Article:

Nazarov, Maxim (2019) On the spin Calogero-Sutherland model at infinity. Progress in Mathematics. pp. 421-439. ISSN 0743-1643

https://doi.org/10.1007/978-3-030-23531-4_11

On the spin Calogero-Sutherland model at infinity

Maxim Nazarov

To Professor Anthony Joseph on the occasion of his 75th birthday

Abstract. For $N = 1, 2, \ldots$ we consider an action of the Yangian $\mathrm{Y}(\mathfrak{gl}_n)$ on Nth symmetric power of the space of polynomials in one variable with coefficients in $\mathbb{C}^n$. This action is given by the Heckman operators [9] via the Drinfeld functor [6]. We describe the limit of this action at $N \to \infty$. This provides another solution to the problem already considered in [11].

Introduction

This quantum Calogero-Sutherland model describes a system of N bosonic particles on a circle $\mathbb{R}/\pi\,\mathbb{Z}$ with the Hamiltonian [3, 19]

$$-\frac{1}{2}\sum_{j}\frac{\partial^2}{\partial q_j^2}+\sum_{i<j}\frac{\beta\,(\beta-1)}{\sin^2(q_i-q_j)} \tag{0.1}$$

where $0 \leqslant q_1, \ldots, q_N < \pi$. After conjugating by the vacuum factor

$$\Big|\prod_{i<j}\sin(q_i-q_j)\Big|^{\beta}$$

and passing to the exponential variables $x_j = \exp(2\mathrm{i}q_j)$ and to the parameter $\alpha = \beta^{-1}$ the Hamiltonian (0.1) becomes

$$\frac{2}{\alpha}\,H+\frac{N^3-N}{6\,\alpha^2}$$

where

$$H=\alpha\sum_i(x_i\,\partial_i)^2+\sum_{i<j}\frac{x_i+x_j}{x_i-x_j}\,(x_i\,\partial_i-x_j\,\partial_j)\,. \tag{0.2}$$

Here ∂_j denotes the derivation with respect to the variable x_j . The operator H acts on the symmetric polynomials in $x_1, \ldots, x_N$. It can be included into a *quantum integrable hierarchy,* that is into a ring of of commuting differential operators with N generators of orders $1, \ldots, N$. The joint eigenfunctions of these commuting differential operators are Jack symmetric polynomials [10].

Two different constructions of generators of this operator ring are known. The first set of generators consists of the coefficients of a certain polynomial of degree N in an auxiliary variable called the Sekiguchi-Debiard determinant [5, 16]. The second set consists of the power sums of degrees $1,\ldots,N$ of the Heckman operators [9], see our Section 2 for their definition. These operators act on all the polynomials in $x_1,\ldots,x_N$ and do not commute, yet their power sums preserve the space of symmetric polynomials. The commuting versions of the Heckman operators were found by Cherednik [4].

It is fascinating to study the limit of the Calogero-Sutherland model when the number N of particles tends to infinity. The limit of the Hamiltonian (0.2) has been known for a long time [18], but explicit description of the limit of the quantum integrable hierarchy was not available until recently. In [14] we described the limits of the generators yielded by the Sekiguchi-Debiard determinant. In [15] we described the limits of the power sums of the Heckman operators, and also identified the resulting integrable hierarchy as that of the quantum counterpart of the classical Benjamin-Ono equation. This equation describes internal waves in fluids of great depth. In [17] the same hierarchy as in [15] was obtained by another approach, namely by describing the limits of the Heckman operators themselves.

The Calogero-Sutherland model has a generalization [8] which describes N bosonic particles on a circle, each particle now having n internal degrees of freedom. Here n is any positive integer. The space of symmetric polynomials used above generalizes now to the subspace in the tensor product

$$(\mathbb{C}^n)^{\otimes N}\otimes\mathbb{C}[x_1,\ldots,x_N] \tag{0.3}$$

consisting of the invariants under the similtaneous permutations of the N tensor factors $\mathbb{C}^n$ and of the variables $x_1,\ldots,x_N$. Remarkably, this subspace comes [2] with an action of the Yangian $\mathrm{Y}(\mathfrak{gl}_n)$. Using either the Cherednik or the Heckman operators on $\mathbb{C}[x_1,\ldots,x_N]$ this action can be obtained as a particular case of a general construction due to Drinfeld [6], see our Section 3. The eigenstates of this model have been studied in [20].

In the present article we consider the limit of this generalization of the Calogero-Sutherland model. This limit was already studied in [1]. Following that work, we identify the limit at $N\to\infty$ of the above mentioned subspace of invariants in (0.3) with the bosonic Fock space $\mathcal{F}$ defined in our Section 1. Using the approach of [17], in Section 2 for any given n we describe the limits at $N\to\infty$ of the Heckman operators now acting on (0.3). This description determines the limiting action of the Yangian $\mathrm{Y}(\mathfrak{gl}_n)$ on $\mathcal{F}$, see our Section 3. This limiting action has been already studied in [11]. However our result has a different form, see the end of Section 3 for an explanation of the difference.

Acknowledgments

I thank Sergey Khoroshkin and Evgeny Sklyanin for helpful conversations. The present work has been supported by the EPSRC grant N023919.

1. Fock space

Fix a positive integer n . Let $\mathcal{F}$ be the commutative algebra over the complex field $\mathbb{C}$ with free generators p_{ck} where $c = 1, \ldots, n$ and $k = 0, 1, 2, \ldots$. We shall refer to $\mathcal{F}$ as to the *Fock space* with n spin degrees of freedom, see [1].

Now take the vector space $\mathbb{C}^n$ with the standard basis vectors $e_1, \ldots, e_n$. Turn $\mathbb{C}^n$ into a commutative ring by setting $e_a e_b = \delta_{ab}\, e_a$ for $a, b = 1, \ldots, n$ and extending this definition of multiplication on $\mathbb{C}^n$ by linearity. The element

$$e = e_1 + \ldots + e_n$$

is a unit of this ring, that is $e\, g = g$ for all $g \in \mathbb{C}^n$. The vector space $V = \mathbb{C}^n[v]$ of all polynomials in the variable v with coefficients in $\mathbb{C}^n$ then also becomes a commutative ring. The element e is a unit of the latter ring as well.

For $N = 1, 2, \ldots$ take the tensor product $V^{\otimes N}$ of N copies of the ring V. This tensor product can be naturally identified with (0.3). The symmetric group $\mathfrak{S}_N$ acts on $V^{\otimes N}$ by permuting the N tensor factors. Consider the subring $(V^{\otimes N})^{\mathfrak{S}_N} \subset V^{\otimes N}$ consisting of the elements invariant under this action. Denote by Λ_N this subring. Define a ring homomorphism

$$\mathcal{F} \to \Lambda_N \tag{1.1}$$

by mapping the identity element $1 \in \mathcal{F}$ to $e^{\otimes N}$ and also mapping the free generators $p_{ck} \in \mathcal{F}$ to the sums

$$\sum_{i=1}^{N} e^{\otimes (i-1)} \otimes e_c\, v^k \otimes e^{\otimes (N-i)} \in \Lambda_N \tag{1.2}$$

respectively. Then the sum

$$\sum_{c=1}^{n} p_{c0} \in \mathcal{F} \tag{1.3}$$

gets mapped to $N e^{\otimes N}$. Our homomorphism (1.1) is surjective due to the next

Proposition 1.1. *The ring Λ_N is generated by the sums* (1.2).

