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Structured Robust Stability and Boundedness of Nonlinear Hybrid Delay Systems

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Abstract

Taking different structures in different modes into account, the paper has developed a new theory on the structured robust stability and boundedness for nonlinear hybrid stochastic differential delay equations (SDDEs) without the linear growth condition. A new Lyapunov function is designed in order to deal with the effects of different structures as well as those of different parameters within the same modes. Moreover, a lot of effort is put into showing the almost sure asymptotic stability in the absence of the linear growth condition.

Key words: Hybrid SDDEs, robust stability, robust boundedness, Brownian motion, Markov chain.

1   Introduction

Systems in many branches of science and industry do not only depend on the present state and the past ones but may also experience abrupt changes in their structures and parameters. Hybrid stochastic differential delay equations (SDDEs, also known as SDDEs with Markovian switching) have been widely used to model these systems (see, e.g., books [23, 24] and the references therein). One of the important issues in the study of hybrid SDDEs is the asymptotic analysis of stability and boundedness (see, e.g., [3, 5, 13, 19]). In the asymptotic analysis, the robust stability and boundedness have been two of most popular topics. For example, Ackermann [1] gave a nice motivation of robust stability. Hinrichsen and Pritchard [7, 8] presented an excellent discussion of the stability radii of linear systems with structured perturbations. Su [26] and Tseng et al. [27] discussed the robust stability for linear delay equations. In the aspect of robustness of stochastic

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stability, Haussmann [6] studied the robust stability for a linear system and Ichikawa [11] for a semilinear system. Mao et al. [21] discussed the robust stability of uncertain linear or semilinear stochastic delay systems. Mao [20] investigated the stability of the stochastic delay interval system with Markovian switching. For more information on the stability and boundedness of hybrid SDDEs please see, e.g., [12, 22, 23, 25]. However, all of the papers, up to 2013, in this area only considered these robust problems where the underlying systems were either linear or nonlinear with the linear growth condition (i.e., the coefficients are bounded by a linear function).

Hu et al. [9] were first to investigate the robust stability and boundedness for non-linear hybrid SDDEs *without the linear growth condition* (i.e., the coefficients are not bounded by a linear function, and we will refer to these coefficients as highly nonlinear functions). The significant contribution of Hu et al. [9] lies in that it shows that a given stable hybrid SDDE can not only tolerate the linear-type perturbation but also the highly nonlinear perturbation without loss of the stability, while the papers up to 2013 could only cope with the linear-type perturbation. In other words, Hu at al. [9] opened a new chapter in the study of robust stability for highly nonlinear hybrid SDDEs. However, the progress in this direction is a little due to the difficulty of high nonlinearity and [9] is the only paper so far, to our best knowledge. The aim of this paper is to make some further progress in this area.

Let us explain our key motivation briefly here though further details will be given in Section 3. As we know, the hybrid SDDEs have been used to model practical systems that may experience abrupt changes in their structures and parameters (see, e.g., [3, 5, 13, 23]). The theory in [9] is good at dealing with the hybrid SDDEs that may experience abrupt changes in their parameters. To explain this, assume that a population system operates in two modes, dry and rain, and it switches from one mode to the other according to a two-state Markov chain with state 1 for dry and 2 for rain. In the dry mode, the system is described by a stochastic delay Lotka-Volterra equation $dx(t) = x(t)([a_1 - b_1 x^2(t)]dt + \sigma_1 x(t-\tau)dB(t))$, while in the rain mode by another equation $dx(t) = x(t)([a_2 - b_2 x^2(t)]dt + \sigma_2 x(t-\tau)dB(t))$, where $\tau > 0$ stands for the time delay, a_1, b_1, a_2, b_2 are all positive numbers, $B(t)$ is a scalar Brownian motion and σ_1, σ_2 represent the intensities of the nonlinear stochastic perturbation. In other words, the population system is described by the hybrid SDDE $dx(t) = x(t)([a_{r(t)} - b_{r(t)} x^2(t)]dt + \sigma_{r(t)} x(t-\tau)dB(t))$. This can be regarded as a stochastically perturbed system of the hybrid delay system $dx(t)/dt = x(t)[a_{r(t)} - b_{r(t)} x^2(t)]$ with the highly nonlinear stochastic perturbation $\sigma_{r(t)} x(t)x(t-\tau)dB(t)$. Given the asymptotic boundedness of the delay system $dx(t)/dt = x(t)[a_{r(t)} - b_{r(t)} x^2(t)]$, the theory in [9] shows the upper bounds on σ_1 and σ_2 for the SDDE to remain asymptotically bounded. We observe that in this example, when the system switches from one mode to the other, *only the system parameters change but the structure of the system remains the same type of Lotka-Volterra.* On the other hand, many practical systems may experience abrupt changes in their structures. For example, a population system may change from a delay geometric Brownian motion $dx(t) = -2x(t)dt + \sigma_1 x(t-\tau)dB(t)$ in the dry mode to a delay Lotka-Volterra equation $dx(t) = x(t)[1 - 2x^2(t)]dt + \sigma_2 x^2(t-\tau)dB(t)$ in the rain mode (see, e.g., [2]); a financial system may switch from a geometric Brownian motion $dx(t) = a_1 x(t)dt + \sigma_1 x(t)dB(t)$ to a constant elasticity of volatility (CEV) process $dx(t) = a_2(\mu - x(t))dt + \sigma_2 x^{1.5}(t)dB(t)$ (see, e.g., [15]). Is the theory in [9] applicable to such hybrid SDDEs? We will show a negative answer in Section 3. This motivates us to develop a new theory on the robust stability and boundedness for highly nonlinear hybrid SDDEs which may experience abrupt changes in

their structures.

To make our theory more general, we consider the case where the space of modes, S, of a given hybrid system can be divided into two proper subspaces, S_1 and S_2, such that the system is described by a same type of SDDEs for modes in S_1 (though different parameters for different modes of course) but by a different type of SDDEs for modes in S_2. For example, for the population system in the second half of last paragraph, we have $S = \{dry, rain\}$, $S_1 = \{dry\}$, $S_2 = \{rain\}$ and the system is described by a delay geometric Brownian motion for mode in S_1 but by a delay Lotka-Volterra equation for mode in S_2. Of course, in our setting, both S_1 and S_2 could contain 2 or more modes (see Example 6.2). We should point out that it is possible to develop our theory to cope with even more general case where S can be divided into more than two subspaces and the structures of the underlying hybrid SDDE are significantly different among these subspaces. However, to avoid our notation becoming too complicated, we will only concentrate on the case of two subspaces in this paper.

The key contributions of our paper are highlighted below:

- This is the first paper that takes the different structures in different modes into account to develop a new theory on the structured robust stability and boundedness for highly nonlinear hybrid SDDEs.

- The new theory established in this paper is applicable to hybrid SDDEs which may experience abrupt changes in both structures and parameters.

- The stabilities discussed in this paper include not only the pth moment and almost sure exponential stability but also the pth moment and almost sure asymptotic stability as well as H_∞ stability. (For the definitions of these stabilities we refer the reader to [9, 23].)

- A significant amount of new mathematics has been developed to deal with the difficulties due to the structured difference and those without the linear growth condition. For example, a new Lyapunov function will be designed in order to deal with the effects of different structures for S_1-modes and S_2-modes as well as the effects of different parameters within S_1 and S_2. A lot of effort has also been put into showing the almost sure asymptotic stability without the linear growth condition.

To develop our new theory, we will introduce some necessary notation in Section 2. We will show in Section 3 that the theory in [9] is not applicable to hybrid SDDEs which may experience abrupt changes in structures and this motivates us to establish a new theory in this paper. Our main results on robust boundedness and stability will be discussed in Sections 4 and 5. We will present some case studies and examples in Section 6 to illustrate our theory. We will finally conclude our paper in Section 7.

2 Notation

Throughout this paper, unless otherwise specified, we use the following notation. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all P-null

sets). Let $B(t) = (B_1(t), \cdots, B_m(t))^T$ be an m-dimensional Brownian motion defined on the probability space. Let $r(t)$, $t \geq 0$, be a right-continuous-left-limit Markov chain on the probability space taking values in a finite state space $S = \{1, 2, \cdots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N \times N}$ given by

$$P\{r(t + \Delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta) & \text{if } i \neq j \\ 1 + \gamma_{ij}\Delta + o(\Delta) & \text{if } i = j \end{cases}$$

where $\Delta > 0$. Here $\gamma_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$ while $\gamma_{ii} = -\sum_{j \neq i} \gamma_{ij}$. We assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $B(\cdot)$. We also denote by $|x|$ the Euclidean norm for $x \in R^n$. If A is a vector or matrix, its transpose is denoted by A^T. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{\text{trace}(A^T A)}$. Let $R_+ = [0, \infty)$ and $\tau > 0$. Denote by $C([-\tau, 0]; R^n)$ the family of continuous functions ξ from $[-\tau, 0]$ to R^n with the norm $\|\xi\| = \sup_{-\tau \leq \theta \leq 0} |\xi(\theta)|$. If both a and b are real numbers, then $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$. If G is a set, its indicator function is denoted by I_G. That is, $I_G(x) = 1$ if $x \in G$ and 0 otherwise.

We also need some notation on M-matrices. For a vector or matrix A, by $A > 0$ we mean all elements of A are positive. A Z-matrix is a square matrix $A = (a_{ij})_{N \times N}$ which has non-positive off-diagonal entries (namely $a_{ij} \leq 0$ for all $i \neq j$). The following lemma provides us with two useful criteria to verify if a given Z-matrix is a nonsingular M-matrix (see, e.g., [4, 9, 23]).

Lemma 2.1 *Let $A = (a_{ij})_{N \times N}$ be a Z-matrix. Then A is a nonsingular M-matrix if and only if one of the following statements holds:*

(1) A^{-1} exists and its elements are all nonnegative.

(2) There exists $x > 0$ in R^N such that $Ax > 0$.

By this lemma, we see, for example, that for any positive numbers ε_i $(i \in S)$,

$$A := \text{diag}(\varepsilon_1, \cdots, \varepsilon_N) - \Gamma$$

is a nonsingular M-matrix as $A(1, \cdots, 1)^T = (\varepsilon_1, \cdots, \varepsilon_N) > 0$. This useful technique will be used quite often when we discuss some special cases in Section 6 below.

3 Motivation

To motivate our new study in this paper, let us recall a key result on robust stability from [9]. Consider an n-dimensional hybrid differential equation

$$\frac{dx(t)}{dt} = F(x(t), t, r(t)), \tag{3.1}$$

on $t \geq 0$ and assume that this hybrid system is subject to a stochastic delay perturbation and the perturbed system is described by a hybrid SDDE

$$dx(t) = F(x(t), t, r(t))dt + G(x(t - \tau), t, r(t))dB(t). \tag{3.2}$$

Here $r(t)$, $B(t)$ and τ have been defined in Section 2, both $F : R^n \times R_+ \times S \to R^n$ and $G : R^n \times R_+ \times S \to R^{n \times m}$ are Borel measurable and locally Lipschitz continuous in the first variable. In [9], the following assumption was imposed.

