THE LIFE AND WORK OF PROF. GEORGE CHRYSTAL (1851-1911)

Mohammad Yousuf

A Thesis Submitted for the Degree of PhD
at the
University of St Andrews

1990

THE LIFE AND WORK OF

PROF. GEORGE CHRYSTAL

(1851 - 1911)

By

MOHAMMAD YOUSUF

A thesis submitted for the degree of Doctor of Philosophy of the Uiversity of

St. Andrews

Jan. 1990

DECLARATIONS

(a) I MOHAMMAD YOUSUF hereby certify that this thesis has been composed by myself, that it is a record of my own work, and that it has not been accepted in any previous application for a higher degree.

Signed Date 3/1/90

(b) I was admitted to the Faculty of Science of the University of St. Andrews under Ordinance General No. 12 on 7/1/86 and as candidate for the degree of Ph.D. on 9/10/86

Signed Date 3/1/90

(c) In submitting this thesis to the University of St. Andrews I understand that I am giving permission for it to be made available for use in accordance with the regulations of the University Library for the time being in force, subject to any copyright vested in the work not being affected thereby. I also understand that the title and abstract will be published, and that a copy of the work may be made and supplied to any <u>bona fide</u> library or research worker.

CERTIFICATION

I hereby certify that the candidate has fulfilled the conditions of the Resolutions and Regulations appropriate to the degree of Ph.D.

Signature of the Supervisor

Date

3/1/90

Acknowledgements

I would like to thank my Supervisor, Dr Edmund F. Robertson, for his guidance at every stage of my research, especially for his encouragement in writing this thesis and his help in final stages of its preparation.

I would also like to thank Professor A. D. D. Craik, for his help and guidance in general and particularly his helpful suggestions in respect of writing the chapter Theory of Seiches.

I would like to record my thanks to Dr J. J. O'Connor, for his valuable suggestions in writing the section Non-Euclidean Geometry in Chapter 4 and to Drs. C. M. Campbell, J. C. Amson, G. M. Phillips for their constant encouragement.

I am grateful to Dr P. M. Harman of Cambridge University for drawing my attention to Chrystal - Maxwell correspondence.

To the following libraries, Institutions, and Public Offices, I am deeply indebted for information, photocopied documents and permission to reproduce letters and other works:

Cambridge University Library; Newnham College Library, Cambridge; Peter House, Cambridge; Corpus Christi College, Cambridge; The Royal Society of London; Department of Entomology, Oxford University; The Royal Society of Edinburgh; Edinburgh Mathematical Society; Edinburgh University Library; Moray House College of Education, Edinburgh; George Watson's College, Edinburgh; General Register Office for Scotland; Aberdeen University Library; and Grammar College, Aberdeen.

In addition I would like to thank all the staff of St. Andrews University Library, in particular, to the staff in the Manuscript Section, and Mr Frank Story, Assistant Librarian for their constant help in locating different material.

I am grateful to the Ministry of Education, Government of Pakistan, for the financial support, without which, all of this would have been impossible.

Last, but by no means least, I am thankful to my family including my parents for their patience and constant support during the entire work.

ABSTRACT

This thesis is principally concerned with George Chrystal's life and his work, mainly in three directions viz., as an experimentalist, a mathematician, and an educationist.

The main object is to bring to light the work of a personality who is representative of many more who are always forgotten. The majority of historians of science consider the works of the giants in science, ignoring totally the contributions made by the less prominent people like Prof. George Chrystal.

In fact their contributions serve as one of the most important factors in propagation of scientific knowledge.

His main contributions: verification of Ohm's Law experimentally; Non-Euclidean geometry; differential equations; text books on algebra; theory of seiches; institution of leaving certificate examination in Scottish education and many more have been discussed in detail.

A survey of Chrystal's general thought is given in so far as it may be gathered from his scattered remarks.

The references are mentioned by numerals in the superscript, details of which are given at the end of each chapter.

The main text consists of six chapters. There are three appendices at the end; Appendix 'A' consists of his correspondence with different scientists, most of which is still unpublished, Appendix 'B' contains a bibliography of his contributions in chronological order, and Appendix 'C' contains his three Promoter's addresses.

Tables and figures are attached at their proper places, including some rarely available photographs.

CONTENTS

Chapter 1

Memoir- George Chrystal

1.1 Introduction

George Chrystal was arguably the most influential Scottish mathematician of his time. He has achieved the rare distinction of one whose influence permeates a whole subject. His view of mathematics as a total system made it difficult for him to concentrate his efforts in any one branch of the subject, although nowadays perhaps he is best remembered for his admired Textbook of Algebra and his Hydrodynamical Theory of Seiches.

It is worth remembering that some areas of mathematics to which he gave an initial impetus have proved useful in fields far removed from pure mathematics. The best known example is the hydrodynamical theory of seiches. Experimental physicists note his name in connection with an experimental verification of Ohm's law, educationists for his work in connection with the institution of Leaving Certificate Examination, and the training of teachers.

No doubt we may arrange the whole horizon; but after all we look from our own point of vantage. What may appear as a towering peak to one may seem an ordinary eminence to another. Nevertheless, incomplete and historically partial though it must be, a sketch of the career of a leader of scientific thought who lined his strenuous mental life through this formative time cannot be without its value as a contribution to the history of the growth of ideas. Such a one, preeminently, was Professor George Chrystal of Edinburgh University. He was influenced by the direct contact and personal friendship of Maxwell, Stokes, Kelvin, Tait, Helmholtz, Cayley, Sylvester and many others. These contemporaries were to him personalities and not mere writers of papers or of books. He got much from them and he gave much to them. As a historian of contemporary developments he takes high rank. He lived at a time when the whole foundations of mathematics were being reexamined.

1.2 Early Life In Aberdeen

George Chrystal, of Gateside, Newhills Aberdeenshire, was born at Mill of Kingoodie in the parish of Bourtie near Old Meldrum, Aberdeenshire on Saturday 8th of March 1851. His father, William Chrystal was first a grain merchant and afterwards a farmer and landed proprietor. His mother, Margaret Burr was daughter of James Burr of Mains of Glaik, Aberdeenshire. He had two brothers and a sister. During the early days of Chrystal's life the family stayed at Mill of Kingoodie, and he received his early education at the parish school of Old Meldrum, some two miles distant from his home.

According to J. S. Black[1], " From an early age he gave marked promise of intellectual distinction, though physically he was far from strong, and was hampered by a lameness which he afterward outgrew, but which precluded him from joining in some of the more boisterous activities of boyhood".

In view of his these intellectual promises his parents decided to send him to a good secondary school, so the family moved to Aberdeen in the early part of the eighteen sixties. According to the record of the Grammar School of Aberdeen he entered in class one. These records also show that in 1866 he gained the Williamson Scholarship, which was given by competition to the best general scholar in class three, in 1867 in his fourth and final class in Grammar school he was awarded the Silver Medal. The same year, at the age of seventeen he passed into Aberdeen University. The University of Aberdeen comprised as today of two colleges, King's and Marischal, each of which was originally a separate foundation, with its own degrees and professors. The two institutions were united in 1860. Professor Chrystal studied for four years in Aberdeen University, where much emphasis was given at the time to Classics in the first two years. Prof. Alexander Bain, Professor of Rhetoric and Logic; and Geddes, Professor of Greek were celebrities in Classics.

Bain's Rhetoric class, of which many amusing reminiscences have been written, together with the less advanced classes in the two learned languages, was then prescribed as the course of Bejans or freshmen during their first year of study.The semis or second-year men proceeded to more advanced classical instruction and began mathematics and science. In their final two years, as Tertians and Magistrands, the undergraduates pursued the exact sciences still further, and, while in the third session they again followed the instruction of Mr. Bain in the class of logic and psychology, in the fourth they made acquaintance for the first time with the Professor of Moral Philosophy.

So both at the beginning and towards the end, of his undergraduate days, Chrystal was to come into contact with Alexander Bain who thus had much influence on Chrystal, an influence which lasted throughout his life.

The staff teaching science courses were hardly less remarkable. This included prominent teachers like Fuller, Professor of Mathematics; Nicol, Professor of Botany and Zoology; Thomson, Professor of Natural Philosophy and others. Professor Fuller in teaching his classes and in examining exercises was aided by his Assistant.

According to the Aberdeen University Calender[2]:

There were two mathematical classes. The junior class met two hours a day for five days in a week. The subjects were Euclid Books I; II; III; IV; VI; and XI. to prop.21; Algebra; and Plane Trigonometry. The first hour each day was devoted to geometry and plane trigonometry, under the superintendence of the professor. The second hour was devoted to algebra, and, when the class had proceeded as far as quadratic equations, it was divided and the less advanced students (usually about one-third of the whole number) pursued a more elementary course in the higher parts of the subject, under the direction of the Assistant. The text-books recommended were Todhunter's "Algebra" and Snowball's "Trigonometry". The senior class met one hour a day for five days a week. The subjects were Geometrical Conic Sections, Spherical Trigonometry, Analytical Geometry, and the Differential and Integral Calculus. This class was also divided about the middle of the session on the same principle as the junior class. The text-books recommended were Drew's "Conic Section"; Todhunter's

"Conic Section" and Todhunter's "Calculus".

Professor Fuller paid special attention to prepare Chrystal for a scholarship in mathematics. His efforts in this respect were strengthened by the private coaching of David Rennet LL.D, in the summer vacation.

In fact among the science teachers in Aberdeen, these two along with Professor Thomson were influential in diverting Chrystal's attention towards mathematical sciences.

Thirty years later , in 1898, he dedicated his book," Introduction to Algebra" to Dr. David Rennet in these words," In memory of happy hours spent in his class room in days of old".

He graduated in 1871 from Aberdeen with first class honours in mathematics and natural philosophy and received the Town's Gold Medal given to the most brilliant student of the year. The efforts of his teachers were not in vain. Chrystal achieved all the scholarships available to a graduate of a Scottish University at the time, which are listed below:

The Simpson Prize in Mathematics; The Arnott Prize of Experimental Physics; Fullerton Scholarship for Mathematics and Natural Philosophy; The Ferguson Mathematical Scholarship open to recent graduates from any of the four universities in Scotland.

In those days Aberdeen had eyes only for Cambridge, where there was an unrivalled gathering of great men in the world of science, and where year by year Aberdonians were winning their way to the highest wranglerships. George Chrystal also got an open scholarship at Peterhouse, Cambridge.

Chrystal describes his own experience at Aberdeen University in his promoter's address to graduates of Arts in Edinburgh University in these words[3]:

"When I entered the University of Aberdeen some eighteen years ago I was a moderate classical scholar, but I had learned practically no mathematics. We used to read the first book of Euclid as far as *pons asinorum*; but regularly as we reached the dreadful pass we were turned back for a revisal. Algebra I had none, not to speak of other mathematical furniture. Yet large demands were made upon me during my second session under Professor Fuller, and I had to work hard during the spare time of my first year to be able to take his junior class with advantage. The fact that mathematical students from Aberdeen had been doing well in the world long before the time I allude to, was due to no exertions on the parts of the schools, but simply to the presence in the Faculty of Arts of two teachers, Professors Fuller and Thomson, of exceptional energy and ability, whose efforts were ably seconded by a private tutor, Mr Rennet, well known and much beloved by all Aberdeen graduates, who combined in a way more happy than common the power of dealing at once with the best and with the worst material that came up to the university. With regard to the rest of the teachers of my first **alma mater,** I have this to say, that I entered their lecture rooms a child intellectually, and that I have emerged as a man, and that during no other part of my mental life have I made so much intellectual progress as I did under their tuition".

1.3 Life at Cambridge and St. Andrews

George Chrystal started his stay in Peterhouse in 1872. From his correspondence of the time it is clear that he was much influenced by Professor James Clerk Maxwell, who himself was appointed as Cavendish Professor in Experimental Physics a year before in 1871, and was not directly involved with the teaching for Mathematical Tripos. Chrystal already an Honours graduate from Aberdeen University was well acquainted with most of the work of the Tripos, so had enough spare time for other academic and extra-mural activities. These included his work on some experiments under the supervision of Maxwell; the wining of the first prize of mathematical junior sophs in 1873, and the same year wining the member's prize for his essay " Wit and Humour as Exhibited in English Poetry".