Proof. Let $g_1, \ldots, g_N \in \mathbb{C}^n$ while $k_1, \ldots, k_N = 0, 1, 2, \ldots$. The vector space Λ_N is spanned by the sums of the tensor products

$$h_1\, v^{l_1} \otimes \ldots \otimes h_N\, v^{l_N}$$

where the summation is over all $N!$ permutations $(h_1, l_1), \ldots, (h_N, l_N)$ of a given sequence of pairs $(g_1, k_1), \ldots, (g_N, k_N)$. Let M be the number of pairs in the latter sequence which are different from $(e, 0)$. We will prove by induction on $M = 0, 1, \ldots, N$ that the sum corresponding to the $(g_1, k_1), \ldots, (g_N, k_N)$ belongs to the image of the homomorphism (1.1). Denote by S this sum. Let

$$\Lambda_N^{(M)} \subset \Lambda_N$$

be the subspace spanned by all the sums S with the given number M .

If $M = 0$ then $S = N!\, e^{\otimes N}$, that is $N!$ times the image of the identity element $1 \in \mathcal{F}$ under (1.1). Now suppose that $M > 0$. Because the sum S does not change when the sequence $(g_1, k_1), \ldots, (g_N, k_N)$ is reordered, we will assume that it is the first M pairs $(g_1,k_1),\ldots,(g_M,k_M)$ of the sequence that differ from $(e,0)$. Then consider the product over $j=1,\ldots,M$ of the sums

$$\sum_{i=1}^{N} e^{\otimes(i-1)}\otimes g_j\,v^{k_j}\otimes e^{\otimes(N-i)}\in\Lambda_N\,. \tag{1.4}$$

The difference between this product and $S\,/\,(N-M)!$ belongs to the subspace

$$\Lambda_N^{(0)}+\ldots+\Lambda_N^{(M-1)}\subset\Lambda_N\,.$$

Since (1.4) is a linear combination of the images (1.2) of the elements $p_{ck}\in\mathcal{F}$ with $c=1,\ldots,n$ and $k=k_j$, we have now made the induction step. $\square$

Proposition 1.2. *The kernels of all homomorphisms* (1.1) *with* $N=1,2,\ldots$ *have the zero intersection.*

Proof. Consider the set of all free generators p_{ck} of the commutative ring $\mathcal{F}$. In this set of free generators we can replace p_{n0} by the sum (1.3), which will be denoted here simply by q. Take any finite linear combination of unordered monomials in the new generators of $\mathcal{F}$. Suppose that it gets mapped to zero by every homomorphism (1.1). Consider the terms in this linear combination which have the maximal total degree in all the new generators but q. Let S be the sum of these terms. Let M be their degree. If $M=0$ then our linear combination is just a polynomial in q with complex coefficients, which for all N vanishes when mapping $q\mapsto N\,e^{\otimes N}$. Hence our linear combination is zero.

Suppose $M>0$. For any $N\geqslant M$ apply to S the homomorphism (1.1). Then apply to the resulting image of S in the subspace $\Lambda_N\subset V^{\otimes N}$ the linear map $V^{\otimes N}\to V^{\otimes M}$ projecting onto the tensor product of the first M tensor factors V of $V^{\otimes N}$. Arguments similar to those of the proof of Proposition 1.1 show that the image of S in $V^{\otimes M}$ must be zero. By letting the number N vary like in the case $M=0$ considered above, one can show that $S=0$ then. But the equality $S=0$ contradicts to the assumption that $M>0$. $\square$

We will regard the Fock space $\mathcal{F}$ as the limit at $N\to\infty$ of the ring Λ_N by using the homomorphism (1.1). The complex general linear Lie algebra $\mathfrak{gl}_n$ acts on the vector space V, and diagonally on the tensor product $V^{\otimes N}$. The latter action commutes with the action of the group $\mathfrak{S}_N$. Hence the action of $\mathfrak{gl}_n$ on $V^{\otimes N}$ preserves the subspace Λ_N. In this section we will describe the corresponding action of the Lie algebra $\mathfrak{gl}_n$ on the vector space $\mathcal{F}$. Namely, for any standard matrix unit $E_{ab}\in\mathfrak{gl}_n$ we will describe its action on $\mathcal{F}$ which makes commutative the following square diagram:

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "E_{ab}"] \arrow[d] & \mathcal{F} \arrow[d] \\
\Lambda_N \arrow[r, "E_{ab}"'] & \Lambda_N
\end{tikzcd}$$

Here the vertical arrows indicate the homomorphism (1.1). It is easy to verify

Lemma 1.3. *The action of E_{ab} on V is a ring endomorphism.*

Note that the endomorphism E_{ab} does *not* preserve the element $e \in V$ unless $a = b$. We will describe the action of E_{ab} on $\mathcal{F}$ using the method of [17]. We will first consider the ring $V \otimes \mathcal{F}$. It contains $\mathcal{F}$ via the embedding

$$\iota : \mathcal{F} \to V \otimes \mathcal{F} : f \mapsto e \otimes f \tag{1.5}$$

for all $f \in \mathcal{F}$. The ring $V \otimes \mathcal{F}$ is generated by the elements $e_c v^k \otimes 1$ and the elements $e \otimes p_{ck}$. Let us extend (1.1) to the ring homomorphism

$$\pi_N : V \otimes \mathcal{F} \to V \otimes \Lambda_{N-1}$$

by mapping

$$e_c v^k \otimes 1 \mapsto e_c v^k \otimes e^{\otimes (N-1)} . \tag{1.6}$$

We have

$$\Lambda_N \subset V \otimes \Lambda_{N-1} \tag{1.7}$$

and our π_N by definition maps the element $e \otimes p_{ck} \in V \otimes \mathcal{F}$ to the sum (1.2). We will describe an operator F_{ab} on $V \otimes \mathcal{F}$ making commutative the diagram

$$\begin{tikzcd} V \otimes \mathcal{F} \arrow[r, "F_{ab}"] \arrow[d, "\pi_N"'] & V \otimes \mathcal{F} \arrow[d, "\pi_N"] \\ V \otimes \Lambda_{N-1} \arrow[r, "E_{ab} \otimes \mathrm{id}"'] & V \otimes \Lambda_{N-1} \end{tikzcd} \tag{1.8}$$