4

Assumption 3.1 *Let $q > p \geq 2$ and assume that for each $i \in S$, there is a real number $\bar{\beta}_{i2}$ and a nonnegative number $\bar{\beta}_{i4}$ such that*

$$x^T F(x, t, i) \leq \bar{\beta}_{i2}|x|^2 - \bar{\beta}_{i4}|x|^{q-p+2} \tag{3.3}$$

for all $(x, t) \in R^n \times R_+$, and

$$\bar{\mathcal{A}} := -\operatorname{diag}(p\bar{\beta}_{12}, \cdots, p\bar{\beta}_{N2}) - \Gamma \tag{3.4}$$

is a nonsingular M-matrix.

It is showed in [9] that this assumption along with the local Lipschtiz condition guarantees the pth moment exponential stability of the given equation (3.1). The study of the robust stability is to investigate how much the stochastic delay perturbation $G(x(t-\tau), t, r(t))dB(t)$ the given stable equation (3.1) can tolerate so that its perturbed system (3.2) remains stable. To measure the stochastic delay perturbation more precisely, the following assumption was then imposed in [9].

Assumption 3.2 *Let $q > p \geq 2$ be the same as in Assumption 3.1 and assume that for each $i \in S$, there are nonnegative numbers $\bar{\beta}_{i3}$ and $\bar{\beta}_{i5}$ such that*

$$|G(y, t, i)|^2 \leq \bar{\beta}_{i3}|y|^2 + \bar{\beta}_{i5}|y|^{q-p+2} \tag{3.5}$$

for all $(y, t) \in R^n \times R_+$.

The study of the robust stability is then to give the bounds on the parameters $\bar{\beta}_{i3}$ and $\bar{\beta}_{i5}$ in order for the perturbed system (3.2) to remain stable. The following theorem describes this situation.

Theorem 3.3 *([9, Theorem 3.4]) Let Assumptions 3.1 and 3.2 hold. Assume that $F(0, t, i) = G(0, t, i) = 0$ for all $t \geq 0$ and $i \in S$. Define*

$$(\bar{\theta}_1, \cdots, \bar{\theta}_N)^T := \bar{\mathcal{A}}^{-1}(1, \cdots, 1)^T, \tag{3.6}$$

(so all $\bar{\theta}_i$'s are positive). If

$$\bar{\beta}_{i3} < \frac{2}{p(p-1)\bar{\theta}_i} \quad and \quad \bar{\beta}_{i5} < \frac{2\min_{j \in S} \bar{\theta}_j \bar{\beta}_{j4}}{(p-1)\bar{\theta}_i} \tag{3.7}$$

for all $i \in S$, then the perturbed system (3.2) is exponentially stable in pth moment.

The significant contribution of this theorem lies in that it does not only show how much the linear perturbation (controlled by $\sqrt{\bar{\beta}_{i3}}|y|$) but also how much the nonlinear perturbation (controlled by $\sqrt{\bar{\beta}_{i5}}|y|^{q-p+2}$) the given stable equation (3.1) can tolerate without loss of the stability, while the existing papers up to 2013 could only cope with the linear perturbation as pointed out in Section 1.

However, we shall now point out its limitation. Recall the population system stated in Section 1: It operates in two modes: dry and rain. Assume that the switching between two modes is modelled by a Markov chain $r(t)$ on the state space $S = \{1, 2\}$ (1 for dry and 2 for rain) with the generator

$$\Gamma = \begin{pmatrix} -1 & 1 \\ 6 & -6 \end{pmatrix}. \tag{3.8}$$

5

The system is modelled by the hybrid SDDE

$$dx(t) = F(x(t), r(t))dt + G(x(t-\tau), r(t))dB(t), \qquad (3.9)$$

where $B(t)$ is a scalar Brownian motion and

$$F(x,1) = -2x, \qquad F(x,2) = x - 2x^3,$$
$$G(y,1) = \sigma_1 y, \qquad G(y,2) = \sigma_2 y^2$$

for $x, y \in R$, in which both σ_1 and σ_2 are positive constants. That is, the system satisfies a delay geometric Brownian motion $dx(t) = -2x(t)dt + \sigma_1 x(t-\tau)dB(t)$ in the dry mode but a delay Lotka-Volterra equation $dx(t) = x(t)[1 - 2x^2(t)]dt + \sigma_2 x^2(t-\tau)dB(t)$ in the rain mode. In other words, the system experiences abrupt changes in their structures when it switches from one mode to the other. If both $\sigma_1 = 0$ and $\sigma_2 = 0$, equation (3.9) becomes

$$\frac{dx(t)}{dt} = F(x(t), r(t)). \qquad (3.10)$$

In other words, equation (3.9) is a stochastically perturbed system of equation (3.10). Noting that

$$xF(x,1) = -2x^2 \quad \text{and} \quad xF(x,2) = x^2 - 2x^4,$$

we see that condition (3.3) holds with $p = 2$, $q = 4$ and

$$\bar{\beta}_{12} = -2, \ \bar{\beta}_{14} = 0, \ \bar{\beta}_{22} = 1, \ \bar{\beta}_{24} = -2.$$

Thus, by (3.4),

$$\bar{\mathcal{A}} = \begin{pmatrix} 5 & -1 \\ -6 & 4 \end{pmatrix} \quad \text{with} \quad \bar{\mathcal{A}}^{-1} = \frac{1}{14}\begin{pmatrix} 4 & 1 \\ 6 & 5 \end{pmatrix}.$$

So $\bar{\mathcal{A}}$ is a nonsingular M-matrix. In other words, Assumption 3.1 is satisfied. This implies that equation (3.10) is exponentially stable in mean square. We expect that equation (3.10) can tolerate a liner perturbation $\sigma_1 x(t-\tau)dB(t)$ in mode 1 while a nonlinear perturbation $\sigma_2 x^2(t-\tau)dB(t)$ in mode 2 given its linear and nonlinear structure in modes 1 and 2, respectively. The aim here is to obtain upper bounds on σ_1 and σ_2 so that the perturbed system (3.9) remains stable. Noting

$$|G(y,1)|^2 = \sigma_1^2 y^2 \quad \text{and} \quad |G(y,2)|^2 = \sigma_2^2 y^4,$$

we see that Assumption 3.2 is satisfied with $p = 2$, $q = 4$ and

$$\bar{\beta}_{13} = \sigma_1^2, \ \bar{\beta}_{15} = 0, \ \bar{\beta}_{23} = 0, \ \bar{\beta}_{25} = \sigma_2^2.$$

To apply Theorem 3.3, we get $\bar{\theta}_1 = 5/14$ and $\bar{\theta}_2 = 11/14$ by (3.6). Hence, condition (3.7) becomes

$$\sigma_1^2 < 14/5 \quad \text{and} \quad \sigma_2^2 < 0. \qquad (3.11)$$

Unfortunately, we never have $\sigma_2^2 < 0$ so Theorem 3.3 is not applicable to the hybrid SDDE (3.9). This indicates that the theory in [9] may not be applicable to the hybrid SDDEs that may experience abrupt changes in their structures.

6

4　Robust Boundedness

Consider an n-dimensional hybrid SDDE

$$dx(t) = f(x(t), x(t-\tau), t, r(t))dt + g(x(t), x(t-\tau), t, r(t))dB(t) \tag{4.1}$$

on $t \geq 0$ with initial data $\{x(\theta) : -\tau \leq \theta \leq 0\} = \xi \in C([-\tau, 0]; R^n)$, where the coefficients $f : R^n \times R^n \times R_+ \times S \to R^n$ and $g : R^n \times R^n \times R_+ \times S \to R^{n \times m}$ are Borel measurable. As a standing hypothesis, we assume the coefficients are locally Lipschitz continuous (see, e.g., [16, 17]).

Assumption 4.1 *For each integer $h \geq 1$ there is a positive constant K_h such that*

$$|f(x, y, t, i) - f(\bar{x}, \bar{y}, t, i)|^2 \vee |g(x, y, t, i) - g(\bar{x}, \bar{y}, t, i)|^2$$
$$\leq K_h(|x - \bar{x}|^2 + |y - \bar{y}|^2)$$

for those $x, y, \bar{x}, \bar{y} \in R^n$ with $|x| \vee |y| \vee |\bar{x}| \vee |\bar{y}| \leq h$ and all $(t, i) \in R_+ \times S$.

It is very easy to verify this local Lipschitz assumption. For example, the assumption is satisfied if f and g are continuously differentiable in x and y or they are differentiable in x and y with locally bounded derivatives. It is known that this classical assumption covers many hybrid SDDEs in the real world (see, e.g., books [23, 24] and the references therein). Of course, this assumption is not enough to guarantee the global solution (i.e., no explosion at a finite time). A standard additional condition for the existence and uniqueness of the global solution of the SDDE (4.1) would be the linear growth condition (see, e.g., [18, 23]). However, our aim here is to study the structured robust stability and boundedness of highly nonlinear SDDEs that do not satisfy the linear growth condition. We hence need to propose alternative assumptions.

Assumption 4.2 *Assume that the state space S of the Markov chain is divided into two proper sub-spaces S_1 and S_2 and we may, without loss of any generality, let $S_1 = \{1, \cdots, N_1\}$ and $S_2 = \{N_1 + 1, \cdots, N\}$, where $1 \leq N_1 < N$. Assume also that there are two constants $q > p \geq 2$. Assume furthermore that for each $i \in S_1$, there are constants $\alpha_{i2} \in R$ and $\alpha_{i1}, \alpha_{i3} \in R_+$ such that, for all $(x, y, t) \in R^n \times R^n \times R_+$,*

$$x^T f(x, y, t, i) + \frac{q-1}{2}|g(x, y, t, i)|^2 \leq \alpha_{i1} + \alpha_{i2}|x|^2 + \alpha_{i3}|y|^2; \tag{4.2}$$

while for each $i \in S_2$, there are constants $\alpha_{i2} \in R$, $\alpha_{i4} > 0$ and $\alpha_{i1}, \alpha_{i3}, \alpha_{i5} \in R_+$ such that

$$x^T f(x, y, t, i) + \frac{p-1}{2}|g(x, y, t, i)|^2$$
$$\leq \alpha_{i1} + \alpha_{i2}|x|^2 + \alpha_{i3}|y|^2 - \alpha_{i4}|x|^{q-p+2} + \alpha_{i5}|y|^{q-p+2}. \tag{4.3}$$

The reason why S is divided into two proper subspaces S_1 and S_2 is because the structure of the underlying hybrid SDDE in S_1-modes differs from that in S_2-modes, as explained in Section 1. In terms of mathematics, conditions (4.2) and (4.3) describethe difference in structure. More understandably, condition (4.2) means that the hybrid SDDE in S_1-modes satisfies the classical Khasminskii-type condition (see, e.g., [14, 23]) while condition (4.2) means that the hybrid SDDE in S_2-modes satisfies the generalised

Khasminskii-type condition (see, e.g., [10]). In layman's terms, the coefficients of the SDDE in S_1-modes may grow linearly in the delay component $x(t-\tau)$ while in S_2-modes it may grow polynomially. It is easy to show if a function grows linearly or polynomially and hence it is not difficult to verify our Assumption 4.2 as demonstrated in our examples in Section 6.

