Introduction to Discrete Mathematics with an Application of Graph Theory

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(1) Setiap orang yang dengan tanpa hak melakukan pelanggaran hak ekonomi sebagaimana dimaksud dalam Pasal 9 ayat (1) huruf i untuk Penggunaan Secara Komersial dipidana dengan pidana penjara paling lama 1 (satu) tahun dan/atau pidana denda paling banyak Rp. 100.000.000,00 (seratus juta rupiah).

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introduction to
DISCRETE MATHEMATICS
with an Application of Graph Theory

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Motto

"Do it with Passion or not at all"

"Intelligence is not the measurement,
but intelligence support all"

Preface

This book is inteded for one for semester introductory course in discrete mathematics. It includes several basic topics of Discrete Mathematics, for examples, exercises, and figures. In addition, it contains some motivating examples aimed to make readers can easily imagine the abstract concepts in this book. It can be said that the realistic mathematics approach is sometimes found in this book.

The book is set sequentially in five chapters. Chapter 1 discusses topic of mathematics induction consisting of "weak mathematics induction" and "strong mathematics induction". Chapter 2 provides topic of combinatorics which widely covers many subtopics, i.e. counting principle, permutation, combination, permutation and combination of multiset, binomial coefficient, principle of inclusion-exclusion, derangement, and pigeonhole principle. Chapter 3 contains topic of generating function which specifically discuss some operations of generating function and the application of generating function in counting. Chapter 4 discusses topic of recurrence relation which mainly discuss how to find the solution of a recurrence relation using both characteritic equation and generating function. The last but not least, chapter 5 discusses several basic things about graph theory and an application of it.

he authors hopes this book is certainly useful for everyone, particularly for mathematics department students in International Class Program in State University of Makassar. However, critiques and advices are emphatically needed for the refinement of this book in future.

Makassar, August 2016

Author

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CHAPTER I

MATHEMATICS INDUCTION

1.1.Preliminary

In mathematics, the natural numbers, N , is the set of all non-negative integers:

$$N = \{1,2,3,\dots\}$$

Frequently, it is a need to prove some mathematical statements related to every member of N . For instance, consider the following problem:

Show that for every $n \geq 1$,

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} \qquad (1.1)$$

In a sense, the equation 1.1 represents a infinity of different statements; for every *n* we care to plug in, we get a different "theorem". Here are the first few:

$$1 = \frac{1(2)}{2} = 1$$

$$1 + 2 = \frac{2(3)}{2} = 3$$

$$1 + 2 + 3 = \frac{3(4)}{2} = 6$$

and so on. Any one of the particular formulas above is easy to prove—just add up the numbers on the left and calculate the product on the right and verify that they are the same. But how do you show that the statement is true for every $n \geq 1$? A very powerful method is known as mathematical induction, often called simply "induction".

A helpful way to think about induction is as follows: Imagine that each of the statements corresponding to a different value of n is a domino standing on end. Imagine also that when a domino's statement is proven, that domino is knocked down. We can prove the statement for every n if we can show that every domino can be knocked over. If we knock them over one at a time, we'll never finish, but imagine that we can somehow set up the dominoes in a line and close enough together that when domino number k falls over, it knocks over domino number $k + 1$ for every value of k. In other words, if domino number 1 falls, it knocks over domino 2. Similarly, 2 knocks over 3, 3 knocks over 4, and so on. If we knock down number 1, it's clear that all the dominoes will eventually fall.

Figure source: http://rapgenius.com

Figure 1.1 Knocking Down Dominoes

So a complete proof of the statement for every value of k can be made in two steps: first, show that if the statement is true for any given value, it will be true for the next, and second, show that it is true for k = 1, the first value.

What follows is a complete proof of statement 1:

Suppose that the statement happens to be true for a particular value of n, say $n = k$. Then we have: 2, 2 knocks over 3, and so on. If we knock down number 1, it's clear that all the dominoes will eventually fall.

Suppose that the statement happens to be true for a particular value of n, say $n = k$. Then we have:

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2} \qquad (1.2)$$

We would like to start from this, and somehow convince ourselves that the statment is also true for the next value: $n = k + 1$. Well, what does statement 1 look like when $n = k + 1$? Just plug in $k + 1$ and see:

$$1 + 2 + \cdots + k + (k+1) = \frac{(k+1)(k+2)}{2} \qquad (1.3)$$

Notice that the left hand side of equation 3 is the same as the left hand side of equation 2 except that thereis an extra $k + 1$ added to it. Further, if equation 2 is true, then we can add $k + 1$ to both sides of it and get:

$$\begin{aligned} 1 + 2 + \cdots + k + (k + 1) &= \frac{k(k + 1)}{2} + (k + 1) \\ &= \frac{k(k + 1) + 2(k + 1)}{2} \\ &= \frac{(k + 1)(k + 2)}{2} \qquad (1.4) \end{aligned}$$

Showing that if we apply a little bit of algebra to the right hand side of equation 4 it is clearly equal to $(k + 1)(k + 2)/2$– exactly what it should be to make equation 3 true. We have effectively shown here that if domino k falls, so does domino $k + 1$.

1.2. Formal Definition of Induction

Here is a more formal definition of induction, but if you look closely at it, you'll see that it's just a restatement of the dominoes definition:

Let $S(n)$ be any statement about a natural number n. If $S(1)$ is true and if we can show that if $S(k)$ is true then $S(k + 1)$ is also true, then $S(n)$ is true for every $n \epsilon N$.

The following figure gives our rule for proof by mathematical induction

Proposition. The statements $S(1), S(2), S(3), S(4)$,... are all true.
Proof. (Induction)

(1) Prove that the first statement $S(1)$ is true.
(2) Given any integer $k \geq 1$, prove that the statement $S(k) \Rightarrow S(k+1)$ is true.

It follows by mathematical induction that every $S(n)$ is true.

In this setup, the first step (1) is called **the basis step**. Because (1) is usually a very simple statement, the basis step is often quite easy to do. The second step (2) is called **the inductive step**. In the inductive step direct proof is most often used to prove.

Example 1.1

Prove that if $n \in N$, then $1 + 3 + 5 + 7 + \cdots + (2n - 1) = n^2$!

Proof :

(1) Observe that if $n = 1$, this statement is $1 = 1^2$, which is obviously true. (**basis step**)
(2) We must now prove $S(k) \Rightarrow S(k+1)$. That is, we must show that if $1 + 3 + 5 + 7 + \cdots + (2k - 1) = k^2$ then

$$1 + 3 + 5 + 7 + \cdots + (2k - 1) + (2(k + 1) - 1) = (k + 1)^2$$

We use direct proof. Suppose $1 + 3 + 5 + 7 + \cdots + (2k - 1) = k^2$. Then

$$1 + 3 + 5 + 7 + \cdots + (2k - 1) + (2(k + 1) - 1)$$
$$= k^2 + (2(k + 1) - 1)$$
$$= k^2 + 2k + 1$$
$$= (k + 1)^2$$

Thus, $1 + 3 + 5 + 7 + \cdots + (2k - 1) + (2(k + 1) - 1) = (k + 1)^2$. This proves $S(k) \Rightarrow S(k + 1)$. It follows by induction that $1 + 3 + 5 + 7 + \cdots + (2n - 1) = n^2$for every $n \epsilon N$

In induction proofs it is not necessarily to start with the first statement $S(1)$ indexed by the natural number 1. Depending on the problem, the first statement could be $S(0)$ or $S(m)$ for any other integer m. In the next example the statements are $S(0), S(1), S(2), S(3), \ldots$. The same rule is used except that the basis step verifies $S(0)$, not $S(1)$.

Example 1.2

If n is a non-negative integer, then $5|(n^5 - n)$

Proof:

We will prove the proposition using mathematical induction. Since it states for non-negative integer, we apply the basis step with $n = 0$.

(1) If $n = 0$, the statement is $5|(0^2 - 0)$, which is indeed true.
(2) Let $k \geq 0$. We need to prove that if $5|(k^5 - k)$, then $5|\left((k + 1)^5 - (k + 1)\right)$.

We use direct proof. Suppose $5|(k^5 - k)$. Thus $k^5 - k = 5a$ for some $a \epsilon Z$.

Observe that

$$(k+1)^5 - (k+1)$$
$$= k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1 - k - 1$$
$$= (k^5 - k) + 5k^4 + 10k^3 + 10k^2 + 5k$$
$$= 5a + 5k^4 + 10k^3 + 10k^2 + 5k$$
$$= 5(a + k^4 + 2k^3 + 2k^2 + k)$$

Since $(a + k^4 + 2k^3 + 2k^2 + k)$ is an integer, it means that $(k+1)^5 - (k+1)$ is an integer multiple of 5, so $5|\left((k+1)^5 - (k+1)\right)$. We have shown that if $5|(k^5 - k)$, then $5|\left((k+1)^5 - (k+1)\right)$. It follows by induction that $5|(n^5 - n)$ for all non-negative integers n.

1.3. Strong Mathematics Induction

Strong mathematics induction is a special case of mathematics induction. To distinguish between what we just discussed in the prevous part and strong mathematics induction, let denote the previously discussed mathematics induction as "weak mathematics induction". As for an analogy, imagine a ladder on which one is stepping on. When you climb up the ladder, you have to step on the lower step and need to go up based on it. After we climb up the several steps, we can go up further by assuming that the step you are stepping on exists.

1. Basis step: The first step in the ladder you are stepping on
2. Inductive step: The steps you are assuming to exist

- Weak Induction: The step that you are currently stepping on
- Strong Induction: The steps that you have stepped on before including the current one

Next, going up further based on the steps we assumed to exist.

The difference between weak mathematics induction and strong mathematics induction only appears in inductive step. In weak induction, we only assume that particular statement holds at k-th step, while in strong induction, we assume that the particular statment holds at all the steps from the base case to k-th step

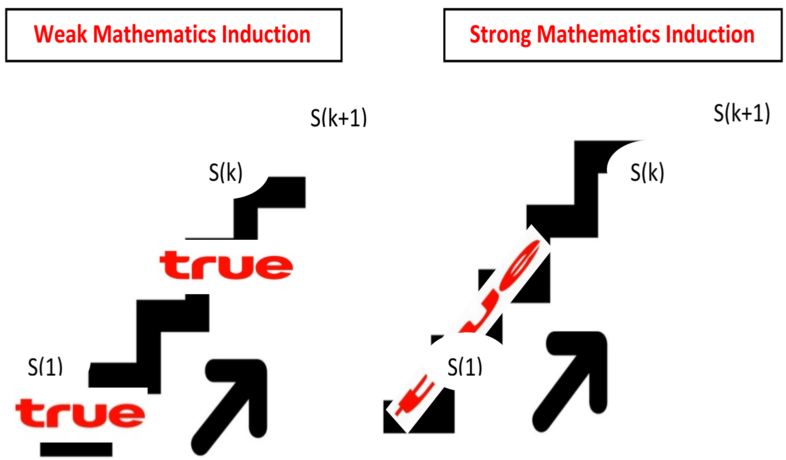


Figure 1.2. "Ladder" for Weak Mathematics Induction and Strong Mathematics Induction

Here are the formal rule for strong induction:

Proposition. The statements $S(1), S(2), S(3), S(4)$,... are all true.

Proof. (Induction)

(1) Prove that the first statement $S(1)$ is true.

(2) Assume that for all *k* in the range $1 \le k < n$, $S(k)$ is true, prove that the statement $S(k) \Rightarrow S(k+1)$ is true.

It follows by mathematical induction that every $S(n)$ is true.

Like in the weak mathematics induction, in strong mathematics induction, the basis step doesn't absolutely start with the first statement $S(1)$. Depending on the problem, the first statement could be $S(0)$ or $S(m)$ for any other integer m.

Example 1.3

Prove by induction that every integer greater than or equal to 2 can be factored into primes!

Proof (*by strong mathematical induction*):

1) Basis step:

The statement is true for n=2 because 2 itself is a prime number, so the prime factorization of 2 is 2. Trivially, the statement S(2) holds.

2) Inductive step:

Assume the statement $S(k)$ is true for all k with $2 \le k < n$ Consider the number $k+1$.

Case 1 : $k+1$ is a prime number.

When k+1 is a prime number, the number is a prime factorization of itself. Therefore, the statement $S(k+1)$ holds.

Case 2 : $k+1$ is not a prime number.

We know that $k+1$ is a composite, so $k+1 = pq\ (p, q \in Z^{+})$. Intuitively, we can conclude that both p and q are less than or equal to $k+1$. From the induction hypothesis stated above, for all integers less than or equal to k, the statement holds, which means both p and q can be expressed as prime factorizations. In this sense, because $k+1$ is a product of p and q, by multiplying the prime factorizations of p and q, we can get the prime factorization for $k+1$ as well.

Therefore, the statement that every integer greater than or equal to 2 can be factored into primes holds for all such integers.

Exercises 1

Prove the following statements by Mathematics Induction!

1. $(n+1)^2+(n+2)^2+(n+3)^2+\cdots+(2n)^2=\frac{n(2n+1)(7n+1)}{6}$ is true for all natural numbers n .
2. $1\ \ n+2(n\ \ 1)+3(n\ \ 2)+\ldots+(n\ \ 2)\ \ 2+n\ \ 1=\frac{1}{6}n(n+1)(n+2)$ is true for all natural numbers n .
3. $n(n+1)(n+2)(n+3)$ is divisible by 24, for all natural numbers n.
4. $n(n+1)(n+2)(n+3)\ldots(n+r-1)$ is divisible by $r\,!$, for all natural numbers n, where $r=1,2,\ldots$.
5. $7|n^7-n$ for any integer $n\geq 1$.
6. $8|3^{2n}-1$ for any integer $n\geq 0$.
7. $n!\leq n^n$ for any integer $n\geq 1$ $(n!=1.2.3\ldots.n)$
8. For any real number $x>-1$ and any positive integer x, $(1+x)^n\geq 1+nx$
9. Let the "Fibonacci sequence" can be defined by $S_1=S_2=S_3=1$ and $S_n=S_{n-1}+S_{n-2}$ for $n\geq 4$. Prove that $S_n<2^n$ for all $n\in Z^+$!

CHAPTER II

COMBINATORICS

2.1. Two Counting Principles

Some proofs concerning finite sets involve counting the number of elements of the sets, so we will look at the basics of counting.

2.1.1. Addition Principle

Let S be a set. A *partition* of S is a collection $S_1, S_2, \dots, S_m$ of subsets of S such that each element of S is in exactly one of those subsets:

$$S = S_1 \cup S_2 \cup \dots \cup S_m,$$
$$S_i \cap S_j = \emptyset, (i \neq j).$$

Theorem 2.1

Suppose that a set S is partitioned into pairwise disjoint parts $S_1, S_2, \dots, S_m$. The number of objects in S can be determined by finding the number of objects in each of the parts, and adding the numbers so obtained:

$$|S| = |S_1| + |S_2| + \cdots |S_m|.$$

Example 2.1

In the faculty of Mathematics and Natural Science of UNM, the Math Department is offering 26 classses, the Biology Department is offering 20 classes, the Physics Department is offering 18 classes, and the Chemistry Department is offering 23 classes. How many classes is the Faculty of Mathematics and Natural Science offering?

Solution :

The classes in the different departments partition the classes of the Mathematics and Natural Science Faculty (we assume no cross listing and that those are the only departments). Thus, we can use the addition principle: 26 + 20 + 18 + 23 = 87:

2.1.2. Multiplication Principle

Theorem 2.2.

Let S be a set of ordered pairs (a, b) of objects, where the first object a comes from a set of size p, and for each choice of object a there are q choices for object b. Then the size of S is $p \times q$

Example 2.2.

Suppose there are three major routes from Makassar to Maros, and four routes from Maros to Soppeng. How many routes are there from Makassar to Soppeng that go through Maros?

Solution :
There are three major from Makassar to Maros. Meanwhile, there are four routes from Maros to Soppeng. By the multiplication principle, the total route is 3 x 4 = 12 routes.

Example 2.3

How many multiples of 5 are there from 10 to 95?

Solution :
As we know, multiples of 5 are integers with two digits having 0 or 5 in the second digit. (i.e. the unit's place). The second digit from the left can be chosen in 2 ways. The first digit can be any one of 1,2,3,4,5,6,7,8,9. i.e. There are 9 choices for the first digit. Thus, there are 2×9 =18 multiples of 5 from 10 to 95.
There are several combinatoric problems, namely:

i) Counting or selecting ordered objects
   - With repetition
   - Without repetition
ii) Counting or selecting unordered objects
   - With repetition
   - Without repetition

To distinguish the objects with repetition and the objects without repetition, we need to differentiate the arrangement or the selection taken from set and multiset. In a multiset, an object can be repeated, meanwhile, in a set, an object can not be repeated. As an example, the multiset *M* = $\{a,a,a,b,c,c,d,d,d,d\}$ has 10 elements. The multiset M can be written as *M* =

$\{3.a, 1.b, 2.c, 4.d\}$. The arrangement of the type i) is called permutation and in the type ii) is called combination.

2.2. Permutation

Figure 2.1. The ways in arranging Comic of Naruto and Conan

Figure Sources : http://blogs.slj.com and http://turnerilmu21.blogspot.com

Suppose we want to arrange our comics on a cupboard. If we have only one comic, there is only one way of arranging it. Suppose we have two comics, Conan and Naruto.

We can arrange the Conan and Naruto comics in two ways. Conan comic first and the Naruto comic next (CN) or Naruto novel first and Conan novel next (NC). In other words, there are two arrangements of the two comics.

Now, suppose we want to add a One Piece (O) comic also to the cupboard. After arranging Conan and Naruto comics in one of the two ways, say CN, we can put One Piece comic in one of the following ways: OCN, CON or CNO. Similarly, corresponding to NC, we have three other ways of arranging the books. So, by the Counting Principle, we can arrange One Piece, Conan, and Naruto comics in 3 × 2 ways = 6 ways.

By permutation we mean an arrangement of objects in a particular order. In the above example, we were discussing the number of permutations of one book or two books. In general, if you want to find the number of permutations of n objects $n \geq 1$, how can you do it? Let us see if we can find an answer to this.

Similar to what we saw in the case of books, there is one permutation of one object, 2×1 permutations of two objects and $3\times2\times1$ permutations of three objects. It may be that, there are $n\times (n-1)\times (n-2)\times\ldots 3\times2\times1$ permutations of n objects. In fact, it is so, as you will see when we prove the following result.

Theorem 2. 3

The total number of permutations of n objects is $n\times(n-1)\times\ldots 2\times1$.

Proof :

The first place in an arrangement can be filled in n different ways. Once it has been done, the second place can be filled by any of the remaining $(n-1)$ objects and so this can be done in $(n-1)$ ways. Similarly, once the first two places have been filled, the third can be filled in $(n-2)$ ways and so on. The last place in the arrangement can be filled only in one way, because in this case we are left with only one object. Using the counting principle, the total number of arrangements of n different objects is $n\times(n-1)\times(n-2)\times\ldots\times 3\times 2\times 1.$

For the sake of the efficiency, the notation of dot product $(\cdot)$ will be sometimes used to replace the sign $\times$ for denoting the multiplication of two numbers. Therefore we will frequently use, for example, $3\cdot 2$ instead of $3\times 2.$ The product $n(n-1)(n-2)\ldots 3\cdot 2\cdot 1$ occurs so often in Mathematics that it deserves a name and notation. It is usually denoted by $n!$ (or by n read as n factorial).

$$n! = n(n-1)\ldots 3\cdot 2\cdot 1$$

Example 2.4.

Find the value of

a) 4! b) 5! + 3!

Solution :

a) 4! = 4 · 3 . 2 · 1 = 24

b) 5! = 5· 4· 3 · 2· 1 = 120

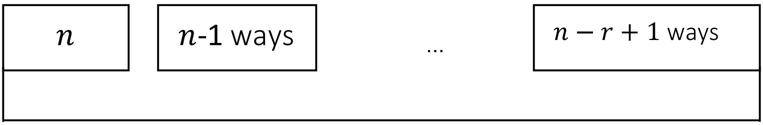
3! = 3· 2· 1 = 6

So, 5! + 3! = 120 + 6 = 126

2.3. Permutation of r Objects out of n Objects

Suppose you have five different cakes and you want to share one each to Hijrah, Irwan, and Ansari. In how many ways can you do it? You can give any one of the five cakes to Hijrah, and after that you can give any one of the remaining four cakes to Irwan. After that, you can give one of the remaining three cakes to Ansari. So, by the Counting Principle, you can distribute the cakes in 5 · 4 · 3 = 60 ways.

More generally, suppose you have to arrange r objects out of n objects. In how many ways can you do it? Let us view this in the following way. Suppose you have n objects and you have to arrange r of these in r boxes, one object in each box.



Suppose there is one box, r = 1. You can put any of the n objects in it and this can be done in n ways. Suppose there are two boxes, r = 2. You can put any of the objects in the first box and after that the second box can be filled with any of the remaining n – 1 objects. So, by the counting principle, the two boxes can be filled in n (n – 1) ways. Similarly, 3 boxes can be filled in n (n – 1) (n – 2) ways. In general, we have the following theorem.

Theorem 2.4

The number of permutations of r objects out of n objects is $n\ (n-1)...(n - r + 1)$.

The number of permutations of r objects out of n objects is usually denoted by $P(n,r)$. Thus,

$P(n,r) = n(n-1)(n-2)...(n-r+1)$ ... (2.1)

Proof :

Suppose we have to arrange *r* objects out of *n* different objects. In fact it is equivalent to filling *r* places, each with one of the objects out of the given *n* objects. The first place can be filled in *n* different ways. Once this has been done, the second place can be filled by any one of the remaining $(n-1)$ objects, in $(n-1)$ ways. Similarly, the third place can be filled in $(n-2)$ ways and so on. The last place, the *r* th place can be filled in $[n-(r-1)]$ i.e. $(n-r+1)$ different ways. You may easily see, as to why this is so. Using the Counting Principle, we get the required number of arrangements of *r* out of *n* objects is *n* $(n-1)(n-2)\ldots\ldots\ldots\ldots(n-r+1)$.

Example 2.5

Evaluate :

a) $P(4,2)$

b) $P(6,3)$

Solution :

a) $P(4,2) = 4(4-1) = 12$

b) $P(6,3) = 6(6-1)(6-2) = 6\cdot 5\cdot 4 = 120.$

Consider the formula for $P(n,r)$, namely $P(n,r) = n(n-1)(n-2)\ldots(n-r+1)$. This can be obtained by removing the terms $n-r, n-r-1, \ldots, 2, 1$ from the product for $n!$. The product of these terms is $(n-r)(n-r-1)\ldots 2\cdot 1$, i.e., $(n-r)!$.

Now

$$\frac{n!}{(n-r)!} = \frac{n(n-1)(n-2)\ldots(n-r+1)(n-r)\ldots 2.1}{(n-r)(n-r-1)\ldots 2.1}$$

$$= n(n-1)(n-2)\dots(n-r+1)$$
$$= P(n,r)$$

So using the factorial notation this formua can be written as follows:

$$P(n,r) = \frac{n!}{(n-r)!} \qquad \dots.(2.2)$$

Example 2.6

Find the value of $P(n,0)$

Solution :

Here $r = 0$. Using equation 2.2. we get

$$P(n,0) = \frac{n!}{n!} = 1$$

Permutation can also be applied to solve several following cases:

Example 2.7

There are 4 Civics books, 5 Chemistry books, and 3 Sport books. In how many ways can we arrange these so that books on Civics are together, Chemistry are together and Sport are together of which we are not asked to arrange the kinds of books in specific order?

Solution :

There are 4 books on Civics and they have to be put together. They can be arranged in 4! ways. Similarly, there are 5 Chemistry books then they can be arranged in 5! ways. And there are 3 Sport books then they can be arranged in 3! ways. So, by the counting principle all of them together can be arranged in $4!\times5!\times3!$ ways = 17280 ways.

Example 2.8

Suppose 5 students who are delegated by Mathematics Department, State University of Makassar to take participation in Mathematics Event are spending night a hotel and they are allotted 5 beds. Among them, Firman does not want a bed next to Yusran because Yusran snores. Then, in how many ways can you allot the beds?

Solution :

Let the beds be numbered 1 to 5.

Case 1 : Suppose Yusran is allotted bed number 1.

Then, Firman cannot be allotted bed number 2. So Firman can be allotted a bed in 3 ways. After alloting a bed to Firman, the remaining 3 students can be allotted beds in 3! ways. So, in this case the beds can be allotted in 3x3! *ways* = 36 ways.

Case 2 : Yusran is allotted bed number 5.

Then, Firman cannot be allotted bed number 4

As in Case 1, the beds can be allotted in 36 ways.

Case 3 : Yusran is allotted one of the beds numbered 2,3, or 4.

Firman cannot be allotted the beds on the right hand side and left hand side of Yusran's bed. For example, if Yusran is allotted bed number 2, beds numbered 1 or 3 cannot be allotted to Firman.

Therefore, Firman can be allotted a bed in 2 ways in all these cases. After allotting a bed to Firman, the other 3 can be allotted a bed in 3! ways. Therefore, in each of these cases, the beds can be allotted in $2\times3!$ = 12 ways. Since Yusran has possibilities to be alloted in three beds, then the total of the ways is $3\times12 = 36$

The beds can be allotted in
$(2\times36) + (3\times12)$ = 108 ways.

Example 2.9

In how many ways can 4 girls and 5 boys be arranged in a row so that all the four girls are together?

Solution :

Let 4 girls be one unit and now there are 6 units in all. They can be arranged in 6! ways. In each of these arrangements 4 girls can be arranged in 4! ways. Total number of arrangements in which girls are always together

= 6! · 4!

= 720·24

= 17280

2.4. Circular Permutation

Theorem 2.5

If n objects are arranged in a circle, then there are $\frac{n!}{(n-1)!}$ or $n!$ permutations of the n objects around the circle. The proof of the theorem 2.5 is given to readers as exercise

2.5. Combination

Suppose Larry has 4 set of shirts and trousers and he wants to take 2 sets to go on a trip to Selayar Island. In how many ways can he do it? Let us denote the sets by $S1$, $S2$, $S3$, $S4$. Then Lerry can choose two pairs in the following ways :

1. $\{S_{12}\}$ 2. $\{S_{13}\}$, 3.$\{S_{14}\}$
4. $\{S_{23}\}$, 5. $\{S_{24}\}$ 6. $\{S_{34}\}$

Observe that $\{S_{12}\}$ is the same as $\{S_{21}\}$. So, there are 6 ways of choosing the two sets that you want to take with you. Of course, if you had 10 pairs and you wanted to take 7 pairs, it will be much more difficult to work out the number of pairs in this way. However, this argument holds good in general as we can see from the following theorem.

