



# A generalisation of the Malgrange–Ehrenpreis theorem to find fundamental solutions to fractional PDEs

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Received XX XXXXXX 201X, appeared XX XXXXXX 20XX

Communicated by Handling Editor

**Abstract.** We present and prove a new generalisation of the Malgrange–Ehrenpreis theorem to fractional partial differential equations, which can be used to construct fundamental solutions to all partial differential operators of rational order and many of arbitrary real order. We demonstrate with some examples and mention a few possible applications.

**Keywords:** fractional derivatives, fundamental solutions, linear partial differential equations, constructive solutions.


**2010 Mathematics Subject Classification:** 26A33, 35E05.

## 1 Introduction

Fractional calculus provides a natural generalisation of ordinary derivatives and integrals to non-integer orders. This field of study has both a long history – dating back to Leibniz himself – and also many applications in mathematics, physics, and engineering [4, 5], including in random walks [28], chaos theory [29], viscoelasticity [14], fluid drainage [17], polymer science [11], biophysics [10], thermodynamics [27], and transport in media [6]. Unfortunately, there are several conflicting formulae for fractional derivatives and integrals, so it is always necessary to make clear which definition is being used. One of the most common is the *Riemann–Liouville* formula, given by Definition 1.1 (see [18, Chapters III–IV]); this definition is the one assumed to be used throughout this paper unless otherwise stated.

Indefinite integration is only well-defined up to an additive constant, and in the fractional context such a constant must also be introduced for differentiation. Note that when the order of differentiation and integration becomes a continuum, the difference between the two is often not clear-cut, and the term *differintegration* is used to cover both. When distinction is necessary, the difference between derivatives and integrals is now defined in terms of the *real part* of the order of differintegration.

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**Definition 1.1 (Riemann–Liouville fractional differintegral).** Let  $x$  and  $\nu$  be complex variables, and  $c$  be a constant in the extended complex plane (usually taken to be either 0 or  $-\infty$ ). The  $\nu$ th derivative, or  $(-\nu)$ th integral, of a function  $f(x)$  with respect to  $x$ , with constant of differintegration  $c$ , is:

$${}_c D_x^\nu f(x) := \frac{1}{\Gamma(-\nu)} \int_c^x (x-y)^{-\nu-1} f(y) dy \quad \text{for } \operatorname{Re}(\nu) < 0 \quad (1.1)$$

$${}_c D_x^\nu f(x) := \frac{d^n}{dx^n} ({}_c D_x^{\nu-n} f(x)), \quad n := \lfloor \operatorname{Re}(\nu) \rfloor + 1, \quad \text{for } \operatorname{Re}(\nu) \geq 0 \quad (1.2)$$

provided these expressions are well-defined.

Since  $x$ ,  $\nu$ , and  $c$  are defined in the complex plane, it is necessary to consider the issue of which path to integrate along from  $c$  to  $x$  and which branch to use for defining the function  $(x-y)^{-\nu-1}$  for  $y$  on this path. Usually the straight line-segment contour  $[c, x]$  is used, meaning that  $\arg(x-y)$  is always equal to  $\arg(x-c)$  independent of  $y$ . The choice of range for  $\arg(x-c)$  usually depends on context, and the essential properties of Riemann–Liouville differintegrals remain unchanged whether we assume  $\arg(x-c) \in [0, 2\pi)$  or  $\arg(x-c) \in (-\pi, \pi)$  or any other range. These issues are covered in [26, §22]. When all variables are real, as is often the case, most of these problems do not have to arise.

The constant of differintegration  $c$  tends to be fixed at either 0 or  $-\infty$ ; other possibilities for  $c$  can usually be covered by the same arguments that work for these two cases. Note in particular that when  $c = -\infty$ , we can always take  $\arg(x-c)$  to be 0, eliminating the problems of the previous paragraph.

The Riemann–Liouville fractional integral (1.1) is a natural generalisation of Cauchy’s formula for repeated integration; see [18, Chapter II]. For analytic functions  $f$ , the Riemann–Liouville fractional derivative (1.2) is exactly the extension of (1.1) by analytic continuation in  $\nu$ ; this follows from the fact that (1.1) gives  $\frac{d}{dx} ({}_c D_x^\nu f(x)) = {}_c D_x^{\nu+1} f(x)$  for  $\operatorname{Re}(\nu) < -1$  already, so (1.2) will preserve analyticity as an extension of (1.1).

The Riemann–Liouville definition of fractional differintegrals is particularly useful because it meshes together well with Fourier and Laplace transforms, as established by the following lemmas, both proved in [26, §7].

