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ASYMPTOTICS FOR RUIN PROBABILITIES IN LÉVY-DRIVEN RISK MODELS WITH HEAVY-TAILED CLAIMS

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Abstract. Consider a bivariate Lévy-driven risk model in which the loss process of an insurance company and the investment return process are two independent Lévy processes. Under the assumptions that the loss process has a Lévy measure of consistent variation and the return process fulfills a certain condition, we investigate the asymptotic behavior of the finite-time ruin probability. Further, we derive two asymptotic formulas for the finite-time and infinite-time ruin probabilities in a single Lévy-driven risk model, in which the loss process is still a Lévy process, whereas the investment return process reduces to a deterministic linear function. In such a special model, we relax the loss process with jumps whose common distribution is long tailed and of dominated variation.

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1. **Introduction.** Consider a bivariate Lévy-driven risk model. In the model, the surplus process $Y_t,\ t>0$, of an insurance company is modelled by the stochastic integral

$$Y_t = x - P_t + \int_0^t Y_{s-}\mathrm{d}\tilde{R}_s,\ t>0, \tag{1}$$

where $Y_0 = x > 0$ represents the initial capital reserve, and $P_t,\ \tilde{R}_t,\ t>0$, are two independent Lévy processes standing for, respectively, the loss process of the insurance business and the return process of the investment. The solution to the stochastic integral (1) is

$$Y_t = \mathrm{e}^{R_t}\left(x - \int_0^t \mathrm{e}^{-R_s}\mathrm{d}P_s\right), \tag{2}$$

where the process $R_t,\ t>0$, called the Doléans-Dade exponential, is the logarithm of the stochastic exponential of $\tilde{R}_t,\ t>0$, and it is also a Lévy process. See, e.g. [24] for more details. If we denote by

$$Z_t = \int_0^t \mathrm{e}^{-R_s}\mathrm{d}P_s \tag{3}$$

the discounted net loss process, then (2) can be simplified to $Y_t = \mathrm{e}^{R_t}(x - Z_t),\ t>0$. For some $0 \le T \le \infty$, introduce

$$M_T = \sup_{0\le t\le T} Z_t,$$

where the supremum is taken over $0 \le t < \infty$ if $T = \infty$. Hence, for the initial capital reserve x, define the finite-time ruin probability within a finite time $T > 0$ and the infinite-time ruin probability, respectively, by

$$\psi(x,T) = \mathbb{P}\Big(\inf_{0\le t\le T} Y_t < 0 \Big| Y_0 = x\Big) = \mathbb{P}(M_T > x),$$

and

$$\psi(x,\infty) = \lim_{T\to\infty}\psi(x,T) = \mathbb{P}\Big(\inf_{0\le t<\infty} Y_t < 0 \Big| Y_0 = x\Big) = \mathbb{P}(M_\infty > x).$$

In this paper, we are interested in the asymptotic behavior of the finite-time and infinite-time ruin probabilities in such a bivariate Lévy-driven risk model. A lot of literature has been devoted to the investigation of the asymptotics for ruin probabilities. Some early works focused on a special case of the bivariate Lévy-driven risk model, in which P_t is a compound Poisson or renewal process and R_t is a deterministic linear function. This means that there exists a constant interest rate and the insurance company invests its wealth only into a risk-free market. See [19], [16], [20], [26], [27], [13], [32] and among others. Further, some dependent risk models have also been considered, where claim sizes are assumed to be a sequence of dependent random variables (r.v.s), see [2], [37], [33] and among others.

Later on, more and more researchers studied some risk models with risky investments by allowing R_t to be a stochastic process but still restricting P_t to be a compound Poisson or renewal process. For example, [23], [8], [17], [22] considered the risk model with a compound Poisson process P_t and a Brownian motion R_t. [18] and [15] obtained some results on ruin probabilities for the case that R_t is a general Lévy process and P_t is a compound Poisson process with regularly varying tailed jumps and long-tailed and dominatedly varying tailed jumps, respectively. [12] further relaxed the compound Poisson process P_t with some dependent and consistently varying tailed jumps. Recently, [29] and [21] established some formulas for finite-time and infinite-time ruin probabilities in the renewal risk models, in which R_t is a general Lévy process and P_t is a compound renewal process with regularly varying (or extendly regularly varying) tailed jumps under the independent and dependent structures, respectively. [36] extended Tang et al.'s results to the dependent case. [35] further studied a non-standard renewal risk model with a càdlàg process R_t and dependent claim sizes, and derived a formula for the finite-time ruin probability.

As for the bivariate Lévy-driven risk model, [10] gave a wealth of examples showing the exact distribution or asymptotic tail probability of Z_∞ defined in (3). An interesting paper [14] derived the asymptotics for the finite-time and infinite-time ruin probabilities in a bivariate Lévy-driven risk model with extendly regularly varying tailed jumps.

Motivated by [14], in the present paper, we aim to investigate the asymptotic behavior for the finite-time ruin probabilities in the bivariate Lévy-driven risk model with consistently varying tailed jumps. Our two other results for the finite-time and infinite-time ruin probabilities are based on a single Lévy-driven risk model, different from the one with risky investment, in which the loss process P_t is still a Lévy process, whereas R_t reduces to a deterministic linear function (corresponding to a constant rate of compound interest). In such a special model, we relax the process P_t with long-tailed and dominatedly varying tailed jumps.

The rest of the paper is organized as follows. In Section 2, after introducing some preliminaries, we present the three main results on the asymptotics for the finite-time and infinite-time ruin probabilities in a bivariate or single Lévy-driven risk model. Sections 3 prepares a series of lemmas and Section 4 gives the proofs of the main results.

2. **Main results.** Throughout the paper, all limit relationships hold for x tending to ∞ unless stated otherwise. For two positive functions $a(x)$ and $b(x)$, we write $a(x) \sim b(x)$ if $\lim a(x)/b(x) = 1$; write $a(x) \prec b(x)$ or $b(x) \succ a(x)$ if $\limsup a(x)/b(x) \le 1$; write $a(x) = o(b(x))$ if $\lim a(x)/b(x) = 0$; write $a(x) = O(b(x))$ if $\limsup a(x)/b(x) < \infty$; and write $a(x) \approx b(x)$ if $0 < \liminf a(x)/b(x) \le \limsup a(x)/b(x) < \infty$. We denote by $\mathbb{1}_A$ the indicator function of an event A. For two real numbers x and y, denote by $x \vee y = \max\{x, y\}$, $x \wedge y = \min\{x, y\}$ and $x^+ = x \vee 0$.