Noting that in Assumption 4.2, we only require $\alpha_{i2} \in R$ for all $i \in S$. According to the Khasminskii-type theorems (see, e.g., [14, 10, 23]), the solution of the hybrid SDDE may grow exponentially. But our aim in this paper is to study the asymptotic boundedness and stability. We therefore need to impose some additional conditions on α_{i2}'s.

Assumption 4.3 *Under Assumption 4.2, assume furthermore that*

$$A := -\mathrm{diag}(p\alpha_{12}, \cdots, p\alpha_{N2}) - \Gamma \tag{4.4}$$

and

$$D := -\mathrm{diag}(q\alpha_{12}, \cdots, q\alpha_{N_12}) - (\gamma_{ij})_{i,j \in S_1} \tag{4.5}$$

are both nonsingular M-matrices.

This assumption means that some α_{i2} must be negative; otherwise A and D could not be nonsingular M-matrices. Hence, the SDDE in mode i with $\alpha_{i2} < 0$ should be asymptotically bounded or stable. Of course, the SDDE in mode i with $\alpha_{i2} \geq 0$ could still grow. However, conditions (4.4) and (4.5) mean that the switchings from those modes with $\alpha_{i2} \geq 0$ to those with $\alpha_{i2} < 0$ are sufficiently fast so that, overall, the underlying hybrid SDDE is still asymptotically bounded or stable. We should also point out that Assumption 4.3 can be verified easily. In fact, compute A^{-1} and D^{-1} easily using Matlab or R and then check if their elements are all nonnegative. If so, by Lemma 2.1, they are nonsingular M-matrices.

When we design our Lyapunov function (see (4.15)), we will need two sets of numbers

$$(\theta_1, \cdots, \theta_N)^T = A^{-1}(1, \cdots, 1)^T \tag{4.6}$$

and

$$(\eta_1, \cdots, \eta_{N_1})^T = D^{-1}(\beta, \cdots, \beta)^T, \tag{4.7}$$

where β is a free positive parameter. Under Assumption 4.3, we see, by Lemma 2.1, that all θ_i ($i \in S$) and η_i ($i \in S_1$) are positive. We will see that β plays a key role in balancing the effects of different structures for S_1-modes and S_2-modes. In particular, if we choose β be sufficiently small, then all η_i will be small too. This means that we can always make condition (4.9) in the following theorem possible by choosing β sufficiently small. In particular, let us make a remark where we show a simple method on how to determine β to guarantee condition (4.9).

Remark 4.4 Let $\tilde{d}$ be the maximum of the row sums of D^{-1} and $\tilde{\gamma} = \max_{i \in S_2} \left(\sum_{j \in S_1} \gamma_{ij} \right)$. Then $\eta_i \leq \beta\tilde{d}$ for all $i \in S_1$ and $\sum_{j \in S_1} \gamma_{ij}\eta_j \leq \beta\tilde{d}\tilde{\gamma}$ for all $i \in S_2$. Hence, if we choose

$$\beta = \frac{\min_{i \in S_2} p\theta_i\alpha_{i4}}{1 + \tilde{d}\tilde{\gamma}} \tag{4.8}$$

then condition (4.9) is guaranteed.

Let us now state our first result in this paper.

Theorem 4.5 *Let Assumptions 4.1, 4.2 and 4.3 hold. Choose $\beta > 0$ sufficiently small for*

$$\alpha_{i4} \geq \frac{\beta + \sum_{j \in S_1} \gamma_{ij}\eta_j}{p\theta_i}, \quad \forall i \in S_2, \tag{4.9}$$

where θ_i and η_i have been defined by (4.6) and (4.7). Assume also that

$$\alpha_{i3} \leq \frac{1}{p\theta_i}, \quad \forall i \in S, \tag{4.10}$$

$$\alpha_{i3} < \frac{\beta}{\eta_i(2q - p)}, \quad \forall i \in S_1, \tag{4.11}$$

and

$$\alpha_{i5} < \frac{\beta q}{\theta_i p(2q - p)}, \quad \forall i \in S_2. \tag{4.12}$$

Then for any initial data $\xi \in C([-\tau, 0]; R^n)$, there is a unique global solution $x(t)$ to the hybrid SDDE (4.1) on $t \in [-\tau, \infty)$. Moreover, the solution has the properties that

$$\limsup_{t \to \infty} \frac{1}{t} \int_0^t E|x(s)|^q ds \leq K_1, \tag{4.13}$$

and

$$\limsup_{t \to \infty} E|x(t)|^p \leq K_2, \tag{4.14}$$

where K_1 and K_2 are positive constants independent of the initial data ξ.

Before the proof, let us give some insight on the relevance of this theorem. We have explained that Assumptions 4.1, 4.2 and 4.3 cover many hybrid SDDEs in the real world while they can be verified easily. Remark 4.4 shows, at least one way, how to determine β to make condition (4.9) hold. The right-hand-side terms of inequalities (4.10)-(4.12) can then computed straightaway and these inequalities give the bounds on the nonlinear perturbation intensities α_{i3} and α_{i5} so that the underlying hybrid SDDE is bounded in L^p asymptotically as well as in time-average of L^q.

Proof. The proof is very technical. To make it more understandable, we will divide it into several steps.

Step 1. In this step, we will define a Lyapunov function $V : R^n \times S \to R_+$ by

$$V(x, i) = \begin{cases} \theta_i|x|^p + \eta_i|x|^q & \text{if } i \in S_1; \\ \theta_i|x|^p & \text{if } i \in S_2 \end{cases} \tag{4.15}$$

and show that it has some nice properties. First of all, it is easy to see that

$$c_1|x|^p \leq V(x, i) \leq c_2(|x|^p + |x|^q), \tag{4.16}$$

where

$$c_1 = \min_{i \in S} \theta_i, \quad c_2 = \left(\max_{i \in S} \theta_i\right) \vee \left(\max_{i \in S_1} \eta_i\right).$$

By the generalized Itô formula (see, e.g., [23, Theorem 1.45 on page 48]), we have that

$$dV(x(t), r(t)) = LV(x(t), x(t - \tau), t, r(t))dt + dM(t) \tag{4.17}$$

9

on $t \geq 0$, where $M(t)$ is a continuous local martingale with $M(0) = 0$ (the explicit form of $M(t)$ is of no use in this paper but can be found in [23]), and the function $LV : R^n \times R^n \times R_+ \times S \to R$ is defined by

$$LV(x, y, t, i) = V_x(x, i)f(x, y, t, i)$$
$$+ \frac{1}{2}\text{trace}[g^T(x, y, t, i)V_{xx}(x, i)g(x, y, t, i)] + \sum_{j \in S} \gamma_{ij} V(x, j),$$

in which

$$V_x(x, i) = \Big(\frac{\partial V(x, i)}{\partial x_1}, \cdots, \frac{\partial V(x, i)}{\partial x_n}\Big) \text{ and } V_{xx}(x, i) = \Big(\frac{\partial^2 V(x, i)}{\partial x_k \partial x_l}\Big)_{n \times n}.$$

Let us first estimate $LV(x, y, t, i)$ for $i \in S_1$. In this case, we have

$$LV(x, y, t, i) = \theta_i p |x|^{p-2} x^T f(x, y, t, i)$$
$$+ \tfrac{1}{2}\theta_i p |x|^{p-2} |g(x, y, t, i)|^2$$
$$+ \tfrac{1}{2}\theta_i p(p-2) |x|^{p-4} |x^T g(x, y, t, i)|^2$$
$$+ \eta_i q |x|^{q-2} x^T f(x, y, t, i)$$
$$+ \tfrac{1}{2}\eta_i q |x|^{q-2} |g(x, y, t, i)|^2$$
$$+ \tfrac{1}{2}\eta_i q(q-2) |x|^{q-4} |x^T g(x, y, t, i)|^2$$
$$+ \sum_{j \in S} \gamma_{ij}\theta_j |x|^p + \sum_{j \in S_1} \gamma_{ij}\eta_j |x|^q.$$

Noting that $|x^T g(x, y, t, i)|^2 \leq |x|^2 |g(x, y, t, i)|^2$, we get

$$LV(x, y, t, i) \leq p\theta_i |x|^{p-2}\Big(x^T f(x, y, t, i) + \frac{p-1}{2}|g(x, y, t, i)|^2\Big)$$
$$+ q\eta_i |x|^{q-2}\Big(x^T f(x, y, t, i) + \frac{q-1}{2}|g(x, y, t, i)|^2\Big)$$
$$+ \sum_{j \in S} \gamma_{ij}\theta_j |x|^p + \sum_{j \in S_1} \gamma_{ij}\eta_j |x|^q. \tag{4.18}$$

By Assumption 4.2, we then have

$$LV(x, y, t, i) \leq p\theta_i |x|^{p-2}\Big(\alpha_{i1} + \alpha_{i2}|x|^2 + \alpha_{i3}|y|^2\Big)$$
$$+ q\eta_i |x|^{q-2}\Big(\alpha_{i1} + \alpha_{i2}|x|^2 + \alpha_{i3}|y|^2\Big)$$
$$+ \sum_{j \in S} \gamma_{ij}\theta_j |x|^p + \sum_{j \in S_1} \gamma_{ij}\eta_j |x|^q. \tag{4.19}$$

But, by (4.6) and (4.7), we have

$$p\alpha_{i2}\theta_i + \sum_{j=1}^{N} \gamma_{ij}\theta_j = -1 \quad \text{and} \quad q\alpha_{i2}\eta_i + \sum_{j \in S_1} \gamma_{ij}\eta_j = -\beta.$$

Hence

$$LV(x, y, t, i) \leq p\theta_i\alpha_{i1}|x|^{p-2} - |x|^p + p\theta_i\alpha_{i3}|x|^{p-2}|y|^2$$
$$+ q\eta_i\alpha_{i1}|x|^{q-2} - \beta|x|^q + q\eta_i\alpha_{i3}|x|^{q-2}|y|^2. \tag{4.20}$$