To this end we will introduce another ring homomorphism

$$\pi'_N : V \otimes \mathcal{F} \to V \otimes \Lambda_{N-1}$$

such that π'_N will map the element $e \otimes p_{ck} \in V \otimes \mathcal{F}$ to the sum

$$\sum_{i=2}^{N} e^{\otimes (i-1)} \otimes e_c v^k \otimes e^{\otimes (N-i)} \tag{1.9}$$

instead of (1.2). The homomorphism π'_N will still map (1.6) as π_N does. So

$$\begin{aligned} \pi_N(e \otimes p_{ck}) &= \pi'_N(e \otimes p_{ck}) + e_c v^k \otimes e^{\otimes (N-1)} \\ &= \pi'_N(e \otimes p_{ck} + e_c v^k \otimes 1) \end{aligned} \tag{1.10}$$

and

$$\pi'_N(e \otimes p_{ck}) = \pi_N(e \otimes p_{ck} - e_c v^k \otimes 1) . \tag{1.11}$$

By the definition of π'_N we immediately obtain commutativity of the diagram

$$\begin{tikzcd} V \otimes \mathcal{F} \arrow[r, "E_{ab} \otimes \mathrm{id}"] \arrow[d, "\pi'_N"'] & V \otimes \mathcal{F} \arrow[d, "\pi'_N"] \\ V \otimes \Lambda_{N-1} \arrow[r, "E_{ab} \otimes \mathrm{id}"'] & V \otimes \Lambda_{N-1} \end{tikzcd}$$

In other words, the limit at $N \to \infty$ of the operator $E_{ab} \otimes \mathrm{id}$ on $V \otimes \Lambda_{N-1}$ relative to the homomorphism π'_N is just the operator $E_{ab} \otimes \mathrm{id}$ on $V \otimes \mathcal{F}$.

Let us now turn to the homomorphism π_N. By Lemma 1.3 the operator $E_{ab} \otimes \mathrm{id}$ on $V \otimes \Lambda_{N-1}$ is a ring endomorphism. Therefore our F_{ab} will be an endomorphism of the ring $V \otimes \mathcal{F}$. Setting

$$F_{ab}\,(e_c\, v^k \otimes 1) = E_{ab}\,(e_c\, v^k) = \delta_{bc}\, e_a\, v^k \otimes 1 \tag{1.12}$$

will make the compositions $\pi_N F_{ab}$ and $(E_{ab} \otimes \mathrm{id})\,\pi_N$ coincide on the element $e_c\, v^k \otimes 1 \in V \otimes \mathcal{F}$, see (1.6) and (1.8). Again according to (1.8) we also need

$$\pi_N F_{ab}\,(e \otimes p_{ck}) = (E_{ab} \otimes \mathrm{id})\,\pi_N\,(e \otimes p_{ck})\,.$$

By (1.10) and (1.11) the right hand side of the above displayed relations equals

$$\begin{gathered}
(E_{ab} \otimes \mathrm{id})\,\pi'_N\,(e \otimes p_{ck} + e_c\, v^k \otimes 1) = \\
\pi'_N\,(E_{ab} \otimes \mathrm{id})\,(e \otimes p_{ck} + e_c\, v^k \otimes 1) = \\
\pi'_N\,(e_a \otimes p_{ck} + \delta_{bc}\, e_a\, v^k \otimes 1) = \pi'_N\,((e_a \otimes 1)\,(e \otimes p_{ck}) + \delta_{bc}\, e_a\, v^k \otimes 1) = \\
\pi_N\,((e_a \otimes 1)\,(e \otimes p_{ck} - e_c\, v^k \otimes 1) + \delta_{bc}\, e_a\, v^k \otimes 1) = \\
\pi_N\,(e_a \otimes p_{ck} - \delta_{ac}\, e_a\, v^k \otimes 1 + \delta_{bc}\, e_a\, v^k \otimes 1)\,.
\end{gathered}$$

Hence

$$F_{ab}\,(e \otimes p_{ck}) = e_a \otimes p_{ck} + (\delta_{bc} - \delta_{ac})\, e_a\, v^k \otimes 1\,. \tag{1.13}$$

So the actions of F_{ab} and $E_{ab} \otimes \mathrm{id}$ on $e \otimes p_{ck}$ differ unless $\delta_{ac} = \delta_{bc}$. We get

Proposition 1.4. *The endomorphism F_{ab} of the ring $V \otimes \mathcal{F}$ defined by* (1.12) *and* (1.13) *makes commutative the diagram* (1.8).

To describe the action of E_{ab} on $\mathcal{F}$ let us now consider the linear map

$$\theta :\, V \otimes \mathcal{F} \to \mathcal{F} :\, e_c\, v^k \otimes f \,\mapsto\, p_{ck}\, f\,. \tag{1.14}$$

This is *not* a ring homomorphism, but is $\mathcal{F}$-linear relative to the embedding $\iota : \mathcal{F} \to V \otimes \mathcal{F}$ defined earlier. Moreover it makes commutative the diagram

$$\begin{tikzcd}
V \otimes \mathcal{F} \arrow[r, "\theta"] \arrow[d, "\pi_N"'] & \mathcal{F} \arrow[d] \\
V \otimes \Lambda_{N-1} \arrow[r, "\theta_N"'] & \Lambda_N
\end{tikzcd} \tag{1.15}$$

where the rightmost vertical arrow indicates the homomorphism (1.1), while θ_N denotes the restriction of the action of the element

$$1 + \sum_{i=2}^{N} (1i) \in \mathbb{C}\mathfrak{S}_N$$

to the subspace $V \otimes \Lambda_{N-1} \subset V^{\otimes N}$. Here $(1i) \in \mathfrak{S}_N$ is the transposition of 1 and i. To prove the commutativity of (1.15) observe that π_N by definition maps the subring $\mathcal{F} \subset V \otimes \mathcal{F}$ to the subring (1.7), while θ_N is Λ_N-linear.

Hence it suffices to chase the element $e_c\, v^k \otimes 1 \in V \otimes \mathcal{F}$ the two ways offered by the diagram (1.15). But both ways yield the same result, the sum (1.2).

Let us now place two more commutative diagrams on the left of (1.15):

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\iota"] \arrow[d] & V \otimes \mathcal{F} \arrow[r, "F_{ab}"] \arrow[d, "\pi_N"'] & V \otimes \mathcal{F} \arrow[r, "\theta"] \arrow[d, "\pi_N"'] & \mathcal{F} \arrow[d] \\
\Lambda_N \arrow[r] & V \otimes \Lambda_{N-1} \arrow[r, "E_{ab}\otimes\mathrm{id}"'] & V \otimes \Lambda_{N-1} \arrow[r, "\theta_N"'] & \Lambda_N
\end{tikzcd}$$

Here we have the diagram (1.8) in the middle. The leftmost vertical arrow is the homomorphism (1.1), the leftmost bottom arrow is the embedding (1.7).