2.1. **A brief review of heavy-tailed distributions.** We shall restrict the loss process P_t to having heavy-tailed jumps. An important class of heavy-tailed distributions is the subexponential distribution class. Recall that a distribution F on $[0, \infty)$ is subexponential, denoted by $F \in \mathbf{S}$, if $\overline{F}(x) = 1 - F(x) > 0$ for all x and $\overline{F^{*2}}(x) \sim 2\overline{F}(x)$, where F^{*2} denotes the convolution of F with itself. More generally, a distribution F on $\mathbb{R}$ is still said to be subexponential if the distribution $F(x)\mathbb{1}_{\{x \ge 0\}}$ is subexponential. Note that every subexponential distribution F is long-tailed, denoted by $F \in \mathbf{L}$, in the sense that the relation $\overline{F}(x + y) \sim \overline{F}(x)$ holds for every fixed y, see, e.g. [6]. Another useful distribution class is $\mathbf{D}$, which consists of all distributions with dominated variation. A distribution F on $\mathbb{R}$ is said to be dominatedly varying tailed, denoted by $F \in \mathbf{D}$, if $\limsup \overline{F}(xy)/\overline{F}(x) < \infty$ for any $0 < y < 1$. A slightly smaller class is $\mathbf{C}$ of consistently varying tailed distributions. A distribution F on $\mathbb{R}$ is said to be consistently varying tailed, denoted by $F \in \mathbf{C}$, if $\lim_{y \uparrow 1} \limsup_{x\to\infty} \overline{F}(xy)/\overline{F}(x) = 1$. Two subclasses are the class $\mathbf{ERV}$ of distributions with extended regularly varying tails, and the class $\mathbf{R}$ of distributions with regularly varying tails. A distribution F on $\mathbb{R}$ is said to be extendly regularly varying tailed, denoted by $F \in \mathbf{ERV}(-\alpha,-\beta)$, if there exist some constants $0<\alpha \leq \beta<\infty$ such that $y^{-\beta} \leq \liminf \overline{F}(xy)/\overline{F}(x) \leq \limsup \overline{F}(xy)/\overline{F}(x) \leq y^{-\alpha}$ for any $y \geq 1$. If $\alpha=\beta$, F is said to be regularly varying tailed, denoted by $F \in \mathbf{R}_{-\alpha}$.

It is well known that

$$\mathbf{R} \subset \mathbf{ERV} \subset \mathbf{C} \subset \mathbf{L} \cap \mathbf{D} \subset \mathbf{S} \subset \mathbf{L},$$

where the class $\mathbf{ERV}$ is the union of $\mathbf{ERV}(-\alpha,-\beta)$ over the range $0<\alpha \leq \beta<\infty$, and the class $\mathbf{R}$ is the union of $\mathbf{R}_{-\alpha}$ over the range $0<\alpha<\infty$.

For a distribution F on $\mathbb{R}$, denote its upper and lower Matuszewska indices, respectively, by

$$J_F^{+}=-\lim_{y \to \infty} \frac{\log \overline{F}_*(y)}{\log y} \quad \text{with } \overline{F}_*(y):=\liminf_{x \to \infty} \frac{\overline{F}(xy)}{\overline{F}(x)} \quad \text{for } y>1,$$

and

$$J_F^{-}=-\lim_{y \to \infty} \frac{\log \overline{F}^*(y)}{\log y} \quad \text{with } \overline{F}^*(y):=\limsup_{x \to \infty} \frac{\overline{F}(xy)}{\overline{F}(x)} \quad \text{for } y>1.$$

The presented definitions yield that the following assertions are equivalent (for details, see [1]):

$$\text{(i) } F \in \mathbf{D}, \quad \text{(ii) } \overline{F}_*(y)>0 \text{ for some } y>1, \quad \text{(iii) } J_F^{+}<\infty.$$

Clearly, for a distribution F, $0<J_F^{-} \leq \infty$ holds if and only if $\overline{F}^*(y)<1$ for some $y>1$, see, e.g. [3]. This condition and $F \in \mathbf{S}$ lead to a subclass $\mathbf{A}$, which was introduced by [20]. The reason for introducing $\mathbf{A}$ is mainly to exclude some very heavy-tailed distributions, such as those that are slowly varying tailed, from the class $\mathbf{S}$.

2.2. **Theorems.** For a general Lévy process L_t, by definition, for each $t \geq 0$, the r.v. L_t has an infinitely divisible distribution with a characteristic function $\mathbb{E}\mathrm{e}^{\mathrm{i}uL_t}=\mathrm{e}^{-t\Psi_L(u)}$, where the characteristic exponent $\Psi_L(\cdot)$ satisfies the following Lévy-Khintchine representation:

$$\Psi_L(u)=\mathrm{i}\mu u+\frac{1}{2}\sigma^2u^2+\int_{-\infty}^{\infty}(1-\mathrm{e}^{\mathrm{i}ux}+\mathrm{i}ux\mathbb{1}_{\{|x|<1\}})\nu(\mathrm{d}x), \tag{4}$$

with $\mu \in \mathbb{R}$, $\sigma \geq 0$, and Lévy measure ν on $\mathbb{R}\backslash\{0\}$ satisfying $\int_{-\infty}^{\infty}(x^2 \wedge 1)\nu(\mathrm{d}x)<\infty$. The triplet (μ,σ,ν) uniquely determines the law of the Lévy process L_t. We further denote by

$$\varphi_L(u)=-\Psi_L(\mathrm{i}u)=\log \mathbb{E}\mathrm{e}^{-uL_1}$$

the Laplace exponent of L_t.

For the Lévy process P_t, when its Lévy measure satisfies $\overline{\nu}_P(1)=\nu_P((1,\infty))>0$, introduce $\Pi_P(\cdot)=\nu_P(\cdot)\mathbb{1}_{(1,\infty)}/\overline{\nu}_P(1)$, which is a proper probability measure on $(1,\infty)$. In our three main results, we shall assume that Π_P belongs to some heavy-tailed distribution classes.

The first main result investigates the asymptotic behavior of the finite-time ruin probability in the bivariate Lévy-driven risk model, which extends the corresponding one in [14].

Theorem 2.1. *Consider the bivariate Lévy-driven risk process Y_t given by (2), where P_t and R_t are two independent Lévy processes. Assume that $\Pi_P \in \mathbf{C}$ with $0 < J^-_{\Pi_P} \le J^+_{\Pi_P} < \infty$. If $\mathbb{E}(\mathrm{e}^{-(J^+_{\Pi_P}+\varepsilon)R_1}) < 1$ for some $\varepsilon > 0$, then it holds that for any $T \in (0, \infty)$,*

$$\psi(x,T) \sim \lambda \int_0^T \mathbb{P}\Big(X\mathrm{e}^{-R_t} > x\Big)\mathrm{d}t, \tag{5}$$

where $\lambda = \overline{\nu}_P(1)$, and X is distributed by Π_P and independent of P_t and R_t.

Clearly, in a special case that $\tilde{R}_t$ in (1) reduces to a deterministic linear function, R_t in (2) is also replaced by a deterministic linear function. Precisely speaking, if $\tilde{R}_s = rs$ for some $r > 0$, then relation (2) can be simplified to

$$Y_t = \mathrm{e}^{rt}\left(x - \int_0^t \mathrm{e}^{-rs}\mathrm{d}P_s\right). \tag{6}$$

This means that the insurance company invests its wealth only into a risk-free market, and the positive constant r represents the constant interest rate. Our next two main results consider such a single Lévy-driven risk model, in which the loss process P_t is still assumed to be a Lévy process, and investigate the asymptotics for the finite-time and infinite-time ruin probabilities, respectively. Denote by $\overline{\Pi_P}(x) = \Pi_P((x,\infty)),\ x \in \mathbb{R}$.

Theorem 2.2. *Consider the single Lévy-driven risk process Y_t given by (6), where P_t is a Lévy process. If $\Pi_P \in \mathbf{L} \cap \mathbf{D}$ with $0 < J^-_{\Pi_P} \le J^+_{\Pi_P} < \infty$, then it holds that for any $T \in (0, \infty)$,*

$$\psi(x,T) \sim \lambda \int_0^T \overline{\Pi_P}(x\mathrm{e}^{rt})\mathrm{d}t, \tag{7}$$

where $\lambda = \overline{\nu}_P(1)$.

Theorem 2.3. *Under the conditions of Theorem 2.2, if $\overline{\Pi_P}^*(\mathrm{e}^r) = \limsup \overline{\Pi_P}(x\mathrm{e}^r)/\overline{\Pi_P}(x) < 1$, then relation (7) holds for $T = \infty$.*

3. **Lemmas.** The following lemma is a combination of Proposition 2.2.1 in [1] and Lemma 3.5 of [28].

