On the Homotopy Exact Sequence
for Log Algebraic Fundamental Groups

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ABSTRACT. We construct a log algebraic version of the homotopy
sequence for a normal crossing log variety over a log point of charac-
teristic zero and prove some exactness properties of it. Our proofs are
purely algebraic.

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1 Introduction

First we recall the classical homotopy exact sequence in the topological setting.
Let $p : E \to B$ be a fibration (a continuous map satisfying the homotopy lifting
property) of topological manifolds with path connected base B. Let b_0 be a
point of B and let $F = p^{-1}(b_0)$. We denote by $i : F \hookrightarrow E$ the canonical
inclusion. Also, we choose a point f_0 in F and we denote by e_0 the point $i(f_0)$.
Then we have the exact sequence

$$
\cdots \longrightarrow \pi_n^{\mathrm{top}}(F, f_0) \longrightarrow \pi_n^{\mathrm{top}}(E, e_0) \longrightarrow \pi_n^{\mathrm{top}}(B, b_0)
$$
$$
\longrightarrow \pi_{n-1}^{\mathrm{top}}(F, f_0) \longrightarrow \pi_{n-1}^{\mathrm{top}}(E, e_0) \longrightarrow \pi_{n-1}^{\mathrm{top}}(B, b_0) \longrightarrow \cdots \qquad (1)
$$

of topological homotopy groups.

In the étale setting an analogous result is proved in [Gro63, corollaire 1.4,
Exposé X]. Let K be a field and let X and S be locally Noetherian connected
K-schemes. Let $f : X \to S$ be a proper separable morphism with geometrically
connected fibers. Also, let x be a geometric point of X, $s = f \circ x$ and let X_s
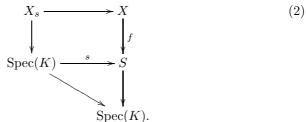be the fiber of X at the geometric point s. Then we have the exact sequence

$$
\pi_1^{\mathrm{et}}(X_s, x) \to \pi_1^{\mathrm{et}}(X, x) \to \pi_1^{\mathrm{et}}(S, s) \to 1
$$

of étale fundamental groups and the first map is injective when $S = \mathrm{Spec}(K)$ [Gro63, théorème 6.1, Exposé IX].

We are interested in proving an analogous result for the log algebraic fundamental group, which is defined as the Tannaka dual of a certain category of modules with integrable connection.

For a smooth scheme X over a field K of characteristic 0 and a K-rational point x of X, the algebraic fundamental group $\pi_1(X/\mathrm{Spec}(K), x)$ is defined as the Tannaka dual of the category of $\mathcal{O}_X$-coherent $\mathcal{D}_X$-modules (which is equivalent to the category of coherent $\mathcal{O}_X$-modules with integrable connection because K is of characteristic 0).

Then the homotopy exact sequence for algebraic fundamental groups is described as follows. Let X and S be smooth connected K-schemes of finite type and let $f : X \to S$ be a proper smooth morphism with geometrically connected fibers. Also, let x be a K-rational point of X, put $s = f(x)$ and denote by X_s the fiber of f over s:

$$
\begin{array}{ccc}
X_s & \longrightarrow & X \\
\downarrow & & \downarrow{\scriptstyle f} \\
\mathrm{Spec}(K) & \xrightarrow{\ s\ } & S \\
& & \downarrow \\
& & \mathrm{Spec}(K).
\end{array}
\qquad (2)
$$

Then the following sequence is exact:
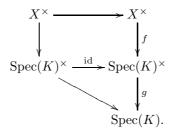
$$\pi_1(X_s/\mathrm{Spec}(K), x) \to \pi_1(X/\mathrm{Spec}(K), x) \to \pi_1(S/\mathrm{Spec}(K), s) \to 1.$$

The homotopy exact sequence in this context is proven by Zhang [Zha13]; another proof is given by Dos Santos [DS15] with a different method which is applicable also to the case where K is of characteristic $p > 0$. Lazda [Laz15] proves the exactness of the homotopy sequence for the relatively pro-unipotent quotient of algebraic fundamental groups when f admits a section, proving also the injectivity of the first map. Moreover, he studies the p-adic version, using the category of relatively unipotent isocrystals as a Tannakian category[1].

The first result of this paper is the analogue of the homotopy exact sequence for certain log schemes. The geometric situation we are interested in is as follows. Let K be a field of characteristic 0 and let X be a normal crossing log variety over K, which means that X is étale locally a finite union of smooth varieties which meet transversally, equipped with a semistable log structure in the sense of [Kat96, definition 11.6]. We denote by $X^\times$ such a log scheme. We denote by $\mathrm{Spec}(K)$ the spectrum of K endowed with the trivial log structure and by $\mathrm{Spec}(K)^\times$ the spectrum of K endowed with the log structure associated to the monoid homomorphism $\mathbb{N} \to K$; $1 \mapsto 0$. Then the

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[1]There is also a work of Lazda and Pál [LP17], which appeared after the first version of this article was written.

map of log schemes $f : X^\times \to \mathrm{Spec}(K)^\times$ induced by the structural morphism is log smooth. On the other hand, the morphisms $g : \mathrm{Spec}(K)^\times \to \mathrm{Spec}(K)$, $h := g \circ f : X^\times \to \mathrm{Spec}(K)$ are not log smooth, but they are still considered to be good morphisms. If we are given a K-rational point x of X, the diagram we obtain as an analogue of the diagram (2) is the following:

$$
\begin{array}{ccc}
X^\times & \longrightarrow & X^\times \\
\downarrow & & \downarrow f \\
\mathrm{Spec}(K)^\times & \xrightarrow{\ \mathrm{id}\ } & \mathrm{Spec}(K)^\times \\
& \searrow & \downarrow g \\
& & \mathrm{Spec}(K).
\end{array}
$$

As in the smooth case described above, we would like to define the algebraic fundamental group as the Tannaka dual of the category of coherent $\mathcal{O}_X$-modules with integrable log connection. However, in this case, it turns out that the category of coherent $\mathcal{O}_X$-modules with integrable log connection is not Tannakian, because, unlike the case of smooth schemes over a field in characteristic 0, it is not true that every coherent module with integrable connection is locally free in the log case. Neither can we consider the category of locally free modules with integrable connection on $X^\times/\mathrm{Spec}(K)^\times$ (or on $X^\times/\mathrm{Spec}(K)$) because it is not abelian. The category we consider is the category of locally free modules with integrable connection having nilpotent residues on $X^\times/\mathrm{Spec}(K)^\times$, $X^\times/\mathrm{Spec}(K)$ and $\mathrm{Spec}(K)^\times/\mathrm{Spec}(K)$, and we define the log algebraic fundamental group as its Tannaka dual, which will be denoted by $\pi_1(X^\times/\mathrm{Spec}(K)^\times, x)$, $\pi_1(X^\times/\mathrm{Spec}(K), x)$ and $\pi_1(\mathrm{Spec}(K)^\times/\mathrm{Spec}(K), \nu)$ respectively (for a suitable choice of the fiber functor ν). We remark that $\pi_1(\mathrm{Spec}(K)^\times/\mathrm{Spec}(K), \nu)$ is indeed isomorphic to $\mathbb{G}_a$.

The first result of the paper is the following.

THEOREM 1.1. *Let the notation be as above. Then the following sequence is exact:*

$$
\begin{aligned}
\pi_1(X^\times/\mathrm{Spec}(K)^\times, x) &\to \pi_1(X^\times/\mathrm{Spec}(K), x) \\
&\to \pi_1(\mathrm{Spec}(K)^\times/\mathrm{Spec}(K), \nu) \to 1.
\end{aligned}
\tag{3}
$$

Moreover, we study the injectivity of the first arrow in (3). We expect that it is injective, by analogy with the topological case. Indeed, the log scheme $\mathrm{Spec}(K)^\times$ over $\mathrm{Spec}(K)$ should be thought as the analogue of the 1-dimensional sphere (note that $\pi_1(\mathrm{Spec}(K)^\times/\mathrm{Spec}(K), \nu) \cong \mathbb{G}_a$ as remarked above). Hence, comparing with (1), we expect that there should be "$\pi_2(\mathrm{Spec}(K)^\times/\mathrm{Spec}(K), x)$", which should be 0 whatever it means, at the left of $\pi_1(X^\times/\mathrm{Spec}(K)^\times, x)$ in the sequence (3).

The study of the injectivity of the first arrow is the hardest part of the work. The result is as follows.

Theorem 1.2. *Let the notations be as above. For a group scheme G we denote by G^{tri} its maximal geometrically protrigonalizable quotient (for definition, see definition 5.3). Then the following sequence is exact:*

$$1 \to \pi_1(X^\times/\mathrm{Spec}(K)^\times, x)^{\mathrm{tri}}$$
$$\to \pi_1(X^\times/\mathrm{Spec}(K), x)^{\mathrm{tri}} \to \pi_1(\mathrm{Spec}(K)^\times/\mathrm{Spec}(K), \nu) \to 1. \tag{4}$$

Here we note that $\pi_1(\mathrm{Spec}(K)^\times/\mathrm{Spec}(K), \nu) = \pi_1(\mathrm{Spec}(K)^\times/\mathrm{Spec}(K), \nu)^{\mathrm{tri}} = \mathbb{G}_a$. If x comes from a section of $f : X^\times \to \mathrm{Spec}(K)^\times$, it induces a splitting $\pi(x^*) : \pi_1(\mathrm{Spec}(K)^\times/\mathrm{Spec}(K), \nu) \to \pi_1(X^\times/\mathrm{Spec}(K), x)^{\mathrm{tri}}$ of the exact sequence (4):

$$1 \to \pi_1(X^\times/\mathrm{Spec}(K)^\times, x)^{\mathrm{tri}}$$
$$\to \pi_1(X^\times/\mathrm{Spec}(K), x)^{\mathrm{tri}} \overset{\pi(x^*)}{\underset{}{\leftrightarrows}} \pi_1(\mathrm{Spec}(K)^\times/\mathrm{Spec}(K), \nu) \to 1.$$

Hence we can define the monodromy action N of $\pi_1(\mathrm{Spec}(K)^\times/\mathrm{Spec}(K), \nu) = \mathbb{G}_a$ on $\pi_1(X^\times/\mathrm{Spec}(K), x)^{\mathrm{tri}}$ as follows: if we denote $\gamma \in \pi_1(X^\times/\mathrm{Spec}(K), x)^{\mathrm{tri}}$ the image of $1 \in \mathbb{G}_a(K) = K$ by $\pi(x^*)$, N is given by

$$N : \pi_1(X^\times/\mathrm{Spec}(K)^\times, x)^{\mathrm{tri}} \to \pi_1(X^\times/\mathrm{Spec}(K)^\times, x)^{\mathrm{tri}}; \quad \tau \mapsto \gamma^{-1}\tau\gamma.$$

We will study (the prounipotent quotient of) this action in a forthcoming work in purely algebraic way, in which we will try to give an algebraic proof for the transcendental part of the proof of a p-adic anabelian criterion of good reduction for curves by Andreatta, Iovita and Kim [AIK15].

The content of each section is as follows. In the second section, we describe our geometric setting and prove that various categories of modules with integrable connection are Tannakian. Then we define the log algebraic fundamental groups. In the third section, we prove the exactness of the homotopy sequence (3) with four terms. The fourth section is devoted to the study of the kernel of the second map of (3): we describe it as the Tannaka dual of another category of modules with integrable connection. In the last section, we prove that the kernel of the maximal geometrically protrigonalizable quotient of the second map in (3) is isomorphic to the maximal geometrically protrigonalizable quotient of the first term of (3), proving the exactness of the homotopy sequence (4) for the maximal geometrically protrigonalizable quotients.

Acknowledgements

2 The categories of representations

Throughout this paper, let K be a field of characteristic 0. Sheaves on schemes are considered on the étale site. Also, throughout this paper, we freely use the notion of log structures developed in [Kat89], [Kat96]. Let N be the log structure on $\mathrm{Spec}(K)$ associated to the pre log structure given by the monoid homomorphism $\mathbb{N} \to K; 1 \mapsto 0$. By [Kat89, (1.3)], N is isomorphic to $\mathbb{N} \times K^\times$ through which the structure morphism $N \cong \mathbb{N} \times K^\times \to K$ of the log structure is written as $(0, a) \mapsto a, (n, a) \mapsto 0\,(n \neq 0)$. We consider $\mathrm{Spec}(K)$ as a log scheme in two ways: either endowed with the trivial log structure, in which case we denote it by $\mathrm{Spec}(K)$, or endowed with the log structure N, in which case we denote it by $\mathrm{Spec}(K)^\times$. The identity map $\mathrm{Spec}(K) \to \mathrm{Spec}(K)$ induces a morphism of log schemes $g : \mathrm{Spec}(K)^\times \to \mathrm{Spec}(K)$ which is not log smooth. By the definition given in [Kat89, (1.7)], we see that the sheaf of 1-differentials $\omega^1_{\mathrm{Spec}(K)^\times/\mathrm{Spec}(K)}$ is a 1-dimensional K-vector space generated by the element $\mathrm{dlog}(1,1)$ with $(1,1) \in \mathbb{N} \times K^\times \cong N$.

DEFINITION 2.1. *Let $X^\times := (X, M)$ be a log scheme such that X is connected. We say that $X^\times$ is a normal crossing log variety if X is a normal crossing variety in the sense of [Kat96, pp.342–343] and M is a log structure of semistable type in the sense of [Kat96, definition 11.6].*

As explained in [Kat96, Section 11], if $X^\times := (X, M)$ is a normal crossing log variety, the structure morphism $X \to \mathrm{Spec}(K)$ is amplified to a morphism of log schemes $f : X^\times \to \mathrm{Spec}(K)^\times$. Also, étale locally on X, the structure morphism $X \to \mathrm{Spec}(K)$ factors as $X \to \mathrm{Spec}(K[x_1, \ldots, x_n]/(x_1 \cdots x_r)) \to \mathrm{Spec}(K)$ for some $1 \leq r \leq n$ such that the first map is étale and that f admits a chart of the form

$$\begin{array}{lll} \mathbb{N}^r \to \mathcal{O}_X, & \mathbb{N} \to K, & \mathbb{N} \to \mathbb{N}^r, \\ e_i \mapsto x_i & 1 \mapsto 0 & 1 \mapsto (1, \ldots, 1) \end{array} \tag{5}$$

where $e_i := (0, \ldots, 1, \ldots, 0)$ (with 1 at the i-th place) and we denoted the image of $x_i \in K[x_1, \ldots, x_n]/(x_1 \cdots x_r)$ in $\mathcal{O}_X$ also by x_i.

The sheaf of 1-differentials $\omega^1_{X^\times/\mathrm{Spec}(K)^\times}$ is a locally free $\mathcal{O}_X$-module which is described étale locally as

$$\omega^1_{X^\times/\mathrm{Spec}(K)^\times} \cong \frac{\bigoplus_{i=1}^r \mathcal{O}_X \mathrm{dlog} x_i}{\mathcal{O}_X \cdot \sum_{i=1}^r \mathrm{dlog} x_i} \oplus \bigoplus_{i=r+1}^n \mathcal{O}_X dx_i. \tag{6}$$

The morphism of log schemes $h := g \circ f : X^\times \to \mathrm{Spec}(K)$ is not log smooth,

but the sheaf of 1-differentials is still locally free: étale locally it is given by

$$\omega^1_{X^\times/\mathrm{Spec}(K)} \cong \bigoplus_{i=1}^{r} \mathcal{O}_X \mathrm{dlog}x_i \oplus \bigoplus_{i=r+1}^{n} \mathcal{O}_X dx_i.$$

Throughout this paper, $X^\times := (X, M)$ will be a normal crossing log variety unless otherwise stated, and assume that the morphism $f : X^\times \to \mathrm{Spec}(K)^\times$ as above is given and fixed. (Thus $h = g \circ f$ is also given.)

We are going to study the relation among the algebraic fundamental groups associated to $f : X^\times \to \mathrm{Spec}(K)^\times$, $h : X^\times \to \mathrm{Spec}(K)$ and $g : \mathrm{Spec}(K)^\times \to \mathrm{Spec}(K)$, defined as the Tannaka duals of the categories of certain coherent modules with integrable connection.

REMARK 2.2. When we define the algebraic fundamental groups associated to f and h, we will assume the existence of a K-rational point of X. This assumption implies geometric connectedness of X.

We recall the definition of a module with integrable connection in the context of log geometry.

DEFINITION 2.3. *Let $(Y, L) \to (S, L')$ be a map of fine log schemes and let E be an $\mathcal{O}_Y$-module. A connection on E is an $\mathcal{O}_S$-linear map*

$$\nabla_E : E \to E \otimes \omega^1_{(Y,L)/(S,L')}$$

which satisfies the Leibniz rule

$$\nabla_E(ae) = a\nabla_E(e) + e \otimes da \quad (a \in \mathcal{O}_Y, e \in E).$$

We can extend ∇_E to a map

$$\nabla_{E,i} : E \otimes \omega^i_{(Y,L)/(S,L')} \xrightarrow{\nabla_{E,i}} E \otimes \omega^{i+1}_{(Y,L)/(S,L')}$$

for $i \geq 1$ by $\nabla_{E,i}(e \otimes \omega) = e \otimes d\omega + \nabla_E(e) \wedge \omega$. We say that ∇_E is integrable if $\nabla_{E,1} \circ \nabla_E = 0$.

Every $\mathcal{O}_Y$-module E endowed with a connection ∇_E induces an $\mathcal{O}_Y$-linear map from the sheaf of log derivations $\mathcal{D}er((Y, L)/(S, L')) = \mathcal{H}om_{\mathcal{O}_Y}(\omega^1_{(Y,L)/(S,L')}, \mathcal{O}_Y)$ [Kat96, definition 5.1, proposition 5.3] to the sheaf of $\mathcal{O}_S$-endomorphisms of E

$$\nabla_E : \mathcal{D}er((Y, L)/(S, L')) \longrightarrow \mathcal{E}nd_{\mathcal{O}_S}(E)$$

defined by the composite

$$\nabla_E(D) : E \xrightarrow{\nabla_E} E \otimes \omega^1_{(Y,L)/(S,L')} \xrightarrow{\mathrm{id}\otimes D} E \quad (D \in \mathcal{D}er((Y, L)/(S, L'))).$$

$\nabla_E(D)$ *satisfies the Leibniz rule*

$$\nabla_E(D)(ae) = a\nabla_E(D)(e) + D(a)e \quad (D \in \mathcal{D}er((Y, L)/(S, L')), a \in \mathcal{O}_Y, e \in E). \tag{7}$$

Conversely, if $\omega^1_{(Y,L)/(S,L')}$ is locally free, every $\mathcal{O}_Y$-linear map

$$\mathcal{D}er((Y,L)/(S,L')) \longrightarrow \mathcal{E}nd_{\mathcal{O}_S}(E)$$

satisfying (7) comes from a unique connection.

We denote by $MIC((Y,L)/(S,L'))$ the category whose objects are pairs (E, ∇_E) consisting of a coherent $\mathcal{O}_Y$-module E and an integrable connection ∇_E and whose morphisms are horizontal morphisms: here, for coherent modules (E, ∇_E), (F, ∇_F) with integrable connection, an $\mathcal{O}_Y$-linear morphism $\gamma : E \to F$ is called horizontal if it satisfies the equality $(\gamma \otimes \mathrm{id}) \circ \nabla_E = \nabla_F \circ \gamma$. If $\omega^1_{(Y,L)/(S,L')}$ is locally free, this condition is equivalent to the condition $\gamma(\nabla_E(D)(e)) = \nabla_F(D)(\gamma(e))$ for every $e \in E$ and $D \in \mathcal{D}er((Y,L)/(S,L'))$.

REMARK 2.4. When $Y = \mathrm{Spec}(B)$ where B is a Noetherian I-adically complete ring for I an ideal of B and $S = \mathrm{Spec}(A)$, we also consider the I-adic completion $\hat{\omega}^1_{(Y,L)/(S,L')}$ of $\omega^1_{(Y,L)/(S,L')}$ (regarded as B-module). Assume that $\hat{\omega}^1_{(Y,L)/(S,L')}$ is finitely generated as B-module. Then a formal connection on a B-module E is an A-linear map

$$\nabla_E : E \to E \otimes \hat{\omega}^1_{(Y,L)/(S,L')}$$

which satisfies the Leibniz rule

$$\nabla_E(be) = b\nabla_E(e) + e \otimes db \quad (b \in B, e \in E).$$

The notion of integrability is defined in the same way as before. We denote by $\widehat{MIC}((Y,L)/(S,L'))$ the category whose objects are pairs (E, ∇_E) consisting of a B-module E and a formal integrable connection ∇_E and whose morphisms are horizontal, with the same meaning as before, morphisms of B-modules. The module of formal log derivations $\widehat{\mathcal{D}er}((Y,L)/(S,L'))$ is defined by $\widehat{\mathcal{D}er}((Y,L)/(S,L')) := Hom_B(\hat{\omega}^1_{(Y,L)/(S,L')}, B)$. (A more intrinsic definition of $\widehat{\mathcal{D}er}((Y,L)/(S,L'))$ is possible but we do not need it.) As before, we have a description of a formal integrable connection in terms of the module of formal log derivations when $\hat{\omega}^1_{(Y,L)/(S,L')}$ is projective as B-module.

Note that the category $\widehat{MIC}((Y,L)/(S,L'))$ depends not only on the morphism $(Y,L) \to (S,L')$ but also on the ideal I. However, we would like to denote it without mentioning I explicitly in the symbol, because the ideal I will be clear in the cases we treat in this paper.

REMARK 2.5. Let $Y = \mathrm{Spec}(B)$, $I \subset B$, $S = \mathrm{Spec}(A)$, $\hat{\omega}^1_{(Y,L)/(S,L')}$ be as in remark 2.4 and let $Y' = \mathrm{Spec}(B') \subset \mathrm{Spec}(B) = Y$ be an affine open subscheme of Y. Then we put $\hat{\omega}^1_{(Y',L)/(S,L')} := B' \otimes_B \hat{\omega}^1_{(Y,L)/(S,L')}$, and using this, we define the notion of formal connection on a B'-module E as an A-linear map

$$\nabla_E : E \to E \otimes \hat{\omega}^1_{(Y',L)/(S,L')}$$

satisfying the Leibniz rule. The notion of integrability is also defined. We denote by $\widehat{MIC}((Y',L)/(S,L'))$ the category whose objects are pairs (E, ∇_E)

consisting of a B'-module E and a formal integrable connection ∇_E and whose morphisms are horizontal morphisms of B'-modules.

Remark 2.6. When $\omega^1_{(Y,L)/(S,L')}$ is locally free, the category $MIC((Y,L)/(S,L'))$ is an abelian tensor category. Given $(E, \nabla_E), (F, \nabla_F) \in MIC((Y,L)/(S,L'))$, the tensor product of them is defined as the pair $(E \otimes F, \nabla)$, where $\nabla = \mathrm{id} \otimes \nabla_F + \nabla_E \otimes \mathrm{id}$. The object $(\mathcal{O}_Y, d)$, where d is the composition of the differential $d : \mathcal{O}_Y \to \Omega^1_{Y/S}$ with the map $\Omega^1_{Y/S} \to \omega^1_{(Y,L)/(S,L')}$, is the unit object of $MIC((Y,L)/(S,L'))$ and is called the trivial connection.

Remark 2.7. The category $MIC(\mathrm{Spec}(K)^\times/\mathrm{Spec}(K))$ is equivalent to the category of finite dimensional K-vector spaces endowed with a K-linear endomorphism.

We will use the theory of Tannakian categories over K to define log algebraic fundamental groups. We recall the definition and the main theorem which relates the fundamental group and the category of its representations.

Definition 2.8. *Let $\mathcal{C}$ be a rigid abelian tensor category with unit object $\underline{1}$. We say that $\mathcal{C}$ is a neutral Tannakian category over K if $\mathrm{End}(\underline{1}) = K$ and if there exists a K-linear faithful exact tensor functor (fiber functor) $\mathcal{C} \to \mathrm{Vec}_K$, where Vec_K is the category of finite dimensional K-vector spaces.*

Theorem 2.9 ([DM82, Theorem 2.11]). *Let $\mathcal{C}$ be a neutral Tannakian category over K endowed with a fiber functor $\omega : \mathcal{C} \to \mathrm{Vec}_K$. Then the functor*

$$(K\text{-algebras}) \longrightarrow (\text{Groups})$$

which sends a K-algebra R to the group of tensor automorphisms $\mathrm{Aut}^\otimes(\mathcal{C} \longrightarrow \mathrm{Vec}_K \longrightarrow \mathrm{Mod}_R)$ (where Mod_R is the category of R-modules) is representable by a pro-algebraic group $G(\mathcal{C}, \omega)$ over K. Moreover, ω induces an equivalence of categories

$$\mathcal{C} \cong \mathrm{Rep}_K(G(\mathcal{C}, \omega)),$$

where $\mathrm{Rep}_K(G(\mathcal{C}, \omega))$ denotes the category of finite dimensional K-representations of $G(\mathcal{C}, \omega)$.

It is also known that, for a pro-algebraic group G over K, $\mathrm{Rep}_K(G)$ is a neutral Tannakian category with forgetful functor $\omega : \mathrm{Rep}_K(G) \to \mathrm{Vec}_K$ as fiber functor and that there is a canonical isomorphism $G \xrightarrow{\cong} G(\mathrm{Rep}_K(G), \omega)$ ([DM82, Proposition 2.8]).

Remark 2.10. We cannot consider the whole categories $MIC(X^\times/\mathrm{Spec}(K)^\times)$ and $MIC(X^\times/\mathrm{Spec}(K))$ to define log algebraic fundamental groups, because they are not Tannakian. We explain this for the category $MIC(X^\times/\mathrm{Spec}(K)^\times)$, but analogous examples exist for $MIC(X^\times/\mathrm{Spec}(K))$. Unlike the case of trivial log structures, it is no longer true that every coherent

module with integrable connection is locally free. To see this, we consider as $X^\times/\mathrm{Spec}(K)^\times$ the log schemes $(\mathrm{Spec}(K[x,y]/(xy)),M)/\mathrm{Spec}(K)^\times$, where M is the log structure associated to $\mathbb{N}^2 \to K[x,y]/(xy)$; $e_1 \mapsto x, e_2 \mapsto y$ and the morphism $(\mathrm{Spec}(K[x,y]/(xy)),M) \to \mathrm{Spec}(K)^\times$ over $\mathrm{Spec}(K)$ is defined by $\mathbb{N} \to \mathbb{N}^2$; $1 \mapsto e_1 + e_2$. In this case, the module of 1-differentials is

$$\omega^1_{(\mathrm{Spec}(K[x,y]/(xy)),M)/\mathrm{Spec}(K)^\times} \cong \frac{(K[x,y]/(xy))\mathrm{dlog}x \oplus (K[x,y]/(xy))\mathrm{dlog}y}{(K[x,y]/(xy))(\mathrm{dlog}x + \mathrm{dlog}y)},$$

which is free of rank 1. Then, $(xK[x,y]/(xy),d)$, where d is the connection induced by the trivial connection on $K[x,y]/(xy)$, is an object of $MIC(X^\times/\mathrm{Spec}(K)^\times)$; indeed, $d(xf) = fdx + xdf = x(f\mathrm{dlog}x + df)$. But $xK[x,y]/(xy)$ is a coherent $K[x,y]/(xy)$-module which is not free. This object is a non-trivial subobject of the unit object $(K[x,y]/(xy),d)$, which is impossible for a Tannakian category over K.

Even if we consider the full subcategory of $MIC(X^\times/\mathrm{Spec}(K)^\times)$ of locally free modules with integrable connection, this is not a Tannakian category because it is not abelian. To see this, we consider the affine 1-dimensional case as before and we look at the map of free modules with connections

$$\varphi : (K[x,y]/(xy), \nabla) \longrightarrow (K[x,y]/(xy), d); \quad f \mapsto xf$$

where $\nabla(f) := df + f\mathrm{dlog}x$. One can check that the map φ is horizontal and that $\mathrm{Coker}(\varphi)$ is given by $(K[y], d')$ (where d' is the induced connection), which is not locally free as $K[x,y]/(xy)$-module.

