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A Derivation of $\pi(n)$ Based on a Stability Analysis of the Riemann-Zeta Function

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The prime-number counting function $\pi(n)$, which is significant in the prime number theorem, is derived by analyzing the region of convergence of the real-part of the Riemann-Zeta function using the unilateral z-transform. In order to satisfy the stability criteria of the z-transform, it is found that the real part of the Riemann-Zeta function must converge to the prime-counting function.

1 Introduction

The Riemann-Zeta function, which is an infinite series in a complex variable s, has been shown to be useful in analyzing nuclear energy levels [1] and the filling of s_1-shell electrons in the periodic table [2]. The following analysis of the Riemann-Zeta function with a z-transform shows the stability zones and requirements for the real and complex variables.

2 Stability with the z-transform

The Riemann-Zeta function is defined as

$$\Gamma(s) = \sum_{n=1}^{\infty} n^{-s}. \tag{1}$$

We start by setting the following equality

$$\Gamma(s) = \sum_{n=1}^{\infty} n^{-s} = \sum_{n=1}^{\infty} e^{-as}. \tag{2}$$

Then by simplifying

$$n^{-s} = e^{-as} = e^{-a(r+j\omega)} \tag{3}$$

and taking natural logarithm of both sides we obtain

$$-s\ln(n) = -as. \tag{4}$$

We then find the constant a such that

$$a = \ln(n). \tag{5}$$

We then apply the unilateral z-transform on (1):

$$\Gamma(s) = \sum_{n=1}^{\infty} n^{-s} z^{-n} = \sum_{n=1}^{\infty} e^{-as} z^{-n} = \sum_{n=1}^{\infty} e^{-a(r+j\omega)} z^{-n}. \tag{6}$$

Substituting (5), the real part of (6) becomes:

$$\text{Re}\left[\Gamma(s)\right] = \sum_{n=1}^{\infty} e^{-ar} z^{-n} = \sum_{n=1}^{\infty} e^{-r\ln(n)} z^{-n}. \tag{7}$$

In order to find the region of convergence (ROC) of (7), we have to factor (7) to the common exponent $-n$, which requires

$$r = n/\ln(n), \tag{8}$$

which is the same as saying that the real part of $\Gamma(s)$ must converge to the prime-number counting function $\pi(n)$. With (8) satisfied, (7) becomes

$$\text{Re}\left[\Gamma(s)\right] = \sum_{n=1}^{\infty} (ez)^{-n}. \tag{9}$$

which has a region of convergence (ROC)

$$\text{ROC} = \frac{1}{1 - \dfrac{1}{ez}}. \tag{10}$$

To be within the region of convergence, z must satisfy the following relation

$$|z| > e^{-1} \quad \text{or} \quad |z| > 0.368. \tag{11}$$

which, places z within the critical strip. It can also be shown that the imaginary part of (6)

$$\text{Im}\left[\Gamma(s)\right] = \sum_{n=1}^{\infty} e^{-aj\omega} z^{-n} = \sum_{n=1}^{\infty} e^{-j\omega\ln(n)} z^{-n}. \tag{12}$$

converges based on the Fourier series of $\sum e^{-j\omega\ln(n)}$.

3 Conclusions

The prime number-counting function $\pi(n)$ has been derived from a stability analysis of the Riemann-Zeta function using the z-transform. It is found that the real part of the roots of the zeta function correspond to $\pi(n)$ under the conditions of stability dictated by the unit-circle of the z-transform. The distribution of prime numbers has been found to be useful in analyzing electron and nuclear energy levels.

References

1. Dyson F. Statistical theory of the energy levels of complex systems. III. *Journal of Mathematical Physics*, 1962, v. 3, 166–175.
2. Harney M. The stability of electron orbital shells based on a model of the Riemann-Zeta function. *Progress in Physics*, 2008, v. 1, 1–5.