**Lemma 1.2 (Fourier transforms).** *If  $f(x)$  is a function with well-defined Fourier transform  $\hat{f}(\lambda)$  and  $\nu \in \mathbb{C}$  is such that  ${}_{-\infty} D_x^\nu f(x)$  is well-defined, then the Fourier transform of  ${}_{-\infty} D_x^\nu f(x)$  is  $(-i\lambda)^\nu \hat{f}(\lambda)$ .*

**Lemma 1.3 (Laplace transforms).** *If  $f(x)$  is a function with well-defined Laplace transform  $\tilde{f}(\lambda)$  and  $\nu \in \mathbb{C}$  is such that  $\operatorname{Re}(\nu) < 0$  and  ${}_0 D_x^\nu f(x)$  is well-defined, then the Laplace transform of  ${}_0 D_x^\nu f(x)$  is  $(-i\lambda)^\nu \tilde{f}(\lambda)$ .*

Some of the standard properties of integer-order differintegrals extend readily to the fractional case. For instance, it is clear (see also [19, Chapter 5]) that in all the above definitions, for any fixed  $\nu$  and  $c$ ,  ${}_c D_x^\nu$  is still a linear operator acting on functions  $f(x)$ .

Other standard properties cannot be extended. For instance, composition of differential operators no longer works in the same way: the fractional derivative of a fractional derivative is not necessarily a fractional derivative, although the fractional integral of a fractional integral is still a fractional integral. A summary of results on the composition of differintegrals is presented in the following two lemmas, both proved in [18, Chapters III–IV].

**Lemma 1.4 (Composition of fractional integrals).** *For any  $x, \mu, \nu \in \mathbb{C}$  with  $\operatorname{Re}(\mu) < 0$ , the identity  ${}_c D_x^\nu ({}_c D_x^\mu f(x)) = {}_c D_x^{\mu+\nu} f(x)$  holds provided these differintegrals exist.*

**Lemma 1.5.** *If  $n \in \mathbb{N}$  and  $f$  is a  $C^n$  function such that one of the three expressions  ${}_c D_x^n({}_c D_x^\mu f(x))$ ,  ${}_c D_x^{n+\mu} f(x)$ ,  ${}_c D_x^\mu({}_c D_x^n f(x))$  exists, then all three exist and*

$${}_c D_x^n({}_c D_x^\mu f(x)) = {}_c D_x^{n+\mu} f(x) = {}_c D_x^\mu({}_c D_x^n f(x)) + \sum_{k=1}^n \frac{(x-c)^{-\mu-k}}{\Gamma(-\mu-k+1)} f^{(n-k)}(c).$$

As a consequence of Lemma 1.5, if the constant of differintegration is  $c = -\infty$  and  $f$  has sufficient decay conditions at infinity (say, if  $f$  is a Schwartz function), then composition of differintegral operators does work as in the classical scenario. In this case, the Riemann–Liouville and Caputo definitions of fractional derivatives are equivalent, although in general they are not.

Still other standard properties *can* be extended from integer-order calculus to fractional calculus, but with more difficulty than simple ones such as linearity, and sometimes yielding less elegant or less general final results. For instance, the product rule [21], the chain rule [22], and Taylor’s theorem [25] all have fractional versions, but most of these are either much messier than the original or require more restrictions on the functions concerned. The following basic generalisation of the product rule is also due to Osler [23].

**Lemma 1.6 (The fractional product rule).** *Let  $u$  and  $v$  be complex functions such that  $u(x)$ ,  $v(x)$ , and  $u(x)v(x)$  are all functions of the form  $x^\lambda \eta(x)$  with  $\operatorname{Re}(\lambda) > -1$  and  $\eta$  analytic on a domain  $R \subset \mathbb{C}$ . Then for any distinct  $x, c \in R$  and any  $\nu \in \mathbb{C}$ , we have*

$${}_c D_x^\nu(u(x)v(x)) = \sum_{n=0}^{\infty} \binom{\nu}{n} {}_c D_x^{\nu-n} u(x) {}_c D_x^n v(x),$$

where all differintegrals are defined using the Cauchy formula.

In [24], Osler proved that the series can also be replaced by an integral to get the following formula:

$${}_c D_x^\nu(u(x)v(x)) = \int_{-\infty}^{\infty} \binom{\nu}{\omega} {}_c D_x^{\nu-\omega} u(x) {}_c D_x^\omega v(x) d\omega.$$

These lemmas and discussion will be used in the proof of our main result below.

## 2 The fractional Malgrange–Ehrenpreis theorem

An important result in the theory of PDEs is the Malgrange–Ehrenpreis theorem, which guarantees the existence of a fundamental solution for any linear partial differential operator with constant coefficients. This is significant because the solution to a given PDE with arbitrary forcing can be generated from the solution with delta-function forcing, i.e. from the fundamental solution, by using convolution of functions. So by proving the existence of fundamental solutions, the Malgrange–Ehrenpreis theorem guarantees the existence of solutions to any linear PDE with constant coefficients on the derivative terms and arbitrary forcing.

Here we seek to extend this theorem in order to find fundamental solutions for fractional partial differential operators defined using the Riemann–Liouville formula.