If $H^0_{\mathrm{dR}}(X^\times/\mathrm{Spec}(K))$ (resp. $H^0_{\mathrm{dR}}(X^\times/\mathrm{Spec}(K)^\times)$) is a field, the full subcategory of $MIC(X^\times/\mathrm{Spec}(K))$ (resp. of $MIC(X^\times/\mathrm{Spec}(K)^\times)$) consisting of the unipotent objects (iterated extensions of the unit object) is a Tannakian category ([Shi00, proposition 3.1.2]). We want to construct a subcategory of $MIC(X^\times/\mathrm{Spec}(K))$ (resp. $MIC(X^\times/\mathrm{Spec}(K)^\times)$) which is Tannakian and properly contains the subcategory of all the unipotent objects. To do this we introduce a notion of nilpotent residues, which will be a punctual notion as in [Ogu03, definition 2.1.1].

The definitions which follow are written for a morphism of fine log schemes of the form $(Y, L) \to (\mathrm{Spec}(K), L')$. This will be mainly applied to $f : X^\times \to \mathrm{Spec}(K)^\times$, $h : X^\times \to \mathrm{Spec}(K)$ and $g : \mathrm{Spec}(K)^\times \to \mathrm{Spec}(K)$.

Let $a : (Y, L) \to (\mathrm{Spec}(K), L')$ be as above. Let y be a geometric point of Y, $\mathfrak{m}_{Y,y}$ the unique maximal ideal of $\mathcal{O}_{Y,y}$ ($:=$ the strict localization of $\mathcal{O}_Y$ at y) and $K(y) = \mathcal{O}_{Y,y}/\mathfrak{m}_{Y,y}$ the residue field. As observed in [Ogu94, lemma 1.3.1], there exists a surjective morphism

$$\omega^1_{(Y,L)/(\mathrm{Spec}(K),L')}(y) \to K(y) \otimes \overline{L}^{\mathrm{gp}}_y/\overline{L'}^{\mathrm{gp}}_y,$$

where $\omega^1_{(Y,L)/(\mathrm{Spec}(K),L')}(y) = \omega^1_{(Y,L)/(\mathrm{Spec}(K),L'),y} \otimes K(y)$, $\overline{L}^{\mathrm{gp}}_y = L^{\mathrm{gp}}_y/\mathcal{O}^*_{Y,y}$, $\overline{L'}^{\mathrm{gp}}_y = (a^*L')^{\mathrm{gp}}_y/\mathcal{O}^*_{Y,y}$ and L^{gp}, $(a^*L')^{\mathrm{gp}}$ are the sheaves of groups associated

to L and a^*L', respectively. The fineness of the log structures L, L' implies that $K(y) \otimes \overline{L}_y^{\mathrm{gp}}/\overline{L'}_y^{\mathrm{gp}}$ is a finite-dimensional K-vector space.

Let (E, ∇_E) be an object of $MIC((Y, L)/(S, L'))$ and we denote by $E(y) = E_y/\mathfrak{m}_{Y,y}E_y \cong E_y \otimes K(y)$ the fiber of E at y, which is a finite dimensional vector space over $K(y)$. Then, ∇_E induces a unique linear morphism called the residue $\rho_y : E(y) \to E(y) \otimes \overline{L}_y^{\mathrm{gp}}/\overline{L'}_y^{\mathrm{gp}}$; indeed, if $e \in E(y)$ and $b \in K(y)$, $\rho_y(be) = b\nabla_E(e) + e \otimes d(b)$ and $d(b) = 0$ in $\overline{L}_y^{\mathrm{gp}}/\overline{L'}_y^{\mathrm{gp}}$.

DEFINITION 2.11. *Let (E, ∇_E) be an object of $MIC((Y, L)/(\mathrm{Spec}(K), L'))$. For a geometric point y of Y, we say that (E, ∇_E) has nilpotent residues at y if, for every $K(y)$-linear map $t_y : K(y) \otimes \overline{L}_y^{\mathrm{gp}}/\overline{L'}_y^{\mathrm{gp}} \to K(y)$, the composite map $(\mathrm{id} \otimes t_y) \circ \rho_y$ is nilpotent:*

$$E(y) \xrightarrow{\rho_y} E(y) \otimes \overline{L}_y^{\mathrm{gp}}/\overline{L'}_y^{\mathrm{gp}}$$
$$\downarrow{\scriptstyle \mathrm{id} \otimes t_y}$$
$$E(y).$$

We say that (E, ∇_E) has nilpotent residues if it has nilpotent residues at any geometric point over a closed point of Y.

We denote by $MIC((Y, L)/(\mathrm{Spec}(K), L'))^{\mathrm{nr}}$ the category of locally free $\mathcal{O}_Y$-modules of finite rank on $(Y, L)/(\mathrm{Spec}(K), L')$ with integrable connection having nilpotent residues.

REMARK 2.12. It is an interesting problem if any object (E, ∇) in $MIC((Y, L)/(\mathrm{Spec}(K), L'))^{\mathrm{nr}}$ has nilpotent residues at any geometric points of Y which is not necessarily over a closed point, namely, the nilpotence of residues at closed points implies that over any points. Unfortunately, we do not have an answer in general case. Nevertheless, we remark here that the above question has the affirmative answer in the case where $(Y, L)/(\mathrm{Spec}(K), L')$ is $X^\times/\mathrm{Spec}(K)$, $X^\times/\mathrm{Spec}(K)^\times$ or $\mathrm{Spec}(K)^\times/\mathrm{Spec}(K)$, namely, the case of our main interest.

We give a proof only in the case $X^\times/\mathrm{Spec}(K)$, because the proof in the other cases is similar. Since the assertion is local, we may assume that there exists an strict etale morphism

$$X^\times \to (\mathrm{Spec}(K[x_1, \ldots, x_n]/(x_1 \cdots x_r)), M),$$

where M is the log structure associated to $\mathbb{N}^r \to K[x_1, \ldots, x_n]/(x_1 \cdots x_r); e_i \mapsto x_i$. For $i \in \{1, \ldots, r\}$, let X_i be the zero locus of x_i and for $I \subseteq \{1, \ldots, r\}$, put $X_I := \bigcap_{i \in I} X_i, X_I^\circ := X_I \setminus \bigcup_{J \supsetneq I} X_J$. Then $X = \sqcup_{I \subseteq \{1, \ldots, r\}} X_I^\circ$ set-theoretically. Also, we have the equality $\overline{M}_{|X_I^\circ}^{\mathrm{gp}} = \mathbb{Z}_{X_I^\circ}^{|I|}$, where $\overline{M}^{\mathrm{gp}} := M^{\mathrm{gp}}/\mathcal{O}_X^*$.

Now take any object (E, ∇) in $MIC(X^\times/\mathrm{Spec}(K))^{\mathrm{nr}}$, a geometric point x of X which is not necessarily over a closed point. Let I be the subset of $\{1, \ldots, r\}$

such that the image of x in X belongs to X_I°. Then ∇ induces a linear map

$$\rho_I : E_{|X_I^\circ} \to E_{|X_I^\circ} \otimes \overline{M}_{|X_I^\circ}^{\mathrm{gp}} = E_{|X_I^\circ} \otimes \mathbb{Z}_{X_I^\circ}^{|I|}.$$

For a geometric point y of X_I°, we denote the specialization of ρ_I to y by

$$\rho_y : E(y) \to E(y) \otimes \overline{M}_y^{\mathrm{gp}} = E(y) \otimes \mathbb{Z}^{|I|}.$$

To prove the assertion, we should prove that, for any $K(x)$-linear map $t_x :$ $K(x) \otimes \overline{M}_x^{\mathrm{gp}} = K(x) \otimes \mathbb{Z}^{|I|} \to K(x)$, the map $(\mathrm{id} \otimes t_x) \circ \rho_x : E(x) \to E(x)$ is nilpotent. By replacing X by a small affine open subscheme containing the image of x in X, we may assume that $E_{|X_I^\circ}$ is free of rank r.

For $1 \leq i \leq |I|$, we denote the i-th projection $\overline{M}_{|X_I^\circ}^{\mathrm{gp}} = \mathbb{Z}_{X_I^\circ}^{|I|} \to \mathbb{Z}_{X_I^\circ}$ by π_i and denote its fiber $\overline{M}_x^{\mathrm{gp}} = \mathbb{Z}^{|I|} \to \mathbb{Z}$ at x by $\pi_{i,x}$. Then, by integrability of ∇, it suffices to prove the nilpotence of $\rho_{x,i} := (\mathrm{id} \otimes \pi_{i,x}) \circ \rho_x : E(x) \to E(x)$ for any i. Since this map is the specialization of the map

$$\rho_{I,i} := (\mathrm{id} \otimes \pi_i) \circ \rho_I : E_{|X_I^\circ} \to E_{|X_I^\circ},$$

we are reduced to proving the nilpotence of the map $\rho_{I,i}$. Choose for any closed point of X_I° a geometric point over it and let S be the set of such geometric points. Then, by assumption, for any $y \in S$ and for any $1 \leq i \leq |I|$, the map $\rho_{y,i}$ is nilpotent. Hence, as an element in $\mathrm{End}(E_{|X_I^\circ}(y))$, $\rho_{y,i}^{r+1}$ is equal to 0. On the other hand, since X_I° is reduced and Jacobson and $E_{|X_I^\circ}$ is free, the natural map $\mathrm{End}(E_{|X_I^\circ}) \to \prod_{y \in S} \mathrm{End}(E_{|X_I^\circ}(y))$ is injective. Hence $\rho_{I,i}^{r+1}$ is equal to 0 in $\mathrm{End}(E_{|X_I^\circ})$. So the proof of the assertion is finished.

REMARK 2.13. When we are in the situation of remark 2.4 (resp. remark 2.5) with $A = K$, we can define the notion of having nilpotent residues at a geometric point y of Y analogously to definition 2.11 for formal integrable connections. We denote the category of finitely generated projective B-modules (resp. B'-modules) with formal integrable connection having nilpotent residues at any geometric point over a closed point of Y by $\widehat{MIC}((Y,L)/(\mathrm{Spec}(K), L'))^{\mathrm{nr}}$ (resp. $\widehat{MIC}((Y',L)/(\mathrm{Spec}(K), L'))^{\mathrm{nr}}$).

REMARK 2.14. The category $MIC(\mathrm{Spec}(K)^\times/\mathrm{Spec}(K))^{\mathrm{nr}}$ is equivalent to the category of finite dimensional K-vector spaces endowed with a nilpotent endomorphism. It is a neutral Tannakian category and the functor

$$\nu : MIC(\mathrm{Spec}(K)^\times/\mathrm{Spec}(K))^{\mathrm{nr}} \longrightarrow \mathrm{Vec}_K; \quad (V, \nabla_V) \mapsto V$$

gives a fiber functor.

DEFINITION 2.15. *We define the log algebraic fundamental group of* $\mathrm{Spec}(K)^\times$ *over* $\mathrm{Spec}(K)$ *with base point* ν *as the Tannaka dual of* $MIC(\mathrm{Spec}(K)^\times/\mathrm{Spec}(K))^{\mathrm{nr}}$, *i.e.*

$$\pi_1(\mathrm{Spec}(K)^\times/\mathrm{Spec}(K), \nu) := G(MIC(\mathrm{Spec}(K)^\times/\mathrm{Spec}(K))^{\mathrm{nr}}, \nu)$$

with the notation as in theorem 2.9.

The group $\pi_1(\mathrm{Spec}(K)^\times/\mathrm{Spec}(K),\nu)$ is isomorphic to $\mathbb{G}_a$: indeed, in [Mil12, example VIII 2.1, example VIII 6.4], it is shown that any object $\rho : \mathbb{G}_a \to GL(V)$ in $\mathrm{Rep}_K(\mathbb{G}_a)$ has the form $\rho(t) = \exp(Nt)$ for a unique nilpotent endomorphism N on V, and so the category $\mathrm{Rep}_K(\mathbb{G}_a)$ is equivalent to the category of finite dimensional K-vector spaces endowed with a nilpotent endomorphism, which is nothing but $MIC(\mathrm{Spec}(K)^\times/\mathrm{Spec}(K))^{\mathrm{nr}}$. Thus we see that

$$\mathbb{G}_a \cong G(MIC(\mathrm{Spec}(K)^\times/\mathrm{Spec}(K))^{\mathrm{nr}}, \nu) =: \pi_1(\mathrm{Spec}(K)^\times/\mathrm{Spec}(K), \nu)$$

by the fact given after theorem 2.9.

We are going to prove now that $MIC(X^\times/\mathrm{Spec}(K)^\times)^{\mathrm{nr}}$ is a neutral Tannakian category over K under suitable assumptions.

PROPOSITION 2.16. *The category $MIC(X^\times/\mathrm{Spec}(K)^\times)^{\mathrm{nr}}$ is abelian.*

Proof. Since $MIC(X^\times/\mathrm{Spec}(K)^\times)^{\mathrm{nr}}$ is a full subcategory of the abelian category $MIC(X^\times/\mathrm{Spec}(K)^\times)$, it is sufficient to prove that the kernel and the cokernel of any morphism in $MIC(X^\times/\mathrm{Spec}(K)^\times)^{\mathrm{nr}}$ are locally free $\mathcal{O}_X$-modules with nilpotent residues.

To prove that a coherent $\mathcal{O}_X$-module E is locally free it is enough to prove that, for every geometric point x over a closed point in X, the stalk E_x is a free $\mathcal{O}_{X,x}$-module. It is moreover enough to prove this at the level of completed local ring $\widehat{\mathcal{O}}_{X,x}$, which has the form $K[[x_1,\ldots,x_n]]/(x_1\cdots x_r)$ with K algebraically closed. Also, to prove the nilpotence of residues, it suffices to check it at the closed point of $\mathrm{Spec}(\widehat{\mathcal{O}}_{X,x})$. Thanks to lemma 2.18 below, we can suppose that $\widehat{\mathcal{O}}_{X,x} \cong K[[x_1,\ldots,x_n]]/(x_1\cdots x_n)$ with $n \geq 2$. Then we proceed by induction on n: in proposition 2.19 below we prove the result for $n = 2$, and in proposition 2.20 below we prove the induction step. □

REMARK 2.17. In what follows we consider on the spectrum of $S = K[[x_1,\ldots,x_n]]/(x_1\cdots x_r)$ the log structure M associated to the monoid homomorphism

$$\mathbb{N}^r \to K[[x_1,\ldots,x_n]]/(x_1\cdots x_r); \quad e_i \mapsto x_i.$$

Then

$$\hat{\omega}^1_{(\mathrm{Spec}(S),M)/\mathrm{Spec}(K)^\times} \cong \frac{\bigoplus_{i=1}^r S\mathrm{dlog}x_i}{S \cdot \sum_i^r \mathrm{dlog}x_i} \oplus \bigoplus_{i=r+1}^n Sdx_i,$$

a basis of it is given by $\{\mathrm{dlog}x_1, \sum_{i=1}^2 \mathrm{dlog}x_i, \ldots, \sum_{i=1}^{r-1} \mathrm{dlog}x_i, dx_{r+1}, \ldots, dx_n\}$. We will use several times in the remaining part of this section its dual basis as a basis of

$$\widehat{\mathcal{D}er}((\mathrm{Spec}(S),M)/\mathrm{Spec}(K)^\times) = Hom(\hat{\omega}^1_{(\mathrm{Spec}(S),M)/\mathrm{Spec}(K)^\times}, S),$$

which we denote by $\{\partial_1,\ldots,\partial_{r-1}, D_{r+1},\ldots,D_n\}$; namely, for $i = 1,\ldots,r-1$ we denote by ∂_i the derivation that sends $\mathrm{dlog}x_i$ to 1, $\mathrm{dlog}x_{i+1}$ to -1, $\mathrm{dlog}x_j$ to 0 for every $j \neq i, i+1$, and dx_j to 0 for every $j = r+1,\ldots,n$, while we denote by

D_i for $i = r+1, \ldots, n$ the derivation which sends dx_i to 1, dx_j to 0 for $j = r+1, \ldots, n$, $j \neq i$ and $\mathrm{dlog}\, x_j$ to 0 for every $j = 1, \ldots, r-1$. For a formal connection ∇_E on a coherent S-module E, the integrability of ∇_E is equivalent to the commutativity of the operators $\nabla_E(\partial_1), \ldots, \nabla(\partial_{r-1}), \nabla_E(D_{r+1}), \ldots, \nabla_E(D_n)$.

LEMMA 2.18. *Let R be the ring $K[[x_1, \ldots, x_r]]/(x_1 \cdots x_r)$ and S be the ring $K[[x_1, \ldots, x_n]]/(x_1 \cdots x_r)$ be as above. If the category $\widehat{MIC}((\mathrm{Spec}(R), M)/\mathrm{Spec}(K)^\times)^{\mathrm{nr}}$ is stable by kernel and cokernel of any morphism, then the category $\widehat{MIC}((\mathrm{Spec}(S), M)/\mathrm{Spec}(K)^\times)^{\mathrm{nr}}$ is stable by kernel and cokernel of any morphism.*

Proof. For an object (E, ∇_E) in $\widehat{MIC}((\mathrm{Spec}(S), M)/\mathrm{Spec}(K)^\times)^{\mathrm{nr}}$, put

$$\overline{E} := \{ e \in E \mid \nabla_E(D_i)(e) = 0 \, (r+1 \leq i \leq n) \}.$$

Then $\overline{E}$ is stable by the action of $\nabla_E(\partial_i) \, (1 \leq i \leq r-1)$. Then, it suffices to prove that $\overline{E}$ with the above action defines an object in $\widehat{MIC}((\mathrm{Spec}(R), M)/\mathrm{Spec}(K)^\times)^{\mathrm{nr}}$ and that the functor

$$\widehat{MIC}((\mathrm{Spec}(S), M)/\mathrm{Spec}(K)^\times)^{\mathrm{nr}} \to \widehat{MIC}((\mathrm{Spec}(R), M)/\mathrm{Spec}(K)^\times)^{\mathrm{nr}}; \quad (8)$$
$$(E, \nabla_E) \mapsto \overline{E}$$

is an equivalence of categories whose quasi-inverse is given by $(\overline{E}, \nabla_{\overline{E}}) \mapsto (E := \overline{E} \otimes_R S, \nabla_E)$ with the action $\nabla_E(\partial_i) \, (1 \leq i \leq r-1), \nabla_E(D_i) \, (r+1 \leq i \leq n)$ defined by $\nabla_E(\partial_i) := \mathrm{id} \otimes d(\partial_i) + \nabla_{\overline{E}}(\partial_i) \otimes \mathrm{id}, \nabla_E(D_i) := \mathrm{id} \otimes d(D_i)$: indeed, if this claim is proved and if we are given a morphism $\varphi : E \to F$ in $\widehat{MIC}((\mathrm{Spec}(S), M)/\mathrm{Spec}(K)^\times)^{\mathrm{nr}}$, it induces a morphism $\overline{\varphi} : \overline{E} \to \overline{F}$ in $\widehat{MIC}((\mathrm{Spec}(R), M)/\mathrm{Spec}(K)^\times)^{\mathrm{nr}}$, and $\mathrm{Ker}(\overline{\varphi}), \mathrm{Coker}(\overline{\varphi})$ are defined as objects in $\widehat{MIC}((\mathrm{Spec}(R), M)/\mathrm{Spec}(K)^\times)^{\mathrm{nr}}$. Then we have $\mathrm{Ker}(\varphi) = \mathrm{Ker}(\overline{\varphi}) \otimes_R S, \mathrm{Coker}(\varphi) = \mathrm{Coker}(\overline{\varphi}) \otimes_R S$ and they are objects in $\widehat{MIC}((\mathrm{Spec}(S), M)/\mathrm{Spec}(K)^\times)^{\mathrm{nr}}$, as required.

We prove the claim in the previous paragraph, following [Kat70, proposition 8.9]. Define the map $P : E \to E$ by

$$P(e) := \sum_{\mathbf{k} \in \mathbb{N}^{n-r}} (-1)^{|\mathbf{k}|} x^{\mathbf{k}} D^{\mathbf{k}}(e)/\mathbf{k}!,$$

where, for $\mathbf{k} = (k_1, \ldots, k_{n-r}) \in \mathbb{N}^{n-r}$, $|\mathbf{k}| = \sum_{i=1}^{n-r} k_i$, $x^{\mathbf{k}} = x_{r+1}^{k_1} \cdots x_n^{k_{n-r}}$, $D^{\mathbf{k}} = D_{r+1}^{k_1} \cdots D_n^{k_{n-r}}$ and $\mathbf{k}! = k_1! \cdots k_{n-r}!$. Then we see that the image of P is contained in $\overline{E}$ and that $P|_{\overline{E}} = \mathrm{id}_{\overline{E}}$. Moreover, if we denote the kernel of the surjection $\pi : S \to R$ defined by $x_i \mapsto x_i \, (1 \leq i \leq r), x_i \mapsto 0 \, (r+1 \leq i \leq n)$ by I, we have $P(e) \equiv e \, (\mathrm{mod}\, IE)$. Thus we see that P induces the isomorphism

$$E/IE \xrightarrow{\cong} \overline{E}. \quad (9)$$

In particular, $\overline{E}$ is a free R-module and it has nilpotent residues. So $\overline{E}$ (endowed with the action of $\nabla_E(\partial_i)\,(1 \leq i \leq r-1)$) defines an object in $\widehat{MIC}((\mathrm{Spec}(R), M)/\mathrm{Spec}(K)^\times)^{\mathrm{nr}}$. To prove the rest of the claim, it suffices to prove that the natural map $\iota : \overline{E} \otimes_R S \to E$ is an isomorphism. Surjectivity follows from the isomorphism (9). To prove injectivity, let $e_1, ..., e_s$ be a basis of $\overline{E}$ and assume $\sum_{i=1}^s f_i e_i = 0$ in E with $f_i \in S$. If f_i is non-zero for some i, we have $\pi(D^{\mathbf{k}}(f_i)) \neq 0$ for some i and some $\mathbf{k} \in \mathbb{N}^{n-r}$. Then we have

$$0 = P(D^{\mathbf{k}}(\sum_{i=1}^s f_i e_i))) = \sum_{i=1}^s \pi(D^{\mathbf{k}}(f_i)) e_i,$$

and this contradicts the linear independence of e_i's over R. Thus $f_i = 0$ for all i and this shows the injectivity of the map ι. So we are done. $\square$

Proposition 2.19. $\widehat{MIC}((\mathrm{Spec}(K[[x,y]]/(xy)), M)/\mathrm{Spec}(K)^\times)^{\mathrm{nr}}$ is an abelian category.

Proof. Let ∂_1 be as in the notation introduced in remark 2.17, which is a basis of

$$\widehat{\mathcal{D}er}((\mathrm{Spec}(K[[x,y]]/(xy)), M)/\mathrm{Spec}(K)^\times)$$
$$= \mathcal{H}om(\hat{\omega}^1_{(\mathrm{Spec}(K[[x,y]]/(xy)), M)/\mathrm{Spec}(K)^\times}, K[[x,y]]/(xy)). \quad (10)$$

For an object (E, ∇_E) in $\widehat{MIC}((\mathrm{Spec}(K[[x,y]]/(xy)), M)/\mathrm{Spec}(K)^\times)^{\mathrm{nr}}$, put

$$\overline{E} := \{e \in E \mid \exists N \in \mathbb{N}, \nabla_E^N(\partial_1)(e) = 0\}.$$

Then $\overline{E}$ is stable by the action of $\nabla_E(\partial_1)$ and this action is locally nilpotent. Then, it suffices to prove that the correspondence $(E, \nabla_E) \mapsto (\overline{E}, \nabla_E(\partial_1))$ defines the functor

$$\widehat{MIC}((\mathrm{Spec}(K[[x,y]]/(xy)), M)/\mathrm{Spec}(K)^\times)^{\mathrm{nr}}$$
$$\to \left\{ (V, N) \;\middle|\; \begin{array}{l} V\text{: a finite-dimensional } K\text{-vector space} \\ N : V \to V\text{: a nilpotent endomorphism} \end{array} \right\}$$

and that it is an equivalence whose quasi-inverse is given by $(V, N) \mapsto (E := V \otimes_K K[[x,y]]/(xy), \nabla_E)$ with the action $\nabla_E(\partial_1)$ defined by $\nabla_E(\partial_1) := \mathrm{id} \otimes d(\partial_1) + N \otimes \mathrm{id}$. (The reason is the same as that in lemma 2.18.)
To prove the above claim, it suffices to construct a basis of $\overline{E}$ over K which is a basis of E over $K[[x,y]]/(xy)$. To do so, first we prove that there exists a basis of E on which $\nabla_E(\partial_1)$ acts as a strictly upper triangular matrix with entries in K. If E has rank n, by hypothesis of nilpotent residues, we can write

$$\nabla_E(\partial_1) = d(\partial_1) + H$$

with respect to some basis, where $H = (a_{i,j}(x, y))_{i,j}$ is an $n \times n$ matrix such that $H_0 = (a_{i,j}(0,0))_{i,j}$ is a strictly upper triangular matrix. We want to prove that there exists a change of basis given by a matrix U such that

$$HU + d(\partial_1)U = UH_0. \tag{11}$$

We proceed as in [Ked07, lemma 3.2.8]. We write U as the sum $U = U_0 + \sum_{i=1}^{\infty} U_i x^i + \sum_{j=1}^{\infty} U_j' y^j$ with U_0, U_i and U_j' matrices with entries in K. If U_0 is invertible, U is invertible.

If we write $H = H_0 + \sum_{i=1}^{\infty} H_i x^i + \sum_{j=1}^{\infty} H_j' y^j$, then the equation (11) is equivalent to the condition that for every $i > 0$

$$H_0 U_i - U_i H_0 + i U_i = -\sum_{k=0}^{i-1} H_{i-k} U_k \tag{12}$$

and that for every $j > 0$

$$H_0 U_j' - U_j' H_0 - j U_j' = -\sum_{k=0}^{j-1} H_{j-k}' U_k'. \tag{13}$$

(Here we put $U_0' := U_0$.) For $i = 0$, the equation (11) is nothing but $H_0 U_0 = U_0 H_0$, hence we can choose U_0 to be the identity matrix.

Next, note that the linear map $\phi : X \mapsto H_0 X - X H_0$ is nilpotent, since so is H_0. Thus the linear map $X \mapsto H_0 X - X H_0 + iX$ (for $i \neq 0$) is the sum of an invertible map $\psi : X \mapsto iX$ and a nilpotent map ϕ. Hence it is invertible, because

$$\frac{\psi}{\psi + \phi} = \sum_{i=0}^{\infty} \left(-\frac{\phi}{\psi} \right)^i.$$

Hence we can construct U_i's $(i > 0)$ and U_j''s $(j > 0)$ uniquely so that (12) and (13) are satisfied.