Theorem 1.5. *The element $E_{ab} \in \mathfrak{gl}_n$ acts on $\mathcal{F}$ as the composition $\theta\, F_{ab}\, \iota$.*

Proof. The composition $\theta_N\,(E_{ab} \otimes \mathrm{id})$ acts on the subspace (1.7) as the sum

$$\sum_{i=1}^{N} \mathrm{id}^{\otimes\,(i-1)} \otimes E_{ab} \otimes \mathrm{id}^{\otimes\,(N-i)}\,.$$

Hence the theorem follows from the commutativity of the latter diagram. □

Now consider the particular case when $a = b$. By (1.13) for $c = 1, \ldots, n$ and $k = 0, 1, 2, \ldots$ we have $F_{aa}\,(e \otimes p_{ck}) = e_a \otimes p_{ck}$. More generally, for any $f \in \mathcal{F}$ we have $F_{aa}\,(e \otimes f) = e_a \otimes f$ because F_{aa} is an endomorphism of the ring $V \otimes \mathcal{F}$. By Theorem 1.5 and by definition of θ we get $E_{aa}(f) = p_{a0}\, f$.

2. Heckman operators

Let α be a complex parameter. For $i = 1, \ldots, N$ consider the *Dunkl operator*

$$Y_i \,=\, \alpha\, \partial_i + \sum_{j \neq i} \frac{1}{x_i - x_j}\,(1 - \sigma_{ij})$$

acting on the ring of all polynomials in the variables $x_1, \ldots, x_N$ with complex coefficients. Here ∂_i is the derivation in this ring relative to the variable x_i , while σ_{ij} is the operator on this ring exchanging the variables x_i and x_j . Note that for any permutation σ of the variables $x_1, \ldots, x_N$ we have the relation

$$\sigma^{-1} Y_i\, \sigma = Y_{\sigma(i)}\,. \tag{2.1}$$

The operators Y_i with $i = 1, \ldots, N$ pairwise commute. This fact is well known, and goes back to the work [7]. Next consider the *Heckman operator* [9]

$$Z_i \,=\, x_i\, Y_i \,=\, \alpha\, x_i\, \partial_i + \sum_{j \neq i} \frac{x_i}{x_i - x_j}\,(1 - \sigma_{ij})\,.$$

The operators Z_i with $i = 1, \ldots, N$ preserve the polynomial degree, but they do not commute if $N > 1$. However, they satisfy the commutation relations

$$[\, Z_i \,, Z_j \,] = \sigma_{ij}\,(Z_i - Z_j)\,. \tag{2.2}$$

Similarly to (2.1), for any permutation σ of the N variables we have

$$\sigma^{-1} Z_i \, \sigma = Z_{\sigma(i)} \, . \tag{2.3}$$

Therefore for every $m = 1, 2, \ldots$ the operator sum

$$H_m = Z_1^m + \ldots + Z_N^m \tag{2.4}$$

commutes with σ . Hence it preserves the space of symmetric polynomials in $x_1, \ldots, x_N$. The joint eigenvectors of operators (2.4) restricted to the latter space are the *Jack polynomials* [10] corresponding to the parameter α .

Let us now regard V as the tensor product $\mathbb{C}^n \otimes \mathbb{C}[v]$ of rings. Then we can identify the ring $V^{\otimes N}$ with the tensor product of $(\mathbb{C}^n)^{\otimes N}$ by the ring of polynomials in N variables with complex coefficients. The Heckman operators act on the latter ring, and we can now extend them to $V^{\otimes N}$ so that they act on $(\mathbb{C}^n)^{\otimes N}$ trivially. More explicitly, then x_i and ∂_i in Z_i become the operators

$$e_c \, v^k \mapsto e_c \, v^{k+1} \quad \text{and} \quad e_c \, v^k \mapsto k \, e_c \, v^{k-1} \tag{2.5}$$

respectively in the ith tensor factor of $V^{\otimes N}$. Note that then σ_{ij} in Z_i acts only on the variables v in the ith and jth tensor factors of $V^{\otimes N}$. This action differs from the permutational action of the transposition $(ij) \in \mathfrak{S}_N$ on the tensor product $V^{\otimes N}$ unless $n = 1$.

However, when regarded as an operator on $V^{\otimes N}$, every sum (2.4) still commutes with the permutational action of the group $\mathfrak{S}_N$. So the action of this sum on $V^{\otimes N}$ preserves the subspace Λ_N . In this section we will describe the limit of the action of the sum (2.4) on Λ_N at $N \to \infty$. This limit will be an operator I_m on the vector space $\mathcal{F}$ making commutative the square diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r, "I_m"] \arrow[d] & \mathcal{F} \arrow[d] \\ \Lambda_N \arrow[r, "H_m"'] & \Lambda_N \end{tikzcd} \tag{2.6}$$

Note that the operator Z_1 on $V^{\otimes N}$ preserves the subspace $V \otimes \Lambda_{N-1}$. We will first describe the limit of the action of Z_1 on this subspace. That will be an operator Z on the vector space $V \otimes \mathcal{F}$ making commutative the diagram

$$\begin{tikzcd} V \otimes \mathcal{F} \arrow[r, "Z"] \arrow[d, "\pi_N"'] & V \otimes \mathcal{F} \arrow[d, "\pi_N"] \\ V \otimes \Lambda_{N-1} \arrow[r, "Z_1"'] & V \otimes \Lambda_{N-1} \end{tikzcd} \tag{2.7}$$

In the case $n = 1$ the operator Z was determined in [17]. We will extend this result to any n . Let D_1 and W_1 be the operators on $V^{\otimes N}$ corresponding to

$$x_1 \partial_1 \quad \text{and} \quad \sum_{j \neq 1} \frac{x_1}{x_1 - x_j} \, (1 - \sigma_{1j}) \tag{2.8}$$

respectively. The latter two operators act on the polynomials in the variables $x_1, \ldots, x_N$ with complex coefficients. Then $Z_1 = \alpha\, D_1 + W_1$ as an operator on $V^{\otimes N}$. Note that both D_1 and W_1 preserve the subspace $V \otimes \Lambda_{N-1}$.