**Theorem 2.1 (Malgrange–Ehrenpreis theorem).** *Every non-zero linear constant-coefficient partial differential operator, i.e. every operator  $P(D)$  where  $P$  is a complex  $n$ -variable  $N$ th-order polynomial and  $D := -i \frac{\partial}{\partial x}$  for the  $n$ -dimensional variable  $x \in \mathbb{R}^n$ , has a fundamental solution, i.e. a distribution  $E \in \mathcal{D}'(\mathbb{R}^n)$  such that  $P(D)E = \delta_0$ .*

The original proofs of this result by Malgrange [16] and Ehrenpreis [9], who proved it independently, were non-constructive and used the Hahn–Banach theorem. But several constructive proofs have since been devised, and some of these can be extended to certain subcases of the fractional context in order to prove generalisations of the theorem.

## 2.1 Proof using Hörmander staircases

One semi-constructive proof due to Hörmander [12] involves building a solution by using complex integration over a Hörmander staircase.

This proof, like many others, relies on the fact that  $P$  is a polynomial: we need to use the Fundamental Theorem of Algebra to factorise  $P(\lambda)$  into linear terms of the form  $\lambda_n - f_j(\lambda_1, \dots, \lambda_{n-1})$  and then analyse these linear factors. If  $P(\lambda)$  is a general function of the form  $\sum_{\alpha} c_{\alpha} \lambda^{\alpha}$  where the exponents  $\alpha$  may be real or complex, then the Fundamental Theorem of Algebra no longer applies.

However, if all the  $\alpha$  are *rational*, the proof can be modified so that it still works. Instead of considering the differential operator  $P(D)$  as a polynomial in  $D$ , we can consider it as a polynomial in  $D^{1/K}$  for some sufficiently large natural number  $K$ , factorise this polynomial using the Fundamental Theorem of Algebra, and proceed more or less as before. In fact, since Hörmander’s proof never uses the fact that  $P(D)$  is a polynomial in  $D_1, D_2, \dots, D_{n-1}$ , it will suffice to assume only that (for example) all the final components  $\alpha_n$  are rational, as in the following theorem.

**Theorem 2.2 (Malgrange–Ehrenpreis theorem: rational-order derivatives).** *Let  $P(\lambda)$  be a function of the complex  $n$ -dimensional parameter  $\lambda$  of the form  $\sum_{\alpha} c_{\alpha} \lambda^{\alpha}$  where the sum is finite, the multi-indices  $\alpha$  are in  $(\mathbb{R}^+)^n$ , and there exists  $j$  such that all the  $j$ th coordinates  $\alpha_j$  of the multi-indices are in  $\mathbb{Q}$ . If  $x \in \mathbb{R}^n$  is an  $n$ -dimensional variable and powers of  $D := -i \frac{\partial}{\partial x}$  are defined using the Riemann–Liouville formula with  $c = -\infty$ , then the partial differential operator  $P(D)$  has a fundamental solution, i.e. a distribution  $E \in \mathcal{D}'(\mathbb{R}^n)$  such that  $P(D)E = \delta_0$ .*

*Proof.* Without loss of generality, say  $j = n$ . Let  $\lambda' \in \mathbb{R}^{n-1}$  denote the vector  $(\lambda_1, \lambda_2, \dots, \lambda_{n-1})$ , and note that  $P(\lambda) = P(\lambda', \lambda_n)$  can be written as a polynomial in  $\lambda_n^{1/K}$  with coefficients depending on  $\lambda'$ , where  $K$  is a fixed natural number (the LCM of the denominators of the exponents  $\alpha_n$ ). More explicitly, write

$$P(\lambda', \lambda_n) = A(\lambda') \left( \lambda_n^{M/K} + \sum_{j=0}^{M-1} a_j(\lambda') \lambda_n^{j/K} \right)$$

where  $M$  is a natural number and the  $A, a_j$  are continuous functions of  $\lambda' \in \mathbb{R}^{n-1}$ . In particular,  $A(\lambda')$  is a product of power functions  $\lambda_i^{\alpha_i}$ , so  $A(\lambda') = 0$  only if  $\lambda' = 0$ . By the Fundamental Theorem of Algebra,  $P(\lambda)$  can then be written as

$$P(\lambda', \lambda_n) = A(\lambda') \prod_{j=1}^M \left( \lambda_n^{1/K} - \tau_j(\lambda') \right)$$

where the  $\tau_j$  are continuous functions on  $\mathbb{R}^{n-1}$ . (If  $\lambda'$  were allowed to be complex, there would be complications with branch cuts, but as it is real, the  $A, a_j, \tau_j$  can be defined to be continuous.)