So we have a basis $e_1, \ldots, e_s$ of E on which $\nabla_E(\partial_1)$ acts as a strictly upper triangular matrix with entries in K. We prove the equality $K e_1 + \cdots + K e_s = \overline{E}$, which implies the claim we want. The inclusion $K e_1 + \cdots + K e_s \subset \overline{E}$ follows from the definition of the basis. We prove the opposite inclusion. Let $e = b_1 e_1 + \cdots + b_s e_s$ be an element of E with $b_i \in K[[x, y]]/(xy)$ for $i = 1, \ldots, s$ such that there exists an integer N with $\nabla_E(\partial_1)^N(e) = 0$. We have to prove that $b_1, \ldots, b_s$ are in fact elements of K. Since $\nabla_E(\partial_1)$ acts as a strictly upper triangular matrix with respect to the basis $e_1, \ldots, e_s$, we can see in the expression of $\nabla_E(\partial_1)^N(e)$ as linear combination of $e_1, \ldots, e_s$ that the coefficient of e_s is $d(\partial_1)^N(b_s)$. Since $\nabla_E(\partial_1)^N(e) = 0$, $d(\partial_1)^N(b_s) = 0$. Writing $b_s = \sum_{i=0}^{\infty} \alpha_i x^i + \sum_{j=1}^{\infty} \beta_j y^j$ with $\alpha_i, \beta_j \in K$, we see that, if $d(\partial_1)^N(b_s) = 0$, then $b_s = \alpha_0 \in K$. We look now at the coefficient of e_{s-1} of $\nabla^N(\partial_1)(e)$; using the strict upper triangularity again (e_{s-1} is sent by $\nabla_E(\partial_1)$ to a K-linear combination of $e_1, \ldots, e_{s-2}$) and the fact that $b_s \in K$, we can prove that $b_{s-1} \in K$. Hence we go on and we prove the inclusion $\overline{E} \subset K e_1 + \cdots + K e_s$. So the proof of the proposition is finished. $\square$

PROPOSITION 2.20. *If the category of free modules with formal integrable connection having nilpotent residues on the log scheme* $(\mathrm{Spec}(K[[x_1,\ldots,x_{n-1}]]/(x_1\cdots x_{n-1})),M)/\mathrm{Spec}(K)^\times$ *is abelian, then the category of free modules with formal integrable connections having nilpotent residues on* $(\mathrm{Spec}(K[[x_1,\ldots,x_n]]/(x_1\cdots x_n)),M)/\mathrm{Spec}(K)^\times$ *is also abelian.*

Proof. To simplify the notation we will denote by B the ring $K[[x_1,\ldots,x_n]]/(x_1\cdots x_n)$ and by A the ring $K[[x_1,\ldots,x_{n-1}]]/(x_1\cdots x_{n-1})$. The map $h : A \to B$ defined by $x_i \mapsto x_i\,(i = 1,\ldots n-2)$, $x_{n-1} \mapsto x_{n-1}x_n$ induces a map of log schemes $(\mathrm{Spec}(B),M_B) \to (\mathrm{Spec}(A),M_A)$, where the log structure M_A, M_B are the log structure M in remark 2.17 for $\mathrm{Spec}(A), \mathrm{Spec}(B)$, respectively. (Note that the above map is not strict.) In the following, A is regarded as a subring of B via the map h.

Let Γ be the set $\{\mathbf{k} = (k_1,...,k_n) \in \mathbb{N}^n \,|\, k_l = 0 \text{ for some } 1 \le l \le n\}$ and let Γ' be the subset of Γ consisting of $\mathbf{k} = (k_1,...,k_n)$ with $k_{n-1} = k_n$. Then any element in B (resp. A) is uniquely written as $\sum_{\mathbf{k}\in\Gamma} b_{\mathbf{k}}x^{\mathbf{k}}\,(b_{\mathbf{k}} \in K)$ (resp. $\sum_{\mathbf{k}\in\Gamma'} b_{\mathbf{k}}x^{\mathbf{k}}\,(b_{\mathbf{k}} \in K)$), where $x^{\mathbf{k}} = x_1^{k_1}\cdots x_n^{k_n}$. Also, let $\partial_1,...,\partial_{n-1} \in \mathcal{H}om(\hat\omega^1_{(\mathrm{Spec}(B),M_B)/\mathrm{Spec}(K)^\times},\mathcal{O}_{\mathrm{Spec}(B)})$ be as in remark 2.17, and put $\partial := \partial_{n-1}$.

For an object (E,∇_E) in $\widehat{MIC}((\mathrm{Spec}(B),M_B)/\mathrm{Spec}(K)^\times)^{\mathrm{nr}}$, put

$$\overline{E} := \{e \in E \,|\, \nabla_E^N(\partial)(e) \to 0\,(N \to \infty)\},$$

where, on the right hand side, E is endowed with $(x_1,...,x_n)$-adic topology. Then $\overline{E}$ is stable by the action of $\nabla_E(\partial_i)\,(1 \le i \le n-2)$ and the action of $\nabla_E(\partial)$ which is locally topologically nilpotent. Then, it suffices to prove the following: firstly, $\overline{E}$ and the actions $\nabla_E(\partial_i)\,(1 \le i \le n-2)$ on it define an object (which we denote by $(\overline{E},\nabla_{\overline{E}})$) in $\widehat{MIC}((\mathrm{Spec}(A),M_A)/\mathrm{Spec}(K)^\times)^{\mathrm{nr}}$. Secondly, the correspondence $(E,\nabla_E) \mapsto ((\overline{E},\nabla_{\overline{E}}),\nabla_E(\partial_1))$ defines the functor

$$\widehat{MIC}((\mathrm{Spec}(B),M_B)/\mathrm{Spec}(K)^\times)^{\mathrm{nr}}$$
$$\to \left\{ ((F,\nabla_F),N) \,\middle|\, \begin{array}{l} (F,\nabla_F) \in \widehat{MIC}((\mathrm{Spec}(A),M_A)/\mathrm{Spec}(K)^\times)^{\mathrm{nr}} \\ N : (F,\nabla_F) \to (F,\nabla_F)\text{: a nilpotent endomorphism} \end{array} \right\}$$

and it is an equivalence whose quasi-inverse is given by $((F,\nabla_F),N) \mapsto (E := F \otimes_A B, \nabla_E)$ with the action $\nabla_E(\partial_i)\,(1 \le i \le n-2)$ and $\nabla_E(\partial)$ defined by $\nabla_E(\partial_i) := \mathrm{id}\otimes d(\partial_i) + \nabla_F(\partial_i)\otimes\mathrm{id},\ \nabla_E(\partial) := \mathrm{id}\otimes d(\partial) + N\otimes\mathrm{id}$. (The reason is the same as that in lemma 2.18.)

To prove the above claim, it suffices to construct a basis of $\overline{E}$ over A which is a basis of E over B. To do so, first we prove the following claim.

CLAIM 1: There exists a basis of E as B-module on which $\nabla_E(\partial)$ acts as a matrix M with entries in A. Moreover, if we write $M = \sum_{\mathbf{k}\in\Gamma'} M_{\mathbf{k}}x^{\mathbf{k}}$ with entries of $M_{\mathbf{k}}$ in K, M_0 is strictly upper triangular.

Proof. Let us fix a basis $\mathbf{e} = (e_1, \ldots, e_s)$ of E and let us denote by H the matrix such that

$$\nabla_E(\partial) = d(\partial) + H$$

with respect to the basis $\mathbf{e}$. We write $H = \sum_{\mathbf{k} \in \Gamma} H_{\mathbf{k}} x^{\mathbf{k}}$ with entries of $H_{\mathbf{k}}$ in K. We may assume that H_0 is strictly upper triangular because we assumed that (E, ∇_E) has nilpotent residues. Hence it suffices to prove the existence of matrices $U = \sum_{\mathbf{k} \in \Gamma} U_{\mathbf{k}} x^{\mathbf{k}}, X = \sum_{\mathbf{k} \in \Gamma} X_{\mathbf{k}} x^{\mathbf{k}}$ satisfying the following conditions:

(a) U_0 is the identity matrix and $U_{\mathbf{k}} = 0$ if $\mathbf{k} \in \Gamma' \setminus \{0\}$.
(b) $X_0 = 0$ and $X_{\mathbf{k}} = 0$ if $\mathbf{k} \in \Gamma \setminus \Gamma'$.
(c) $HU + d(\partial)U = U(H_0 + X)$.

In fact, U is invertible by the condition (a) and then the conditions (b) and (c) imply that, after some change of basis, the action of $\nabla_E(\partial)$ on the new basis is described by the matrix $M := H_0 + X$, whose entries belong to A with its constant term H_0 strict upper triangular.

The equality (c) is equivalent to the condition that, for any $\mathbf{k} \in \Gamma$,

$$H_0 U_{\mathbf{k}} - U_{\mathbf{k}} H_0 + (k_{n-1} - k_n) U_{\mathbf{k}} = -\sum_{\mathbf{i} < \mathbf{k}} H_{\mathbf{k}-\mathbf{i}} U_{\mathbf{i}} + \sum_{\mathbf{i} < \mathbf{k}} U_{\mathbf{i}} X_{\mathbf{k}-\mathbf{i}}. \qquad (14)$$

(Here, for $\mathbf{i} = (i_1, ..., i_n)$ and $\mathbf{k} = (k_1, ..., k_n)$, we write $\mathbf{i} < \mathbf{k}$ if $i_j \leq k_j$ $(1 \leq j \leq n)$ and $\mathbf{i} \neq \mathbf{k}$.) We construct (uniquely) the matrices $U_{\mathbf{k}}, X_{\mathbf{k}}$ by induction on $|\mathbf{k}| = \sum_i k_i$ so that the conditions (a), (b) and the equality (14) are satisfied. If $\mathbf{k} = 0$, U_0 should be the identity matrix and X_0 should be zero by (a) and (b). In this case, the equality (14) for $\mathbf{k} = 0$ is obviously satisfied. For general $\mathbf{k}$ with $k_{n-1} = k_n$, $U_{\mathbf{k}}$ should be zero by (a) and so the equality (14) is written as

$$0 = -\sum_{\mathbf{i} < \mathbf{k}} H_{\mathbf{k}-\mathbf{i}} U_{\mathbf{i}} + \sum_{0 < \mathbf{i} < \mathbf{k}} U_{\mathbf{i}} X_{\mathbf{k}-\mathbf{i}} + X_{\mathbf{k}}.$$

Then we can take a unique matrix $X_{\mathbf{k}}$ satisfying this equality. For general $\mathbf{k}$ with $k_{n-1} \neq k_n$, $X_{\mathbf{k}}$ should be 0 by (b) and so the equality (14) is written as

$$H_0 U_{\mathbf{k}} - U_{\mathbf{k}} H_0 + (k_{n-1} - k_n) U_{\mathbf{k}} = -\sum_{\mathbf{i} < \mathbf{k}} H_{\mathbf{k}-\mathbf{i}} U_{\mathbf{i}}.$$

We can take a unique matrix $U_{\mathbf{k}}$ satisfying this equality because the map $Y \mapsto H_0 Y - Y H_0 + (k_{n-1} - k_n) Y$, being the sum of a nilpotent map $Y \mapsto H_0 Y - Y H_0$ and an invertible map $Y \mapsto (k_{n-1} - k_n) Y$, is invertible. So we proved the claim. $\square$

Let $e_1, \ldots, e_s$ be a B-basis of E which satisfies the condition in the statement of CLAIM 1. We prove the following claim, which implies the claim we want:

CLAIM 2: $Ae_1 + \cdots + Ae_s = \overline{E}$.

Proof. Let us prove first the inclusion $Ae_1 + \cdots + Ae_s \subset \overline{E}$. For an element $e := a_1e_1 + \cdots + a_se_s$ with $a_i \in A$, we calculate $\nabla_E(\partial)^N(a_1e_1 + \cdots + a_se_s) = a_1\nabla_E(\partial)^N(e_1) + \cdots + a_s\nabla_E(\partial)^N(e_s)$. Let us denote by v the $(x_1, \ldots, x_n)$-adic valuation on B. Then this induces a valuation v on E defined by $v(\sum_{i=1}^s b_ie_i) := \min_i v(b_i)$. To prove that e is in $\overline{E}$, we need to prove that for every $m \in \mathbb{N}$ there exists an $N \in \mathbb{N}$ such that $v(\nabla_E(\partial)^N(e)) \geq m$. Since v is non-archimedean $v(\nabla_E(\partial)^N(e)) \geq \min_i v(a_i\nabla_E(\partial)^N(e_i)) = \min_i\{v(a_i) + v(\nabla_E(\partial)^N(e_i))\}$, hence it is enough to prove that for every $m \in \mathbb{N}$ there exists an $N \in \mathbb{N}$ such that $v(\nabla_E(\partial)^N(e_i)) \geq m$. The action of $\nabla_E(\partial)$ on $Ae_1 + \cdots + Ae_s$ is written by some matrix $\sum_{\mathbf{k}} M_{\mathbf{k}}\mathbf{x}^k$ with M_0 nilpotent. Hence each entry of the matrix $\nabla_E(\partial)^s$ has strictly positive $(x_1, \ldots, x_n)$-adic valuation. Then we see that $v(\nabla_E(\partial)^{ms}e_i) \geq m$. Hence, if we choose $N \geq ms$, we are done.

Next we prove the inclusion $\overline{E} \subset Ae_1 + \cdots + Ae_s$. Let $e = a_1e_1 + \cdots + a_se_s \in E$ with $a_j \in B$ for every j, such that for all $m \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that $\nabla_E(\partial)^N(e) = b_1e_1 + \cdots + b_se_s$ with $b_j = \sum_{\mathbf{k},|\mathbf{k}|\geq m} \beta_{\mathbf{k}}\mathbf{x}^k$, namely, $v(\nabla_E(\partial)^N(e)) \geq m$. We want to prove that $a_j \in A$ for every j. Let us calculate

$$\nabla_E(\partial)^N(a_1e_1 + \cdots + a_se_s)$$
$$= \sum_{l=0}^N \binom{N}{l}(d(\partial)^{N-l}(a_1)\nabla_E(\partial)^l(e_1) + \cdots + d(\partial)^{N-l}(a_s)\nabla_E(\partial)^l(e_s)). \tag{15}$$

Let us fix i and write $a_i = \sum_{\mathbf{k}} \alpha_{\mathbf{k}}\mathbf{x}^k$; we are interested in how $d(\partial)$ acts on a_i. It acts on monomials of the form $\mathbf{x}^k = x_1^{k_1}\cdots x_{n-1}^{k_{n-1}}x_n^{k_n}$ by $d(\partial)(\mathbf{x}^k) = (k_{n-1} - k_n)\mathbf{x}^k$. We define a function $\delta : B \to \mathbb{N} \cup \{\infty\}$ as follows; if $a_i = \sum_{\mathbf{k}} \alpha_{\mathbf{k}}\mathbf{x}^k \in B$, then $\delta(a_i) := \min\{|\mathbf{k}|\,|\,\alpha_{\mathbf{k}} \neq 0, k_{n-1} - k_n \neq 0\}$. (We define that $\delta(a_i) = \infty$ if the set on the right hand side is empty.) Then $\delta(a_i) = \infty$ if and only if $a_i \in A$. Using the function δ we prove the claim that, for every $l \geq 1$, the element $d(\partial)^l(a_i)$ has $(x_1, \ldots, x_n)$-adic valuation equal to $\delta(a_i)$. If $a_i \in A$, the claim is true because $d(\partial)^l(a_i) = 0$ and $\delta(a_i) = \infty$. If $a_i \in B$ but not in A, then there exists $\overline{\mathbf{k}} = (\overline{k}_1, \ldots, \overline{k}_n)$ such that $\overline{k}_{n-1} \neq \overline{k}_n$, $|\overline{\mathbf{k}}| = \delta(a_i) < \infty$. We can write a_i as

$$a_i = \sum_{\mathbf{k},|\mathbf{k}|<|\overline{\mathbf{k}}|} \alpha_{\mathbf{k}}\mathbf{x}^k + \alpha_{\overline{\mathbf{k}}}\mathbf{x}^{\overline{\mathbf{k}}} + \sum_{\mathbf{k}\neq\overline{\mathbf{k}},|\mathbf{k}|\geq|\overline{\mathbf{k}}|,} \alpha_{\mathbf{k}}\mathbf{x}^k.$$

By definition of $\overline{\mathbf{k}}$, we have the equality

$$d(\partial)^l(a_i) = \alpha_{\overline{\mathbf{k}}}\mathbf{x}^{\overline{\mathbf{k}}}(\overline{k}_{n-1} - \overline{k}_n)^l + \sum_{\mathbf{k}\neq\overline{\mathbf{k}},|\mathbf{k}|\geq|\overline{\mathbf{k}}|,} \alpha_{\mathbf{k}}\mathbf{x}^k(k_{n-1} - k_n)^l.$$

So we conclude that $d(\partial)^l(a_i)$ has $(x_1, \ldots, x_n)$-adic valuation equal to $\delta(a_i)$, as we wanted.

We come back to $e = a_1e_1 + \cdots + a_se_s$. We know by hypothesis that for $m \in \mathbb{N}$, there exists N such that $v(\nabla_E(\partial)^N(e)) \geq m$. We have to show that a_i is in A for $i = 1, \ldots, s$. Let us suppose by absurd that there exists some

j such that $\delta(a_j) \neq \infty$. We can suppose that $\delta(a_j) \leq \delta(a_i)$ for every $i = 1, \ldots, s$ and that $\delta(a_j) < \delta(a_i)$ for all $i > j$. We look at the e_j-component of $\nabla_E(\partial)^N(e)$ for $N \geq 1$, using the expression of (15). Thanks to what we have shown $v(d(\partial)^N(a_j)e_j) = \delta(a_j)$; for $1 \leq l \leq N-1$, the e_j-component of $d(\partial)^{N-l}(a_i)\nabla_E(\partial)^l(e_i)$, denoted by $[d(\partial)^{N-l}(a_i)\nabla_E(\partial)^l(e_i)]_j$, has the following $(x_1, \ldots, x_n)$-adic valuation for $i > j$:

$$v([d(\partial)^{N-l}(a_i)\nabla_E(\partial)^l(e_i)]_j) \geq \delta(a_i) > \delta(a_j),$$

while for $i \leq j$

$$v([d(\partial)^{N-l}(a_i)\nabla_E(\partial)^l(e_i)]_j) \geq \delta(a_i) + 1 > \delta(a_j)$$

because e_i is sent via the matrix M_0 to a K-linear combination of $e_1, \ldots, e_{i-1}$ due to the assumption that M_0 is strictly upper triangular. Also, the e_j-component $[a_i\nabla_E(\partial)^N(e_i)]_j$ of $a_i\nabla_E(\partial)^N(e_i)$ has the following $(x_1, \ldots, x_n)$-adic valuation, by the argument in the first paragraph of the proof:

$$v([a_i\nabla_E(\partial)^N(e_i)]_j) \geq \lfloor N/s \rfloor.$$

Therefore, if $N \geq s(\delta(a_j) + 1)$, the $(x_1, \ldots, x_n)$-adic valuation of the e_j-component $[\nabla_E(\partial)^N(a)]_j$ of $\nabla_E(\partial)^N(a)$ is given as

$$v([\nabla_E(\partial)^N(a)]_j) = \delta(a_j),$$

and this does not go to infinity as $N \to \infty$. This is a contradiction. Hence $a_i \in A$ for all i, as we wanted. $\square$

Since CLAIM 2 is proved, the proof of the proposition is finished. $\square$

PROPOSITION 2.21. *Let us suppose that there exists a K-rational point $x \in X$. Then $MIC(X^\times/\mathrm{Spec}(K)^\times)^{\mathrm{nr}}$ is a neutral Tannakian category over K.*

Proof. As we saw in proposition 2.16 the category $MIC(X^\times/\mathrm{Spec}(K)^\times)^{\mathrm{nr}}$ is abelian; moreover with the tensor structure defined in $MIC(X^\times/\mathrm{Spec}(K)^\times)$ and unit object given by $(\mathcal{O}_X, d)$ (see remark 2.6) it is a rigid abelian tensor category, thanks to the fact that, for every object (E, ∇_E) in $MIC(X^\times/\mathrm{Spec}(K)^\times)^{\mathrm{nr}}$, E is a locally free $\mathcal{O}_X$-module.
Taking the fiber at x gives a map $\mathrm{End}((\mathcal{O}_X, d)) \to K$, which is injective because, thanks to [Shi00, Proposition 3.1.6], we know that $\mathrm{End}((\mathcal{O}_X, d)) \cong H^0_{\mathrm{dR}}(X^\times/\mathrm{Spec}(K)^\times)$ is a field. But since $K \subset \mathrm{End}((\mathcal{O}_X, d))$, we have indeed that $\mathrm{End}((\mathcal{O}_X, d)) \cong K$.
We define a functor

$$\omega_x : MIC(X^\times/\mathrm{Spec}(K)^\times)^{\mathrm{nr}} \to \mathrm{Vec}_K$$

which sends every pair (E, ∇_E) to the K-vector space given by the fiber $E(x)$ of E at x. Since E is locally free for every object (E, ∇_E) in $MIC(X^\times/\mathrm{Spec}(K)^\times)^{\mathrm{nr}}$, ω_x is an exact tensor functor and by [Del90, Corollaire 2.10] ω_x is faithful. So it is a fiber functor. $\square$

Definition 2.22. *Let x be a K-rational point of X and let*

$$\omega_x : MIC(X^\times/\mathrm{Spec}(K)^\times)^{\mathrm{nr}} \to \mathrm{Vec}_K$$

be the fiber functor introduced in the above proposition. We define the log algebraic fundamental group of $X^\times$ over $\mathrm{Spec}(K)^\times$ with base point x as the Tannaka dual of $MIC(X^\times/\mathrm{Spec}(K)^\times)^{\mathrm{nr}}$, i.e.

$$\pi_1(X^\times/\mathrm{Spec}(K)^\times, x) := G(MIC(X^\times/\mathrm{Spec}(K)^\times)^{\mathrm{nr}}, \omega_x)$$

with the same notation as in theorem 2.9.

Next we prove that $MIC(X^\times/\mathrm{Spec}(K))^{\mathrm{nr}}$ is also a neutral Tannakian category over K under some suitable assumptions.

Proposition 2.23. *The category $MIC(X^\times/\mathrm{Spec}(K))^{\mathrm{nr}}$ is abelian.*

Proof. Note first that, by restricting the derivations, a functor

$$r : MIC(X^\times/\mathrm{Spec}(K))^{\mathrm{nr}} \to MIC(X^\times/\mathrm{Spec}(K)^\times)^{\mathrm{nr}}$$

is defined. To see this, it suffices to check that, for an object (E, ∇_E) in $MIC(X^\times/\mathrm{Spec}(K))^{\mathrm{nr}}$, the module with integrable connection $r((E, \nabla_E)) := (E, \nabla_{E|\mathrm{Spec}(K)^\times})$ has nilpotent residues. This is true because, for every geometric point y over a closed point of X and every $K(y)$-linear map $t_y : K(y) \otimes \overline{M}_y^{\mathrm{gp}}/\overline{N}_y^{\mathrm{gp}} \to K(y)$, the composite map

$$E(y) \xrightarrow{\rho_y} E(y) \otimes \overline{M}_y^{\mathrm{gp}}/\overline{N}_y^{\mathrm{gp}} \xrightarrow{\mathrm{id}\otimes t_y} E(y)$$

(where the notation is as in definition 2.11 for $r((E, \nabla_E))$) coincides with the nilpotent map

$$E(y) \xrightarrow{\rho'_y} E(y) \otimes \overline{M}_y^{\mathrm{gp}} \xrightarrow{\mathrm{id}\otimes t'_y} E(y),$$

where ρ'_y is the map induced by (E, ∇_E) and t'_y is the composite of the projection $K(y) \otimes M_y^{\mathrm{gp}} \longrightarrow K(y) \otimes \overline{M}_y^{\mathrm{gp}}/\overline{N}_y^{\mathrm{gp}}$ and t_y.
For any morphism $\varphi : E \to F$ in $MIC(X^\times/\mathrm{Spec}(K))^{\mathrm{nr}}$, the underlying $\mathcal{O}_X$-module of $\mathrm{Ker}(\varphi)$ (resp. $\mathrm{Coker}(\varphi)$) is the same as that of $\mathrm{Ker}(r(\varphi))$ (resp. $\mathrm{Coker}(r(\varphi))$), and the latter is locally free by proposition 2.16. Also, since the same is true for the image and the coimage of φ, we have the natural injection $\mathrm{Ker}(r(\varphi))(y) \hookrightarrow E(y)$ and the natural surjection $F(y) \twoheadrightarrow \mathrm{Coker}(\varphi)(y)$ of fibers at any geometric point y of X. Since these maps are compatible with residues, we conclude that $\mathrm{Ker}(\varphi)$ and $\mathrm{Coker}(\varphi)$ have nilpotent residues. So the proof is finished. $\qquad\square$

Proposition 2.24. *Suppose that there exists a K-rational point x of X. Then the category $MIC(X^\times/\mathrm{Spec}(K))^{\mathrm{nr}}$ is a neutral Tannakian category over K.*

Proof. The proof is analogous to that of proposition 2.21. We saw in proposition 2.23 that $MIC(X^\times/\mathrm{Spec}(K))^{\mathrm{nr}}$ is an abelian category. Moreover, with the tensor structure defined in $MIC(X^\times/\mathrm{Spec}(K))$ and the unit object given by $(\mathcal{O}_X, d)$ (see remark 2.6) it is a rigid abelian tensor category, thanks to the fact that, for every object (E, ∇_E) in $MIC(X^\times/\mathrm{Spec}(K))^{\mathrm{nr}}$, E is a locally free $\mathcal{O}_X$-module.

To prove that it is a neutral Tannakian category over K we have to check that $\mathrm{End}((\mathcal{O}_X, d)) \cong K$ and to construct a fiber functor. Every $\alpha \in \mathrm{End}((\mathcal{O}_X, d))$ is indeed an element of $\mathrm{End}_{MIC(X^\times/\mathrm{Spec}(K)^\times)}((\mathcal{O}_X, d_{|\mathrm{Spec}(K)^\times}))$ which is isomorphic to K as we saw in proposition 2.21. Since $K \subset \mathrm{End}((\mathcal{O}_X, d))$, $\mathrm{End}((\mathcal{O}_X, d)) \cong K$ as required. We define a functor

$$\eta_x : MIC(X^\times/\mathrm{Spec}(K))^{\mathrm{nr}} \to \mathrm{Vec}_K$$

which sends a pair $(E, \nabla_E) \in MIC(X^\times/\mathrm{Spec}(K))^{\mathrm{nr}}$ to the fiber $E(x)$ of E at x. Since the underlying $\mathcal{O}_X$-module E is locally free for every object (E, ∇_E) in $MIC(X^\times/\mathrm{Spec}(K))^{\mathrm{nr}}$, η_x is an exact tensor functor and by [Del90, Corollaire 2.10] it is faithful. Hence it is a fiber functor. □

DEFINITION 2.25. *Let x be a K-rational point of X and let*

$$\eta_x : MIC(X^\times/\mathrm{Spec}(K))^{\mathrm{nr}} \to \mathrm{Vec}_K$$

be the fiber functor introduced in the above proposition. We define the log algebraic fundamental group of $X^\times$ over $\mathrm{Spec}(K)$ with base point x as the Tannaka dual of $MIC(X^\times/\mathrm{Spec}(K))^{\mathrm{nr}}$, i.e.

$$\pi_1(X^\times/\mathrm{Spec}(K), x) := G(MIC(X^\times/\mathrm{Spec}(K))^{\mathrm{nr}}, \eta_x)$$

with the same notation as in theorem 2.9.

3 The exact sequence with four terms

In this section we suppose that there exists a K-rational point x of X.
The morphism of log schemes $f : X^\times \to \mathrm{Spec}(K)^\times$ induces a functor

$$f_{\mathrm{dR}}^* : MIC(\mathrm{Spec}(K)^\times/\mathrm{Spec}(K))^{\mathrm{nr}} \to MIC(X^\times/\mathrm{Spec}(K))^{\mathrm{nr}} :$$

indeed, if (E, ∇_E) is an object of $MIC(\mathrm{Spec}(K)^\times/\mathrm{Spec}(K))^{\mathrm{nr}}$, the module with integrable connection $f_{\mathrm{dR}}^*((E, \nabla_E))$ is defined as the pair $(f^*(E), f^*(\nabla_E))$, where $f^*(E)$ is the $\mathcal{O}_X$-module $\mathcal{O}_X \otimes_{f^{-1}K} f^{-1}(E)$ and $f^*(\nabla_E)$ is the unique connection which extends $f^{-1}(E) \xrightarrow{f^{-1}(\nabla_E)} f^{-1}(E) \otimes_{f^{-1}K} f^{-1}\omega^1_{\mathrm{Spec}(K)^\times/\mathrm{Spec}(K)} \to f^*(E) \otimes_{\mathcal{O}_X} \omega^1_{X^\times/\mathrm{Spec}(K)}$. The module with integrable connection $f_{\mathrm{dR}}^*((E, \nabla_E))$ has nilpotent residues: indeed, for every geometric point y over a closed point of X and every $K(y)$-linear map $t_y : K(y) \otimes \overline{M}_y^{\mathrm{gp}} \to K(y)$, the composite map

$$f^*(E)(y) \xrightarrow{\rho_y} f^*(E)(y) \otimes \overline{M}_y^{\mathrm{gp}} \xrightarrow{\mathrm{id} \otimes t_y} f^*(E)(y)$$

(where the notation is as in definition 2.11) is nilpotent because it coincides with the nilpotent map $\nabla_E : E \to E \otimes \overline{N}^{\mathrm{gp}} \cong E$ up to scalar after we tensor it with $K(y)$.