Theorem 2.6

Let $n, n \geq 1$ be an integer and $r \leq n$. Let us denote the number of ways of choosing r objects out of n objects by $C(n,r)$. Then

$$C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{(n-r)!\,r!} \quad \ldots \qquad (2.3)$$

Proof :

We can choose *r* objects out of *n* objects in $C(n,r)$ ways. Each of the *r* objects chosen can be arranged in *r*! ways. Thus, by the counting principle, the number of ways of choosing *r* objects and arranging the *r* objects chosen can be done in $C(n,r)r!$ ways. But, this is precisely $P(n,r)$. In other words, we have

$$P(n,r) = r!\,C(n,r) \quad \ldots. \qquad (2.4)$$

Dividing both sides by $r!$, we get the result in the theorem.

Corollary 2.1

$$C(n,r) = C(n,n-r)$$

Example 2.10

Find the number of subsets of the set {1,2,3,4,5,6,7,8,9,10,11} having 4 elements.

Solution :

Here the order of choosing the elements doesn't matter and this is a problem in combinations. We have to find the number of ways of choosing 4 elements of this set which has 11 elements. By relation (2.3), this can be done in $C(11,4) = 330$ ways.

Theorem 2.7

The number of subsets in a set S containing *n* elements is

$$2^n = C(n,0) + C(n,1) + \cdots + C(n,n)$$

Proof :

The aim of this proof is to show that the two sides of the equation is counting the number of the subsets of a set with n elements. As a matter of fact, each subset of S is a subset with r elements, for $r = 0, 1, 2, \dots, n.$ Since $C(n, r)$ is equal to the number of subset with r elements in S which satisfies the addition rule namely $C(n, 0) + C(n, 1) + \cdots + C(n, n)$which is equal to the number of subsets in S.

Let H be a subset of S. Then the first element could be or could not be in H. It also holds for the second element, the third element, ..., and the n-th element as well. Therefore, by using multiplication rule, there are 2.2.2. ...2 = 2^n.

2.6. Permutation of Multiset

Let M be a multiset. An r-permutation of M is an ordered arrangement of r objects of M. If $|M|$ = n, then an *n*-permutation of M is called a permutation of M .

Theorem 2.8.

Let M be a multiset of k different types where each type has infinitely many elements. Then the number of r-permutations of M equals k^r

Example 2.11

Let $S = \{\infty.0, \infty.1, \infty.2\}$. The number of 4-permutation of multiset is $3^4 = 81$

Suppose we want to permute the letters of the word DARWAN. There would be 6! ways to permute DARWIN since all of the letters are different. How do we deal with the repeated A? Let's pretend they're different: DA_1RWA_2N. Now there are 6! ways, but we counted both of these, but in the original problem they should only be counted once:

DA_1RWA_2N $\qquad\qquad$ DA_2RWA_1N

In fact, we counted every permutation twice: with each possible ordering of the As. The real solution is $\frac{6!}{2!}$ = 360

Example 2.12:

How many ways are there to permute the letters of the word DARWAN?

Solution:

Let's first decide where to put the As, in $C(6,2)$ ways. Then in the remaining 4 positions, permute the remaining 4 elements. Final answer:

$$C(6,2).P(4,4) = \frac{6!}{2!\,4!}\frac{4!}{0!} = \frac{6!}{2!}$$

Theorem 2.9

There are $n!/k!$ ways to permute n objects where k are identical (but the other $n-k$ are different).

Proof idea: Exactly as the previous example, with n= 5 and k=2

Theorem 2.10

Suppose we have n items, where there are $n_1, n_2, \ldots, n_k$ that are identical. The number of ways to permute them is

$$\frac{n!}{n_1!\,n_2!\,\ldots n_k!}$$

Proof:

As before, first select positions for the n_1 identical items in $C(n, n1)$ ways. Then place the n_2 items in $C(n-n_1, n_2)$ ways, and so on. The total number of ways to arrange the items is

$C(n,n_1)C(n-n_1,n_2)C(n-n_1-n_2,n_3)\cdots C(n-n_1-\cdots-n_k-1,n_k)$

$$= \frac{n!}{n_1!(n-n_1)!}\ \frac{(n-n_1)!}{n_1!(n-n_1-n_2)!}\ \frac{(n-n_1-n_2)!}{n_1!(n-n_1-n_2-n_3)!}\cdots\frac{(n-n_1-\cdots-n_{k-1})}{n_k!0!}$$

$$= \frac{n!}{n_1!n_2!\ldots n_k!}$$

Example 2.13

How many ways to order the letters of MAKASSAR?

Solution :

There are 8 letters, but three As and two Ss, so there are

$$\frac{8!}{3!\,2!} = 3360$$

2.7. Combination of Multiset

A motivating Example:

How many ways can you select 15 kinds of cakes from a cake store containing large quantities of cakes, Jalangkote, Barongko, Baruasa', Taripang, and Panada?

You may need to model this problem using a chart:

	Jalangkote	Barongko	Baruasa'	Taripang	Panada	
A:	111	111	111	111	111	=15
B:	11		111111	111111	1	=15
C:		1111	1111111	1111		=15

Here, we set an order of the categories and just count how many from each category are chosen. Now, each event will contain fifteen 1's, but we need to indicate where we transition from one category to the next. If we use 0 to mark our transitions, then the events become:

A: 1110111011101110111
B: 1100111111011111101
C: 0011110111111101111

Thus, associated with each event is a binary string with number of 1's = number of things to be chosen and number of 0's = number of transitions between categories. From this example we see that the number of ways to select 15 cakes from a collection of 5 types of cake is C(15 + 4,15) = C(19,15) = C(19,4). Note that number of zeros = number of transitions = number of categories – 1.

Theorem 2.11

The number of ways to fill r slots from n categories with repetition allowed is: C(r + n – 1, r) = C(r + n – 1, n – 1). In words, the counts are: C(number of slots + number of transitions, number of slots) or C(number of slots + number of transitions, number of transitions).

Example 2.14

How many ways can we fill a box holding 100 pieces of candy from 30 different types of candy?

Solution:

Here number of slots = 100, number of transitions = 30 – 1, so there are $C(100+29,9)=\frac{129!}{100!29!}$ different ways to fill the box.

Example 2.15

How many non-negative integer *solution*s are there to the equation *a* + *b* + *c* + *d* = 100.

Solution :

In this case, we could have 100 *a*'s or 99 *a*'s and 1 *b*, or 98 *a*'s and 2 *d*'s, etc.We see that the number of slots = 100 and we are ranging over 4 categories, so number of transitions = 3. Therefore, there are C(100+3,100) = 103!/(100!3!) non-negative *solution*s to *a* + *b* + *c* + *d* = 100.

2.8. Binomial Coefficient

$C(n,k)$ or $\binom{n}{k}$ represents the combination of k from a set n . In this section, we will explore various properties of binomial coefficients.

Pascal's Triangle

Table 2.1 contains the values of the binomial coefficients $\binom{n}{k}$for n = 0 to 6 and all relevant k values. The table begins with a 1 for n = 0 and k = 0, because the empty set, the set with no elements, has exactly one 0-element subset, namely itself. We have not put any value into the table for a value of k larger than n, because we haven't defined what we mean by the binomial coefficient $\binom{n}{k}$ in that case. However, since there are no subsets of an n-element set that have size larger than n, it is natural to define $\binom{n}{k}$ to be zero when $k > n$, and so we define $\binom{n}{k}$ to be zero when $k > n$. Thus we could could fill in the empty places in the table with zeros. The table is easier to read if we don't fill in the empty spaces, so we just remember that they are zero.

Table 2.1: A table of binomial coefficients

n/k	0	1	2	3	4	5	6
0	1						
1	1	1					
2	1	2	1				
3	1	3	3	1			
4	1	4	6	4	1		
5	1	5	10	10	5	1	
6	1	6	15	20	15	6	1

Several properties of binomial coefficients are apparent in Table 2.1. Each row begins with a 1, because $\binom{n}{k}$ is always 1, as it must be because there is just one subset of an n-element set with 0 elements, namely the empty set. Similarly, each row ends with a 1, because an n-element set S has just one n-element subset, namely S itself. Each row increases at first, and then decreases.

Further the second half of each row is the reverse of the first half. The array of numbers called Pascal's Triangle emphasizes that symmetry by rearranging the rows of the table so that they line up at their centers. In Table 2.1, each entry is the sum of the one above it and the one above it and to the left. In algebraic terms, then, the Pascal Relationship says

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \qquad (2.5)$$

whenever $n > 0$ and $0 < k < n$. Notice that It is possible to give a purely algebraic (and rather dreary) proof of this formula by plugging in our earlier formula for binomial coefficients into all three terms and verifying that we get an equality. A guiding principle of discrete mathematics is that when we have a formula that relates the numbers of elements of several sets, we should find an explanation that involves a relationship among the sets.

A proof using sets

As we know that the expression $\binom{n}{k}$ is the number of k-element subsets of an n element set. Each of the three terms in Equation 2.5 therefore represents the number of subsets of a particular size chosen from an appropriately sized set. In particular, the three sets are the set of k-element subsets of an n-element set, the set of $(k - 1)$-element subsets of an $(n - 1)$-element set, and the set of k-element subsets of an $(n - 1)$-

element set. We should, therefore, be able to explain the relationship between these three quantities using the addition principle.

The number of k-element subsets of an n-element set is called a *binomial coefficient* because of the role that these numbers play in the algebraic expansion of a binomial $x+y$.

Theorem 2.12 (Binomial Theorem)

Let $x, y \in R$ For any integer n ≥ 0

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n$$

Combinatorial Proof :

Consider how to get a term of the form $x^{n-j}y^{k}$ out of the product of n terms each $(x+y)$: $(x+y)\ (x+y) \ldots (x+y)$. Such terms are formed by picking k y's and (n - k) x's. Since once the y's are picked, there is really no choice for the x's, there are $\binom{n}{k}$ such terms. So

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{0} x^k y^{n-k}$$

The proof of the Binomial Theorem can also be obtained by using mathematics induction. We leave it to readers as an exercise. The Binomial Theorem can be written in the several equivalent forms as follows:

a) $(x+y)^n = \sum_{k=0}^{n} \binom{n}{n-k} x^k y^{n-k}$

b) $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$

c) $(x+y)^n = \sum_{k=0}^{n} \binom{n}{n-k} x^{n-k} y^k$

Corollary 2.2

Let n be a positive integer, then for all $x \in R$,

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k = \sum_{k=0}^{n} \binom{n}{n-k} x^k$$

In addition, there are also several identities that can be stated based on the binomial coefficient for n and k element positive integer as follows:

1) $\binom{n}{k} = \binom{n-1}{k-1} \frac{n}{k}$

2) $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$

3) $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n \binom{n}{n} = 0$

4) $1\binom{n}{1} + 2\binom{n}{2} + \cdots + n\binom{n}{n} = n2^{n-1}$

5) $n(n+1)2^{n-2} = \sum_{k=0}^{n} k^2 \binom{n}{k}$

The proof of the identity 2 and 4 is given here. For the proof of other identities, we give them to readers as exercise.

Proof :

2) From the Binomial theorem (Theorem 2.11), put $x = y = 1$. Then the equation becomes

$$(1+1)^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n}$$

4) Let $x = 0\binom{n}{0} + 1\binom{n}{1} + 2\binom{n}{2} + \cdots + n\binom{n}{n} \quad \ldots(1)$

Then by corollary 2.1, x can be written as

$$x = 0\binom{n}{n} + 1\binom{n}{n-1} + 2\binom{n}{n-2} + \cdots + n\binom{n}{0}$$

or

$$x = n\binom{n}{0} + (n-1)\binom{n}{1} + (n-2)\binom{n}{2} + \cdots + 0\binom{n}{0} \quad \ldots(2)$$

by adding (1) by (2), we obtain

$$2x = n\binom{n}{n} + n\binom{n}{n-1} + n\binom{n}{n-2} + \cdots + n\binom{n}{0}$$

$$2x = n\left[\binom{n}{n} + \binom{n}{n-1} + \binom{n}{n-2} + \cdots + \binom{n}{0}\right]$$

By referring to the identity no. 2, then we have

$$2x = n2^n$$

Replacing x with $0\binom{n}{0} + 1\binom{n}{1} + 2\binom{n}{2} + \cdots + n\binom{n}{n}$, we obtain

$$2\left[0\binom{n}{0} + 1\binom{n}{1} + 2\binom{n}{2} + \cdots + n\binom{n}{n}\right] = n2^n$$

$$\Leftrightarrow \quad 2\left[1\binom{n}{1} + 2\binom{n}{2} + \cdots + n\binom{n}{n}\right] = n2^n$$

$$\Leftrightarrow \quad \binom{n}{1} + 2\binom{n}{2} + \cdots + n\binom{n}{n} = \frac{n2^n}{2}$$

$$\Leftrightarrow \quad 1\binom{n}{1} + 2\binom{n}{2} + \cdots + n\binom{n}{n} = n2^{n-1}$$

The extention of the notation $\binom{n}{k}$ for arbitrary real number n and positive integer$k$, is denoted by

$$\binom{n}{k} = \frac{r(r-1)\ldots(r-k+1)}{k(k-1)\ldots 1}$$

Where r is real number

Example 2.15

$$\binom{\frac{7}{2}}{5} = \frac{\frac{7}{2}\times\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\times\frac{-1}{2}}{5\times4\times3\times2\times1}$$

Moreover, Pascal relationship also holds in the following formula:

$$\binom{r}{k} = \binom{r-1}{k-1} + \binom{r-1}{k}$$

Furthermore, by expanding the binomial coefficient, we obtain

$$\binom{r}{0} + \binom{r+1}{1} + \binom{r+2}{2} + \cdots + \binom{r+k}{k} = \binom{r+k+1}{k}$$

For every real number r and non-negative integer k

Binomial theorem provides formula $(x+y)^n$, where n is positive integer. The formula can be expanded to get formula $(x+y+z)^n$. More general, it can be expanded for a sum of real numbers as many as t, i.e. $(x_1+x_2+\cdots+x_t)^n$.

In the general formula, the coefficient of binomial is in the form of

$$\frac{n!}{n_1!n_2!..n_t!}$$

where $n_1, n_2, \ldots, n_t$ are non-negative integers and $n_1+n_2+\cdots+n_t = n$. That form is called **multinomial number** and denoted by

$$\binom{n}{n_1\ n_2\ \ldots\ n_t}$$

Theorem 2.12 (Multinomial Theorem)

Let *n* be positive integers. For all $x_1, x_2, \ldots, x_t$ satisfies

$$(x_1+x_2+\cdots+x_t)^n = \sum \binom{n}{n_1\ n_2\ \ldots\ n_t} {x_1}^{n_1}{x_2}^{n_2}\ldots{x_t}^{n_t}$$

Where $n_1+n_2+\cdots+n_t = n$

Example 2.16

The coefficient of ${x_1}^3 x_2 {x_3}^2$ in $(2x_1 - 3x_2 + 5x_3)^6$ is

$$\binom{6}{3\ 1\ 2}(2^3)(-3)(5^2) = -36000$$

2.9. Inclusion-Exclusion Principle

A motivating example:

At AIHS (Australia International High School) there are

_ 28 students in algebra class,

_ 30 students in biology class, and

_ 8 students in both classes.

How many students are in either algebra or biology class?

Solution :

Let A denote the set of students in algebra class and B denote the set of students in biology class. To find the number of students in either class, we first add up the students in each class:

$$|A| + |B|$$

However, this counts the students in both classes twice. Thus we have to subtract them once:

$$-|A \cap B|$$

This shows

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$|A \cup B| = 28 + 30 - 8 = 50$$

so there are 50 students in at least one of the two classes. In the above example, we firstly do addition or "inclusion" and next do subtraction or "exclusion" The same reasoning works with three sets.

From an observation of the favorite teams of students in SMA Negeri 5 Makassar, it is obtained that there are

- 55 students who like either Real Madrid, Manchester City, or Schalke
- 28 students who like Real Madrid
- 30 students who like Manchester city
- 24 students who like Schalke
- 8 students who like both Real Madrid and Manchester City
- 16 students like both Real Madrid and Schalke
- 5 students like both Manchester City and Schalke
- How many students who like Real Madrid, Manchester City, and Schalke?

Solution:

Let's denote Real Madrid as R, Manchester City as M, and Schalke as S. Next, let A, B, and C denote the set of students who like R, M, and S respectively. Then $A \cup B \cup C$ is the set of students who like one of the three teams, $A \cap B$ is the set of students who like R and M, $A \cap C$ is the set of students who like R and S, and $B \cap C$ is the set of students who like M and S To count the number of students who like all three teams, i.e. count$|A \cup B \cup C|$, we can first add all the number of students who like R, who like M, and who like S:

$$|A| + |B| + |C|$$

However, now we've counted the students who like two teams too many times. So we subtract out the students who like each pair of teams:

$$|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C|$$

However, for students who like two teams, we've counted them twice, then subtracted them once, so they're counted once. But for students who like all three teams, we counted them 3 times, then subtracted them 3 times. So we counted them 0 time. Thus we need to add them again

$$|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Thus

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$$55 = 28 + 30 + 24 - 8 - 16 - 5 + |A \cap B \cap C|$$

Thus

$|A \cap B \cap C| = 2$

Therefore, the number of students who like Real Madrid, Manchester City, and Schalke is 2. The same reasoning works with an arbitrary number of sets; we state the general result in the following theorem.

Theorem 2.13 (Inclusion-Exclusion Principle)

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = \sum_{1 \le i \le n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| - \cdots + (-1)^n |A_1 \cap A_2 \cap \ldots \cap A_n|$$

Proof:

We will prove the formula by showing that an element in the union is counted exactly once by the right-hand side of the equation.

Suppose that a is a member of exactly r of the sets $A_1, A_2, \ldots, A_n$. Where $1 \le r \le n$. This element is counted C (r, 1) times by $\sum |A_i|$. It is counted C (r, 2) times by $\sum |A_i \cap A_j|$. In general, it is counted C (r, m) times by the summation involving m of the sets A_i .

Thus, this element is counted exactly C (r, 1) - C (r, 2) + C (r, 3) - ... + $(-1)^{r+1}C(r, r)$. times by the expression on the right-hand side of this equation. Our goal is to evaluate this quantity.

From binomial theorem, we have $C\ (r, 0) - C$ (r, 1) + C (r, 2) - C (r, 3) - ... + $(-1)^{r+1}C(r, r) = 0$

Hence, $1 = C\ (r, 0) = C(r, 1) - C$ (r, 2) + C (r, 3) - ... + $(-1)^{r+1}C(r, r)$. Therefore, each element in the union is counted exactly once by the expression on the right-hand side of the equation.

Example 2.17

Find the number of positive integers less than or equal to 1000 that are divisible by 7, 10, or 15!

Solution:

For a positive integer k, let A_k denote the set of integers in $r\{1,2,\ldots,1000\}$; that are divisible by k. We want to find $|A_7 \cup A_{10} \cup A_{15}|$. Note that

$$A_k = \left\lfloor \frac{1000}{k} \right\rfloor$$

Where $\lfloor y \rfloor$ denotes the greatest integer less than y. Indeed, the multiples of k. Note also that $A_k \cap A_l = A_{lcm(k,l)}$ since a number is divisible by both k and l if and only if it is divisible by lcm(k, l). Using the property of Inclusion-Exclusion, we get

$|A_7 \cup A_{10} \cup A_{15}|$

$= |A_7| + |A_{10}| + |A_{15}| - |A_7 \cap A_{10}| - |A_7 \cap A_{15}| - |A_{10} \cap A_{15}| + |A_7 \cap A_{10} \cap A_{15}|$

$= |A_7| + |A_{10}| + |A_{15}| - |A_{70}| - |A_{105}| - |A_{30}| + |A_{210}|$

$= \left\lfloor \frac{1000}{7} \right\rfloor + \left\lfloor \frac{1000}{10} \right\rfloor + \left\lfloor \frac{1000}{15} \right\rfloor - \left\lfloor \frac{1000}{70} \right\rfloor - \left\lfloor \frac{1000}{105} \right\rfloor - \left\lfloor \frac{1000}{30} \right\rfloor + \left\lfloor \frac{1000}{210} \right\rfloor$

= 142 + 100 + 83 – 14 – 9 – 33 + 4

= 273

2.10. Derangements

A *derangement* (or *complete permutation*) of a set is a permutation that leaves no element in its original position. Let three objects of *A* 1,2,3. When we permute A, we will obtain six kinds of permutations, namely :

1, 2,3	2,1, 3	3,1, 2
1,3, 2	2,3,1	3, 2,1

Of the six permutations of *A*, there are only two derangements, i.e. 2,3,1 and 3,1, 2. In 2,3,1 for example, there is no element in its original position since 2 takes the 1's position, 3 takes the 2's position, and 1 takes the 3's position.

A Motivating Example: Serving Meal Context

As another illustration for derangement, suppose Hisyam, Zaki, and Uni, who live in different apartments, respectively order Burger, Cronut, and Steak in a chain restaurant. However, the deliverer of the restaurant delivers Cronut to Hisyam, Steak to Zaki, and Burger to Uni. Another derangement that could happen is that the deliverer delivers Steak to Hisyam, Burger to Zaki, and Cronut to Uni.

Figure 2.2. Cronut Cake

In this section, we will discuss the number of derangements that is possible for n objects. However, before it, let we try to find the number of derangements in the case of delivering meal above.

Let us refer to a meal by a number and to a person by a number. Our task is to determine the number of ways to pair the meal and the persons so that no meal numbers match person numbers. When we have only 1 kind of meal ordered by 1 person, there is no way to derange the meal, for there is one meal to deliver to one person. When we have two kinds of meals ordered by two persons, the deliverer may deliver meal #2 to person #1 and meal #1 to person #2.

Now, let us denote burger as meal #1, cronut as meal #2, and steak as meal #3. Let us also denote Hisyam as person #1, Zaki as person #2, and Uni as person #3. When we have those three meals, there are 3! = 6 ways to distribute them. The deliverer now writes the meal numbers in the order they are delivered, such as 1 3 2, indicating burger is delivered to Hisyam, steak is delivered to Zaki, and cronut is delivered to Uni.

The 6 possible distributions for 3 meals are

1 2 3	2 1 3	3 1 2
1 3 2	2 3 1	3 2 1

Suppose there is an additional person, namely Agus who orders Pizza in the same restaurant. For the purpose of efficiency, let we denote Agus as person #4 and Pizza as meal #4.

We know there are 4! = 24 ways the deliverer could deliver the 4 meals. Rather than list the 24 cases, let us consider how the Inclusion-Exclusion Principle may help us. We seek the number of ways to place the numbers in the set {1,2,3,4} in line such that no value occurs in its natural position. Let $X(1)$ represent the property that 1 is delivered to the right person when 1,2,3,4 are permuted. Then $|X(1)| = 1.3!$. The 1 represents the 1 way to place the 1 in its natural position and the 3! shows the number of ways to permute the remaining 3 values. Note that we are not considering whether any of 2,3,4 wind up in their respective natural positions. We could argue similarly that $X(2) = X(3) = X(4)$. Therefore, there are 4.3! ways for a value to occur in its natural position.

About $X(1) \cap X(2)$ which means both 1 and 2 are delivered to the right persons, there is 1 way to place 1,2 in their natural order, and then 2! ways to place the remaining values. This will be true for any pair of values we wish to restrict to their natural positions. How many pairs are there? This is just $C(4,2)$ = 6. Therefore, there are $C(4,2)2!$ ways for two values to simultaneously occur in their natural positions. So, $|X(1) \cap X(2)|$ = $C(4,2)2!$

Then, for $|X(1) \cap X(2) \cap X(3)|$, there is 1 way to place 1,2,3 in their natural order, and then 1! way to place the remaining value. This will be true for any set of three values we wish to restrict to their natural positions. How many 3-element sets are there? This is just $C(4,3)$ = 4. Therefore, there are $C(4,3)1!$ ways for three values to simultaneously occur in their natural positions.

Finally, $|X(1) \wedge X(2) \wedge X(3) \wedge X(4)| = 1$, since there is only one way for all 4 values to be in their natural positions.

Now apply the Inclusion-Exclusion Principle :

$$|{\sim}X(1) \wedge {\sim}X(2) \wedge {\sim}X(3) \wedge {\sim}X(4)| = 4! - 4(3!) + 6(2!) - 4(1!) + 1$$
$$= 9$$

In words, using the Inclusion Exclusion Principle, we are suggesting that to determine the number of derangements of the values 1,2,3,4, first calculate the number of permutations of those values (4!), subtract the number of ways to keep at least one element in its natural position, add back the number of ways to keep at least two values in their natural positions, subtract the number of ways to keep at least three values in their natural positions, and finally add back the number of ways to keep all values in their natural positions.

If we denote $D(4)$ as a derangement of four objects, then

$$D(4) = |{\sim}X(1) \wedge {\sim}X(2) \wedge {\sim}X(3) \wedge {\sim}X(4)|$$

$$= 4! - 4(3!) + 6(2!) - 4(1!) + 1$$

$$= C(4,0)4! - C(4,1)3! + C(4,2)2! - C(4,3)1! + C(4,4)0!$$

$$= \frac{4!}{0!\,4!}.4! - \frac{4!}{1!\,3!}.3! + \frac{4!}{2!\,2!}.2! - \frac{4!}{3!\,1!}.1! + \frac{4!}{4!\,0!}.0!$$

$$= \frac{4!}{0!} - \frac{4!}{1!} + \frac{4!}{2!} - \frac{4!}{3!} + \frac{4!}{4!}$$

$$= 4!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right)$$

$$= 9$$

Therefore, the number of possible derangements, i.e. Hisyam, Zaki, Uni, and Agus don't receive their own ordered meal is 9.