Fix  $\mu \in \mathbb{R}^{n-1} \setminus \{0\}$ ; we wish to bound  $P(\mu, \lambda_n)$  below, in order to get an upper bound on its reciprocal. Now let  $R = R(\mu) := \max_j |\tau_j(\mu)| + |A(\mu)|^{-1/M} + 1$  (this is in  $\mathbb{R}^+$  since

$\mu \neq 0 \Rightarrow A(\mu) \neq 0$ ). By continuity of the  $A, \tau_j$ , there exists an open neighbourhood  $N(\mu) \subset \mathbb{R}^{n-1} \setminus \{0\}$  of  $\mu$  such that for all  $\lambda' \in N(\mu)$ ,  $\max_j |\tau_j(\lambda')| + |A(\lambda')|^{-1/M} < R$ . Now whenever  $|\lambda_n^{1/K}| \geq R(\mu)$  and  $\lambda' \in N(\mu)$ , we have

$$|\lambda_n^{1/K} - \tau_j(\lambda')| > |A(\lambda')|^{-1/M}$$

for each  $j$ , and therefore

$$|P(\lambda', \lambda_n)| = |A(\lambda')| \prod_{j=1}^M |\lambda_n^{1/K} - \tau_j(\lambda')| > |A(\lambda')| \prod_{j=1}^M |A(\lambda')|^{-1/M} = 1.$$

In particular, define  $\gamma = \gamma(\mu)$  for  $\mu \in \mathbb{R}^{n-1}$  to be the black contour shown in Figure 2.1, i.e.

$$\gamma = \{re^{i\pi} : \infty > r > R^K\} \cup \{R^K e^{i\theta} : -\pi < \theta < \pi\} \cup \{r : R^K < r < \infty\}.$$

Since  $\lambda_n^{1/K}$  is on the red contour shown in Figure 2.1 when  $\lambda_n$  is on the black one, we have  $|\lambda_n^{1/K}| \geq R$  for all  $\lambda_n \in \gamma$  and therefore

$$|P(\lambda', \lambda_n)| > 1 \text{ for } \lambda' \in N(\mu), \lambda \in \gamma(\mu). \quad (2.1)$$

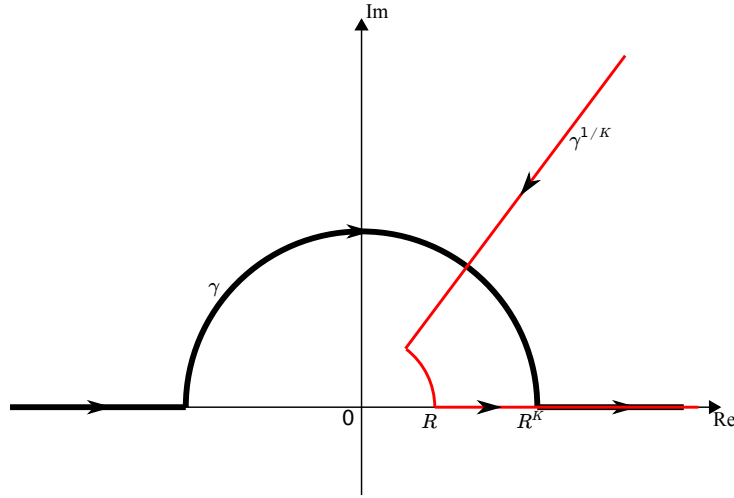


Figure 2.1: The contours for  $\gamma$  and  $\gamma^{1/K}$

The sets  $N(\mu)$  form an open cover of  $\mathbb{R}^{n-1} \setminus \{0\}$ . But  $\mathbb{R}^{n-1} \setminus \{0\}$  is an open subset of  $\mathbb{R}^{n-1}$  and therefore locally compact, and it is also  $\sigma$ -compact, so it must be a Lindelöf space, i.e. every open cover has a countable subcover. So there is a countable sequence  $\mu_1, \mu_2, \mu_3, \dots$  such that the open sets  $N(\mu_k)$  cover  $\mathbb{R}^{n-1} \setminus \{0\}$ . Let  $\Delta_k := N(\mu_k) \setminus \bigcup_{j=1}^{k-1} N(\mu_j)$  for all  $k$ ; these sets are open and disjoint and  $\bigcup_{k=1}^{\infty} \overline{\Delta_k} = \mathbb{R}^{n-1}$ .