Note that $p\theta_i\alpha_{i3} \leq 1$ by condition (4.10), while by the well-known Young inequality (see [23, p52]), we have

$$|x|^{p-2}|y|^2 \leq \frac{p-2}{p}|x|^p + \frac{2}{p}|y|^p$$

and similarly for $|x|^{q-2}|y|^2$. We hence obtain from (4.20) that, for $i \in S_1$,

$$
\begin{aligned}
LV(x,y,t,i) \leq\ & p\theta_i\alpha_{i1}|x|^{p-2} - (2/p)|x|^p + (2/p)|y|^p \\
& + q\eta_i\alpha_{i1}|x|^{q-2} - \beta|x|^q \\
& + q\eta_i\alpha_{i3}\Big(\frac{q-2}{q}|x|^q + \frac{2}{q}|y|^q\Big).
\end{aligned}
\tag{4.21}
$$

Similarly, for $i \in S_2$, we can show that

$$
\begin{aligned}
LV(x,y,t,i) \leq\ & p\theta_i\alpha_{i1}|x|^{p-2} - |x|^p + p\theta_i\alpha_{i3}|x|^{p-2}|y|^2 \\
& + \Big(-p\theta_i\alpha_{i4} + \sum_{j \in S_1}\gamma_{ij}\eta_j\Big)|x|^q \\
& + p\theta_i\alpha_{i5}|x|^{p-2}|y|^{q-p+2}.
\end{aligned}
\tag{4.22}
$$

But, by condition (4.9), we have

$$-p\theta_i\alpha_{i4} + \sum_{j \in S_1}\gamma_{ij}\eta_j \leq -\beta. \tag{4.23}$$

Consequently

$$
\begin{aligned}
LV(x,y,t,i) \leq\ & p\theta_i\alpha_{i1}|x|^{p-2} - |x|^p + p\theta_i\alpha_{i3}|x|^{p-2}|y|^2 \\
& - \beta|x|^q + p\theta_i\alpha_{i5}|x|^{p-2}|y|^{q-p+2}.
\end{aligned}
\tag{4.24}
$$

By condition (4.10) and the Young inequality, we then obtain that, for $i \in S_2$,

$$
\begin{aligned}
LV(x,y,t,i) \leq\ & p\theta_i\alpha_{i1}|x|^{p-2} - (2/p)|x|^p + (2/p)|y|^p \\
& - \beta|x|^q + p\theta_i\alpha_{i5}\Big(\frac{p-2}{q}|x|^q + \frac{q-p+2}{q}|y|^q\Big).
\end{aligned}
\tag{4.25}
$$

Combining (4.21) and (4.25), we see that, for all $i \in S$,

$$
\begin{aligned}
LV(x,y,t,i) \leq\ & c_3(|x|^{p-2} + |x|^{q-2}) - (2/p)|x|^p + (2/p)|y|^p \\
& - \beta|x|^q + \hat{\beta}\Big(\frac{q-2}{q}|x|^q + \frac{q-p+2}{q}|y|^q\Big),
\end{aligned}
\tag{4.26}
$$

where

$$\hat{\beta} := \Big(\max_{i \in S_1} q\eta_i\alpha_{i3}\Big) \vee \Big(\max_{i \in S_2} p\theta_i\alpha_{i5}\Big),$$

$$c_3 := \Big(\max_{i \in S} p\theta_i\alpha_{i1}\Big) \vee \Big(\max_{i \in S_1} q\eta_i\alpha_{i1}\Big).$$

By conditions (4.11) and (4.12), we have $\hat{\beta} < \frac{\beta q}{2q-p}$. Define

$$2\beta_1 := \beta - \frac{\hat{\beta}(2q-p)}{q} \quad \text{and} \quad \beta_2 := \frac{\hat{\beta}(q-p+2)}{q}.$$

Then both β_1 and β_2 are positive numbers. Noting that

$$\beta - \frac{\hat{\beta}(q-2)}{q} = 2\beta_1 + \beta_2,$$

we obtain from (4.26) that, for all $i \in S$,

$$LV(x, y, t, i) \leq c_3(|x|^{p-2} + |x|^{q-2}) - (2/p)|x|^p + (2/p)|y|^p$$
$$- (2\beta_1 + \beta_2)|x|^q + \beta_2|y|^q. \tag{4.27}$$

Step 2. In this step, we will show the existence and uniqueness of the global solution of the SDDE (4.1) given any initial data $\xi \in C([-\tau, 0]; R^n)$. Under Assumption 4.1, it is known (see, e.g., [23, Theorem 7.12 on page 278]) that there is a unique maximal local solution $x(t)$ on $t \in [-\tau, \sigma_\infty)$, where σ_∞ is the explosion time. To show this is a unique global solution, we need to show $\sigma_\infty = \infty$ a.s. Let $k_0 > 0$ be a sufficiently large integer such that $\|\xi\| < k_0$. For each integer $k \geq k_0$, define the stopping time

$$\tau_k = \inf\{t \geq 0 : |x(t)| \geq k\},$$

where throughout this paper we set $\inf \emptyset = \infty$ (as usual $\emptyset$ denotes the empty set). It is easy to see that τ_k is increasing as $k \to \infty$ and $\tau_\infty := \lim_{k\to\infty} \tau_k \leq \sigma_\infty$ a.s. Hence the aim of this step will be done if we can show that $\tau_\infty = \infty$ a.s.

We can rearrange (4.27) as

$$LV(x, y, t, i) \leq c_3(|x|^{p-2} + |x|^{q-2}) - \beta_1|x|^q - (2/p)|x|^p + (2/p)|y|^p$$
$$- (\beta_1 + \beta_2)|x|^q + \beta_2|y|^q \tag{4.28}$$

for all $(x, y, t, i) \in R^n \times R^n \times R_+ \times S$. Set

$$c_4 := \sup_{x \in R^n} \left(c_3(|x|^{p-2} + |x|^{q-2}) - \beta_1|x|^q \right) < \infty.$$

Substituting this into (4.28) yields

$$LV(x, y, t, i) \leq c_4 - (2/p)|x|^p + (2/p)|y|^p - (\beta_1 + \beta_2)|x|^q + \beta_2|y|^q. \tag{4.29}$$

Applying the generalized Itô formula, we then have

$$EV(x(t \wedge \tau_k), r(t \wedge \tau_k)) \leq EV(x(0), r(0))$$
$$+ E \int_0^{t\wedge\tau_k} \left(c_4 - (2/p)|x(s)|^p + (2/p)|x(s-\tau)|^p \right.$$
$$\left. - (\beta_1 + \beta_2)|x(s)|^q + \beta_2|x(s-\tau)|^q \right) ds \tag{4.30}$$

for all $t \geq 0$. Noting that

$$\int_0^{t\wedge\tau_k} |x(s-\tau)|^p ds \leq \int_{-\tau}^0 |\xi(s)|^p ds + \int_0^{t\wedge\tau_k} |x(s)|^p ds$$

and

$$\int_0^{t\wedge\tau_k} |x(s-\tau)|^q ds \leq \int_{-\tau}^0 |\xi(s)|^q ds + \int_0^{t\wedge\tau_k} |x(s)|^q ds,$$

we have

$$E \int_0^{t\wedge\tau_k} \left[\beta_2(|x(s-\tau)|^q - |x(s)|^q) + \frac{2}{p}(|x(s-\tau)|^p - |x(s)|^p) \right] ds$$

$$\leq \int_{-\tau}^0 \left[\frac{2}{p}|\xi(s)|^p + \beta_2|\xi(s)|^q \right] ds.$$

This, along with (4.16) and (4.30), implies that

$$c_1 E|x(t \wedge \tau_k)|^p \leq c_5 + c_4 t - \beta_1 E \int_0^{t\wedge\tau_k} |x(s)|^q ds, \qquad (4.31)$$

where

$$c_5 = c_2(|\xi(0)|^p + |\xi(0)|^q) + \int_{-\tau}^0 \left((2/p)|\xi(s)|^p + \beta_2|\xi(s)|^q \right) ds.$$

Consequently

$$c_1 k^p P(\tau_k \leq t) \leq c_5 + c_4 t.$$

Letting $k \to \infty$ gives that $P(\tau_\infty \leq t) = 0$. This means that $\tau_\infty > t$ a.s. Letting $t \to \infty$, we get the desired result $\tau_\infty = \infty$ a.s.

Step 3. We shall show assertion (4.13). It follows from (4.31) that

$$\beta_1 E \int_0^{t\wedge\tau_k} |x(s)|^q ds \leq c_5 + c_4 t.$$

Letting $k \to \infty$ and then using the Fubini theorem, we get

$$\beta_1 \int_0^t E|x(s)|^q ds \leq c_5 + c_4 t.$$

Dividing both sides by $\beta_1 t$ and then letting $t \to \infty$, we see

$$\limsup_{t\to\infty} \frac{1}{t} \int_0^t E|x(s)|^q ds \leq \frac{c_4}{\beta_1},$$

which is the desired assertion (4.13).

Step 4. In this final step we shall prove assertion (4.14). Choose a positive constant δ sufficiently small for

$$\beta_1 > \delta c_2 + \beta_2(e^{\delta\tau} - 1). \qquad (4.32)$$

By the generalized Itô formula again, we have that for any $t \geq 0$,

$$e^{\delta t} EV(x(t), r(t)) = EV(x(0), r(0))$$

$$+ E \int_0^t e^{\delta s} \left[\delta V(x(s), r(s)) + LV(x(s), x(s-\tau), s, r(s)) \right] ds. \qquad (4.33)$$

By (4.16) and (4.29), we then have

$$c_1 e^{\delta t} E|x(t)|^p \leq c_2(|\xi(0)|^p + |\xi(0)|^q)$$

$$+ E \int_0^t e^{\delta s} \Big[\delta c_2(|x(s)|^p + |x(s)|^q)$$

$$+ c_4 - (2/p)|x(s)|^p + (2/p)|x(s-\tau)|^p$$

$$- (\beta_1 + \beta_2)|x(s)|^q + \beta_2|x(s-\tau)|^q \Big] ds. \qquad (4.34)$$

13

Noting that

$$\int_0^t e^{\delta s}|x(s-\tau)|^p ds \le \tau e^{\delta \tau}\|\xi\|^p + \int_0^t e^{\delta(s+\tau)}|x(s)|^p ds$$

etc., we get

$$c_1 e^{\delta t} E|x(t)|^p \le c_6 + E\int_0^t e^{\delta s}H(|x(s)|)ds, \tag{4.35}$$

where $c_6 = (c_2 + 2\tau e^{\delta \tau}/p)\|\xi\|^p + (c_2 + \beta_2 \tau e^{\delta \tau})\|\xi\|^q$ and $H : R_+ \to R$ is defined by

$$H(u) = c_4 + [\delta c_2 + (2/p)(e^{\delta \tau} - 1)]u^p - [\beta_1 - \delta c_2 - \beta_2(e^{\delta \tau} - 1)]u^q.$$

But, by (4.32), we have

$$c_7 := \sup_{u \ge 0} H(u) < \infty.$$

It then follows from (4.35) that

$$c_1 e^{\delta t} E|x(t)|^p \le c_6 + (c_7/\delta)e^{\delta t}. \tag{4.36}$$

This implies

$$\limsup_{t \to \infty} E|x(t)|^p \le c_7/(c_1\delta),$$

which is the desired assertion (4.14). The proof is therefore complete. □

5   Robust Stability

In this section we will discuss the robust stability of the SDDE (4.1). For this purpose, we will assume that $f(0, 0, t, i) = 0$ and $g(0, 0, t, i) = 0$ for all $(t, i) \in R_+ \times S$. Hence the SDDE (4.1) admits a trivial solution $x(t) = 0$ for all $t \ge 0$ when the initial data $\xi = 0$. It is also natural to let $\alpha_{i1} = 0$ for all $i \in S$ in Assumption 4.2. The following theorem gives a criterion on the H_∞-stability in L^q.