He also mentioned his friendship with Robert Neil from Aberdeen. This has been described by Dr. J. S. Black as follows[4]:

"Writing long afterwards to the Late Dr Adam of Emmanuel, who was engaged on a memoir of that eminent scholar Mr R. A. Neil of Pembroke, he said:-

' The happiest days of my life were my undergraduate days at Peterhouse, and the chief joy of that time was my friendship with Neil''.

Robert Neil also from Aberdeen, a teacher and a close friend of Chrystal died in 1901.

In his last Promoter's Address of 1908, Chrystal in the light of his own experience gives a comparison of his student days at Aberdeen and Cambridge. He says:

"When I went to the University of Cambridge, I found that the course there for the ordinary degree in Arts was greatly inferior in educational quality to the Scottish one. On the other hand, the courses in honours were on a very much higher standard, although they suffered greatly from the chaotic organisation of the English universities which, about that time, were, to use a mathematical phrase, passing through a minimum turning point in their history. I might liken the difference between the English and Scottish university courses at that time to the difference that then existed between their national styles of cookery. The Scottish cuisine was characterised by lightness and variety, the English cuisine was noted for plenty and excellence of material, but lacked variety, and the defective preparation of its dishes often left them heavy and indigestible. I have frequently been tempted to think that the three years I spent as an undergraduate at Cambridge were wasted years of my life, if they were to be valued merely by the amount of new knowledge acquired, no doubt they were largely wasted, but, on the other hand, they were of great advantage to me in other respects. I made the acquaintance of a large number of the ablest young men of my generation........ .

Cambridge at that time presented strange contrasts. Although almost decadent as an educational institution, it numbered among its members, perhaps the greatest galaxy of intellectual stars that ever illustrated any period of the history of a university. It was doubtless these great men who sowed, it may be unconsciously, the seeds of that great resurrection which has again raised my second **Alma Mater,** in spite of many picturesque absurdities, to her present high position she now holds, not because of stars in her intellectual firmament, although such are not wanting, but because she possesses a great body of devoted teachers and investigators, all inspired by their various ways by high ideals of the work of a university".

Chrystal graduated from Cambridge in 1875 and according to Mathematical Tripos on 28 January 1875, he was bracketed as second Wrangler with W. Burnside of Pembroke, whereas John William Lord of Trinity was declared as first Wrangler. On 10 February 1875, he was declared Second Smith's Prizeman, and the First Smith's Prize was given to W. Burnside of Pembroke with whom he shared second wranglership, the examiner for Smith's Prizes being G. G. Stokes. William Burnside(1852-1927) and George Chrystal were latter intimate friends and together they did the proof reading of the second volume of the admirable book "Elements Of Natural Philosophy" by Tait and Kelvin, most popularly known as T &T′ and also checked the answers to the numerical exercises in the book, which has been fully acknowledged in the preface.This is also mentioned in one of Chrystal's letters to Sir William Thomson[5].

William Burnside who became Professor of mathematics at the Royal Naval college, Greenwich was awarded the Honorary degree of LL. D in 1909 by the University of Edinburgh. He published many papers on group theory and published a famous book "Theory of Groups of Finite Order", Cambridge, First edition, 1897.

According to Patrick Bury in his " History of Corpus Christi College"[6], Chrystal was elected Fellow and Lecturer of Corpus Christi College on 30 April 1875, which at first he declined because of a misrepresentation, but after a long discussion at a College meeting on 3 May 1875, the election was confirmed. His duties included lecturing in mathematics and physics to students of different colleges in Cambridge.

This was a great start for young Chrystal and he soon established himself as a successful teacher. Besides his teaching activities he also took part in some university and college reform.

In the meantime William L. F. Fisher, occupant of the Regius Chair of Mathematics in the University of St. Andrews resigned in the summer of 1877, thus leaving the post vacant. Chrystal came to know about this through a letter from Prof. J. C. Maxwell which is given below:[7]

"Dear Chrystal

I send on to you what Professor Swan sent me.

The main thing to get evidence of is your teaching powers. Have you lectured or coached any men who could either testify themselves or be testified so by examiners that before they came to you they were ignorant of facts and incapable of actions which after you had operated on them became familiar?

They are afraid of getting a learned man who cannot teach but his learning remains with him".

Professor Swan was Professor of Natural Philosophy in University of St. Andrews and the letter referred to above is his letter of 13 July 1877 which is reproduced as follows:

"The chair of mathematics in the University of St. Andrews will shortly be vacant by the resignation of Professor Fisher; and it is most desirable that a thoroughly good man should be got to succeed him. For that end it is obviously of the greatest importance that the impending vacancy should be made known extensively as possible in quarters where fit candidates are likely to be found; and I shall esteem it a favour if you can kindly direct the attention of any thoroughly competent persons whom the appointment may concern to the following statement".

At the end of the letter he describes the work of a Professor in these words:

"......The professor's yearly work extends over a session not exceeding six months in Nov-April. The Crown is patron. The appointment ought to be made before November at latest, and probably will be much earlier".

Chrystal applied for the post and strengthened his application by testimonials from many of the scientists with whom he had been in contact at Cambridge. These included Prof. J. C. Maxwell, Sir William Thomson, Prof. Tait, Sir G. G. Stokes, and Mr E. Routh, his private coach at Cambridge. They all praised his capabilities as a teacher, which was the main requirement for the post. But despite this, he was not very hopeful, because he had very little experience and expected tough competition.

At the same time he was also a candidate for a similar post in Sydney, which he latter refused. He mentioned all this in a letter to Professor G. G. Stokes on 30 July 1877 as follows[8]:

"Sir

I have become a candidate for the St. Andrews chair of mathematics which is to be vacant soon.

The candidates are I believe expected to produce evidence of fitness in some way. I have therefore taken the liberty of asking whether as Smith's Prize Examiner for 1875 you formed such an opinion of my abilities as would enable you to testify that I am a fit person to hold the above office.

Since I refused to become a candidate for a similar office at Sydney my prospects of being able to remain at Cambridge have changed for the worse. If I were fortunate enough to get the St. Andrews appointment. I should have several months free for independent work".

His refusal to become a candidate for the professorship in Sydney was also due to the reluctance from his mother and sister to allow him to leave the country, which he mentions in another letter to Professor G. G. Stokes as follows[9]:

"I have had a letter from my people this forenoon. My mother and sister are extremely averse to my leaving the country and I am not clear in my own mind that I should be exactly the sort of man wanted in Sydney.

I have to thank you for your great kindness in the matter, not without a feeling of regret that I should have occupied your time in the way I have done lately".

To his surprise, he got a telegram on Saturday, 3 November 1877, followed by a letter from the Home Secretary confirming his appointment as Regius Professor at St. Andrews. As the appointment was already very late and the session had also started, so he was asked to join in his earliest covenience. This he did and took up the post the following Monday.

The long awaited news was welcomed in St. Andrews and Principal Shairp in his welcome address said[10]:

"I am glad to say that the Crown, in whose hands the patronage of the mathematical chair is vested, has given the university as Professor Fisher's successor one of the most distinguished of the young mathematicians of Cambridge, of whose future the chief men of science there and elsewhere have expressed very high anticipations. I trust that these prognostications may be amply fulfilled by what he may be able to do for this university, and for the sciences to which he has devoted to himself".

As is clear from Prof. Swan's letter the work of a Professor extended over a session from November through April, so he had enough time to complete his article" Electricity" for the ninth edition of the Encyclopædia Britannica.

He married Miss Ann Balfour on 26 June 1879 in Bonn (on Rhine) in Germany.

1.4 Life in Edinburgh

In May 1879 the chair of mathematics in the University of Edinburgh became vacant due to the death of Rev. Professor Kelland, a man of Cambridge, senior Wrangler and first Smith's Prizeman, who had occupied the mathematical chair from the time of the resignation of Professor Wallace in 1838. Although an Englishman, however as Sir Alexander Grant, the Principal of the University of Edinburgh said, he came to know the Scottish universities better even than Scotsmen themselves. Professor George Chrystal together with Professor Tait later writing Kelland's obituary added the following[11]:

"He knew also, as few have ever known them, the characteristics and the wants of Scottish students........ . He who has in person instructed, alike by clear precept and noble example, many thousands of the youth of a nation, cannot fail to have a happy and lasting influence on that nation's progress".

Philip Kelland was, in the very highest sense, a benefactor to Scotland. An idea of Professor Kelland's scientific activity will be obtained by looking at the list of papers under his name in the Royal Society's catalogue of scientific memoirs.

Chrystal offered himself as a candidate for the post and made his application complete in all respects before leaving to go to Germany for his marriage ceremony. The other candidates were Professor Aldis, Physical Science College, Newcastle; Professor Barker, Owens College, Manchester; Mr Norman Macleod Ferrers, Gonville and Caius College, Cambridge; Mr William Lees, Mathematical examiner, Edinburgh; Dr Alexander Macfarlane, Edinburgh; Mr Munn, Glasgow and Professor Niven, Queen's College, Cork. There is no difficulty now about placing these men in their appropriate niches ; but in 1879, when the best work of most of them was still to be done, it could not have been an easy matter to discriminate among them. In the *Scotsman* of Saturday, 19 July 1879 we find a remarkably sane and prescient discussion of the choice which the Curators had made. Some of the sentences are well worth quoting as showing that even in these days the characteristics of some of the men had been clearly diagnosed:

"Professor Chrystal of the University of St. Andrews, was yesterday appointed to the Chair of Mathematics, in the University of Edinburgh, vacant by the death of Professor Kelland. We understand that, although no actual vote was taken the names of Mr Munn of the High School of Edinburgh; Professor Barker of Manchester; and the Rev. Norman Macleod Ferrers of Cambridge were also highly spoken of, and their claims strongly urged by several of the Curators".

The Edinburgh Courant of the day welcomed the appointment but described it as unexceptional, most probably because of Professor Chrystal's youth and less experience and expressed it as follows:

"........ yesterday's appointment in the Chair of Mathematics is in all respects unexceptional, and we hail it as an additional augury of a time of real progress. We welcome to Edinburgh, in the place of the late venerable professor, a young hard-headed Aberdonian- the most brilliant, perhaps, of the northern band which was trained by Professor Fuller to sweep off the spoils from the Saxon in the Cambridge Mathematical Tripos. Taught in Scotland by an Englishman, one who like (Professor Kelland) came to know the Scottish universities better almost than do Scots themselves, and who was one of the most successful teachers of mathematics whom this century has produced, Professor Chrystal has added to his store from English and from foreign sources alike; and, coming back to Scotland, he has already proved himself to be an admirable teacher. His testimonials are almost the strongest we have ever seen, and they come from men like Stokes, Thomson, and Clerk Maxwell, whose verdict in such matters is final.

In the face of a candidature which embraced no less than four senior wranglers, besides several others high on Cambridge lists, what accounts could the curators take of the Scotsman's mystic....? The vulgar Scottish use of the words mathematics and mathematician is alone responsible for this very curious specimen of a blunder. In some parts of Scotland and in the North-East of Ireland (which is practically a part of Scotland) it is not unusual to hear of a man who is a 'grand mathematician' and who 'also knows some Algebra!' The people who use language in this way, six books of excellent but altogether elementary work of Euclid represent 'mathematics'; quaternions, quantics, Invariants, and Transcendents of all kinds are alike unknown. What if the mystic candidate from cock burn street had got the chair, and a student had come seeking information - not in 'mathematics' or algebra, but in calculus of variations - and had said (in scotlandised Latin) 'Dominie dirige nos?' It is too awful to think of......... ".

Professor Chrystal's inaugural address in Edinburgh was delivered on Thursday 30 October 1879. He chose as his subject "The History of Mathematics", with special reference to the former occupants of the Edinburgh chair; and for the occasion the chemistry class room was densely crowded, and a large number of professors were present, as reported by the *Edinburgh Courant* the following day.

It was also reported there that in the unavoidable absence of Sir Alexander Grant, Bart; Professor Campbell Fraser, the Dean of the Faculty of Arts, presided. The lecture was interesting and a major part of it was latter quoted by Sir Alexander Grant, in his two volume book "The story of the University of Edinburgh: its first three hundred years"[12], who acknowledges Professor Chrystal along with some other professors for helping him in preparing this book.