Theorem 2.14

For

$$n \geq 1, D_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right); n \in N$$

Proof:

Let S_i be the set of permutations of n items which fix item i. Then the number of permutation in k of the Si would be the permutations that fix k items. There are $\binom{n}{k}$ ways to choose the k items to fix, and $(n-k)!$ ways to arrange the other $n-k$ items. Thus, the number of permutations that fix at least 1 item would be

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}(n-k)! = \sum_{k=1}^{n}(-1)^{k-1}\frac{n!}{k!}$$

Since there are $n!$ permutations in total, the number of permutations that don't fix any items is

$$\begin{aligned} D_n &= n! - \sum_{k=1}^{n}(-1)^{k-1}\frac{n!}{k!} \\ &= \sum_{k=0}^{n}(-1)^{k}\frac{n!}{k!} \\ &= n!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\cdots+(-1)^n\frac{1}{n!}\right) \end{aligned}$$

Note that, the series of e^{-1} is

$$e^{-1} = 1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\cdots$$

Therefore, we may write

$$e^{-1} = \frac{D_n}{n!}+(-1)^{n+1}\frac{1}{(n+1)!}+(-1)^{n+2}\frac{1}{(n+2)!}+\cdots$$

There are other properties of derangement as follows:

1) $D_n = (n-1)(D_{n-1} + D_{n-2}),\ \ n = 3,4,5, \dots$ (2.7)

2) $D_n = nD_{n-1} + (-1)^n, \quad n = 2,3,4, \dots$ (2.8)

Proof :

1) For any derangment $(j_1, j_2, \dots, j_n)$, we have $j_n \neq n$. Let $j_n = k$, where $k \in \{1, 2, \dots, n-1\}$. We now break the derangements on n element is two cases

Case 1: $j_k = n$ (so k and n map to each other). By removing elements *k* and *n* from the permutation we have a derangement on *n*–2 elements, and so, for fixed *k*, there are D_{n-2}derangements in this case.

Case 2: $j_k \neq n$. Swap the valued of j_k and j_n, so that we have a new permutation with *jk*=*k* and *jn*≠*n*. By removing element *k* we have a derangement on *n*–1 elements, and so, for fixed *k*, there are *Dn*–1 derangements in this case.

Thus, with *n*–1 choices for *k*, we have, for *n*≥3,

$$D_n = (n-1)(D_{n-1} + D_{n-2})$$

The proof of the property number 2 is left to readers as an exercise.

Example 2.18

Aulia, Budi, Catur, Dinda, and Eka are siblings. Each of the siblings has a toy which is different one to another. After playing with their own toy, a room in their house is messy because of the toys. Mr. Jumaris, as their father, wants to clean up the room by putting each toy to its box (each child has its own box for his toy). How many possible occurrence that Mr. Jumaris puts the toys of which there is no toy is put in its own box?

Solution:

The number of objects is 5. So the total derangements of five objects D_5 is :

$$D_5 = 5!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right) = 44$$

Or the value D_5 can be obtained by using one of the properties of derangement formula, i.e. the equation (2.6) : $D_n = (n-1)(D_{n-1} + D_{n-2}),\ \ n = 3,4,5,\ldots$

By substituting n with 5, we get

$$D_5 = (5-1)(D_4 + D_3)$$

Since $D_4 = 9$ and $D_3 = 2$, then

$$D_5 = (5-1)(9+2) = 4(11) = \ 44$$

Therefore, the number of total possibilities of which Mr. Jumaris put no toy in its own box is 44.

2.11 Pigeonhole Principle

Consider there are four pigeonholes and five pigeons. When the pigeons go to the pigeonholes, then there exist pigeonhole that contain at least two pigeons.

The pigeonhole principle is sometimes useful in answering the question: is there an item having a given property? When the pigeonhole principle is successfully applied, the principle tells us only that the object exists; the principle will not tell us how to find the object or how many there are. We will discuss the first version of the pigeonhole principle.

Pigeonhole Principle (First Form) :

If n pigeons fly into k pigeonholes and $k<n$, some pigeonhole contains at least two pigeons.

We note that the Pigeonhole Principle tells us nothing about how to locate the pigeonhole that contains two or more pigeons. It only asserts the existence of a pigeonhole containing two or more pigeons.

To apply the pigeonhole principle, we must decide which objects will play the roles of the pigeons and which objects will play the roles of the pigeonholes. Our beginning examples illustrate the application.

Example 2.19

Ten persons have first names Andi, Budi, and Charlie and last names Didi, Eman, and Fatur. Show that at least two persons have the same first and the last names.

Solution:

There are nine possible names, derived from $3^2 = 9$, for example, Andi Didi, Charlie Eman, Budi Fatur, etc, for the 10 persons. If we think of the person as pigeons and the names as pigeonholes, we can consider the assignment of names to people to be that of assigning pigeonholes to the pigeons. By the pigeonhole principle, some name (pigeonhole) is assigned to at least two persons (pigeons).

The simplest form of the pigeonhole principle is the following fairly obvious assertion.

Example 2.20

If Messi has ten black socks and ten white socks, and he is picking socks randomly, how many socks, at least, he needs to take to find a matching pair?

Solution:

He will only need to pick three to find a matching pair. The three socks (pigeons) can be one of two colors (pigeonhole). By the pigeonhole principle, at least two must be of the same color.

Another way of seeing this is by thinking sock by sock. If the second sock matches the first, then we are done. Otherwise, pick the third sock. Now the first two socks already cover both color cases. The third sock must be one of those and form a matching pair.

Theorem 2.15

If $n + 1$ objects are distributed into n boxes, then at least one box contains two or more of the objects.

Proof :

The proof is by contradiction. If each of the *n* boxes contains at most one

of the objects, then the total number of objects is at most $1 + 1 + \cdots + 1 (n\ 1s) = n$. Since we distribute $n + 1$ objects, some box contains at least two of the objects. Notice that neither the pigeonhole principle nor its proof gives any help in finding a box that contains two or more of the objects. They simply assert that if we examine each of the boxes, we will come upon a box that contains more than one object. The pigeonhole principle merely guarantees the existence of such a box. Thus, whenever the pigeonhole principle is applied to prove the existence of an arrangement or some phenomenon, it will give no indication of how to construct the arrangement or find an instance of the phenomenon other than to examine all possibilities.

Notice also that the conclusion of the pigeonhole principle cannot be *guaranteed* if there are only n (or fewer) objects. This is because we may put a different object in each of the n boxes. Of course, it is possible to distribute as few as two objects among the boxes in such'a way that a box contains two objects, but there is no guarantee that a box will contain two or more objects unless we distribute at least $n+1$ objects. The pigeonhole principle asserts that, no matter how we distribute $n+1$ objects among n boxes, we cannot avoid putting two objects in the same box. Instead of putting objects into boxes, we may think of coloring each object with one of n colors. The pigeonhole principle asserts that if $n+1$ objects are colored with n colors, then two objects have the same color.

Example 2.21

Among 13 people there are at least two who have their birthdays in the same month.

Example 2.22

There are n married couples. How many of the $2n$ people must be selected to guarantee that a married couple has been selected?

Solution:

To apply the pigeonhole principle in this case, think of n boxes, one corresponding to each of the n couples. If we select $n + 1$ people and put each of them in the box corresponding to the couple to which they belong, then some box contains two people; that is, we have selected a married couple. Two of the ways to select *n* people without getting a married couple are to select all the husbands or all the wives. Therefore, *n* + 1 is the smallest number that will guarantee a married couple has been selected.

Pigeonhole Principle (Second Form):

If f is a function from a finite set X to a finite set Y and $|X| > |Y|$, then $f(x_1) = f(x_2)$ for some $x_1, x_2 \in X, x_1 \neq x_2$.

The second form of the pigeonhole can be reduced to the first form by letting X be the set of pigeons and Y be the set of pigeonholes. We assign pigeon x to pigeonhole $f(x)$. By the first form of the Pigeonhole Principle, at least two pigeons, $x_1, x_2 \in X$, are assigned to the same pigeonhole; that is $f(x_1) = f(x_2)$ for some $x_1, x_2 \in X$, $x_1 \neq x_2$.

Our next example illustrates the use of the second version of the Pigeonhole Principle.

Example 2.23

An inventory in International Class Program consists of a list of 89 items, each marked "available" or "unavailable". There are 45 available items. Show that there are at least two available items in the list exactly nine items apart. (For example, available items at positions 13 and 22 or positions 69 and 78 satisfy the condition).

Solution:

Let a_i denote the position of the *i*th available item. We must show that $a_i - a_j = 9$ for some *i* and *j*. Consider the numbers

$$a_{1,} \quad a_2, \quad \ldots, \quad a_{45} \qquad (2.9)$$

and

$$a_1 + 9, \quad a_2 + 9, \quad \ldots, \quad a_{45} + 9 \qquad (2.10)$$

The 90 numbers in (2.9) and (2.10) have possible values only from 1 to 89. By the second form of the Pigeonhole Principle, two of the numbers must coincide. We cannot have two of (2.9) or two of (2.10) identical; thus some number in (2.9) is equal to some number in (2.10). Therefore, $a_i - a_j = 9$ for some *i* and *j*, as desired.

Exercises 2

1. In the examination of Real Analysis conducted in Mathematics Department, there are 6 true - false questions. How many responses are possible ?

2. There are six oranges and eight apples. How many non-empty subsets that can be formed by the two kinds of those fruits?

3. Four couples are sitting in a row. Find the number of arrangements in which no person is sitting next to his or her partner?. What if they are sitting in a circle?

4. In how many ways a march leader arrange his team consisting of six men and five women so that, in one column march, no two men are together?

5. Find the number of permutations of the letters of the word 'PARANGTAMBUNG', in each of the following cases :
   (i) beginning with A and ending with R.
   (ii) vowels are always together.
   (iii) vowels are never together.

6. "Warung Bu Bety" provides 3 kinds of vegetables, 2 kinds of fish and 2 types of rices. If Mr. Rahman wants 1 vegetable, 1 fish and 1 rice, how many choices does he have ?

7. A group of 12 friends meet at a party. Each person shake hands once with all others. How many hand shakes will be there ?

8. Use binomial theorem to prove that

$$3^n = \sum_{k=0}^{n} \binom{n}{k} 2^k$$

9. In Serie-A league consisting of 38 matches in one season, how many different ways the team Inter Milan are there to have 20 wins, 12 draws, and 6 loses?

10. How many integer solutions are there to: $a + b + c + d = 15$, when $a \geq 3$, $b \geq 0$, $c \geq 2$ and $d \geq 1$?

11. How many integer solutions are there to: $a + b + c + d = 15$, when $a \geq$ --3, $b \geq 0$, $c \geq -2$ and $d \geq -1$?

12. Find the coefficient of ${x_1}^2{x_2}^3x_3{x_4}^2$ in $(x_1 - x_2 + 2x_3 - 2x_4)^8$!

13. At SMP Negeri 24 Makassar, there are

- 44 students in either mathematics, biology, or physics class
- 25 students in biology class
- 23 students in physics class
- 13 students in both mathematics and biology
- 9 students in both biology and physics
- 10 students in both algebra and physics

- 6 students in all three classes.

  How many students are in mathematics class?

14. Use the principle of inclusion-exclusion to find the number of primes not exceeding 100!

15. A new employee checks the hats of n people at a restaurant, forgetting to put claim check numbers on the hats. When customers return for their hats, the checker gives them back hats chosen at random from the remaining hats. What is the probability that no one receives the correct hat?

16. Prove that D_n is even if and only if n is odd!

17. Suppose that each person in a class A consisting 32 students receives scholarship in January. Prove that at least two students receive scholarship on the same day!

18. Eighteen persons have first names Ayu, Bani, Cacha and last names Dian and Eput. Show that at least three persons have the same first and last names!

19. In a birthday party, with two or more people, show that there must be at least two people who have the same number of friends.

CHAPTER III

GENERATING FUNCTIONS

Generating function is a very "beautiful" way to work with a sequence of numbers. In simple words, it transforms problems about sequences into problems about functions.

We'll begin this chapter by introducing the notion of ordinary generating function and next we'll describe several operations involving generating function.

3.1. Ordinary Generating Functions

Let $(a_0, a_1, a_2, a_3, \ldots)$ be infinite sequence of real numbers. Generating function of the sequence is the power series

$$G(x) = \sum_{n=0}^{\infty} a_n x^n$$

That generating function can also be written as

$$G(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots.$$

If a sequence is finite, we can still construct a generating function by taking all the terms after the last to be zero. For a more convenient way, we'll frequently indicate the correspondence between a sequence and its generating function with a double-sided arrow as follows:

$(a_0, a_1, a_2, a_3 \dots) \leftrightarrow a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$.

For example, here are several sequences and their generating functions of the sequence:

(0,0,0,0,...) $\leftrightarrow$ $0 + 0x + 0x^2 + 0x^3 + \cdots = 0$

(3,2,1,0,0,...) $\leftrightarrow 3 + 2x + 1x^2 + 0x^3 + 0x^4 = 3 + 2x + 1x^2$

(1,3,3,1,0,0,0,0,...) $\leftrightarrow 1 + 3x + 3x^2 + x^3$

In the last form, we enable to change it into the form $(1 + x)^3$

Here, we may see that a generating function is a "formal" power series in the sense that we usually regard x as a placeholder rather than a number. Only in rare cases will we actually evaluate a generating function by letting x take a real number value, so we generally ignore the issue of convergence.

Recall the sum of an infinite geometric series is:

$$1 + c + c^2 + c^3 + \cdots = \frac{1}{1 - c}$$

This equation does not hold when $|z| \geq 1$, but as remarked, we don't mind with about convergence issues. This formula gives closed-form generating functions for a whole range of sequences. For example:

$(1,1,1,1,...) \quad \leftrightarrow \quad 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$

$(1,0,1,0,...) \quad \leftrightarrow \quad 1 + x^2 + x^4 + x^6 + \cdots = \frac{1}{1-x^2}$

$(1,-1,1,-1,...) \leftrightarrow 1 - x + x^2 - x^3 + x^4 + \cdots = \frac{1}{1+x}$

3.2. Some Operations on Ordinary Generating Functions

a. Scaling

A generating function can be multiplied by a constant to scale every term in the associated sequence by the same constant. For example, Multiplying a generating function by a constant scales every term in the associated sequence by the same constant.

Example 3.1

$$(1, 0, 1, 0, 1, 0,...) \quad \leftrightarrow \quad 1 + x^2 + x^4 + x^6 \ldots = \frac{1}{1-x^2}$$

If we multiply the generating function by 2, we obtain

$$\frac{2}{1-x^2} = 2 + 2x^2 + 2x^4 + 2x^6 + \cdots$$

Which generates the sequence :

(2, 0, 2, 0, 2, 0,...)

Theorem 3.1

If

$$(f_0, f_1, f_2, \ldots) \leftrightarrow F(x)$$

Then

$$(cf_0, cf_1, cf_2, \ldots) \leftrightarrow c.F(x)$$

Proof :

$$(cf_0, cf_1, cf_2, \ldots) \leftrightarrow cf_0 + cf_1x + cf_2x^2 + \cdots$$

$$= c(f_0 + f_1x + f_2x^2 + \cdots)$$

$$= c\,F(x)$$

b. Addition

We may also do addition on generating functions by adding the two sequences term by term.

Example 3.2

(1,1,1,1,...) + (1,−1,1,−1,...) $\leftrightarrow$

$$\frac{1}{1-x} + \frac{1}{1+x} = \frac{(1+x)+(1-x)}{(1-x)(1+x)} = \frac{2}{1-x^2}$$

Theorem 3.2

If $(f_0, f_1, f_2, \ldots) \leftrightarrow F(x)$ and

$(g_0, g_1, g, \ldots) \leftrightarrow G(x)$

then

$$(f_0 + g_0, f_1 + g_1, f_2 + g_2, \ldots) \leftrightarrow F(x) + G(x)$$

Proof :

$(f_0 + g_0, f_1 + g_1, f_2 + g_2, ...) \quad \leftrightarrow \quad \sum_{n=0}^{\infty}(f_n+g_n)x^n$

$= \sum_{n=0}^{\infty}(f_n x^n) + \sum_{n=0}^{\infty}(g_n x^n)$

$= F(x) + G(x)$

c. Right Shifting

We may add *k* leading terms in a sequence

Example 3.3

$(1,1,1,1,...) \quad \leftrightarrow \quad \frac{1}{1-x}$

For that sequence, we may right-shift it by adding *k* leading zeros:

$(0,0,0,0,...1,1,1,1,...) \quad \leftrightarrow \quad x^k + x^{k+1} + x^{k+2} + x^{k+3} + \cdots$

$$= x^k(1 + x + x^2 + x^3 + \cdots)$$

$$= \frac{x^k}{1-x}$$

Theorem 3.3

If

$(f_0, f_1, f_2, ...) \leftrightarrow \quad F(x)$

Then

$(0,0,0,0,...0, f_0, f_1, f_2,...) \leftrightarrow \quad x^k F(x)$

We let the readers to prove the theorem 3.3

d. **Multiplication**

Multiplication can also be performed on generating functions.

Theorem 3.4

If $(a_0, a_1, a_2, a_3, \dots) \leftrightarrow A(x)$ and $(b_0, b_1, b_2, b_3, \dots) \leftrightarrow B(x)$

Then $(c_0, c_1, c_2, c_3, \dots) \leftrightarrow A(x) \cdot B(x)$

where $c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_n b_0$

Proof:

Let $C(x) = A(x) \cdot B(x) = \sum_{n=0}^{\infty} c_n x^n$

To evaluate the product $A(x) \cdot B(x)$, it can be used a table to list all the cross-terms from the multiplication of the sums:

Table 4.1. The croos-terms product of $A(x) \cdot B(x)$

	$b_0 x^0$	$b_1 x^1$	$b_2 x^2$	$a_3 3$	...
$a_0 x^0$	$a_0 b_0 x^0$	$a_0 b_1 x^1$	$a_0 b_2 x^2$	$a_0 b_3 x^3$	...
$a_1 x^1$	$a_1 b_0 x^1$	$a_1 b_1 x^2$	$a_1 b_2 x^3$	...	
$a_2 x^2$	$a_2 b_0 x^2$	$a_2 b_1 x^3$	...		
$a_3 x^3$	$a_3 b_0 x^3$	...			
....	....				

Notice that all terms involving the same power of x lie on a /-sloped diagonal. Collecting these terms together, we find that the coefficient of x^n in the product is the sum of all the terms on the $(n+1)$st diagonal, i.e.

$$a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_n b_0$$

e. Differentiation and Integration

We may take the derivative of a generating function.

Example 3.4

$$\frac{d}{dx}(1 + x + x^2 + x^3 + x^4 + \cdots) = \frac{d}{dx}\left(\frac{1}{1-x}\right)$$

$$1 + 2x + 3x^2 + 4x^3 + \cdots = \frac{1}{(1-x)^2}$$

$$(1,2,3,4,\ldots) \leftrightarrow \frac{1}{(1-x)^2}$$

In general, differentiating a generating function has two effects on the corresponding sequence: each term is multiplied by its index and the entire sequence is shifted left one place.

Theorem 3.5

If

$(f_0, f_1, f_2, \ldots.) \quad \leftrightarrow \quad F(x)$

Then

$(f_1, 2f_2, 3f_3, \ldots.) \quad \leftrightarrow \quad F'(x)$

Proof.

$$(f_1, 2f_2, 3f_3, \ldots.) \leftrightarrow f_1 + 2f_2x + 3f_3x^2 + \cdots$$

$$= \frac{d}{dx}(f_0 + f_1x + f_2x^2 + f_3x^3 \ldots)$$

$$= \frac{d}{dx}F(x)$$

The Derivative Rule is very useful. In fact, there is frequent, independent need for each of differentiation's two effects, multiplying terms by their index and left-shifting one place. Typically, we want just one effect and must somehow cancel out the other. For example, let's try to find the generating function for the sequence of squares, (0, $1, 4, 9, 16, \ldots.$). If we could start with the sequence (1,1,1,1,...) multiply each term by its index two times, then we'd have the desired result:

(0, $0 \cdot\ 1 \cdot 1, 2 \cdot 2, 3 \cdot 3, \ldots)$ = (0, $1, 4, 9,\ \ldots.$)

A challenge is that differentiation not only multiplies each term by its index, but also shifts the whole sequence left one place. However, the Right-Shift Rule tells how to cancel out this unwanted left-shift: multiply the generating function by x. Our procedure, therefore, is to begin with the generating function for (1,1,1,1,...), differentiate, multiply by x, and then differentiate and multiply by x once more.

(1,1,1,1,...) $\leftrightarrow$ $\left(\frac{1}{1-x}\right)$

(1,2,3,4,...) $\leftrightarrow$ $\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}$

(0,1,2,3,...) $\leftrightarrow$ $x\ \frac{1}{(1-x)^2} = \frac{x}{(1-x)^2}$

(1,4,9,16,...) $\leftrightarrow$ $\frac{d}{dx}\frac{x}{(1-x)^2} = \frac{1+x}{(1-x)^3}$

(0, $1, 4, 9, \ldots.$) $\leftrightarrow$ $x\frac{1+x}{(1-x)^3} = \frac{x(1+x)}{(1-x)^3}$

Thus the generating function for squares is :

$$\frac{x(1+x)}{(1-x)^3}$$

As we may expect, we can also perform integration on generating functions.

Example 3.5

$(0,1,-\frac{1}{2},\frac{1}{3},-\frac{1}{4}...) \leftrightarrow x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4 ...$

$$= \int (1-x+x^2-x^3+\cdots)dx$$

$$= \int \frac{dx}{(1+x)}$$

$$= \ln(1+x) + C$$

To find the constant *C*, we put in *x*=0 to get *C*=*G*(0). If we write $G(x) = a_0 + a_1x+a_2x^2 + a_3x^3 + \cdots$.

Then *G*(0) is simply equal to a_0, which is 0 in this case. Hence the answer is

$$G(x) = \ln(1+x)$$

3.3. Finding the Coefficient of Generating Functions

Generating functions are particularly useful for solving counting problems. In particular, problems involving choosing items from a set often lead to nice generating functions by letting the coefficient of x^n be the number of ways to choose n items. Consider the following sequence and its generating function :

$$\left(\binom{k}{0},\binom{k}{1},\binom{k}{2},\ldots,\binom{k}{k},0,0,0,\ldots\right) \leftrightarrow \binom{k}{0}+\binom{k}{1}x+\binom{k}{2}x^2+\cdots+\binom{k}{k}x^k$$

Here, we can see that the coefficient of, for example, x^2 is $\binom{k}{2}$ that is the number of ways to choose 2 items from a set with k elements. Thus, the coefficient of x^n in $(1+x)^k$ is $\binom{k}{n}$. Similarly, the coefficient of x^{k+1} is the number of ways to choose $k+1$ items from a size k set, which is zero.

Some motivating examples :

- Suppose there is a single-element set $\{a_1\}$. Then the generating function for the number of ways to select *n* elements from this set is simply $1 + x$: we have 1 way to select zero elements, 1 way to select one element, and 0 ways to select more than one element. Similary the number of ways to select *n* elements from the set $\{a_2\}$is also given by the generating function $1 + x$.
- To find the generating function for the number of ways to select *n* elements from the $\{a_1, a_2\}$ is multiplying the generating function for choosing from each set, i.e.

$(1+ x).(1+ x)$	$=(1+ x)^2$	$= 1+ 2x + x^2$
Gen func for Selecting an a_1	Gen func for Selecting an a_2	Gen func for Selecting an $\{a_1, a_2\}$

Sure enough, for the set $\{a_1, a_2\}$, we have 1 way to select zero elements, 2 ways to select one element, 1 way to select two elements, and 0 ways to select more than two elements. Repeated application of this rule gives the generating function for selecting n items from a *k*-element set $\{a_1, a_2,$ ...., $a_k\}$:

$(1+ x)$	$(1+ x)$ ...	$(1+ x)$	$= (1+ x)^k$
Gen func for Selecting an a_1	Gen func for Selecting an a_1	Gen func for Selecting an a_k	Gen func for Selecting an $\{a_1, a_2,$ ...., $a_k\}$

- There is a boy named Anto. He has 3 shirts and 2 trousers. That Anto owns 3 shirts means he has 1 way to choose no shirt and 3 ways to choose 1 shirt. Therefore, the model of the generating function of this context is $1+3x$. Similarly, that Anto owns 2 trousers means he has 1 way to choose no trouser and 2 ways to choose 1 trouser. Then, the model of the generating function of that case is $1+2x$. When we seek for the combination in which Anto uses shirt or trouser, we may multiply the models of the generating functions each other that is

  $(1+3x)(1+2x) = 1+5x+6x^2$ meaning that he has 1 way to choose neither shirt nor trouser, 5 ways to exactly choose either one shirt or one trouser and 6 ways to choose both one shirt and one trouser.

This section is also about developing algebraic techniques for calculating the coefficients of generating functions. All methods seek to reduce a given generating function to a simple binomial –type generating function, or a product of binomial-type generating functions. The followings are several polynomial identities and polynomial expantions :

1. $\frac{1-x^{n+1}}{1-x} = 1+x+x^2+\cdots+x^n$

2. $\frac{1}{1-x} = 1+x+x^2+\cdots$

3. $(1+x)^n = 1+\binom{n}{1}x+\binom{n}{1}x^2+\cdots+\binom{n}{r}x^r+\cdots+\binom{n}{n}x^n$

4. $(1-x^m)^n = 1 - \binom{n}{1} x^m + \binom{n}{2} x^{2m} + \cdots + (-1)^r \binom{n}{r} x^{rm} + \cdots + (-1)^n \binom{n}{n} x^{nm}$
5. $\frac{1}{(1-x)^n} = \left(\sum_{i=0}^{\infty} x^i\right)^n = \sum_{i=0}^{\infty} \binom{n+i-1}{i} x^i$
6. if $h(x) = f(x)g(x)$, where $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$ and

   $g(x) = b_0 + b_1x + b_2x^2 + b_3x^3 + \cdots$

   Then $h(x) = a_0b_0 + (a_1b_0 + a_0b_1)x + (a_2b_0 + a_1b_1 + a_0b_2)x^2 + \cdots + (a_rb_0 + a_{r-1}b_1 + a_{r-2}b_2 + \cdots + a_0b_r)x^r + \cdots$
7. The coefficient of x^r in $(1+x+x^2+\cdots)^n$ is $C(r+n-1, r) = \binom{r+n-1}{r}$

Here are some proofs of the polynomial identities:

1. $(1+x+x^2+\cdots+x^n)(1-x)$

   $= (1+x+x^2+\cdots+x^n) + (-x-x^2-\cdots x^n - x^{n+1})$

   $= 1 - x^{n+1}$

   By dividing both sides with $(1-x)$, we get the identitiy 1.