Also, as we have seen in the proof of proposition 2.23, the restriction functor

$$r : MIC(X^\times/\mathrm{Spec}(K))^{\mathrm{nr}} \to MIC(X^\times/\mathrm{Spec}(K)^\times)^{\mathrm{nr}}$$

is defined.

Let $\pi_1(\mathrm{Spec}(K)^\times/\mathrm{Spec}(K), \nu)$ be as in definition 2.15, let $\pi_1(X^\times/\mathrm{Spec}(K)^\times, x)$ be as in definition 2.22 and let $\pi_1(X^\times/\mathrm{Spec}(K)^\times, x)$ be as in definition 2.25. Then the functors f_{dR}^* and r induce respectively the homomorphisms $\pi(f_{\mathrm{dR}}^*)$ and $\pi(r)$ in the sequence

$$\pi_1(X^\times/\mathrm{Spec}(K)^\times, x) \xrightarrow{\pi(r)} \pi_1(X^\times/\mathrm{Spec}(K), x) \tag{16}$$
$$\xrightarrow{\pi(f_{\mathrm{dR}}^*)} \pi_1(\mathrm{Spec}(K)^\times/\mathrm{Spec}(K), \nu).$$

In the remaining part of the section we prove that the sequence (16) is exact and that $\pi(f_{\mathrm{dR}}^*)$ is faithfully flat, using a theorem by H. Esnault, P. H. Hai and X. Sun ([EHS08, Theorem A1]) which translates the exactness of the sequence of group schemes in terms of the corresponding categories of representations. We recall the theorem here for the convenience of the reader.

THEOREM 3.1. *Let $L \xrightarrow{\pi(q)} G \xrightarrow{\pi(p)} A$ be a sequence of homomorphisms of affine group schemes over a field K. It induces a sequence of functors*

$$\mathrm{Rep}(A) \xrightarrow{p} \mathrm{Rep}(G) \xrightarrow{q} \mathrm{Rep}(L),$$

where Rep *is the category of finite dimensional representations over K.*

(i) *The map $\pi(p)$ is faithfully flat if and only if p is fully faithful and the subcategory of $\mathrm{Rep}(G)$ given by $p(\mathrm{Rep}(A))$ is closed by subobjects.*

(ii) *The map $\pi(q)$ is a closed immersion if and only if any object of $\mathrm{Rep}(L)$ is a subquotient of an object of the form $q(V)$ for $V \in \mathrm{Rep}(G)$.*

(iii) *Let us assume that $\pi(p)$ is faithfully flat and that $\pi(q)$ is a closed immersion. The sequence $L \xrightarrow{\pi(q)} G \xrightarrow{\pi(p)} A$ is exact if and only if the following conditions are satisfied:*

    (a) *For every $E \in \mathrm{Rep}(G)$, $q(E)$ in $\mathrm{Rep}(L)$ is trivial if and only if there exists V in $\mathrm{Rep}(A)$ such that $p(V) \cong E$.*

    (b) *Let E be an object in $\mathrm{Rep}(G)$ and let W_0 be the maximal trivial subobject of $q(E)$ in $\mathrm{Rep}(L)$. Then there exists $F \subset E$ in $\mathrm{Rep}(G)$ such that $q(F) \cong W_0$.*

    (c) *Any W in $\mathrm{Rep}(L)$ is a quotient of (hence, by taking duals, embeddable in) $q(E)$ for some $E \in \mathrm{Rep}(G)$.*

Proposition 3.2. *Let (E, ∇_E) be an object in $MIC(X^\times/\mathrm{Spec}(K))^{\mathrm{nr}}$. Then $H^0_{\mathrm{dR}}(X^\times/\mathrm{Spec}(K)^\times, r((E, \nabla_E)))$ comes equipped with a nilpotent endomorphism ∇, in such a way that $(H^0_{\mathrm{dR}}(X^\times/\mathrm{Spec}(K)^\times, r((E, \nabla_E))), \nabla)$ is an object of $MIC(\mathrm{Spec}(K)^\times/\mathrm{Spec}(K))^{\mathrm{nr}}$.*

Proof. We need to prove that $H^0_{\mathrm{dR}}(X^\times/\mathrm{Spec}(K)^\times, r((E, \nabla_E)))$ is a finite dimensional K-vector space endowed with a nilpotent endomorphism. If we denote by $E^{\nabla_E|\mathrm{Spec}(K)^\times}$ the sheaf of horizontal sections of $r((E, \nabla_E))$, then by definition $H^0_{\mathrm{dR}}(X^\times/\mathrm{Spec}(K)^\times, r((E, \nabla_E))) = f_*(E^{\nabla_E|\mathrm{Spec}(K)^\times})$. The sheaf $E^{\nabla_E|\mathrm{Spec}(K)^\times}$ is equipped with an action of $\mathcal{D}er(\mathrm{Spec}(K)^\times/K)$: indeed by [Kat89, proposition 3.12] we have an exact sequence of sheaves

$$
\begin{aligned}
0 \to \mathcal{D}er(X^\times/\mathrm{Spec}(K)^\times) &\to \mathcal{D}er(X^\times/\mathrm{Spec}(K)) \\
&\to \mathcal{O}_X \otimes \mathcal{D}er(\mathrm{Spec}(K)^\times/\mathrm{Spec}(K)) \to 0
\end{aligned}
\tag{17}
$$

which locally splits. We denote by τ a splitting

$$
\mathcal{O}_X \otimes \mathcal{D}er(\mathrm{Spec}(K)^\times/\mathrm{Spec}(K)) \to \mathcal{D}er(X^\times/\mathrm{Spec}(K)).
$$

For every $D \in \mathcal{D}er(\mathrm{Spec}(K)^\times/\mathrm{Spec}(K))$ we define the action of $\nabla(D)$ on $E^{\nabla_E|\mathrm{Spec}(K)^\times}$ by

$$
\nabla(D)(e) := \nabla_E(\tau(1 \otimes D))(e) \quad (e \in E^{\nabla_E|\mathrm{Spec}(K)^\times}).
$$

The action is well-defined and it does not depend on the splitting because two different splittings differ by an element of $\mathcal{D}er(X^\times/\mathrm{Spec}(K)^\times)$. Thus we have an action ∇ of $\mathcal{D}er((\mathrm{Spec}(K)^\times)/\mathrm{Spec}(K))$ on $E^{\nabla_E|\mathrm{Spec}(K)^\times}$ and hence on $f_*(E^{\nabla_E|\mathrm{Spec}(K)^\times})$, which defines the structure of a connection (see [Kat70, remark 3.1] and [Hai13, lemma 2.1] for the analogous action in the smooth case).

We denote by $f_{\mathrm{dR}*}((E, \nabla_E))$ the K-vector space $f_*(E^{\nabla_E|\mathrm{Spec}(K)^\times})$ endowed with the action ∇ described above. Then there is a morphism of $\mathcal{O}_X$-modules with integrable connection

$$
f^*_{\mathrm{dR}}(f_{\mathrm{dR}*}((E, \nabla_E))) \longrightarrow (E, \nabla_E),
\tag{18}
$$

because the natural map between the sheaves $f^*(f_*(E^{\nabla_E|\mathrm{Spec}(K)^\times})) \longrightarrow E^{\nabla_E|\mathrm{Spec}(K)^\times}$ composed with the inclusion $E^{\nabla_E|\mathrm{Spec}(K)^\times} \hookrightarrow E$ is indeed horizontal. (Here f^*_{dR} denotes the pullback of quasi-coherent modules with integrable connection. Note that we have not yet proved that $f_{\mathrm{dR}*}((E, \nabla_E))$ belongs to the category $MIC(\mathrm{Spec}(K)^\times/\mathrm{Spec}(K))^{\mathrm{nr}}$, which is the purpose of this proposition.) To prove that $f_{\mathrm{dR}*}((E, \nabla_E))$ belongs to $MIC(\mathrm{Spec}(K)^\times/\mathrm{Spec}(K))^{\mathrm{nr}}$, it suffices to prove that the morphism (18) is injective and that it has nilpotent residues. Let us denote the neutral Tannakian category $MIC(X^\times/\mathrm{Spec}(K)^\times)^{\mathrm{nr}}$ by $\mathcal{T}$ and denote the unit object $(\mathcal{O}_X, d)$ of

$\mathcal{T}$ simply by $\mathbf{1}$. Also, put $V := r((E, \nabla_E))$. Then the morphism we obtain by applying r to (18) is the canonical map

$$\mathrm{Hom}_{\mathcal{T}}(\mathbf{1}, V) \otimes_K \mathbf{1} \to V, \tag{19}$$

and this map is injective in any neutral Tannakian category. So we have shown the required injectivity. Also, since the cokernel of the map (19) is also locally free, if we pull back the injection (18) to a smooth closed point s of X, we obtain an injection

$$(f_{\mathrm{dR}*}((E, \nabla_E))) \otimes_K K(s) \longrightarrow s^*(E, \nabla_E),$$

and by functoriality of the nilpotence of residues, we see that $f_{\mathrm{dR}*}((E, \nabla_E))$ has nilpotent residues. So we are done. $\square$

DEFINITION 3.3. *Let* (E, ∇_E) *be an object in* $MIC(X^\times/\mathrm{Spec}(K))^{\mathrm{nr}}$. *The object*

$$(H^0_{\mathrm{dR}}(X^\times/\mathrm{Spec}(K)^\times, r((E, \nabla_E))), \nabla)$$

of $MIC(\mathrm{Spec}(K)^\times/\mathrm{Spec}(K))^{\mathrm{nr}}$ *will be denoted by* $f_{\mathrm{dR}*}((E, \nabla_E))$.

We denote by H the group scheme $\pi(r)(\pi_1(X^\times/\mathrm{Spec}(K)^\times, x))$.

PROPOSITION 3.4. *The category* $\mathrm{Rep}(H)$ *can be described as the full subcategory of* $MIC(X^\times/\mathrm{Spec}(K)^\times)^{\mathrm{nr}}$ *whose objects are subquotients of objects of the type* $r((E, \nabla_E))$ *with* $(E, \nabla_E) \in MIC(X^\times/\mathrm{Spec}(K))^{\mathrm{nr}}$.

Proof. The proof is analogous to [EH06, proposition 3.1]. Let C be the full subcategory of $MIC(X^\times/\mathrm{Spec}(K)^\times)^{\mathrm{nr}}$ whose objects are subquotients of objects of the form $r((E, \nabla_E))$ with $(E, \nabla_E) \in MIC(X^\times/\mathrm{Spec}(K))^{\mathrm{nr}}$. Let i be the inclusion functor $C \to MIC(X^\times/\mathrm{Spec}(K)^\times)^{\mathrm{nr}}$ and let $G(C, x)$ be the Tannaka dual of C as in theorem 2.9. The morphism $\pi(r) : \pi_1(X^\times/\mathrm{Spec}(K)^\times, x) \to \pi_1(X^\times/\mathrm{Spec}(K), x)$ factors through $G(C, x)$ with a faithfully flat morphism $\pi(i) : \pi_1(X^\times/\mathrm{Spec}(K)^\times, x) \to G(C, x)$ followed by a closed immersion $G(C, x) \to \pi_1(X^\times/\mathrm{Spec}(K), x)$: indeed, by theorem 3.1 (i), $\pi(i)$ is faithfully flat because i is clearly fully faithful and $i(C)$ is closed by subobjects. Also, the map $G(C, x) \to \pi_1(X^\times/\mathrm{Spec}(K), x)^{\mathrm{nr}}$ is a closed immersion because by definition every object of C is a subquotient of an object of $r(MIC(X^\times/\mathrm{Spec}(K))^{\mathrm{nr}})$, hence the condition in theorem 3.1 (ii) is satisfied. $\square$

PROPOSITION 3.5. *The map of fundamental groups*

$$\pi(f^*_{\mathrm{dR}}) : \pi_1(X^\times/\mathrm{Spec}(K), x) \longrightarrow \pi_1(\mathrm{Spec}(K)^\times/\mathrm{Spec}(K), x)$$

is faithfully flat.

Proof. By theorem 3.1 (i), to prove the proposition we are asked to prove that f^*_{dR} is fully faithful and that $f^*_{\mathrm{dR}}(MIC(\mathrm{Spec}(K)^\times/\mathrm{Spec}(K))^{\mathrm{nr}})$ is closed by

taking subobjects. Let $(V, \nabla_V) \in MIC(\mathrm{Spec}(K)^\times/\mathrm{Spec}(K))^{\mathrm{nr}}$. To prove that f_{dR}^* is fully faithful, it suffices to prove that f_{dR}^* induces an isomorphism

$$
\begin{aligned}
H_{\mathrm{dR}}^0(\mathrm{Spec}(K)^\times&/\mathrm{Spec}(K), (V, \nabla_V)) \\
&\cong H_{\mathrm{dR}}^0(X^\times/\mathrm{Spec}(K), f_{\mathrm{dR}}^*((V, \nabla_V))).
\end{aligned}
\tag{20}
$$

Note that we have the isomorphism

$$
f_{\mathrm{dR}*}(f_{\mathrm{dR}}^*((V, \nabla_V))) \cong (V, \nabla_V);
\tag{21}
$$

indeed, $f_{\mathrm{dR}*}(f_{\mathrm{dR}}^*((V, \nabla_V)))$ is by definition the K-vector space $f_*(f^*(V)^{f^*(\nabla_V)_{|\mathrm{Spec}(K)^\times}})$ endowed with the nilpotent endomorphism introduced in the proof of proposition 3.2. By definition of the connection $f^*(\nabla_V)$ and the isomorphism $f_{\mathrm{dR}*}(\mathcal{O}_X, d) = f_*(\mathcal{O}_X^{d_{|\mathrm{Spec}(K)^\times}}) \cong K$, we have the isomporphism
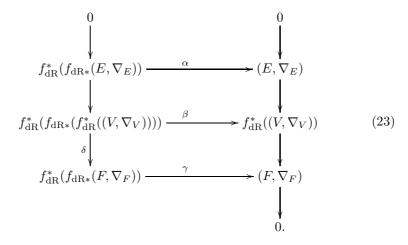
$$
f_*(f^*(V)^{f^*(\nabla_V)_{|\mathrm{Spec}(K)^\times}}) \cong f_*((V \otimes \mathcal{O}_X, \mathrm{id} \otimes d)^{(\mathrm{id} \otimes d)_{|\mathrm{Spec}(K)^\times}}) \cong V
$$

as vector spaces. Moreover, the action of the nilpotent operator on $f_*((V \otimes \mathcal{O}_X, \mathrm{id} \otimes d)^{(\mathrm{id} \otimes d)_{|\mathrm{Spec}(K)^\times}})$ coincides with the action of ∇_V on V by the description of the former one given in proposition 3.2. By applying $H_{\mathrm{dR}}^0(\mathrm{Spec}(K)^\times/\mathrm{Spec}(K), -)$ to the isomorphism (21), we obtain the isomorphism (20). Thus the proof of the full faithfulness of f_{dR}^* is finished.

Next we prove that $f_{\mathrm{dR}}^*(MIC(\mathrm{Spec}(K)^\times/\mathrm{Spec}(K))^{\mathrm{nr}})$ is closed by taking subobjects. Let (V, ∇_V) be an object of $MIC(\mathrm{Spec}(K)^\times/\mathrm{Spec}(K))^{\mathrm{nr}}$, and let $(E, \nabla_E) \subset f_{\mathrm{dR}}^*((V, \nabla_V))$ and $(F, \nabla_F) := f_{\mathrm{dR}}^*((V, \nabla_V))/(E, \nabla_E)$. Then we have the exact sequence in $MIC(X^\times/\mathrm{Spec}(K))^{\mathrm{nr}}$ defining (F, ∇_F)

$$
0 \to (E, \nabla_E) \to f_{\mathrm{dR}}^*((V, \nabla_V)) \to (F, \nabla_F) \to 0.
\tag{22}
$$

Applying the functor $f_{\mathrm{dR}}^* \circ f_{\mathrm{dR}*}$ to the sequence (22), we obtain the following commutative diagram with exact columns

$$
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
f_{\mathrm{dR}}^*(f_{\mathrm{dR}*}(E, \nabla_E)) & \xrightarrow{\;\alpha\;} & (E, \nabla_E) \\
\downarrow & & \downarrow \\
f_{\mathrm{dR}}^*(f_{\mathrm{dR}*}(f_{\mathrm{dR}}^*((V, \nabla_V)))) & \xrightarrow{\;\beta\;} & f_{\mathrm{dR}}^*((V, \nabla_V)) \\
{\scriptstyle\delta}\downarrow & & \downarrow \\
f_{\mathrm{dR}}^*(f_{\mathrm{dR}*}(F, \nabla_F)) & \xrightarrow{\;\gamma\;} & (F, \nabla_F) \\
& & \downarrow \\
& & 0.
\end{array}
\tag{23}
$$

The map β is an isomorphism for what we have seen in the last paragraph and α, γ are injective as shown in the proof of proposition 3.2. Then we see easily the surjectivity of δ, and by snake lemma α is an isomorphism, *i.e.* $(E, \nabla_E) \cong f_{\mathrm{dR}}^*(f_{\mathrm{dR}*}(E, \nabla_E))$. So the proof is finished. $\square$

PROPOSITION 3.6. *Let (E, ∇_E) be an object in $MIC(X^\times/\mathrm{Spec}(K))^{\mathrm{nr}}$ and let (W_0, ∇_{W_0}) be the maximal trivial subobject of $r((E, \nabla_E))$ in $\mathrm{Rep}(H)$. Then there exists $(F, \nabla_F) \subset (E, \nabla_E)$ in $MIC(X^\times/\mathrm{Spec}(K))^{\mathrm{nr}}$ such that $r((F, \nabla_F)) \cong (W_0, \nabla_{W_0})$.*

Proof. The maximal trivial subobject of $r((E, \nabla_E))$ in $\mathrm{Rep}(H)$ is $(H_{\mathrm{dR}}^0(X^\times/\mathrm{Spec}(K)^\times, r((E, \nabla_E))) \otimes_K \mathcal{O}_X, 1 \otimes d)$ because, as we saw in the proof of proposition of 3.2, the map $\gamma : H_{\mathrm{dR}}^0(X^\times/\mathrm{Spec}(K)^\times, r((E, \nabla_E))) \otimes_K \mathcal{O}_X \to E$ is injective, $(H_{\mathrm{dR}}^0(X^\times/\mathrm{Spec}(K)^\times, r((E, \nabla_E))) \otimes_K \mathcal{O}_X, 1 \otimes d)$ is trivial as subobject of $r((E, \nabla_E))$, and any trivial subobject of $r((E, \nabla_E))$ must be generated by horizontal sections. Since

$$(H_{\mathrm{dR}}^0(X^\times/\mathrm{Spec}(K)^\times, r((E, \nabla_E))) \otimes_K \mathcal{O}_X, 1 \otimes d)$$
$$\cong r(f_{\mathrm{dR}}^*((H_{\mathrm{dR}}^0(X^\times/\mathrm{Spec}(K)^\times, r((E, \nabla_E))), \nabla))), \quad (24)$$

one can prove the proposition by defining

$$(F, \nabla_F) := f_{\mathrm{dR}}^*((H_{\mathrm{dR}}^0(X^\times/\mathrm{Spec}(K)^\times, r((E, \nabla_E))), \nabla)).$$

$\square$

PROPOSITION 3.7. *Conditions (a) and (b) of theorem 3.1(iii) are satisfied for the sequence*

$$H \to \pi_1(X^\times/\mathrm{Spec}(K), x) \to \pi_1(\mathrm{Spec}(K)^\times/\mathrm{Spec}(K), \nu).$$

Proof. Proposition 3.6 gives (b) of theorem 3.1. The if part of (a) of theorem 3.1 is clear by the fact that the functor $r \circ f_{\mathrm{dR}}^*$ sends every object to a finite sum of the trivial object. Let (E, ∇_E) be an object of $MIC(X^\times/\mathrm{Spec}(K))^{\mathrm{nr}}$ such that $r((E, \nabla_E))$ is trivial. Then, because $r((E, \nabla_E))$ itself is the maximal trivial subobject of $r((E, \nabla_E))$, $r((E, \nabla_E)) \cong r(f_{\mathrm{dR}}^*((H_{\mathrm{dR}}^0(X^\times/\mathrm{Spec}(K)^\times, r((E, \nabla_E))), \nabla)))$ by the proof of proposition 3.6. This implies that the inclusion $f_{\mathrm{dR}}^*((H_{\mathrm{dR}}^0(X^\times/\mathrm{Spec}(K)^\times, r((E, \nabla_E))), \nabla)) \hookrightarrow (E, \nabla_E)$ is an isomorphism. $\square$

The following theorem is analogous to [EH06, theorem 5.10].

THEOREM 3.8. *Let (G, ∇_G) be an object of $MIC(X^\times/\mathrm{Spec}(K)^\times)^{\mathrm{nr}}$ and suppose that there exists an object (E, ∇_E) in $MIC(X^\times/\mathrm{Spec}(K))^{\mathrm{nr}}$ such that $(G, \nabla_G) \subset r((E, \nabla_E))$. Then there exists an object (F, ∇_F) in $MIC(X^\times/\mathrm{Spec}(K))^{\mathrm{nr}}$ such that $r((F, \nabla_F))$ surjects to (G, ∇_G).*

Proof. Suppose first that (G, ∇_G) has rank 1. In this case, we construct (F, ∇) as the (G, ∇_G)-isotypical component of $(E, \nabla_{E|\mathrm{Spec}(K)^\times}) := r((E, \nabla_E))$, in the following way. Let us consider $(E', \nabla') := (E, \nabla_{E|\mathrm{Spec}(K)^\times}) \otimes (G, \nabla_G)^\vee$: it is an object of $MIC(X^\times/\mathrm{Spec}(K)^\times)^{\mathrm{nr}}$ and the inclusion $(G, \nabla_G) \subset (E, \nabla_{E|\mathrm{Spec}(K)^\times})$ corresponds to a non-trivial section of $H^0_{\mathrm{dR}}(X^\times/\mathrm{Spec}(K)^\times, (E', \nabla'))$. We denote by (E'_0, ∇'_0) the object $(H^0_{\mathrm{dR}}(X^\times/\mathrm{Spec}(K)^\times, (E', \nabla')) \otimes \mathcal{O}_X, \mathrm{id} \otimes d)$ and consider the $\mathcal{O}_X$-module $E_1 = E'/E'_0$ with the induced connection ∇_1. If $H^0_{\mathrm{dR}}(X^\times/\mathrm{Spec}(K)^\times, (E_1, \nabla_1)) = 0$ we define (F, ∇) as $(E'_0, \nabla'_0) \otimes (G, \nabla_G)$. Otherwise, we denote by (E'_1, ∇'_1) the inverse image of $(H^0_{\mathrm{dR}}(X^\times/\mathrm{Spec}(K)^\times, (E_1, \nabla_1)) \otimes \mathcal{O}_X, \mathrm{id} \otimes d)$ in E' and we consider $E_2 = E'/E'_1$ with the induced connection ∇_2 and we proceed as before. If $H^0_{\mathrm{dR}}(X^\times/\mathrm{Spec}(K)^\times, (E_2, \nabla_2)) = 0$ we define (F, ∇) as $(E'_1, \nabla'_1) \otimes (G, \nabla_G)$. Otherwise, we go on with this process until we find the maximal subobject (F, ∇) of $(E, \nabla_{E|\mathrm{Spec}(K)^\times})$ which is a successive extension of (G, ∇_G). In the following, we denote (F, ∇) also by $(F, (\nabla_{E|\mathrm{Spec}(K)^\times})_{|F})$.

We remark that $(F, \nabla) \in MIC(X^\times/\mathrm{Spec}(K)^\times)^{\mathrm{nr}}$ is a subobject of $(E, \nabla_{E|\mathrm{Spec}(K)^\times})$. Thus the $\mathcal{O}_X$-linear map

$$F \xrightarrow{\nabla_E} E \otimes \omega^1_{X^\times/\mathrm{Spec}(K)} \longrightarrow (E/F) \otimes \omega^1_{X^\times/\mathrm{Spec}(K)} \tag{25}$$

takes values in $(E/F) \otimes f^*(\omega^1_{\mathrm{Spec}(K)^\times/\mathrm{Spec}(K)})$, because the composition with the projection $(E/F) \otimes \omega^1_{X^\times/\mathrm{Spec}(K)} \to (E/F) \otimes \omega^1_{X^\times/\mathrm{Spec}(K)^\times}$ is the zero map. We take an isomorphism $\varphi : f^*(\omega^1_{\mathrm{Spec}(K)^\times/\mathrm{Spec}(K)}) \xrightarrow{\cong} \mathcal{O}_X$ and let $l : F \to E/F$ be the composition

$$F \longrightarrow (E/F) \otimes f^*(\omega^1_{\mathrm{Spec}(K)^\times/\mathrm{Spec}(K)}) \xrightarrow{\mathrm{id} \otimes \varphi} E/F,$$

where the first map is the one induced by (25).

We have the following local description of l which is also useful. Let $D_0 \in \mathcal{H}om(\omega^1_{X^\times/\mathrm{Spec}(K)}, \mathcal{O}_X)$ be an element with $D_{0|f^*(\omega^1_{\mathrm{Spec}(K)^\times/\mathrm{Spec}(K)})} = \varphi$. (There exists such an element D_0 locally on X.) Then l is equal to the composition

$$F \hookrightarrow E \xrightarrow{\nabla_E(D_0)} E \twoheadrightarrow E/F. \tag{26}$$

Now we prove that the diagram

$$\begin{array}{ccc} F & \xrightarrow{\quad l \quad} & (E/F) \\ {\scriptstyle (\nabla_{E|\mathrm{Spec}(K)^\times})_{|F}} \downarrow & & \downarrow {\scriptstyle (\nabla_{E|\mathrm{Spec}(K)^\times})_{|E/F}} \\ F \otimes \omega^1_{X^\times/\mathrm{Spec}(K)^\times} & \xrightarrow{\quad l \otimes \mathrm{id} \quad} & (E/F) \otimes \omega^1_{X^\times/\mathrm{Spec}(K)^\times} \end{array} \tag{27}$$

is commutative. To see this, it suffices to work locally and to prove the com-

mutativity of the diagram

$$
\begin{array}{ccc}
F & \xrightarrow{\;l\;} & (E/F) \\
{\scriptstyle (\nabla_{E|\mathrm{Spec}(K)^\times})_{|F}(D)}\Big\downarrow & & \Big\downarrow{\scriptstyle (\nabla_{E|\mathrm{Spec}(K)^\times})_{|E/F}(D)} \\
F & \xrightarrow{\;l\;} & (E/F)
\end{array}
$$

for any $D \in \mathcal{H}om(\omega^1_{X^\times/\mathrm{Spec}(K)^\times}, \mathcal{O}_X)$. By the description of the map l given in (26), the commutativity of the above diagram follows from that of the diagram

$$
\begin{array}{ccc}
E & \xrightarrow{\nabla_E(D_0)} & E \\
{\scriptstyle \nabla_E(D)}\Big\downarrow & & \Big\downarrow{\scriptstyle \nabla_E(D)} \\
E & \xrightarrow{\nabla_E(D_0)} & E
\end{array}
$$

(where we denoted the composition $\omega^1_{X^\times/\mathrm{Spec}(K)} \to \omega^1_{X^\times/\mathrm{Spec}(K)^\times} \xrightarrow{D} \mathcal{O}_X$ also by D), which follows from the integrability of (E, ∇_E). So the diagram (27) is commutative, as required.