Now introduce an operator on the vector space $V \otimes \mathcal{F}$

$$D = v\,\partial \otimes \mathrm{id} + \sum_{d=1}^{n} \sum_{l=1}^{\infty} e_d\, v^{\,l} \otimes p_{dl}^{\perp} \tag{2.9}$$

where v and ∂ are the operators (2.5) on V respectively, while $p_{dl}^{\perp}$ denotes the product of l by the derivation in the free commutative ring $\mathcal{F}$ relative to p_{dl}. We claim that commutative is the diagram obtained by replacing Z and Z_1 in (2.7) by D and D_1 respectively. To prove this claim, observe that the operator $v\,\partial$ is a derivation of the ring V. So it suffices to show that the compositions $\pi_N D$ and $D_1 \pi_N$ coincide on any generator of the ring $V \otimes \mathcal{F}$:

$$e_c\, v^k \otimes 1 \xmapsto[D]{} k\, e_c\, v^k \otimes 1 \xmapsto[\pi_N]{} k\, e_c\, v^k \otimes e^{\otimes(N-1)},$$

$$e_c\, v^k \otimes 1 \xmapsto[\pi_N]{} e_c\, v^k \otimes e^{\otimes(N-1)} \xmapsto[D_1]{} k\, e_c\, v^k \otimes e^{\otimes(N-1)};$$

$$e \otimes p_{ck} \xmapsto[D]{} k\, e_c\, v^k \otimes 1 \xmapsto[\pi_N]{} k\, e_c\, v^k \otimes e^{\otimes(N-1)},$$

$$e \otimes p_{ck} \xmapsto[\pi_N]{} \sum_{i=1}^{N} e^{\otimes(i-1)} \otimes e_c\, v^k \otimes e^{\otimes(N-i)} \xmapsto[D_1]{} k\, e_c\, v^k \otimes e^{\otimes(N-1)}.$$

Consider W_1. For $j \neq 1$ let U_j be the operator on $V^{\otimes N}$ corresponding to the summand in (2.8) with index j. Then $W_1 = U_2 + \ldots + U_N$. Observe that the restriction of the operator W_1 to the subspace $V \otimes \Lambda_{N-1}$ coincides with that of the composition $(\mathrm{id} \otimes \theta_{N-1})\, U_2$. This is because for $j = 3, \ldots, N$ the conjugation of U_2 by the action of $(2j) \in \mathfrak{S}_N$ on $V \otimes \Lambda_{N-1}$ yields the operator U_j, while the action of $(2j)$ on this subspace is trivial.

Now consider the ring $V \otimes V \otimes \mathcal{F}$. It contains $V \otimes \mathcal{F}$ as a subring via the embedding $\mathrm{id} \otimes \iota$. In particular, it contains $\mathcal{F}$ via the natural mapping $f \mapsto e \otimes e \otimes f$ for every $f \in \mathcal{F}$. Let us extend (1.1) to a homomorphism

$$\rho_N : V \otimes V \otimes \mathcal{F} \to V \otimes V \otimes \Lambda_{N-2}$$

similarly to π_N. Namely, our ρ_N maps

$$e_c\, v^k \otimes e_d\, v^l \otimes 1 \mapsto e_c\, v^k \otimes e_d\, v^l \otimes e^{\otimes(N-2)} \tag{2.10}$$

and also maps $e \otimes e \otimes p_{ck}$ to the sum (1.2). We get a commutative diagram

$$\begin{tikzcd} V \otimes \mathcal{F} \arrow[r, "\mathrm{id} \otimes \iota"] \arrow[d, "\pi_N"'] & V \otimes V \otimes \mathcal{F} \arrow[d, "\rho_N"] \\ V \otimes \Lambda_{N-1} \arrow[r] & V \otimes V \otimes \Lambda_{N-2} \end{tikzcd} \tag{2.11}$$

where the bottom horizontal arrow represents the natural embedding.

Further let

$$\omega : V \otimes V \otimes \mathcal{F} \to V \otimes \mathcal{F}$$

be a linear map defined by the assignment

$$e_c\, v^k \otimes e_d\, v^l \otimes f \;\mapsto\; (e_c\, v^k \otimes f)\,(e \otimes p_{dl} - e_d\, v^l \otimes 1)$$

for every $f \in \mathcal{F}$. The map ω is different from the more straightforward map

$$\mathrm{id} \otimes \theta : V \otimes V \otimes \mathcal{F} \to V \otimes \mathcal{F}\,.$$

Under the latter

$$e_c\, v^k \otimes e_d\, v^l \otimes f \;\mapsto\; e_c\, v^k \otimes p_{dl}\, f\,.$$

Later on we will also use the map $\mathrm{id} \otimes \theta$ due to the *equalizing property* below.

Lemma 2.1. *The action of* ω *and* $\mathrm{id} \otimes \theta$ *is the same on any element of the ring* $V \otimes V \otimes \mathcal{F}$ *divisible by* $e_c \otimes e \otimes 1 - e \otimes e_c \otimes 1$ *for some index* c *.*

Proof. Any element of $V \otimes V \otimes \mathcal{F}$ is a linear combination of tensor products $e_a\, v^r \otimes e_b\, v^s \otimes f$ where $a, b = 1, \ldots, n$ and $r, s = 0, 1, 2, \ldots$ and $f \in \mathcal{F}$. Take

$$(e_c \otimes e \otimes 1 - e \otimes e_c \otimes 1)\,(e_a\, v^r \otimes e_b\, v^s \otimes f) =$$
$$\delta_{ac}\, e_a\, v^r \otimes e_b\, v^s \otimes f - e_a\, v^r \otimes \delta_{bc}\, e_b\, v^s \otimes f\,.$$

By applying the difference of maps $\mathrm{id} \otimes \theta - \omega$ to the last displayed line we get

$$\delta_{ab}\, \delta_{ac}\, v^{r+s} \otimes f - \delta_{ab}\, \delta_{bc}\, e_a\, v^{r+s} \otimes f = 0\,. \qquad \square$$

However, it is the map ω that makes commutative the diagram

$$\begin{tikzcd} V \otimes V \otimes \mathcal{F} \arrow[r, "\omega"] \arrow[d, "\rho_N"'] & V \otimes \mathcal{F} \arrow[d, "\pi_N"] \\ V \otimes V \otimes \Lambda_{N-2} \arrow[r, "\mathrm{id} \otimes \theta_{N-1}"'] & V \otimes \Lambda_{N-1} \end{tikzcd} \tag{2.12}$$

To prove the commutativity of (2.12) observe that π_N and ρ_N map $\mathcal{F}$, as a subring of respectively $V \otimes \mathcal{F}$ and $V \otimes V \otimes \mathcal{F}$, to the ring Λ_N . But the map ω is $\mathcal{F}$-linear, while the map $\mathrm{id} \otimes \theta_{N-1}$ is Λ_N-linear. The maps at all four sides of the diagram (2.12) also commute with multiplication by the elements of V in the first tensor factor of their source and target vector spaces. So it suffices to chase the element $e \otimes e_c\, v^k \otimes 1 \in V \otimes V \otimes \mathcal{F}$ the two ways offered by the diagram (2.12). Both ways yield the same result, which is the sum (1.9).

We will employ the operator U on the vector space $V \otimes V \otimes \mathcal{F}$ making commutative the diagram

$$\begin{tikzcd} V \otimes V \otimes \mathcal{F} \arrow[r, "U"] \arrow[d, "\rho_N"'] & V \otimes V \otimes \mathcal{F} \arrow[d, "\rho_N"] \\ V \otimes V \otimes \Lambda_{N-2} \arrow[r, "U_2"'] & V \otimes V \otimes \Lambda_{N-2} \end{tikzcd} \tag{2.13}$$

Namely, we will set

$$W = \omega\, U\,(\,\mathrm{id} \otimes \iota\,)\,. \tag{2.14}$$

Then commutative will be the diagram, obtained by replacing Z and Z_1 in (2.7) by W and W_1 respectively. To prove this claim, it suffices to place the diagrams (2.11) and (2.12) respectively on the left and on the right of (2.13). It will then follow that $Z = \alpha\, D + W$ makes commutative the diagram (2.7).