2. If *n* is made infinitely large, so that $1+x+x^2+\cdots+x^n$ becomes the infinite series $1+x+x^2+\cdots$ then the multiplication process will yield a power series in which the coefficient of each $x^k, k>0$ is zero. We conclude that $(1+x)(1+x+x^2+\cdots+x^n) = 1$

   By dividing both sides, we obtain the identitiy 2.

The identity 3 is binomial coefficient, which was explained in the chapter 2. Meanwhile the identity 4 is the application of the binomial coefficient by replacing x in the identity 3 with x^m.

For the identity 5, $(1-x)^{-n} = \left(\frac{1}{1-x}\right)^n = (1+x+x^2+\cdots)^n$

Since $\left(\frac{1}{1-x}\right) = (1 + x + x^2 + \cdots)$

Let us determine the coefficient x^r in the identity 7 by counting the number of formal products whose sum of exponents is *r*, if *ei* represents the exponent of the *i*th term in a formal product , the the number of formal products $x^{e_1}x^{e_2}x^{e_3} \dots x^{e_n}$ whose exponents sum to *r* is the same as the number of integer solution to the equation

$e_1 + e_2 + e_3 + \cdots + e_n = r, e_i \geq 0$

In the chapter 2, we have explained that the number of nonnegative integers solutions to this equation is $C(r + n - 1, r)$, so the coefficient x^r in the identity 7 is $C(r + n - 1, r) = \binom{r + n - 1}{r}$. This verifies identity 5.

Example 3.6

Find the coefficient of x^{16} in $(x^2 + x^3 + x^4 + \cdots)^5$

Solution:

To simplify the expression, we extract x^2 from each polynomial factor and then apply the identity 2.

$$\begin{aligned}(x^2 + x^3 + x^4 + \cdots)^5 &= [x^2(1 + x + x^2 + \cdots)]^5 \\ &= x^{10}(1 + x + x^2 + \cdots)^5 \\ &= x^{10}\frac{1}{(1-x)^5}\end{aligned}$$

Thus the coefficient of x^{16} in $(x^2 + x^3 + x^4 + \cdots)^5$ is the coefficient of x^{16} in $x^{10}\frac{1}{(1-x)^5}$ (i.e. the x^6 term in $(1 - x)^{-5}$ is multiplied by x^{10} to become the x^{16} in $x^{10}\frac{1}{(1-x)^5}$).

From identity 5, we see that the coefficient of x^6 in $(1 - x)^{-5}$ is $\binom{6 + 5 - 1}{6} = 210$

Exercises 3

1. Let $p = 1 + x + x^2 + x^3$, $q = 1 + x + x^2 + x^3 + x^4$, and $r = \frac{1}{1-x}$.
   a. Find the coefficient of x^3 in p^2; in p^3; in p^4
   b. Find the coefficient of x^3 in q^2; in q^3; in q^4
   c. Find the coefficient of x^3 in r^2; in r^3; in r^4
   d. Give a simple explanation for the fact that $p, q,$ and r all gave the same answers?
   e. Repeat the problem of 1.a, 1.b, and 1.c to but the instruction is finding the coefficient of x^4

2. Find the coefficient of x^2 in each of the following.
   a. $(2 + x + x^2)(1 + 2x + x^2)(1 + x + 2x^2)$
   b. $(2 + x + x^2)(1 + 2x + x^2)^2(1 + x + 2x^2)^3$
   c. $x(1 + x)^{43}(2 - x)^5$

3. Find the coefficient of x^{21} in $(x^2 + x^3 + x^4 + x^5 + x^6)^8$

4. Find the coefficient of x^5 in the following functions:
   a. $f(x) = (1 - 2x)^{-8}$
   b. $g(x) = (1 + x + x^2 + \cdots)^{-8}$

5. How many ways in distributing 25 identical balls into 7 different boxes, if the first box can be filled at most 10 balls and the other balls can be put in the remain 6 boxes ?

CHAPTER IV

RECURRENCE RELATION

4.1. Introduction

"*Sometimes we value someone based only on other's perspective, however, it is good when we value someone after being close with him*"

The quote above is related to the material in this chapter, i.e. recurrence relation. Specifically, we will show how to determine a specific term of a sequence after recognizing several preceding terms. A recurrence relation for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, a_1, a_2, ..., a_{n-1}, for all integers n with $n \geq n_0$, where n_0 is a non-negative integer. To completely describe the sequence, the first few values are needed, where few" depends on the recurrence. These are called the initial conditions. When we are given a recurrence relation and initial conditions, then you can write down as many terms of the sequence we please: just keep applying the recurrence. For example, $a_0 = a_1 = 1, a_n = a_{n-1} + a_{n-2}, n \geq 2$, defines the *Fibonacci Sequence* 1,1,2,3,5,8,13,... where each subsequent term is the sum of the preceding two terms.

A sequence is called a solution of a recurrence relation if it terms satisfy the recurrence relation. In other words, a recurrence relation is like a recursively defined sequence, but without specifying any initial values (initial conditions). Therefore, the same recurrence relation can have (and usually has) multiple solutions. If both the initial conditions and the recurrence relation are specified, then the sequence is uniquely determined.

Consider the recurrence relation $a_n = 2a_{n-1} - a_{n-2}$ for *n* = 2, 3, 4, ... The sequence $a_n = 3n$ is a solution of the recurrence relation since for $n \geq 2$ we see that $2a_{n-1} - a_{n-2} = 2\big(3(n-1)\big) - 3(n-2) = 3n = a_n$. In addition, the sequence $a_n = 5$, is a solution of the recurrence relation since for $n \geq 2$, $2a_{n-1} - a_{n-2} = 2.5 - 5 = a_n$.

Example 4.1

Fira deposits Rp 10.000.000,- in a savings account at a bank yielding 5% per year with interest compounded annually. How much money will be in the account after 30 years?

Solution:

Let P_n denote the amount in the account after *n* years. Based on the condition, we can state P_n in the term of P_{n-1} by deriving the following recurrence relation

$P_n = P_{n-1} + 0{,}05P_{n-1} = 1{,}05P_{n-1}$.

The initial condition is $P_0 = 10.000$

Then we have:

$P_1 = 1{,}05P_0$
$P_2 = 1{,}05P_1 = (1{,}05)^2P_0$
$P_3 = 1{,}05P_2 = (1{,}05)^3P_0$
...
$P_n = 1{,}05P_{n-1} = (1{,}05)^nP_0$

We now have a formula to calculate P_n for any natural number *n* and can avoid the iteration. So the value of P_{30} denoting the amount of money after 30 years is $P_{30} = (1{,}05)^{30}10.000.000 = 43.219.420$

4.2. Solving Recurrence Relation

On this section we discuss how to obtain the solution of linear recurrence relation. There are two reasons on the selection of linear recurrence relation. First, it generally has systematic steps to solve. Secondly, it often occurs in modelling of several problems.

4.2.1. Solving Homogeneous Linear Recurrence Relation with Constant Coefficient

Linear Recurrence Relation (LRR) of degree *k* with constant coefficient is a recurrence relation of the form

$$a_n = c_1a_{n-1} + c_2a_{n-2} + \cdots + c_ka_{n-k} + f(n) \qquad (4.1)$$

where $c_1, c_2, \dots, c_k$ are real numbers, and $c_k \neq 0$. Meanwhile $f(n)$ is a function of n.

The recurrence relation in the definition is linear since the right-hand side is a sum of multiples of the previous terms of the sequence. On this section, we firstly discuss LRR which is homogeneous.

Linear Homogeneous Recurrence Relation (LHRR) with constant coefficient is a linear recurrence relation in the equation 4.1 with $f(n) = 0$. So it is written in the form : $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$. It is called homogeneous since no term occur that are not multiples of the a_js. The coefficients of the terms of the sequence are all constants, rather than functions that depend on *n*. The degree is *k* because a_n is expressed in terms of the previous *k* terms of the sequence. In other words, the order of a recurrence relation is the di_erence between the greatest and lowest sub-scripts of the terms of the sequence in the equation.

To understand comprehensively about the linearity, homogenity, and degree concepts, we provide several examples:

Example 4.2

- The recurrence relation $P_n = 1{,}05 P_{n-1}$ is a linear homogeneous recurrence relation of degree one.
- The recurrence relation $f_n = f_{n-1} + f_{n-2}$ is a linear homogeneous recurrence relation of degree two.
- The recurrence relation $a_n = a_{n-1} + a^2{}_{n-2}$ is not linear.

- The recurrence relation $E_n = E_{n-1} + 1$ is not homogeneous.
- The recurrence relation $D_n = D_{n-1}$ does not have constant coefficients.

The basic approach for solving LHRR is to look for solutions of theform $a_n = r^n$, where r is a constant. Note that $a_n = r^n$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

if and only if

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k}$$

When both sides of late equation are divided by r^{n-k} and the right-hand side is subtracted from the left, we obtain the equivalent equation

$$r^k - c_1 r^{k-1} + c_2 r^{k-2} + \cdots + c_{k-1} r - c_k = 0,$$

which is called the characteristic equation of the recurrence relation. The solutions of this equation are called the characteristic roots of the recurrence relation.

We will first develop results that deal with LHRR with constant coefficients of degree two. Then corresponding general results when the degree may be greater than two will be stated.

Theorem 4.1

Let c_1 and c_2 be real numbers. Suppose that $r^2 - c_1 r - c_2 = 0$ has two distinct roots r_1 and r_2. Then the sequence $\{a_n\}$ is a a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$

for $n = 0,1,2,\ldots$ where α_1 and α_2 are constants.

Proof:

It should be firstly shown that if r_1 and $r_2$are the root of the characteristic equation, and α_1 and α_2 are constants, then the sequence $\{a_n\}$ with $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ is a solution of he recurrence relation. Then, it must be shown that if the sequence $\{a_n\}$ is a solution, then $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for some constants α_1 and α_2. Now we will show that $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ is a solution of the recurrence relation. Since r_1 and r_2 are roots of $r^2 - c_1 r - c_2 = 0$, it follows that $r_1^2 = c_1 r_1 + c_2$, and $r_2^2 = c_1 r_1 + c_2$.

From these equations, we see that

$$\begin{aligned} c_1 a_{n-1} + c_2 a_{n-2} &= c_1(\alpha_1 r_1^{n-1} + \alpha_2 r_2^{n-1}) + c_2(\alpha_1 r_1^{n-2} + \alpha_2 r_2^{n-2}) \\ &= \alpha_1 r_1^{n-2}(c_1 r_1 + c_2) + \alpha_2 r_2^{n-2}(c_1 r_2 + c_2) \\ &= \alpha_1 r_1^n + \alpha_2 r_2^n \\ &= a_n \end{aligned}$$

To show that every solution $\{a_n\}$ of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ has $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n = 0,1,2,\ldots$ for some constants α_1 and α_2, suppose $\{a_n\}$ is a solution of the recurrence relation, and the initial conditions $a_0 = C_0$ and $a_1 = C_1$ hold.

It will be shown that there are constants α_1 and α_2 so that the sequence $\{\breve{a}_n\}$ with $\breve{a}_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ satisfies these same initial conditions.

This requires that

$\breve{a}_0 = C_0 = \alpha_1 + \alpha_2$ and $\breve{a}_1 = C_1 = \alpha_1 r_1 + \alpha_2 r_2$

From these equations it follows that

$\alpha_1 = (C_1 - C_0 r_2)/(r_1 - r_2)$, $\alpha_2 = (C_0 r_1 - C_0)/(r_1 - r_2)$,
where these expressions for α_1 and α_2 depend on the fact that $r_1 \neq r_2$
since this recurrence relation and these initial conditions uniquely determine the sequence, it follows that $a_n = \breve{a}_n = \alpha_1 r_1^n + \alpha_2 r_2^n$.

Example 4.3

What is the solution of the recurrence $a_n = a_{n-1} + 2a_{n-2}$ with $a_0 = 2$ and $a_1 = 7$?

Solution:

The characteristic equation of the recurrence relation is $r^2 - r - 2 = 0$
Its roots are $r = 2$ and $r = -1$.
Hence, the sequence $\{a_n\}$ is a solution to the recurrence relation if and only if: $a_n = \alpha_1 2^n + \alpha_2(-1)^n$, for some constants α_1 and α_2.
Given the equation $a_n = \alpha_1 2^n + \alpha_2(-1)^n$ and the initial conditions $a_0 = 2$ and $a_1 = 7$, it follows that

$a_0 = 2 = \alpha_1 + \alpha_2$

$a_1 = 7 = \alpha_1 . 2 + \alpha_2 . (-1)$

The solution of the equation system is $\alpha_1 = 3$ and $\alpha_2 = -1$.
Therefore, the solution to the recurrence relation and initial conditions is the sequence $\{a_n\}$ with

$a_n = 3.2^n - (-1)^n$.

Check: We can check our answer quickly and easily. The recurrence formula gives us

$a_2 = a_1 + 2a_0 = 7 + 2.2 = 11$

$a_3 = a_2 + 2a_1 = 11 + 2.7 = 25$
$a_4 = a_3 + 2a_2 = 25 + 2.11 = 47$

Based on the solution $a_n = 3.2^n - (-1)^n$, it appears that the sequence is indeed giving us numbers {2,7,11,25,47,...}. So the formula of the solution seems to be correct.

Example 4.4

Solve the recurrence relation satisfied by the Fibonacci sequence:

$a_n = a_{n\ 1} + a_{n\ 2}, \quad n \quad 2$, with $a_0 = 0$ and $a_1 = 1$

Solution:

The characteristic equation of the recurrence relation

$r^2 \quad r \quad 1 = 0$ and its characteristic roots are

$$r_1 = \frac{1+\sqrt{5}}{2}, \quad r_2 = \frac{1 \quad \sqrt{5}}{2}.$$

Therefore

$$a_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^{\mathrm{n}} + \alpha_2 \left(\frac{1+\sqrt{5}}{2}\right)^{\mathrm{n}}$$

Substituting the initial conditions we get a system of linear equation which are uniquely solvable giving

$\alpha_1 = \frac{1}{\sqrt{5}}$, $\alpha_2 = -\frac{1}{\sqrt{5}}$.

The theorem 4.1 does not apply when there is one characteristic root of multiplicity two. This case can be handled using the following theorem.

Recurrence relation with degree *k* having *k* characteristic roots $r_1, r_2, \dots, r_k$ of which $r_1 \neq r_2 \neq \cdots \neq r_k$, has general solution:

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_k r_k^n$$

Theorem 4.2

Let c_1 and c_2 be real numbers with $c_2 \neq 0$. Suppose that $r^2 - c_1 r - c_2 = 0$ has only one root r_0. A sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$ for $n = 0,1,2,\ldots$ where α_1 and α_2 are constants.

Example 4.5

What is the solution of the recurrence relation

$$a_n = 6a_{n-1} - 9a_{n-2}$$

with initial conditions $a_0 = 1$ and a_1 = 6?

Solution:

The only root of $r^2 - 6r + 9$ is $r = 3$. Hence the solution to this recurrence relation is $a_n = \alpha_1 3^n + \alpha_2 n 3^n$ for some constants α_1 and α_2. Using the initial conditions, it follows that $\alpha_1 = \alpha_2 = 1$.

Recurrence relation with degree *k* having one root r_0 (with multiplicity *k)* has general solution:

$$a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n + \alpha_3 n^2 r_0^n \ldots + \alpha_k n^{k-1} r_0^n$$

4.2.2. Solving Inhomogeneous Linear Recurrence Relation with Constant Coefficient

Inhomogeneous linear recurrence relation with constant coefficient is a recurrence relation of the form of

$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + f(n)$, where $c_k \neq 0$ and $f(n) \neq 0$.

It includes *k* initial conditions and for $1 \leq i \leq k,\ c_i$ is constant.

Frequently, we denote the homogeneous part of the inhomogeneous recurrence relation as "the associated homogeneous recurrence relation". There haven't been yet general procedures to determine solution for linear recurrence relation which is not homogeneous. However, we firstly may determine general form of the particular solution based on $f(n)$, and then we are able to determine the exact solution based on a given recurrence relation.

Case I

If $f(n)$ is a polynom of degree *t* in *n*, i.e.

$A_1n^t + A_2n^{t-1} + \cdots + A_tn + A_{t+1}$

Then the general form of its particular solution is

$$B_1n^t + B_2n^{t-1} + \cdots + B_tn + B_{t+1}$$

Example 4.6

What is the particular solution of $a_n + 5a_{n-1} + 6a_{n-2} = 3n^2 - 2n + 1$

Solution :

The particular solution, based on the polynom, is

$$B_1n^2 + B_2n + B_3 \qquad (4.2)$$

By substituting (4.2) to the recurrence relation, we obtain

$$(B_1n^2 + B_2n + B_3) + 5(B_1(n-1)^2) + B_2(n-1) + B_3)$$
$$+ 6(B_1(n-2)^2 + B_2(n-2) + B_3) = 3n^2 - 2n + 1$$
$$12B_1n^2 - (34B_1 - 12B_2)n + (29B_1 - 17B_2 + 12B_3)$$
$$= 3n^2 - 2n + 1$$

By equating the coefficient of the two sides in the latest equation, we obtain the equation system

$12B_1 = 3$
$34B_1 - 12B_2 = 2$
$29B_1 - 17B_2 + 12B_3 = 1$
Solving the equation system, we obtain

$B_1 = 1/4$; $B_2 = 13/24$; $B_3 = 71/288$

Then, the exact solution of the recurrence relation is

$$a_n = \frac{1}{4}n^2 + \frac{13}{24}n + \frac{71}{288}$$

Case 2

If $f(n)$ is in the form of β^n, then the particular solution will be in the form of $B\beta^n$, providing that β is not the characteristic root of the recurrence relation given.

Example 4.7

Find the particular solution of the recurrence relation

$a_n + 5a_{n-1} + 6a_{n-2} = 42.4^n$!

Solution :

The particular solution of $a_n + 5a_{n-1} + 6a_{n-2} = 42.4^n$ has general form of $B4^n$.

By replacing a_n with $B4^n$ in $a_n + 5a_{n-1} + 6a_{n-2} = 42.4^n$, we obtain

$$B.4^n + 5B.4^{n-1} + 6B.4^{n-2} = 42.4^n$$
$$\Leftrightarrow \quad B.4^n + 5B.4^n.4^{-1} + 6B.4^n.4^{-2} = 42.4^n$$
$$\Leftrightarrow \quad B.4^n + \frac{5}{4}.B.4^n + \frac{6}{16}.B.4^n = 42.4^n$$
$$\Leftrightarrow \quad \frac{42}{16}.B.4^n = 42.4^n$$
$$\Leftrightarrow \quad B = 16$$

Therefore, the particular solution of the recurrence relation $a_n + 5a_{n-1} + 6a_{n-2} = 42.4^n$ is $a_n = 16.4^n$

Case 3

If $f(n)$ is in the form of β^n, then the particular solution will be in the form of $Bn^t\beta^n$, providing that β is the characteristic root with multiplicity *t* of the associated homogeneous recurrence relation given.

Example 4.8

Find the particular solution of the recurrence relation $a_n - 6a_{n-1} + 9a_{n-2} = 3^n$!

Solution :

The recurrence relation $a_n - 6a_{n-1} + 9a_{n-2} = 3^n$ has characteristic equation i.e.

$$r^2 - 3r + 9 = 3^n$$

$$\Leftrightarrow \quad (r-3)^2 = 3^n$$

Its characteristic root is 3, which is the same as the base part of the exponent 3^n. It occurs in multiplicity 2, hence, the general form of the particular solution of the recurrence relation is $Bn^2 3^n$. By replacing a_n with $Bn^2 3^n$ in $a_n - 6a_{n-1} + 9a_{n-2} = 3^n$, we obtain

$$Bn^2 3^n - 6B(n-1)^2.3^{n-1} + 9B(n-2)^2.3^{n-2} = 3^n$$

$$\Leftrightarrow Bn^2 3^n - 6B(n^2-2n+1).3^n 3^{-1} + 9B(n^2-4n+4).3^n 3^{-2} = 3^n$$

$$\Leftrightarrow Bn^2 3^n - 2Bn^2.3^n + 4Bn.3^n - 2B3^n + Bn^2.3^n - 4Bn.3^n + 4B.3^n = 3^n$$

$$\Leftrightarrow 2B3^n = 3^n$$

$$\Leftrightarrow B = 1/2.$$

Then the particular solution of the recurrence relation is $a_n = \frac{1}{2}.n^2.3^n$

Case 4

If $f(n)$ is the product of a polynom and an exponent, then the general form of the particular solution of the recurrence relation given is the product of the particular solution in the case 1 and the particular solution in the case 2, i.e. if $f(n)$ is in the form of

$(A_1n^t + A_2n^{t-1} + \cdots + A_tn + A_{t+1})\beta^n$

Then the general form of the particular solution of a recurrence relation given is

$(B_1n^t + B_2n^{t-1} + \cdots + B_tn + B_{t+1})\beta^n$

Example 4.9

Find the particular solution of $a_n = -a_{n-1} + 3n.2^n$

Solution :

The characteristic equation of the recurrence relation $a_n = -a_{n-1} + 3n.2^n$ is

$$r + 1 = 3n.2^n$$

The general form of the particular solution is $(B_1n + B_0).2^n$

By replacing a_n with $(B_1n + B_0).2^n$ in the recurrence relation $a_n = -a_{n-1} + 3n.2^n$, we obtain

$$(B_1n + B_0).2^n + (B_1(n-1) + B_0).2^{n-1} = 3n.2^n$$

$$\Leftrightarrow \quad B_1n.2^n + B_0n.2^n + B_1n.2^{n-1} - B_1.2^{n-1} + B_0.2^{n-1} = 3n.2^n$$

$$\Leftrightarrow \quad B_1n.2^n + B_0n.2^n + B_1n.2^n.2^{-1} - B_1.2^n.2^{-1} + B_0.2^n.2^{-1} = 3n.2^n$$

$$\Leftrightarrow \quad B_1n.2^n + B_0n.2^n + (B_1/2)n.2^n - (B_1/2).2^n + (B_0/2).2^n = 3n.2^n$$

$$\Leftrightarrow \quad \left(B_1 + \frac{B_1}{2}\right)n.2^n + \left(\left(\frac{3B_0}{2}\right) - \left(\frac{B_1}{2}\right)\right).2^n = 3n.2^n$$

By equating the coefficient of the corresponding terms, we obtain linear equation system:

$B_1 + \frac{B_1}{2} = 3$ and $\left(\frac{3B_0}{2}\right) - \left(\frac{B_1}{2}\right) = 0$

Solving the linear equation system gives

$B_1 = 2$ and $B_0 = 2/3$

Then, the particular solution of the recurrence relation $a_n = -a_{n-1} + 3n.2^n$ is $(2n + 2/3).2^n$

The solution of a nonhomogeneous linear recurrence relation with constant coefficient is the sum of the particular solution of the associated recurrence relation and the particular solution that satisfies the inhomogeneous recurrence relation given including $f(n)$ in the right side. If the roots of the characteristic equation, as many as *k*, are all different, then the total solution of a nonhomogeneous linear recurrence relation given is

$a_n = a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_k r_k^n + p(n)$, where $p(n)$ is the particular solution of a nonhomogeneous linear recurrence relation given.

Example 4.10

Find the solution of the inhomogeneous recurrence relation $a_n - 6a_{n-1} + 9a_{n-2} = 3^n$!

Solution :

In the example 4.8, 3 is the only characteristic root, then the general solution of its associated homogeneous recurrence relation is

$a_n = A_0.3^n + A_1.n.3^n$

meanwhile, the particular solution of the recurrence relation is $a_n = (1/2).n^2 3^n$

Hence, the total solution of the recurrence relation $a_n - 6a_{n-1} + 9a_{n-2} = 3^n$ is $a_n = A_0.3^n + A_1.n.3^n + (1/2).n^2 3^n$.

Example 4.11

Find the solution of $a_n + 5a_{n-1} + 6a_{n-2} = 42.4^n$, $a_2 = 278$ and $a_3 = 962$!

Solution :

The solution of the recurrence relation given is

$$a_n = A_1.(-2)^n + A_1.(-3)^n + 16.4^n \qquad (4.3)$$

Substituting a_2 and a_3 for a_n in the equation (4.3) gives

$278 = 4A_1 + 9A_2 + 256$

$962 = -8A_1 - 27A_2 + 1024$

By solving this linear equation system, we obtain

$A_1 = 1$ and $A_2 = 2$

Then, the solution of the recurrence relation for $a_n - 6a_{n-1} + 9a_{n-2} = 3^n$ is $a_n = (-2)^n + 2(-3)^n + 16.4^n$

4.3. Using Generating Functions to solve Recurrence Relations

Generating function is also a useful tool to find the solution of a recurrence relation. The process of solving recurrence relation may take following systematic ways:

Step 1: Assume that $G(x)$ is the generating function for the sequence $a_0, a_1, a_2, \cdots, a_n, \cdots$. That is $G(x) = \sum_{n=0}^{\infty} a_n x^n$.

(Note that if we can somehow find the coefficients in the expansion of $G(x)$, then we can map them into $a_0, a_1, a_2, \cdots, a_n, \cdots$ and hence can get the value of a_n).

Step 2: Multiply both sides of the recurrence relation by x^n to get $a_n x^n = C_1 a_{n-1} x^n + C_2 a_{n-2} x^n + \cdots + C_k a_{n-k} x^n$.

Step 3: Sum over n from k to ∞ on both sides to get

$$\sum_{n=k}^{\infty} a_n x^n = \sum_{n=k}^{\infty} C_1 a_{n-1} x^n + \sum_{n=k}^{\infty} C_2 a_{n-2} x^n + \cdots + \sum_{n=k}^{\infty} C_k a_{n-k} x^n$$

Step 4: Rearrange the indices to get

$$\sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{k-1} a_n x^n = C_1 x \left(\sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{k-2} a_n x^n \right)$$

$$+ C_2 x^2 \left(\sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{k-3} a_n x^n \right) + \cdots$$

$$+ C_{k-1} x^{k-1} \left(\sum_{n=0}^{\infty} a_n x^n - a_0 \right) + C_k x^k \sum_{n=0}^{\infty} a_n x^n .$$

Step 5: Substitute $G(x)$ for $\sum_{n=0}^{\infty} a_n x^n$ and put the k initial values of a_is in the above equation and then solve for $G(x)$.