Lemma 3.1. *For a distribution $F \in \mathbf{D}$ on $\mathbb{R}$ with the Matuszewska indices $0 < J^-_F \le J^+_F < \infty$, the following assertions hold.*

(1) For any $0 < \epsilon < J^-_F \le J^+_F < \infty$ and some $C > 1$, there exists some positive constant D, such that for all $x \ge D$ and $xy \ge D$,

$$C^{-1}\left(y^{-(J^-_F-\epsilon)} \vee y^{-(J^+_F+\epsilon)}\right) \le \frac{\overline{F}(xy)}{\overline{F}(x)} \le C\left(y^{-(J^-_F-\epsilon)} \vee y^{-(J^+_F+\epsilon)}\right).$$

(2) For any $p > J^+_F$, it holds that

$$x^{-p} = o(\overline{F}(x)).$$

The next Lemma 3.2 is a direct consequence of Theorem 3.3 of [5], which can also be found in [4].

Lemma 3.2. *Let X be a real-valued r.v. with distribution F, and Y be a nonnegative r.v. independent of X. Denote by H the distribution of the product r.v. XY. Then the following assertions hold.*

(1) If $F \in \mathbf{C}$, then $H \in \mathbf{C}$.

(2) If $F \in \mathbf{D}$ and $\mathbb{E}Y^{J^+_F+\epsilon} < \infty$ for some $\epsilon > 0$, then $\overline{F}(x) \approx \overline{H}(x)$.

The first assertion of Lemma 3.3 below is a slight extension of Lemma 3.3 of [14]. Note that in this assertion, X and Y can be arbitrarily dependent. The second assertion is due to Corollary 3.18 of [7], which gives a refinement in the independence case.

Lemma 3.3. *Let (X,Y) be a random vector with marginal distributions F and G, respectively.*

(1) If $F\in\mathbf{C}$ and $\mathbb{P}(|Y|>x)=o(\overline{F}(x))$, then

$$\mathbb{P}(X+Y>x)\sim\overline{F}(x). \tag{8}$$

(2) Assume that X and Y are independent. If $F\in\mathbf{S}$ and $\overline{G}(x)=o(\overline{F}(x))$, then (8) holds.

Proof. We only prove the first assertion along the line of the proof of Lemma 4.4.2 of [25]. For any $0<v<1$, on one hand, by $F\in\mathbf{C}\subset\mathbf{D}$,

$$\begin{aligned}
&\limsup_{v\downarrow 0}\limsup_{x\to\infty}\frac{\mathbb{P}(X+Y>x)}{\overline{F}(x)}\\
\le\ &\limsup_{v\downarrow 0}\limsup_{x\to\infty}\left(\frac{\overline{G}(vx)}{\overline{F}(x)}+\frac{\mathbb{P}(X+Y>x,Y\le vx)}{\overline{F}(x)}\right)\\
\le\ &\limsup_{v\downarrow 0}\limsup_{x\to\infty}\left(\frac{\overline{G}(vx)}{\overline{F}(vx)}\cdot\frac{\overline{F}(vx)}{\overline{F}(x)}+\frac{\overline{F}((1-v)x)}{\overline{F}(x)}\right)\\
=\ &1.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\liminf_{v\downarrow 0}\liminf_{x\to\infty}\frac{\mathbb{P}(X+Y>x)}{\overline{F}(x)} &\ge \liminf_{v\downarrow 0}\liminf_{x\to\infty}\frac{\mathbb{P}(X+Y>x,Y\ge -vx)}{\overline{F}(x)}\\
&\ge \liminf_{v\downarrow 0}\liminf_{x\to\infty}\frac{\mathbb{P}(X>(1+v)x,Y\ge -vx)}{\overline{F}(x)}\\
&\ge \liminf_{v\downarrow 0}\liminf_{x\to\infty}\left(\frac{\overline{F}((1+v)x)}{\overline{F}(x)}-\frac{\mathbb{P}(Y<-vx)}{\overline{F}(x)}\right)\\
&= 1.
\end{aligned}$$

It ends the proof of the lemma. □

The next lemma shows that the inequality (9) holds uniformly for all nonnegative r.v.s independent of X. Denote by $\mathbf{Y}_X=\{Y\ge 0:\ \mathbb{E}(Y^{J_F^- -\epsilon}\vee Y^{J_F^+ +\epsilon})<\infty$, for some $0<\epsilon<J_F^-\}$ the set with all nonnegative r.v.s independent of X with distribution F.

Lemma 3.4. *Let X be a real-valued r.v. with distribution $F\in\mathbf{D}$ and $0<J_F^-\le J_F^+<\infty$. Then, for any $0<\epsilon<J_F^-$ and some $C>1$, there exists a positive $D(\epsilon,C)$ such that for all $Y\in\mathbf{Y}_X$,*

$$\mathbb{P}(XY>x)\le C\mathbb{E}\left(Y^{J_F^- -\epsilon}\vee Y^{J_F^+ +\epsilon}\right)\overline{F}(x). \tag{9}$$

Proof. By Lemma 3.1 (1) and Markov's inequality, it holds that for all $Y \in \mathbf{Y}_X$ and all $x \geq D$,

$$\begin{aligned}\mathbb{P}(XY > x) &\leq \int_0^{x/D} \overline{F}\Big(\frac{x}{u}\Big)\mathbb{P}(Y \in \mathrm{d}u) + \mathbb{P}\Big(Y > \frac{x}{D}\Big) \\ &\leq C\overline{F}(x)\mathbb{E}\left(Y^{J_F^- - \epsilon} \vee Y^{J_F^+ + \epsilon}\right) + \Big(\frac{x}{D}\Big)^{-(J_F^+ + \epsilon)} \mathbb{E}Y^{J_F^+ + \epsilon}, \quad (10)\end{aligned}$$

which, together with Lemma 3.1 (2), leads to relation (9). □

The following lemma considers the tail behavior of the randomly weighted sums with heavy-tailed primary r.v.s, which comes from [9] and [30], respectively. We remark that in Gao and Wang's result it requires the technical condition $\mathbb{P}(X < -x) = o(\mathbb{P}(X > x))$, because of the certain dependence among $\{X_i,\ i \geq 1\}$; while in the independence structure, such a restriction can be easily dropped.

Lemma 3.5. *Let $\{X_i,\ i \geq 1\}$ be a sequence of independent and identically distributed (i.i.d.) real-valued r.v.s with common distribution F; and let $\{\theta_i,\ i \geq 1\}$ be another sequence of nonnegative and non-degenerate at zero r.v.s, which are independent of $\{X_i,\ i \geq 1\}$.*

(1) If $F \in \mathbf{C},\ 0 < J_F^- \leq J_F^+ < \infty$ and for some $0 < \epsilon < J_F^-$,

$$\begin{cases} \sum_{i=1}^{\infty} \mathbb{E}\Big(\theta_i^{J_F^- - \epsilon} \vee \theta_i^{J_F^+ + \epsilon}\Big) < \infty, & \text{for } \ 0 < J_F^+ < 1, \\ \sum_{i=1}^{\infty} \Big(\mathbb{E}\Big(\theta_i^{J_F^- - \epsilon} \vee \theta_i^{J_F^+ + \epsilon}\Big)\Big)^{\frac{1}{J_F^+ + \epsilon}} < \infty, & \text{for } \ J_F^+ \geq 1, \end{cases} \qquad (11)$$

then

$$\mathbb{P}\Big(\max_{n\geq 1} \sum_{i=1}^{n} \theta_i X_i > x\Big) \sim \mathbb{P}\Big(\sum_{i=1}^{\infty} \theta_i X_i^+ > x\Big) \sim \sum_{i=1}^{\infty} \mathbb{P}(\theta_i X_i > x). \qquad (12)$$

Further, the distributions of $\max_{n\geq 1} \sum_{i=1}^{n} \theta_i X_i$ and $\sum_{i=1}^{\infty} \theta_i X_i^+$ both belong to the class $\mathbf{C}$.