Theorem 5.1 *Let all the conditions in Theorem 4.5 hold and, moreover, $\alpha_{i1} = 0$ for all $i \in S$. Then for any initial data $\xi \in C([-\tau, 0]; R^n)$, the unique global solution $x(t)$ of the SDDE (4.1) has the property that*

$$\int_0^\infty E|x(t)|^q dt < \infty. \tag{5.1}$$

Proof.   We use the same notation as in the proof of Theorem 4.5. Clearly, everything we showed there is correct. In particular, $c_3 = 0$ in (4.27) given that $\alpha_{i1} = 0$ for all $i \in S$. Hence, (4.27) becomes

$$LV(x, y, t, i) \le -(2/p)|x|^p + (2/p)|y|^p - (2\beta_1 + \beta_2)|x|^q + \beta_2|y|^q. \tag{5.2}$$

It is then easy to show by the generalized Itô formula that

$$2\beta_1 \int_0^t E|x(s)|^q ds \le (c_2 + 2\tau/p + \beta_2\tau)(\|\xi\|^p + \|\xi\|^q).$$

Letting $t \to \infty$ yields assertion (5.1). □

In general it is not possible to imply $\lim_{t\to\infty} E|x(t)|^q = 0$ from (5.1). On the other hand, You et al. [28] showed this is possible if both coefficients f and g of the SDDE (4.1) satisfy the linear growth condition. However, we are interested in the SDDEs which do not satisfy the linear growth condition in this paper. It is therefore useful if we can show $\lim_{t\to\infty} E|x(t)|^q = 0$ from (5.1) without the linear growth condition. The following theorem describes this possibility which is one of our new contributions in this paper.

Theorem 5.2 *In addition to the same conditions as in Theorem 5.1 , assume that there is a positive constant K such that*

$$x^T f(x, y, t, i) + \frac{q-1}{2}|g(x, y, t, i)|^2 \le K(|x|^2 + |y|^2) \tag{5.3}$$

for all $(x, y, t) \in R^n \times R^n \times R_+$. Then for any initial data $\xi \in C([-\tau, 0]; R^n)$, the unique global solution $x(t)$ of the SDDE (4.1) has the property that

$$\lim_{t\to\infty} E|x(t)|^q = 0. \tag{5.4}$$

Proof. Fix any initial data $\xi \in C([-\tau, 0]; R^n)$. If (5.4) were not true, there must exist a positive number ε and a sequence of positive numbers $\{t_k\}_{k\ge1}$ such that $t_k \to \infty$ as $k \to \infty$ and

$$E|x(t_k)|^q \ge 2\varepsilon, \quad \forall k \ge 1. \tag{5.5}$$

Without loss of generality, we may let $t_1 \ge 2\tau$ and $t_{k+1} > t_k + 2\tau$. By (5.1), we hence have

$$\sum_{k=1}^{\infty} \int_{t_k-2\tau}^{t_k} E|x(s)|^q ds \le \int_0^\infty E|x(s)|^q ds < \infty.$$

Consequently, there exists a k_0 such that

$$\int_{t_k-2\tau}^{t_k} E|x(s)|^q ds \le \frac{\varepsilon}{2qK}, \quad \forall k \ge k_0. \tag{5.6}$$

On the other hand, for any $k \ge k_0$ and $t \in [t_k - \tau, t_k]$, it is easy to show by the Itô formula that

$$E|x(t_k)|^q - E|x(t)|^q \le \mathbb{E} \int_t^{t_k} q|x(s)|^{q-2} \Big(x^T(s) f(x(s), x(s-\tau), s, r(s))$$
$$+ \frac{q-1}{2}|g(x(s), x(s-\tau), s, r(s))|^2 \Big) ds. \tag{5.7}$$

By condition (5.3) and inequality (5.6), we derive

$$E|x(t_k)|^q - E|x(t)|^q \le E \int_t^{t_k} qK|x(s)|^{q-2}(|x(s)|^2 + |x(s-\tau)|^2) ds$$
$$\le E \int_t^{t_k} 2qK(|x(s)|^q + |x(s-\tau)|^q) ds$$
$$\le 2qK \int_{t_k-\tau}^{t_k} E(|x(s)|^q + |x(s-\tau)|^q) ds$$
$$= 2qK \int_{t_k-2\tau}^{t_k} E|x(s)|^q ds$$
$$\le \varepsilon. \tag{5.8}$$

15

This, together with (5.5), implies

$$\varepsilon \leq E|x(t_k)|^q - \varepsilon \leq E|x(t)|^q, \quad \forall t \in [t_k - \tau, t_k]. \tag{5.9}$$

Thus

$$\int_0^\infty E|x(t)|^q dt \geq \sum_{k=k_0}^\infty \int_{t_k-\tau}^{t_k} E|x(t)|^q dt \geq \sum_{k=k_0}^\infty \varepsilon\tau = \infty. \tag{5.10}$$

But, this contradicts (5.1). The desired assertion (5.4) must therefore hold. □

In general it is not possible to imply $\lim_{t\to\infty} |x(t)| = 0$ a.s. from (5.1). However, this is possible in our case and we will show this under the same conditions of Theorem 5.1 without any additional condition, unlike Theorem 5.2 which needs the additional condition (5.3). We should also point out that You et al. [28] showed $\lim_{t\to\infty} |x(t)| = 0$ a.s. from $E \int_0^\infty |x(t)|^2 dt < \infty$ (please note it is 2 but not q) under the linear growth condition. Our new proof given below not only overcomes the difficulty without the linear growth condition but is also much simplified.

Theorem 5.3 *Under the same conditions of Theorem 5.1, for any initial data $\xi \in C([-\tau, 0]; R^n)$, the unique global solution $x(t)$ of the SDDE (4.1) has the property that*

$$\lim_{t\to\infty} |x(t)| = 0 \quad a.s. \tag{5.11}$$

Proof. Again fix any initial data $\xi \in C([-\tau, 0]; R^n)$. We first observe that (5.1) is equivalent to that

$$c_8 := E \int_0^\infty |x(t)|^q dt < \infty \tag{5.12}$$

by the well-known Fubini theorem. This implies that $\int_0^\infty |x(t)|^q dt < \infty$ a.s. and hence

$$\liminf_{t\to\infty} |x(t)| = 0 \quad a.s. \tag{5.13}$$

But this is not assertion (5.11) yet. Let us now assume that the assertion were not true. There is then a positive number $\varepsilon \in (0, 1/4)$ such that

$$P\left(\limsup_{t\to\infty} |x(t)| > 2\varepsilon\right) \geq 4\varepsilon. \tag{5.14}$$

Let τ_k be the same stopping time as defined in the proof of Theorem 4.5. We can easily show from (5.2) that

$$c_1 k^p P(\tau_k \leq t) \leq c_1 E|x(t \vee \tau_k)|^p \leq c_9, \quad \forall t > 0,$$

where c_1 was defined before and c_9 is a positive constant dependent on the initial data only. Letting $t \to \infty$ and then choosing k sufficiently large for $c_9/c_1 k^p \leq \varepsilon$, we get $P(\tau_k < \infty) \leq \varepsilon$. This means that

$$P(|x(t)| < k \text{ for } \forall t \geq -\tau) \geq 1 - \varepsilon. \tag{5.15}$$

Combining (5.14) and (5.15) together gives

$$P(\bar{\Omega}) \geq 3\varepsilon, \tag{5.16}$$

16

where
$$\bar{\Omega} = \left\{ \limsup_{t \to \infty} |x(t)| > 2\varepsilon \text{ and } |x(t)| < k \text{ for } \forall t \geq -\tau \right\}.$$

Fix k from now on and define the stopped process $y(t) = x(t \wedge \tau_k)$ for $t \geq 0$. Clearly, $y(t)$ is an Itô process of the form

$$dy(t) = \bar{f}(t)dt + \bar{g}(t)dB(t), \tag{5.17}$$

where

$$\bar{f}(t) = f(x(t), x(t - \tau), t, r(t))I_{[0,\tau_k)}(t),$$
$$\bar{g}(t) = g(x(t), x(t - \tau), t, r(t))I_{[0,\tau_k)}(t).$$

By Assumption 4.1 as well as $f(0, 0, t, i) = 0$ and $g(0, 0, t, i) = 0$, we see that $\bar{f}(t)$ and $\bar{g}(t)$ are bounded processes, say

$$|\bar{f}(t)| \vee |\bar{g}(t)| \leq c_{10} \quad a.s. \tag{5.18}$$

for all $t \geq 0$. Let us now define a sequence of stopping times

$$\rho_1 = \inf\{t \geq 0 : |y(t)| \geq 2\varepsilon\},$$
$$\rho_{2i} = \inf\{t \geq \rho_{2i-1} : |y(t)| \leq \varepsilon\}, \quad i = 1, 2, \cdots,$$
$$\rho_{2i+1} = \inf\{t \geq \rho_{2i} : |y(t)| \geq 2\varepsilon\}, \quad i = 1, 2, \cdots.$$

By (5.13) and the definition of $\bar{\Omega}$, we have

$$\bar{\Omega} \subset \{\rho_i < \infty\}, \quad i = 1, 2, \cdots. \tag{5.19}$$

Choose a positive number δ and a positive integer j such that

$$c_{10}(\delta + 4\sqrt{2\delta}) \leq \varepsilon^2 \quad \text{and} \quad c_8 < \varepsilon^{q+1}\delta j. \tag{5.20}$$

By (5.16) and (5.19), we can further choose a sufficiently large number T for

$$P(\rho_{2j} \leq T) \geq 2\varepsilon. \tag{5.21}$$

In particular, if $\rho_{2j} \leq T$, $|y(\rho_{2j})| = \varepsilon$ and hence $\rho_{2j} < \tau_k$ by the definition of $y(t)$ (otherwise $|y(\rho_{2j})| = |y(\tau_k)| = k$, a contradiction). In other words, we have

$$y(t, \omega) = x(t, \omega) \text{ for all } 0 \leq t \leq \rho_{2j} \text{ and } \omega \in \{\rho_{2j} \leq T\}. \tag{5.22}$$