Step 6: Expand the closed form of $G(x)$ as a series and state the value of a_n as the coefficient of x^n in that series.

We will apply those steps to get the soluion of recurrence relation as in the following examples.

Example 4.12

Solve the recurrence relation $a_n = 7a_{n-1}$ with the initial condition $a_0 = 5$.

Solution:

Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$, so that a_n = Coefficient of x^n in $G(x)$.

Multiplying both sides of the recurrence relation by x^n, we get $a_n x^n = 7a_{n-1}x^n$.

Summing over n from 1 to ∞ on both sides, we get,

$$\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} 7a_{n-1}x^n.$$

or, $\sum_{n=0}^{\infty} a_n x^n - a_0 = \sum_{n=1}^{\infty} 7a_{n-1} x \cdot x^{n-1}$,

or, $G(x) - a_0 = 7x \sum_{n=1}^{\infty} a_{n-1}x^{n-1}$

or, $G(x) - 5 = 7xG(x)$, Or, $(1 - 7x)G(x) = 5$

or, $G(x) = \dfrac{5}{1 \quad 7x} = 5 \sum_{n=0} (7x)^n$

Hence, a_n = Coefficient of x^n in $G(x)$, which is $5 \quad 7^n$.

Example 4.13

Solve the recurrence relation $a_n = 5a_{n \ 1} \quad 6a_{n \ 2}$ with the initial conditions $a_0 = 6, a_1 = 30$ using generating function method !

Solution:

Let $G(x) = \sum_{n=0} a_n x^n$, so that a_n = Coefficient of x^n in $G(x)$

Multiplying both sides of the recurrence relation by x^n, we get $a_n x^n = 5a_{n \ 1} x^n \quad 6a_{n \ 2} x^n$.

Summing over n from 2 to on both sides, we get,

$$\sum_{n=2} a_n x^n = \sum_{n=2} 5a_{n \ 1} x^n \quad \sum_{n=2} 6a_{n \ 2} x^n .$$

$$\Leftrightarrow \sum_{n=0} a_n x^n \quad a_0 \quad a_1 x = \sum_{n=2} 5a_{n \ 1} \quad x \ x^{n \ 1} \quad \sum_{n=2} 6a_{n \ 2} x^2 x^{n \ 2}$$

$$\Leftrightarrow G(x) \quad 6 \quad 30x = 5x \sum_{n=2} a_{n \ 1} x^{n \ 1} \quad 6x^2 \sum_{n=2} a_{n \ 2} x^{n \ 2}$$

$$\Leftrightarrow G(x) \quad 6 \quad 30x = 5x(G(x) \quad a_0) \quad 6x^2 G(x),$$

$$\Leftrightarrow G(x) \quad 6 \quad 30x = 5x(G(x) \quad 6) \quad 6x^2 G(x)$$

$$\Leftrightarrow (1 \quad 5x + 6x^2)G(x) = 6$$

$$\Leftrightarrow G(x) = \frac{6}{1 \quad 5x + 6x^2} = \frac{6}{(1 \quad 3x)(1 \quad 2x)} = \frac{18}{1 \quad 3x} \quad \frac{12}{1 \quad 2x}$$

$$\Leftrightarrow G(x) = 18 \sum_{n=0} (3x)^n \quad 12 \sum_{n=0} (2x)^n$$

Hence, a_n = Coefficient of x^n in $G(x)$, which is $18\ \ 3^n \quad 12\ \ 2^n$.

Example 4.14

Solve the recurrence relation $a_n = 2a_{n-1} + 4^{n-1}, n \geq 2;\ a_0 = 1, a_1 = 3$ using generating function method !

Solution :

Let $G(x) = \quad a_n x^n$, so that a_n = Coefficient of x^n in $G(x)$.

$n=0$

Multiplying both sides of the recurrence relation by x^n, we get.

$$a_n x^n = 2a_{n-1}x^n + 4^{n-1}x^n$$

$$\Leftrightarrow \quad a_n x^n = (2a_{n-1} + 4^{n-1})x^n$$

Summing over n from 2 to   on both sides, we get,

$$\sum_{n=2}^{\infty} a_n x^n = \sum_{n=2}^{\infty} (2a_{n-1} + 4^{n-1})x^n \ ,$$

$$\Leftrightarrow \quad \sum_{n=2}^{\infty} a_n x^n = \sum_{n=2}^{\infty} (2a_{n-1} + 4^{n-1})x^n$$

$$\Leftrightarrow \quad \sum_{n=2}^{\infty} a_n x^n = 2\sum_{n=2}^{\infty} a_{n-1}x^n + \sum_{n=2}^{\infty} 4^{n-1}\, x^n \qquad (4.4)$$

From the left side of the equation (4.4), we obtain

$$\sum_{n=2}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^n - a_0 - a_1 x$$

$$= G(x) - 1 - 3x$$

From the first term of the right side of the equation (4.4), we get

$$2\sum_{n=2}^{\infty} a_{n-1}x^n = 2x\sum_{n=2}^{\infty} a_{n-1}x^{n-1}$$

$$= 2x\left(\sum_{n=1}^{\infty} a_{n-1}x^{n-1} - a_0\right)$$

$$= 2x(G(x) - 1)$$

$$= 2xG(x) - 2x$$

From the first term of the right side of the equation (4.4), we get

$$\sum_{n=2}^{\infty} 4^{n-1} = x\sum_{n=2}^{\infty} 4^{n-1}\, x^{n-1}$$

$$= x\left(\sum_{n=2}^{\infty}(4x)^{n-1} - 1\right)$$

$$= x\left(\frac{1}{1-4x} - 1\right)$$

Then the equation (4.4) becomes

$$G(x) - 1 - 3x = 2xG(x) - 2x + \frac{1}{1-4x} - x$$

Which is equivalent to

$$G(x) = \frac{1-3x}{(1-4x)(1-2x)}$$

Since

$\frac{1-3x}{(1-4x)(1-2x)} = \frac{1/2}{(1-4x)} + \frac{1/2}{(1-2x)}$, then

$$G(x) = \frac{1}{2}\left(\frac{1}{(1-4x)} + \frac{1}{(1-2x)}\right)$$

$$= \sum_{n=0}^{\infty} \frac{1}{2}(4^n + 2^n)x^n$$

Since $\frac{1}{2}(4^n + 2^n)$ is also the coefficient of x^n in $G(x)$, then we can conclude that $a_{n=} \frac{1}{2}(4^n + 2^n)$.

In summary, the procedures of solving a nonhomogeneous linear recurrence relation i,e.

1. Write down the associated homogeneous recurrence and its general solution.

2. Find a particular solution the non-homogeneous recurrence. This may involve solving several simpler non-homogeneous recurrences (using this same procedure).

3. Add all of the above solutions together to obtain the general solution to the non-homogeneous recurrence.

4. Use the initial conditions to get a system of *k* equations in *k* unknowns, then solve it to obtain the solution you want.

Exercises 4

1. Solve the following recurrence relations by using characteristic root method :

   i. $a_1 = a_2 = 1;\ a_n = a_{n-1} + a_{n-2},\ n \geq 3$

   ii. $a_0 = 0;\ a_1 = -1;\ a_n = 7a_{n-1} - 12a_{n-2},\ n \geq 2$.

   iii. $a_0 = a_1 = 1;\ a_n = 2a_{n-1} + 3a_{n-2},\ n \geq 2$

   iv. $a_1 = 2,\ a_2 = 6;\ a_n - 4a_{n-1} + 4a_{n-2} = 0,\ n \geq 3$

   v. $a_0 = 0,\ a_1 = 1,\ a_2 = 2;\ a_n = 9a_{n-1} - 15a_{n-2} + 7a_{n-3},\ n \geq 3$

   vi. $a_0 = 0, a_1 = 1, a_2 = 2, a_3 = 3;\ a_n + 2a_{n-2} - 15a_{n-2},\ n \geq 4$

   vii. $a_0 = 1;\ a_n = 3a_{n-1} + 2^n, n > 0$

   viii. $a_0 = 0;\ a_1 = 1;\ a_n - 4a_{n-1} + 4a_{n-2} + n2^n + 3^n + 4, n \geq 2$

2. Akbar and Hasrawan flip their coins. If the coins are both heads or both tails, Akbar wins. If one coin is a head and the other a tail, Hasrawan wins. Akbar starts with *T* coins, and Hasrawan starts with *S* coins.

   a. Let p_n denote the probability that Akbar wins all of Hasrawan's coins if Akbar starts with *n* coins. Write a recurrence relation for p_n.

   b. What is the value of p_0 ?

   c. What is the value of p_{S+T} ?

   d. Find the solution of p_n

3. Referring to the problems number 1, solve each recurrence relation using generating function.

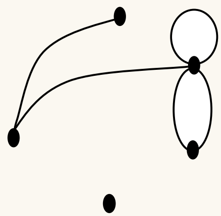
CHAPTER V

INTRODUCTION AND THE APPLICATION OF GRAPH THEORY

5.1. Definitions and Fundamental Concepts

Conceptually, a *graph* is formed by *vertices* and *edges* connecting the vertices.

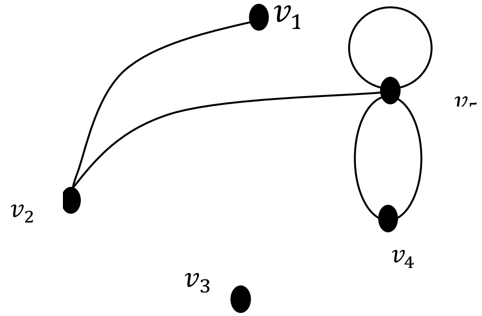
Example 5.1



Formally, a graph is a pair of sets (*V*,*E*), where V is the set of vertices and E is the set of edges, formed by pairs of vertices. E is a multiset, in other words, its elements can occur more than once so that every element has a multiplicity. Often, we label the vertices with letters (for example: a, b, c . . . or v_1, v_2 , . . .) or numbers 1, 2, . . . Throughout this chapter, we will label the elements of *V* in this way.

Example 5.2

(Continuing from example 5.1) We label the vertices as follows:



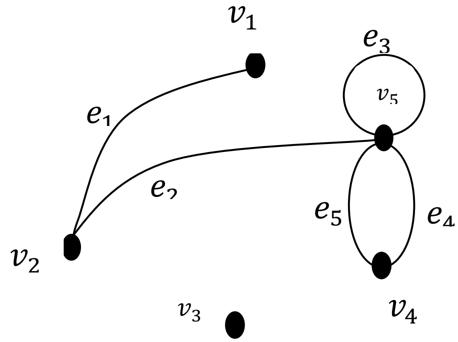
We have $V = \{v_1, \ldots, v_5\}$ for vertices and $E = \{(v_1, v_2), (v_2, v_5), (v_5, v_5), (v_5, v_4), (v_5, v_4)\}$ for the edges.

Similarly, we often label the edges with letters (for example: : a, b, c . . . or e_1, e_2 , . . .) or numbers 1, 2, . . . for simplicity.

Remark. *The two edges* (u, v) *and* (v, u) *are the same. In other words, the pair is not* ordered.

Example 5.3

Continuing from the previous example. We label the edges as follows:



So $E = \{e_1, e_2, e_3, e_4, e_5\}$

We have the following terminologies:

1. The two vertices u and v are *end vertices* of the edge (u,v).
2. Edges that have the same end vertices are *parallel*.
3. An edge of the form (v, v) is a *loop*.
4. A graph is *simple* if it has no parallel edges or loops.
5. A graph with no edges (i.e. E is empty) is *empty*.
6. A graph with no vertices (i.e. V and E are empty) is a *null graph*.
7. A graph with only one vertex is *trivial*.
8. Edges are *adjacent* if they share a common end vertex.

Example 5.4

- v_4 and v_5 are end vertices of e_5.
- e_4 and e_5 are parallel.
- e_3 is a loop.
- The graph is not simple.
- e_1 and e_2 are adjacent.
- v_1 and v_2 are adjacent.
- The degree of v_1 is 1 so it is a pendant vertex.
- e_1 is a pendant edge.
- The degree of v_5 is 5.
- The degree of v_4 is 2.
- The degree of v_3 is 0 so it is an isolated vertex.

In the future, we will label graphs with letters, for example:

$$G = (V, E)$$

The *minimum degree* of the vertices in a graph G is denoted $\delta(G)$ (= 0 if there is an isolated vertex in G). Similarly, we write $\Delta(G)$ as the *maximum degree* of vertices in G.

Example 5.5

(Continuing from the previous example) $\delta(G) = 0$ *and* $\Delta(G) = 5$.

Remark. *In this course, we only consider* finite *graphs, i.e.* v *and* e *are finite sets.*

That every edge has two end and vertices, we get

Theorem 5.1

The graph $G = (V, E)$, where $V = \{v_1, \dots., v_n\}$ and $V = \{e_1, \dots., e_m\}$ satisfies

$$\sum_{i=1}^{n} d(v_i) = 2m.$$

Corollary 5.1

Every graph has an even number of vertices of odd degree.

Proof :

If the vertices $v_1, \dots., v_k$ have odd degrees and the vertices $v_{k+1}, \dots, v_n$ have even degrees, then (Theorem 5.1) $d(v_1) + \cdots + d(v_k) = 2m - \cdots - d(v_n)$ is even. Therefore, k is even.

Example 5.6

(Continuing from the previous example). *Now the sum of the degrees is* 1 + 2 + 0 +2 + 5 = 10 = 2 · 5*. There are two vertices of odd degree, namely* v_1 *and* v_5*.* A simple graph that contains every possible edge between all the vertices is called a *complete graph*. A complete graph with *n* vertices is denoted as K_n. The first four complete graphs are given as examples:

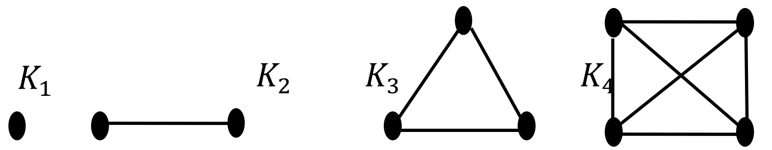


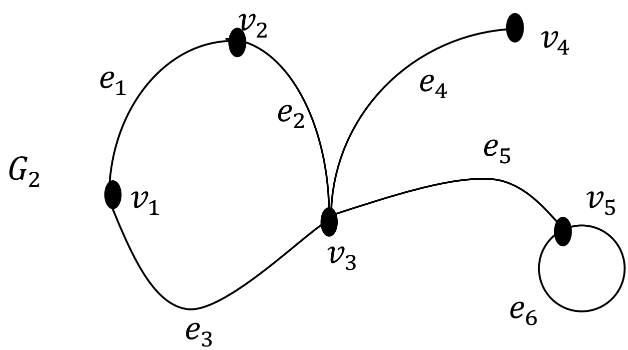
Figure 5.1. Complete graphs with certain number of vertices

The graph $G_1 = (V_1, E_1)$ is a *subgraph* of $G_2 = (V_2, E_2)$ if

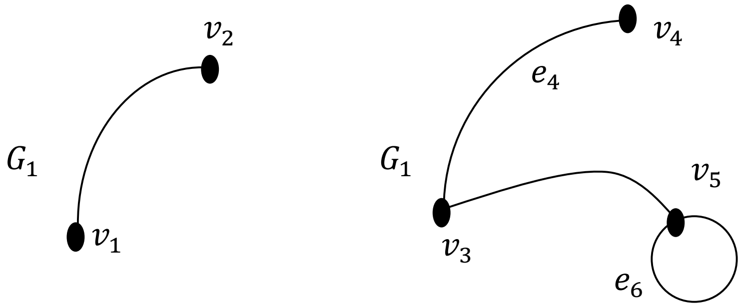
1. $V_{1,} \subseteq V_2$ and
2. Every edge of G_1 is also an edge of G_2

Example 5.7

We have the graph



and some of its subgraphs are



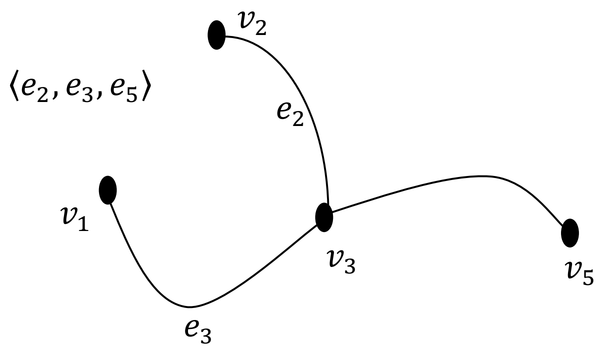
The subgraph of $G = (V, E)$ induced by the edge set $E_1 \subseteq E_2$ is

$$G_1 = (V_{1,} E_1) =_{def.} \langle E_1 \rangle$$

where $V_{1,}$ consists of every end vertex of the edges in $E_{1,}$.

Example 5.8

(Continuing from above) From the original graph G, the edges E_2, E_3 and E_5 induce the subgraph



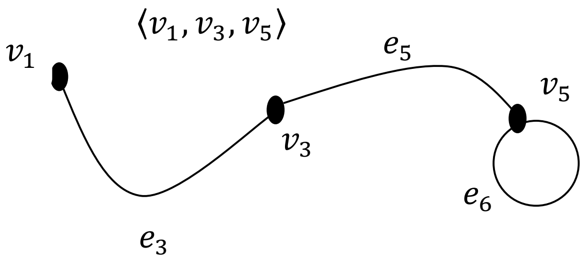
The subgraph of $G = (V, E)$ induced by the vertex set $V_1 \subseteq V$ is

$$G_1 = (V_{1,} E_1) =_{def.} \langle V_1 \rangle$$

where $E_{1,}$ consists of every end vertex of the edges in $V_{1,}$.

Example. 5.9

(Continuing from the previous example) From the original graph G, the vertices $v_{1,} v_3$, and $v_{5,}$ and induce the subgraph



5.2. Walks, Trails, Paths, Circuits, Connectivity, Components

Remark. There are many different variations of the following terminologies. We will adhere to the definitions given here. A walk in the graph $G = (V, E)$ is a finite sequence of the form

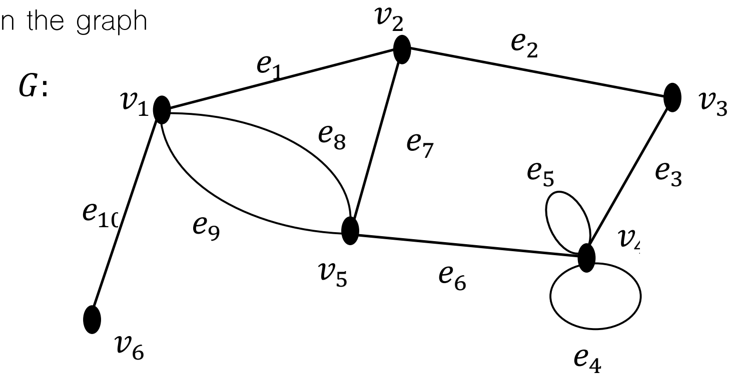
$$v_{i0}, e_{j1}, e_{i1}, e_{j2}, \dots, e_{jk}, v_{ik},$$

which consists of alternating vertices and edges of G. The walk starts at a vertex. Vertices v_{it-1} and v_{it} are end vertices of $e_{jt}, (t = 1, \dots, k)$. v_{i0} is the *initial vertex* and v_{ik} is the *terminal vertex*. k is the *length* of the walk. A zero length walk is just a single vertex

v_{i0} . It is allowed to visit a vertex or go through an edge more than once. A walk is *open* if $v_{i0} \neq v_{ik}$. Otherwise it is *closed*.

Example 5.10

In the graph



the walk

$$v_2, e_7, v_5, e_8, v_1, e_8, v_5, e_6, v_4, e_5, v_4, e_5, v_4$$

is open. On the other hand, the walk

$$v_4, e_5, v_4, e_3, v_3, e_2, v_2, e_7, v_5, e_6, v_4$$

is closed.

A walk is a *trail* if any edge is traversed at most once. Then, the number of times that the vertex pair u, v can appear as consecutive vertices in a trail is at most the number of parallel edges connecting u and v.

Example 5.11

(Continuing from the previous example) The walk in the graph

$$v_1, e_8, v_5, e_9, v_1, e_1, v_2, e_7, v_5, e_6, v_4, e_5, v_4, e_4, v_4$$

is a trail.

A trail is a *path* if any vertex is visited at most once except possibly the initial and terminal vertices when they are the same. A closed path is a *circuit*. For simplicity, we will assume in the future that a circuit is not empty, i.e. its length ≥ 1. We identify the paths and circuits with the subgraphs induced by their edges.

Example 5.12

(Continuing from the previous example) The walk

$$v_2, e_7, v_5, e_6, v_4, e_3, v_3$$

is a path and the walk

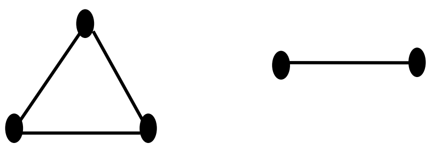
$$v_2, e_7, v_5, e_6, v_4, e_3, v_3, e_2, v_2$$

is a circuit.

The walk starting at u and ending at v is called an $u - v$ *walk*. u and v are *connected* if there is a $u - v$ walk in the graph (then there is also a $u - v$ path!). If u and v are connected and v and w are connected, then u and w are also connected, i.e. if there is a $u - v$ walk and a $v - w$ walk, then there is also a $u - w$ walk. A graph is *connected* if all the vertices are connected to each other. (A trivial graph is connected by convention.)

Example 5.13

The graph



is not connected

The subgraph G_1 (not a null graph) of the graph G is a *component* of G if

1. G_1 is connected and
2. Either G_1 is trivial (one single isolated vertex of G) or G_1 is not trivial and G_1 is the subgraph induced by those edges of G that have one end vertex in G_1.

Different components of the same graph do not have any common vertices because of the following theorem.

Theorem 5.2.

If the graph G has a vertex v that is connected to a vertex of the component G_1 of G, then v is also a vertex of G_1.

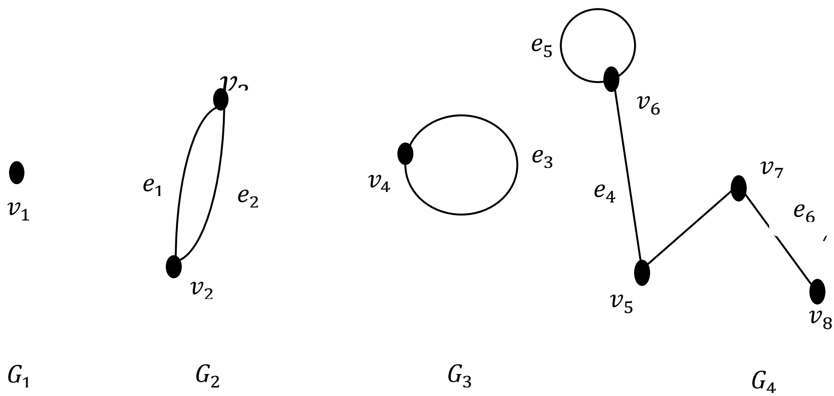
Proof :

If v is connected to vertex v of G_1, then there is a walk in G

$v = v_{i0}, e_{j1}, v_{i1}, \ldots., v_{ik-1}, e_{jk}, v_{ik} = v'$.

Since v' is a vertex of G_1, then (condition #2 above) e_{jk} is an edge of G_1 and v_{ik-1} is a vertex of G_1. We continue this process and see that v is a vertex of G_1.

Example 5.14



Theorem 5.3

Every vertex of G belongs to exactly one component of G. Similarly, every edge of G belongs to exactly one component of G.

Proof :

We choose a vertex v in G. We do the following as many times as possible starting with $V_1 = \{v\}$ =

($*$) If v' is a vertex of G such that $v' \notin V_1$ and v' is connected to some vertex of V_1, then

$$V_1 \leftarrow V_1 \cup \{v'\}$$

Since there is a finite number of vertices in G, the process stops eventually. The last V_1 induces a subgraph G_1 of G that is the component of G containing v. G_1 is connected because its vertices are connected to v so they are also connected to each other. Condition #2 holds because we can not repeat ($*$). By Theorem 5.2, v does not belong to any other component. The edges of the graph are incident to the end vertices of the components.

Theorem 5.3 divides a graph into distinct components. The proof of the theorem gives an algorithm to do that. We have to repeat what we did in the proof as long as we have free vertices that do not belong to any component. Every isolated vertex forms its own component. A connected graph has only one component, namely, itself.

A graph G with n vertices, m edges and k components has the *rank*

$$\rho(G) = n - k$$

The *nullity* of the graph is

$$\mu(G) = m - n + k$$

We see that $\rho(G) \geq 0$ and $\rho(G) + \mu(G) = m$. In addition, $\mu(G) \geq 0$ because

Theorem 5.4

$$\rho(G) \leq m$$

Proof :

We will use the principle of strong of induction for m.

1. Basis step: $m = 0$ The components are trivial and $n = k$.

We make Induction Hypothesis: The theorem is true for $m < p.(p \geq 1)$

2. Inductive step: the theorem is true for m = p.

   Proof: We choose a component G_1 of G which has at least one edge.

   We label that edge e and the end vertices u and v. We also label G_2 as the subgraph of G and G_1, obtained by removing the edge e from G_1 (but not the vertices u and v). We label G' as the graph obtained by removing the edge e from$G$ (but not the vertices u and v) and let k'be the number of components of G'. We have two cases:

a. G_2 is connected. Then, $k' = k$ = k. We use the Induction Hypothesis on G': $n - k = n - k' = \rho(G') \leq m - 1 < m.$

b. G_2 is not connected. Then there is only one path between u and v:

$$u, e, v$$

and no other path. Thus, there are two components in G_2 and $k' = k + 1$. We use the Induction Hypothesis on G':

$$\rho(G') = n - k' = n - k - 1 \leq m - 1$$

Hence $n - k \leq m$

These kinds of combinatorial results have many consequences. For example:

Theorem 5.5

If G is a connected graph and $k \geq 2$ is the maximum path length, then any two paths in G with length k share at least one common vertex.

