{X}$-condition 5 (indep. premises), via the path $v - p_1' - s - p_3 - b - c - p_1'' - w$. □

Lemma 12. *Let R be in $\mathcal{X}$, and r a rule occurrence in R. Let s be one of the premises of r. Then there is an alternative cut operation Cut^* such that the following property holds: Let C be the connected component of s in $\mathrm{Cut}^*(\mathrm{Drop}(R, r))$, that is $C = [s]_{\mathrm{Cut}^*(\mathrm{Drop}(R,r))}$.*
(*) For any contraction c in R with premises a and b we have $c \in C$ iff $a, b \in C$.

Proof. We construct a series of alternative cuts, where at each step exactly one cut of a contraction is switched. Let C be the connected component of s at the current step in $\mathrm{Cut}(\mathrm{Drop}(R, r))$. Let $c_1, \ldots, c_n$ be the set of contractions at the current step that do not fulfill the property that $c_i \in C$ iff both premises of c_i are in C. We show by induction on $\prec$ that we can remove the contraction with the smallest label amongst $c_1, \ldots, c_n$ by changing the cut for this contraction. Without loss of generality assume c_1 carries the minimal label of all the c_i. In particular, by $\mathcal{X}$-condition 6 (contr. premise), the property (*) for any other contraction $c \prec c_1$ is not effected by the choice which premise of c_1 is cut.

By Lemma 11, we can change the cut side at c_1, which we denote by Cut', obtaining again a $\mathcal{X}$ deduction. Using Lemma A.6, changing to Cut' removes c_1 from the connected component C' of s in $\mathrm{Cut}'(\mathrm{Drop}(R, r))$. Thus we obtain a new set of contractions not satisfying (*) w.r.t. C' where the minimal label in this set has increased. By finiteness this procedure terminates. □