(2) If $F \in \mathbf{L} \cap \mathbf{D}$ and $\mathbb{E}\theta_i^{J_F^+ + \epsilon} < \infty,\ i \geq 1$, then for each fixed $n \geq 1$,

$$\mathbb{P}\Big(\sum_{i=1}^{n} \theta_i X_i > x\Big) \sim \sum_{i=1}^{n} \mathbb{P}(\theta_i X_i > x).$$

Proof. In claim (1), relation (12) is proved in Theorem 2.1 (b) of [9] by slightly modification and noting the independence among $\{X_i,\ i \geq 1\}$. Denote the distributions of $\sum_{k=1}^{\infty} \theta_k X_k^+$ and $\theta_i X_i,\ i \geq 1$, by H_∞^+ and H_i, respectively. We only check $H_\infty^+ \in \mathbf{C}$. Indeed, by Lemma 3.2, for each $i \geq 1$, $H_i \in \mathbf{C}$ and $\overline{H_i}(x) \approx \overline{F}(x)$. By Lemma 3.4, for any $0 < \epsilon < J_F^-$, some $C > 1$ and each $i \geq 1$, there is some $D(\epsilon, C) > 0$ such that

$$\overline{H_i}(x) \leq C\overline{F}(x)\mathbb{E}\left(\theta_i^{J_F^- - \epsilon} \vee \theta_i^{J_F^+ + \epsilon}\right)$$

holds for all $i \geq 1$ and all $x \geq D$. By (11) and $\overline{H_1}(x) \approx \overline{F}(x)$, for any small $\delta > 0$, sufficiently large $x > D$ and sufficiently large n,

$$\begin{aligned}\sum_{i=n+1}^{\infty} \overline{H_i}(x) &\leq C\overline{F}(x) \sum_{i=n+1}^{\infty} \mathbb{E}\left(\theta_i^{J_F^- - \epsilon} \vee \theta_i^{J_F^+ + \epsilon}\right) \\ &\leq \delta\overline{H_1}(x). \qquad (13)\end{aligned}$$

Thus, by (12), (13) and $H_i \in \mathbf{C}$, for sufficiently large n,

$$\begin{aligned} & \limsup_{y\uparrow 1}\limsup_{x\to\infty}\frac{\overline{H_\infty^+}(xy)}{\overline{H_\infty^+}(x)} && (14)\\ \le\ & \limsup_{y\uparrow 1}\limsup_{x\to\infty}\left(\frac{\sum_{i=1}^n \overline{H_i}(xy)}{\sum_{i=1}^n \overline{H_i}(x)}+\frac{\sum_{i=n+1}^\infty \overline{H_i}(xy)}{\overline{H_1}(xy)}\cdot\frac{\overline{H_1}(xy)}{\overline{H_1}(x)}\right)\\ \le\ & \limsup_{y\uparrow 1}\limsup_{x\to\infty}\left(\max_{1\le i\le n}\frac{\overline{H_i}(xy)}{\overline{H_i}(x)}+\delta\cdot\frac{\overline{H_1}(xy)}{\overline{H_1}(x)}\right)\\ =\ & 1+\delta. && (15)\end{aligned}$$

Similarly, we have that for sufficiently large n,

$$\begin{aligned}\liminf_{y\uparrow 1}\liminf_{x\to\infty}\frac{\overline{H_\infty^+}(xy)}{\overline{H_\infty^+}(x)} &\ge \liminf_{y\uparrow 1}\liminf_{x\to\infty}\frac{\sum_{i=1}^n \overline{H_i}(xy)}{(1+\delta)\sum_{i=1}^n \overline{H_i}(x)}\\ &\ge \frac{1}{1+\delta}\liminf_{y\uparrow 1}\liminf_{x\to\infty}\min_{1\le i\le n}\frac{\overline{H_i}(xy)}{\overline{H_i}(x)}\\ &= \frac{1}{1+\delta}. \qquad (16)\end{aligned}$$

Therefore, $H_\infty^+ \in \mathbf{C}$ follows from (15), (16) and the arbitrariness of $\delta > 0$.

It can be easily obtained that the distribution of $\max_{n\ge 1}\sum_{i=1}^n \theta_i X_i$ also belongs to the class $\mathbf{C}$, by the first relation of (12).

The proof of claim (2) can be found in Theorem 3.3 of [30]. □

Applying Lemma 3.5 and going along the lines of the proof of Lemma 3.5 of [14], we can prove that

Lemma 3.6. *Let $C_t = \sum_{k=1}^{N_t} X_k$ be a compound renewal process, where $\{X_i,\ i\ge 1\}$ are i.i.d. real-valued r.v.s with common distribution F, N_t is a renewal counting process with intensity function $\lambda(t)$; and let L_t be a Lévy process independent of C_t. If $F\in\mathbf{C}$, $0<J_F^-\le J_F^+<\infty$ and $\varphi_L(J_F^+ + \epsilon)<0$ for some $0<\epsilon<J_F^-$, then it holds that for any $0<T\le\infty$,*

$$\mathbb{P}\Big(\sup_{0\le t\le T}\int_0^T \mathrm{e}^{-L_s}\mathrm{d}C_s > x\Big) \sim \int_0^T \mathbb{P}(X_1\mathrm{e}^{-L_t}>x)\lambda(\mathrm{d}t),$$

and the distribution of $\sup_{0\le t\le T}\int_0^T \mathrm{e}^{-L_s}\mathrm{d}C_s$ belongs to the class $\mathbf{C}$.

Recently, [13] investigated the uniform asymptotics for the finite-time ruin probability in a risk model with i.i.d. claim sizes and constant interest rate $r\ge 0$, see also [33]. We restate their result as follows.

Lemma 3.7. *Let N_t be a renewal counting process with intensity function $\lambda(t)$, $\{X_i,\ i\ge 1\}$, independent of N_t, be a sequence of i.i.d. nonnegative r.v.s with common distribution F, and $r\ge 0$ be a constant interest rate. If $F\in\mathbf{L}\cap\mathbf{D}$, then for any $T>0$,*

$$\mathbb{P}\Big(\sup_{0\le t\le T}\sum_{k=1}^{N_t} X_k\mathrm{e}^{-r\tau_k}>x\Big)\sim\int_0^T \overline{F}(x\mathrm{e}^{rt})\lambda(\mathrm{d}t),$$

where $\tau_k,\ k\ge 1$, are the arrival times of the renewal counting process $N_t,\ t\ge 0$.

We remark that in Lemma 3.7, the restriction $J_F^- > 0$ is dropped and the constant interest rate can be allowed to zero.

The following lemma is due to Corollary 3.16 of [7].