Define  $E \in \mathcal{D}'(\mathbb{R}^n)$  by

$$\langle E, \phi \rangle = (2\pi)^{-n} \sum_{k=1}^{\infty} \int_{\Delta_k} \int_{\gamma(\mu_k)} \frac{\hat{\phi}(-\lambda', -\lambda_n)}{P(\lambda', \lambda_n)} d\lambda_n d\lambda'; \quad (2.2)$$

this is well-defined as a distribution, since (2.1) tells us that  $|P| > 1$  on all regions integrated

over. Now for any  $\phi \in \mathcal{D}(\mathbb{R}^n)$ , (2.2) implies

$$\begin{aligned} \langle P(D)E, \phi \rangle &= \langle E, P(D)\phi \rangle = (2\pi)^{-n} \sum_{k=1}^{\infty} \int_{\Delta_k} \int_{\gamma(\mu_k)} \frac{\widehat{P(D)\phi}(-\lambda', -\lambda_n)}{P(\lambda', \lambda_n)} d\lambda_n d\lambda' \\ &= (2\pi)^{-n} \sum_{k=1}^{\infty} \int_{\Delta_k} \int_{\gamma(\mu_k)} \hat{\phi}(-\lambda', -\lambda_n) d\lambda_n d\lambda' \\ &= (2\pi)^{-n} \sum_{k=1}^{\infty} \int_{\Delta_k} \int_{\mathbb{R}} \hat{\phi}(-\lambda', -\lambda_n) d\lambda_n d\lambda' \\ &= (2\pi)^{-n} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \hat{\phi}(-\lambda', -\lambda_n) d\lambda_n d\lambda' = \phi(0), \end{aligned}$$

so  $P(D)E = \delta_0$  as required. (To get from the second line to the third above, we used Cauchy's theorem and the fact that the Fourier transform  $\hat{\phi}(\lambda)$  of a test function  $\phi \in \mathcal{D}(\mathbb{R}^n)$  is analytic in each coordinate of  $\lambda$ .)  $\square$

In this way we have proved the Malgrange–Ehrenpreis theorem for all non-zero linear constant-coefficient fractional partial differential operators which are literally “fractional”, i.e. contain only rational-order differintegrals.

## 2.2 Proof using Wagner construction

A recent proof of the Malgrange–Ehrenpreis theorem due to Ortner and Wagner [20] involves constructing explicit fundamental solutions using inverse Fourier transforms.

In this proof, the fact that  $P$  is a polynomial is relevant because the binomial theorem is used to turn an expression of the form  $P(\partial + \lambda\eta)$  into a finite sum, and also because the residue theorem is used to cancel out most terms in this finite sum. The binomial theorem, in a more complicated form involving *infinite* series, can still be applied when  $P$  is not a polynomial; the residue theorem is harder to apply in this case, and so we again require an extra assumption.

**Theorem 2.3 (Malgrange–Ehrenpreis theorem: real-order derivatives with integer differences).** *Let  $P(\lambda)$  be a function of the complex  $n$ -dimensional parameter  $\lambda$  of the form  $\sum_{\alpha} c_{\alpha} \lambda^{\alpha}$  where the sum is finite, the multi-indices  $\alpha$  are in  $(\mathbb{R}^+)^n$ , and there exists  $A \in \mathbb{R}$  such that all the magnitudes  $|\alpha| = \sum_j \alpha_j$  of the multi-indices are of the form  $A - m$  for some integer  $m \geq 0$ . If  $x \in \mathbb{R}^n$  is an  $n$ -dimensional variable and powers of  $D := -i \frac{\partial}{\partial x}$  are defined using the Riemann–Liouville formula with  $c = -\infty$ , then the partial differential operator  $P(D)$  has a fundamental solution, i.e. a distribution  $E \in \mathcal{D}'(\mathbb{R}^n)$  such that  $P(D)E = \delta_0$ .*

*Proof.* We define the fundamental solution  $E$  by

$$E(x) := \frac{1}{2\pi i P_A(-i\eta)} \int_{S^1} \lambda^{A-1} e^{\lambda\eta x} \psi_{\lambda}(x) d\lambda$$

where  $\eta \in \mathbb{R}^n$  is a fixed real vector,  $P_A(\lambda) := \sum_{|\alpha|=A} c_{\alpha} \lambda^{\alpha}$  is the ‘maximum order’ part of  $P$ , and the Schwartz distribution  $\psi_{\lambda}$  is defined by its Fourier transform being

$$\widehat{\psi_{\lambda}}(\xi) = \frac{P(\xi - i\lambda\eta)}{P(\xi - i\lambda\eta)}.$$

Now there are two things we need to prove: firstly that  $E$  is a well-defined distribution, and secondly that  $P(D)E = \delta_0$ .

Firstly, note that the zero set of  $P$  in  $\mathbb{R}^n$  has Lebesgue measure zero, so  $\frac{\overline{P(\xi - i\lambda\eta)}}{P(\xi - i\lambda\eta)}$  is an  $L^\infty$  function of  $\xi \in \mathbb{R}^n$ , and therefore a Schwartz distribution, for any fixed  $\lambda \in \mathbb{C}, \eta \in \mathbb{R}$ . So  $\psi_\lambda \in \mathcal{S}'(\mathbb{R}^n)$  is well-defined. Also the map

$$\begin{aligned} \mathcal{S}^1 &\rightarrow \mathcal{S}'(\mathbb{R}^n) \\ \lambda &\mapsto \frac{\overline{P(\xi - i\lambda\eta)}}{P(\xi - i\lambda\eta)} \end{aligned}$$

is continuous, so  $E$  is the integral over a compact set of a continuous function taking values in  $\mathcal{D}'(\mathbb{R}^n)$  and therefore well-defined as an element of  $\mathcal{D}'(\mathbb{R}^n)$ .