By the Burkholder-Davis-Gundy inequality (see, e.g., [23, Theorem 2.13 on page 70]), we can then derive from (5.17) that, for $1 \leq i \leq j$,

$$E\left(\sup_{0 \leq t \leq \delta} \left| |y(\rho_{2i-1} \wedge T + t)| - |y(\rho_{2i-1} \wedge T)| \right| \right)$$
$$\leq E\left(\sup_{0 \leq t \leq \delta} |y(\rho_{2i-1} \wedge T + t) - y(\rho_{2i-1} \wedge T)| \right)$$
$$\leq E\int_{\rho_{2i-1} \wedge T}^{\rho_{2i-1} \wedge T + \delta} |\bar{f}(s)|ds$$
$$+ 4\sqrt{2}E\left(\int_{\rho_{2i-1} \wedge T}^{\rho_{2i-1} \wedge T + \delta} |\bar{g}(s)|^2 ds \right)^{1/2}$$
$$\leq c_{10}(\delta + 4\sqrt{2\delta}).$$

This, together with (5.20), implies

$$P\left(\sup_{0\leq t\leq\delta}\big|\,|y(\rho_{2i-1}\wedge T+t)|-|y(\rho_{2i-1}\wedge T)|\,\big|\geq\varepsilon\right)\leq\varepsilon.$$

Noting that $\rho_{2i-1}\leq T$ if $\tau_{2j}\leq T$, we can derive from (5.21) and the above inequality that

$$P\left(\{\rho_{2j}\leq T\}\cap\left\{\sup_{0\leq t\leq\delta}\big|\,|y(\rho_{2i-1}+t)|-|y(\rho_{2i-1})|\,\big|<\varepsilon\right\}\right)$$

$$=P(\rho_{2j}\leq T)-P\Big(\{\rho_{2j}\leq T\}$$

$$\cap\left\{\sup_{0\leq t\leq\delta}\big|\,|y(\rho_{2i-1}\wedge T+t)|-|y(\rho_{2i-1}\wedge T)|\,\big|\geq\varepsilon\right\}\Big)$$

$$\geq P(\rho_{2j}\leq T)$$

$$-P\left(\sup_{0\leq t\leq\delta}\big|\,|y(\rho_{2i-1}\wedge T+t)|-|y(\rho_{2i-1}\wedge T)|\,\big|\geq\varepsilon\right)$$

$$\geq\varepsilon.$$

This implies easily that

$$P\Big(\{\rho_{2j}\leq T\}\cap\{\rho_{2i}-\rho_{2i-1}\geq\delta\}\Big)\geq\varepsilon. \tag{5.23}$$

Finally, by (5.12), (5.22) and (5.23), we derive

$$c_8=E\int_0^\infty|x(t)|^q dt$$

$$\geq\sum_{i=1}^{j}E\left(I_{\{\rho_{2j}\leq T\}}\int_{\rho_{2i-1}}^{\rho_{2i}}|y(t)|^q dt\right)$$

$$\geq\varepsilon^q\sum_{i=1}^{j}E\left(I_{\{\rho_{2j}\leq T\}}(\rho_{2i}-\rho_{2i-1})\right)$$

$$\geq\varepsilon^q\delta\sum_{i=1}^{j}P\Big(\{\rho_{2j}\leq T\}\cap\{\rho_{2i}-\rho_{2i-1}\geq\delta\}\Big)\Big)$$

$$\geq\varepsilon^{q+1}\delta j.$$

But this contradicts the second inequality in (5.20). Therefore the desired assertion (5.11) must hold. $\square$

The theorems above do not show how fast the solution will tends to the equilibrium state 0 as $t\to\infty$. It is more desirable if we could describe the rate of this asymptotic convergence. The exponential stability meets this desire. Let us now discuss the robustness of pth moment and almost sure exponential stability to close this section.

Theorem 5.4 *Let all the conditions in Theorem 4.5 hold except condition (4.10) which is strengthened by*

$$\alpha_{i3}<\frac{1}{p\theta_i},\quad\forall i\in S \tag{5.24}$$

and, moreover, $\alpha_{i1}=0$ for all $i\in S$. Then there is a positive number λ such that for any initial data $\xi\in C([-\tau,0];R^n)$, the unique global solution $x(t)$ of the SDDE (4.1) satisfies

$$\limsup_{t\to\infty}\frac{1}{t}\log(E|x(t)|^p)\leq-\lambda \tag{5.25}$$

18

and

$$\limsup_{t\to\infty} \frac{1}{t}\log(|x(t)|) \le -\frac{\lambda}{p} \quad a.s. \tag{5.26}$$

Proof. In the same way as (4.27) was proved, we can show from (4.20) and (4.24) that

$$LV(x, y, t, i) \le -|x|^p + \hat{\alpha}|x|^{p-2}|y|^2 \\ - (2\beta_1 + \beta_2)|x|^q + \beta_2|y|^q, \tag{5.27}$$

where

$$\hat{\alpha} := \max_{i\in S} p\theta_i \alpha_{i3} < 1$$

by condition (5.24). This implies

$$LV(x, y, t, i) \le -(1 - \hat{\alpha}(p-2)/p)|x|^p + (2\hat{\alpha}/p)|y|^p \\ - (2\beta_1 + \beta_2)|x|^q + \beta_2|y|^q. \tag{5.28}$$

Let $\lambda > 0$ be sufficiently small for

$$1 - \hat{\alpha}(p-2)/p \ge c_2\lambda + 2\hat{\alpha}e^{\lambda\tau}/p \tag{5.29}$$

and

$$2\beta_1 + \beta_2 \ge c_2\lambda + \beta_2 e^{\lambda\tau}. \tag{5.30}$$

By the generalized Itô formula, we have that

$$e^{\lambda t}V(x(t), r(t)) - V(x(0), r(0)) \\ = \int_0^t e^{\lambda s}\Big(\lambda V(x(s), r(s)) \\ + LV(x(s), x(s-\tau), s, r(s))\Big)ds + M(t) \tag{5.31}$$

on $t \ge 0$, where $M(t)$ is a continuous local martingale with $M(0) = 0$. Making use of (4.16), (5.28)–(5.30), we can then easily show

$$c_1 e^{\lambda t}|x(t)|^p \le c_{11} + M(t), \tag{5.32}$$

where c_{11} is a positive number dependent on the initial data only. Since $M(t)$ is a local martingale, there is a sequence $\{\tilde{\tau}_k\}_{k=1}^\infty$ of stopping times such that $\tilde{\tau}_k \to \infty$ as $k \to \infty$ while for each k, $M(t \wedge \tilde{\tau}_k)$ is a martingale on $t \ge 0$. It follows from (5.32) that, for each $k \ge 1$,

$$c_1 e^{\lambda(t\wedge\tilde{\tau}_k)}|x(t \wedge \tilde{\tau}_k)|^p \le c_{11} + M(t \wedge \tilde{\tau}_k). \tag{5.33}$$

Taking the expectations on both sides yields

$$c_1 E\left[e^{\lambda(t\wedge\tilde{\tau}_k)}|x(t \wedge \tilde{\tau}_k)|^p\right] \le c_{11}. \tag{5.34}$$

Letting $k \to \infty$, we get assertion (5.25) immediately. Moreover, by the nonnegative semimartingale convergence theorem (see, e.g., [23, Theorem 1.10 on page 18]), we have

$$\limsup_{t\to\infty}\left(c_1 e^{\lambda t}|x(t)|^p\right) < \infty \quad a.s.$$

which implies another assertion (5.26). □

6 Special Cases and Examples

In this section we will discuss a number of special cases of hybrid SDDEs in order to demonstrate how our new theory established in the previous two sections can be applied to show the robustness of boundedness and stability of a given hybrid systems subject to various types of nonlinear stochastic perturbations. As a standing hypothesis in this section, we will assume that all coefficients of SDDEs in this section will satisfy the local Lipschitz condition and, moreover, $q > p \geq 2$. To make our cases a bit simple, we assume that the given hybrid system is described by a hybrid differential equation

$$dx(t)/dt = F(x(t), t, r(t)). \tag{6.1}$$

Its structured differences and various stochastic perturbations will be discussed in the following cases. We leave the situation to the reader where the given hybrid system is described by a hybrid differential delay equation $dx(t)/dt = f(x(t), x(t - \tau), t, r(t))$.

6.1 Case 1

Assume that

$$x^T F(x, t, i) \leq a_{i1} |x|^2 - a_{i2} |x|^{q-p+2} \tag{6.2}$$

for $(x, t, i) \in R^n \times R_+ \times S$. Here $a_{i2} > 0$ for $i \in S$ but, for the structured difference, we let $a_{i1} < 0$ for $i \in S_1$ and $a_{i1} \in R$ for $i \in S_2$. This means that the differential equation in mode $i \in S_1$ is stable but may not in mode $i \in S_2$. In order for the hybrid equation (6.1) to be stable, we assume moreover that

$$\mathcal{A} := -\mathrm{diag}(pa_{11}, \cdots, pa_{N1}) - \Gamma \tag{6.3}$$

is a nonsingular M-matrix. It is then known (see, e.g., [9]) that equation (6.1) is exponentially stable in pth moment. Suppose that this equation is subject to a stochastic perturbation and the perturbed system is described by

$$dx(t) = F(x(t), t, r(t))dt + G(x(t), x(t - \tau), t, r(t))dB(t), \tag{6.4}$$

and the perturbation has its structured difference: when mode $i \in S_1$, the perturbation is independent of $x(t - \tau)$, namely

$$G(x, y, t, i) = G_1(x, t, i), \quad i \in S_1;$$

but when mode $i \in S_2$, the perturbation is independent of $x(t)$, namely

$$G(x, y, t, i) = G_2(y, t, i), \quad i \in S_2.$$

Assume furthermore that

$$|G_1(x, t, i)| \leq a_{i3} |x|^{q-p+2}, \quad i \in S_1 \tag{6.5}$$

and

$$|G_2(y, t, i)| \leq a_{i3} |y|^{q-p+2}, \quad i \in S_2, \tag{6.6}$$

where $a_{i3} > 0$. Our aim here is to give a bound on a_{i3} so that the perturbed system (6.4) remains stable. Note that for $i \in S_1$

$$x^T F(x, t, i) + 0.5(q - 1)|G_1(x, t, i)|^2$$
$$\leq a_{i1} |x|^2 - (a_{i2} - 0.5(q - 1)a_{i3})|x|^{q-p+2};$$

while for $i \in S_2$

$$
x^T F(x,t,i) + 0.5(p-1)|G_2(y,t,i)|^2
$$
$$
\leq a_{i1}|x|^2 - a_{i2}|x|^{q-p+2} + 0.5(p-1)a_{i3}|y|^{q-p+2}.
$$