Proof :

We only consider the case where the paths are not circuits (Other cases can be proven in a similar way.). Consider two paths of G with length k:

$$v_{i_0}, e_{j_1}, v_{i_1}, e_{j_2}, \ldots, e_{j_k}, v_{i_k} \text{ (path } p_1\text{)}$$

and

$$v_{i'_0}, e_{j'_1}, v_{i'_1}, e_{j'_2}, \ldots, e_{j'_k}, v_{i'_k} \text{ (path } p_2\text{).}$$

Let us consider the counter hypothesis: The paths p1 and p2 do not share a common vertex. Since G is connected, there exists an $v_{i_0} - v_{i'_k}$ path. We then find the last vertex on this path which is also on p_1 (at least v_{i_0} is on p_1) and we label that vertex v_{i_t} We find the first vertex of the $v_{i_t} - v_{i'_k}$ path which is also on p_2 (at least $v_{i'_k}$ is on p_2) and we label that vertex $v_{i'_s}$. So we get a $v_{i_t} - v_{i_s}$ path

The situation is as follows:

$$v_{i_0}, e_{j_1}, v_{i_1}, \dots, v_{i_t}, e_{j_{t+1}}, \dots, e_{j_k}, v_{i_k}$$
$$e_{j''_1}$$
$$.$$
$$.$$
$$.$$
$$e_{j''_l}$$
$$v_{i'_0}, e_{j'_1}, v_{i'_1}, e_{j'_2}, \dots, e_{j'_k}, v_{i'_k}$$

From here we get two paths: $v_{i_0} - v_{i'_k}$ path and $v_{i'_0} - v_{i_k}$ path. The two cases are:

- $t \geq s$: Now the length of the $v_{i_0} - v_{i'_k}$ path is more than $k + 1$
- $t \leq s$: Now the length of the $v_{i'_0} - v_{i_k}$ path is more than $k + 1$

A graph is *circuitless* if it does not have any circuit in it.

Theorem 5.6.

If G is a connected graph and $k \geq 2$ is the maximum path length, then any two paths in G with length k share at least one common vertex.

A graph is circuitless exactly when there are no loops and there is at most one path between any two given vertices.

Proof :

First let us assume G is circuitless. Then, there are no loops in G. Let us assume the counter hypothesis: There are two different paths between distinct vertices u and v in G:

$$u = v_{i_0}, e_{j_1}, v_{i_1}, e_{j_2}, \dots, e_{j_k}, v_{i_k} = v \text{ (path } p_1)$$

and

$$u = v_{i'_0}, e_{j'_1}, v_{i'_1}, e_{j'_2}, \dots, e_{j'_k}, v_{i'_k} = v \text{ (path } p_2)$$

(here we have $i_0 = i'_0$ and $i_k = i'_l$ where $k \geq l$. We choose the smallest index t such that $v_{i_t} \neq v_{i'_t}$.There is such a t because otherwise

1. $k > l$ and $v_{i_k} = v = v_{i_l} = v_{i_l}$ or
2. $k = l$ and $v_{i_0} = v_{i'_0}, \dots, v_{i_l} = v_{i'_l}$. Then, there would be two parallel edges between two consecutive vertices in the path. That would imply the existence of a circuit between two vertices in G

We choose the smallest index s such that $s \geq t$ and $v_{i_s}$is in the path p_2 (at least v_{i_k} is in p_2). We choose an index r such that $r \geq t$ and $v_{i'_r} = v_{i_s}$ (it exists because p_1 is a path). Then,

$$v_{i_{t-1}}, e_{j_t}, \dots, e_{j_s}, v_{i_s}(= v_{i'_r}), e_{j'_r}, \dots, e_{j'_t}, v_{i'_{t-1}}(= v_{i'_{t-1}})$$

is a circuit. (verify the case $t = s = r$.)

Let us prove the reverse implication. If the graph does not have any loops and no two distinct vertices have two different paths between them, then there is no circuit. For example, if

$$v_{i_0}, e_{j_1}, v_{i_1}, e_{j_2}, \dots, e_{j_k}, v_{i_k} = v_{i_0}$$

is a circuit, then either $k = 1$ and e_{j_1} is a loop, or $k \geq 2$ and then two vertices v_{i_0} and v_{i_1} are connected by two disctinct paths

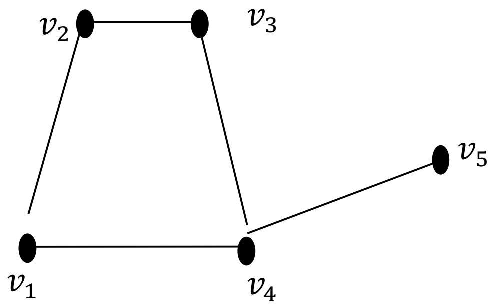
$$v_{i_0}, e_{j_1}, v_{i_1} \text{ and } v_{i_1}, e_{j_2}, \ldots, e_{j_k}, v_{i_k} = v_{i_0}$$

5.3 Graph Operations

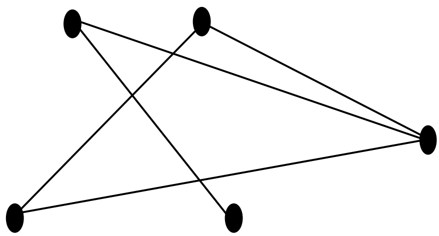
The *complement* of the simple graph $G = (V, E\,)$is the simple graph $\overline{G} = (V, \overline{E}\,)$, where the edges in $\overline{E}$ are exactly the edges not in G.

Example 5.14

G:



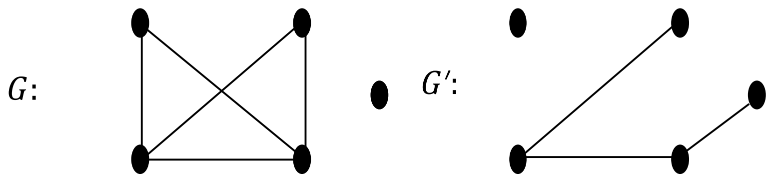
$\overline{G}$:

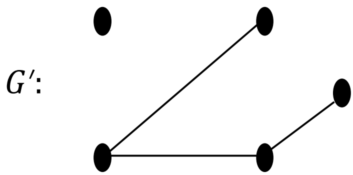


Example 5.15

The complement of the complete graph K_n is the empty graph with n vertices. Obviously, $\overline{\overline{G}} = G$. If the graphs $G = (V, E)$ and $G' = (V', E')$ are simple and $V' \subseteq V$ then the difference graph is $G - G' = (V, E'')$, where E'' contains those edges from G that are not in G'' (simple graph).

Example 5.16





Here are some binary operations between two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$:

- The union is $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$ (simple graph)
- The intersection is $G_1 \cap G_2 = (V_1 \cap V_2, E_1 \cap E_2)$ (simple graph)
- The ring sum $G_1 \oplus G_2$ is the subgraph of $G_1 \cup G_2$ induced by the edge set $E_1 \oplus E_2$ (simple graph), where $\oplus$ is the symmetric difference, i.e.

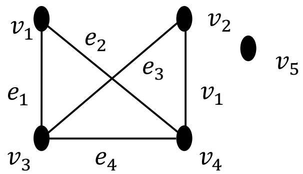
$$E_1 \oplus E_2 = (E_1 - E_2) \cup (E_2 - E_1)$$

Since the ring sum is a subgraph induced by an edge set, there are no isolated vertices. All three operations are commutative and associative.

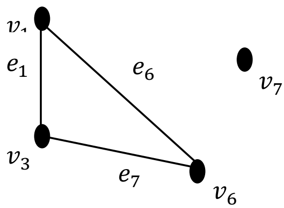
Example 5.17

For the graphs

G_1:

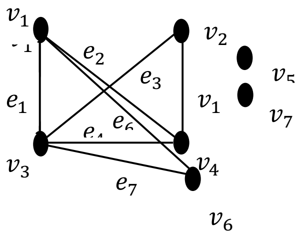


G_2:

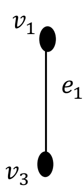


We have

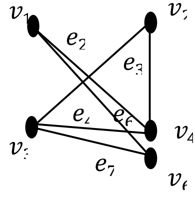
$G_1 \cup G_2$:



$G_1 \cap G_2$:



G_1



The operations $\cup, \cap,$ and $\oplus$ can also be defined for more general graphs other than simple graphs. Naturally, we have to "keep track" of the multiplicity of the edges:

$\cup$: The multiplicity of an edge in $G_1 \cup G_2$ is the larger of its multiplicities in G_1 and G_2.

$\cap$: The multiplicity of an edge in $G_1 \cap G_2$ is the smaller of its multiplicities in G_1 and G_2.

$\oplus$: The multiplicity of an edge in $G_1 \oplus G_2$ is $|m_1 - m_2|$, where m_1 is its multiplicity in G_1 and m_2 is its multiplicity in G_2 .

(We assume zero multiplicity for the absence of an edge.) In addition, we can generalize the difference operation for all kinds of graphs if we take account of the multiplicity. The multiplicity of the edge e in the difference $G - G'$ is

$m_1 - m_2 = \begin{cases} m_1 - m_2, \text{if } m_1 \geq m_2 \\ 0, \text{if } m_1 < m_2 \end{cases}$ (also known as the proper difference),

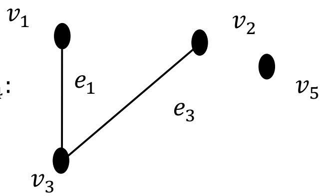
where m_1 and m_2 are the multiplicities of e in G_1 and G_2, respectively.

If v is a vertex of the graph $G = (V, E)$, then $G - v$ is the subgraph of G induced by the vertex set $V - \{v\}$. We call this operation the *removal of a vertex*.

Example 5.18

(Continuing from the previous example)

$G_1 - v_4$:

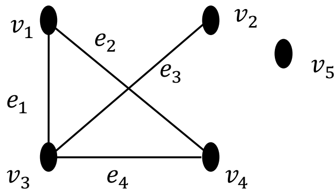


Similarly, if e is an edge of the graph $G = (V, E)$, then $G - e$ is graph (V, E'), where E' is obtained by removing e from E. This operation is known as *removal of an edge*. We remark that we are not talking about removing an edge as in Set Theory, because the edge can have nonunit multiplicity and we only remove the edge once.

Example 5.19

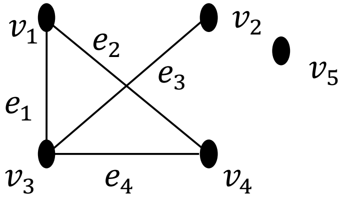
(Continuing from the previous example)

$G_1 - e_5$:



(Continuing from the previous example)

$G_1 - e_5$:



If u and v are two distinct vertices of the graph $G = (V, E)$, then we can *short circuit* the two vertices u and v and obtain the graph(V', E'), where

$V' = (V - \{u, v\}) \cup \{w\}$ $\quad (w \notin V$is the "new" vertex)

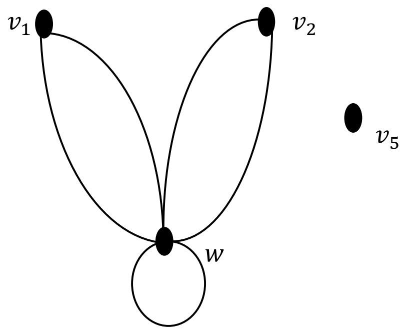
and

$$E' = (E - \{(v', u), (v', v) | v' \in V\})$$
$$\cup \{(v', w) | (v'u \in E \text{ or } (v', v) \in E\}$$
$$\cup \{(w, w) | (u, u) \in E \text{ or } (v, v) \in E\}$$

(Recall that the pair of vertices corresponding to an edge is not ordered). *Note!* We have to maintain the multiplicity of the edges. In particular, the edge (u, v) becomes a loop.

Example 5.20

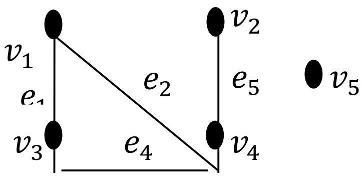
(Continuing from the previous example) Short-circuit v_3 and v_4 in the graph G_1:

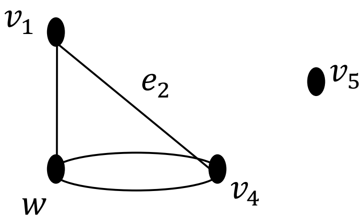


In the graph $G = (V, E)$, *contracting* the edge $e = (u, v)$ (not a loop) means the operation in which we first remove e and then short-circuit u and v. (Contracting a loop simply removes that loop.)

Example 5.21

(Continuing from the previous example) We contract the edge e_3 in G_1 by first removing e_3 and then short-circuiting v_2 and v_3.





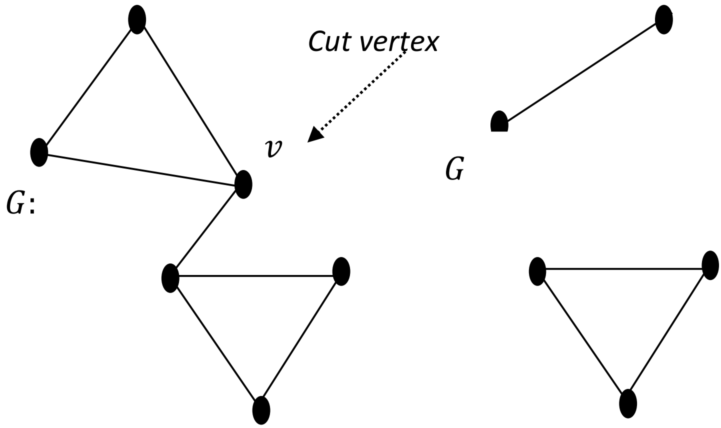
Remark. If we restrict short-circuiting and contracting to simple graphs, then we remove loops and all but one of the parallel edges between end vertices from the results.

5.4 Cuts

A vertex v of a graph G is a *cut vertex* or an *articulation vertex* of G if the graph $G - v$ consists of a greater number of components than G.

Example 5.22

v is a cut vertex of the graph below:

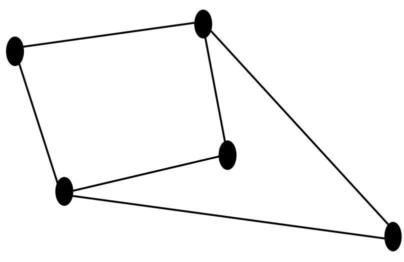


(*Note!* Generally, the only vertex of a trivial graph is not a cut vertex, neither is an isolated vertex.)

A graph is *separable* if it is not connected or if there exists at least one cut vertex in the graph. Otherwise, the graph is *nonseparable*. For example, *The graph* $\boldsymbol{G}$ *in the previous example is separable.*

Example 5.23

The graph below is nonseparable.

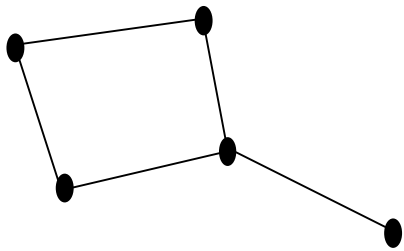


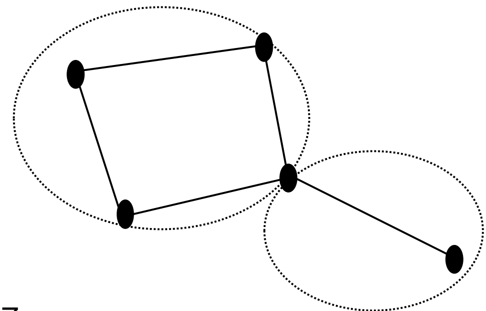
A *block* of the graph G is a subgraph G_1 of G (not a null graph) such that

- G_1 is nonseparable, and
- if G_2 is any other subgraph of G, then $G_1 \cup G_2 = G_1$ or $G_1 \cup G_2$ is separable (think about that!).

Example 5.24

The graph below is separable:





Theorem 5.7

The vertex v is a cut vertex of the connected graph G if and only if there exist two vertices u and w in the graph G such that

i. $v \neq u, v \neq w$ and $u \neq w$, but
ii. v is on every $u - w$ path.

Proof:

First, let us consider the case that v is a cut-vertex of G. Then, $G - v$ is not connected and there are at least two components $G_1 = (V_1, E_1)$. We choose $u \in V_1$ and $w \in V_2$. The $u - w$ path is in G because it is connected. If v is not on this path, then the path is also in $G - v$. The same reasoning can be used for all the $u - w$ paths in G. If v is in every $u - w$ path, then the vertices u and w are not connected in $G - v$.

Theorem 5.8

A nontrivial simple graph has at least two vertices which are not cut vertices.

Proof:

We will use induction for the graph G with n vertices.

1. Basis step: The case n = 2 is obviously true.

   We make Induction Hypothesis: The theorem is true for $n \leq k. (k \geq 2)$
2. Inductive Step: The theorem is true for $n = k + 1$.

Proof: If there are no cut vertices in G, then it is obvious. Otherwise, we consider a cut vertex v of G. Let G_1, . . . , G_m be the components of $G - v$ $(\text{so } m \geq 2)$. Every component G_i falls into one of the two cases:

i. G_i is trivial so the only vertex of G_i is a pendant vertex or an isolated vertex of G but it is not a cut vertex of G.

ii. G_i is not trivial. The Induction Hypothesis tells us that there exist two vertices u and w

in G_i which are not cut vertices of G_i. If v and u (respectively v and w) are not adjacent in G, then u (respectively w) is not a cut vertex in G. If both v and u as well as u and w are adjacent in G, then u and w can not be cut vertices of G.

A *cut set* of the connected graph $G = (V, E)$ is an edge set $F \subseteq E$ such that

1. $G - F$ (remove the edges of F one by one) is not connected, and
2. $G - H$ is connected whenever $H \subset F$.

Theorem 5.9.

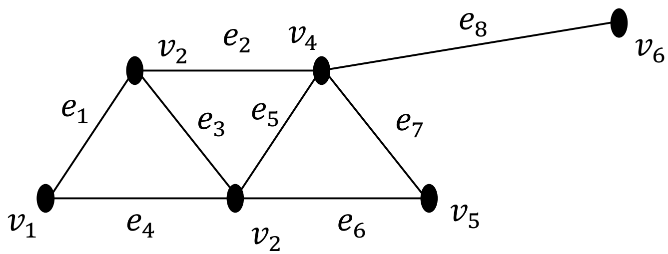
If F is a cut set of the connected graph G, then $G - F$ has two components.

Proof.

Let F = $\{e_1, \ldots, e_k\}$. The graph $G - \{e_1, \ldots, e_{k-1}\}$ is connected (and so is G if $k = 1$) by condition #2. When we remove the edges from the connected graph, we get at most two components.

Example 5.25

In the graph



$\{e_1, e_4\}$, $\{e_6, e_7\}$, $\{e_1, e_2, e_3\}$, $\{e_8\}$, $\{e_3, e_4, e_5, e_6\}$, $\{e_2, e_5, e_7\}, \{e_2, e_5, e_6\}$ and $\{e_2, e_3, e_4\}$ are cut sets. Are there other cut sets?

In a graph $G = (V, E)$, a pair of subsets V_1 and V_2 of V satisfying

$$V = V_1 \cup V_2,\ V_1 \cap V_2 = \emptyset,\ V_1 \neq \emptyset,\ V_2 \neq \emptyset$$

is called a *cut* (or a *partition*) of G, denoted $\langle V_1, V_2 \rangle$,. Usually, the cuts $\langle V_1, V_2 \rangle$ and $\langle V_2, V_1 \rangle$ are considered to be the same.

Example 5.26

(Continuing from the previous example) $\langle \{v_1, v_2, v_3\} \rangle$, $\langle \{v_4, v_5, v_6\} \rangle$ is a cut.

We can also think of a cut as an edge set:

cut $\langle V_1, V_2 \rangle$= {those edges with one end vertex in V_1 and the other end vertex in V_2}.

(*Note!* This edge set does not define V_1 and V_2 uniquely so we can not use this for the definition of a cut.)

Using the previous definitions and concepts, we can easily prove the following:

1. The cut $\langle V_1, V_2 \rangle$, of a connected graph$G$ (considered as an edge set) is a cut set if and only if the subgraphs induced

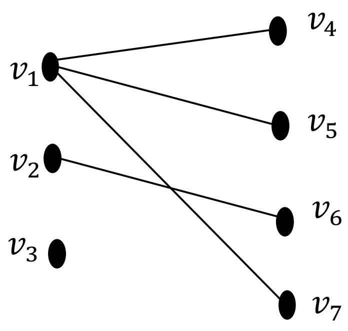
by V_1 and V_2 are connected, i.e. $G - \langle V_1, V_2 \rangle$, has two components.

2. If F is a cut set of the connected graph G and V_1 and V_2 are the vertex sets of the two components of $G - F$, then $\langle V_1, V_2 \rangle$, is a cut and $F = \langle V_1, V_2 \rangle$,
3. If v is a vertex of a connected (nontrivial) graph $G = (V, E)$, then $\langle \{v\}, V - \{v\} \rangle$ is a cut of G. It follows that the cut is a cut set if the subgraph (i.e. $G - v$v) induced by $V - \{v\}$ is connected, i.e. if v is *not* a cut vertex.

If there exists a cut $\langle V_1, V_2 \rangle$ for the graph $G = (V, E)$ so that $E = \langle V_1, V_2 \rangle$ i.e. the cut (considered as an edge set) includes every edge, then the graph G is *bipartite*.

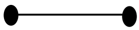
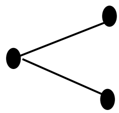
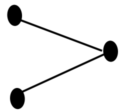
Example 5.27

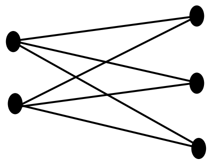
The graph



is *bipartite*. $V_1 = \{v_1, v_2, v_3\}$, and $V_{12} = \{v_4, v_5, v_6, v_7\}$, A simple bipartite graph is called a *complete bipartite graph* if we can not possibly add any more edges to the edge set $\langle V_1, V_2 \rangle$, i.e. the graph contains exactly all edges that have one end vertex in V_1 and the other end vertex in V_2. If there are n vertices in V_1 and m vertices in V_2, we denote it as $K_{n,m}$ (cf. complete graph).

Example 5.28

$K_{1,1}$  $K_{1,2}$  $K_{2,1}$ 

$K_{2,3}$ 

(Usually, $K_{n,m}$ and $K_{m,n}$ are considered to be the same)

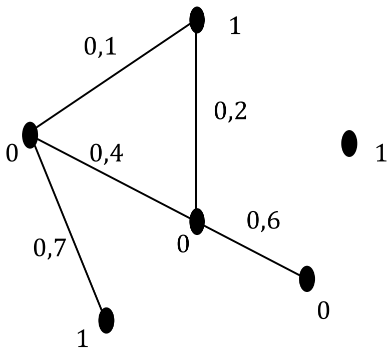
5.5 Labeled Graphs and Isomorphism

By a *labeling of the vertices* of the graph $G = (V, E)$, we mean a mapping $\alpha: V \to A,$ where A is called the *label* set. Similarly, a *labeling of the edges* is a mapping $\beta: E \to B$, where B is the label set. Often, these labels are numbers. Then, we call them *weights* of vertices and edges. In a weighted graph, the weight of a path is the sum of the weights of the edges traversed.

The labeling of the vertices (respectively edges) is *injective* if distinct vertices (respectively edges) have distinct labels. An injective labeling is *bijective* if there are as many labels in α (respectively in B) as the number of vertices (respectively edges)

Example 5.29

If $A = \{0,1\}$ and $B = \mathbb{R},$ then in the graph,



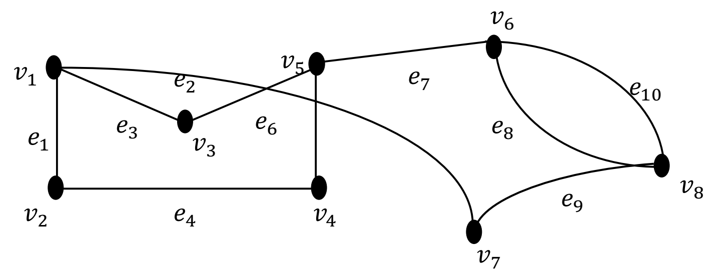
the labeling of the edges (weights) is injective but not the labeling of the vertices.

The two graphs $G_1 = (V_1, E_1)$ and$G_2 = (V_2, E_2)$ are *isomorphic* if labeling the vertices of G_1 bijectively with the elements of V_2 gives G_2. (*Note!* We have to maintain the multiplicity of the edges.)

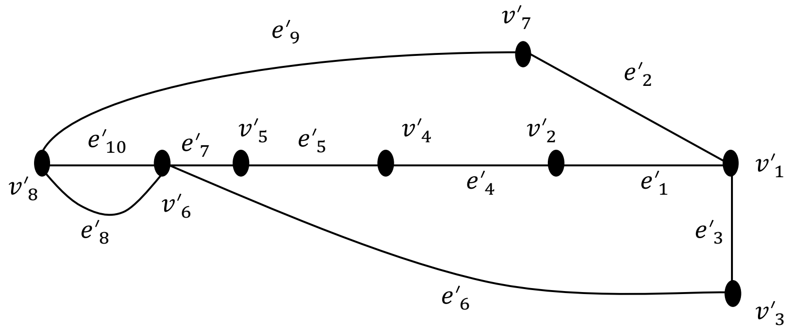
Example 5.30

The graphs G_1 and $G_2$are isomorphic and the vertex labeling $v_i \longmapsto v'_i$ and edge labeling $e_j \longmapsto e'_j$ define the isomorphism.

G_1:



G_2:



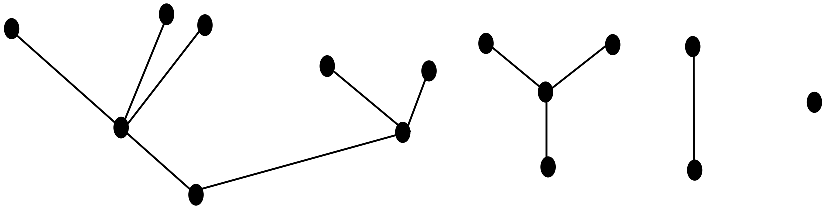
5.6. Trees

5.6.1. Trees and Forests

A *forest* is a circuitless graph. A *tree* is a connected forest. A *subforest* is a subgraph of a forest.A connected subgraph of a tree is a *subtree*. Generally speaking, a subforest (respectively subtree) of a graph is its subgraph, which is also a forest (respectively tree).