Thus we have constructed a morphism

$$
l : (F, (\nabla_{E|\mathrm{Spec}(K)^\times})_{|F}) \longrightarrow (E/F, (\nabla_{E|\mathrm{Spec}(K)^\times})_{|E/F})
$$

in the category $MIC(X^\times/\mathrm{Spec}(K)^\times)^{\mathrm{nr}}$. Since all irreducible subquotients of $(F, (\nabla_{E|\mathrm{Spec}(K)^\times})_{|F})$ are isomorphic to (G, ∇_G) and $(E/F, (\nabla_{E|\mathrm{Spec}(K)^\times})_{|E/F})$ has no irreducible subobject isomorphic to (G, ∇_G), the morphism l must be zero. This proves that ∇_E stabilizes F. Hence $(F, \nabla_{E|F})$ is a subobject of (E, ∇_E) such that $r((F, \nabla_{E|F}))$ has a surjection to (G, ∇_G). So the proof is finished in the rank 1 case.

If G has rank r we use the isomorphism

$$
(G, \nabla_G) \cong \det(G, \nabla_G) \otimes \wedge^{r-1}(G, \nabla_G)^\vee.
$$

We define $(F, \nabla_F) := (F', \nabla_{F'}) \otimes \wedge^{r-1}(E, \nabla_E)^\vee$, where $(F', \nabla_{F'})$ is the object in $MIC(X^\times/\mathrm{Spec}(K))^{\mathrm{nr}}$ we constructed above for the rank 1 object $\det(G, \nabla_G)$. Then we see that $r((F, \nabla_F)) = (F', \nabla_{F'|\mathrm{Spec}(K)^\times}) \otimes \wedge^{r-1}(E, \nabla_{E|\mathrm{Spec}(K)^\times})^\vee$ surjects to (G, ∇_G). So the proof is finished. $\qquad\square$

PROPOSITION 3.9. *Condition (c) of theorem 3.1(iii) is satisfied for the sequence*

$$
H \to \pi_1(X^\times/\mathrm{Spec}(K), x) \to \pi_1(\mathrm{Spec}(K)^\times/\mathrm{Spec}(K), \nu).
$$

Proof. The proof is analogous to the proof of [EH06, theorem 5.11]. Take any object (W, ∇_W) in $\mathrm{Rep}(H)$. Then, by proposition 3.4, there exist $(E, \nabla_E) \in MIC(X^\times/\mathrm{Spec}(K))^{\mathrm{nr}}$ and $(G_1, \nabla_{G_1}), (G, \nabla_G) \in MIC(X^\times/\mathrm{Spec}(K)^\times)^{\mathrm{nr}}$ such

that $(G_1, \nabla_{G_1}) \subset (G, \nabla_G) \subset r((E, \nabla_E))$ and $(W, \nabla_W) \cong (G, \nabla_G)/(G_1, \nabla_{G_1})$. Thanks to theorem 3.8 there exists an object $(F, \nabla_F) \in MIC(X^\times/\mathrm{Spec}(K))^{\mathrm{nr}}$ such that $r((F, \nabla_F))$ surjects to (G, ∇_G). Hence (c) of theorem 3.1 is verified. $\square$

Putting together all the results, we have the main theorem of this section

THEOREM 3.10. *The sequence*

$$\pi_1(X^\times/\mathrm{Spec}(K)^\times, x) \xrightarrow{\pi(r)} \pi_1(X^\times/\mathrm{Spec}(K), x)$$
$$\xrightarrow{\pi(f^*_{\mathrm{dR}})} \pi_1(\mathrm{Spec}(K)^\times/\mathrm{Spec}(K), \nu) \to 1 \qquad (28)$$

is exact.

Proof. The map $\pi(f^*_{\mathrm{dR}})$ is faithfully flat (proposition 3.5) and the sequence is exact at $\pi_1(X^\times/\mathrm{Spec}(K), x)$ because of proposition 3.7 and proposition 3.9. $\square$

4   DESCRIPTION OF THE KERNEL OF THE SECOND MAP

In this section, we introduce an auxiliary neutral Tannakian category denoted by $MIC(X^\times/\mathrm{Spec}(K)[u], x)^{\mathrm{nr}}$ and its corresponding fundamental group denoted by $\pi_1(X^\times/\mathrm{Spec}(K)[u], x)$, and we will prove (proposition 4.19) that $\pi_1(X^\times/\mathrm{Spec}(K)[u], x)$ is isomorphic to $\mathrm{Ker}(\pi(f^*_{\mathrm{dR}}))$. As before, we consider the map of log schemes $g : \mathrm{Spec}(K)^\times = (\mathrm{Spec}(K), N) \to \mathrm{Spec}(K)$ and denote simply by $\mathrm{dlog}1 \in \omega^1_{\mathrm{Spec}(K)^\times/\mathrm{Spec}(K)}$ the element $\mathrm{dlog}(1, 1)$ (in the notation of [Kat89, (1.7)]) with $(1, 1) \in \mathbb{N} \times K^* \cong N$. (We remark that $\mathrm{dlog}1$ is independent of the choice of the isomorphism $\mathbb{N} \times K^* \cong N$.)

DEFINITION 4.1. *Let* $z : (Y, L) \to \mathrm{Spec}(K)^\times$ *be a morphism of fine log schemes and let* u *be a variable. We define an action of the exterior differential on* u^i $(i > 0)$ *in such a way that* du *is the image of* $\mathrm{dlog}1$ *under the map* $z^{-1}(\omega^1_{\mathrm{Spec}(K)^\times/\mathrm{Spec}(K)}) \to \omega^1_{(Y,L)/\mathrm{Spec}(K)}$ *and that* du^i *is equal to* $iu^{i-1}du$. *We denote by* $\mathcal{O}_Y[u]$ *the sheaf of algebras* $\bigoplus_{i=0}^\infty \mathcal{O}_Y u^i$ *and by* $d : \mathcal{O}_Y[u] \to \mathcal{O}_Y[u] \otimes \omega^1_{(Y,L)/\mathrm{Spec}(K)}$ *the* K-*linear extension of the derivation on* $\mathcal{O}_Y$ *such that* $d(cu^i) = d(c)u^i + ciu^{i-1}du$ *for* $c \in \mathcal{O}_Y$ *and* $i \geq 1$.
For a coherent $\mathcal{O}_Y[u]$-*module* E, *a connection* ∇_E *on* E *is a* K-*linear map*

$$\nabla_E : E \longrightarrow E \otimes \omega^1_{(Y,L)/\mathrm{Spec}(K)}$$

which satisfies the Leibniz rule

$$\nabla_E(ae) = a\nabla_E(e) + e \otimes da \quad (a \in \mathcal{O}_Y[u], e \in E).$$

We can extend ∇_E *to the map*

$$\nabla_{E,i} : E \otimes \omega^i_{(Y,L)/\mathrm{Spec}(K)} \to E \otimes \omega^{i+1}_{(Y,L)/\mathrm{Spec}(K)} \quad (i \geq 1)$$

by $\nabla_{E,i}(e \otimes \omega) = e \otimes d\omega + \nabla_E(e) \wedge \omega$. We say that ∇_E is integrable if $\nabla_{E,1} \circ \nabla_E = 0$.

If (E, ∇_E) and (F, ∇_F) are two coherent $\mathcal{O}_Y[u]$-modules endowed with integrable connection, and if $\gamma : E \to F$ is a morphism of $\mathcal{O}_Y[u]$-modules, we say that γ is horizontal if $(\gamma \otimes \mathrm{id}) \circ \nabla_E = \nabla_F \circ \gamma$. Given (E, ∇_E) and (F, ∇_F), two coherent $\mathcal{O}_Y[u]$-modules endowed with integrable connection, the tensor product is defined as the pair $(E \otimes F, \nabla)$, where $\nabla = \mathrm{id} \otimes \nabla_F + \nabla_E \otimes \mathrm{id}$.

We denote by $MIC((Y, L)/\mathrm{Spec}(K)[u])$ the category whose objects are pairs (E, ∇_E) consisting of a coherent $\mathcal{O}_Y[u]$-module E and an integrable connection ∇_E on E. The morphisms between two objects (E, ∇_E) and (F, ∇_F) are morphisms of $\mathcal{O}_X[u]$-modules which are horizontal.

REMARK 4.2. We can give the definition of a connection on a coherent $\mathcal{O}_Y[u]$-module E using the sheaf of log derivations $\mathcal{D}er((Y, L)/\mathrm{Spec}(K))$, as in definition 2.3. The details are left to the reader.

DEFINITION 4.3. *We denote by $MIC((Y, L)/\mathrm{Spec}(K)[u])^{\mathrm{nr}}$ the full subcategory of $MIC((Y, L)/\mathrm{Spec}(K)[u])$ consisting of objects (E, ∇_E) which satisfy the following conditions:*

(i) *for any geometric point y over a closed point of Y, $\hat{E}_y := E_y \otimes_{\mathcal{O}_{Y,y}[u]} \mathcal{O}_{Y,y}[u]^\wedge$ is free as $\mathcal{O}_{Y,y}[u]^\wedge$-module, where $\mathcal{O}_{Y,y}[u]^\wedge$ is the $\mathfrak{m}_{Y,y}[u]$-adic completion of $\mathcal{O}_{Y,y}[u]$ and $\mathfrak{m}_{Y,y}$ is the maximal ideal of $\mathcal{O}_{Y,y}$.*

(ii) *if we define the residue $\rho_y : E(y) \to E(y) \otimes \overline{L}_y^{\mathrm{gp}}$ for a geometric point y over a closed point of Y in the same way as the paragraph before definition 2.11, $E(y)$ can be written as a union $E(y) = \bigcup_{i \in \mathbb{N}} E(y)_i$ of finite dimensional subspaces $E(y)_i$ such that*

  (ii-1) *$\rho_y(E(y)_i) \subset E(y)_i \otimes \overline{L}_y^{\mathrm{gp}}$.*

  (ii-2) *for every $K(y)$-linear map $t_y : K(y) \otimes \overline{L}_y^{\mathrm{gp}} \to K(y)$, the composite map $(\mathrm{id} \otimes t_y) \circ (\rho_{y|E(y)_i}) : E(y)_i \to E(y)_i$ defined as in definition 2.11 is nilpotent.*

(iii) *as a quasi-coherent $\mathcal{O}_Y$-module with integrable connection, (E, ∇_E) is a colimit of objects (E_i, ∇_{E_i}) in $MIC((Y, L)/\mathrm{Spec}(K))^{\mathrm{nr}}$ indexed by $\mathbb{N}$.*

Also, we denote by $MIC((Y, L)/\mathrm{Spec}(K)[u])'$ the full subcategory of $MIC((Y, L)/\mathrm{Spec}(K)[u])$ consisting of objects (E, ∇_E) which satisfy the conditions (i), (ii) above.

REMARK 4.4. The condition (iii) implies the condition (ii) in definition 4.3; indeed, for any y as in (ii), if we denote the image of $E_i(y)$ (where E_i is as in (iii)) in $E(y)$ by $E(y)_i$, the subspaces $E(y)_i$ $(i \in \mathbb{N})$ of $E(y)$ satisfy the condition (ii). We write the conditions (ii), (iii) separately due to technical reasons in the proofs below.

Next we give a 'formal version' of definitions 4.1, 4.3.

Definition 4.5. *Let B be a Noetherian I-adically complete ring for an ideal I and let $Y = \mathrm{Spec}(B)$. Let $z : (Y, L) \to \mathrm{Spec}(K)^{\times}$ be a morphism of fine log schemes and assume that the completed differential module $\hat{\omega}^1_{(Y,L)/\mathrm{Spec}(K)}$ associated to the composite $(Y, L) \xrightarrow{z} \mathrm{Spec}(K)^{\times} \to \mathrm{Spec}(K)$ (defined in remark 2.4) is finitely generated as B-module. Let u be a variable and we define an action of the exterior differential on u^i $(i > 0)$ in the same way as in definition 4.1. Let $B[u]^{\wedge}$ be the $I[u]$-adic completion of the polynomial ring $B[u]$ and let $d : B[u]^{\wedge} \to B[u]^{\wedge} \otimes \hat{\omega}^1_{(Y,L)/\mathrm{Spec}(K)}$ the K-linear continuous extension of the derivation on B such that $d(cu^i) = d(c)u^i + ciu^{i-1}du$ for $c \in B$ and $i \geq 1$.*

For a finitely generated $B[u]^{\wedge}$-module E, a formal connection ∇_E on E is a K-linear map

$$\nabla_E : E \longrightarrow E \otimes \hat{\omega}^1_{(Y,L)/\mathrm{Spec}(K)}$$

which satisfies the Leibniz rule as before. We can define the integrability of a formal connection and the horizontality of a morphism of formal integrable connections as in definition 4.1.

We denote by $\widehat{MIC}((Y, L)/\mathrm{Spec}(K)[u])$ the category whose objects are pairs (E, ∇_E) consisting of a finitely generated $B[u]^{\wedge}$-module E and an integrable connection ∇_E on E. The morphisms between two objects (E, ∇_E) and (F, ∇_F) are morphisms of $B[u]^{\wedge}$-modules which are horizontal.

Definition 4.6. *We denote by $\widehat{MIC}((Y, L)/\mathrm{Spec}(K)[u])'$ the full subcategory of $\widehat{MIC}((Y, L)/\mathrm{Spec}(K)[u])$ consisting of objects (E, ∇_E) which satisfy the following conditions:*

   (i) *E is free as $B[u]^{\wedge}$-module.*

   (ii) *if we define the residue $\rho_y : E(y) \to E(y) \otimes \overline{L}_y^{\mathrm{gp}}$ (here $E(y) := E \otimes_{B[u]^{\wedge}} K(y)[u]$) for a geometric point y over a closed point of Y in the same way as the paragraph before definition 2.11, $E(y)$ can be written as a union $E(y) = \bigcup_i E(y)_i$ of finite dimensional subspaces $E(y)_i$ such that the conditions (ii-1), (ii-2) in definition 4.3 are satisfied.*

Remark 4.7. Definitions 4.5, 4.6 are not a naive analogue of definitions 4.1, 4.3 in the sense that E in definitions 4.5, 4.6 is a $B[u]^{\wedge}$-module (not a $B[u]$-module). Our definition is designed in order that an induction argument works well later. We will not define a formal analogue of the category $MIC((Y, L)/\mathrm{Spec}(K)[u])^{\mathrm{nr}}$ because we will not need it.

Also, we remark that the condition (i) in definition 4.6 is a global condition, but this will not cause any problem because we will use this definition only when B is a complete local ring with maximal ideal I.

Lemma 4.8. *The category $MIC(\mathrm{Spec}(K)^{\times}/\mathrm{Spec}(K)[u])'$ is abelian.*

Proof. To prove the assertion, we only need to check that property (i) of definition 4.1 is stable by kernel and cokernel of any morphism. We will prove that every finitely generated $K[u]$-module endowed with an integral connection is in

fact free. Since $K[u]$ is a PID, it is enough to prove the torsion-freeness. The proof is analogous to [Cre98, proposition 6.1]. Let (E, ∇_E) be a finitely generated $K[u]$-module endowed with an integrable connection. The annihilator of the torsion submodule of E is an ideal of $K[u]$. Assume it is generated by $h \in K[u]$. Then, for e in the torsion submodule of E, $he = 0$. If we denote by ∂_u the derivation which sends du to 1, then $d(\partial_u)(h)e + h\nabla_E(\partial_u)(e) = 0$, and multiplying by h we obtain that $h^2 \nabla_E(\partial_u)(e) = 0$. Hence $\nabla_E(\partial_u)(e)$ is in the torsion submodule of E, so that $h\nabla_E(\partial_u)(e) = 0$, hence $d(\partial_u)(h)e = 0$. This implies that $d(\partial_u)(h) \in (h)$, hence $1 \in (h)$. Hence E does not have any nonzero torsion element. □

PROPOSITION 4.9. *The category* $MIC(X^\times/\mathrm{Spec}(K)[u])'$ *is abelian.*

Proof. It is sufficient to prove that the kernel and the cokernel of any morphism in $MIC(X^\times/\mathrm{Spec}(K)[u])'$ belongs to $MIC(X^\times/\mathrm{Spec}(K)[u])'$. Take any geometric point x over a closed point of X and consider the restriction functor

$$MIC(X^\times/\mathrm{Spec}(K)[u]) \to \widehat{MIC}(\mathrm{Spec}(\hat{\mathcal{O}}_{X,x}), M)/\mathrm{Spec}(K)[u]).$$

Since the map $\mathcal{O}_{X,x}[u] \to \hat{\mathcal{O}}_{X,x}[u]^\wedge$ is flat, the functor is exact. Also, it is easy to see that it induces the functor

$$MIC(X^\times/\mathrm{Spec}(K)[u])' \to \widehat{MIC}(\mathrm{Spec}(\hat{\mathcal{O}}_{X,x}), M)/\mathrm{Spec}(K)[u])' \qquad (29)$$

and that an object in $MIC(X^\times/\mathrm{Spec}(K)[u])$ belongs to the category $MIC(X^\times/\mathrm{Spec}(K)[u])'$ if the restriction of it to the category $\widehat{MIC}(\mathrm{Spec}(\hat{\mathcal{O}}_{X,x}), M)/\mathrm{Spec}(K)[u])$ belongs to the category $\widehat{MIC}(\mathrm{Spec}(\hat{\mathcal{O}}_{X,x}), M)/\mathrm{Spec}(K)[u])'$ for any x. So it suffices to prove that $\widehat{MIC}(\mathrm{Spec}(\hat{\mathcal{O}}_{X,x}), M)/\mathrm{Spec}(K)[u])'$ is an abelian category for the completed local ring $\hat{\mathcal{O}}_{X,x} = K[[x_1, \ldots, x_n]]/(x_1 \cdots x_r)$. By lemma 4.11 below, we can reduce to the case $r = n$. Then we proceed by induction on n. The case $n = 2$ is proven in proposition 4.12 below and the induction step is proven in proposition 4.13 below. □

REMARK 4.10. In what follows we consider on the spectrum of $S = K[[x_1, \ldots, x_n]]/(x_1 \cdots x_r)$ the log structure M as in remark 2.17. Then we have

$$\hat{\omega}^1_{(\mathrm{Spec}(S),M)/\mathrm{Spec}(K)} \cong \bigoplus_{i=1}^{r} S \mathrm{dlog} x_i \oplus \bigoplus_{i=r+1}^{n} S dx_i.$$

We will use a basis $\{\partial_1, \ldots, \partial_{r-1}, \partial_r, D_{r+1}, \ldots, D_n\}$ of its dual

$$\widehat{\mathcal{D}er}((\mathrm{Spec}(S), M)/\mathrm{Spec}(K)) = Hom(\hat{\omega}^1_{(\mathrm{Spec}(S),M)/\mathrm{Spec}(K)}, S),$$

defined in the following way: $\partial_1, \ldots, \partial_{r-1}, D_{r+1}, \ldots, D_n$ are as in remark 2.17 and ∂_r is the derivation that sends $\mathrm{dlog} x_j$ to 0 for $j = 1, \ldots, r-1$, $\mathrm{dlog} x_r$ to 1 and dx_j to 0 for every $j = r+1, \ldots, n$.

LEMMA 4.11. *Let* $R = K[[x_1, \ldots, x_r]]/(x_1 \cdots x_r)$ *and* $S = K[[x_1, \ldots, x_n]]/(x_1 \cdots x_r)$ *be as above. If* $\widehat{MIC}((\mathrm{Spec}(R), M)/\mathrm{Spec}(K)[u])'$ *is stable by kernel and cokernel of any morphism, then the category* $\widehat{MIC}((\mathrm{Spec}(S), M)/\mathrm{Spec}(K)[u])'$ *is stable by kernel and cokernel of any morphism.*

Proof. Note that $R[u]^\wedge = K[u][[x_1, \ldots, x_r]]/(x_1 \cdots x_r)$ and $S[u]^\wedge = K[u][[x_1, \ldots, x_n]]/(x_1 \cdots x_r)$. For an object in the category (E, ∇_E) in $\widehat{MIC}((\mathrm{Spec}(S), M)/\mathrm{Spec}(K)[u])'$, put

$$\overline{E} := \{ e \in E \mid \nabla_E(D_i)(e) = 0 \, (r + 1 \leq i \leq n) \}.$$

Then $\overline{E}$ is stable by the action of $\nabla_E(\partial_i) \, (1 \leq i \leq r)$. It suffices to prove that $\overline{E}$ with the above action defines an object in $\widehat{MIC}((\mathrm{Spec}(R), M)/\mathrm{Spec}(K)[u])'$ and that the functor

$$\widehat{MIC}((\mathrm{Spec}(S), M)/\mathrm{Spec}(K)[u])' \to \widehat{MIC}((\mathrm{Spec}(R), M)/\mathrm{Spec}(K)[u])';$$
$$(E, \nabla_E) \mapsto \overline{E}$$

is an equivalence of categories whose quasi-inverse is given by $(\overline{E}, \nabla_{\overline{E}}) \mapsto (E := \overline{E} \otimes_{R[u]^\wedge} S[u]^\wedge, \nabla_E)$ with the action $\nabla_E(\partial_i) \, (1 \leq i \leq r), \nabla_E(D_i) \, (r + 1 \leq i \leq n)$ defined by $\nabla_E(\partial_i) := \mathrm{id} \otimes d(\partial_i) + \nabla_{\overline{E}}(\partial_i) \otimes \mathrm{id}, \nabla_E(D_i) := \mathrm{id} \otimes d(D_i)$. (The reason is the same as that in lemma 2.18.) We can prove the above claim in the same way as lemma 2.18. So we are done. □

PROPOSITION 4.12. *Let* M *be the log structure on* $\mathrm{Spec}(K[[x, y]]/(xy))$ *defined in remark 4.10. Then* $\widehat{MIC}((\mathrm{Spec}(K[[x, y]]/(xy)), M)/\mathrm{Spec}(K)[u])'$ *is an abelian category.*

Proof. Let ∂_1, ∂_2 be as in the notation introduced in remark 4.10. For an object (E, ∇_E) in $\widehat{MIC}((\mathrm{Spec}(K[[x, y]]/(xy)), M)/\mathrm{Spec}(K)[u])'$, put

$$\overline{E} := \{ e \in E \mid \exists N \in \mathbb{N}, \nabla_E^N(\partial_1)(e) = 0 \}.$$

Then $\overline{E}$ is stable by the action of $\nabla_E(\partial_i) \, (i = 1, 2)$ and the action of $\nabla_E(\partial_1)$ is locally nilpotent. Then, it suffices to prove that the correspondence $(E, \nabla_E) \mapsto (\overline{E}, \nabla_E(\partial_1))$ defines the functor

$$\widehat{MIC}((\mathrm{Spec}(K[[x, y]]/(xy)), M)/\mathrm{Spec}(K)[u])'$$
$$\to \left\{ ((F, \nabla_F), N) \,\middle|\, \begin{array}{l} (F, \nabla_F) \in MIC(\mathrm{Spec}(K)^\times/\mathrm{Spec}(K)[u])' \\ N : (F, \nabla_F) \to (F, \nabla_F) \text{: a nilpotent endomorphism} \end{array} \right\}$$

and that it is an equivalence whose quasi-inverse is given by $((F, \nabla_F), N) \mapsto (E := F \otimes_{K[u]} K[u][[x, y]]/(xy), \nabla_E)$ with the action $\nabla_E(\partial_i) \, (i = 1, 2)$ defined by $\nabla_E(\partial_1) := \mathrm{id} \otimes d(\partial_1) + N \otimes \mathrm{id}, \nabla_E(\partial_2) := \mathrm{id} \otimes d(\partial_2) + \nabla_F(\partial_2) \otimes \mathrm{id}$. (The reason is the same as that in lemma 2.18.)

To prove the above claim, it suffices to construct a basis of $\overline{E}$ over $K[u]$ which is a basis of E over $K[u][[x,y]]/(xy)$. To do so, first we prove that there exists a basis of E on which $\nabla_E(\partial_1)$ acts as a strictly upper triangular matrix with entries in $K[u]$. If E has rank n, by hypothesis of nilpotent residues, we can write

$$\nabla_E(\partial_1) = d(\partial_1) + H$$

with respect to some basis, where $H = (a_{i,j}(x,y))_{i,j}$ is an $n \times n$ matrix such that $H_0 = (a_{i,j}(0,0))_{i,j}$ is a nilpotent matrix with entries in $K[u]$. Moreover, by changing the basis, we may assume that H_0 is strictly upper triangular. Using the fact $d(\partial_1)(u) = 0$, the same argument as proposition 2.19 works to deduce that there exists a basis $e_1, \ldots, e_s$ of E with respect to which the action of $\nabla_E(\partial_1)$ is given by the matrix H_0. Also, we can prove the equality $K[u]e_1 + \cdots + K[u]e_s = \overline{E}$ in the same way as the proof of proposition 2.19. Thus we have the claim we want and so the proof is finished. $\qquad\square$

PROPOSITION 4.13. *Let B be the ring $K[[x_1, \ldots, x_n]]/(x_1 \cdots x_n)$ and let A be the ring $K[[x_1, \ldots, x_{n-1}]]/(x_1 \cdots x_{n-1})$. If $\widehat{MIC}((A, M)/\mathrm{Spec}(K)[u])'$ is an abelian category, $\widehat{MIC}((B, M)/\mathrm{Spec}(K)[u])'$ is also an abelian category.*

Proof. The proof is analogous to the proof of proposition 2.20. We use the same notation as in that proposition. The map $h : A \to B$ defined by $x_i \mapsto x_i \, (i = 1, \ldots, n-2), x_{n-1} \mapsto x_{n-1}x_n$ induces a map of log schemes $(\mathrm{Spec}(B), M_B) \to (\mathrm{Spec}(A), M_A)$, where M_A, M_B are as in the proof of proposition 2.20. Moreover, h naturally induces the map $A[u]^\wedge = K[u][[x_1, \ldots, x_{n-1}]]/(x_1 \cdots x_{n-1}) \to K[u][[x_1, \ldots, x_n]]/(x_1 \cdots x_n) = B[u]^\wedge$. Also, let $\partial_1, \ldots, \partial_n$ be as in remark 4.10 and put $\partial := \partial_{n-1}$.

For an object (E, ∇_E) in $\widehat{MIC}((\mathrm{Spec}(B), M_B)/\mathrm{Spec}(K)[u])'$, put

$$\overline{E} := \{ e \in E \mid \nabla_E^N(\partial)(e) \to 0 \, (N \to \infty) \},$$

where, on the right hand side, E is endowed with $(x_1, ..., x_n)$-adic topology. Then $\overline{E}$ is stable by the action of $\nabla_E(\partial_i) \, (1 \leq i \leq n-2, i = n)$ and the action of $\nabla_E(\partial)$ which is locally topologically nilpotent. Then, it suffices to prove the following: firstly, $\overline{E}$ and the actions $\nabla_E(\partial_i) \, (1 \leq i \leq n-2, i = n)$ on it define an object (which we denote by $(\overline{E}, \nabla_{\overline{E}})$) in $\widehat{MIC}((\mathrm{Spec}(A), M_A)/\mathrm{Spec}(K)[u])'$. Secondly, the correspondence $(E, \nabla_E) \mapsto ((\overline{E}, \nabla_{\overline{E}}), \nabla_E(\partial_1))$ defines the functor

$$\widehat{MIC}((\mathrm{Spec}(B), M_B)/\mathrm{Spec}(K)[u])'$$
$$\to \left\{ ((F, \nabla_F), N) \,\middle|\, \begin{array}{l} (F, \nabla_F) \in \widehat{MIC}((\mathrm{Spec}(A), M_A)/\mathrm{Spec}(K)[u])' \\ N : (F, \nabla_F) \to (F, \nabla_F) \text{: a nilpotent endomorphism} \end{array} \right\}$$

and it is an equivalence whose quasi-inverse is given by $((F, \nabla_F), N) \mapsto (E := F \otimes_A B, \nabla_E)$ with the action $\nabla_E(\partial_i) \, (1 \leq i \leq n-2, i = n)$ and $\nabla_E(\partial)$ defined by $\nabla_E(\partial_i) := \mathrm{id} \otimes d(\partial_i) + \nabla_F(\partial_i) \otimes \mathrm{id}, \nabla_E(\partial) := \mathrm{id} \otimes d(\partial) + N \otimes \mathrm{id}$. (The reason is the same as that in lemma 2.18.)