Similarly to π'_N let us introduce another ring homomorphism

$$\rho'_N : V \otimes V \otimes \mathcal{F} \to V \otimes V \otimes \Lambda_{N-2}$$

such that ρ'_N will map the element $e \otimes e \otimes p_{ck} \in V \otimes V \otimes \mathcal{F}$ to the sum

$$\sum_{i=3}^{N} e^{\otimes\,(i-1)} \otimes e_c\, v^k \otimes e^{\otimes\,(N-i)}$$

instead of (1.2). The homomorphism ρ'_N will still map (2.10) as ρ_N does. So

$$\begin{aligned}
\rho_N\,(e \otimes e \otimes p_{ck}\,) &= \rho'_N\,(e \otimes e \otimes p_{ck} + e_c\, v^k \otimes e \otimes 1 + e \otimes e_c\, v^k \otimes 1\,)\,,\\
\rho'_N\,(e \otimes e \otimes p_{ck}\,) &= \rho_N\,(e \otimes e \otimes p_{ck} - e_c\, v^k \otimes e \otimes 1 - e \otimes e_c\, v^k \otimes 1\,)\,.
\end{aligned}$$

For short let x and y denote the operators of multiplication by v respectively in the first and the second tensor factors of $V \otimes V \otimes \mathcal{F}$. Let τ be operator on $V \otimes V \otimes \mathcal{F}$ exchanging the variables v in these two tensor factors. By the definition of ρ'_N we immediately obtain commutativity of the diagram

$$\begin{tikzcd}
V \otimes V \otimes \mathcal{F} \arrow[r, "\frac{x}{x-y}(1-\tau)"] \arrow[d, "\rho'_N"'] & V \otimes V \otimes \mathcal{F} \arrow[d, "\rho'_N"] \\
V \otimes V \otimes \Lambda_{N-2} \arrow[r, "U_2"'] & V \otimes V \otimes \Lambda_{N-2}
\end{tikzcd} \tag{2.15}$$

For the purpose of determining the operator W on $V \otimes \mathcal{F}$ via (2.14) it suffices to find the action of U on the image of $\mathrm{id} \otimes \iota$, that is on the subspace

$$V \otimes e \otimes \mathcal{F} \subset V \otimes V \otimes \mathcal{F}\,.$$

Furthermore, the maps ρ_N and U_2 commute with multiplication by elements of the subspace $\mathbb{C}^n \subset V$ in the first tensor factors of their source and target vector spaces. Hence the operator U will have the same commuting property. Therefore it suffices to find for $l = 0, 1, 2, \ldots$ the action of U on the elements

$$e\, v^l \otimes e \otimes \prod_{(c,k)\in\mathcal{P}} p_{ck} \;=\; x^l \prod_{(c,k)\in\mathcal{P}} (\,e \otimes e \otimes p_{ck}\,) \tag{2.16}$$

where $\mathcal{P}$ is any finite collection of pairs of $c = 1, \ldots, n$ and $k = 0, 1, 2, \ldots$. This collection is unordered, but may contain same pairs with multiplicity.

By the commutativity of the diagrams (2.13) and (2.15) we have

$$\rho_N\, U\Bigl(\,x^{\,l} \prod_{(c,k)\in\mathcal{P}} (\,e\otimes e\otimes p_{ck}\,)\Bigr) = U_2\, \rho_N\Bigl(\,x^{\,l} \prod_{(c,k)\in\mathcal{P}} (\,e\otimes e\otimes p_{ck}\,)\Bigr) =$$

$$U_2\, \rho_N^{\,\prime}\Bigl(\,x^{\,l} \prod_{(c,k)\in\mathcal{P}} (\,e\otimes e\otimes p_{ck} + e_c\, v^{\,k}\otimes e\otimes 1 + e\otimes e_c\, v^{\,k}\otimes 1)\Bigr) =$$

$$\rho_N^{\,\prime}\Bigl(\,\frac{x}{x-y}\,\Bigl(\,x^{\,l} \prod_{(c,k)\in\mathcal{P}} (\,e\otimes e\otimes p_{ck} + e_c\, v^{\,k}\otimes e\otimes 1 + e\otimes e_c\, v^{\,k}\otimes 1)$$

$$-\,y^{\,l} \prod_{(c,k)\in\mathcal{P}} (\,e\otimes e\otimes p_{ck} + e_c\otimes e\, v^{\,k}\otimes 1 + e\, v^{\,k}\otimes e_c\otimes 1)\Bigr)\Bigr) =$$

$$\rho_N\Bigl(\,\frac{x}{x-y}\,\Bigl(\,x^{\,l} \prod_{(c,k)\in\mathcal{P}} (\,e\otimes e\otimes p_{ck}\,) - y^{\,l} \prod_{(c,k)\in\mathcal{P}} (\,e\otimes e\otimes p_{ck}\, +$$

$$e_c\otimes e\, v^{\,k}\otimes 1 + e\, v^{\,k}\otimes e_c\otimes 1 - e_c\, v^{\,k}\otimes e\otimes 1 - e\otimes e_c\, v^{\,k}\otimes 1)\Bigr)\Bigr) =$$

$$\rho_N\Bigl(\,\frac{x}{x-y}\,\Bigl(\,x^{\,l} \prod_{(c,k)\in\mathcal{P}} (\,e\otimes e\otimes p_{ck}\,) - y^{\,l} \prod_{(c,k)\in\mathcal{P}} (\,e\otimes e\otimes p_{ck}\, +$$

$$(\,y^{\,k} - x^{\,k}\,)\,(\,e_c\otimes e\otimes 1 - e\otimes e_c\otimes 1\,)\,)\Bigr)\Bigr)\,.$$

This calculation shows that the operator U maps the element (2.16) to

$$\frac{x}{x-y}\,\Bigl(\,x^{\,l} \prod_{(c,k)\in\mathcal{P}} (\,e\otimes e\otimes p_{ck}\,) - y^{\,l} \prod_{(c,k)\in\mathcal{P}} (\,e\otimes e\otimes p_{ck}\, +$$

$$(\,y^{\,k} - x^{\,k}\,)\,(\,e_c\otimes e\otimes 1 - e\otimes e_c\otimes 1\,)\,)\Bigr)\,. \tag{2.17}$$