Lemma 3.8. *Let F and G be two distributions. If $F \in \mathbf{S}$, $\overline{F}(x)+\overline{G}(x)$ is long-tailed and $\overline{G}(x) = O(\overline{F}(x))$, then $\overline{F*G}(x) \sim \overline{F}(x)+\overline{G}(x)$.*

Lemma 3.9. *Let (ξ,η) be a random vector satisfying $\mathbb{E}\log(|\xi|\vee 1)<\infty$, $\mathbb{P}(\eta \geq 0)=1$ and $-\infty \leq \mathbb{E}\log\eta < 0$. Let Q be a r.v. independent of (ξ,η).*

(1) There exists exactly one distribution for Q satisfying the stochastic difference equation

$$Q \overset{\mathrm{d}}{=} \xi + Q\eta. \tag{17}$$

(2) Furthermore, if $F_\xi \in \mathbf{L}\cap\mathbf{D}$ with $\overline{F_\xi}^*(\mathrm{e}^r)<1$ and $\eta=\mathrm{e}^{-r}$ for some $r>0$, then

$$\mathbb{P}(Q>x) \sim \sum_{i=1}^{\infty}\overline{F_\xi}(x\mathrm{e}^{r(i-1)}). \tag{18}$$

Note that $\overline{F_\xi}^*(\mathrm{e}^r) < 1$ implies $J_{F_\xi}^- > 0$. The second assertion of this lemma extends Lemma 3.7 of [14] from the class **ERV** to the intersection $\mathbf{L}\cap\mathbf{D}$ in a special case, where η reduces to a constant e^{-r}.

Proof. (1) The existence and uniqueness of the weak solution of (17) are given by Theorem 1.6(b, c) and Theorem 1.5(i) of [31].

(2) We follow the line of the proof of Lemma 3.7 of [14], whose method is developed by [11]. We shall conduct a sequence of r.v.s $\{Q_i,\ i\geq 1\}$ defined recursively by

$$Q_i = \xi_i + Q_{i-1}\mathrm{e}^{-r},\ i\geq 1, \tag{19}$$

where Q_0 is a starting r.v. independent of $\{\xi_i,\ i\geq 1\}$, which are independent copies of ξ. Then, Theorem 1.5(i) of [31] showed that the sequence $\{Q_i,\ i\geq 1\}$ weakly converges with a limit distribution, which does not depend on Q_0 and coincides with the distribution of Q in (17).

We firstly consider the upper bound of (18). Introduce a nonnegative r.v. Q_0' independent of ξ with the tail distribution

$$\mathbb{P}(Q_0'>x) \sim c\overline{F_\xi}(x\mathrm{e}^{-r}) \tag{20}$$

for some $c > \overline{F_\xi}^*(\mathrm{e}^r)(1-\overline{F_\xi}^*(\mathrm{e}^r))^{-1} > 0$ because of $F_\xi \in \mathbf{D}$. Clearly, $\mathbb{P}(Q_0'\mathrm{e}^{-r} > x) \sim c\overline{F_\xi}(x)$. Then, by Lemma 3.8 and (20), we have that

$$\begin{aligned}\mathbb{P}(\xi+Q_0'\mathrm{e}^{-r}>x) &\sim (1+c)\overline{F_\xi}(x)\\ &\prec c\overline{F_\xi}(x\mathrm{e}^{-r})\\ &\sim \mathbb{P}(Q_0'>x),\end{aligned}$$

where in the second step we used the fact

$$\limsup_{x\to\infty}\frac{\overline{F_\xi}(x)}{\overline{F_\xi}(x\mathrm{e}^{-r})} = \overline{F_\xi}^*(\mathrm{e}^r) < c(1+c)^{-1}. \tag{21}$$

This means that there exists some $x_0>0$ such that for all $x\geq x_0$,

$$\mathbb{P}(\xi+Q_0'\mathrm{e}^{-r}>x) \leq \mathbb{P}(Q_0'>x). \tag{22}$$

Construct a conditional r.v. $Q_0 = (Q_0'|Q_0' > x_0)$, independent of ξ, which is the starting r.v. in the recursive equation (19). Clearly, for sufficiently large x,

$$\mathbb{P}(Q_0\mathrm{e}^{-r} > x) = \frac{\mathbb{P}(Q_0' > x\mathrm{e}^r)}{\mathbb{P}(Q_0' > x_0)} \sim \frac{c\overline{F_\xi}(x)}{\mathbb{P}(Q_0' > x_0)},$$

which implies that the distribution of $Q_0\mathrm{e}^{-r}$ belongs to the class $\mathbf{L}\cap\mathbf{D}$. Thus, by Lemma 3.8,

$$\mathbb{P}(Q_1 > x) = \mathbb{P}(\xi_1 + Q_0\mathrm{e}^{-r} > x) \sim \overline{F_\xi}(x) + \mathbb{P}(Q_0\mathrm{e}^{-r} > x).$$

Meanwhile, Q_1 is stochastically not greater than Q_0, denoted by $Q_1 \le_{st} Q_0$, in the sense that $\mathbb{P}(Q_1 > x) \le \mathbb{P}(Q_0 > x)$ for all x. Indeed, by (22), for $x \ge x_0$,

$$\mathbb{P}(Q_1 > x) \le \frac{\mathbb{P}(\xi + Q_0'\mathrm{e}^{-r} > x)}{\mathbb{P}(Q_0' > x_0)} \le \mathbb{P}(Q_0 > x);$$

and for $x < x_0$, $\mathbb{P}(Q_1 > x) \le 1 = \mathbb{P}(Q_0 > x)$. Now, we inductively assume that

$$\mathbb{P}(Q_n > x) \sim \sum_{i=1}^{n} \overline{F_\xi}(x\mathrm{e}^{r(n-i)}) + \mathbb{P}(Q_0\mathrm{e}^{-rn} > x), \tag{23}$$

and

$$Q_n \le_{st} Q_{n-1} \tag{24}$$

hold. We shall use the mathematical induction to prove the above two relations (23) and (24) hold for $n+1$. Clearly, by (23), $F_\xi \in \mathbf{L}\cap\mathbf{D}$ and $F_{Q_0} \in \mathbf{L}\cap\mathbf{D}$, we know that the distributions of Q_n and $Q_n\mathrm{e}^{-r}$ both belong to the class $\mathbf{L}\cap\mathbf{D}$, and $\mathbb{P}(Q_n\mathrm{e}^{-r} > x) = O(\overline{F_\xi}(x))$. Then, by Lemma 3.8 and the induction assumption (23), we have that

$$\begin{aligned}\mathbb{P}(Q_{n+1} > x) &= \mathbb{P}(\xi_{n+1} + Q_n\mathrm{e}^{-r} > x)\\ &\sim \overline{F_\xi}(x) + \mathbb{P}(Q_n > x\mathrm{e}^r)\\ &\sim \sum_{i=1}^{n+1} \overline{F_\xi}(x\mathrm{e}^{r(n+1-i)}) + \mathbb{P}(Q_0\mathrm{e}^{-r(n+1)} > x).\end{aligned}$$

Since ξ_{n+1} is independent of Q_n, and by (24), we have that for all x,

$$\begin{aligned}\mathbb{P}(Q_{n+1} > x) &= \int_{-\infty}^{\infty} \mathbb{P}(Q_n > (x-u)\mathrm{e}^r) F_\xi(\mathrm{d}u)\\ &\le \int_{-\infty}^{\infty} \mathbb{P}(Q_{n-1} > (x-u)\mathrm{e}^r) F_\xi(\mathrm{d}u)\\ &= \mathbb{P}(Q_n > x),\end{aligned} \tag{25}$$

which means $Q_{n+1} \le_{st} Q_n$. Hence, it follows that

$$\begin{aligned}\mathbb{P}(Q > x) &\prec \sum_{i=1}^{n} \overline{F_\xi}(x\mathrm{e}^{r(n-i)}) + \mathbb{P}(Q_0 > x\mathrm{e}^{rn})\\ &= \sum_{i=1}^{n} \overline{F_\xi}(x\mathrm{e}^{r(i-1)}) + \mathbb{P}(Q_0 > x\mathrm{e}^{rn}).\end{aligned} \tag{26}$$