Let us use  $\mathcal{F}$  to denote the Fourier transform from variable  $x \in \mathbb{R}^n$  to variable  $\xi \in \mathbb{R}^n$ , so that  $\psi_\lambda(x) = \mathcal{F}^{-1}\left(\frac{\overline{P(\xi - i\lambda\eta)}}{P(\xi - i\lambda\eta)}\right)$ . Now consider how the fractional partial differential operator  $P(D)$  works on a function of the form  $e^{\lambda\eta x} \mathcal{F}^{-1}S$  for some Schwartz distribution  $S$ . By Osler's product rule, Theorem 1.6, we have

$$\begin{aligned} D^\alpha(e^{\lambda\eta x} \mathcal{F}^{-1}S) &= (-i)^\alpha {}_{-\infty}D_x^\alpha(e^{\lambda\eta x} \mathcal{F}^{-1}S) \\ &= (-i)^\alpha \sum_k \binom{\alpha}{k} {}_{-\infty}D_x^{\alpha-k}(e^{\lambda\eta x}) {}_{-\infty}D_x^k(\mathcal{F}^{-1}S) \\ &= e^{\lambda\eta x} \sum_k \binom{\alpha}{k} (-i\lambda\eta)^{\alpha-k} D^k(\mathcal{F}^{-1}S) \\ &= e^{\lambda\eta x} \sum_k \binom{\alpha}{k} (-i\lambda\eta)^{\alpha-k} \mathcal{F}^{-1}(\xi^k S) \\ &= e^{\lambda\eta x} \mathcal{F}^{-1}\left(\sum_k \binom{\alpha}{k} (-i\lambda\eta)^{\alpha-k} \xi^k S\right) \\ &= e^{\lambda\eta x} \mathcal{F}^{-1}\left(\xi^\alpha \sum_k \binom{\alpha}{k} \left(\frac{-i\lambda\eta}{\xi}\right)^{\alpha-k} S\right) \\ &= e^{\lambda\eta x} \mathcal{F}^{-1}\left(\xi^\alpha \left(1 + \frac{-i\lambda\eta}{\xi}\right)^\alpha S\right) = e^{\lambda\eta x} \mathcal{F}^{-1}\left((\xi - i\lambda\eta)^\alpha S\right) \end{aligned}$$

for any multi-index  $\alpha \in (\mathbb{R}_0^+)^n$ , where the sums are taken over all multi-indices  $k \in (\mathbb{Z}_0^+)^n$ , and where we use Lemma 1.2 between the third and fourth lines. So by finite summation, it follows that

$$P(D)(e^{\lambda\eta x} \mathcal{F}^{-1}S) = e^{\lambda\eta x} \mathcal{F}^{-1}(P(\xi - i\lambda\eta)S).$$

In particular, setting  $S(\xi) = \widehat{\psi}_\lambda(\xi) = \frac{\overline{P(\xi - i\lambda\eta)}}{P(\xi - i\lambda\eta)}$  gives

$$\begin{aligned} P(D)(e^{\lambda\eta x} \psi_\lambda(x)) &= e^{\lambda\eta x} \mathcal{F}^{-1}(\overline{P(\xi - i\lambda\eta)}) = e^{\lambda\eta x} \mathcal{F}^{-1}(\overline{P(\xi + i\bar{\lambda}\eta)}) \\ &= e^{\lambda\eta x} \mathcal{F}^{-1}\left(\sum_\alpha \bar{c}_\alpha (\xi + i\bar{\lambda}\eta)^\alpha\right) \\ &= e^{\lambda\eta x} \mathcal{F}^{-1}\left(\sum_\alpha \bar{c}_\alpha \sum_k \binom{\alpha}{k} (i\bar{\lambda}\eta)^{\alpha-k} \xi^k\right) \\ &= e^{\lambda\eta x} \sum_\alpha \bar{c}_\alpha \sum_k \binom{\alpha}{k} (i\bar{\lambda}\eta)^{\alpha-k} D^k(\mathcal{F}^{-1}(\mathbb{1})) \\ &= e^{\lambda\eta x} \sum_\alpha \bar{c}_\alpha \sum_k \binom{\alpha}{k} (i\bar{\lambda}\eta)^{\alpha-k} \delta_0^{(k)}(x). \end{aligned}$$