Hence, if we impose the following bounds

$$
a_{i3} \leq \frac{2a_{i2}}{q-1}, \quad i \in S_1, \tag{6.7}
$$

then Assumption 4.2 is satisfied with

$$
\alpha_{i1} = 0, \ \alpha_{i2} = a_{i1}, \ \alpha_{i3} = 0 \quad \text{for } i \in S;
$$
$$
\alpha_{i4} = a_{i2}, \ \alpha_{i5} = 0.5(p-1)a_{i3} \quad \text{for } i \in S_2.
$$

Hence the matrix A defined by (4.4) is the same as the matrix $\mathcal{A}$ defined by (6.3) and hence A is a nonsingular M-matrix. Moreover, the matrix D defined by (4.5) becomes

$$
D := -\text{diag}(qa_{11}, \cdots, qa_{N_11}) - (\gamma_{ij})_{i,j \in S_1} \tag{6.8}
$$

which is a nonsingular M-matrix too by Lemma 2.1 and the note below it as $a_{i1} < 0$ for all $i \in S_1$. In other words, Assumption 4.3 is satisfied too. To Apply Theorem 5.4, we choose β by (4.8) so condition (4.9) is satisfied by Remark 4.4. Compute θ_i's by (4.6). Conditions (4.11) and (5.24) are satisfied of course as $\alpha_{i3} = 0$ for all $i \in S$. If we further impose the following bounds

$$
a_{i3} < \frac{2q\beta}{p(p-1)(2q-p)\theta_i}, \quad i \in S_2, \tag{6.9}
$$

then condition (4.12) is satisfied as well. We can therefore conclude by Theorem 5.4 that the perturbed system (6.4) is both pth moment and almost surely exponentially stable provided the perturbation parameters a_{i3} satisfy conditions (6.7) and (6.9).

6.2   Case 2

Assume that for each $i \in S_1$, there is a number $a_{i1} < 0$ such that

$$
x^T F(x,t,i) \leq a_{i1}|x|^2 \tag{6.10}
$$

while for each $i \in S_2$, there ia a pair of numbers $a_{i1} \in R$ and $a_{i2} > 0$ such that

$$
x^T F(x,t,i) \leq a_{i1}|x|^2 - a_{i2}|x|^{q-p+2} \tag{6.11}
$$

for $(x,t) \in R^n \times R_+$. We also assume that the matrix $\mathcal{A}$ defined by (6.3) is a nonsingular M-matrix. Suppose that equation (6.1) is subject to a stochastic perturbation dependent on the delay state $x(t - \tau)$ and the perturbed system is described by

$$
dx(t) = F(x(t),t,r(t))dt + G(x(t-\tau),t,r(t))dB(t), \tag{6.12}
$$

and the perturbation has its structured difference in the sense that

$$
|G(y,t,i)| \leq a_{i3}|y|^2, \quad i \in S_1 \tag{6.13}
$$

21

and

$$|G(y, t, i)| \le a_{i3}|y|^{q-p+2}, \quad i \in S_2 \tag{6.14}$$

for $(y, t) \in R^n \times R_+$, where $a_{i3} > 0$ for all $i \in S$. Once again, we wish to obtain upper bounds on a_{i3}'s for the perturbed system (6.12) to remain stable. Noting that for $i \in S_1$

$$x^T F(x, t, i) + 0.5(q-1)|G(y, t, i)|^2$$
$$\le a_{i1}|x|^2 + 0.5(q-1)a_{i3}|y|^2$$

while for $i \in S_2$

$$x^T F(x, t, i) + 0.5(p-1)|G(y, t, i)|^2$$
$$\le a_{i1}|x|^2 - a_{i2}|x|^{q-p+2} + 0.5(p-1)a_{i3}|y|^{q-p+2},$$

we see that Assumption 4.2 is satisfied with

$$\alpha_{i1} = 0, \ \alpha_{i2} = a_{i1} \ \text{ for } i \in S;$$
$$\alpha_{i3} = 0.5(q-1)a_{i3} \ \text{ for } i \in S_1;$$
$$\alpha_{i3} = 0, \ \alpha_{i4} = a_{i2}, \ \alpha_{i5} = 0.5(p-1)a_{i3} \ \text{ for } i \in S_2.$$

It is also easy to see that Assumption 4.3 is satisfied with $A = \mathcal{A}$ defined by (6.3) and D is the same as defined by (6.8). To apply Theorems 5.1 and 5.3, we again choose β by (4.8) so condition (4.9) is satisfied by Remark 4.4. Compute θ_i's and by η_i's by (4.6) and (4.7), respectively. Conditions (4.10)–(4.12) yield the following bounds

$$a_{i3} \le \frac{2}{p(q-1)\theta_i} \ \text{ and } a_{i3} < \frac{2\beta}{(q-1)(2q-p)\eta_i} \ \text{ for } i \in S_1 \tag{6.15}$$

while

$$a_{i3} < \frac{2q\beta}{p(p-1)(2q-p)\theta_i} \ \text{ for } i \in S_2. \tag{6.16}$$

By Theorems 5.1, 5.3 and 5.4, we can therefore conclude that if the perturbed parameters a_{i3} satisfy (6.15) and (6.16), then for any initial data $\xi \in C([-\tau, 0]; R^n)$, the solution $x(t)$ of the SDDE (6.12) has the properties that $\int_0^\infty E|x(t)|^q dt < \infty$ and $\lim_{t\to\infty} |x(t)| = 0$ a.s. If, moreover, condition (6.15) is slightly strengthened by

$$a_{i3} < \min \left\{ \frac{2}{p(q-1)\theta_i}, \ \frac{2\beta}{(q-1)(2q-p)\eta_i} \right\} \ \text{ for } i \in S_1, \tag{6.17}$$

the SDDE (6.12) is both pth moment and almost surely exponentially stable.

Example 6.1 Let us now return to the population system discussed in Section 3, namely the hybrid SDDE (3.9) which is the stochastically perturbed system of equation (3.10). This is a special example of Case 2 discussed above. Here we have $S = \{1, 2\}$ with $S_1 = \{1\}$ and $S_2 = \{2\}$ and Γ given by (3.8). Moreover, we have the following system parameters:

$$p = 2, \ q = 4,$$
$$a_{11} = -2, \ a_{21} = 1, \ a_{22} = 2,$$
$$a_{13} = \sigma_1^2, \ a_{23} = \sigma_2^2.$$

We then have

$$\mathcal{A} = \begin{pmatrix} 5 & -1 \\ -6 & 4 \end{pmatrix} \quad \text{with} \quad \mathcal{A}^{-1} = \frac{1}{14} \begin{pmatrix} 4 & 1 \\ 6 & 5 \end{pmatrix}.$$

So $\mathcal{A}$ is a nonsingular M-matrix. We can then further compute

$$\theta_1 = 5/14, \ \theta_2 = 11/14, \ D = 9, \ \tilde{d} = 1/9,$$
$$\tilde{\gamma} = 6, \ \beta = 66/35, \ \eta_1 = 22/105.$$

Conditions (6.16) and (6.17) become

$$\sigma_1 < 0.966, \ \sigma_2 < 1.265. \tag{6.18}$$

We hence conclude that under condition (6.18), the SDDE (3.9) is both mean square and almost surely exponentially stable.

To perform computer simulations, we set $\sigma_1 = 0.8$, $\sigma_2 = 1.2$, $\tau = 0.1$ and let the initial data $\xi(t) = 2 + \sin(t)$ on $t \in [-0.1, 0]$ and $r(0) = 2$. The following computer simulations (Figure 6.1) support our theoretical results clearly.
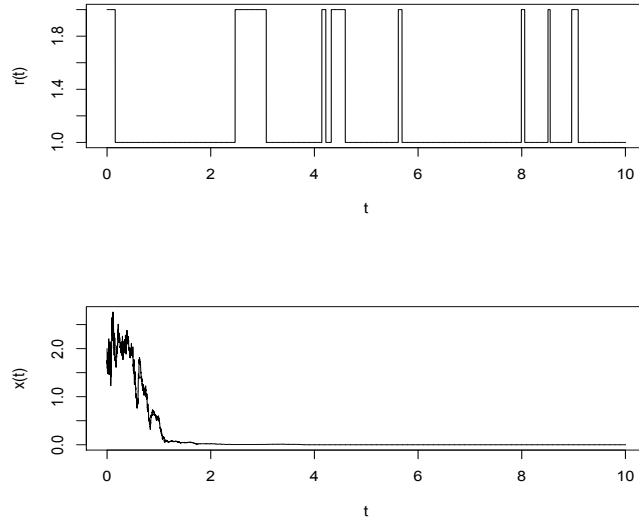

Figure 6.1: The computer simulations of the sample paths of the Markov chain and the solution of equation (3.9) with the parameters and initial data specified above using the Euler–Maruyama method with step size 10^{-3}.

6.3 Case 3

In this case we will discuss the robust boundedness. Assume that

$$x^T F(x, t, i) \leq a_{i1} - a_{i2}|x|^2, \quad i \in S_1 \tag{6.19}$$

and

$$x^T F(x, t, i) \leq a_{i1} - a_{i2}|x|^{q-p+2}, \quad i \in S_2, \tag{6.20}$$

where all a_{i1} and a_{i2} are positive numbers. Suppose that the perturbed system is described by

$$dx(t) = F(x(t), t, r(t))dt + G(x(t - \tau), t, r(t))dB(t), \qquad (6.21)$$

and the perturbation coefficients satisfy

$$|G(y, t, i)| \leq a_{i3}|y|^2, \quad i \in S_1 \qquad (6.22)$$

and

$$|G(y, t, i)| \leq a_{i3}|y|^2 + a_{i4}|y|^{q-p+2}, \quad i \in S_2, \qquad (6.23)$$

where a_{i3} and a_{i4} are all nonnegative numbers. We aim to obtain upper bounds on them so that the perturbed system (6.21) remains asymptotically bounded. It follows from these conditions that for $i \in S_1$

$$\begin{aligned}
& x^T F(x, t, i) + 0.5(q - 1)|G(y, t, i)|^2 \\
& \leq a_{i1} - a_{i2}|x|^2 + 0.5(q - 1)a_{i3}|y|^2;
\end{aligned} \qquad (6.24)$$

while for $i \in S_2$

$$\begin{aligned}
& x^T F(x, t, i) + 0.5(p - 1)|G(y, t, i)|^2 \\
& \leq a_{i1} - a_{i2}|x|^{q-p+2} + 0.5(p - 1)a_{i3}|y|^2 \\
& + 0.5(p - 1)a_{i4}|y|^{q-p+2}.
\end{aligned} \qquad (6.25)$$

If we compare these with (4.2) and (4.3) in Assumption 4.2, we might attempt to have

$$\alpha_{i2} = -a_{i2} \text{ for } i \in S_1 \text{ and } 0 \text{ for } i \in S_2.$$