Repeating (21), we can prove

$$\begin{aligned} \mathbb{P}(Q_0 > x\mathrm{e}^{rn}) &\sim \frac{c\overline{F_\xi}(x\mathrm{e}^{r(n-1)})}{\mathbb{P}(Q_0' > x_0)} \\ &\prec \frac{c(\overline{F_\xi}^*(\mathrm{e}^r))^{n-1}}{\mathbb{P}(Q_0' > x_0)} \cdot \overline{F_\xi}(x). \end{aligned} \tag{27}$$

Plugging this estimate into (26) and letting $n \to \infty$, we can derive that

$$\mathbb{P}(Q > x) \prec \sum_{i=1}^{\infty} \overline{F_\xi}(x\mathrm{e}^{r(i-1)}). \tag{28}$$

We return to the lower bound of (18). In the recursive equation (19), construct another starting r.v. Q_0, independent of ξ, with the tail distribution

$$\mathbb{P}(Q_0 > x) = \mathbb{P}(Q > 0)\overline{F_\xi}(x)\mathbb{1}_{\{x\geq 0\}} + \mathbb{P}(Q > x)\mathbb{1}_{\{x<0\}},$$

where $\mathbb{P}(Q > 0) > 0$, see the proof of Theorem 1 of [11]. Clearly, the distribution of Q_0 belongs to the class $\mathbf{L} \cap \mathbf{D}$ (so, the distribution of $Q_0\mathrm{e}^{-r}$ is also in $\mathbf{L} \cap \mathbf{D}$) and $Q_0 \leq_{st} Q$. Indeed, for $x \geq 0$, $\mathbb{P}(Q > x) \geq \mathbb{P}(\xi + Q\mathrm{e}^{-r} > x, Q > 0) \geq \mathbb{P}(Q > 0)\overline{F_\xi}(x)$; and for $x < 0$, $\mathbb{P}(Q_0 > x) = \mathbb{P}(Q > x)$. Similarly to (25), $Q_i \leq_{st} Q$, $i \geq 1$. As the proof of (23) and (27), for each $n \geq 1$,

$$\begin{aligned} \mathbb{P}(Q > x) &\geq \mathbb{P}(Q_n > x) \\ &\sim \sum_{i=1}^{n} \overline{F_\xi}(x\mathrm{e}^{r(i-1)}) + \mathbb{P}(Q_0 > x\mathrm{e}^{rn}) \\ &\geq \left(\sum_{i=1}^{\infty} - \sum_{i=n+1}^{\infty}\right) \overline{F_\xi}(x\mathrm{e}^{r(i-1)}) \\ &\succ \sum_{i=1}^{\infty} \overline{F_\xi}(x\mathrm{e}^{r(i-1)}) - \overline{F_\xi}(x) \sum_{i=n+1}^{\infty} (\overline{F_\xi}^*(\mathrm{e}^r))^{i-1}, \end{aligned}$$

which, by $\overline{F_\xi}^*(\mathrm{e}^r) < 1$ and letting $n \to \infty$, yields that

$$\mathbb{P}(Q > x) \succ \sum_{i=1}^{\infty} \overline{F_\xi}(x\mathrm{e}^{r(i-1)}). \tag{29}$$

Therefore, the desired relation (18) follows from (28) and (29). □

Finally, we cite two useful martingale inequalities, which can be found in [14]. The first one is related to Doob's inequality.

Lemma 3.10. *Let L_t be a Lévy process with Laplace exponent $\varphi_L(\cdot)$. If $\varphi_L(u) < \infty$ for some $u > 0$, then $\mathbb{E}(\sup_{0\leq t\leq T} \mathrm{e}^{-uL_t}) < \infty$ for every fixed $T \in (0, \infty)$.*

The following lemma recalls the Burkholder-Gundy-Davis inequality.

Lemma 3.11. *For a local martingale U_t, denote by $U_T^* = \sup_{0\leq t\leq T} |U_t|$ for $0 \leq T \leq \infty$.*

(1) For any $q > 1$, there are two positive constants C_q and C_q' such that, uniformly for all local martingales U_t with $U_o = 0$ and all $0 \leq T \leq \infty$,

$$C_q'\mathbb{E}[U, U]_T^{q/2} \leq \mathbb{E}(U_T^*)^q \leq C_q\mathbb{E}[U, U]_T^{q/2}, \tag{30}$$

where $[U, U]$ is the quadratic variation of U_t. Moreover, if U_t is continuous, then (30) holds for all $0 < q < \infty$.

(2) If U_t is a local square integrable martingale with $U_0 = 0$, then it holds that for any $q \in (0, 2)$,

$$\mathbb{E}(U_T^*)^q \le \frac{4-q}{2-q}\mathbb{E}\langle U, U\rangle_T^{q/2},$$

where $\langle U, U\rangle$ is the predictable quadratic variation of U_t.

4. **Proofs of the main results.** For the Lévy process P_t, by the Lévy-Khintchine representation (4), its Lévy-Itô decomposition is given by

$$P_t = pt + \sigma_P W_t + U_t + C_t, \tag{31}$$

where W_t is a standard Wiener process, U_t is a square integrable martingale with almost surely countably many jumps of magnitude less than 1, and $C_t = \sum_{k=1}^{N_t} X_k$ is a compound Poisson process in which N_t is a Poisson process with the Poisson intensity $\lambda^* = \nu_P(\mathbb{R} \setminus (-1, 1))$, and $\{X_i,\ i \ge 1\}$ are i.i.d. r.v.s with common distribution F given by $\nu_P(\cdot)\mathbb{1}_{\mathbb{R}\setminus(-1,1)}/\lambda^*$. In particular, W_t, U_t and C_t are three independent Lévy processes.

4.1. **Proof of Theorem 2.1.** The proof of Theorem 2.1 is parallel with that of Theorem 2.1 (1) of [14], and we only show the difference. By (31),

$$\sum_{i=1}^{3} \inf_{0\le t\le T} I_{i,t} + \sup_{0\le t\le T} I_{4,t} \le M_T \le \sum_{i=1}^{4} \sup_{0\le t\le T} I_{i,t}, \tag{32}$$

where $I_{1,t} = p\int_0^t \mathrm{e}^{-R_s}\mathrm{d}s$, $I_{2,t} = \sigma_P \int_0^t \mathrm{e}^{-R_s}\mathrm{d}W_s$, $I_{3,t} = \int_0^t \mathrm{e}^{-R_s}\mathrm{d}U_s$ and $I_{4,t} = \int_0^t \mathrm{e}^{-R_s}\mathrm{d}C_s$. By Lemma 3.6, the distribution of $\sup_{0\le t\le T} I_{4,t}$ belongs to the class $\mathbf{C}$, and

$$\begin{aligned}
\mathbb{P}\Big(\sup_{0\le t\le T} I_{4,t} > x\Big) &\sim \lambda^* \int_0^T \mathbb{P}(X^* \mathrm{e}^{-R_t} > x)\mathrm{d}t \\
&= \overline{\nu}_P(1)\int_0^T \int_0^\infty \Pi_P\Big(\Big(\frac{x}{u}, \infty\Big)\Big)\mathbb{P}(\mathrm{e}^{-R_t} \in \mathrm{d}u)\mathrm{d}t \\
&= \lambda \int_0^T \mathbb{P}(X\mathrm{e}^{-R_t} > x)\mathrm{d}t,
\end{aligned} \tag{33}$$

where X^* and X, independent of R_t, are distributed by F and Π_P, respectively. As done in [14], by Lemmas 3.10, 3.11 and noting $J_F^+ = J_{\Pi_P}^+$, we can obtain that for each $i = 1, 2, 3$,

$$\mathbb{E}\Big(\sup_{0\le t\le T} |I_{i,t}|^{J_F^+ + \varepsilon}\Big) < \infty,$$

which implies that

$$\mathbb{P}\Big(\sup_{0\le t\le T} I_{i,t} > x\Big) = o(\overline{F}(x)), \tag{34}$$