To prove the above claim, it suffices to construct a basis of $\overline{E}$ over $A[u]^\wedge$ which is a basis of E over $B[u]^\wedge$. First, we construct a basis $e_1, \ldots, e_s$ of E as $B[u]^\wedge$-module on which $\nabla_E(\partial)$ acts as a matrix H_0 with entries in $A[u]^\wedge$ such that, if we write $H_0 = \sum_{\mathbf{k}} M_{\mathbf{k}} \mathbf{x}^{\mathbf{k}}$ with entries of $M_{\mathbf{k}}$ in $K[u]$, M_0 is strictly upper triangular. Noting the fact $d(\partial)(a) = 0$ for any $a \in A[u]^\wedge$, the proof of this claim is perfectly analogous to the proof of CLAIM 1 of proposition 2.20. Next, we can proceed as in the proof of CLAIM 2 of proposition 2.20 and prove that $A[u]^\wedge e_1 + \cdots + A[u]^\wedge e_s = \overline{E}$. Hence we have the claim we want and so the proof is finished. $\qquad \square$

Now we consider the condition (iii) in definition 4.3.

PROPOSITION 4.14. *The category $MIC(X^\times/\mathrm{Spec}(K)[u])^{\mathrm{nr}}$ is abelian.*

Proof. It suffices to prove that, for a horizontal morphism $\varphi : (E, \nabla_E) \to (F, \nabla_F)$ in $MIC(X^\times/\mathrm{Spec}(K)[u])^{\mathrm{nr}}$, the kernel and the cokernel of φ belong to $MIC(X^\times/\mathrm{Spec}(K)[u])^{\mathrm{nr}}$. By proposition 4.9 (ii), they belong to $MIC(X^\times/\mathrm{Spec}(K)[u])'$. Moreover, by definition, $(E, \nabla_E), (F, \nabla_F)$ are written as the colimit $\varinjlim_{i \in \mathbb{N}}(E_i, \nabla_{E_i})$, $\varinjlim_{i \in \mathbb{N}}(F_i, \nabla_{F_i})$ of objects in $MIC(X^\times/\mathrm{Spec}(K))^{\mathrm{nr}}$ indexed by $\mathbb{N}$, and by changing the index set suitably, the morphism φ is written as the colimit of the morphisms $\varphi_i : (E_i, \nabla_{E_i}) \to (F_i, \nabla_{F_i})$. So the kernel of φ is written as the colimit $\varinjlim_{i \in \mathbb{N}} \mathrm{Ker}\, \varphi_i$ of objects in $MIC(X^\times/\mathrm{Spec}(K))^{\mathrm{nr}}$ indexed by $\mathbb{N}$, and the same is true for the cokernel. Hence the kernel and the cokernel belong to $MIC(X^\times/\mathrm{Spec}(K)[u])^{\mathrm{nr}}$, as desired. $\qquad \square$

PROPOSITION 4.15. *Let us suppose that there exists a K-rational point $x \in X$. Then $MIC(X^\times/\mathrm{Spec}(K)[u])^{\mathrm{nr}}$ is a neutral Tannakian category over K.*

Proof. The category $MIC(X^\times/\mathrm{Spec}(K)[u])^{\mathrm{nr}}$ is an abelian category by proposition 4.14. We define the unit object of the category as the pair $(\mathcal{O}_X[u], d)$ and considering the tensor structure induced by the tensor structure of the category of the coherent $\mathcal{O}_X[u]$-modules with integrable connection, the category $MIC(X^\times/\mathrm{Spec}(K)[u])^{\mathrm{nr}}$ is an abelian tensor category. Moreover, it is a rigid abelian tensor category, thanks to condition (i) of definition 4.3.
We will prove in proposition 5.10 below that $\mathrm{End}((\mathcal{O}_X[u], d)) \cong H^0_{\mathrm{dR}}(X^\times/\mathrm{Spec}(K)^\times, (\mathcal{O}_X, d))$. Then, because there exists a K-rational point of X, $\mathrm{End}((\mathcal{O}_X[u], d)) \cong K$ (as proven in proposition 2.21).
Let x be the K-rational point which exists by hypothesis. We define a functor

$$\gamma_x : MIC(X^\times/\mathrm{Spec}(K)[u])^{\mathrm{nr}} \to \mathrm{Vec}_K$$

which is the composition of the functor which sends an object (E, ∇_E) to $E(x)/uE(x)$, where $E(x)$ is the fiber of E at x. To prove that γ_x is a fiber functor, it is enough to prove that it is an exact tensor functor because it will be automatically faithful thanks to [Del90, Corollaire 2.10], and we see the exactness of the functor by the condition (i) of definition 4.3. So the proof is finished. $\qquad \square$

Definition 4.16. *Let x be a K-rational point of X and let*

$$\gamma_x : MIC(X^\times/\mathrm{Spec}(K)[u])^{\mathrm{nr}} \to \mathrm{Vec}_K$$

be the fiber functor introduced in the above proposition. We define the log algebraic fundamental group of $X^\times/\mathrm{Spec}(K)[u]$ with base point x as the Tannaka dual of $MIC(X^\times/\mathrm{Spec}(K)[u])^{\mathrm{nr}}$, i.e.

$$\pi_1(X^\times/\mathrm{Spec}(K)[u], x) := G(MIC(X^\times/\mathrm{Spec}(K)[u])^{\mathrm{nr}}, \gamma_x)$$

with the same notation as in theorem 2.9.

Let $t_u : MIC(X^\times/\mathrm{Spec}(K))^{\mathrm{nr}} \to MIC(X^\times/\mathrm{Spec}(K)[u])^{\mathrm{nr}}$ be the functor which sends $(E, \nabla_E) \in MIC(X^\times/\mathrm{Spec}(K))^{\mathrm{nr}}$ to $(E, \nabla_E)[u]$ in $MIC(X^\times/\mathrm{Spec}(K)[u])^{\mathrm{nr}}$, where $(E, \nabla_E)[u]$ consists of the $\mathcal{O}_X[u]$-module $E \otimes_{\mathcal{O}_X} \mathcal{O}_X[u]$ and the connection which sends eu^i ($e \in E, i \in \mathbb{N}$) to $\nabla(e)u^i + eidu^{i-1}$. For a K-rational point x of X, the functor t_u induces an homomorphism of fundamental groups

$$\pi(t_u) : \pi_1(X^\times/\mathrm{Spec}(K)[u], x) \longrightarrow \pi_1(X^\times/\mathrm{Spec}(K), x).$$

Lemma 4.17. *The map $\pi(t_u)$ is a closed immersion.*

Proof. It is enough to check that theorem 3.1 (ii) is satisfied. Every object (E, ∇_E) in $MIC(X^\times/\mathrm{Spec}(K)[u])^{\mathrm{nr}}$ is written as the colimit $(E, \nabla_E) = \varinjlim_{i \in \mathbb{N}}(E_i, \nabla_{E_i})$ with $(E_i, \nabla_{E_i}) \in MIC(X^\times/\mathrm{Spec}(K))^{\mathrm{nr}}$. Then we have the surjection $\varinjlim_{i \in \mathbb{N}} t_u((E_i, \nabla_{E_i})) \to (E, \nabla_E)$ and so there exists some $i \in \mathbb{N}$ such that the map $t_u((E_i, \nabla_{E_i})) \to (E, \nabla_E)$ is surjective. In particular every object $(E, \nabla_E) \in MIC(X^\times/\mathrm{Spec}(K)[u])^{\mathrm{nr}}$ is a subquotient of an object in $MIC(X^\times/\mathrm{Spec}(K))^{\mathrm{nr}}$, as required by theorem 3.1 (ii). □

Moreover, let $p_u : MIC(X^\times/\mathrm{Spec}(K)[u])^{\mathrm{nr}} \to MIC(X^\times/\mathrm{Spec}(K)^\times)^{\mathrm{nr}}$ be the functor induced by the morphism which sends u and du to 0. Then p_u induces an homomorphism of fundamental groups:

$$\pi(p_u) : \pi_1(X^\times/\mathrm{Spec}(K)^\times, x) \longrightarrow \pi_1(X^\times/\mathrm{Spec}(K)[u], x).$$

Proposition 4.18. *The map $\pi(p_u)$ is faithfully flat.*

Proof. Thanks to theorem 3.10 the sequence

$$\pi_1(X^\times/\mathrm{Spec}(K)^\times, x) \xrightarrow{\pi(r)} \pi_1(X^\times/\mathrm{Spec}(K), x)$$
$$\xrightarrow{\pi(f_{\mathrm{dR}}^*)} \pi_1(\mathrm{Spec}(K)^\times/\mathrm{Spec}(K), \nu) \longrightarrow 1$$

is exact. Moreover, the first map factors as follows:

$$\pi_1(X^\times/\mathrm{Spec}(K)^\times, x) \xrightarrow{\ \pi(r)\ } \pi_1(X^\times/\mathrm{Spec}(K), x) \qquad (30)$$

$$\Big\downarrow \pi(p_u) \qquad \qquad \pi(t_u) \nearrow$$

$$\pi_1(X^\times/\mathrm{Spec}(K)[u], x).$$

If we prove the inclusion $\pi(t_u)(\pi_1(X^\times/\mathrm{Spec}(K)[u], x)) \subset \mathrm{Ker}(\pi(f_{\mathrm{dR}}^*))$, we see from the above diagram the following sequence of inclusions

$$\mathrm{Ker}(\pi(f_{\mathrm{dR}}^*)) \subset (\pi(t_u) \circ \pi(p_u))((\pi_1(X^\times/\mathrm{Spec}(K)^\times, x)) \subset$$
$$\pi(t_u)(\pi_1(X^\times/\mathrm{Spec}(K)[u], x)) \subset \mathrm{Ker}(\pi(f_{\mathrm{dR}}^*)).$$

Since $\pi(t_u)$ is a closed immersion, this implies $\pi(p_u)(\pi_1(X^\times/\mathrm{Spec}(K)^\times, x)) = \pi_1(X^\times/\mathrm{Spec}(K)[u], x)$, hence the faithful flatness of $\pi(p_u)$.
So, we prove in what follows that $\pi(t_u)(\pi_1(X^\times/K[u], x)) \subset \mathrm{Ker}(\pi(f_{\mathrm{dR}}^*))$. The composition of functors

$$MIC(\mathrm{Spec}(K)^\times/\mathrm{Spec}(K))^{\mathrm{nr}} \xrightarrow{f_{\mathrm{dR}}^*} MIC(X^\times/\mathrm{Spec}(K))^{\mathrm{nr}}$$
$$\xrightarrow{r} MIC(X^\times/\mathrm{Spec}(K)^\times)^{\mathrm{nr}}$$

sends every object of $MIC(\mathrm{Spec}(K)^\times/\mathrm{Spec}(K))^{\mathrm{nr}}$ to a finite sum of trivial ones. This implies that the composition

$$MIC(\mathrm{Spec}(K)^\times/\mathrm{Spec}(K))^{\mathrm{nr}} \xrightarrow{f_{\mathrm{dR}}^*} MIC(X^\times/\mathrm{Spec}(K))^{\mathrm{nr}}$$
$$\xrightarrow{t_u} MIC(X^\times/\mathrm{Spec}(K)[u])^{\mathrm{nr}} \xrightarrow{p_u} MIC(X^\times/\mathrm{Spec}(K)^\times)^{\mathrm{nr}}$$

also sends every object of $MIC(\mathrm{Spec}(K)^\times/\mathrm{Spec}(K))^{\mathrm{nr}}$ to a finite sum of trivial ones. Since p_u, restricted to the image of t_u is fully faithful (proposition 5.10 below), the composition

$$MIC(\mathrm{Spec}(K)^\times/\mathrm{Spec}(K))^{\mathrm{nr}} \xrightarrow{f_{\mathrm{dR}}^*} MIC(X^\times/\mathrm{Spec}(K))^{\mathrm{nr}}$$
$$\xrightarrow{t_u} MIC(X^\times/\mathrm{Spec}(K)[u])^{\mathrm{nr}}$$

sends every object of $MIC((K, N)/K)^{\mathrm{nr}}$ to a finite sum of the trivial ones. Hence $\pi(t_u)(\pi_1(X^\times/\mathrm{Spec}(K)[u]))$ is contained in $\mathrm{Ker}(\pi(f_{\mathrm{dR}}^*))$. $\qquad \square$

PROPOSITION 4.19. *Let x be a K-rational point of X. Then the sequence*

$$1 \longrightarrow \pi_1(X^\times/\mathrm{Spec}(K)[u], x) \xrightarrow{\pi(t_u)} \pi_1(X^\times/\mathrm{Spec}(K), x)$$
$$\xrightarrow{\pi(f_{\mathrm{dR}}^*)} \pi_1(\mathrm{Spec}(K)^\times/\mathrm{Spec}(K), \nu) \longrightarrow 1 \qquad (31)$$

is exact.

Proof. The homomorphism $\pi(t_u)$ is a closed immersion (lemma 4.17) and $\pi(f_{\mathrm{dR}}^*)$ is faithfully flat (proposition 3.5). Moreover, by theorem 3.10, $\mathrm{Ker}(\pi(f_{\mathrm{dR}}^*)) = \pi(r)(\pi_1(X^\times/\mathrm{Spec}(K)^\times, x))$ and since the diagram (30) is commutative, $\mathrm{Ker}(\pi(f_{\mathrm{dR}}^*)) = \pi(t_u)(\pi(p_u)(\pi_1(X^\times/\mathrm{Spec}(K)^\times, x)))$. Since $\pi(p_u)$ is faithfully flat (proposition 4.18), $\mathrm{Ker}(\pi(f_{\mathrm{dR}}^*)) = \pi(t_u)(\pi_1(X^\times/\mathrm{Spec}(K)[u], x))$ and the sequence (31) is exact. $\qquad\square$

5   Injectivity of the first map

In this section we study the injectivity of $\pi(r)$. The log point should be thought as a log algebraic analogue of the punctured open disk in the complex plane. It is then reasonable to expect that, if we can define the higher homotopy groups for the log point, they should be trivial, *i.e.* it is natural to expect that $\pi(r)$ is injective. To make this intuition more precise, we recall how we can characterize the topological spaces for which the only non-vanishing homotopy group is the first.

Proposition 5.1. *Let T be a connected topological manifold and t a point. Then T is a $K(\pi, 1)$ space, i.e., the fundamental group is the only non-vanishing homotopy group if and only if the map*

$$H^i(\pi_1^{\mathrm{top}}(T, t), A) \to H^i(T, \mathcal{A}), \tag{32}$$

where A is a representation of $\pi_1^{\mathrm{top}}(T, t)$ and $\mathcal{A}$ is the local system associated to A, is an isomorphism for every i.

We would like to translate the notion of $K(\pi, 1)$ space in our context and prove that the log point $\mathrm{Spec}(K)^\times/\mathrm{Spec}(K)$ is a kind of $K(\pi, 1)$ space in log algebraic geometry. We proceed as in [EH06, section 2]. Let $\mathcal{V}$ be a representation of $\pi_1(\mathrm{Spec}(K)^\times/\mathrm{Spec}(K), \nu)$ (which can be regarded also as an object in $MIC(\mathrm{Spec}(K)^\times/\mathrm{Spec}(K))^{\mathrm{nr}}$) and let $\mathcal{K}$ be the trivial representation (which can be regarded also as the trivial object in $MIC(\mathrm{Spec}(K)^\times/\mathrm{Spec}(K))^{\mathrm{nr}}$); then we consider

$$H^i(\pi_1(\mathrm{Spec}(K)^\times/\mathrm{Spec}(K), \nu), \mathcal{V}) := \mathrm{Ext}^i_{\mathrm{Rep}(\pi_1(\mathrm{Spec}(K)^\times/\mathrm{Spec}(K), \nu))}(\mathcal{K}, \mathcal{V}).$$

Let ϵ be an i-extension of $\mathcal{K}$ by $\mathcal{V}$ in $\mathrm{Rep}(\pi_1(\mathrm{Spec}(K)^\times/\mathrm{Spec}(K), \nu))$. Then it induces an i-extension of the trivial object by $\mathcal{V}$ in $MIC(\mathrm{Spec}(K)^\times/\mathrm{Spec}(K))^{\mathrm{nr}}$, and so we have the connecting homomorphism

$$\delta_\epsilon : H^0_{\mathrm{dR}}(\mathrm{Spec}(K)^\times/\mathrm{Spec}(K), \mathcal{K}) \cong K \longrightarrow H^i_{\mathrm{dR}}(\mathrm{Spec}(K)^\times/\mathrm{Spec}(K), \mathcal{V}).$$

This induces the map of K-vector spaces

$$\delta^i : H^i(\pi_1(\mathrm{Spec}(K)^\times/\mathrm{Spec}(K), \nu), \mathcal{V}) \longrightarrow H^i_{\mathrm{dR}}(\mathrm{Spec}(K)^\times/\mathrm{Spec}(K), \mathcal{V})$$

defined by $\epsilon \mapsto \delta_\epsilon(1)$. The map δ^i is the analogue of (32), and the next proposition says that the log point is a kind of $K(\pi, 1)$ space in log algebraic geometry.

LEMMA 5.2. *Let $\mathcal{V}$ be a representation of $\pi_1(\mathrm{Spec}(K)^\times/\mathrm{Spec}(K), \nu)$. The map*

$$\delta^i : H^i(\pi_1(\mathrm{Spec}(K)^\times/\mathrm{Spec}(K), \nu), \mathcal{V}) \longrightarrow H^i_{\mathrm{dR}}(\mathrm{Spec}(K)^\times/\mathrm{Spec}(K), \mathcal{V})$$

defined as above is an isomorphism for every i.

Proof. For $i = 0, 1$ the same proof as in [EH06, proposition 2.2] works, so that δ^i is an isomorphism. For $i \geq 2$ $H^i_{\mathrm{dR}}(\mathrm{Spec}(K)^\times/\mathrm{Spec}(K), \mathcal{V}) = 0$ because $\omega^1_{\mathrm{Spec}(K)^\times/\mathrm{Spec}(K)}$ is of dimension 1. On the other hand, $H^i(\mathbb{G}_a, \mathcal{K}) = 0$ for $i \geq 2$, thanks to [Jan87, remark 2 in page 71]. Since every $\mathcal{V} \in \mathrm{Rep}(\mathbb{G}_a)$ is a successive extension of $\mathcal{K}$, we can conclude that $H^i(\mathbb{G}_a, \mathcal{V}) = 0$ for every $i \geq 2$ and for every $\mathcal{V}$ by long exact sequence of cohomologies. □

In the previous section, we described the kernel of the second map in (28) as $\pi_1(X^\times/\mathrm{Spec}(K)[u], x)$. In this section we study the relation of $\pi_1(X^\times/\mathrm{Spec}(K)[u], x)$ with $\pi_1(X^\times/\mathrm{Spec}(K)^\times, x)$. In particular we prove that $\pi(p_u)$ induces an isomorphism

$$\pi(p_u)^{\mathrm{tri}} : \pi_1(X^\times/\mathrm{Spec}(K)^\times, x)^{\mathrm{tri}} \xrightarrow{\cong} \pi_1(X^\times/\mathrm{Spec}(K)[u], x)^{\mathrm{tri}},$$

where $\pi_1(X^\times/\mathrm{Spec}(K)^\times, x)^{\mathrm{tri}}$ and $\pi_1(X^\times/\mathrm{Spec}(K)[u], x)^{\mathrm{tri}}$ are the maximal geometrically protrigonalizable quotients of $\pi_1(X^\times/\mathrm{Spec}(K)^\times, x)$ and $\pi_1(X^\times/\mathrm{Spec}(K)[u], x)$ respectively, which we define now.

DEFINITION 5.3. *(i)([Mil15, definition 17.1]) An algebraic group G over K is called trigonalizable if every nonzero representation of G over K is an iterated extension of 1-dimensional representations.*
(ii) An algebraic group G over K is called geometrically trigonalizable if $G \otimes_K \overline{K}$ (where $\overline{K}$ is an algebraic closure of K) is trigonalizable.
(iii) For an affine group scheme G over K, we define G^{tri} by $G^{\mathrm{tri}} := \varprojlim_N G/N$, where N runs through normal subgroups of G such that G/N is geometrically trigonalizable, and call it the maximal geometrically protrigonalizable quotient of G.
(iv) An affine group scheme G over K is called geometrically protrigonalizable if G itself is the maximal geometrically protrigonalizable quotient of G.

In definition 5.3(iii), the naturally induced morphism $G \to G^{\mathrm{tri}}$ is indeed surjective, because the Hopf algebra of G^{tri}, which is the filtered direct limit of the Hopf algebras of G/N's, injects into the Hopf algebra of G by [Wat79, theorem 14.1] and [Wat79, theorem 13.3]. So it is reasonable to call G^{tri} as the 'quotient'.
For every affine group scheme Γ over K which is the inverse limit of geometrically trigonalizable algebraic groups and every morphism $\gamma : G \to \Gamma$ of group schemes over K, γ uniquely factors through G^{tri}, because the trigonalizability is stable under subquotients [Mil15, corollary 17.3]. Also, every map of affine group schemes $\gamma : G \to H$ naturally induces a map between their maximal geometrically protrigonalizable quotients $\gamma^{\mathrm{tri}} : G^{\mathrm{tri}} \to H^{\mathrm{tri}}$, because $G \xrightarrow{\gamma} H \to H^{\mathrm{tri}}$ factors uniquely through G^{tri}.

We prove the following proposition which we need later.

PROPOSITION 5.4. *Let $K \subset L$ be an extension of fields of characteristic zero. Then, for an affine group scheme G over K, G is geometrically protrigonalizable if and only if $G \otimes_K L$ is geometrically protrigonalizable.*

Proof. By using the fact that the trigonalizability is stable under subquotients [Mil15, corollary 17.3], we see that it suffices to prove the following: for an algebraic group G over K, G is geometrically trigonalizable if and only if so is $G \otimes_K L$. Moreover, we may replace K, L by their algebraic closures and it suffices to prove that G is trigonalizable if and only if so is $G \otimes_K L$.
When G is trigonalizable, so is $G \otimes_K L$ by [Mil15, corollary 17.4]. We prove the converse. So assume that $G \otimes_K L$ is trigonalizable and take a representation V of G. Then the action of G on the set of 1-dimensional subspaces of V defines the action $G \times \mathbb{P}(V) \to \mathbb{P}(V)$. Let $R \subset G \times \mathbb{P}(V) \times \mathbb{P}(V)$ be the graph of this action and let $\pi : R \to \mathbb{P}(V) \times \mathbb{P}(V)$ be the natural projection. Let $R' \subset G \times \mathbb{P}(V)$ be $\pi^{-1}(\Delta(\mathbb{P}(V)))$ (where $\Delta : \mathbb{P}(V) \hookrightarrow \mathbb{P}(V) \times \mathbb{P}(V)$ is the diagonal map) and let $\pi' : R' \to \mathbb{P}(V)$ be the natural projection. Let $G = \coprod_\alpha G_\alpha$ be the decomposition into connected components of G and let $\pi'_\alpha : R'_\alpha := R' \cap (G_\alpha \times \mathbb{P}(V)) \to \mathbb{P}(V)$ be the restriction of π'. Then the set $C_\alpha := \{x \in \mathbb{P}(V) \,|\, \dim {\pi'_\alpha}^{-1}(x) = \dim G_\alpha\}$ is closed in $\mathbb{P}(V)$ and so $C = \bigcap_\alpha C_\alpha$ (with reduced subscheme structure) is a closed subscheme of $\mathbb{P}(V)$. By construction, we have the equalities

$$
\begin{aligned}
C(K) &= \{x \in \mathbb{P}(V)(K) \,|\, \forall \alpha, \dim {\pi'_\alpha}^{-1}(x) = \dim G_\alpha\} \\
&= \{x \in \mathbb{P}(V)(K) \,|\, {\pi'}^{-1}(x) = G\} \\
&= \left\{ x \in \mathbb{P}(V)(K) \left| \begin{array}{l} \text{the action } G \times \mathbb{P}(V) \to \mathbb{P}(V) \\ \text{induces the map } G \times x \to x \end{array} \right. \right\} \\
&= \left\{ x \in \mathbb{P}(V)(K) \left| \begin{array}{l} \text{the 1-dimensional subspace of } V \\ \text{corresponding to } x \text{ is stable by } G \end{array} \right. \right\},
\end{aligned}
$$

where the second equality follows from the smoothness of G over K, which is always true because K is of characteristic zero. By the same argument, we see the equality

$$
C(L) = \left\{ x \in \mathbb{P}(V)(L) \left| \begin{array}{l} \text{the 1-dimensional subspace of } V \otimes_K L \\ \text{corresponding to } x \text{ is stable by } G \otimes_K L \end{array} \right. \right\}.
$$

Hence G (resp. $G \otimes_K L$) is trigonalizable if and only if $C(K)$ is nonempty (resp. only if $C(L)$ is nonempty) for any V. Also, if $C(L)$ is nonempty, C is nonempty as a scheme and so $C(K)$ is nonempty because C is of finite type over an algebraically closed field K. So the proof is finished. $\qquad\square$

REMARK 5.5. For an affine group scheme G over K, the full subcategory $\mathrm{Rep}(G)^{\mathrm{tri}}$ of $\mathrm{Rep}(G)$ consisting of representations V such that $V \otimes_K \overline{K}$ is

an iterated extension of 1-dimensional representations of $G \otimes_K \overline{K}$ is a Tannakian subcategory of $\mathrm{Rep}(G)$. By definition, for every representation (V, ρ) which factors through $\rho^{\mathrm{tri}} : G^{\mathrm{tri}} \to GL(V)$, $V \otimes_K \overline{K}$ is an iterated extension of 1-dimensional representations of $G \otimes_K \overline{K}$, and so V belongs to $\mathrm{Rep}(G)^{\mathrm{tri}}$. Conversely, if (V, ρ) is a representation of G which belongs to $\mathrm{Rep}(G)^{\mathrm{tri}}$, $\rho(G) \otimes_K \overline{K}$ is trigonalizable by [Mil15, proposition 17.2]. Hence $\rho(G)$ is geometrically trigonalizable and so ρ factors through G^{tri}. Therefore, we have an equivalence $\mathrm{Rep}(G^{\mathrm{tri}}) \cong \mathrm{Rep}(G)^{\mathrm{tri}}$, namely, G^{tri} is the Tannaka dual of $\mathrm{Rep}(G)^{\mathrm{tri}}$.

By describing the above fact in terms of neutral Tannakian categories, we obtain the following: For a neutral Tannakian category $\mathcal{C}$ over K, the full subcategory $\mathcal{C}^{\mathrm{tri}}$ of $\mathcal{C}$ consisting of objects V such that $V_{\overline{K}} \in \mathcal{C}_{\overline{K}}$ (where $\overline{K}$ denotes the scalar extension to $\overline{K}$ [Del89, §4]) is an iterated extension of rank 1 objects of $\mathcal{C}_{\overline{K}}$ is a Tannakian subcategory of $\mathcal{C}$, and the Tannaka dual of $\mathcal{C}^{\mathrm{tri}}$ is equal to the maximal geometrically protrigonalizable quotient of the Tannaka dual of $\mathcal{C}$.