Let us apply ω to the latter element. By applying the difference $\mathrm{id}\otimes\theta-\omega$ to

$$\frac{x}{x-y}\,\Bigl(\,x^{\,l} \prod_{(c,k)\in\mathcal{P}} (\,e\otimes e\otimes p_{ck}\,) - y^{\,l} \prod_{(c,k)\in\mathcal{P}} (\,e\otimes e\otimes p_{ck}\,)\Bigr) =$$

$$\frac{x\,(\,x^{\,l}-y^{\,l}\,)}{x-y} \prod_{(c,k)\in\mathcal{P}} (\,e\otimes e\otimes p_{ck}\,)$$

we get the element

$$e\, v^{\,l} \otimes \prod_{(c,k)\in\mathcal{P}} p_{ck} \in V\otimes\mathcal{F} \tag{2.18}$$

multiplied by l. This multiplication by l amounts to applying to (2.18) the operator $v\,\partial\otimes\mathrm{id}$. The element (2.16) is just the image of (2.18) under $\mathrm{id}\otimes\iota$. Now by repeatedly using Lemma 2.1 we get the operator equality on $V\otimes\mathcal{F}$

$$\omega\, U\,(\,\mathrm{id}\otimes\iota) = (\,\mathrm{id}\otimes\theta\,)\, U\,(\,\mathrm{id}\otimes\iota) - v\,\partial\otimes\mathrm{id}\;. \tag{2.19}$$

Here we also used the fact that the map ω commutes with multiplication by elements of the subspace $\mathbb{C}^n\subset V$ in the first tensor factor of its source and target vector spaces, like the operator U does.

Now let $p^{*}_{dl} = \alpha\, p^{\perp}_{dl}$, that is the product of $\alpha\, l$ by the derivation in $\mathcal{F}$ relative to the generator p_{dl} . Then we can recall that our $Z = \alpha\, D + W$ and combine (2.9),(2.14) and (2.19) to get the following principal result.

Theorem 2.2. *The diagram* (2.7) *is made commutative by the operator*

$$Z = (\alpha - 1)\, v\, \partial \otimes \mathrm{id} + \sum_{d=1}^{n} \sum_{l=1}^{\infty} e_d\, v^{\,l} \otimes p^{*}_{dl} + (\,\mathrm{id} \otimes \theta\,)\, U\, (\,\mathrm{id} \otimes \iota\,)$$

where ι *and* θ *are defined by* (1.5) *and* (1.14). *The operator* U *on* $V \otimes V \otimes \mathcal{F}$ *commutes with multiplication by elements of the subspace* $\mathbb{C}^n \subset V$ *in the first tensor factor, and maps* (2.16) *to the element displayed in two lines* (2.17).

Corollary 2.3. *For* $m = 1, 2, \ldots$ *the diagram* (2.6) *is made commutative by*

$$I_m = \theta\, Z^{m}\, \iota\,.$$

Proof. For any $i = 2, \ldots, N$ the conjugation of the operator Z_1^m by the action of $(1i) \in \mathfrak{S}_N$ on $V^{\otimes N}$ yields the operator Z_i^m . Therefore the composition $\theta_N\, Z_1^m$ acts on the subspace $\Lambda_N \subset V^{\otimes N}$ as the operator sum (2.4). Now the required statement follows from the commutativity of the composite diagram

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\iota"] \arrow[d] & V \otimes \mathcal{F} \arrow[r, "Z^m"] \arrow[d, "\pi_N"'] & V \otimes \mathcal{F} \arrow[r, "\theta"] \arrow[d, "\pi_N"'] & \mathcal{F} \arrow[d] \\
\Lambda_N \arrow[r] & V \otimes \Lambda_{N-1} \arrow[r, "Z_1^m"'] & V \otimes \Lambda_{N-1} \arrow[r, "\theta_N"'] & \Lambda_N
\end{tikzcd}$$

Here we use the commutativity of the diagrams (1.15) and (2.7). ☐

3. Yangian action

Consider the *Yangian* $\mathrm{Y}(\mathfrak{gl}_n)$. This is a complex unital associative algebra with an infinite family of generators $T^{(1)}_{ab}, T^{(2)}_{ab}, \ldots$ where $a, b = 1, \ldots, n$. Now let u be another variable. Introduce the formal power series in u^{-1}

$$T_{ab}(u) \;=\; \delta_{ab} + T^{(1)}_{ab} u^{-1} + T^{(2)}_{ab} u^{-2} + \ \ldots \tag{3.1}$$

with the coefficients in $\mathrm{Y}(\mathfrak{gl}_n)$. Using both the variables u and v , the defining relations in the algebra $\mathrm{Y}(\mathfrak{gl}_n)$ can be written as

$$(u - v)\, [\, T_{ab}(u), T_{cd}(v)\,] \;=\; T_{cb}(u) T_{ad}(v) - T_{cb}(v) T_{ad}(u)\,. \tag{3.2}$$

If $n = 1$ then the algebra $\mathrm{Y}(\mathfrak{gl}_n)$ is commutative by this definition. The next proposition is a particular case of a general construction due to Drinfeld [6].

Proposition 3.1. *The algebra* $\mathrm{Y}(\mathfrak{gl}_n)$ *acts on vector space* Λ_N *so that* $T^{(m+1)}_{ab}$ *with* $m = 0, 1, 2, \ldots$ *acts as the operator sum*

$$\sum_{i=1}^{N} \mathrm{id}^{\otimes\,(i-1)} \otimes E_{ab} \otimes \mathrm{id}^{\otimes\,(N-i)} \cdot (\,-Z_i\,)^{m}\,. \tag{3.3}$$

Note that the operator Z_i on $V^{\otimes N}$ by its definition commutes with the action of the Lie algebra $\mathfrak{gl}_n$ on any of the N tensor factors V. Further, due to the relations (2.3) the operator (3.3) commutes with the permutational action of the group $\mathfrak{S}_N$ on $V^{\otimes N}$. So the operator (3.3) preserves the subspace Λ_N. To prove Proposition 3.1 it now remains to verify that the restrictions of these operators to Λ_N satisfy the relations (3.2). To this end one employs the series

$$\delta_{ab}+\sum_{m=0}^{\infty}\sum_{i=1}^{N}\mathrm{id}^{\otimes(i-1)}\otimes E_{ab}\otimes\mathrm{id}^{\otimes(N-i)}\cdot(-Z_i)^m\,u^{-m-1}\;=$$

$$\delta_{ab}+\sum_{i=1}^{N}\mathrm{id}^{\otimes(i-1)}\otimes E_{ab}\otimes\mathrm{id}^{\otimes(N-i)}\cdot(u+Z_i)^{-1}$$

with operator coefficients (3.3) and applies the commutation relations (2.2). For the details of the verification of (3.2) see [12, Section 1].

Note that for any fixed $m=0,1,2,\ldots$ the operator (2.4) on $V^{\otimes N}$ equals $(-1)^m$ times the sum of operators (3.3) over $a=b=1,\ldots,n$. By using the results of the previous sections, we can now describe the limit of the action of $\mathrm{Y}(\mathfrak{gl}_n)$ on Λ_N defined in Proposition 3.1 at $N\to\infty$. This limit is an action of the algebra $\mathrm{Y}(\mathfrak{gl}_n)$ on the Fock space $\mathcal{F}$ determined by the next theorem.