Thus he began his long career as Professor of Mathematics, which lasted until his death in 1911, without any interruption. During these long years he did not waste any moment in raising the standards of mathematics, not only in the University of Edinburgh, but in all the Scottish universities and schools. This also provided him an opportunity to work for proper educational reforms to improve the standards of education as a whole.

The main work of a professor at the time was teaching, as had been made clear earlier, but this did not mean that they were not undertaking any research. Indeed Prof. Chrystal did promote both teaching and research keeping the balance, and kept alive the old tradition of Edinburgh as a centre of mathematical sciences.

Mathematics was taught at the time as one of the seven compulsory subjects in the Faculty of Arts, and the students coming from different secondary schools did not have a uniform level of mathematics, so raising the standards in mathematics and for that matter in any of the seven compulsory subjects was not an easy task.

Professor Chrystal from the very beginning tried to introduce in his teaching gradually but steadily the new developments in the subject, which proved to be difficult for some of his students; whereas the students who had a special interest in mathematics really profited from this experience.

From different accounts of the students of the time given in different places it is clear that Prof. Chrystal changed the whole atmosphere of teaching mathematics in Edinburgh, yet at the time of the examinations he used to pay due consideration to every category of students.

J. M. Barrie in his " An Edinburgh Eleven "[13] gives a graphic picture of Prof. Chrystal, which is unsatisfactory if not misleading although it does suggest that Chrystal's teaching talents were wasted on the weakest students.

Sir J. S. Flett in his account of his student days describes the abilities of Professor Chrystal in these words:[14]

"A sharp little man, with piercing eyes and a very quiet manner, but very prompt and businesslike, and a good and earnest teacher...... belonged to a school of modern mathematicians that had quite a vogue in France and Germany and have done important work in clearing up the essentials of mathematical reasoning........"

In the **Student,** the Edinburgh University magazine, many interesting accounts of his personality are given. It is a fact that the students are the real judge of a teacher. In the **Student** of 3 December 1890[15], it is written:

"As a professor in his own class-room, no one could be more courteous and considerate than Professor Chrystal. He grudges neither time nor labour in the elucidation of the numerous difficulties of his students. The gulf which is said to be fixed between professors and students cannot be said to have existence in the mathematical class-room. Professor Chrystal is perhaps the most approachable of all our professors".

Professor Campbell Fraser, Professor of Logic and Metaphysics resigned from his chair and from Dean Faculty of Arts in 1891. Professor Chrystal was appointed to succeed him. He therefore took charge of the office at a time when the new scheme for Arts degree was under consideration, as a consequence of Universities (Scotland) Act, 1889, followed by many ordinances passed in 1892 and later. This at last finished the monopoly of the seven compulsory subjects in Arts and many new options were introduced thus broadening the Arts curriculum. It was Professor Chrystal who by his administrative experience and broad knowledge of problems brought these changes in to practice. He led the Faculty through this transitional period successfully, and worked hard for the introduction of three terms sessions into Scottish universities under Universities(Scotland) Act, 1908.

In the **Student** of 8 February 1907, another account of his personality is given in the following words[16]:

"......our clearest recollection of Professor Chrystal stands out from a mist of dingy lecture-rooms in the old buildings. It consists of a grey-headed but erect figure, dashing, with marvellous speed, cohorts and battalions of graphs, theorems, triangles, and symbols upon a silent, but secretly suffering black-board. Then, when every available square inch of space had been filled, there was a triumphant swing round to his astounded audience, as the professor sped on to some other colossal piece of mathematical architecture.

We recall the voice, convincing and sustained, soaring intrepidly through a mass of stupefying calculation. We recall the genial face, flushed with a victorious effort over the obstinate powers of x and y, the glasses twinkling with the success of a clear piece of demonstration. These we would see from under our heavy matutinal eyelids between the hours of 9 and 10 a.m.; and we would emerge from the class-room thoroughly awake, refreshed, and suitably ashamed of our appalling ignorance and slowness of wit.

The subject of mathematics is necessarily a dry, precise, punctilious one, but Chrystal contrives to handle it with grace and artistry. In our denser moments we would unmathematically admire the sober structure of the determinant, the perfect curve of the hand made circle, the swift Gothic Swerve of the asymptote, as they grew under the hand of a master.

A strong personality is his. There are few classes so completely dominated by their teachers as Professor Chrystal's; he has a fine and tactful sarcasm which he knows well how to use....... .

But those who only know the Chrystal of the class-room know little of him. There is the Chrystal of the private interview - a kindly, sympathetic, helpful teacher. There is the Chrystal who, as Dean of the Faculty of Arts, advises the timid urchin hesitating on the threshold of his academic career, or guides the inexperienced footsteps of students as they face out into the unknown world. And there are many who owe to him more than they themselves are aware of".

The following extract from the university calender of the time gives an outline of the courses given:

"**First Class:** Theory of Arithmetic; Plane Geometry equivalent to Euclid, books I; II; III; IV; and VI; (in the lectures the arrangement of the *syllabus of the plane geometry,* prepared by the Association for the Improvement of Geometrical Teaching, is followed more or less closely); Solid Geometry, equivalent to Euclid, book XI; Geometrical Conic Sections; and Elementary Algebra.

Second Class: Algebra, including the elements of the theory of equations, and of determinants; Plane Trigonometry; Conic Sections, treated geometrically and analytically; and the Elements of Modern Geometry.

Third Class: Higher Algebra, Analytical Geometry, Differential and Integral Calculus, Calculus of Finite Differences.

Additional lectures on quaternions, and other special subjects may be given occasionally".

After the Universities (Scotland) Act 1889 was passed, new Ordinances were enforced under which the outline of courses in mathematics were changed, in the Second Class Algebra, the Theory of Limits and Elements of Curve Tracing were added, the Third Class then extended over two years, being in alternate years more elementary and more advanced. The subjects of the elementary courses were the more elementary parts of the Differential and Integral Calculus, and the calculus of Finite Differences, and the Analytical Geometry of Conic Sections. It was assumed that the students had acquired a knowledge of the Theory of Equations, and of Elements of the Theory of Determinants before they joined the class.

The subjects of the advanced course were the higher parts of the Differential and Integral Calculus; Differential Equations; Definite Integrals with special reference to Fourier's Theorem; Functions of Laplace and Bessel; Higher Plane Curves; Analytical Geometry of Three Dimensions.

Additional lectures on Quaternions, Projective Geometry, and other special subjects were given occasionally.

From the lecture notes taken by his students in Edinburgh University[17], it is noted that Chrystal does not seem to have kept notes of his course, but simply to have prepared his ideas the night before the lecture. As a guide he used all the available material both Continental and British. He never seemed to hurry; and yet the ground covered was enormous. He used many examples to make the concept clear to the class. Though broadly the same, his advanced course varied in detail from year to year. His idea was to attract those who wished to familiarise themselves with the methods of research. This he did by giving every encouragement to anyone who had thought of some mathematical question worthy of investigation, or by suggesting some line of research to the students at large.

Away from his official university work his tireless energies were finding fields for exercise. He wrote many of the longer and more important scientific articles as well as the biographical notes of famous mathematicians for the Ninth Edition of the Encyclopaedia Britannica. Moreover, he was making use of his free time in the summer vacation in the early part of his professoriate, in experimental and mathematical researches, which will be described in detail in the forthcoming chapters.

1.4.1 The Royal Society of Edinburgh

When he started his career as Professor of Mathematics in Edinburgh, his friend and colleague Professor P. G. Tait, Professor of Natural Philosophy was General Secretary of the Royal Society of Edinburgh, and so his connection with the society started right from 1879 when he was asked to give an address to the society. He was elected Fellow of the Society in its meeting held on Monday, 2 February 1880, and in November of the same year he was elected as a Counsellor. For this office he was elected three times and served the society in this capacity in the years 1880-3, 1884-7, 1895-1901. In 1901 when Professor Tait died, the Society in view of his experience and broad knowledge of the affairs of the society elected him as the General Secretary of the Society, the post which he held till his death in 1911. He was very loyal to the Society and with the exception of a few papers, notes, and some other articles, all Chrystal's original contributions to science are to be found in the publications of the Royal Society of Edinburgh. When the council of the society decided to arrange special obituary notices of famous scientists, he contributed by writing an extensive notice of Joseph Liouville (1802-1882)[18].

By this time he had proved himself to be a successful administrator not only in the University but all over Scotland. In running the affairs of the Society he successfully made use of this, and helped the President and the Council in every matter where his help was needed.

The most important achievement of his tenure of this office was to provide the Society with a permanent place for its offices and library, which is the home of the Society even today. The Society occupied the west wing of the Royal Institution on Princess Street in 1826 as a tenant of the Government, but due to the increasing volume of the Society's valuable library it was felt in 1903 that the place was insufficient to accommodate the Society any more. An attempt was made to get the whole building of the Royal Institution for the Society, but there was no alternative accommodation for the National Art Galleries of

Scotland and the Royal Scottish Academy, who occupied the other half of the building.

In 1906 when the Liberal Government came into power, they introduced the National Galleries Bill of Scotland in the parliament, under which the entire building of the Royal Institution was given for the purpose of Arts. But surprisingly there was no mention in the bill how to accommodate the Royal Society of Edinburgh. So the Society for some time found itself without a home.

This was a real time of emergency for the Society and a time of test for Chrystal how to handle this emergency and provide the Society with a proper abode. A committee from amongst the Fellows of the Society was formed to promote the cause, headed by the President of the Society Lord Kelvin. The first deputation was received by the then Secretary for Scotland in Edinburgh, who favoured the cause, but objected that there was not enough support from members of parliament and other public bodies.

Professor Chrystal immediately started the campaign to get the desired support, he convinced all the Scottish members of the parliament, irrespective of their parties, and obtained their support. He also tried to get the support of the Fellows of the Royal Society of London, which can be noted from his letters to Sir Joseph Larmor FRS; in his first letter on the subject of 13 November 1906, he writes[19]:-

" I hope you will by your presence support a deputation next week to the Secretary for Scotland to make a last appeal for justice to the Royal Society of Edinburgh in the matter of its accommodation. We are to be expelled without any previous consultation from the rooms that were built for us and which we have occupied for eighty years; and now government proposes to put us into a miserably inconvenient house in a bad situation and to give for our installation a sum which at the highest computation is less than half of what is necessary. The only compensation being a shadowy promise to remedy another grievance of thirty years standing by giving us a publication grant of £300.

Which amounts to enlarging the blanket by cutting off the top and sewing it to the tail. The result of course would be the financial ruin of the Society. Our annual deficit is already about £300 and we are paying for our publications partly out of capital.

The whole thing is the result of an intrigue by some friends of the R. S. A. you will have similar trick played on you in London some day".

Two days latter in another letter to Sir Joseph Larmor he writes[20]:-

" The deputation is fixed for Thursday, 22nd at 12.30 in Dover House.

I can not see that you have no locus standi in a matter affecting one of the oldest scientific societies in the kingdom.

If I heard of a proposal to evict the Royal Society of London to make way for the R. S. A. and to transplant the former to inferior rooms, say, in Bloomsbury or London Tower, I should certainly come up to London and join a deputation to Government to protect against such an enormity, although I am not a member of the R. S. L. and not an Englishman.

We are going to the Secretary who is not likely to know you by right. All we want is the outward expression of your sympathy; and that would be of great value to us just now. I hope you will think better of it.

I shall be in London from Tuesday morning early till Thursday night, and my address will be my son's rooms 78 St. George's Square".

The next day he again writes to Sir Joseph Larmor in reply to his letter of 15 November 1906, in which he writes[21]:-

" Many thanks for your kind letter of 15th and promise to countenance our demonstration. Thursday 22nd at 12.30 in the Scottish Office Whitehall is the hour and place.

It is kind of the R. S. L. to take up our cause. It is taking its proper place in so doing for it is the mother of scientific societies and therefore leader of them all".

The second deputation was scheduled to meet the Secretary of Scotland, Lord Pentland,on Thursday 22 November 1906 in London. The deputation this time included the Fellows of both the Royal Societies of Edinburgh and London, Scottish members of parliament and representatives of other public bodies. It was headed by Lord Kelvin, President of the Society. As a result of this deputation a clause was introduced into the National Galleries Bill of Scotland allowing the Secretary for Scotland to allocate sums for the purchase and equipment of the buildings for the Society. The National Galleries Bill of

Scotland after this amendment was passed and became law on 21 December 1906.