We consider as before the functor $p_u : MIC(X^\times / \mathrm{Spec}(K)[u])^{\mathrm{nr}} \to MIC(X^\times / \mathrm{Spec}(K)^\times)^{\mathrm{nr}}$ induced by the morphism which sends u and du to 0. Then p_u induces a faithfully flat (proposition 4.18) homomorphism of fundamental groups

$$\pi(p_u) : \pi_1(X^\times / \mathrm{Spec}(K)^\times, x) \longrightarrow \pi_1(X^\times / \mathrm{Spec}(K)[u], x)$$

which induces a faithfully flat homomorphism between the maximal geometrically protrigonalizable quotients:

$$\pi(p_u)^{\mathrm{tri}} : \pi_1(X^\times / \mathrm{Spec}(K)^\times, x)^{\mathrm{tri}} \longrightarrow \pi_1(X^\times / \mathrm{Spec}(K)[u], x)^{\mathrm{tri}}.$$

To see that $\pi(p_u)^{\mathrm{tri}}$ is an isomorphism, it is enough to prove that it is a closed immersion: according to 3.1 (ii) we need to prove that every $(E, \nabla_E) \in MIC(X^\times / \mathrm{Spec}(K)^\times)^{\mathrm{nr, tri}}$ (where $^{\mathrm{tri}}$ is as in remark 5.5) is isomorphic to a subquotient of an object of the form $p_u(F, \nabla_F)$ with $(F, \nabla_F) \in MIC(X^\times / \mathrm{Spec}(K)[u])^{\mathrm{nr, tri}}$. We will show more: we will show that every $(E, \nabla_E) \in MIC(X^\times / \mathrm{Spec}(K)^\times)^{\mathrm{nr, tri}}$ is isomorphic to $p_u(F, \nabla_F)$ with $(F, \nabla_F) \in MIC(X^\times / \mathrm{Spec}(K)[u])^{\mathrm{nr, tri}}$. We look first at the case of $(E, \nabla_E) \in MIC(X^\times / \mathrm{Spec}(K)^\times)^{\mathrm{nr}}$ of rank 1.

THEOREM 5.6. *Let (E, ∇_E) be an object of $MIC(X^\times / \mathrm{Spec}(K)^\times)^{\mathrm{nr}}$ of rank 1. Then, there exists a unique object $(E, \widetilde{\nabla_E})$ in $MIC(X^\times / \mathrm{Spec}(K))^{\mathrm{nr}}$ up to canonical isomorphism such that the restriction functor $r : MIC(X^\times / \mathrm{Spec}(K))^{\mathrm{nr}} \to MIC(X^\times / \mathrm{Spec}(K)^\times)^{\mathrm{nr}}$ sends $(E, \widetilde{\nabla_E})$ to (E, ∇_E).*

Proof. We say that $(E, \widetilde{\nabla_E})$ is a lift of (E, ∇_E) or that $(E, \widetilde{\nabla_E})$ lifts (E, ∇_E) if $(E, \widetilde{\nabla_E})$ is an object of $MIC(X^\times / \mathrm{Spec}(K))^{\mathrm{nr}}$ such that the functor $r : MIC(X^\times / \mathrm{Spec}(K))^{\mathrm{nr}} \to MIC(X^\times / \mathrm{Spec}(K)^\times)^{\mathrm{nr}}$ sends $(E, \widetilde{\nabla_E})$ to (E, ∇_E). We start proving the uniqueness of the lift. Let (E, ∇_E) be an object of $MIC(X^\times / \mathrm{Spec}(K)^\times)^{\mathrm{nr}}$ of rank 1 and let $(E, \widetilde{\nabla_i}) \in MIC(X^\times / \mathrm{Spec}(K))^{\mathrm{nr}}$ for

$i = 1, 2$ be two different lifts of (E, ∇_E). Then $(\mathcal{O}_X, \widetilde{\nabla}) := (E, \widetilde{\nabla_1})^\vee \otimes (E, \widetilde{\nabla_2})$ is a lift of the unit object $(\mathcal{O}_X, d)$ in $MIC(X^\times/\mathrm{Spec}(K)^\times)^{\mathrm{nr}}$. To prove the uniqueness, it is enough to prove that $\widetilde{\nabla} = \tilde{d}$, where $\tilde{d}$ is the exterior derivative $\mathcal{O}_X \to \omega^1_{X^\times/K}$: indeed, if $\widetilde{\nabla} = \tilde{d}$, then

$$\mathrm{Hom}((E, \widetilde{\nabla_1}), (E, \widetilde{\nabla_2})) = H^0_{\mathrm{dR}}(X^\times/\mathrm{Spec}(K), (\mathcal{O}_X, \widetilde{\nabla})) = K$$
$$= H^0_{\mathrm{dR}}(X^\times/\mathrm{Spec}(K)^\times, (\mathcal{O}_X, d))$$
$$= \mathrm{Hom}((E, \nabla_1), (E, \nabla_2))$$

and so there exists a unique isomorphism $\varphi : (E, \widetilde{\nabla_1}) \to (E, \widetilde{\nabla_2})$ such that $r(\varphi)$ is the identity morphism on (E, ∇_E).

To prove that $\widetilde{\nabla} = \tilde{d}$, it is sufficient to prove that $\widetilde{\nabla}(1) = 0$ in $\hat{\mathcal{O}}_{X,P}$ for any geometric point P over a closed point of X. Thus we can work on $\hat{\mathcal{O}}_{X,P} = K[[x_1, \ldots, x_n]]/(x_1 \cdots x_r)$ for some $r \geq 1$, where K here is an algebraic closure of our original field K.

Since $(\mathcal{O}_X, \widetilde{\nabla})$ is a lift of $(\mathcal{O}_X, d)$, we can write $\widetilde{\nabla}(1) = \alpha \sum_{i=1}^r \mathrm{dlog} x_i$ with $\alpha \in K[[x_1, \ldots, x_n]]/(x_1 \cdots x_r)$. Since $\widetilde{\nabla}$ is integrable,

$$0 = \widetilde{\nabla} \circ \widetilde{\nabla}(1) = \left(\sum_{i=1}^r x_i \frac{\partial \alpha}{\partial x_i} \mathrm{dlog} x_i + \sum_{i=r+1}^n \frac{\partial \alpha}{\partial x_i} dx_i \right) \wedge \left(\sum_{i=1}^r \mathrm{dlog} x_i \right). \quad (33)$$

The equation (33) implies that

$$x_i \frac{\partial \alpha}{\partial x_i} = x_j \frac{\partial \alpha}{\partial x_j} \quad (1 \leq i < j \leq r), \qquad \frac{\partial \alpha}{\partial x_i} = 0 \quad (i > r). \quad (34)$$

The former equalities in (34) imply that

$$x_i \frac{\partial \alpha}{\partial x_i} \in \bigcap_{i=1}^r x_i K[[x_1, \ldots, x_n]]/(x_1 \cdots x_r) = 0 \quad (1 \leq i \leq r),$$

thus α is constant with respect to the variables $x_i \, (1 \leq i \leq r)$. Also, the latter equalities in (34) imply that α is constant with respect to the variables $x_i \, (i > r)$. Hence $\alpha \in K$. Then, since $\widetilde{\nabla}$ has nilpotent residues, $\alpha = 0$.

Since we proved the uniqueness of the lift, we can work étale locally to prove the existence of the lift. So we suppose that $X = \mathrm{Spec}(A)$ and we fix an étale morphism $X = \mathrm{Spec}(A) \to \mathrm{Spec}(K[x_1, \ldots, x_n]/(x_1 \cdots x_r))$ such that the log structure on X is induced by the monoid homomorphism $\mathbb{N}^r \to A$ which sends e_i to the image of x_i in A for $i = 1, \ldots, r$. Also, we may assume that E is free of rank 1 on X.

We introduce some notation which are in force along the proof. We put $[r] := \{1, \ldots, r\}$, $[n] := \{1, \ldots, n\}$. For a subset I of $[r]$, we denote by $|I|$ the number of elements contained in I. For nonempty subsets I, J of $[r]$, we will use the

following subsets of X:

$$X_I := \{P \in X \,|\, x_i(P) = 0 \; \forall i \in I\}, \quad X^J := \{P \in X \,|\, x_j(P) \neq 0 \; \forall j \in J\},$$
$$X_I^J := X_I \cap X^J,$$

$$X_I^0 := X_I^{[r] \setminus I} = \{P \in X \,|\, x_i(P) = 0 \; \forall i \in I, \; x_i(P) \neq 0 \; \forall i \in [r] \setminus I\}.$$

The X_I's are closed in X and the X^J's are open. The X_I^0's are smooth, locally closed in X and set-theoretically $X = \bigsqcup_I X_I^0$. For any fixed non empty $I \subseteq [r]$, we have that $\bigsqcup_{\emptyset \neq I' \subseteq I} X_{I'}^0 = X^{[r] \setminus I}$. We now construct a special cover of X: for $1 \leq k \leq r$, we define the open subscheme $X^{(k)}$ of X by $X^{(k)} := \bigsqcup_{1 \leq |I| \leq k} X_I^0 = \bigcup_{|I| = k} X^{[r] \setminus I}$. Then $X^{(0)} = \emptyset$ and $X^{(r)} = X$. We prove the existence of a lift of (E, ∇_E) on $X^{(k)}$ by induction on k.

First we prove the case $k = 1$. Since $f^*(N)_x \cong M_x$ for any geometric point x of $X^{(1)}$, the composition $\omega^1_{X/\mathrm{Spec}(K)|X^{(1)}} \to \omega^1_{X^\times/\mathrm{Spec}(K)|X^{(1)}} \to \omega^1_{X^\times/\mathrm{Spec}(K)^\times|X^{(1)}}$ is an isomorphism. Hence a lift of $\nabla_E : E_{|X^{(1)}} \to E_{|X^{(1)}} \otimes \omega^1_{X^\times/\mathrm{Spec}(K)^\times|X^{(1)}}$ is given by $\widetilde{\nabla}_E : E_{|X^{(1)}} \to E_{|X^{(1)}} \otimes \omega^1_{X^\times/\mathrm{Spec}(K)^\times|X^{(1)}} \cong E_{|X^{(1)}} \otimes \omega^1_{X/\mathrm{Spec}(K)|X^{(1)}} \to E_{|X^{(1)}} \otimes \omega^1_{X^\times/\mathrm{Spec}(K)|X^{(1)}}$.

Then we proceed to prove that, if (E, ∇_E) is liftable on $X^{(k-1)}$, then it is liftable on $X^{(k)}$. To prove this claim we decompose $X^{(k)}$ further. We order the subsets of $[r]$ of cardinality k as $\{I \subseteq [r] | |I| = k\} = \{I_1, \ldots, I_m\}$, and we define $X_l^{(k)} := X^{(k-1)} \sqcup \bigsqcup_{i=1}^l X_{I_i}^0 = X^{(k-1)} \cup \bigcup_{i=1}^l X^{[r] \setminus I_i}$ for $0 \leq l \leq m$. The set $X_l^{(k)}$ is open in X and $X_0^{(k)} = X^{(k-1)}$, $X_m^{(k)} = X^{(k)}$. We proceed again by induction, proving that if (E, ∇_E) is liftable on $X_{l-1}^{(k)}$, then it is liftable on $X_l^{(k)}$.

LEMMA 5.7. *The set $X_{I_l}^0$ is closed in $X_l^{(k)}$.*

Proof. The set $X_l^{(k)}$ can be covered by $\bigcup_J X^J$ where J runs through the sets of non empty $J \subseteq [r]$ with $|J| \geq r - k + 1$ or $J = [r] \setminus I_i$ for $1 \leq i \leq l$.

If $|J| \geq r - k + 1$ or $J = [r] \setminus I_i$ for $1 \leq i \leq l - 1$, then $J \not\subseteq [r] \setminus I_l$ which implies that $J \cap I_l \neq \emptyset$. Hence $X_{I_l}^0 \cap X^J = \emptyset$ which is closed in X^J. If $J = [r] \setminus I_l$, then $X_{I_l}^0 \cap X^J = X_{I_l}^J$ which is closed in X^J. $\square$

Since $X_{I_l}^0$ is locally closed in X and closed in $X_l^{(k)}$, which is open in X, the completion $\widehat{X_{|X_{I_l}^0}}$ is isomorphic to $\widehat{X_{l|X_{I_l}^0}^{(k)}}$. We denote by $Y_{I_l}^0$ the scheme

$$Y_{I_l}^0 := \mathrm{Spec}(\Gamma(\widehat{X_{|X_{I_l}^0}}, \mathcal{O})) \cong \mathrm{Spec}\left(\frac{\bar{A}[x_i^{-1}]_{i \in [r] \setminus I_l}[[x_i]]_{i \in I_l}}{\prod_{i=1}^r x_i} \right)$$

where $\bar{A} := A/(x_i)_{i \in I_l}$. There exists a canonical map $Y_{I_l}^0 \to X_l^{(k)}$, so that we have an fpqc covering of $X_l^{(k)}$ given by $X_{l-1}^{(k)} \coprod Y_{I_l}^0 = \bigcup_J X^J \coprod Y_{I_l}^0$ where J

runs through the sets of non empty $J \subseteq [r]$ with $|J| \geq r - k + 1$ or $J = [r] \setminus I_i$ for $1 \leq i \leq l - 1$.

By induction hypothesis for each $J \subseteq [r]$ with $|J| \geq r - k + 1$ or $J = [r] \setminus I_i$ for $1 \leq i \leq l - 1$, (E, ∇_E) is liftable on X_J and the lifts are compatible. On $Y_{I_l}^0 \cong \mathrm{Spec}\left(\frac{\bar{A}[x_i^{-1}]_{i \in [r] \setminus I_l}[[x_i]]_{i \in I_l}}{\prod_{i \in I_l} x_i} \right)$ with log structure defined as the pullback of that on X, (E, ∇_E) is the pullback of a module with integrable connection on $\mathrm{Spec}\left(\bar{A}[x_i^{-1}]_{i \in [r] \setminus I_l} \right) / \mathrm{Spec}(K)$ (with trivial log structures) by lemma 5.8 below, and so (E, ∇_E) is liftable on $Y_{I_l}^0$.

Hence it suffices to prove that the lifts are canonically isomorphic on $X^J \times_{X_l^{(k)}} Y_{I_l}^0$ (with log structure defined as the pullback of that on X). As in the proof of the uniqueness, it suffices to prove that any lift $(\mathcal{O}, \widetilde{\nabla})$ of $(\mathcal{O}, d)$ is trivial. Note that

$$
\begin{aligned}
\Gamma(X^J \times_{X_l^{(k)}} Y_{I_l}^0, \mathcal{O}) &= \left(\frac{\bar{A}[x_i^{-1}]_{i \in [r] \setminus I_l}[[x_i]]_{i \in I_l}}{\prod_{i=1}^r x_i} \right) [x_i^{-1}]_{i \in J} \\
&= \left(\frac{\bar{A}[x_i^{-1}]_{i \in [r] \setminus I_l}[[x_i]]_{i \in I_l}}{\prod_{i \in I_l} x_i} \right) [x_i^{-1}]_{i \in I_l \cap J}.
\end{aligned}
\tag{35}
$$

Moreover $Z = \mathrm{Spec}(\bar{A}[x_i^{-1}]_{i \in [r] \setminus I_l})$ is étale over $\mathrm{Spec}(K[x_i]_{i \in [n] \setminus I_l}[x_i^{-1}]_{i \in [r] \setminus I_l})$. Let $P \in Z$ be a closed point. Then the completion $\hat{\mathcal{O}}_{Z,P}$ has the form $K(P)[[y_{P,i}]]_{i \in [n] \setminus I_l}$, where $K(P)$ is the residue field of Z at P, $y_{P,i}$'s ($1 \leq i \leq n, i \notin I_l$) are local parameters with $dy_{P,i} = u_i \mathrm{dlog}\, x_i$ for $1 \leq i \leq r$, $i \notin I_l$ with some $u_i \in \hat{\mathcal{O}}_{Z,P}^\times$ and $dy_{P,i} = dx_i$ for $r + 1 \leq i \leq n$.

Using these notations, the ring in (35) can be embedded in

$$
\begin{aligned}
&\left(\prod_{P \in Z} \frac{K(P)[[y_{P,i}]]_{i \in [n] \setminus I_l}[[x_i]]_{i \in I_l}}{\prod_{i \in I_l} x_i} \right) [x_i^{-1}]_{i \in I_l \cap J} \hookrightarrow \\
&\quad \prod_{P \in Z} \frac{K(P)[[y_{P,i}]]_{i \in [n] \setminus I_l}[[x_i]]_{i \in I_l}[x_i^{-1}]_{i \in I_l \cap J}}{\prod_{i \in I_l \setminus J} x_i}.
\end{aligned}
\tag{36}
$$

We will prove that $\widetilde{\nabla}(1) = 0$ in $\frac{K(P)[[y_{P,i}]]_{i \in [n] \setminus I_l}[[x_i]]_{i \in I_l}[x_i^{-1}]_{i \in I_l \cap J}}{\prod_{i \in I_l \setminus J} x_i}$. To simplify the notation, we put $y_i := y_{P,i}$ in the following.

Let us suppose that $\widetilde{\nabla}(1) = \alpha \sum_{i=1}^r \mathrm{dlog}\, x_i$; the integrability $\widetilde{\nabla} \circ \widetilde{\nabla}(1) = 0$ implies that $d\alpha \wedge (\sum_{i=1}^r \mathrm{dlog}\, x_i) = 0$. We calculate $d\alpha$ in terms of basis of 1-differentials:

$$
\begin{aligned}
d\alpha &= \sum_{i \in I_l} x_i \frac{\partial \alpha}{\partial x_i} \mathrm{dlog}\, x_i + \sum_{i \in [r] \setminus I_l} u_i \frac{\partial \alpha}{\partial y_i} u_i^{-1} dy_i + \sum_{i=r+1}^n \frac{\partial \alpha}{\partial y_i} dy_i = \\
&= \sum_{i \in I_l} x_i \frac{\partial \alpha}{\partial x_i} \mathrm{dlog}\, x_i + \sum_{i \in [r] \setminus I_l} u_i \frac{\partial \alpha}{\partial y_i} \mathrm{dlog}\, x_i + \sum_{i=r+1}^n \frac{\partial \alpha}{\partial y_i} dx_i.
\end{aligned}
$$

Since $d\alpha \wedge (\sum_{i=1}^{r} \mathrm{dlog} x_i) = 0$, we see that the elements $x_i \frac{\partial \alpha}{\partial x_i}$ $(i \in I_l), u_i \frac{\partial \alpha}{\partial y_i}$ $(i \in [r] \setminus I_l)$ are the same and $\frac{\partial \alpha}{\partial y_i} = 0$ for $i > r$. From the latter assertion, we see that α is constant with respect to the variables y_i $(i > r)$. From the former assertion for $i \in I_l$, we see that

$$x_i \frac{\partial \alpha}{\partial x_i} \in \bigcap_{j \in I_l} x_j \frac{K(P)[[y_i]]_{i \in [n] \setminus I_l}[[x_i]]_{i \in I_l}[x_i^{-1}]_{i \in I_l \cap J}}{\prod_{i \in I_l \setminus J} x_i} = 0 \quad (i \in I_l),$$

and so α is constant with respect to the variables x_i $(i \in I_l)$. Thus $\alpha \in K(P)[[y_i]]_{i \in [r] \setminus I_l}$ and $u_i \frac{\partial \alpha}{\partial y_i} = 0$ $(i \in [r] \setminus I_l)$, hence $\alpha \in K(P)$. Then, thanks to the nilpotent residues condition we conclude that $\alpha = 0$. So the proof is finished (modulo lemma 5.8 below). $\qquad \square$

LEMMA 5.8. *Let* $C = \frac{\bar{A}[x_i^{-1}]_{i \in [r] \setminus I_l}[[x_i]]_{i \in I_l}}{\prod_{i \in I_l} x_i}$, $B = \bar{A}[x_i^{-1}]_{i \in [r] \setminus I_l}$ *be as above and let* M *be the log structure on* $\mathrm{Spec}(C)$ *associated to* $\mathbb{N}^r \to C; e_i \mapsto x_i$. *Let* (E, ∇_E) *be an object in* $\widehat{MIC}((\mathrm{Spec}(C), M)/\mathrm{Spec}(K)^\times)^{\mathrm{nr}}$ *such that* E *is free of rank 1. Then there exists* (E_1, ∇_{E_1}) *in* $MIC(\mathrm{Spec}(B)/\mathrm{Spec}(K))$ *with* E_1 *free of rank 1 such that* $(h, g)^*_{\mathrm{dR}}(E_1, \nabla_{E_1}) \cong (E, \nabla_E)$, *where* $(h, g)^*_{\mathrm{dR}}$ *is the pullback with respect to the morphisms* h, g *in the diagram below:*

$$
\begin{array}{ccc}
(\mathrm{Spec}(C), M) & \xrightarrow{\ h\ } & \mathrm{Spec}(B) \\
\downarrow & & \downarrow \\
\mathrm{Spec}(K)^\times & \xrightarrow{\ g\ } & \mathrm{Spec}(K).
\end{array}
$$

In particular, the object (E, ∇_E) *is liftable to an object in the category* $\widehat{MIC}((\mathrm{Spec}(C), M)/\mathrm{Spec}(K))^{\mathrm{nr}}$.

Proof. We can suppose that $I_l = \{1, \ldots, k\}$. For every $i \in I_l$, we denote by C_i the ring $B[[x_1, \ldots, x_i]]/(x_1 \cdots x_i)$ and by M_i the log structure on $\mathrm{Spec}(C_i)$ induced by $\mathbb{N}^i \to C_i; e_j \mapsto x_j$ $(1 \le j \le i)$. (Then $(\mathrm{Spec}(C_k), M_k) = (\mathrm{Spec}(C), M)$.) We regard C_i as a C_{i-1}-algebra via the map $h_i : C_{i-1} \to C_i$ defined by $x_j \mapsto x_j$ $(j = 1, \ldots, i-2), x_{i-1} \mapsto x_{i-1} x_i$. This induces a morphism of log schemes $(\mathrm{Spec}(C_i), M_i) \to (\mathrm{Spec}(C_{i-1}), M_{i-1})$.
Note that $\Omega^1_{B/K}$ is a free B-module of finite rank. So we take an isomorphism $\Omega^1_{B/K} \cong \bigoplus_{t=1}^{s} Bb_t$ by fixing a basis $\{b_t\}_{t=1,\ldots,s}$ of $\Omega^1_{B/K}$. Then $\hat{\omega}^1_{(\mathrm{Spec}(C_i), M_i)/\mathrm{Spec}(K)^\times}$ is isomorphic to the C_i-module

$$\bigoplus_{t=1}^{s} C_i b_t \oplus \frac{C_i \mathrm{dlog} x_1 \oplus \cdots \oplus C_i \mathrm{dlog} x_i}{C_i \cdot \sum_{j=1}^{i} \mathrm{dlog} x_j}.$$

We consider the derivations $\{\partial_i\}_{i=1,\ldots,k-1}$ of $(C_k, M_k)/\mathrm{Spec}(K)^\times$ as in remark 2.17, *i.e.* ∂_i is an element of $\mathrm{Hom}(\hat{\omega}^1_{(\mathrm{Spec}(C_k), M_k)/\mathrm{Spec}(K)^\times}, C_k)$ which sends

dlogx_i to 1, dlogx_{i+1} to -1, dlogx_j to 0 for every $j \neq i, i+1$ and b_t to 0 for every $t = 1, \ldots, s$. The derivation ∂_{i-1} can be seen as a generator of the rank 1 C_i-module $\mathrm{Hom}(\widehat{\omega}^1_{(\mathrm{Spec}(C_i), M_i)/(\mathrm{Spec}(C_{i-1}), M_{i-1})}, C_i)$.

To prove the lemma, it suffices to prove that, for an object (E, ∇_E) in $\widehat{MIC}((C_k, M_k)/\mathrm{Spec}(K)^{\times})^{\mathrm{nr}}$ such that E is free of rank 1, the following two assertions hold by induction on k:

(1) There exists (E_1, ∇_{E_1}) in $MIC(\mathrm{Spec}(B)/\mathrm{Spec}(K))$ with E_1 free of rank 1 such that (E, ∇_E) is the pullback of (E_1, ∇_{E_1}) to $(C_k, M_k)/\mathrm{Spec}(K)^{\times}$.

(2) Any endomorphism $\varphi : (E, \nabla_E) \to (E, \nabla_E)$ is the multiplication by some element in B.

(In fact, the assertion (1) is the same as the statement of the lemma. However, we need to prove also the assertion (2) in order that the induction works.) In the case $k = 1$, the assertions are true because $B = C_1$ as rings and $MIC((\mathrm{Spec}(C_1), M_1)/\mathrm{Spec}(K)^{\times}) \cong MIC(\mathrm{Spec}(B)/\mathrm{Spec}(K))$. So we prove the assertions in general case. Put $\partial := \partial_{k-1}$. Take $H \in C_k$ with $\nabla_E(\partial) = d(\partial) + H$ and write $H = \sum_{\mathbf{k} \in \Gamma} H_{\mathbf{k}} x^{\mathbf{k}}$ ($H_{\mathbf{k}} \in B$) with Γ the index set as in the proof of proposition 2.20. Then, by the assumption that (E, ∇_E) has nilpotent residues, H_0 is sent to 0 by any homomorphism $B \to \overline{K}$ (where $\overline{K}$ is the algebraic closure of K) over K. Because B is reduced and Jacobson, we conclude that $H_0 = 0$. Then, by the same argument as the proof of CLAIM 1 in proposition 2.20, we can prove the existence of a basis e of E as C_k-module such that $\nabla_E(\partial)$ acts on it by the multiplication of an element $M = \sum_{\mathbf{k} \in \Gamma'} M_{\mathbf{k}} x^{\mathbf{k}} \in C_{k-1}$ ($M_{\mathbf{k}} \in B$) with $M_0 = 0$. Also, by the same argument as the proof of CLAIM 2 in proposition 2.20, we see that

$$\overline{E} := C_{n-1} e = \{ e' \in E \mid \nabla_E^N(\partial)(e') \to 0 \, (N \to \infty) \}.$$

As in the proof of proposition 2.20, $\overline{E}$ is stable by the action of $\nabla_E(\partial_i)$ ($1 \leq i \leq k-2$) and with this action, $\overline{E}$ defines an object $(\overline{E}, \nabla_{\overline{E}})$ in $MIC((\mathrm{Spec}(C_{k-1}), M_{k-1})/\mathrm{Spec}(K)^{\times})^{\mathrm{nr}}$. Moreover, $\nabla_E(\partial)$ induces an endomorphism of $(\overline{E}, \nabla_{\overline{E}})$ which is the multiplication by M. By induction hypothesis (the assertion (2) for $(\overline{E}, \nabla_{\overline{E}})$), M belongs to B. Since $M_0 = 0$, we conclude that $M = 0$. Thus (E, ∇_E) is the pullback of $(\overline{E}, \nabla_{\overline{E}})$ to $(\mathrm{Spec}(C_k), M_k)/\mathrm{Spec}(K)^{\times}$. Since $(\overline{E}, \nabla_{\overline{E}})$ is the pullback of an object (E_1, ∇_{E_1}) in $MIC(\mathrm{Spec}(B)/\mathrm{Spec}(K))$ with E_1 free of rank 1 to $(C_{k-1}, M_{k-1})/\mathrm{Spec}(K)^{\times}$ by induction hypothesis, we conclude that (E, ∇_E) is the pullback of (E_1, ∇_{E_1}) to $(C_k, M_k)/\mathrm{Spec}(K)^{\times}$. So the proof of the assertion (1) is finished.

We prove the assertion (2). By definition of $(\overline{E}, \nabla_{\overline{E}})$, we see that any endomorphism $\varphi : (E, \nabla_E) \to (E, \nabla_E)$ induces an endomorphism on $(\overline{E}, \nabla_{\overline{E}})$, and it is the multiplication of some element in B by induction hypothesis. Hence so is φ. So the proof of the assertion (2) is also finished. $\qquad \square$

To prove the property (ii) of theorem 3.1 for iterated extensions of rank 1 objects, we need the following two results: first we prove that the category

$MIC(X^\times/\mathrm{Spec}(K)[u])^{\mathrm{nr}}$ is closed under extensions (theorem 5.9) and then prove that we can compare extensions of objects in $MIC(X^\times/\mathrm{Spec}(K)[u])^{\mathrm{nr}}$ with those in $MIC(X^\times/\mathrm{Spec}(K)^\times)^{\mathrm{nr}}$ in certain case (proposition 5.10).