$i = 1, 2, 3$. By (33),

$$\begin{aligned}
\mathbb{P}\Big(\sup_{0\le t\le T} I_{4,t} > x\Big) &\sim \sum_{k=1}^{\infty} \mathbb{P}(X_k^* \mathrm{e}^{-R_{\tau_k}} \mathbb{1}_{\{\tau_k \le T\}} > x) \\
&\ge \mathbb{P}(X_1^* \mathrm{e}^{-R_{\tau_1}} \mathbb{1}_{\{\tau_1 \le T\}} > x) \\
&\succ C\overline{F}(x),
\end{aligned}$$

for some $C > 0$, where $\{X_k^*,\ k \ge 1\}$ are the independent copies of X^*, $\{\tau_k,\ k \ge 1\}$ are the arrival times of the Poisson process N_t, and in the last step we used Lemma

3.3 (ii) of [34] by noting $F \in \mathbf{D}$ (equivalently, $\Pi_P \in \mathbf{D}$). Hence, (34) yields that for each $i = 1, 2, 3$,

$$\mathbb{P}\Big(\sup_{0\le t\le T} I_{i,t} > x\Big) = o(1)\mathbb{P}\Big(\sup_{0\le t\le T} I_{4,t} > x\Big). \tag{35}$$

Therefore, plugging (33) and (35) into (32), Lemma 3.3 (1) gives the desired relation (5).

4.2. **Proof of Theorem 2.2.** In this special case, $I_{i,t}$, $i = 1, 2, 3, 4$, in relation (32) can be simplified to $I_{1,t} = pr^{-1}(1 - \mathrm{e}^{-rt})$, $I_{2,t} = \sigma_P \int_0^t \mathrm{e}^{-rs}\mathrm{d}W_s$, $I_{3,t} = \int_0^t \mathrm{e}^{-rs}\mathrm{d}U_s$ and $I_{4,t} = \int_0^t \mathrm{e}^{-rs}\mathrm{d}C_s = \sum_{k=1}^{N_t} X_k \mathrm{e}^{-r\tau_k}$. Clearly, by $\Pi_P \in \mathbf{L} \cap \mathbf{D}$, the distribution F given by $\nu_P(\cdot)\mathbb{1}_{\mathbb{R}\setminus(-1,1)}/\lambda^*$ is also in the class $\mathbf{L} \cap \mathbf{D}$. We firstly consider $\sup_{0\le t\le T} I_{4,t}$. Note that

$$\begin{aligned}\sup_{0\le t\le T} I_{4,t} &= \sup_{0\le t\le T} \sum_{k=1}^{N_t} X_k \mathrm{e}^{-r\tau_k} \\ &= \sup_{n\ge 0} \sum_{k=1}^{n} X_k \mathrm{e}^{-r\tau_k} \mathbb{1}_{\{\tau_k\le T\}}. \end{aligned}\tag{36}$$

By Lemma 3.7,

$$\begin{aligned}\mathbb{P}\Big(\sup_{0\le t\le T} I_{4,t} > x\Big) &\le \mathbb{P}\Big(\sup_{0\le t\le T} \sum_{k=1}^{N_t} X_k^+ \mathrm{e}^{-r\tau_k} > x\Big) \\ &\sim \lambda \int_0^T \overline{F}(x\mathrm{e}^{rt})\mathrm{d}t. \end{aligned}\tag{37}$$

As for the lower bound of $\sup_{0\le t\le T} I_{4,t}$, we use the method on the tail behavior of randomly weighted sums. By Lemma 3.1 (1) and Lemma 3.2 (2), for any $0 < \epsilon < J_F^-$, all $x \ge D$ and sufficiently large n,

$$\begin{aligned}\sum_{k=n+1}^{\infty} \mathbb{P}(X_k \mathrm{e}^{-r\tau_k}\mathbb{1}_{\{\tau_k\le T\}} > x) &\le \sum_{k=n+1}^{\infty} \int_0^1 \overline{F}\Big(\frac{x}{u}\Big)\mathbb{P}(\mathrm{e}^{-r\tau_k} \in \mathrm{d}u) \\ &\le C\overline{F}(x) \sum_{k=n+1}^{\infty} \mathbb{E}\mathrm{e}^{-(J_F^- -\epsilon)r\tau_k} \\ &= C\overline{F}(x) \sum_{k=n+1}^{\infty} \Big(\mathbb{E}\mathrm{e}^{-(J_F^- -\epsilon)r\tau_1}\Big)^k \\ &\le \epsilon\overline{F}(x) \\ &\le \epsilon C_0 \mathbb{P}(X_1 \mathrm{e}^{-r\tau_1}\mathbb{1}_{\{\tau_1\le T\}} > x), \end{aligned}\tag{38}$$

for some $C_0 > 0$. Then, by Lemma 3.5 (2) and (38), we obtain that for sufficiently large n and x,

$$\begin{aligned}\mathbb{P}\Big(\sup_{0\le t\le T} I_{4,t} > x\Big) &\ge \mathbb{P}\Big(\sum_{k=1}^{n} X_k \mathrm{e}^{-r\tau_k}\mathbb{1}_{\{\tau_k\le T\}} > x\Big) \\ &\sim \Bigg(\sum_{k=1}^{\infty} - \sum_{k=n+1}^{\infty}\Bigg)\mathbb{P}(X_k \mathrm{e}^{-r\tau_k}\mathbb{1}_{\{\tau_k\le T\}} > x) \end{aligned}$$

$$\begin{aligned}&\geq (1-\epsilon C_0)\sum_{k=1}^{\infty}\mathbb{P}(X_k\mathrm{e}^{-r\tau_k}\mathbb{1}_{\{\tau_k\leq T\}}>x)\\ &= (1-\epsilon C_0)\lambda\int_0^T\overline{F}(x\mathrm{e}^{rt})\mathrm{d}t. \qquad (39)\end{aligned}$$

Hence, by (37), (39) and the arbitrariness of $\epsilon>0$, we derive that

$$\mathbb{P}\Big(\sup_{0\leq t\leq T}I_{4,t}>x\Big)\sim\lambda\int_0^T\overline{F}(x\mathrm{e}^{rt})\mathrm{d}t, \qquad (40)$$

which, together with $F\in\mathbf{L}\cap\mathbf{D}$, implies that the distribution of $\sup_{0\leq t\leq T}I_{4,t}$ belongs to the intersection $\mathbf{L}\cap\mathbf{D}$. Indeed, for any $a\in\mathbb{R}$,

$$\int_0^T\overline{F}((x+a)\mathrm{e}^{rt})\mathrm{d}t\sim\int_0^T\overline{F}(x\mathrm{e}^{rt})\mathrm{d}t,$$

and for any $0<v<1$,

$$\limsup_{x\to\infty}\frac{\int_0^T\overline{F}(vx\mathrm{e}^{rt})\mathrm{d}t}{\int_0^T\overline{F}(x\mathrm{e}^{rt})\mathrm{d}t}\leq\limsup_{x\to\infty}\sup_{0\leq t\leq T}\frac{\overline{F}(vx\mathrm{e}^{rt})}{\overline{F}(x\mathrm{e}^{rt})}\leq\limsup_{x\to\infty}\sup_{z\geq x}\frac{\overline{F}(vz)}{\overline{F}(z)}<\infty.$$