Professor Chrystal then tried to find proper buildings for the Society to purchase, and discovered that 22-24 George Street buildings could be purchased for the Society. This is also stated in " The Royal Society of Edinburgh: 1783-1983", as follows:[22]

"It was Chrystal who ascertained that the present rooms in George Street could be purchased, and it was Turner who so persuaded the Secretary for Scotland, Lord Pentland, that the Treasury granted the necessary £25,000 for purchase of 22-24 George Street and £3,000 to cover the cost of removal and equipment".

John Horne in the **Student**, 11 July 1916 stated:

"The present rooms may not inaptly be regarded as a monument to two distinguished men, Chrystal and Turner".

Professor Chrystal did all this not for personal praise but for the benefit of the Society. In fact Professor Chrystal devoted himself wholeheartedly for the promotion of science and for that to be in the welfare of the Royal Society of Edinburgh, which was at the time the main promoter of scientific investigations in Scotland. Many instances are to be found in his correspondence showing his efforts. In his letter to Sir Joseph Larmor on 23 November 1901, he writes:

" I am directed by the council of the R. S. E to ask you to give them a confidential report on the paper which has been submitted by Mr J. Fraser, entitled 'A Theory of the Constitution of Matter etc.'

The author is not known to us in any way; and at first sight the paper struck me as paradoxical merely a paradoxer of the ordinary sort by any means; and it struck some of us that it would be well to ask one like yourself, to whom the Ether has whispered of its secrets, whether there was anything of value or real interest in the paper. You will lay the council under great obligation if you will read the paper and give us your views regarding it".

In his next letter to Sir Joseph Larmor on 7 September 1902, he thanks for his prompt report on the paper by Mr J. Fraser and then asks his help in another matter of the Society and writes:

"I mean presently to overhaul our printing arrangements at the R. S. E. It is not an easy matter to get expert criticism of printers' accounts; and as you and I are "two corbies", a big corbie and a little one, and "corbies deuna pyke out corbie's een", you might perhaps help me with advice and information.

Could you, without committing an indiscretion, tell me what the printing of the Transactions and Proceedings of the R. S. L cost per sheet, exclusive of illustrations! How do you arrange your estimate with the printers so as to keep control of extras - such as change for small type; tabular matter; making up into page; printing legend lines;etc? I have laid down a number of pretty strict rules for contributors; and now I want to get a grip of the printers as well; so that, when there is bungling or extravagance, the burden may be laid on the right shoulders.

The main purpose served by the scientific societies nowadays, (at least by the R. S. E.) is the publication of scientific memoirs. The funds available for the purpose are not great, and I am very anxious to make the most of them. I think you said in your last letter to me that it was better to publish too much than too little; and I cordially agree. The notion of making a corps d'elite of scientific memoirs is to my mind as absurd as the idea of constituting a corps d'elite of scientific men. The work of most of the men of science in any period is but a midge-dance, that lasts a summers' day and then is gone forever".

In fact all his correspondence with Sir Joseph Larmor FRS is full of exchange of ideas about betterment of the R. S. E.

Sir William Turner was elected president of the Society after Lord Kelvin's death in 1907, and had the honour to open the new home of the Society on 8 November 1907. In the opening address, besides other things he had the following to say:

"It is due to Lord Pentland that we should record our sense of his courtesy at our interviews with him, as well as our hearty thanks for the effective advocacy of our claim to obtain the requisite funds from the Treasury, both for the purchase and equipment of our habitation and for an annual grant of £600 to assist in the discharge of our scientific work. We are now, therefore, no longer tenants-at-will of apartments, to be dispossessed on short notice; we sit rent free in a handsome and commodious building, and with our occupancy ensured by a parliamentary title. We have a lecture theatre equipped with modern appliances for the illustration of the subjects from time to time discussed in our meetings; we are provided with ample library accommodation and with storage and safety for our publications and manuscripts of value. We have reading and other rooms for the use of the Fellows and the officials, and a house of residence for the caretaker. It is sometimes said that history repeats itself, a saying which in one particular applies to that of our Society. In 1810 the Society purchased No.40 George Street, in which house it was accommodated until 1826, when it removed to the Royal Institution Buildings. George Street again provides us with a home, larger, more dignified, and more fully adapted to our present needs than the house purchased by the Society a hundred years ago, and with much more accommodation than was at our disposal in the rooms in the Royal Institution which we have just vacated.......".

1.4.2 Family Life

After spending some time in Germany, the Chrystals returned to St. Andrews; and in the meantime he got his appointment in Edinburgh. The couple, therefore, moved to Edinburgh to spend rest of their lives there.

On 28 August 1880, one year and two months after they had married, the Chrystals' first child, a son was born. They named him George William. He took his early education at George Watson's College before going to Edinburgh University. After graduation from Edinburgh, he moved on to Balliol College, Oxford and graduated in 1904. The same year he entered the Admiralty and was transferred to the Home Office in 1906. He held different offices before his appointment as Assistant Secretary, Ministry of National Service in 1917, and then Secretary of the same ministry in 1918-19. He was knighted in 1922, at that time he was Secretary of the Ministry of Pensions, a post which he held up to 1935, and was then appointed Permanent Secretary, Ministry of Health, in which capacity he served up to 1940.

He remained unmarried and had published some books. He died on 1 November 1944 at the Manor House, Madingley, Cambridge, where he was living with his two sisters.

On 23 December 1882, they had another son, whom they named Francis Maxwell, who after his early education at George Watson's College, graduated in Medicine from Edinburgh University in 1912, then served as Ship's Surgeon, Blue Funnel Line and took part in the First World War. His main recreation was photography, a liking which he inherited from his father. Some of the portraits made by him are in the special collections of the Edinburgh University library.

On 20 March 1884, Margaret Ann Chrystal gave birth to a daughter, whom they named Marjorie Janet Margaret. She will be best remembered as a Musician and from an early age she showed unmistakable musical ability, and was encouraged by her teacher, M. Chollet, to aim high. Accordingly, in 1901, she entered as a student the Brussels Conservatoire under the distinguished teacher M. Cornelis. Here, after two years' study, she was awarded the Van Hal Prize, or first prize with the highest distinction, in the open competition for violin playing. She then went on a musical tour through Belgium (1903-4) and received warm commendation from the critical press as the following extract testifies:- "Mdlle. Chrystal est une jeune ecossaise violoniste de haute valeur. Elle se characterise par une belle virtuosité et par un coup d'archet tout personnel. Mdlle. Chrystal est une violoniste de temperament et d'avenir......" and much more to the same effect.

She gave her first recital in Edinburgh in 1906, and received very appreciative notices from the press. The critics commended the "delightful refinement and charm" of her playing, her combined "daintiness of execution with a masterly grip of technique," and the "exceptional musical feeling and intelligence" of her interpretation.

It is further reported in **the Student,** 5 March 1909, " it is appropriate that the daughter of one of our professors should make her first appearance in the M'Evan Hall at the annual concert of the University Musical Society. Last year Miss Copeland, the daughter of the late Professor of Astronomy, delighted us with her violin; this year it is the daughter of the Professor of Mathematics. Sylvester said in one of his foot notes to an abstruse mathematical paper that 'Music was the mathematic of sense, and mathematic the music of the reason - the soul of both the same!'

Perhaps he was not so very far wrong!"

She was the longest living member of the Chrystal family and was living in Cambridge up to 1963 and even after that, of which no account is available.

On 28 December 1886, they had their third son, who was named Walter Macdonald, no details about him are available and most probably he died at an early age.

On 25 November 1887, the Chrystals had their second daughter, named Edith Margaret, who was educated at St. George's School Edinburgh, at Edinburgh University and at Newnham College, Cambridge, where she read Economics. On leaving Newnham College in 1916, she became welfare officer in a munitions factory on the Clyde, and then house property manager at Coventry. This brought her into contact with the rough side of life and sometimes involved her in really alarming experiences; but it also gave her a profound respect for the working-man and his wife and family which no experience of riots, strikes and picketings could diminish.

In 1920 Edith returned to Cambridge and was appointed tutor of Sidgwick Hall and latter transferred to Clough as senior tutor of the College, a position which she held until her retirement in 1953. Other offices held by her at various times were those of Junior Bursar, Librarian, Prælector, Director of Studies in Economics, Theology and Oriental Languages, and finally Vice-Principal. In all these capacities her wisdom, versatility and executive ability served the College in good stead. She died on 24 July 1963. In her obituary notice in the Newnham College Roll of Letter-i.e alumnæ annual circular, the following details are mentioned:-

In some characteristically extroverted autobiographical notes found among her papers she gives a vivid description of her early upbringing with its "curious mixture of discipline and unusual freedom." As her father disbelieved in the regime of the nursery, she led in many respects an untrammelled existence, running about with her brothers and sisters and their friends, as one of themselves. But her childhood had its sterner side."unheated rooms, cold baths and practice before breakfast at the grand piano in a vast, frozen drawing-room were enforced.... Religious practice was strict......My father and mother headed a procession of six children walking nearly two miles to St. Giles Cathedral, where we sat in the family pew and had to bring home the text and the heads of the discourse to cold Sunday dinner..... The fundamental religious idea was one of worship and dignity and order and a strong basis of moral conduct was part of our equipment of life". Professor Chrystal "was not a modern parent, but rather a dignified exemplar to us children - and yet I remember many happy weeks of fishing holidays, many hours of reading French books in his study - he correcting examination papers, but acting as a dictionary as often as I stuck on a word.... His relations with his children were a combination of austerity of the old Scottish type and liberal ideas such as one finds common at the present day..... He did in fact regard women and men on the same plane and helped as much as he could in women's education".

This is the best explanation of Chrystal's family life we can get.

To complete the family they had another two sons one Robert Neil born on 18 August 1891, and the other John Murray born on 17 March 1894. Some excerpts from Robert Neil's obituary are mentioned below which was published in the Journal of the Oxford University Forest Society:

Robert Neil Chrystal graduated in Forestry at Edinburgh University; received an M. A. from Oxford and a D.Sc from Edinburgh University; he then joined the Entomological Branch of the Department of Agriculture in Canada, returned to Britain in 1921 and worked for three years with the Forestry Commission.

In 1925 Dr. Chrystal succeeded Dr. Munro as head of the Entomology Section of the Department of Forestry at Oxford. He retired from the department in 1951, but continued to teach at the forestry commission schools. He published many papers and in 1937 he published his famous book "Insects of the British Woodlands." In 1954 he spent a year in Cyprus Forest Department. He died on 16 August 1956, and left behind a loving wife.

The other son John Murray Chrystal was a science student from 1912-14 of Edinburgh University and served in the first world war as Second Lieutenant in Royal Engineering. No further account of him is available.

It is interesting to note that none of the six children inherited their father's mathematical talent, but they grew up to do remarkably well in various ways. He cultivated family peace as if it were a tender plant, claiming it was one of God's greatest gifts and he thought nothing so important to children's welfare as harmony between their parents.

Mrs Chrystal died on 22 September 1903, at the age of fifty-one, leaving behind a family of four sons and two daughters to the care of Professor Chrystal to whom it was a great shock, he not only lost his loving wife but he had now the responsibility of taking care of his family. In November 1903 he met with a somewhat serious cycling accident and the injury he sustained was a broken arm, due to which he was unable to attend to his university duties for quite some time. In a letter to Sir Joseph Larmor on 30 December 1903[23], which was in reply to his letter written to condole the death of his wife, he starts with these words:

"I was glad to get your letter and good wishes. These are returned very heartily by all those of my family that you knew and that are now left.

As to the couple of grumbles, I have been of two minds to say something or nothing. I respect you so highly, and crave for liking, so much (as every human being worthy of regard must do,) that I am tempted to a word of defence at the risk of boring you. If I do so, set it down to a broken arm, the effects of which have not yet passed away......"