THEOREM 5.9. *The category $MIC(X^\times/\mathrm{Spec}(K)[u])^{\mathrm{nr}}$ is closed under extensions in the category of coherent $\mathcal{O}_X[u]$-modules with integrable log connections.*

Proof. Let (F, ∇_F) and (G, ∇_G) be objects in $MIC(X^\times/\mathrm{Spec}(K)[u])^{\mathrm{nr}}$ and let us suppose that there exists an exact sequence of $\mathcal{O}_X[u]$-modules with integrable connection:

$$0 \longrightarrow (F, \nabla_F) \longrightarrow (E, \nabla_E) \longrightarrow (G, \nabla_G) \longrightarrow 0. \tag{37}$$

It is easy to see that (E, ∇_E) satisfies the conditions (i), (ii) of definition 4.3. Hence, it suffices to see that (E, ∇_E) is written as a colimit of objects in $MIC(X^\times/\mathrm{Spec}(K))^{\mathrm{nr}}$ indexed by $\mathbb{N}$. Let us write $(F, \nabla_F), (G, \nabla_G)$ as colimits of objects in $MIC(X^\times/\mathrm{Spec}(K))^{\mathrm{nr}}$ indexed by $\mathbb{N}$:

$$(F, \nabla_F) = \varinjlim_{i \in \mathbb{N}}(F_i, \nabla_{F_i}), \quad (G, \nabla_G) = \varinjlim_{j \in \mathbb{N}}(G_j, \nabla_{G_j}).$$

Also, let

$$0 \longrightarrow (F, \nabla_F) \longrightarrow (E_j, \nabla_{E_j}) \longrightarrow (G_j, \nabla_{G_j}) \longrightarrow 0. \tag{38}$$

be the pullback of (37) by $(G_j, \nabla_{G_j}) \to (G, \nabla_G)$. The exact sequence (38) defines the extension class $[(E_j, \nabla_{E_j})]$ in

$$\mathrm{Ext}^1((G_j, \nabla_{G_j}), (F, \nabla_F)) = H^1_{\mathrm{dR}}(X^\times/\mathrm{Spec}(K), (G_j, \nabla_{G_j})^\vee \otimes (F, \nabla_F))$$
$$= \varinjlim_i H^1_{\mathrm{dR}}(X^\times/\mathrm{Spec}(K), (G_j, \nabla_{G_j})^\vee \otimes (F_i, \nabla_{F_i}))$$
$$= \varinjlim_i \mathrm{Ext}^1((G_j, \nabla_{G_j}), (F_i, \nabla_{F_i})).$$

Here, the first and the third equalities follow from the local freeness of G_j. Also, the second equality is straightforward in the case X is affine and in general it is reduced to the affine case by using Čech cohomology for an open affine cover. Hence there exists a system of extension classes $[(E_{j,i}, \nabla_{E_{j,i}})] \in \mathrm{Ext}^1((G_j, \nabla_{G_j}), (F_i, \nabla_{F_i}))$ for some $i = i(j) \in \mathbb{N}$ which induces $[(E_j, \nabla_{E_j})]$. Also, we can take the indices $i = i(j) \in \mathbb{N}$ so that the map $j \mapsto i(j)$ is strictly increasing and that the extension classes
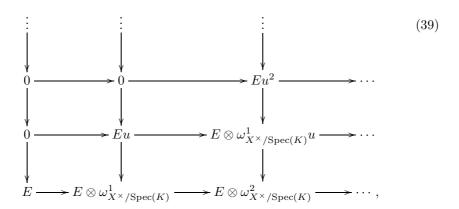
$$[(E_{j,i(j)}, \nabla_{E_{j,i(j)}})] \in \mathrm{Ext}^1((G_j, \nabla_{G_j}), (F_{i(j)}, \nabla_{F_{i(j)}})),$$
$$[(E_{j+1,i(j+1)}, \nabla_{E_{j+1,i(j+1)}})] \in \mathrm{Ext}^1((G_{j+1}, \nabla_{G_{j+1}}), (F_{i(j+1)}, \nabla_{F_{i(j+1)}}))$$

are compatible in the sense that they have the same image in $\mathrm{Ext}^1((G_j, \nabla_{G_j}), (F_{i(j+1)}, \nabla_{F_{i(j+1)}}))$. Then we see that $\{(E_{j,i(j)}, \nabla_{E_{j,i(j)}})\}_{j \in \mathbb{N}}$ forms an inductive system in $MIC(X^\times/\mathrm{Spec}(K))^{\mathrm{nr}}$ whose colimit is (E, ∇_E). Hence we are done. $\qquad\square$

Proposition 5.10. *Let (E, ∇_E) be an object of $MIC(X^\times/\mathrm{Spec}(K))^{\mathrm{nr}}$ and let $(E, \nabla_E)[u] = t_u((E, \nabla_E))$ as defined in section 4. Then the functor $p_u : MIC(X^\times/\mathrm{Spec}(K)[u])^{\mathrm{nr}} \to MIC(X^\times/\mathrm{Spec}(K)^\times)^{\mathrm{nr}}$ defined in section 4 induces a map from the de Rham complex of $(E, \nabla_E)[u]$, denoted by $\mathrm{DR}((E, \nabla_E)[u])$, to the de Rham complex of $p_u((E, \nabla_E)[u])$, denoted by $\mathrm{DR}(p_u((E, \nabla_E)[u]))$, and it is a quasi-isomorphism.*

Proof. The proof is inspired by [KH04, lemma 6]. We can regard the de Rham complex $\mathrm{DR}((E, \nabla_E)[u])$ as the total complex of the bicomplex

$$
\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & Eu^2 & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Eu & \longrightarrow & E \otimes \omega^1_{X^\times/\mathrm{Spec}(K)}u & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
E & \longrightarrow & E \otimes \omega^1_{X^\times/\mathrm{Spec}(K)} & \longrightarrow & E \otimes \omega^2_{X^\times/\mathrm{Spec}(K)} & \longrightarrow & \cdots ,
\end{array}
\tag{39}
$$

where the vertical maps are given locally by $\alpha u^i \mapsto \alpha \wedge i u^{i-1} du$ and the horizontal maps are induced by the differentials of the de Rham complex of (E, ∇_E). The cohomology of the total complex of (39) gives the cohomology of $\mathrm{DR}((E, \nabla_E)[u])$. We first prove that all the columns of (39) are exact except at the last term. Let us consider the column

$$
E \otimes \omega^{j-1}_{X^\times/\mathrm{Spec}(K)}u^{i+1} \xrightarrow{\gamma_{j-1}} E \otimes \omega^j_{X^\times/\mathrm{Spec}(K)}u^i \xrightarrow{\gamma_j} E \otimes \omega^{j+1}_{X^\times/\mathrm{Spec}(K)}u^{i-1}.
$$

It is enough to study the exactness of the maps

$$
\omega^{j-1}_{X^\times/\mathrm{Spec}(K)}u^{i+1} \xrightarrow{\delta_{j-1}} \omega^j_{X^\times/\mathrm{Spec}(K)}u^i \xrightarrow{\delta_j} \omega^{j+1}_{X^\times/\mathrm{Spec}(K)}u^{i-1},
$$

because $\gamma_j = \mathrm{id} \otimes \delta_j$ for every j and E is locally free. An element βu^i of $\omega^j_{X^\times/K}u^i$ is in the kernel of δ_j if and only if $\beta \wedge i u^{i-1} du = 0$. If we work locally and choose $\mathrm{dlog}x_1, \ldots, \mathrm{dlog}x_{n-1}, du$ as a basis for $\omega^1_{X^\times/\mathrm{Spec}(K)}$, β can be written uniquely as an $\mathcal{O}_X$-linear combination of the j-th exterior powers of the elements of the basis of $\omega^1_{X^\times/\mathrm{Spec}(K)}$ we chose. Hence if $\beta \wedge i u^{i-1} du = 0$, then β is an $\mathcal{O}_X$-linear combination of j-th exterior powers which contain du. But this happens if and only if βu^i is in the image of δ_{j-1}, *i.e* the columns are exact.

Hence, to calculate the cohomology of the total complex of (39), we only need to calculate the cohomology of the complex given by the cokernels of the last

vertical maps. It is given by

$$E \longrightarrow E \otimes \omega^1_{X^\times/\mathrm{Spec}(K)^\times} \longrightarrow E \otimes \omega^2_{X^\times/\mathrm{Spec}(K)^\times} \longrightarrow \cdots,$$

i.e. the complex $\mathrm{DR}(p_u((E, \nabla_E)[u]))$. Hence the proof is finished. □

We are ready to compare $\pi_1(X^\times/\mathrm{Spec}(K)^\times, x)^{\mathrm{tri}}$ and $\pi_1(X^\times/\mathrm{Spec}(K)[u], x)^{\mathrm{tri}}$. First we treat the case where K is algebraically closed.

PROPOSITION 5.11. *Suppose that K is algebraically closed. Then the map $\pi(p_u)^{\mathrm{tri}}$ is an isomorphism:*

$$\pi(p_u)^{\mathrm{tri}} : \pi_1(X^\times/\mathrm{Spec}(K)^\times, x)^{\mathrm{tri}} \xrightarrow{\cong} \pi_1(X^\times/\mathrm{Spec}(K)[u], x)^{\mathrm{tri}}.$$

Proof. We will show that, for every $(E, \nabla_E) \in MIC(X^\times/\mathrm{Spec}(K)^\times)^{\mathrm{nr,tri}}$, there exists an object (F, ∇_F) in $MIC(X^\times/\mathrm{Spec}(K)[u])^{\mathrm{nr,tri}}$ with $p_u(F, \nabla_F) = (E, \nabla_E)$ (where $^{\mathrm{tri}}$ is as in remark 5.5), and that the morphisms of de Rham cohomologies

$$H^i_{\mathrm{dR}}(X^\times/\mathrm{Spec}(K), (F, \nabla_F)) \to H^i_{\mathrm{dR}}(X^\times/\mathrm{Spec}(K)^\times, (E, \nabla_E)) \ \ (i \in \mathbb{N}) \quad (40)$$

induced by p_u are isomorphisms.
We prove the above claim by induction on the rank of (E, ∇_E). When (E, ∇_E) is of rank 1, thanks to theorem 5.6, we know that there exists $(E, \widetilde{\nabla}_E) \in MIC(X^\times/\mathrm{Spec}(K))^{\mathrm{nr}}$ such that $p_u((E, \widetilde{\nabla}_E)[u]) = (E, \nabla_E)$. Moreover, the morphisms

$$H^i_{\mathrm{dR}}(X^\times/\mathrm{Spec}(K), (E, \widetilde{\nabla}_E)[u]) \to H^i_{\mathrm{dR}}(X^\times/\mathrm{Spec}(K)^\times, (E, \nabla_E)) \ \ (i \in \mathbb{N})$$

are isomorphisms by proposition 5.10. So we are done in rank 1 case.
In general case, we have an exact sequence

$$0 \to (E', \nabla_{E'}) \to (E, \nabla_E) \to (E'', \nabla_{E''}) \to 0 \quad (41)$$

in $MIC(X^\times/\mathrm{Spec}(K)^\times)^{\mathrm{nr,tri}}$ with E'' of rank 1. Also, there exist $(F', \nabla_{F'}), (F'', \nabla_{F''})$ in $MIC(X^\times/\mathrm{Spec}(K)[u])^{\mathrm{nr,tri}}$ with $p_u(F', \nabla_{F'}) = (E', \nabla_{E'}), p_u(F'', \nabla_{F''}) = (E'', \nabla_{E''})$, by induction hypothesis. The above exact sequence defines an element in

$$\mathrm{Ext}^1((E'', \nabla_{E''}), (E', \nabla_{E'})) \cong H^1_{\mathrm{dR}}(X^\times/\mathrm{Spec}(K)^\times, (E'', \nabla_{E''})^\vee \otimes (E', \nabla_{E'})).$$

On the other hand, the group of extension classes

$$\mathrm{Ext}^1_{MIC(X^\times/\mathrm{Spec}(K)[u])^{\mathrm{nr}}}((F'', \nabla_{F''}), (F', \nabla_{F'}))$$

is isomorphic to $H^1_{\mathrm{dR}}(X^\times/\mathrm{Spec}(K), (F'', \nabla_{F''})^\vee \otimes (F', \nabla_{F'}))$ by theorem 5.9. Because the map

$$H^1_{\mathrm{dR}}(X^\times/\mathrm{Spec}(K), (F'', \nabla_{F''})^\vee \otimes (F', \nabla_{F'})) \to$$
$$H^1_{\mathrm{dR}}(X^\times/\mathrm{Spec}(K)^\times, (E'', \nabla_{E''})^\vee \otimes (E', \nabla_{E'}))$$

induced by p_u is an isomorphism by induction hypothesis, we obtain an exact sequence

$$0 \to (F', \nabla_{F'}) \to (F, \nabla_F) \to (F'', \nabla_{F''}) \to 0$$

in $MIC(X^\times/\mathrm{Spec}(K)[u])^{\mathrm{nr,tri}}$ which gives rise to the exact sequence (41) when we apply p_u to it. Moreover, we see that the morphisms (40) induced by p_u are isomorphisms by induction hypothesis and five lemma. So we are done. $\square$

Next we compare $\pi_1(X^\times/\mathrm{Spec}(K)^\times, x)^{\mathrm{tri}}$ and $\pi_1(X^\times/\mathrm{Spec}(K)[u], x)^{\mathrm{tri}}$ in the general case by reducing to the previous case. In the following, let $\overline{K}$ be an algebraic closure of K. Also, for a field L containing K, let X_L (resp. $X_L^\times$) be $X \otimes_K L$ (resp. $X^\times \otimes_K L$), let $\mathrm{Spec}(L)^\times$ be $\mathrm{Spec}(K)^\times \otimes_K L$ and let $\alpha_L : X_L^\times \to X^\times$ be the natural projection.

COROLLARY 5.12. *The map $\pi(p_u)^{\mathrm{tri}}$ is an isomorphism:*

$$\pi(p_u)^{\mathrm{tri}} : \pi_1(X^\times/\mathrm{Spec}(K)^\times, x)^{\mathrm{tri}} \xrightarrow{\cong} \pi_1(X^\times/\mathrm{Spec}(K)[u], x)^{\mathrm{tri}}.$$

Proof. We prove that $\pi(p_u)^{\mathrm{tri}}$ is a closed immersion. To do so, it suffices to prove the essential surjectivity of $p_u : MIC(X^\times/\mathrm{Spec}(K)[u])^{\mathrm{nr,tri}} \to MIC(X^\times/\mathrm{Spec}(K)^\times)^{\mathrm{nr,tri}}$. Let $(E, \nabla_E) \in MIC(X^\times/\mathrm{Spec}(K)^\times)^{\mathrm{nr,tri}}$ and let $(V, \rho : \pi_1(X^\times/\mathrm{Spec}(K), x) \to GL(V))$ be the corresponding representation. Then the base change $\rho_{\overline{K}} : \pi_1(X^\times/\mathrm{Spec}(K), x) \otimes_K \overline{K} \to GL(V \otimes_K \overline{K})$ of ρ to $\overline{K}$ is an iterated extension of rank 1 representations. The morphism $\alpha_{\overline{K}} : X_{\overline{K}} \to X$ induces the morphism $\pi(\alpha_{\overline{K}}) : \pi_1(X_{\overline{K}}^\times/\mathrm{Spec}(\overline{K}), x) \to \pi_1(X^\times/\mathrm{Spec}(K), x) \otimes_K \overline{K}$ and the pullback $\alpha_{\overline{K}}^*(E, \nabla_E) \in MIC(X_{\overline{K}}^\times/\mathrm{Spec}(\overline{K})^\times)^{\mathrm{nr}}$ of (E, ∇_E) by $\alpha_{\overline{K}}$ corresponds to the representation $\rho_{\overline{K}} \circ \pi(\alpha_{\overline{K}})$. Hence it is an iterated extension of rank 1 representations and so $\alpha_{\overline{K}}^*(E, \nabla_E)$ is an object in $MIC(X_{\overline{K}}^\times/\mathrm{Spec}(\overline{K})^\times)^{\mathrm{nr,tri}}$. Hence, by proposition 5.11, there exists an object $(F_{\overline{K}}, \nabla_{F_{\overline{K}}}) \in MIC(X_{\overline{K}}^\times/\mathrm{Spec}(\overline{K})[u])^{\mathrm{nr}}$ such that $p_u(F_{\overline{K}}, \nabla_{F_{\overline{K}}}) \cong \alpha_{\overline{K}}^*(E, \nabla_E)$. By standard argument, we see that there exists a finite Galois subextension L of K in $\overline{K}$ such that $(F_{\overline{K}}, \nabla_{F_{\overline{K}}})$ is the pullback of some object (F_L, ∇_{F_L}) in $MIC(X_L^\times/\mathrm{Spec}(L)[u])^{\mathrm{nr}}$ and that $p_u(F_L, \nabla_{F_L})$ is isomorphic to $\alpha_L^*(E, \nabla_E)$. By full faithfulness of $p_u : MIC(X^\times/\mathrm{Spec}(K)[u])^{\mathrm{nr}} \to MIC(X^\times/\mathrm{Spec}(K)^\times)^{\mathrm{nr}}$, which follows from proposition 4.18 and Galois descent, (F_L, ∇_{F_L}) descends to an object (F, ∇_F) in $MIC(X^\times/\mathrm{Spec}(K)[u])^{\mathrm{nr}}$ and we have an isomorphism $p_u(F, \nabla_F) \cong (E, \nabla_E)$. Thus it suffices to prove that (F, ∇_F) belongs to $MIC(X^\times/\mathrm{Spec}(K)[u])^{\mathrm{nr,tri}}$. Let

$$(W, \tau : \pi_1(X^\times/\mathrm{Spec}(K)[u], x) \to GL(W))$$

be the representation corresponding to (F, ∇_F). Then the composition

$$\pi_1(X^\times/\mathrm{Spec}(K)^\times, x) \otimes_K \overline{K} \xrightarrow{\pi(p_u) \otimes \mathrm{id}}$$
$$\pi_1(X^\times/\mathrm{Spec}(K)[u], x) \otimes_K \overline{K} \xrightarrow{\tau_{\overline{K}}} GL(W \otimes_K \overline{K})$$

(where $\tau_{\overline{K}}$ is the base change of τ to $\overline{K}$) is isomorphic to $\rho_{\overline{K}}$ by construction. Since the latter representation is an iterated extension of rank 1 representations and the map $\pi(p_u) \otimes \mathrm{id}$ is faithfully flat, we see that $\tau_{\overline{K}}$ is also an iterated extension of rank 1 representations. Hence (F, ∇_F) belongs to $MIC(X^\times/\mathrm{Spec}(K)[u])^{\mathrm{nr,tri}}$ and the proof is finished. $\qquad\square$

As a lemma for the proof of the main theorem, we prove a base change property of our algebraic fundamental groups.

LEMMA 5.13. *Let L be a finite extension of K. Then we have an isomorphism*

$$\pi_1(X_L^\times/\mathrm{Spec}(L), x) \cong \pi_1(X^\times/\mathrm{Spec}(K), x) \otimes_K L. \qquad (42)$$

Proof. It is easy to check that the push-forward by α_L induces an equivalence

$$\alpha_{L,*} : MIC(X_L^\times/\mathrm{Spec}(L))^{\mathrm{nr}} \overset{\cong}{\to}$$
$$\{(E, \nabla_E) \in MIC(X^\times/\mathrm{Spec}(K))^{\mathrm{nr}} \text{ with a } K\text{-linear structure of } L\text{-module}\},$$

by the fact that any object (E, ∇) in $MIC(X_L^\times/\mathrm{Spec}(L))^{\mathrm{nr}}$ is a direct summand of $\alpha_L^* \alpha_{L,*}(E, \nabla)$. The claim follows from this equivalence (see [Del89, §4]). $\quad\square$

Now we have all the ingredients to prove the main theorem.

THEOREM 5.14. *The following sequence is exact:*

$$1 \longrightarrow \pi_1(X^\times/\mathrm{Spec}(K)^\times, x)^{\mathrm{tri}} \longrightarrow \pi_1(X^\times/\mathrm{Spec}(K), x)^{\mathrm{tri}}$$
$$\longrightarrow \pi_1(\mathrm{Spec}(K)^\times/\mathrm{Spec}(K), \nu)^{\mathrm{tri}} \longrightarrow 1. \qquad (43)$$

Proof. First note that, for an affine group scheme G over K, $N := \mathrm{Ker}(G \to G^{\mathrm{tri}})$ is stable under any automorphism σ on G. Indeed, for any surjection $\rho : G \to H$ with H geometrically trigonalizable, $\rho \circ \sigma$ has the same property and so $\rho(\sigma(N))$ is trivial. Hence $\sigma(N)$ is contained in N. Then, by applying this argument also to σ^{-1}, we see that $\sigma(N) = N$.
Now consider the exact sequence

$$1 \longrightarrow \pi_1(X^\times/\mathrm{Spec}(K)[u], x) \xrightarrow{\pi(t_u)} \pi_1(X^\times/\mathrm{Spec}(K), x)$$
$$\xrightarrow{\pi(f_{\mathrm{dR}}^*)} \pi_1(\mathrm{Spec}(K)^\times/\mathrm{Spec}(K), \nu) \longrightarrow 1$$

proven in proposition 4.19. By the remark in the previous paragraph, $N := \mathrm{Ker}(\pi_1(X^\times/\mathrm{Spec}(K)[u], x) \to \pi_1(X^\times/\mathrm{Spec}(K)[u], x)^{\mathrm{tri}})$ is a normal subgroup of $\pi_1(X^\times/\mathrm{Spec}(K), x)$. If we put $\pi_1(X^\times/\mathrm{Spec}(K), x)' := \pi_1(X^\times/\mathrm{Spec}(K), x)/N$, we obtain the exact sequence

$$1 \longrightarrow \pi_1(X^\times/\mathrm{Spec}(K)[u], x)^{\mathrm{tri}} \cong \pi_1(X^\times/\mathrm{Spec}(K)^\times, x)^{\mathrm{tri}} \longrightarrow$$
$$\pi_1(X^\times/\mathrm{Spec}(K), x)' \xrightarrow{\pi(f_{\mathrm{dR}}^*)'} \pi_1(\mathrm{Spec}(K)^\times/\mathrm{Spec}(K), \nu) \longrightarrow 1$$

and there exists a natural surjection π : $\pi_1(X^\times/\mathrm{Spec}(K),x)' \to \pi_1(X^\times/\mathrm{Spec}(K),x)^{\mathrm{tri}}$. Hence it suffices to prove that π is an isomorphism, namely, $\pi_1(X^\times/\mathrm{Spec}(K),x)'$ is geometrically protrigonalizable. This is reduced to proving the same property for $\pi_1(X^\times/\mathrm{Spec}(K),x)' \otimes_K \widetilde{L}$ for some field extension $\widetilde{L}$ over K, by proposition 5.4. For some finite extension L of K, $f_L : X_L^\times \to \mathrm{Spec}(L)^\times$ admits a section y and so the map $\pi(f_{L,\mathrm{dR}}^*)$: $\pi_1(X_L^\times/\mathrm{Spec}(L),y) \to \pi_1(\mathrm{Spec}(L)^\times/\mathrm{Spec}(L),\nu)$ induced by f_L admits a section. On the other hand, we have a map

$$\pi_1(X_L^\times/\mathrm{Spec}(L),x) \cong \pi_1(X^\times/\mathrm{Spec}(K),x) \otimes_K L$$
$$\xrightarrow{\pi(f_{\mathrm{dR}}^*)\otimes\mathrm{id}} \pi_1(\mathrm{Spec}(K)^\times/\mathrm{Spec}(K),\nu) \otimes_K L \cong \pi_1(\mathrm{Spec}(L)^\times/\mathrm{Spec}(L),\nu).$$

We would like to identify these two maps after some field extension of L. Take a tensor isomorphism γ between fiber functors

$$MIC(X_L^\times/\mathrm{Spec}(L))^{\mathrm{nr}} \xrightarrow{\omega_x} \mathrm{Vec}_L \to \mathrm{Vec}_{\widetilde{L}},$$
$$MIC(X_L^\times/\mathrm{Spec}(L))^{\mathrm{nr}} \xrightarrow{\omega_y} \mathrm{Vec}_L \to \mathrm{Vec}_{\widetilde{L}}$$

associated to x and y, which exists for some field extension $\widetilde{L}$ of L. The tensor isomorphism γ induces an isomorphism

$$i_\gamma : \pi_1(X_L^\times/\mathrm{Spec}(L),x) \otimes_L \widetilde{L} \xrightarrow{\cong} \pi_1(X_L^\times/\mathrm{Spec}(L),y) \otimes_L \widetilde{L}$$

of 'conjugation by γ'. The isomorphism i_γ induces an inner automorphism

$$\overline{i}_\gamma : \pi_1(\mathrm{Spec}(L)_L^\times/\mathrm{Spec}(L),\nu) \otimes_L \widetilde{L} \xrightarrow{\cong} \pi_1(\mathrm{Spec}(L)_L^\times/\mathrm{Spec}(L),\nu) \otimes_L \widetilde{L},$$

but this is necessarily the identity because $\pi_1(\mathrm{Spec}(L)_L^\times/\mathrm{Spec}(L),\nu) \otimes_L \widetilde{L} \cong \mathbb{G}_a$ is abelian. Hence the isomorphism i_γ is compatible with maps

$$\pi(f_{L,\mathrm{dR}}^*) \otimes \mathrm{id} : \pi_1(X_L^\times/\mathrm{Spec}(L),y) \otimes_L \widetilde{L} \to \pi_1(\mathrm{Spec}(L)^\times/\mathrm{Spec}(L),\nu) \otimes_L \widetilde{L},$$

$$\pi(f_{\mathrm{dR}}^*) \otimes \mathrm{id} : \pi_1(X_L^\times/\mathrm{Spec}(L),x) \otimes_L \widetilde{L} = \pi_1(X^\times/\mathrm{Spec}(K),x) \otimes_K \widetilde{L}$$
$$\to \pi_1(\mathrm{Spec}(K)^\times/\mathrm{Spec}(K),\nu) \otimes_K \widetilde{L} = \pi_1(\mathrm{Spec}(L)^\times/\mathrm{Spec}(L),\nu) \otimes_L \widetilde{L},$$

as required. Hence

$$\pi(f_{\mathrm{dR}}^*) \otimes \mathrm{id} : \pi_1(X^\times/\mathrm{Spec}(K),x) \otimes_K \widetilde{L} \to \pi_1(\mathrm{Spec}(K)^\times/\mathrm{Spec}(K),\nu) \otimes_K \widetilde{L}$$

admits a section, and so does the map

$$\pi(f_{\mathrm{dR}}^*)' \otimes \mathrm{id} : \pi_1(X^\times/\mathrm{Spec}(K),x)' \otimes_K \widetilde{L} \to \pi_1(\mathrm{Spec}(K)^\times/\mathrm{Spec}(K),\nu) \otimes_K \widetilde{L}.$$

Therefore, the proof of the theorem is reduced to the following claim: For a split exact sequence

$$1 \to H \to G \to \mathbb{G}_a \to 1$$

of algebraic groups over a field K of characteristic zero with H geometrically trigonalizable, G is also geometrically trigonalizable.

We prove the above claim. By [Mil15, proposition 17.5], H has a unique (maximal) unipotent normal subgroup U such that $M := H/U$ is of multiplicative type. By uniqueness, U is normal in G. Let P be G/U. Then we have a split exact sequence

$$1 \to M \to P \to \mathbb{G}_a \to 1.$$

Since the action of $\mathbb{G}_a$ on M induced by the splitting is trivial by [Mil15, theorem 14.29], $P \cong M \times \mathbb{G}_a$. If we denote the kernel of $G \to P \cong M \times \mathbb{G}_a \to M$ by U', we have exact sequences

$$1 \to U' \to G \to M \to 1, \qquad 1 \to U \to U' \to \mathbb{G}_a \to 1.$$

From the second exact sequence above, U' is unipotent. Then, by the first exact sequence above and [Mil15, proposition 17.2], G is geometrically trigonalizable. So the proof is finished. □

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