By Lemma 3.11, there exist two positive constants C_1 and C_2 such that for any $\epsilon>0$,

$$\mathbb{E}\Big(\sup_{0\leq t\leq T}|I_{2,t}|^{J_F^++\epsilon}\Big)\leq C_1\Big(\int_0^T\mathrm{e}^{-2rs}\mathrm{d}s\Big)^{\frac{J_F^++\epsilon}{2}}<\infty,$$

and

$$\mathbb{E}\Big(\sup_{0\leq t\leq T}|I_{3,t}|^{J_F^++\epsilon}\Big)\leq C_2\Big(\int_{|x|\leq1}x^2\nu_P(\mathrm{d}x)\Big)^{\frac{J_F^++\epsilon}{2}}<\infty.$$

By using Markov's inequality, Lemma 3.1 (2) and Lemma 3.2 (2), these two inequalities imply that

$$\begin{aligned}\mathbb{P}(\sup_{0\leq t\leq T}|I_{i,t}|>x) &\leq x^{-(J_F^++\epsilon)}\mathbb{E}\Big(\sup_{0\leq t\leq T}|I_{i,t}|^{J_F^++\epsilon}\Big)\\ &= o(\overline{F}(x))\\ &= o(1)\mathbb{P}(X_1\mathrm{e}^{-r\tau_1}\mathbb{1}_{\{\tau_1\leq T\}}>x)\\ &= o(1)\int_0^T\overline{F}(x\mathrm{e}^{rt})\mathrm{d}t, \qquad (41)\end{aligned}$$

$i=2,3$. Since the distribution of $\sup_{0\leq t\leq T}I_{4,t}$ is in the class $\mathbf{L}\cap\mathbf{D}=\mathbf{S}\cap\mathbf{D}$, by (40), (41) and using Lemma 3.3 (2), it follows that

$$\mathbb{P}(M_T>x)\sim\mathbb{P}\Big(\sup_{0\leq t\leq T}I_{4,t}>x\Big)\sim\lambda\int_0^T\overline{F}(x\mathrm{e}^{rt})\mathrm{d}t.$$

This ends the proof of Theorem 2.2.

4.3. **Proof of Theorem 2.3.** Along the line of [14], for the upper bound of relation (7) with $T=\infty$,

$$M_\infty\leq M_1+\mathrm{e}^{-r}\sup_{t\geq1}\int_1^t\mathrm{e}^{-r(s-1)}\mathrm{d}P_s\overset{\mathrm{d}}{=}M_1+\mathrm{e}^{-r}M_\infty, \qquad (42)$$

where on the right-hand side M_∞ is independent of M_1. Consider the stochastic difference equation

$$Q^* \overset{\mathrm{d}}{=} M_1 + Q^* \mathrm{e}^{-r}, \tag{43}$$

where on the right-hand side Q^* is independent of M_1. By Theorem 2.2, we have that $F_{M_1} \in \mathbf{L} \cap \mathbf{D}$ and

$$\mathbb{P}(M_1 > x) \sim \lambda \int_0^1 \overline{F}(x\mathrm{e}^{rt})\mathrm{d}t. \tag{44}$$

Let $\{Q_k,\ k \geq 1\}$ be a sequence of r.v.s defined by the recursive equation

$$Q_k = M_{1,k} + Q_{k-1}\mathrm{e}^{-r},\ k \geq 1,$$

where $Q_0 = M_\infty$ and $M_{1,k},\ k \geq 1$, independent of Q_0, are the independent copies of M_1. By (42), we can prove that $\{Q_k,\ k \geq 1\}$, starting with $Q_0 = M_\infty$, are stochastically nondecreasing, and weakly converges with a limit distribution of Q^* in (43), by Theorem 1.5 (i) of [31]. Thus, by Lemma 3.9 (2),

$$\begin{aligned}
\mathbb{P}(M_\infty > x) &\leq \mathbb{P}(Q^* > x) \\
&\sim \sum_{k=1}^{\infty} \mathbb{P}(M_1 > x\mathrm{e}^{r(k-1)}) \\
&\sim \lambda \int_0^\infty \overline{F}(x\mathrm{e}^{rt})\mathrm{d}t,
\end{aligned} \tag{45}$$

here we used the fact

$$\begin{aligned}
\overline{F_{M_1}}^*(\mathrm{e}^r) &= \limsup_{x\to\infty} \frac{\mathbb{P}(M_1 > x\mathrm{e}^r)}{\mathbb{P}(M_1 > x)} \\
&= \limsup_{x\to\infty} \frac{\int_0^1 \overline{F}(x\mathrm{e}^{rt} \cdot \mathrm{e}^r)\mathrm{d}t}{\int_0^1 \overline{F}(x\mathrm{e}^{rt})\mathrm{d}t} \\
&\leq \limsup_{x\to\infty} \sup_{0\leq t\leq 1} \frac{\overline{F}(x\mathrm{e}^{rt} \cdot \mathrm{e}^r)}{\overline{F}(x\mathrm{e}^{rt})} \\
&\leq \limsup_{x\to\infty} \sup_{z\geq x} \frac{\overline{F}(z\mathrm{e}^r)}{\overline{F}(z)} = \overline{F}^*(\mathrm{e}^r) < 1.
\end{aligned}$$

For the lower bound of relation (7) with $T = \infty$, by Theorem 2.2, we have that for any $T > 0$,

$$\begin{aligned}
\mathbb{P}(M_\infty > x) &\geq \mathbb{P}(M_T > x) \\
&\sim \lambda \left(\int_0^\infty - \int_T^\infty \right) \overline{F}(x\mathrm{e}^{rt})\mathrm{d}t.
\end{aligned} \tag{46}$$

For any $0 < \epsilon < J_F^-$, by Lemma 3.1 (1), the dominated convergence theorem and Fatou's lemma, respectively, give that

$$\begin{aligned}
\limsup_{x\to\infty} \int_T^\infty \frac{\overline{F}(x\mathrm{e}^{rt})}{\overline{F}(x)}\mathrm{d}t &\leq C \int_T^\infty \mathrm{e}^{-r(J_F^- - \epsilon)t}\mathrm{d}t \\
&= \frac{C}{r(J_F^- - \epsilon)} \to 0, \quad T \to \infty,
\end{aligned}$$

and

$$\liminf_{x\to\infty}\int_0^\infty \frac{\overline{F}(x\mathrm{e}^{rt})}{\overline{F}(x)}\mathrm{d}t \geq \frac{1}{C}\int_0^\infty \mathrm{e}^{-r(J_F^+ +\epsilon)t}\mathrm{d}t = \frac{1}{Cr(J_F^+ +\epsilon)} > 0.$$

Plugging these two estimates into (46) and letting $T\to\infty$, we derive that

$$\mathbb{P}(M_\infty > x) \succ \lambda\int_0^\infty \overline{F}(x\mathrm{e}^{rt})\mathrm{d}t. \tag{47}$$

Therefore, the desired relation (7) holds for $T=\infty$ from (45) and (47), immediately.

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