Chrystal started as a reformer when he joined as a lecturer in Corpus Christi College, Cambridge, and never lost his interest in this direction. He worked on the governing body of the Heriot Trust for sixteen years from 1886-1902, worked as inspector of schools on behalf of the Scottish Education Department and recommended, in 1888, the institution of the Leaving Certificate Examination. As Dean of the Faculty of Arts he worked hard from 1891-1911 to remodel the curricula of Scottish Universities.

In the last ten years of his life he also worked as a member of a committee, which had been appointed by the War Office to advise in regard to the education of officers. This is mentioned in his letters to Sir Joseph Larmor. It is worth quoting from one of these letters[24]:

"I have been struck with the expense which my attendance on the A. B. M. E is to the country. I claim only my actual out of pocket expenses, and there are items unavoidable which no honest man can put into such a claim, so that after all I am a loser. Yet I have recovered something over £50 already.

I do not believe for a moment that my services on the Board are worth that sum; and yet the ignorance of English people about the commonest things outside their own parishes is such that some representative from the north had to be there.

Still we should have as little of that kind of waste of time and money as possible......."

In 1905 he was nominated a representative of the University in Edinburgh Provincial Committee for training of teachers. Later he was elected not only the chairman of this committee but also of the joint committee of the four provincial committees. In this capacity he served for four years and was asked to continue but he refused because of his ill health. The details of all his educational reforms are given in chapter 3.

But the most important activity of his last ten years was his involvement in the hydrodynamical theory of seiches, which will be discussed in detail in Chapter 5. For this work he was awarded the Gunning Victoria Prize of the Royal Society of Edinburgh and a Royal Medal by the Royal Society of London. At the start he was much excited by the results obtained and their publication and he expresses his feelings in a letter to Sir J. Larmor FRS as follows:[25]

"Herewith your proofs most interesting to me they were, precious touches of a vanished hand. I have read nothing so interesting since I came across the correspondence de quelques célèbres géomètres du XVIII^{me} siècle, and the letters of Gauss & Schumacher.........

Can you give me any reference to observations on wave and wave group velocity in ocean waves during and after storm.i.e. Rising sea, Risen sea, and swell. The ordinary theories of the textbooks barely lick the surface of the question.

Some recent papers by Kelvin in Proc. R. S. E on ship waves etc. have an interesting bearing on the question.

But I have been so carried by the duties of the chairmanship of a committee on training of teachers that I have not had a moment to think of really amusing things such as seiches.

Our work on Earn and Tay was successful on the whole. We had many disappointments; but on the other hand, we found some things we had not started to look for. When I shall get the results published, Heaven knows. I wish there was no such thing as publication; it is tedious and expensive".

His last published paper was in a sense, a continuation of his investigations into the causes of seiches.

He talks of his ill health himself in his letter to Sir Joseph Larmor in these words[26]:

"I am domiciled here with my son George, doing absolutely nothing for a time, in the hope of recovering some portion of my health, which has been considerably shaken by the over work and strain of recent years. I went for a seven week holiday to Norway in summer; but, as the weather was abominable, the whole time, I came back worse than I went away".

Professor Chrystal tried hard as a teacher, reformer and experimentalist, but this did not mean that he had no other interest. Indeed he was very interested in travelling. His first visit to the Continent was as a student in 1874, when he studied during the summer in Tübingen. This he mentioned in his letter to Professor Maxwell. Two years later in 1876 when he was a lecturer at Corpus Christi College, he led a Cambridge reading party which according to Dr J. S. Black [27]" included Martin Conway (later Sir Martin Conway) and F. O. Bower (later Professor F.O. Bower), then undergraduates of Trinity, to Sterzing, Tyrol, and he found a new recreation in mountaineering".

In later life he frequently visited France, Germany, Norway and Italy.

He also visited the western states of America to give advice as to the organisation of Stanford University, Palo Alto, California in December 1892. The University which was opened to students in October 1891 is now one of the leading universities in the western states. One of his former Assistants from the University of Edinburgh, R. E. Allardice, was Professor of Mathematics at the time in Stanford University. Professor Chrystal came back at the end of January 1893. In contrast his colleague Professor Tait did not leave Scotland at all after 1875.

His travels abroad and his interest to probe new developments in science and otherwise, in different parts of the world made him interested to learn some Continental languages; these included German, French, Norse, and Italian.

His hobbies were reading, angling, cycling, the art and science of photography.

He worked very hard in the last ten years of his life, and the strain of this hard work was mainly responsible for his ill health which he mentioned in his letter to Sir J. Larmor.

He made some last attempts to recover from it; these according to Dr J. S. Black included his visit to Italy, Northumberland, and treatment at Harrogate, but his health showed no signs of improvement.

His last days have been described by Dr J. S. Black as follows:[28]

" He continued to fight bravely on, and proved equal to the discharge of his professorial duties to the end of the winter session 1910-11. When the spring had far advanced there remained no room for doubt in the minds of those who were nearest him that his trouble was incurable, and that all the highest professional skill could now do was to mitigate the inevitable suffering incident to a distressing and mortal illness. Yet he continued to find pleasure and refreshment in his work; and though at the beginning of the winter session of 1911-12 the university court had granted him extended leave of absence, his enthusiasm and strength of purpose enabled him to attend at the university and award the bursaries as late as 21 October".

He died after a serious operation on Friday, 3 November 1911, at his home, 5 Belgrave Crescent, Edinburgh in his sixty-first year.

The following Sunday two separate services were held at St. Giles, Cathedral, where he used to go each Sunday along with his family for worship. These were reported in The Daily Scotsman, 6 November 1911 as follows:

" The forenoon service at St Giles' Cathedral on Sunday 5 November was attended by members of the Town Council and the Edinburgh University professors. The Rev. S. J. Ramsay Scibbald, M. V. O; B. D; of Craithie preached from the second verse of Revelations, 21, and at the close made the following reference to the late Professor Chrystal:- I must not close today without making reference to a loss which both church and city will join with our university in deploring. The subject of which the late Professor Chrystal was so brilliant an exponent was not calculated to bring him prominently into public view; but the clearness of intellect, the capacity for concentrated thought, and rapidity of method which characterised his teaching were lent without grudge to many of the higher departments of public life. In an age like our own, prone to loose thought and vague sentimentalism, a man such as he with an innate love of truth for its own sake, and a faculty of accurate statement, is a steadying and strengthening influence, whose value we are apt to learn only when we lose it".

"At the fortnightly university service in St Giles' cathedral on Sunday 5 November afternoon, the Rev. Professor Kennedy, D. D; who had conducted the devotional exercises, at the close of the service paid a tribute to the memory of the late Professor Chrystal. Today, he said, the university mourned the loss of one of its distinguished teachers. Of him we might truly apply the words of the Hebrew King - 'Know ye not that there is a prince and a great man fallen this day in Israel?' For the teacher who had been taken from them was indeed 'a prince and a great man' - great as a scholar, great as a teacher, and great as an administrator.

The Dead March was played at the close of both services".

He was buried on 8 November in the churchyard of Foveran, Aberdeenshire, along side his parents, and according to Dr J. S. Black[29]" at the same hour an impressive service, attended by a large congregation, which included many students as well as representatives of the various public bodies with which he had been associated, was held in St. Giles Cathedral, Edinburgh".

The news of the award of the Royal Medal came just two hours after his death and the Council of the Royal Society recommended that the Medal be handed over to his family as a visible token of his services. The King approved the recommendation and also directed the following message to be sent to his family:

" The King trusts that you will be so good to convey the family the assurance of His Majesty's sincere sympathy in the terrible loss that they have sustained, through which so distinguished a career has been brought to a close".

His main obituary notice appeared in the Proceedings of the Royal Society of Edinburgh written by J. S. Black, Assistant Editor, Encyclopaedia Britannica, and Professor C.G. Knott, who was elected General Secretary of the Royal Society of Edinburgh in place of Professor Chrystal. He also wrote Professor Chrystal's obituary for Nature. His other obituary notices appeared in the Scotsman, The Times, Edinburgh Courant and Educational News.

References chapter 1

1.Dr J. S. Black and Professor C. G. Knott," Obituary Notice of Professor G. Chrystal",*Proceedings Royal Society Of Edinburgh* (1911-12); Vol.xxxii, P.477.

2.*Aberdeen University Calender (1867).*

3.Promoter's Address to Graduates of Arts, *University of Edinburgh* (1885).

4.Dr J. S. Black and Professor C. G. Knott," Obituary Notice of Professor G. Chrystal",*Proceedings Royal Society Of Edinburgh* (1911-12); Vol.xxxii, P.478.

5.Letter from Professor G. Chrystal to Sir William Thomson, 9 October 1878; *Cambridge University Library*; Add MSS 7342 c86.

6.Patrick Bury,"*A History of Corpus Christi College 1822-1952*"; Cambridge (1952).P.226.

7.Letter from Professor J. C. Maxwell to George Chrystal, 17 July 1877; *Cambridge University Library*; Add MSS8375.

8.Letter from G. Chrystal to Professor G. G. Stokes, 30 July 1877; *Cambridge University LIbrary*, Add MSS7656c529.

9.Letter from G. Chrystal to Professor G. G. Stokes, ~August 1877; *Cambridge University Library*, Add MSS 7656c535.

10.*St. Andrews Citizen*, Saturday, 3 November 1877; (Also reported by the *St. Andrews Gazette* and *Fifeshire News*, Saturday, 3 November 1877).

11.Obituary Notice of Rev. Professor Kelland; *Proceedings Royal Society of Edinburgh* (1879); Vol.x, p.323.

12.Sir Alexander Grant,"*The Story of University of Edinburgh: Its First Three Hundred Years*"; London (1884), Vols. I, II.

13.J. M. Barrie,"*An Edinburgh Eleven*"; Holder and Stroughton Ltd, London (1924).

14.Sir J. S. Flett,"Notice of Account of His Student Days, 1886-1894"; *Edinburgh University Library*, MSS DC.6.116, ff.10-11.

15.*The Student* (New Series), 3 December 1890; Vol.iv, No. 7, p. 98.

16.*The Student* (1906-07), 8 February 1907; No. 409, pp. 486-88.

17.(a)C. C. Easterbrook M. A. 1887,"Lectures on Senior Mathematics"
Edinburgh University Library, MSS E72/26, Gen. 1903.
(b)Lectures on Mathematics (1894-95), Edinburgh University Library,
MSS. E75/59, Gen. 1992/2.
(c)Robert Russel," 1887-88 Class Notes";
St. Andrews University Library, Special Collections.

18.Obituary Notice of Joseph Liouville (1802-1882);
Proceedings of the Royal Society of Edinburgh (1886-87); Vol. xiv, pp. 83-91.

19.Letter form Professor G. Chrystal to Sir J. Larmor, 13 November 1906;
Royal Society of London Library, MSS. Lm263.

20.Letter from Professor G. Chrystal to Sir J. Larmor, 15 November 1906;
Royal Society of London Library, MSS. Lm264.

21.Letter from Professor G. Chrystal to Sir J. Larmor, 16 November 1906;
Royal Society of London Library, MSS. Lm265.

22.Neil Campbell, R. Martin, and S. Smellie,"*The Royal Society of Edinburgh 1783-1983*"; Edinburgh (1983); p.49.

23.Letter from Professor G. Chrystal to Sir J. Larmor, 30 December 1903;
Royal Society of London Library, MSS. Lm258.

24.Letter from Professor G. Chrystal to Sir J. Larmor, 13 February 1904;
Royal Society of London Library, MSS. Lm260.

25.Letter from Professor G. Chrystal to Sir J. Larmor, 5 January 1906;
Cambridge University Library, Add MSS. 7656L83.

26.Letter from Professor G. Chrystal to Sir J. Larmor, 30 March 1909;
Royal Society of London Library, MSS. Lm270.

27. Dr J. S. Black and Professor C. G. Knott," Obituary Notice of Professor G. Chrystal",*Proceedings Royal Society Of Edinburgh* (1911-12); Vol.xxxii, P.485.

28. Ibid; P.487.

29. Ibid.

Chapter 2

EXPERIMENTAL WORK

2.1 Introduction

There are clear indications that from his early student days Chrystal's main interest was in physical science; this was shown in his student days at Aberdeen; where as Tertian in 1869-70 he shared the first prize for the Junior Natural Philosophy class with Alex Anderson and as Magistrand in 1870-71 he got first prize in the Senior Natural Philosophy class. Just after graduation from Aberdeen in 1871 he won the Arnott Prize for experimental physics along with all the distinctions available to any graduate of a Scottish university. He had indeed all the instincts of the born experimenter.