Putting this together with the formula for  $E$ , and using the fact that  $\lambda$  is a scalar of modulus 1, we get

$$\begin{aligned} P(D)E &= \frac{1}{2\pi i P_A(-i\eta)} \int_{S^1} \lambda^{A-1} P(D)(e^{\lambda\eta x} \psi_\lambda(x)) d\lambda \\ &= \frac{1}{2\pi i P_A(-i\eta)} \int_{S^1} \lambda^{A-1} e^{\lambda\eta x} \sum_{\alpha} \bar{c}_\alpha \sum_k (i\bar{\lambda})^{|\alpha|-|k|} \binom{\alpha}{k} \eta^{\alpha-k} \delta_0^{(k)}(x) d\lambda \\ &= \frac{1}{2\pi i P_A(-i\eta)} \sum_{\alpha} \bar{c}_\alpha \sum_k \binom{\alpha}{k} (i\eta)^{\alpha-k} \left( \int_{S^1} \lambda^{A-1-|\alpha|+|k|} e^{\lambda\eta x} d\lambda \right) \delta_0^{(k)}(x). \end{aligned}$$

Now we use the hypothesis that all  $|\alpha|$  are of the form  $A - m$  for non-negative  $m \in \mathbb{Z}$ . So the residue theorem enables us to eliminate all terms except those where  $|\alpha|$  is maximal and  $k = 0$ , resulting in:

$$\begin{aligned} P(D)E &= \frac{1}{2\pi i P_A(-i\eta)} \sum_{\alpha:|\alpha|=A} \bar{c}_\alpha (i\eta)^\alpha \left( \int_{S^1} \lambda^{A-1-A+0} e^{\lambda\eta x} d\lambda \right) \delta_0(x) \\ &= \frac{1}{P_A(-i\eta)} \sum_{\alpha:|\alpha|=A} \bar{c}_\alpha (i\eta)^\alpha \delta_0(x) = \frac{1}{P_A(-i\eta)} \sum_{\alpha:|\alpha|=A} c_\alpha (-i\eta)^\alpha \delta_0(x) \\ &= \delta_0(x),. \end{aligned}$$

as required. □

In this way we have proved the Malgrange–Ehrenpreis theorem for all non-zero linear constant-coefficient fractional partial differential operators all of whose terms are of order differing by an integer from a fixed number. Perhaps the most useful sub-case of this is where all the terms have the same order, i.e.  $P = P_A$ .

### 3 Further discussion

#### 3.1 Remarks on the proofs

In the proof of Theorem 2.2, the argument is roughly based on that of Hörmander [12], with the important difference that we need to consider the function  $\lambda_n^{1/K}$  as well as just  $\lambda_n$ . This makes things more complicated at a few points in the proof.

For one thing, since we can no longer make a linear change of coordinates in the vector variable  $\lambda$ , we now need to account for the function  $A(\lambda)$  in our estimates, whereas in the non-fractional proof this function could be assumed without loss of generality to be constant. For another, we need to be more careful about the bounds we set on the variables  $\lambda_n$  and  $\lambda_n^{1/K}$ , and we end up looking at both of the two curves shown in Figure 2.1 rather than just a single type of curve as in the non-fractional proof.

In the proof of Theorem 2.3, the argument is roughly based on that of Ortner and Wagner [20], with the important difference that the series throughout the proof which were finite in the non-fractional proof have now become infinite. This is partly because of the fact that when fractional powers are involved, we need to use the more general form of the binomial theorem rather than the simple finite-series form that works for natural-number exponents. It also relates to the fact that we must now use Osler’s infinite-series version of the product rule rather than the standard Leibniz rule.



### 3.2 Examples and applications

As a basic example, let us consider the operator  $P(D) = \frac{\partial^\alpha}{\partial x^\alpha}$ , i.e. the operator given by the power function  $P(\lambda) = (i\lambda)^\alpha$ , where  $\alpha$  is fixed and rational. We shall assume  $\alpha > 1$ , i.e. all components of  $\alpha$  are greater than 1, for reasons which will become clear later. Now  $P$  is analytic on the whole complex plane except for the negative real axis, so the contour  $\gamma(\mu)$  can be deformed, regardless of  $\mu$ , to the same contour with  $R = 1$ . Call this contour  $\gamma_1$ , i.e.

$$\gamma_1 = \{re^{i\pi} : \infty > r > 1\} \cup \{e^{i\theta} : -\pi < \theta < \pi\} \cup \{r : 1 < r < \infty\}.$$

So we can write

$$\begin{aligned} \langle E, \phi \rangle &= (2\pi)^{-n} \sum_{k=1}^{\infty} \int_{\Delta_k} \int_{\gamma_1} (i\lambda)^{-\alpha} \hat{\phi}(-\lambda) \, d\lambda_n \, d\lambda' \\ &= (2\pi)^{-n} \int_{\mathbb{R}^{n-1}} \int_{\gamma_1} (i\lambda)^{-\alpha} \hat{\phi}(-\lambda) \, d\lambda_n \, d\lambda' \end{aligned}$$

as a possibility for the fundamental solution  $E$ . By symmetry (or indeed by induction on  $n$ ), we can therefore define  $E$  as follows:

$$\begin{aligned} \langle E, \phi \rangle &= (2\pi)^{-n} \int_{\gamma_1} \dots \int_{\gamma_1} (i\lambda)^{-\alpha} \hat{\phi}(-\lambda) \, d\lambda_1 \dots d\lambda_n \\ &= (2\pi)^{-n} \int_{\gamma_1} \dots \int_{\gamma_1} (i\lambda)^{-\alpha} \left( \int_{\mathbb{R}^n} e^{i\lambda \cdot x} \phi(x) \, dx \right) \, d\lambda_1 \dots d\lambda_n \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} \phi(x) \left( \int_{\gamma_1} \dots \int_{\gamma_1} (i\lambda)^{-\alpha} e^{i\lambda \cdot x} \, d\lambda_1 \dots d\lambda_n \right) \, dx, \end{aligned}$$

where Fubini's theorem was used to get from the second line to the third, this being valid because  $\phi$  is a test function,  $|\lambda| \geq 1$  for all  $\lambda \in \gamma_1$ , and  $\lambda^{1-\alpha}$  decays at infinity (here we use the assumption that  $\alpha > 1$ ).

So the distribution  $E$  can be identified with the function

$$\begin{aligned} E(x) &= (2\pi)^{-n} \int_{\gamma_1} \dots \int_{\gamma_1} (i\lambda)^{-\alpha} e^{i\lambda \cdot x} \, d\lambda_1 \dots d\lambda_n \\ &= (2\pi)^{-n} \prod_{k=1}^n \left( \int_{\mathbb{R}} (i\lambda)^{-\alpha_k} e^{i\lambda x_k} \, d\lambda \right) \\ &= (2\pi)^{-n} \prod_{k=1}^n \left( \frac{2\pi}{\Gamma(\alpha_k)} H(x_k) x_k^{\alpha_k-1} \right) = \frac{H(x) x^{\alpha-1}}{\Gamma(\alpha)} \end{aligned}$$

where  $H$  is the Heaviside step function defined by  $H(x) = 1$  if  $x > 0$ ,  $H(x) = 0$  if  $x < 0$ , and the functions  $H, \Gamma$  applied to the vector variable  $x$  are defined by taking the product over the individual coordinates of  $x$ .  $\square$

### 3.3 Possible extensions

Since any real-order differintegral operator can be approximated arbitrarily closely by rational-order ones, Theorem 2.2 is sufficient to get accurate numerical approximations to fundamental solutions of any non-zero linear constant-coefficient fractional partial differential operator which contains only *real*-order differintegrals. So from the point of view of applications, we have got as far as necessary with this theorem.

It may also be possible to use a continuity argument to extend to the  $\mathbb{R}$  case properly. More explicitly, if  $(P_i)$  is a sequence of rational-order finite series of power functions which

converges to an irrational-order one  $P$ , then there exists a fundamental solution  $E_i$  for each  $P_i(D)$ , and these  $E_i$  should converge to a distribution  $E$  which is a fundamental solution for  $P(D)$ . But we have not yet managed to construct a completely rigorous argument to prove this.

Another useful way of extending our result would be to consider differintegration methods different from the Riemann–Liouville one. Note that the only property of Riemann–Liouville fractional derivatives used in the proof of Theorem 2.2 is that they work well with Fourier transforms, i.e. the result of Lemma 1.2. So Theorem 2.2 at least should be easy to extend, although the proof of Theorem 2.3 also uses the Osler product rule (Lemma 1.6).

Extending to the well-known Caputo formula for fractional derivatives should be straightforward, since the Riemann–Liouville and Caputo definitions are equivalent when  $c = -\infty$  and  $f$  is a Schwartz function, as discussed in the paragraph following Lemma 1.5.

Using the more recent Caputo–Fabrizio definition (introduced in [7] and further developed in [15], with applications demonstrated in e.g. [2] and [8]), we should be able to construct fundamental solutions to PDEs involving fractional Laplacians, since the Fourier transforms of these using the Caputo–Fabrizio formula are known, but the resulting formulae for the fundamental solutions will of course be much more complicated.

For the definition introduced in [1], defining fractional derivatives using an integral with non-singular Mittag-Leffler kernel, the formula for Fourier transforms is even more complicated and was established in [3]. But even in this case, the Fourier transform of the fractional derivative of a function  $f$  can be written as the Fourier transform of  $f$  multiplied by some function independent of  $f$ . So despite its complexity, we should be able to establish an analogue at least of Theorem 2.2 for PDEs defined using these derivatives.

Work on all of the above-described potential extensions of the results contained herein is ongoing and may appear in future publications.

## Acknowledgements

The second author wishes to thank Anthony Ashton and Thanasis Fokas for helpful discussions and recommendations to the literature. Both authors would like to thank the anonymous referee for their very useful comments and remarks. The second author is funded by a grant from the Engineering and Physical Sciences Research Council, UK.

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