Consequently, the matrix A defined by (4.4) becomes

$$A = \text{diag}(pa_{12}, \cdots, pa_{N_12}, 0, \cdots, 0) - \Gamma.$$

But A might not be a nonsingular M-matrix. To avoid this, we can simply choose a pair of constants $\delta_1 > 0$ and $\delta_2 \in (0, 1)$ and re-arrange (6.25) as

$$\begin{aligned}
& x^T F(x, t, i) + 0.5(p - 1)|G(y, t, i)|^2 \\
& \leq \alpha_{i1} - \delta_1|x|^2 + 0.5(p - 1)a_{i3}|y|^2 \\
& - (1 - \delta_2)a_{i2}|x|^{q-p+2} + 0.5(p - 1)a_{i4}|y|^{q-p+2},
\end{aligned} \qquad (6.26)$$

where

$$\alpha_{i1} = \sup_{u \geq 0} \left(a_{i1} + \delta_1 u^2 - \delta_2 a_{i2} u^{q-p+2} \right).$$

As a result, Assumption 4.2 is satisfied with

$$\alpha_{i1} = a_{i1}, \quad \alpha_{i2} = -a_{i2}, \quad \alpha_{i3} = 0.5(q - 1)a_{i3}$$

for $i \in S_1$ while

$$\begin{aligned}
& \alpha_{i2} = -\delta_1, \quad \alpha_{i3} = 0.5(p - 1)a_{i3}, \\
& \alpha_{i4} = (1 - \delta_2)a_{i2}, \quad \alpha_{i5} = 0.5(p - 1)a_{i4}
\end{aligned}$$

for $i \in S_2$ (and α_{i1} has been defined above). Hence, the matrices A and D in Assumption 4.3 become

$$A = \text{diag}(pa_{12}, \cdots, pa_{N_12}, \delta_1, \cdots, \delta_1) - \Gamma$$

and
$$D = \text{diag}(qa_{12}, \cdots, qa_{N_12}) - (\gamma_{ij})_{i,j \in S_1}.$$

By Lemma 2.1 and the note below it, both A and D are nonsingular M-matrices. In other words, Assumption 4.3 is satisfied too. To apply Theorem 4.5, we once again choose β by (4.8) so condition (4.9) is satisfied by Remark 4.4. Compute θ_i's and by η_i's by (4.6) and (4.7), respectively. Conditions (4.10)–(4.12) then become

$$a_{i3} \leq \frac{2}{p(q-1)\theta_i}, \quad a_{i3} < \frac{2\beta}{(q-1)(2q-p)\eta_i} \quad \text{for } i \in S_1 \qquad (6.27)$$

and

$$a_{i3} \leq \frac{2}{p(p-1)\theta_i}, \quad a_{i4} < \frac{2q\beta}{p(p-1)(2q-p)\theta_i} \quad \text{for } i \in S_2. \qquad (6.28)$$

By Theorems 4.5, we can therefore conclude that if the perturbed parameters α_{i3} satisfy (6.15) and (6.16), then for any initial data $\xi \in C([-\tau, 0]; R^n)$, the solution $x(t)$ of the SDDE (6.21) has properties (4.13) and (4.14).

Example 6.2 Consider a scalar stochastically perturbed hybrid system

$$dx(t) = F(x(t), t, r(t))dt + G(x(t-\tau), t, r(t))dB(t), \qquad (6.29)$$

where $B(t)$ is a scalar Brownian, $r(t)$ is a Markov chain with the state space $S = \{1, 2, 3, 4\}$ and the generator

$$\Gamma = \begin{pmatrix} -8 & 1 & 4 & 3 \\ 1 & -6 & 2 & 3 \\ 1 & 1 & -3 & 1 \\ 1 & 1 & 0 & -2 \end{pmatrix},$$

and the coefficients are defined by

$$F(x, t, i) = \begin{cases} \cos t - 2x, & i = 1, \\ \sin t - 3x, & i = 2, \\ \cos t - 2x^3, & i = 3, \\ \sin t - 3x^3, & i = 4 \end{cases} \quad \text{and} \quad G(y, t, i) = \begin{cases} \sigma_1 y, & i = 1, \\ \sigma_2 y, & i = 2, \\ \sigma_3 y^2, & i = 3, \\ \sigma_4 y^2, & i = 4. \end{cases}$$

Let $S_1 = \{1, 2\}$, $S_2 = \{3, 4\}$ and $p = 2$, $q = 4$. It is straightforward to show that conditions (6.19), (6.20), (6.22) and (6.23) are satisfied with

$$a_{12} = 1.9, \ a_{22} = 2.9, \ a_{32} = 1.9, \ a_{42} = 2.9,$$
$$a_{13} = \sigma_1^2, \ a_{23} = \sigma_2^2, \ a_{33} = 0, \ a_{43} = 0,$$
$$a_{34} = \sigma_3^2, \ a_{44} = \sigma_4^2,$$

and a_{i1}'s are all positive numbers but their values are of no further use so we do not specify them. Choose two free parameters $\delta_1 = 10$ and $\delta_2 = 0.1$. Then

$$A = \begin{pmatrix} 11.8 & -1 & -4 & -3 \\ -1 & 11.8 & -2 & -3 \\ -1 & -1 & 13 & -1 \\ -1 & -1 & 0 & 12 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 15.6 & -1 \\ -1 & 17.6 \end{pmatrix}.$$

Noting that

$$A^{-1} = \begin{pmatrix} 0.091 & 0.013 & 0.030 & 0.028 \\ 0.011 & 0.089 & 0.017 & 0.027 \\ 0.009 & 0.009 & 0.081 & 0.011 \\ 0.009 & 0.009 & 0.004 & 0.088 \end{pmatrix}$$

and

$$D^{-1} = \begin{pmatrix} 0.064 & 0.004 \\ 0.004 & 0.057 \end{pmatrix},$$

we see, by Lemma 2.1, that both A and D are nonsingular M-matrices. We can then compute

$$\theta_1 = 0.162, \ \theta_2 = 0.144, \ \theta_3 = 0.110, \ \theta_4 = 0.110,$$

$$\tilde{d} = 0.068, \ \tilde{\gamma} = 2, \beta = 0.331, \ \eta_1 = 0.023, \ \eta_2 = 0.020.$$

Conditions (6.27) and (6.28) then become

$$\sigma_1 \le 1.264, \ \sigma_2 \le 1.356, \ \sigma_3 < 1.416, \ \sigma_4 < 1.416. \tag{6.30}$$

We can therefore conclude that if the perturbed parameters σ_i satisfy (6.30), then for any initial data $\xi \in C([-\tau, 0]; R)$, the solution $x(t)$ of the SDDE (6.29) has the properties that

$$\limsup_{t \to \infty} \frac{1}{t} \int_0^t E|x(s)|^4 ds \le K_1,$$

and

$$\limsup_{t \to \infty} E|x(t)|^2 \le K_2,$$

where K_1 and K_2 are positive constants independent of the initial data ξ.

To perform a computer simulation for the second moment of the solution, we set $\sigma_1 = 1, \sigma_2 = \sigma_3 = \sigma_4 = 1.3, \tau = 0.1$ and let the initial data $\xi(t) = 1 + \sin(t)$ on $t \in [-0.1, 0]$ and $r(0) = 1$. The computer simulations in Figure 6.2 show a single sample path of the Markov chain and that of the solution, from which we can see how the Markov chain jumps from one mode to another and also the solution evolves in a bounded domain. To illustrate the boundedness of the second moment, we perform 200-sample-path simulations and then compute the average of their squares to form the approximation of $E|x(t)|^2$. This is showed in Figure 6.3.

7   Conclusion

To distinguish the difference in structures of the underlying hybrid system, we have considered the case where the space of modes, S, can be divided into two subspaces, S_1 and S_2, such that the system is described by a same type of SDDEs for modes in S_1 but by a different type of SDDEs for modes in S_2. Taking these different structures into account we have successfully developed our new theory on the structured robust stability and boundedness for highly nonlinear hybrid SDDEs. A significant amount of new techniques has been developed to deal with the difficulties due to the structured difference and those without the linear growth condition. The proofs of Theorems 4.5 and 5.3 represent typically our new techniques. We have also discussed three special cases and two examples plus some computer simulations to illustrate our theory.
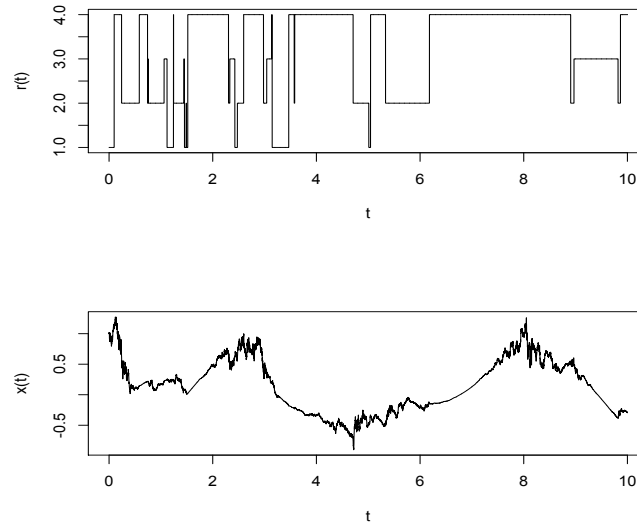
Figure 6.2: The computer simulations of the sample paths of the Markov chain and the solution of equation (6.29) using the Euler–Maruyama method with step size 10^{-3}.
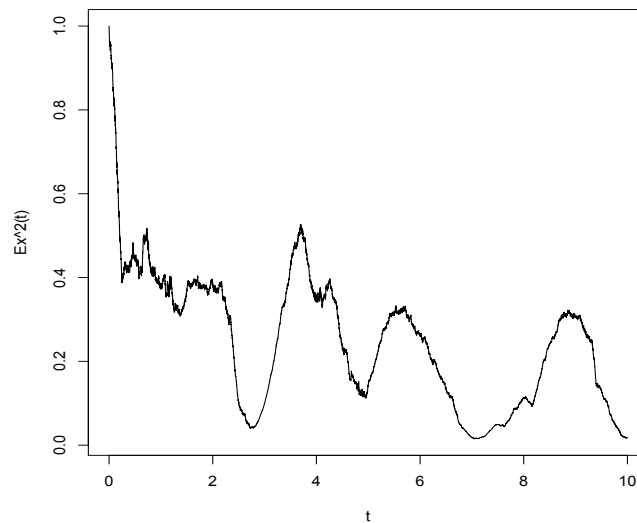

Figure 6.3: The computer simulation of the second moment of the solution of equation (6.29) using the Euler–Maruyama method with step size 10^{-3} and sample size 200.

Acknowledgements

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