But it was not until he went to Cambridge that he found the opportunity for real scientific work. While in Cambridge he was lucky enough to be associated with Professor James Clerk Maxwell, who had been Professor of Natural Philosophy in Marischal College, Aberdeen(1856-60); Professor King's College, London(1860-65); and in 1865 he retired to private life until his appointment in October 1871 as Cavendish Professor of Experimental Physics, had published his Treatise on Electricity and Magnetism in 1873, and had opened the Cavendish Laboratory in June 1874.

His association with Professor J. C. Maxwell was so strong that even on his first visit to the continent in the summer of 1874 when he studied for some months in Tübingen he remained in contact with Professor Maxwell as is clear from his letter written to Professor Maxwell on 7 July 1874.[1]

Just after his graduation from Cambridge in the spring of 1875, a series of experiments was made by him with a view to comparing the different resistance coils of the set of British

Association Units formerly deposited at Kew Observatory and now in the Cavendish Laboratory. In view of possible change in resistance of any or all of them, it was important to compare them with each other at frequent intervals. The comparison was difficult because accurate temperature determinations were almost impossible owing to the wires being embedded in solid paraffin.

In the month of October a final set of experiments was made, which was the work of both George Chrystal and S.A.Saunder, sometimes working together and sometimes separately. The results of these experiments were submitted by them to the British Association and the General Committee of the British Association ordered these to be printed in extenso among the reports of the British Association meeting at Glasgow.[2]

In the work initiated by Chrystal, and subsequently so ably continued by Fleming and Glazebrook, we have the first stage of the series of investigations which in Lord Rayleigh's time covered the whole range of electrical measurements and made the Cavendish Laboratory the chief centre for establishing and maintaining accurate standards of electrical units, a position it held until the foundation of the National Physical Laboratory.[3]

2.2 Experimental Verification of Ohm's Law

In 1875 a small committee consisting of Professor J .C. Maxwell, Professor J. D. Everett, and Dr. A. Schuster was appointed by the British Association, for testing experimentally Ohm's Law in electricity. Ohm's Law can be described as follows:

If the electromotive force between two points of a uniform linear conductor measured in appropriate units by means of an electrometer be E, and the quantity of electricity that passes through any section of the conductor in unit time, measured either by a galvanometer or by a voltmeter, be C, then according to Ohm's Law, E/C is directly proportional to the length of the conductor, and inversely proportional to the area of its section.

The coefficient of proportionality for a definite substance depends merely on the temperature of the substance; for unit length and unit section of a given substance the value of the ratio E/C for a given temperature is called the specific resistance of the substance for the temperature, and is one of the most important of its physical constants.

This law had been directly verified by its discoverer, and by Becquerel, Davy, Fechner, Kohlraush, and others;[4] and indirectly it had been verified for a great variety of substances with a degree of accuracy approached in few physical measurements.

In discussing some experiments of his own, Dr. Schuster raised the question whether after all Ohm's Law is only an approximation[5], the limit of whose accuracy lies within experimental error. He supposes that the ratio E/C was some function of C^2, say

$$E/C = R - SC^2$$

where R is a constant very nearly equal to the specific resistance, and S is a small constant which according to Dr. Schuster's suggestion, would be positive. It is clear that E/C can only be an even function of C, unless we admit unilateral conductivity, for which there is no evidence in a purely metallic circuit.

The committee appointed were of the opinion that it was of importance to attempt a further experimental verification of Ohm's Law. Professor Clerk Maxwell undertook to have the test made in the Cavendish Laboratory, and to Chrystal was entrusted the task of making the experiments. It provided him a great chance to prove his competency of doing any work given to him.

The main difficulty he faced while testing this law was the fact that the current generates heat and alters the temperature of the conductor, so that it was extremely difficult to ensure that the conductor was at the same temperature when currents of different strength were passed through it.

Since the resistance of a conductor was the same in whichever direction the current passes through it, the resistance, if it was not constant, must depend upon even powers of the intensity of the current through each element of the conductor. Hence if we could cause a current to pass in succession through two conductors of different sections, the deviations from Ohm's Law would be greater in the conductor of smaller section; and if the resistances of the conductors were equal for small currents, they would be no longer equal for large currents.

The first method proposed by the committee was to prepare a set of five resistance coils of such a kind that their resistance could be very accurately measured. Mr.Hockin, who had greater experience in measuring resistances, suggested 30 ohms as a convenient magnitude of the resistance to be measured. The five coils and two others to complete the bridge were therefore constructed, each of 30 ohms, by Messrs.Warden, Muirhead, and Clark, and it was found that a difference of one in four millions in the ratio of the resistance of two such coils could be detected.

The experiments were conducted in the Cavendish Laboratory by George Chrystal with great success, and he wrote the report which was presented to the Glasgow meeting of the British Association in 1876. In the report it was remarked that "According to Ohm's Law, the resistance of a system consisting of four equal resistance coils joined in two series of two should be equal to that of any one of the coils. The current in the single coil was, however, of double the intensity of that in any of the four coils. Hence if Ohm's Law was not true, and if the five coils when compared in pairs with the same current were found to have equal resistances, the resistance of the four coils combined would no longer be equal to that of a single coil.

A system of mercury cups was arranged so that when the system of five coils was placed with its electrodes in the cups, any one of the coils might be compared with the other four combined two and two. After this comparison had been made, the system of five coils was moved forward a fifth of a revolution, so as to compare the second coil with a combination of the other four, and so on".

A very small deviation from Ohm's Law was observed. But the defect of this method of experiment was that it was impossible to pass a current of great intensity through a conductor without heating it rapidly. There was then no time to make an observation before the resistance had been considerably increased by the rise in temperature.

A second method was therefore adopted, in which the resistances were compared by means of strong and weak currents, which were passed alternately through the wires many times in a second. The resistances to be compared were those of a very fine and short wire enclosed in a glass tube, and a long thick wire of nearly the same resistance. When the same current was passed through both wires, its intensity was many times greater in the thin wire than in the thick wire, so that the deviation, if any, from Ohm's law would be much greater in the thin wire than in the thick one.

Hence, if these two wires were combined with two equal large resistances in Wheatstone's bridge, the condition of equilibrium for the galvanometer would be different for weak currents and for strong ones. But since a strong current heats the fine wire much more than the thick wire, Ohm's law could not be tested by any ordinary observation, first with a weak current and then with a strong one, for before the galvanometer could give an indication the thin wire would be heated to an unknown extent.

In the experiment therefore, the weak and strong currents were made to alternate 30 and sometimes 60 times in a second, so that the temperature of the wire could not sensibly alter during the interval between one current and the next.

If the galvanometer was observed to be in equilibrium, then, if Ohm's law is true, this must be because no current passes through the galvanometer, derived either from the strong current or the weak one. But if Ohm's law is not true, the apparent equilibrium of the galvanometer needle must arise from a succession of alternate currents through its coil, these being in one direction when the strong current is flowing, and in the opposite direction when the weak current is flowing. To ascertain whether this is the case, we have only to reverse the direction of the weak current.This will cause the alternate currents through the galvanometer coil to flow both in the same direction, and the galvanometer will be deflected if Ohm's law is not true.

Professor Chrystal in his report of this experiment, gave an account of the way in which the various difficulties were surmounted.

"In the first place the galvanometer indications in a Wheatstone's bridge as described in the experiment, are somewhat peculiar.

Suppose we are somewhere near a balance for some temperature of the thin wire above that of the room; then on turning the current there is a sharp kick in one direction, say to the right, then a slower but still tolerably quick swing over to the left, and then a gradual subsidence back to zero or thereabouts, which may last for half an hour or longer".

Chrystal explained these movements of the galvanometer in these words:

"The first sharp short kick was due to the fact that before the thin wire was heated its resistance was much smaller than that corresponding to a balance; the quick swing in the opposite direction was due to the sudden rise of temperature causing a corresponding increase of resistance; the slow return movement was due to the increase of the balancing resistance owing to the gradual development of heat in the thick wire".

Commenting further on the sensitivity of the wire the report says:

"It had been found that the thin wire was very sensitive to air currents, merely blowing towards it from a considerable distance sending the spot off the galvanometer scale; in fact to get any approach to steadiness the wire had to be enclosed in a box, and latterly it was enclosed in a narrow tube, and that again loosely rolled in a silk pocket-handkerchief, and the whole enclosed in a box. It was therefore at first suspected that the peculiarity in question was due to air currents; but some experiments with the wire in an exhausted tube showed that it was due to some other cause. This cause was found in the slow heating of the thick wire against which the thin wire was balanced; and some obvious experiments were made confirming this conclusion.

This slow variation of the balance was sometimes avoided by letting the batteries work until it had died away, and sometimes it was allowed for by suitably arranging the order of experiment".

In his report Chrystal commented on the success of the second experiment as follows:

"Of the two experiments the second is by far the most conclusive. It not only avoids the difficulty of eliminating temperature effects, which to a certain extent interfere with the first experiment, but it pushes the verification of Ohm's law very near the natural limit of all such verifications, viz. the limit of the solid continuity of the conductor. It has thus been rendered probable that experiment cannot detect any deviation from Ohm's law, either in the direction indicated by Dr. Schuster, or in the opposite direction as suggested by Weber, even in the wires that have been brought by the electric current to a temperature beyond red heat".

Dr. Schuster latter writing on the history of the Cavendish Laboratory writes [7]:

"Previous to Chrystal's experiments there was no evidence that the law was more than a rough approximation, nor is there any theoretical reason why it should hold accurately......"

He then mentions his own experiments and describes the comments of Professor Maxwell on the accuracy of the experiments conducted by Chrystal as follows:

"It is seldom, if ever, that so searching a test has been applied to a law which was originally established by experiment, and which must still be considered a purely empirical law, as it has not hitherto been deduced from the fundamental principles of dynamics. But the mode in which it has borne this test not only warrants our entire reliance on its accuracy within the limit of ordinary experimental work, but encourages us to believe that the simplicity of an experimental law may be an argument for its exactness, even when we are not able to show that the law is a consequence of elementary dynamical principles".

Maxwell, writing to Tait on 5 February 1876, put the results obtained by Chrystal in these words :[8]

"Ohm's Law has now been tested with currents that make the wires swag and swelter, and it is now at least 10^5 to 1 that if Schuster observed anything it was not an error of Ohm's law".

In another letter to Professor Lewis Campbell on 4 March 1876, Professor Maxwell writes :[9]

"Two Aberdonians, Chrystal and Mollison, are working at the Cavendish Laboratory. I think Chrystal's work is of a kind not comparable with that done in " a third - class German university", which was the charitable hope of **Nature** as to what we might aspire to in ten year's time. He has worked steadily at the testing of Ohm's law since October, and Ohm has come out triumphant, though in some experiments the wire was kept bright red hot by the current".

Another example, indicating the impression which Chrystal's personality had made on Maxwell, is the following quotation from a letter to Tait. In the summer of 1878 Tait had evidently asked Maxwell for some help in a conduction of heat calculation, and Maxwell replied :

"If you mean that I am, by the aid of Fourier, to get up the theory of a square box, and let you have it before the Edinburgh University library opens, then in that case also you will not bother me, for I will not do it. Nevertheless, I heard Chrystal say that the variable state of a parallelepiped was more tolerable than that of a cylinder, and he therefore cut his paraffin into a square prism.He also said that in this matter Poisson was of more use than Fourier".

According to Professor C. G. Knott[10]

"The most direct expression we have of Maxwell's opinion of Chrystal's capacity as an experimentalist is contained in the testimonial with which, on 10 July 1877, he supported Chrystal's application for the chair of mathematics in St. Andrews. 'Of Mr. Chrystal's papers', he wrote, 'the most important is that on the 'testing and verification of Ohm's law......The difficulties which he encountered and overcame in the course of this work can be appreciated only by one who, like myself, has had opportunity of watching his progress through all its stages'. The testimonial ends with a reference to his 'extensive and thorough culture, his original and penetrating intellect, and his untiring energy'".

His experimental verification of Ohm's law ranks as classical research, well planned and brilliantly executed.