Example 5.31

Four trees which together form a forest:



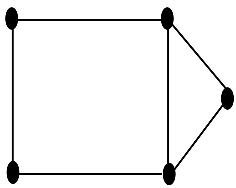
A *spanning tree* of a connected graph is a subtree that includes all the vertices of that graph. If T is a spanning tree of the graph G, then

$$G - T =_{\mathrm{def.}} T^*$$

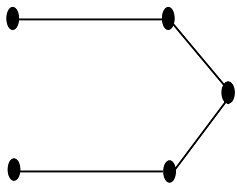
is the *cospanning tree*.

Example 5.32

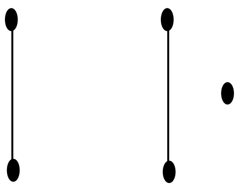
G:



$spanning\ tree$:



$cospanning\ tree$:



If the graph G *has* n *vertices and* m *edges, then the following statements are equivalent:*

i) G is a tree.

ii) There is exactly one path between any two vertices in G and G has no loops.

iii) G is connected and $m = n - 1$

iv) G is circuitless and $m = n - 1$

v) G is circuitless and if we add any new edge to G, then we will get one and only one circuit.

Proof :

$i) \Rightarrow ii)$ if G is a tree, then it is connected and circuitless. Thus, there are no loops in G. There exists a path between any two vertices of G. By Theorem 5.6, we know that there is only one such path.

$ii) \Rightarrow iii)$: G is connected. Let us use induction on m.

1. Basis Step: $m = 0$, G is trivial and the statement is obvious. We set Inductive Hypothesis: $m = n - 1$ when $m \leq l, (l \geq 0)$.
2. Inductive step: $m = n - 1$ when $m = l + 1$.

Proof: Let e be an edge in G. Then $G - e$ has l edges. If $G - e$ is connected, then there exist two different paths between the end vertices of e so (ii) is false. Therefore, $G - e$ has two components G_1 and G_2. Let there be n_1 vertices and m_1 edges in G_1. Similarly, let there be n_2 vertices and m_2 vertices in G_2. Then, $n = n_1 + n_2$ and $m = m_1 + m_2 + 1$.

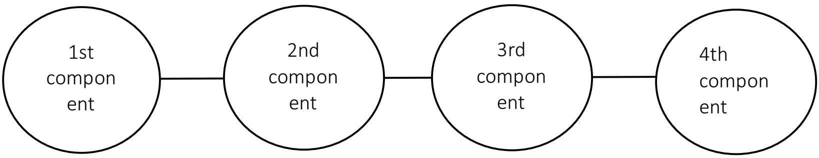
The Inductive Hypothesis states that

$$m_1 = n_1 - 1 \text{ and } m_2 = n_2 - 1,$$

so $m = n_1 + n_2 - 1 = n - 1$.

$iii) \Rightarrow iv)$: consider the counter hypothesis: There is a circuit in G. Let e be some edge in that circuit. Thus, there are n vertices and $n-2$ edges in the connected graph $G - e$.

$iv) \Rightarrow v)$: If G is circuitless, then there is at most one path between any two vertices (Theorem 5.6). If G has more than one component, then we will not get a circuit when we draw an edge between two different components. By adding edges, we can connect components without creating circuits:



If we add $k \geq 1$ edges, then (because $i \Rightarrow iii$)

$m + k = n - 1$ (because $m = n - 1$).

So G is connected. When we add an edge between vertices that are not adjacent, we get only one circuit. Otherwise, we can remove an edge from one circuit so that other circuits will not be affected and the graph stays connected, in contradiction to $iii \Rightarrow iv$. Similarly, if we add a parallel edge or a loop, we get exactly one circuit.

$v) \Rightarrow i)$: Consider the counter hypothesis: G is not a tree, i.e. it is not connected. When we add edges as we did previously, we do not create any circuits (see figure).

Since spanning trees are trees, Theorem 5.10 is also true for spanning trees.

Theorem 5.11.

A connected graph has at least one spanning tree.

Proof. Consider the connected graph G with n vertices and m edges. If $m = n - 1$, then G is a tree. Since G is connected, $m \geq n - 1$ (Theorem 5.4). We still have to consider the case $m \geq n$, where there is a circuit in G. We remove an edge e from that circuit. $G - e$ is now connected. We repeat until there are $n - 1$ edges. Then, we are left with a tree.

Remark. *We can get a spanning tree of a connected graph by starting from an arbitrary subforest M (as we did previously). Since there is no circuit whose edges are all in M, we can remove those edges from the circuit which are not in M.*

By Theorem 5.10, the subgraph G_1 of G with n vertices is a spanning tree of G (thus G is connected) if any three of the following four conditions hold:

1. G_1 has n vertices.
2. G_1 is connected.
3. G_1 has $n - 1$ edges.
4. G_1 is circuitless.

Actually, conditions #3 and #4 are enough to guarantee that G_1 is a spanning tree. If conditions #3 and #4 hold but G_1 is not connected, then the components of G_1 are trees and the number of edges in G_1 would be:

number of vertices - number of components < $n-1$

Theorem 5.12.

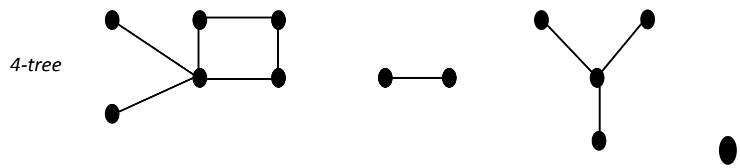
If a tree is not trivial, then there are at least two pendant vertices.

Proof :

If a tree has $n \geq 2$ vertices, then the sum of the degrees is $2(n-1)$. If every vertex has a degree ≥ 2, then the sum will be $\geq 2n$. On the other hand, if all but one vertex have degree ≥ 2, then the sum would be $\geq 1 + 2(n-1) = 2n-1$. (This also follows from Theorem 5.8 because a cut vertex of a tree is not a pendant vertex!)

A forest with k components is sometimes called a k *-tree*. (So a 1-tree is a tree.)

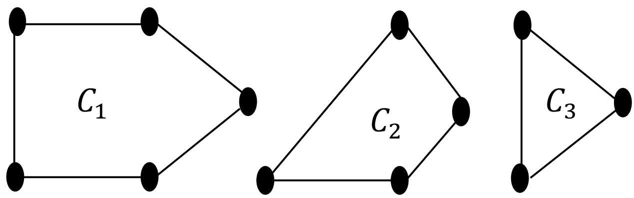
Example 5.33



5.6.2. (Fundamental) Circuits and (Fundamental) Cut Sets

If the branches of the spanning tree T of a connected graph G are b_1, . . . , $b_{n-1}$and the corresponding links of the cospanning tree T$*$ are c_1, . . . , c_{m-n+1}, then there exists one and only one circuit C_i in $T + c_i$ (which is the subgraph of G induced by the branches of T and c_i) (Theorem 2.1). We call this circuit a *fundamental circuit*. Every spanning tree defines $m - n + 1$ fundamental circuits c_1, . . . , c_{m-n+1}, which together form a *fundamental set of circuits*. Every fundamental circuit has exactly one link which is not in any other fundamental circuit in the fundamental set of circuits. Therefore, we can not write any fundamental circuit as a ring sum of other fundamental circuits in the same set. In other words, the fundamental set of circuits is linearly independent under the ring sum operation.

Example 5.34

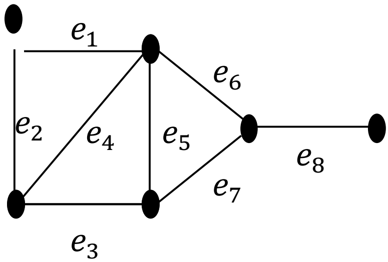


The graph $T - b_i$ has two components T_1 and T_2. The corresponding vertex sets are V_1 and V_2. Then, $\langle V_1, V_2 \rangle$, is a cut of G. It is also a cut set of G if we treat it as an edge set because $G - \langle V_1, V_2 \rangle$ has two components. Thus, every branch b_i of T has a corresponding cut set I_t. The cut sets I_1, . . . , I_{n-1} are also known as *fundamental cut sets* and they form a *fundamental set of cut sets*. Every fundamental cut set includes exactly one branch of T and every branch of T belongs to exactly one fundamental cut set. Therefore, every spanning tree defines a unique fundamental set of cut sets for G.

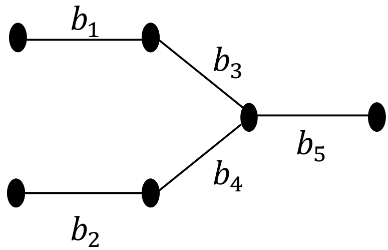
Example. 5.35

(Continuing from the previous example) The graph

G:



has the spanning tree



that defines these fundamental cut sets:

b_1 : $\{e_1, e_2\}$ b_2 : $\{e_2, e_3, e_4\}$ b_3 : $\{e_2, e_4, e_5, e_6\}$

b_4 : $\{e_2, e_4, e_5, e_7\}$ b_5 : $\{e_8\}$

Next, we consider some properties of circuits and cut sets:

(a) Every cut set of a connected graph G includes at least one branch from every spanning tree of G. (Counter hypothesis: Some cut set F of G does not include any branches of a spanning tree T. Then, T is a subgraph of $G - F$ and $G - F$ is connected.

(b) Every circuit of a connected graph G includes at least one link from every cospanning tree of G. (Counter hypothesis: Some circuit C of G does not include any link of a cospanning tree T^*. Then, $T = G - T^*$ has a circuit and T is not a tree.

Theorem 5.12

The edge set F of the connected graph G is a cut set of G if and only if

(i) F includes at least one branch from every spanning tree of G, and

(ii) *if* $H \subset F$, then there is a spanning tree none of whose branches is in H.

Proof :

Let us first consider the case where F is a cut set. Then, (i) is true (previous proposition (a)). If $H \subset F$ then $G - H$ is connected and has a spanning tree T. This T is also a spanning tree of G. Hence, (ii) is true.

Let us next consider the case where both (i) and (ii) are true. Then $G - F$ is disconnected. If $H \subset F$ there is a spanning tree T none of whose branches is in H . Thus T is a subgraph of $G - H$ and $G - H$ is connected. Hence, F is a cut set.

Similarly:

Theorem 5.13

The subgraph C of the connected graph G is a circuit if and only if

(i) C includes at least one link from every cospanning tree of G, and

(ii) if D is a subgraph of C and $D \neq C$, then there exists a cospanning tree none of whose links is in D.

Proof:

Let us first consider the case where C is a circuit. Then, C includes at least one link from every cospanning tree (property (b) above) so (i) is true. If D is a proper subgraph of C, it obviously does not contain circuits, i.e. it is a forest. We can then supplement D so that it is a spanning tree of G ,i.e. some spanning tree T of G includes D and D does not include any link of T∗. Thus, (ii) is true.

Now we consider the case where (i) and (ii) are both true. Then, there has to be at least one circuit in C because C is otherwise a forest and we can supplement it so that it is a spanning tree of G. We take a circuit C' in C. Since (ii) is true, $C' \neq C$ is not true, because C ' is a circuit and it includes a link from every cospanning tree (see property (b) above). Therefore, $C = C'$ is a circuit.

Theorem 5.14

A circuit and a cut set of a connected graph have an even number of common edges

Proof :

We choose a circuit C and a cut set F of the connected graph G. $G - F$ has two components $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. If

C is a subgraph of G_1 or G_2, then the theorem is obvious because they have no common edges. Let us assume that C and F have common edges. We traverse around a circuit by starting at some vertex v of G_1. Since we come back to v, there has to be an even number of edges of the cut $\langle V_1, V_2 \rangle$ in C.

5.7. An Application : Scheduling Serie-A Competition

5.7.1. Introduction

It can be seen in the name, the graph is represented as a diagram graphically to simplify to know its properties. The diagram visualizes points including lines connecting them. A graph can represent something. For example, points represent people and connecting lines represents friendship or love relationship. Moreover, graph can be broadly applied to a wide range of disciplines. It can be found in the field of biochemistry (genomics), electrical engineering (communication network and coding theory), computer science (algorithm and computation) and operations research (scheduling), etc.

In addition graph specifically can be used to set schedule of a full season soccer league competition. Full season competition means that each team has to play both home match and away match against every other team. One of the popular full season-soccer competitions is Italy League Serie-A. The number of teams in Serie A is 20 teams. It is then consequently assumed that each team has a home-stadium to have a home match. In this study, according to regulations of soccer league competition in common, there are several basic conditions which should be maintained. Firstly, a team has to meet all other teams once before having a match against another team for the second time. Such a thing is called half-season competition. Secondly, the schedule should be arranged such that each team has a home-match and away match alternately as frequent as possible. If a team has a home-match or an away match consecutively, then it is called that a team has a *break*. The number of breaks should be minimized as few as possible. Thirdly, in the competition of Serie-A, there are some couples of teams which have the same home-stadium, e.g. Milan and Inter in San Siro/Giuseppe Meazza stadium and Roma and Lazio in Olimpico stadium. Each of these couple of teams should be kept such that the two teams don't play a home-match or an away-match in the same match-daty. For the term, the two teams are called complementary teams. The notable thing from a complementary team is that. Consequently, if Milan has a home-match in a certain match-day, then Inter has an away-match. Therefore, if one of the teams has a break of home-match, for example, then the other team has a break of away-match. Then the challenge is that how to construct Serie-A competition with the most minimum number of breaks and particularly arrange the schedule of the complementary teams such that they don't have the same home match and the same away match in all match-days of the competition

5.7.2. Several Theoretical Definitions

Conceptually, graph is formed by vertices and edges. Formally, a graph is a pair of sets (V, E) where V is the set of vertices and E is the set of edges. Each edge $e = (x, y)$ is an unordered pair of vertices of which x and y are the end points of e. Two vertices x and y in a graph *G* is said to be adjacent each other if they are directly connected by an edge (Yulianti, 2008). In other words, x is adjacent to y if (x, y) is an adge in a graph *G*. An edge e or (x, y) can be directed, for instance, from x to y. On this case $e = (x \to y)$. For arbitrary $e = (x, y)$, then e is incident to vertices x and y.

The degree $d_G(x)$ of a vertex x in a graph G is the number of edges in G incident to vertex x. Graph G is said d-reguler if $d_G(x) = d$ for every x. The subfamily of F of the edges G of which there is no two edges adjacent is called *matching*. 1-reguler matching is called factor and the partition of the edge family of G into factor is called factorization of G.

Some other important related-graph concepts are simple graph, bipartite graph and complete graph. Simple graph doesn't contain loop which is an edge connecting the same vertex. Bipartite graph is a simple graph which can be partitioned into two sets V_1 and V_2 with the following properties:

1. If $v \in V_1$, then v can only be adjacent to the vertices in V_2
2. If $v \in V_2$, then v can only be adjacent to the vertices inV_1
3. $V_1 \cap V_2 = \emptyset$
4. $V_1 \cup V_2 = V$.

Meanwhile, a graph is said to be a complete graph if each vertex adjacent to all vertices in the graph. The symbol of a complete graph K_n, where n is the number of vertices in the graph. Then, it can be drawn that the degree $d_G(x)$ for every vertex x of a complete graph is the same, that is if G is a complete graph and the number of vertices in G is n, then $d_G(x) = n - 1$, for every x.

In the competition of Italia soccer league, every team should meet evey other team exactly one teame till the half of the season. Then the competition can be represented as a complete graph if every team is represented as a vertex and the match between any two teams is represented as edge. Therefore all of the matches in each match-day including the teams competing is called matching or factor.

Furthermore, the other important concepts are *walk*, *trail*, *path*, and *cycle*. A walk of a graph *G* is a non-null finite sequence $W = v_0, e_1, v_1, e_2, \dots, v_{n-1}, e_n, v_n$ whose terms are of vertices and edges of which each vertex and each edge alternate such that v_{i-1} and v_i is connected by e_i where $1 \leq i \leq n$. If each edge in walk *W* is transversed at most once, then the walk *W* is said a trail. A walk W is a path if for any two vertices v_i and v_j in the sequence W satisfies $v_i \neq v_j$. Meanwhile, cycle is a walk which starts and ends at the same vertex.

5.7.3. Some Important Concepts and Theorems

In this section, it will be explained several important concepts related to a tournament construction and some theorems functioning to prove the feasibility of the concepts to use in the soccer league schedule.

Theorem 5.15. There will be breaks in a full-season soccer league with home-away system consisting of $2n$ teams, $n > 1$, $n \in N$

Proof :

Let the elements of a set $H = \{1,2,3, \dots, n\}$ be the teams beginning the competition as home-teams. Conversely, each of the other teams in a set $A = \{n+1, n+2, n+3, \dots., 2n\}$ begins the competition as away-teams. If every team is represented as a vertex, then there are $2n$ vertices in a graph G. Then, matching can be formed from a bipartite graph G which is partitioned into two sets K for home teams and T for away teams. To avoid break, for match-day p, where *p* is odd, all of the teams in H are in the set K meanwhile those of A are in the set T. On the other hand, for match-day *p*, where *p* is even, all teams in H is in the set T and those of A are in the set K. Since $d_G(x)$ for each vertex x in the complete bipartite graph is n then this system can only be maintained till the *n*-th match-day since till that match-day. In other words, each of the team in H has had a match against all teams in A till the *n*-th match-day. In the *n+1*-th match-day, each of teams both in H and in A will have a match against a team in H and in A respectively. Without loss of generality, let in the *n*-th match-day, the teams in H has a home-match. Then in the *n+1*-th match-day, there will be $\frac{n}{2}$ teams of *H* having match against the other $\frac{n}{2}$ teams in their own home-stadium. It then implies that breaks occur in the soccer league.

One of the competitions related to the proof of the theorem 2.1. is the group phase of Europe Champion League of which there are four teams in every group and the breaks start in the third match-day of game.

Then, consider a graph G = K_{2n} and define the schedule with oriented coloring $(\overrightarrow{F_1}, \overrightarrow{F_2}, \dots, \overrightarrow{F_{2n-1}})$ where each $\overrightarrow{F_p}$ is a factor such that if edge (j, k) meaning that team j has a match against team k in the home-stadium of the team k is in $\overrightarrow{F_p}$ then the edge can be notated as $(j \to k)$ which is on the *p*-th match-day. In a schedule S, pattern of home-away = $H(S)$ is defined by :

$h_{jp}(S) = \begin{cases} K \\ T \\ - \end{cases}$ if team j has a match $\begin{cases} \text{Home} \\ \text{Away} \\ - \end{cases}$, in the *p*-th match-day

For example, for the first half season of a competition consisting of four teams, the oriented coloring for the competition is as follows : $\overrightarrow{F_1}$: $\overrightarrow{41}$ $\overrightarrow{32}$, $\overrightarrow{F_2}$: $\overrightarrow{24}$ $\overrightarrow{13}$, $\overrightarrow{F_3}$: $\overrightarrow{21}$ $\overrightarrow{43}$

From the oriented coloring, the pattern of home-away can be shown in the table 5.1. :

Table 5.1. Home-away pattern for each factor $\overrightarrow{F_i}$

Team/Match-day	1	2	3
1	K	T	K
2	K	T	T
3	T	K	K
4	T	K	T

The profile of the *i*-th team is the *i*-th row of $H(S)$. The following theorem describes the properties of the competition which has the minimum number of breaks.

Lemma 5.1.

There are at most $\alpha(G)$ vertices which have the same profile.

Proof :

Suppose there is a set T, $|T| > \alpha(G)$ with the elements i.e. vertices with the same profile. It implies that in T there are at least two vertices i, j adjacent in G. Since (i, j) can be oriented from i to j or from j to i, there exist day k where $h_{ik}(S) \neq h_{jk}(S)$ which is a contradiction. Then $|T| \leq \alpha(G)$.

In addition, it can be concluded from lemma 3.1. that there are at most $2\alpha(G)$ teams which have the same profile without any break. Specifically, there are as many as $\alpha(G)$ teams which have a profile without any break beginning the competition with a home-match and there are as many as $\alpha(G)$ teams which have a profile without any break beginning the competition with an away-match.

Theorem 5.16. Let G is *d*-reguler graph with $2n$ vertices and $\alpha(G)$ is the maximum size of a set of independent vertex in G, then there are at least $2(n-\alpha(G))$ in the oriented coloring $(\overrightarrow{F_1},\overrightarrow{F_2},\ldots,\overrightarrow{F_d})$ of G.

Proof :

From the proof of lemma 3.1, it can be concluded that there are at least $2\big(n-\alpha(G)\big)$ teams with the profile at least 1 break, therefore the number of the minimum breaks are $2\big(n-\alpha(G)\big)$.

Corollari 5.1. The oriented coloring $(\overrightarrow{F_1},\overrightarrow{F_2},\ldots,\overrightarrow{F_{2n-1}})$ of K_{2n} has at least $2n-2$ breaks.

Proof :

The maximum size for the independent set of vertices of K_{2n} is 1 from the theorem 3.1., so the number of the mimimum breaks is $2(n-1)=2n-2$.

The schedule corresponding to factorization with oriented coloring $(\overrightarrow{F_1},\overrightarrow{F_2},\ldots,\overrightarrow{F_d})$ means that each team plays exactly one match in every match-day for d match-days. In other words, for the entire match-days, there is no team which has no match. Werra (1988) states that the schedule is compact. In a compact schedule, if there is a team having a home-match in the *k*-th match-day and in the *k*+1-th match-day, then there is another team with an away match in those consecutive weks. Besides that, the compact schedule of the competition with $2n$ teams has n couples of complementary teams (Werra, 1988). The example of a compact schedule can be seen in the table 2.1. of which there are two couples of complement teams i.e. team 1 with team 4 and team 2 with team 3.

5.7.4. Kirkman Tournament Construction

Kirkman tournament construction is a scheduling method firstly introduced by Reverend T.P. Kirkman in 1846. There are several kinds of constructions of Kirkman, however, the kind of construction used in this study concerns the competition which will be resulted. In other words, the construction is appropriate to a soccer league with $2n$ teams with full season system. In relation to graph, all matches and teams are represented as a complete graph with $2n$ vertices. It can then be factorized into some matchings resulting the oriented coloring $\overrightarrow{F_1}, \overrightarrow{F_2}, \dots, \overrightarrow{F_{2n-1}}$. Therefore, the schedule covers half-season of a competition. The following procedure of constructing the schedule is based on the review of Froncek (2010).

To begin, we give a label l for each team, where l : $1, 2, 3, \dots, 2n$. The labels can be considered as numbers so it can be mathematically operated. Indeed, every team is represented as a vertex and arbitrary match between two teams is represented as an edge. Then we set a formation for the position from the vertex 1 to the vertex $2n - 1$ consecutively in a circle at a similar distance such that if we connect each edge to the next edge using a segment it will form a regular $2n - 1$-gon. Meanwhile, vertex $2n$ is placed in the center of the circle. We define a match $(k \rightarrow j)$ as a match between team j and team k in the home-stadium of team j. In the first match-day, we form an edge connecting the vertex $2n$ and vertex 1 by setting the team 1 as the home team notated by edge $(2n \rightarrow 1)$. Moreover, the other edges are perpendicular to the edge $(2n \rightarrow 1)$, i.e. the edges respectively are incident to the vertex 2 and $2n-1$, the vertex 3 and the vertex $2n-2$, and so on till the vertex n and the vertex $n+1$. In the matches represented by these edges, teams 2, 3, ..., n play a home-match then it results $\overrightarrow{F_1}$: $(2n \rightarrow 1), \left((2n-1) \rightarrow 2\right), \left((2n-2) \rightarrow 3\right), \ldots, ((n+1) \rightarrow n)$. In the second match-day, the label of the opponent of the team $2n$ is obtained by adding the label of the opponent team in the first match-day by n using mod $(2n-1)$ system. In general, the label of the opponent team of team $2n$ in the $p+1$-th match-day is obtained by adding its opponent team label in the p-th match-day by n using mod $(2n-1)$ system. For any two teams matching in the other matches, each label of any two vertices connected by an edge in the p-th match-day is added by n using mod $(2n-1)$ system for the $p+1$-th match-day. This system has to be set such that $h_{(2n)p}(S) \neq h_{(2n)(p+1)}(S)$. From this condition, the team $n+1$ having an away match in the first match-day has an away match again in the second match-day since its opponent is the team $2n$.

For the illustration of the method using graph, eight teams are taken as the samples. The teams are respectively given labels 1, 2, 3, 4, 5, 6, 7, and 8 (Froncek, 2010). Consecutively, the graphs for the first match-day, the second match-day, and the third match-day are shown in the figure 5.1., figure 5.2., and figure 5.3.

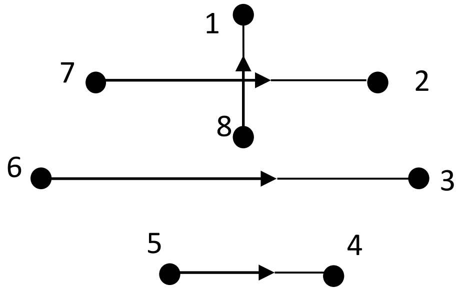


Figure 5.1 Graph representation for the first match-day

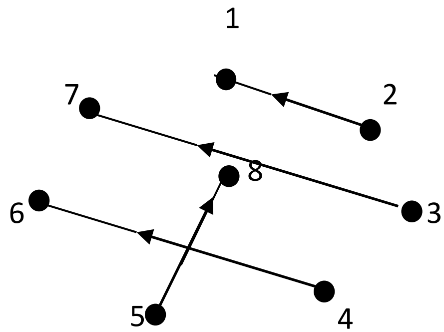


Figure 5.2 Graph representation for the second match-day

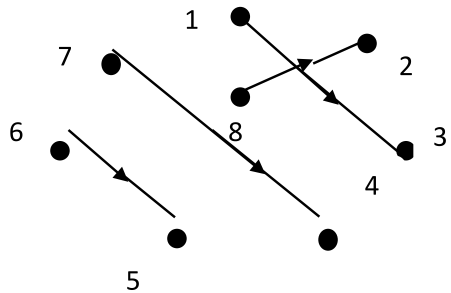


Figure 5.3 Graph representation for the third match-day

In a full season competition with $2n$ teams, the total of match-days is $4n - 2$. Furthermore, there are n matches in every match-day. Therefore, there are totally $4n^2 - 2n$ matches in the competition. If it is counted till the half-season, then there are $2n - 1$ match-days and $2n^2 - n$ matches.