Theorem 3.2. *The algebra* $\mathrm{Y}(\mathfrak{gl}_n)$ *acts on the vector space* $\mathcal{F}$ *so that* $T_{ab}^{(m+1)}$ *with* $m=0,1,2,\ldots$ *acts as the composition* $\theta\,(-Z)^m\,F_{ab}\,\iota$.

Proof. The composition $\theta_N\,(E_{ab}\otimes\mathrm{id})\,(-Z_1)^m$ acts on the subspace (1.7) as the operator sum (3.3). Hence the theorem follows from Proposition 3.1 by using the commutativity of the diagrams (1.8),(1.15) and (2.7). ☐

Other limits at $N\to\infty$ of the operators $E_{ab}\otimes\mathrm{id}$ and Z_1 on $V\otimes\Lambda_{N-1}$ were computed in [11]. Comparing our Theorems 1.5 and 2.2 with the results of [11] shows that these limits were defined by the homomorphism π_N' instead of π_N used in (2.7). This however entails changing our ι to the homomorphism

$$\iota'\,:\,\mathcal{F}\to V\otimes\mathcal{F}\,:\,p_{ck}\mapsto e\otimes p_{ck}+e_c\,v^k\otimes 1\,.$$

Further, once π_N is changed to π_N' in (1.15), our linear map θ also needs to be changed, to keep the latter diagram commutative. The changed linear map

$$\theta'\,:\,(e_d\,v^l\otimes 1)\prod_{(c,k)\in\mathcal{P}}(e\otimes p_{ck}+e_c\,v^k\otimes 1)\;\mapsto\;p_{dl}\prod_{(c,k)\in\mathcal{P}}p_{ck}$$

for any pair (d,l) and for any collection $\mathcal{P}$ of pairs (c,k) as in (2.16) above.

Indeed, after receiving a preliminary version of the present article which included the above remark, Sergey Khoroshkin verified that the counterparts from [11] of our operators F_{ab} and Z on $V\otimes\mathcal{F}$ can be rewritten as

$$F_{ab}'=\varepsilon\,F_{ab}\,\varepsilon^{-1}\quad\text{and}\quad Z'=\varepsilon\,Z\,\varepsilon^{-1}$$

where ε is the ring automorphism of $V\otimes\mathcal{F}$ identical on $V\otimes 1$ such that

$$\varepsilon\,:\,e\otimes p_{ck}\mapsto e\otimes p_{ck}+e_c\,v^k\otimes 1\,.$$

Since $\iota' = \varepsilon\,\iota$ and $\theta' = \theta\,\varepsilon^{-1}$ by the definition of the automorphism ε, then for any $m = 0, 1, 2, \ldots$ we get the equalities of operators on $\mathcal{F}$

$$\theta'(Z')^m\,\iota' = \theta\,Z^m\,\iota$$

and

$$\theta'(Z')^m F'_{ab}\,\iota' = \theta\,Z^m F_{ab}\,\iota\,.$$

By Corollary 2.3 and by Theorem 3.2, these equalities show that the limits at $N \to \infty$ of the operators H_m on Λ_N, and of the action of the algebra $\mathrm{Y}(\mathfrak{gl}_n)$ on Λ_N, are the same in [11] as in the present article. This should be the case, because the mapping (1.1) which defined the limits in [11] is the same as ours.

References

[1] H. Awata, Y. Matsuo and T. Yamamoto, *Collective field description of spin Calogero-Sutherland models*, J. Phys. **A29** (1996), 3089–3098.

[2] D. Bernard, M. Gaudin, F. D. M. Haldane and V. Pasquier, *Yang-Baxter equation in long-range interacting systems*, J. Phys. **A26** (1993), 5219–5236.

[3] F. Calogero, *Ground state of a one-dimensional N-body system*, J. Math. Phys. **10** (1969), 2197–2200.

[4] I. Cherednik, *A unification of Knizhnik-Zamolodchikov and Dunkl operators via affine Hecke algebras*, Invent. Math. **106** (1991), 411–431.

[5] A. Debiard, *Polynômes de Tchébychev et de Jacobi dans un espace euclidien de dimension* p, C. R. Acad. Sc. Paris **I296** (1983), 529–532.

[6] V. G. Drinfeld, *Degenerate affine Hecke algebras and Yangians*, Funct. Analysis Appl. **20** (1986), 56–58.

[7] C. F. Dunkl, *Differential-difference operators associated to reflection groups*, Trans. Amer. Math. Soc. **311** (1989), 167–183.

[8] Z. N. C. Ha and F. D. M. Haldane, *Models with inverse-square exchange*, Phys. Rev. **B46** (1992), 9359–9368.

[9] G. J. Heckman, *An elementary approach to the hypergeometric shift operators of Opdam*, Invent. Math. **103** (1991), 341–350.

[10] H. Jack, *A class of symmetric polynomials with a parameter*, Proc. Royal Soc. Edinburgh **A69** (1970), 1–18.

[11] S. M. Khoroshkin, M. G. Matushko and E. K. Sklyanin, *On spin Calogero-Moser system at infinity*, J. Phys. **A50** (2017), 115203.

[12] S. Khoroshkin and M. Nazarov, *Yangians and Mickelsson algebras I*, Transformation Groups **11** (2006), 625–658.

[13] S. Khoroshkin and M. Nazarov, *On the functor of Arakawa, Suzuki and Tsuchiya*, Adv. Stud. Pure Math. **76** (2018), 275–302.

[14] M. L. Nazarov and E. K. Sklyanin, *Sekiguchi-Debiard operators at infinity*, Commun. Math. Phys. **324** (2013), 831–849.

[15] M. L. Nazarov and E. K. Sklyanin, *Integrable hierarchy of the quantum Benjamin-Ono equation*, Symmetry, Integrability and Geometry: Methods and Applications **9** (2013), 078.

[16] J. Sekiguchi, *Zonal spherical functions on some symmetric spaces*, Publ. Res. Inst. Math. Sci. **12** (1977), 455–459.

[17] A. N. Sergeev and A. P. Veselov, *Dunkl operators at infinity and Calogero-Moser systems*, Int. Math. Res. Notices (2015), 10959–10986.

[18] R. Stanley, *Some combinatorial properties of Jack symmetric functions*, Adv. Math. **77** (1989), 76–115.

[19] B. Sutherland, *Exact results for a quantum many-body problem in one dimension II*, Phys. Rev. **A5** (1972), 1372–1376.

[20] K. Takemura and D. Uglov, *The orthogonal eigenbasis and norms of eigenvectors in the spin Calogero-Sutherland model*, J. Phys. **A30** (1997), 3685–3717.

Maxim Nazarov
Department of Mathematics
University of York
York YO10 5DD
United Kingdom