Thus within a year of graduation from Cambridge Chrystal gave to the scientific world his first paper, and this work actually proved to be a milestone for his entire career. It is this work which impressed Clerk Maxwell, Sir William Thomson, Professor Tait and Sir George Stokes, the most eminent of the Cambridge men of science.

In "The Electrician" 25 July 1890[11], Oliver J. Lodge reported an objection to the complete validity of the theory of the experimental method of verifying Ohm's Law with twelve-figure accuracy, devised by Professor Clerk Maxwell and carried out by Chrystal as suggested by Professor George F. Fitzgerald. This was done by Professor Fitzgerald in a circular to the electrolysis committee of the British Association dated 24 June 1886 in the following words:-

"There is an objection to this method that I have not seen noticed. Maxwell assumes that you can expand in powers of C^2/S^2. Now, if the law were the positive value of $(C/S)^n$ where n differs very slightly from unity, the method would fail, for the current would vanish both in the numerator and in the denominator of Maxwell's expansion".

According to Oliver J. Lodge Dr. Fison seems to have promulgated the same objection at a latter date, and consequently Professor Fitzgerald wrote to Professor Chrystal about it. In reply he received a letter, which he passed on to Professor Oliver J. Lodge, and from which an extract of the portion referring to this subject was published by "The Electrician".

In the beginning he describes, while proving Ohm's Law experimentally he in fact showed that when a Wheatstone's bridge is balanced for any electromotive force in the battery circuit, it is balanced for every, or, to be on the safe side for widely varying e.m.f.

He also clarified that theoretical part of the experiments was due to Professor Maxwell and neither he nor any body else at the time examined it from skeptical point of view. He then gives his own reasoning for the general approach towards the main theory of the experiments which is as follows:

"In order to find necessary condition upon that resistance-function E/C, let us make matter as simple as possible by considering a bridge in which two arms, R, R, are of equal resistance, of the same metal, and *alike in every respect* . Let the two other resistances S and T be made of two different metals, say of Cu and Fe. Let the length and section of S be l and w; and the length and section of T be l' and w'. The specific resistance must in each case be a function of the current intensity(current per unit of section). Temperature is kept constant, of course. Let the whole current flowing through S and T when there is balance be ι, the specific resistances of S and T will be $\phi(\iota/w)$ and $\psi(\iota/w')$ respectively.

The condition for balance will therefore be

$(l/w)\phi(\iota/w) = (l'/w')\psi(\iota/w') \ldots . (1);$

and this equation must, by the result of the experiment, *hold for all value of* ι.

Let us suppose that we alter the length of the iron wire S to l'', then there will be a corresponding section w'', for which there will again be a balance; so that we must have

$(l''/w'')\phi(\iota/w'') = (l'/w')\psi(\iota/w') \ldots . (2);$

and this again must hold for all values of ι.

Combining (1) and (2) we get

$(l/w)\phi(\iota/w) = (l''/w'')\phi(\iota/w'') \ldots . (3);$

from this equation we can readily determine the form of the function ϕ.

If we put $\mu=w''/w$, $\lambda=l''w/lw''$, $x=\iota/w''$, we get $\phi(\mu x) = \lambda\phi(x)$; whence, putting $x=\mu x$, we get $\phi(\mu^2x)=\lambda\phi(\mu x)= \lambda^2\phi(x)$; $\phi(\mu^3x) = \lambda^3\phi(x)$; and, in general $\phi(\mu^n x) = \lambda^n\phi(x)$. Hence putting x=1, we get $\phi(\mu^n) = \lambda^n\phi(1)$. Now μ is unrestricted, we may put $z=\mu^n$, $n=\log z/\log\mu$.

whence, finally, $\phi(z)=\lambda^{\log z/\log\mu}\phi(1)= z^{\log\lambda/\log\mu}\phi(1)$.

The general form of $\phi(z)$ is, therefore $\phi(z) = Az^B$, where A and B are constants, the physical meanings of which are obvious from what precedes".

He concludes that the like holds for the specific resistance of every metal which has the property indicated in the experiment. Moreover he remarks that such a law of specific resistance is sufficient to secure the result of the experiment as pointed out by Professor Fitzgerald and further says:

"We conclude, therefore, that what the experiment really proves is that the specific resistance of metal varies as a power of the current intensity, *which power is the same for* *all metals*. This is a good deal, but not quite so much as is included in the paper in which the experiment was originally described. The deviation spoken of in the paper must, therefore be regarded as deviations not from absolutely *constant resistance,* but from the *resistance calculated according to the above simple law.*

To establish that the constant B is zero will not be quite so simple a matter. Many ways might be suggested, and will, doubtless, occur to you. The most direct and satisfactory would be to get the resistance for different current intensities, in Joule's way, by measuring the heat evolved.

Should the above sophistry be right, it is curious that you and Dr. Fison should each have suggested not *a way*, but *the only possible way* in which the resistance may vary with the current, and Wheatstone's bridge still remains the ideal instrument that electricians have always considered it to be".

He thus established the authenticity of Professor Maxwell's theory and his own experimental verification of Ohm's Law.

A third form of experiment was devised by Chrystal himself, being a modification of one already tried by Schuster[12]. It was based upon the fact that in an induction coil the induced current at break of the primary has a higher maximum intensity than the induced current at make. If, then, the induction currents from the secondary circuit of an induction coil, whose primary is made and broken by a tuning-fork, are passed through a galvanometer, the induced currents will not balance in their effects if the resistance depends on strength of current. Certain effects, which at first (as in Schuster's experiments) seemed to indicate a departure from Ohm's law, were traced by Chrystal to the galvanometer.

The explanation of these peculiar effects was given by Chrystal in a paper "Bi- and Unilateral Galvanometer Deflections".[13]

Under the conditions as described above, Chrystal found that the indication of a galvanometer is a function of the ratio of the strengths of the magnetic field when there is no current and when the current is passing, and also of the position of equilibrium of the needle when there is no current. He thought that some of the results he had found were new, but later discovered these not to be so, yet he certainly had the lead in getting the results published.

He made many observations and found that both theory and observation give certain similar results of which two were specifically unique. These are as follows:

1. If the ratio of the magnetic forces due to the currents to that acting on the needle when there is no current does not exceed a certain quantity, then if the position of rest of axis of the needle is inclined at an angle α (<90°) to the plane of the coil-windings, the effect of the alternating currents is to increase that angle, so that, according as the needle is deflected one way or the other by means of the deflecting magnet, one gets opposite effects.

The effect is zero when α is zero.

2. If the above mentioned ratio exceeds a certain value, the position of the needle parallel to the windings (i.e., for α = 0) becomes unstable, and there now appear two positions of equilibrium of equal inclination either way to the coil-windings. Either of these the needle will take up and keep if brought there with sufficient small velocity.

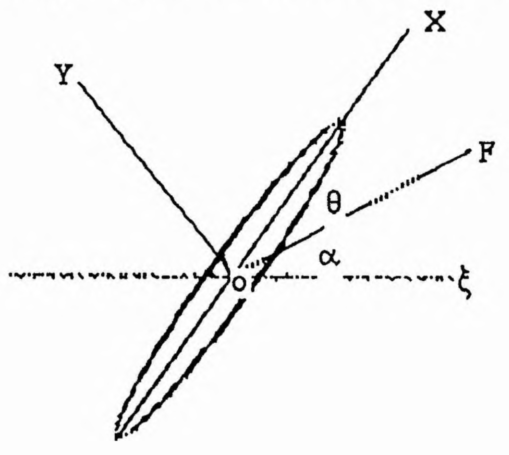
The greater the ratio, the more nearly these positions approach to parallelism with the plane of coil-windings.

The first of these phenomena he called Unilateral Deflection,which was previously noted by Lord Rayleigh, but he did not publish it until it was published by Chrystal in this paper.

The second of the phenomena was originally observed by Poggendorff, who called it "Doppelsinnige Ablenkung"[14], whereas Chrystal called it Bilateral Deflection. Poggendorff made many experiments to confirm the phenomenon and attributed this to the effect of the alternating currents, which led many others to suppose the same way including Dr. Schuster as is clear from his paper of 1874[15]. However Chrystal in this paper showed that this is not so. Unilateral deflection remained un-noticed by Poggendorff as claimed by Chrystal.

Chrystal carried out many experiments with three different galvanometers to make sure that unilateral deflection could be produced and was able to confirm it with all the three galvanometers. Moreover, he was able to notice that the needle was elongated in all cases where the effect was strong.

A closer examination into the behaviour of an elongated magnet further confirms the above theory. Let us take as the type of such a magnet a very elongated ellipsoid of revolution magnetised parallel to its axis and suspended from a point in the equator. let $O\xi$ be taken along the line parallel to the plane of the galvanometer-windings, OF the direction of the resultant magnetic force(F) when there is no current, OX the direction of the axis of the magnet at any time. Let α be the inclination of the needle to the plane of the galvanometer-coil windings when no current passes, θ its inclination at any time , m the permanent magnetic moment, κ the coefficient of induced magnetisation, i the current at any time t, g the galvanometer constant and n the number of breaks per second in primary.



The component forces parallel to the axes at time t tending to magnetise the needle inductively are

$$X = gi \sin\theta + F \cos(\theta - \alpha),$$

$$Y = gi \cos\theta - F\sin(\theta - \alpha).$$

Hence the couple tending to increase θ is

$$V(\kappa - \frac{\kappa}{1+2\pi\kappa})XY = \frac{\pi\kappa^2 V}{1+2\pi\kappa}\{g^2 i^2 \sin 2\theta + 2Fgi\cos(2\theta - \alpha) - F^2 \sin 2(\theta - \alpha)\}.$$

Considering now what happens during a complete oscillation. Let P be the uniform force whose action during that time is equivalent to the action of the varying force due to induced magnetism, then

$$P\frac{1}{n} = \frac{\pi\kappa^2 V}{1+2\pi\kappa}\{g^2 \sin 2\theta \int_0^{\frac{1}{n}} i^2 dt - F^2 \frac{1}{n} \sin 2(\theta - \alpha)\}.$$

Hence, if I^2 denotes

$$\int_0^{\frac{1}{n}} i^2 dt \div \frac{1}{n}.$$

i.e. the mean square of the induced currents, then

$$P = A' I^2 \sin 2\theta - B' \sin 2(\theta - \alpha)$$

where $A' = (\pi\kappa^2 V g^2) \div (1 + 2\pi\kappa)$, $B' = (\pi\kappa^2 V F^2) \div (1 + 2\pi\kappa)$.

Adding now the couple due to the permanent moment of the needle, we get for the whole force tending to decrease θ,

$$mF \sin(\theta - \alpha) - A' I^2 \sin 2\theta + B' \sin 2(\theta - \alpha).$$

The permanent field on the galvanometer was very much weakened by properly adjusting permanent magnets, so that B' became very small compared with mF and $A' I^2$.

The expression for the couple therefore becomes

$$mF\sin(\theta - \alpha) - A' I^2 \sin 2\theta.$$

If we make a series of observations of unilateral deflection, and if α defines the position of the needle for no current, θ the position when the currents are flowing, then other things being equal, couple will become equal to zero, thus giving

$$\frac{\sin(\theta - \alpha)}{\sin 2\theta} = \frac{A I^2}{mF} = C, \text{ say.}$$

This Chrystal verified experimentally. Moreover he made some experiments to determine whether, other things being equal, C varies as I^2, and nearly succeeded in proving this, but due to unavailability of a sine-inductor or a sufficiently delicate electrodynamometer at that time in the Cavendish Laboratory, he was unable to prove it perfectly.

George Chrystal concluded that possibly by increasing the speed of revolution we might with a sine inductor be able to introduce the element of time into magnetic measurements, and thereby throw a new light on the difficult subject of magnetic induction.

2.3 Articles "Electricity" And "Magnetism"

Professor Maxwell, quite happy with the performance of young enthusiastic Chrystal, recommended him for yet another difficult but important task. In a letter to George Chrystal on 9th January 1877, he writes :

"Professor T. Spencer Baques Editor Ninth Edition of the Encyclopaedia Britannica has asked me if I knew any one who would give him an article on Electricity. I suggested you and Garnett for any work of the sort. If you think you would have time to make an article 3 or 4 times as long as 'Atom'[16] let me know........"