5.7.5. Discussions

Each factor $\overrightarrow{F_p}$ in the competition with $2n$ teams is a graph H in the form of matching with n edges. Let n edges formed in each factor are $e_1, e_2, \dots, e_n$. In the $\overrightarrow{F_i}$, e_1 is the edge incident to vertex $2n$ dengan vertex 1 i.e. $e_1 = ((2n) \rightarrow 1), e_2,$ connect vertex $2n$-1 with vertex 2, and so forth till e_n connecting vertex $n + 1$ and vertex n. Now, define v_{ps} as the team incident to e_s in the match-day *p* which is related to the following theorems.

Theorem 5.17. For arbitrary e_s, where $2 \leq s \leq n$, and for each match-day *p* where $1 \leq p \leq 2n - 1$, there is no two v_{ps} which are alike both as a home-team and as away team in the construction of Kirkman.

Proof :

Specifically, theorem 5.17. means that every team except team $2n$ is incident to an edge e_s twice, $2 \leq s \leq n$, namely once when the team has a home-match and once when it has an away-match.

(i) Without loss of generality, consider in the match-day *p* = 1, the team with a home-match incident to e_s is *n*. As the method of the schedule construction states that in each next match-day, every number/label is added by n using mod $(2n-1)$ system, let a function $\tau(x) \equiv (n+\ xn) \equiv k \bmod (2n-1)$, where $x \geq 1, x \in N$. By mathematics induction it can be shown that for all odd $x \geq 1$ then $\tau(x) = (n+\ xn)\ \equiv \frac{x+1}{2} \bmod (2n-1)$

For the base case $x = 1$, $\tau(1) \equiv 2n \equiv 1 \bmod (2n-1)$ which is true.

If x is odd then $\tau(x)$ can be written as $\tau(2m-1), m \geq 1, m \in N$.

It is assumed that for $x = 2k-1$,

$\tau(2k-1) \equiv (n + (2k-1)n) \equiv \frac{(2k-1)+1}{2} \ \equiv k \bmod(2\mathrm{n}-1)$ is true

For the inductive step, $x = 2(k+1)-1$,

$\tau(2(k+1)-1) \equiv (n + (2k+1)n \equiv \frac{(2k+1)+1}{2} \equiv k + 1 \bmod(2\mathrm{n}-1)$ is true.

In addition, it can be shown that for all even $x \geq 2$, $n + xn \equiv n + \frac{x}{2} \bmod (2n-1)$

For the base case $x = 2$, $\tau(2) \equiv 3n \equiv n + 1 \bmod (2n-1)$ which is true.

For the induction hypothesis, it is assumed that for $x = 2k$,

$\tau(2k) \equiv (n + 2kn) \equiv n(2k+1) \equiv n + k \bmod(2n-1)$ is true.

For the inductive step which is for $x = 2k+2$

$\tau(2k+2) \equiv n + (2k + 2n) \equiv n + k + 1 \ \bmod(2n-1)$ is true.

Then, one can verify that the list of teams with a home match incident to e_s are as follows :

$v_{1s} = n$, $v_{2s} = 1$,

$v_{3s} = n + 1$, $v_{4s} = 2$,

$v_{5s} = n + 2$, $v_{6s} = 3$,

It can be seen in the pattern that for *p* is odd $v_{ps} = n + k, k = 0,1,2, \dots, c$ with bound *c* and for *p* is even, $v_{ps} = m, m = 1,2,3, \dots, d$ with bound *d* which are all different.

Since $v_{2ns} = n$, then $v_{(2n-2)s} = n - 1$ and $v_{(2n-1)s} = 2n - 1$, then $c = d = n - 1$.

(ii) From (i), it can be seen that $v_{(2n-1)s} = 2n - 1$ and $v_{2ns} = n$, then if we form a graph $W = v_{1s}\,\widehat{e_1}\,v_{2s}\,\widehat{e_2}\,v_{3s} \dots v_{(2n-1)s}\,\widehat{e_{2n-1}}\,v_{2ns}$ where $\widehat{e_k}$ connects v_{ks} and $v_{(k+1)s}$ then *W* is a walk in the form of *cycle* since $v_{1s} = v_{2ns} = n$.

(iii) The section (ii) means that for arbitrary team *j* with a home-match for example connected by e_s in the first match-day implies $v_{1s} = j$. Since $v_{1s} = j$ then $v_{2ns} = j$. furthermore, a walk W can be formed, i.e. $W = v_{1s}\,\widehat{e_1}\,v_{2s}\,\widehat{e_2}\,v_{3s} \dots v_{(2n-1)s}\,\widehat{e_{2n-1}}\,v_{2ns}$ where $v_{1s} = v_{2ns} = j$. If $W = (V, E)$, a path then can be set from W by eliminating the elements v_{2ns} and $\widehat{e_{2n-1}}$ from respectively V and E. Therefore, there is no two v_{ps} which are alike where $1 \leq p \leq 2n - 1$ for arbitrary e_s, where $2 \leq s \leq n$.

A path set from vertex v_{ps} both as a home team and away team connected by e_s and connecting edges $\widehat{e_k}$ where $\widehat{e_k}$ incident to v_{ks} and $v_{(k+1)s}$ is said e_s*-generated open path*

Corollary 5.2. In the Kirkman construction, there are different $2n-2$ *generated open path* set from edges e_s, where $2 \leq s \leq n$.

Proof :

There are $n-1$ edges i.e. $e_2, e_3, e_4, \dots, e_n$ and each edge connects one home-team and one away-team. It then implies that there are $2n-2$ different teams in the first match-day connected by $e_2, e_3, e_4, \dots, e_n$. By the theorem 5.17, for arbitrary team in the first match-day, a generated open path can be formed. Since $v_{12} \neq v_{13} \neq \cdots \neq v_{1n}$, there are different $2n-2$ *generated open path* which can be formed.

Theorem 5.18. In the tournament construction of Kirkman for a half-season competition, each team meets another team exactly once. (Froncek, 2010)

Proof :

(i) In the edge e_1, the opponent of the team $2n$ in the first match-day is the team 1. Since the label of the opponent team should be added by n with mod$(2n-1)$ system to determine the opponent of the team $2n$ in each next match-day, then one can verify that from the 1stmatch-day to the 2*n*-1-th match-day, the opponents of the team $2n$ are respectively $1, n+1,\ 2,\ n+2,\ 3,\ n+3, \dots,\ 2n-1,\ n$. Therefore, the team c plays against the other teams exactly once. Besides that, because of the vertices position of the team 2*n*-1's opponents are different on graph, there is no two edges in e_1 e.g. $e_{1x} = (2n, x)$ and $e_{1y} = (2n, y)$ for respectively a match between *2n* and *x* and a match between *2n* and *y* which are parallel or coincidence in the graph construction

(ii) For the other edges besides e_1, if team j has a match against team k in the match-day *x*, then the edge (j, k) is perpendicular to an edge connecting the team $2n$ and another team a i.e. $(2n, a)$. If the team j has a match against team k in another match-day, e.g. in the match-day *y*, then the edge (j, k) is perpendicular to an edge connecting the team $2n$ and another team b i.e. $(2n, b)$. If the edge (j, k) is perpendicular to the edges $(2n, a)$ and $(2n, b)$ then the edge $(2n, a)$ and the edge $(2n, b)$ is parralel or coincide which implies a contradiction. It follows that team j meets team k exactly once.

Theorem 5.19 For $2 \leq s \leq n$, if v_{1s} is a team with a home-match, then the order of the team team *n* in an e_s*-generated open path* is odd, meanwhile if v_{1s} is a team with an away-match, the order of the team *n* in an e_s*-generated open path* is even.

Proof :

In other words, the theorem *5.19.* means that the team *n* plays a home-match and an away- match in the *g*-th match-day where *g* is odd and even respectively is an e_s*-generated open path*. Trivially, in the edge e_n, the team *n* is a team with a home-match in the first match-day. Then the order of the team *n* in the e_n*-generated open path* is odd. For the edges besides e_n, let each team v_{1s} with a home-match be $n - x$, where $1 \leq x \leq n - 2$. It follows then:

$n - x + cn \equiv n \bmod (2n - 1)$, where $c \in N$, meaning that $(2n - 1)|cn - x$. Indeed $cn - x \geq 0$. Since a number $cn - x$ which is more than 0 divisible by $2n - 1$ can be represented in the form of $z(2n - 1),\ z \in N$, then :

$cn - x = z(2n - 1) = 2zn - z$.

It means that $c = 2x$ which implies *c* is an even number. Since *c* is even, then the match-day for $n - x + cn$ using modulo $2n - 1$ which is equal *n* is odd.

Furthermore, for each team v_{1s} with an away-match can be represented as $n + y,$ where $1 \leq y \leq n - 1$. Since $2n - 1 + n \equiv n \bmod (2n - 1)$, then for each *y*, $(n + y + n) \equiv (n - x) \bmod (2n - 1),$ where $0 \leq x \leq n - 1$. It means that each team v_{2s} with an away math is a team with label $n - x \leq n$. Since the vertex of each team $n - x$ occupies the second order in arbitrary e_s-*generated open path*, $2 \leq s \leq n$, then *n* has an away match in the even match-day..

Q.E.D.

The theorem 5.19 also signify that, for the edge e_1 of which v_{1s} with a home-match i.e. $v_{1s} = 1$, the team *n* also plays a home-match in the odd match-day since $(1 + (2n - 2)n) \equiv n \bmod (2n - 1)$ meaning that $1 + 2n^2 - 2n - n = 1 + 2n^2 - 3n$ divisible by $2n - 1$. The following theorem relates to complementary teams.

Theorem 5.20 In the construction of Kirkman tournament, for arbitrary team j with label $j \leq n$, has a complement team $j + n$.

Proof :

(i) Case $j = n$

Based on the description of Kirkman tournament, for the team $2n$ has regular home-away pattern without break. Therefore, in the $2n - 1$-th match-day, the team $2n$ has a home match in each match-day *p*, where *p* is even and an away match in each match-day p where p is odd. Based on the theorem 3.5., the team *n* has a home-match in each match-day *p*, where *p* is odd and an away-match in each match-day *p* where *p* is even. Therefore, the complement team of the team *n* is the team $2n$.

(ii) Case $j < n$, *j* is a home team.

In the 1-st match-day, each team $j < n$ plays a home-match. Conversely, each of its complement $j + n$ plays an away-match. For the other match-days, suppose a team $k < n$ is the initial vertex of e_t-*generated open path.* If $(k + cn) \equiv j \bmod (2n - 1),\ c \in N$ then there exist an e_u-*generated open path* with initial vertex $l \neq k$ such that $(l + cn) \equiv (j + n) \bmod (2n - 1)$. Suppose l is in the range $[1, n]$. Then l can be written as $l = k + m \leq n$ where $m \leq n - k$ implying $m \neq n$. Based on the equivalence form (1), it is equivalent to $(k + m + cn) \equiv l + cn \equiv (j + m) \bmod (2n - 1)$ which is a contradiction since $j + m \neq j + n$. It means that l is not in the range $[1, n]$ implying l is an away team. Therefore, team $j + n$ plays an away match when team *j* plays a home match.

(iii) Case $j < n$, $j + n$ is a home team.

If team j plays an away match in the match-day *p*, $2 \leq p \leq 2n - 1$, then team j is at an e_t-*generated open path* with an initial vertex $k > n$. Therefore, there exist $c \in N$ such that $(k + cn) \equiv j \bmod (2n - 1), c \ \in N$ (2). In the same match-day for team $j + n < 2n$, there exist initial vertex $l \neq k$ of an e_u-*generated open path* such that $(l + cn) \equiv j + n \bmod (2n - 1)$. Suppose l is in the range $(n, 2n)$ then $l = k + m \leq 2n - 1$ where $m \leq 2n - k - 1$ implying that $m < n$. Based on the equivalence form (2), it is equivalent to $(k + m + cn) \equiv l + cn \equiv (j + m) \bmod (2n - 1)$ which is a contradiction since $j + m \neq j + n$. It means that team *l* is a team with a home match. In conclusion, in the same match-day, team $j < n$ plays an away match.

Theorem 5.21. There are $2n - 2$ breaks in the Kirkman tournament construction.

Proof :

(i) Based on the theorem 5.20., in the edge e_1, each team $j \leq n$ plays a home-match against the team $2n$ in the odd match-day i.e. team *j* plays against team $2n$ in the match-day $2j - 1$. It is known that v_{1n} with a home-match and an away match are respectively the team n and the team $n + 1$. Based on the proof of the theorem 3.1. it can be showed that in the match-day $2j$, a team j plays a match against a team $j+1$ in the home-stadium of the team j where $1 \leq j \leq n-1$. The schedules cause $n-1$ breaks. Since team $j+n$ is the complement of team $j \leq n$, then the total of the breaks is $2n-2$. The addition of these $n-1$ breaks is obtained in the edge e_1, exactly in the even match-day where team $j+n<2n$ has a match against the team $2n$ in the home-stadium of the team $2n$ i.e. in the match-day $2j$ because in the previous match-day in the edge e_n, a team $j+n$ plays an away-match in the match-day $2j-1$.

(ii) As previously described, $\overrightarrow{F_1}$ consists of the edges $(2n \rightarrow 1), \left((2n-1) \rightarrow 2\right), \left((2n-2) \rightarrow 3\right), \ldots,$ dan $((n+1) \rightarrow n)$. Thus, for arbitrary edge e_s, v_{1s} with a home-match is team $j \leq n$. Based on the theorem 3.1. part (i), $v_{(2n-1)s}$, for $s=n$ is $2n-1$ meaning

$$(n+2n^2-2n) \equiv (2n-1) \bmod (2n-1)$$

Hence, for $j=n-k$, where $1 \leq k \leq n-1$ it satisfies:

$$(n-k+2n^2-2n) \equiv (2n-1-k) \bmod (2n-1)$$

Therefore, $v_{(2n-1)s}$ with a home match for each e_s, $s=1,2,3,\ldots,n$ are consecutively $n, n+1, n+2, \ldots, 2n-1$. For the edges except the edge e_1, these $v_{(2n-1)s}$ are the team with away match in the first match-day. Let $v_{(2n-1)t}$ be a team with a home-match in an edge e_t. In addition, $v_{(2n-1)t}$ is v_{1u} with an away match in the edge e_u which is likely $u=t$. Since $v_{1t}, \widehat{e_{1t}}, v_{2t}, \widehat{e_{2t}}, v_{3t}, \ldots, e_{\widehat{(2n-2)t}}, v_{(2n-1)t}, e_{\widehat{(2n-1)t}}, v_{1t}$ is a cycle where $\widehat{e_{rt}}$ connects v_{rt} and $v_{(r+1)t}$, it means that v_{1t} is v_{2u} in the edge e_u. Then the form of the e_u-*generated* *open path* where $v_{1u} = v_{(2n-1)t}$ is $v_{(2n-1)t}$, $\widehat{e_{1u}}, v_{1t}\widehat{e_{2u}}, v_{2t}, \dots, \widehat{e_{(2n-2)u}}, v_{(2n-2)t}$. Therefore, each team with a home-match in the edge e_s, $2 \le s \le n$ in a match-day will become a team with an away match in the next match-day. In conclusion, there is no break.

(iii) Team v_{1s} with an away match in an edge e_s, $2 \le s \le n$ is team k, $n+1 \le k \le 2n-1$. One can verify that, for example, for $k = 2n-1$ as v_{1y} then $v_{2y} = n$. Therefore, if $v_{(2n-1)y} = x$, it satisfies:

$$(2n-1 + (2n-2)n) \equiv (n + (2n-3)n) \equiv x \bmod (2n-1)$$

It is known that $v_{1n} = n$ with a home-match and $(v_{1n} + cn) \equiv v_{(1+c)n} \bmod (2n-1)$ Then in $(n + (2n-3)n) \equiv x \bmod (2n-1)$, x is $v_{(1+2n-3)n} = v_{(2n-2)n}$. Based on the proof of the theorem 3.3., $v_{(2n-2)n} = n-1$. Thus, $v_{(2n-1)y} = x = n-1$. It follows that for team $k = 2n-1-l$, where $1 \le l \le n-2$ it satisfies :

$(2n-1-l + (2n-2)n) \equiv (x-l) \bmod (2n-1)$. Therefore each team $v_{(2n-1)s}$ with an away match for $2 \le s \le n$ are $n-1, n-2, n-3, \dots, 3, 2, 1$. These teams play a home-match in the first match-day in edge e_s, $1 \le s \le n-1$. Besides the edge e_1, for example team $v_{(2n-1)z}$ is the team with an away-match in edge e_z. $v_{(2n-1)z}$ is v_{1w} with a home-match in edge e_w which is likely that $z = w$. Since

$$v_{1z}, \widehat{e_{1z}}, v_{2z}, \widehat{e_{2z}}, v_{3z}, \dots, \widehat{e_{(2n-2)z}}, v_{(2n-1)z}, \widehat{e_{(2n-1)z}}, v_{1z}$$

is a cycle where $\widehat{e_{rz}}$ connects v_{rz} and $v_{(r+1)z}$, it implies that v_{1z} is v_{2w} in the edge e_w. Then the form of e_w-*generated open path* with $v_{1w} = v_{(2n-1)z}$ is $v_{(2n-1)z}$, $\widehat{e_{1w}}, v_{1z}\widehat{e_{2w}}, v_{2z}, \dots, e_{\widehat{(2n-2)}w}, v_{(2n-2)z}$. Therefore each team with an away match in edge e_s, $2 \leq s \leq n-1$ in a matchday will become a team with a home-team in the next matchday. Therefore, in this case, no break occurs.

Based on (i), (ii), and (iii) the total of breaks in the Kirkman tournament is $2n-2$.

Application to the Construction of Italia Serie A Soccer League Schedule

Since there are twenty teams competing in the soccer league, in this case n=10, and particularly in the season of 2015/2016, there are five complementary teams, i.e. Milan-Inter, Roma-Lazio, Chievo-Hellas Verona, Sampdoria-Genoa, Torino-Juve then one of the team at each complementary team should be attributed distinct labels $j_i \leq n$, for example $j_1, j_2, j_3, j_4$and j_5 respectively. Conversely each of j_i's complements is automatically attributed with labels $j_i + n$. However, the explanation about Kirkman construction in the previous section covers only the half of the season. Consequently, the second half of the season schedules should be determined. This study proposes two distinct systems i.e. *repetition system* and *two-way around system* to regulate the schedule in the second half of the season.

The *repetition* system means that for $\overrightarrow{F_i}$ where $1 \leq i \leq 2n-1$, arbitray two teams, e.g. team *j* and *k* set an edge $(k \rightarrow j)$, then team *j* and *k* set an edge $(j \rightarrow k)$ in $\overrightarrow{F_{\iota+(2n-1)}}$. This kind of way causes the addition of $2n-2$ breaks. Moreover, since $h_{jp}(S) = h_{j(p+1)}(S) = K$ for *j* : $1, 2, \ldots., n-1$ respectively in the match-day - $2, 4, \ldots., 2n-2$ and $h_{jp}(S) = h_{j(p+1)}(S) = T$ for *j* : $n+1, n+2, \ldots 2n-1$ also respectively in the match-day- $2, 4, \ldots., 2n-2$, then the addition of breaks occurs since $h_{jp}(S) = h_{j(p+1)}(S) = T$ for *j* : $1, 2, \ldots., n-1$ and $h_{jp}(S) = h_{j(p+1)}(S) = K$ for *j* : $n+1, n+2, \ldots 2n-1$ respectively in the match-day $2n+1, 2n+3, \ldots, 4n-3$. Then the total of breaks using this system is $2(2n-2) + 2n-2 = 6n-6$. Specifically, using this system, in the competition of Italia Serie-A soccer league, there are 54 breaks in total.

On the other hand, *two-way around* system means for each $\overrightarrow{F_\iota}$ where $1 \leq i \leq 2n-1$, if team *j* and *k* sets an edge $(k \rightarrow j)$ then in $\overrightarrow{F_{2n+(2n-1-\iota)}}$ they set an edge $(j \rightarrow k)$. By this rule, there is no break in the match-day *2n* since the teams which play a home-match (or away-match) in the match-day- *2n-1* will play an away-match (or a home-match) in the match-day $2n$. Unlike the repetition system, for the second half of the season, the breaks take place only in the match-day $2n+1, 2n+3, \ldots,$ and $4n-3$. Consequently, the total of breaks for this system is $2(2n-2) = 4n-4$ which is less than those of *repetition* system. In specific, the total of breaks occurring in the Italia Serie-A soccer league using this system is $4 \times 10 - 4 = 36$.

Based on the data from http://www.flashscore.com/soccer/italy/serie-a/, the teams competing in the Serie-A soccer league in the season 2015-2016 can be shown in the table 5.2.

Table 5.2. Teams of Serie A 2015-2016

Bologna	Empoli
Carpi	Juventus
Roma	Genoa
Atalanta	Napoli
Fiorentina	Chievo
Verona	Sassuolo
Frosinone	AC Milan
Sampdoria	Palermo
Torino	Udinese
Inter	Lazio

For one of the alternatives, Milan, Roma, Chievo, Genoa, and Juventus are subsequently given labels 1, 2, 3, 4, and 5. Meanwhile Inter, Lazio, Verona, Sampdoria, and Torino are assigned to labels 11, 12, 13, 14, dan 15. Concurrently, each of the other teams are given label besides $1, 2, 3, 4, 5, 11, 12, 13, 14,$ and 15. The example of label distribution for the 20 teams is shown in the table 5.3.

Table 5.3. Example of label distribution for Serie A Italia Soccer League Teams

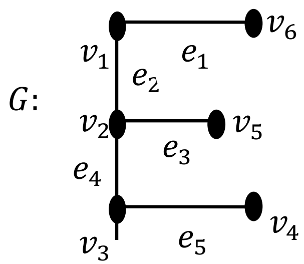
Label	Team	Label	Team
1	Milan	11	Inter
2	Roma	12	Lazio
3	Chievo	13	Verona

4	Genoa	14	Sampdoria
5	Juventus	15	Torino
6	Carpi	16	Napoli
7	Frosinone	17	Sassuolo
8	Atalanta	18	Empoli
9	Fiorentina	19	Udinese
10	Bologna	20	Palermo

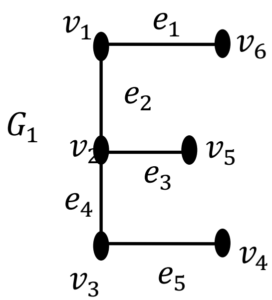
This application suggests that the concept of graph and the method of Kirkman tournament construction can be applied in constructing a compact schedule used for specially the Serie-A Italia Soccer League. Notably, the method creates a schedule of which for each matchday, every team plays exactly one match and each team of the complementary teams never plays a home match and an away match in the same time with its complement. Furthermore, based on the construction, it just results $2n-2$ breaks until the half of the season which is precisely the same as the minimum number of breaks. Therefore, till the end of the entire season, the number of breaks can be optimized up to $4n-4$. The number of the minimum breaks can take place if the scheduling for the second half of the season apply, one of them, *two-way around* system.

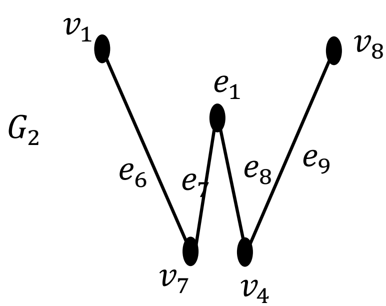
Exercises 5

1. What does it mean for v and w to be adjacent vertices ?
2. What does it mean for e_1 and e_2 to be adjacent edges? give an example!
3. What are parallel edges?
4. What is a loop? Give an example!
5. In the example 5.7, depict several other subgraphs.
6. Give two examples for open walk and closed walk respectively in the following figure!



7. Draw the union and the intersection for the two following graphs:





8. Give an example for each separable graph and nonseparable graph!
9. Give and example of two graphs which is isomorphic to each other
10. Give an example of tree and forest!

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GLOSSARY OF IMPORTANT TOPICS

Circuit : a path that begins and ends at the same vertex.

Combination: an arrangement of objects without considering the order.

Derangement: a kind of permutation of which there is no object in its original position.

Edge: a connection between two vertices.

Forest: a circuitless graph / disjoint union of trees.

Graph: a pair of sets (V,E), where V is the set of vertices and E is the set of edges, formed by pairs of vertices.

Inclusion Exclusion Principle: A technique of counting used to determine the number of elements in the union of any number of sets.

Loop: an edge that links a vertice to itself.

Mathematics Induction: a method of proof consisting of basis step and inductive step used to prove mathematical statements related to series involving natural numbers.

Ordinary Generating Function: a power series in the form of

$$G(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$$

Permutation: an arrangement of objects in a particular order.

Pigeonhole Principle: a principle that states If *n* pigeons fly into *k* pigeonholes and *k*<*n*, some pigeonhole contains at least two pigeons.

Spanning Tree: minimum set of edges that can connect all vertices in a graph.

Strong Mathematics Induction: a kind of mathematics induction of which, when we want to prove for the truth of a particular statement $S(k+1)$ in inductive step, we assume that the particular statment holds at all the steps from the base case to *k*-th step.

Recurrence Relation: an equation that states a term using one or more some previous terms.

Trails : a walk which has edges which are all different.

Tree: a graph in which any two vertices are connected by one simple path.

Walks : an alternating sequence of vertices and edges, beginning and ending with a vertex.

BIOGRAPHY

Dr. Djadir, M.Pd. born in Buton, South-East Sulawesi, on July 10,1956 graduated his bachelor degree in State University of Makassar. He then continued his academic study to obtain his master in Institute of Teacher and Training (IKIP) Malang and doctoral degree in Surabaya State University. As a mathematics lecturer in State University of Makassar, he has many experiences in teaching both in several high schools and universities. Besides that, he has plenty of researches and community services fund by the government. Recently, he is the head of Mathematics Department, State University of Makassar

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