To which he clearly consented, and so Professor Maxwell recommended his name for writing the article "Electricity" for the ninth edition of the Encyclopædia Britannica. Prof. Maxwell also give him broad guide lines for writing the article. The article was published in volume eight of the Encyclopædia in 1878, which means that he must have handed over the material to the publishers by the time he left Cambridge to take up his new appointment as

Regius Professor of mathematics in November 1877. In the same volume another article "Electrometer" was published.

The article "Electricity" with the introduction of some later developments can be considered as a hand book for those who do not have much time to go into details of everything but yet they want to have a complete knowledge of the subject. It starts with an introduction where Chrystal says that the word electricity is derived from the Greek word ηλεκτρου, meaning amber. The science of electricity according to him can be divided into three branches Electrostatics, which deals with electricity at rest; Electrokinetics, which considers the passage of electricity from place to place; and Electromagnetism, which treats of the relation of electricity to magnetism.

He then gives a complete historical sketch, a portion of which was taken from Sir David Brewster's introduction to his article "Electricity" in the eighth edition of the Encyclopaedia. It was modified by suppressions and alterations here and there and by large additions at the end which were thought necessary to make it suit the state of science at that time. For the sake of the student in search of original sources of information, he added throughout copious references. According to him the most valuable of these are Riess's Reibungselectricität, Young's Natural Philosophy, Wiedemann's Galvanismus and work on electricity of Professor Mascart of Collège de France. None of the later articles on "Electricity" in the Encyclopaedia Britannica contains such an extensive history of the subject.

Chrystal next gives a general sketch of phenomena in which he mentions the fundamental experiments, and defines various terms in electricity before moving on to the provisional theory of electricity where he says "Before going further into detail, it will be convenient to give a working theory of electrical phenomena, so far as we have considered them. The use of such a theory at the present stage is to enable us to co-ordinate and classify the results of experiment, and to furnish a few leading principles under which we may group results which appear to be due to a common cause. Such a theory is invaluable as a *memoria technica* for experimental results, and is useful in suggesting directions for experimental inquiry; but in framing it we must be careful to make it contain as little as possible beyond the results of actual experiment, and in using it we must be on our guard against allowing it to prepossess our minds as to what may be the ultimate explanation of the phenomena we are considering".

Chrystal then deals with experimental investigation of electrical quantity, distribution, and force. About electric quantity he says:

"We have to explain how the introduction of the term quantity into electrical science is justified by experiment, and how we can multiply and sub-divide quantities of electricity. Although it is no doubt possible to introduce the notion of quantity independently of the *measure* of electric force. Yet the most convenient and *practical measure* of quantity depends on the measurement of the force, and the absolute electrostatic unit of quantity is stated in this way. We are naturally led, therefore, to combine with the study of quantity and distribution the experimental study of the laws of electric force.

We shall have occasion to allude to two leading experimental methods that have been used in investigating the present subject. These might be called the old method and the new.

The old method, which did so much for electrical science in the master hand of Coulomb, depended on the use of the torsion balance and proof plane, both invented by Coulomb himself. This method was used by Reiss and others up to Faraday's time".

The "new method" was given by Faraday using Coulomb's balance, which he presented in the second volume of his *Experimental Researches* . The success of the method depended much on the use of some delicate instrument for measuring potential differences, and was provided by Sir William Thomson (later Lord Kelvin) in the form of his quadrant electrometer.

Writing about electrical distribution he says:

"Experiments had been made before Coulomb's time to determine what effect the nature of a body has on electric distribution. Gray and White concluded, from an experiment with two cubes of oak, one hollow and the other solid,'that it was the surface of the cubes only which attracted.' Le Monnier showed that a sheet of lead gave a better spark when extended than when rolled together. These experiments point to the conclusion that electrical distribution in conducting bodies depends merely on the shape of the bounded surface".

He then gave the laws of electric force deduced by Coulomb and proceeded on to mathematical theory of electrical equilibrium or electrostatical theory in a form accessible to students of moderate mathematical acquirements and writes as follows:

"In our mathematical outline we have in view the requirements of physical more than mathematical student, and shall pass over many points of great interest and importance to the latter, for full treatment of which we must refer him to original sources, such as the classical papers of Green, the papers of Sir William Thomson, and the works of Gauss. Of peculiar interest mathematically is the elegant and powerful memoir of the last- Allgemeine Lehrsätze in Beziehung auf die in verkehrten Verhältnisse des Quadrats der Entfernung wirkenden Anziehungs-und Abstossungskrafte, in which will be found detailed discussions of continuity of the integrals used in the potential theory, &c. The works of Green and Thomson are too well known in this country to require further remark".

He further deals with dielectric and residual discharge of electricity, before going on to Ohm's law and its application to electrolytes, and writes on resistance in general.

Next he takes up transformations of energy accompanying the electric current in which he considers the heating effect and disruptive discharge with special developments about systems of linear conductors, thermoelectrics and electric machines.

Last but not the least he considers the idea of electromagnetism and electrodynamics. He first takes the vector potential, which according to him has the property that its line integral taken round any circuit is equal to the surface integral of magnetic induction taken over any surface bounded by the circuit. The mathematical idea concerned here seems to have originated with Professor Stokes; it is deeply involved in the improvements effected in the theories of hydrodynamics, elasticity, electricity, etc; by Stokes, Thomson, Helmholtz, and Maxwell.

Also he gives Ampère's theory of electromagnetic rotations, then he moves on to electromagnetic induction. He first gives Faraday's laws of induction then Maxwell's statement of Faraday's laws, the theory of Helmholtz and Thomson, the theory of Neumann and the law of Lenz. At the end he mentions references for anyone who wishes to pursue the subject further. He then takes up electromotive force, in which he first deals with the origin of electromotive force then gives experiments by Kohlrausch, Gerland's results, contact force from polarization, the theory of Sir William Thomson about the dynamical theory of electrolysis. This theory investigates, for any circuit, the relation between the electromotive force, the electrochemical equivalents of the substances operated on, and the dynamical equivalent of the chemical effect produced in the consumption of a given amount of materials, and by means of this relation to determine in absolute measure from experimental data the electromotive force of a single cell of Daniell's battery, and the electromotive force required for the electrolysis of water. Also he takes up predictions of Thomson's effect and its experimental verification, Tait's conjecture about σ, the coefficient of the Thomson effect, the thermoelectric diagram developed by Tait and his experiments, comparison of heat and electric measurements, absolute measurements.

In the last paragraph he says:

"Throughout this article we have limited ourselves as much as possible to an exposition of the experimental facts of electricity. Where mathematical developments have occurred, they have in most cases been simply deductions from some principles well established by experience".

Throughout the article Chrystal followed the views expounded in Professor Maxwell's treatise on electricity and magnetism. Moreover, the correspondence between the two provides evidence of Professor Maxwell's advice on many points in the preparation of the article, which Professor Chrystal has properly acknowledged. At the end he has given a general index which serves as a ready reference of what has been dealt with in the article.

Professor Maxwell commented on Chrystal's article "Electricity" in the testimonial which he wrote to support his application for the professorship in Edinburgh. He says

".....that Professor Chrystal is well qualified to maintain the old reputation of the

University is amply shown by the article 'Electricity' which, since his appointment to St. Andrews, has been published in the Encyclopædia Britannica.

I have reason to know something of the amount of matter which must be gone through in order to write such an article, and of the difficulty of coordinating it, and I can confidently assert, that the manner in which Professor Chrystal has made use of this mass of matter shows that he has the power, invaluable in a professor, of giving such an account of what has been done in any subject as will give his students the greatest advantage in dealing with it themselves".

In 1879 his other two articles "Galvanometer" and "Goniometer" were published, and thus he became a permanent contributor to the Ninth Edition of Encyclopaedia Britannica. His other main article "Magnetism" was published in 1883. He starts the article giving the origin of the word magnetism and then gives a general plan of the article which is as follows:

"In the first place we shall give a sketch of the leading phenomena of strongly magnetic bodies. We shall then describe a provisional theory sufficient to render a general account of these phenomena, and shall afterwards proceed to render this theory more precise, to develop it to its necessary conclusions, and to compare these with experiment, indicating where the theory is either incorrect or incomplete. Then we shall discuss the paramagnetic and diamagnetic properties of all bodies, as expounded by Faraday; an account will be given of the connection between the magnetic and other physical properties of bodies; and, lastly, we shall endeavour to give some idea of the different physical theories that have been proposed in order to give something more than a mere short hand record of the facts of observation".

In the first section he precisely gives the basic properties of magnetism. In the provisional theory he takes the mathematical theory of the action of permanently magnetised bodies, the experimental foundation for the law of the inverse square and magnetic measurements, relative and absolute. He makes some historical remarks on the progress of magnetic measurements which are worth quoting :

"The method of vibrations came very early into use in magnetic measurements. Whiston and Graham made vibration observations with a dipping needle. Musschenbroek and Mallet also used a horizontal needle. Lambert appears, however, to have been the first to thoroughly understand and appreciate the method. For long it was the only accurate process in use for obtaining relative measures of the earth's force. It was so used by Rossel, D'Entrecasteaux, and Humbolt. Coulomb, Hansteen, and Poisson, all contributed more or less to its improvement; and it finally reached perfection in the hands of Gauss (in his memoir,"Anleitung zur Bestimmung der Schwingungsdauer einer Magnetnadel," in Res. d. Mag. Ver; 1837), who gave the experimental process for obtaining the moment of inertia, investigated the correction for resistance, and, by the introduction of the mirror and scale method, imparted astronomical accuracy to the determination of the period of vibration.

The method of deflexion, in one form or another, is very old. Its existence as a thoroughly scientific method, however, dates from Hansteen. The essential improvement of eliminating the constants depending on the magnetic distribution by observation at different distances is due to Gauss. The advantages of sine method were first pointed out by Lamont (in his,"Handbuch des Magnetismus," P. 309) in 1841.

Poisson seems to have been the first to conceive the idea of absolute magnetic measurement. In a short but luminous article at the end of the *Connaissance des Temps* for 1828, he describes a method for obtaining the value of the horizontal intensity of the earth's force. The first absolute measure of earth's horizontal force was made by Gauss in Göttingen on 18 September 1832; the value found was 0.17821 C. G. S units".

Professor Chrystal then moves on to the mathematical theory of magnetic induction, in which he makes some historical remarks on the history of the mathematical theory of magnetism, according to which "Although the *Tentamen* of Æpinus, published in 1759, and the discoveries of Mayer and Lambert did much to make clear the exact nature of the problems involved in the modern mathematical theory of magnetism, yet the origin of that theory is usually, and with justice, dated from Coulomb. Not only did the results of his careful and judicious experiments afford the means of bringing the mathematical theory to the test, but the marvellous sagacity he displayed in analysing the phenomena enabled him actually to lay the foundations upon which such a theory could be constructed. After him, Biot and Hansteen, are to be reckoned as pioneers. The theory as it now stands was virtually created by Poisson in four of the most admirable memoirs[Mém. de l'Inst; v; 1821(two memoirs); vi; 1823; and xvi; 1838] to be found in the whole literature of physics..... After Poisson the most important investigators are Green and Gauss......

In Crelle's journal for 1848 J. Neumann worked out the solution of the induction problem for an ellipsoid of revolution under the action of any conservative system; and six years later, in the same journal, Kirchhoff worked out the case of a circular cylinder of infinite length. We are not aware that the solution of Poisson's equations in particular cases has been carried any farther, unless we include as new the case of a hollow ellipsoid treated by A. G. Greenhill in the Journal de Physique for 1881.

The most important contributions to the general theory of magnetism since Poisson are to be found in a series of memoirs (Reprinted in 1872 under the title of papers on Electrostatics and Magnetism) by Sir William Thomson....... The value of his theory was fully recognised by Plücker, and apparently also by Faraday; indeed one of his ablest expositors was Beer the friend and coadjutor of Plücker. Plücker's theory was later further developed by Helmholtz......"

At the end of this section Chrystal appends a list of the more important papers on the mathematical theory of magnetism that had appeared by that time and are not quoted elsewhere in the article. Then he deals with induction in strongly magnetic bodies, at the end of which he gives an extensive list of references for anyone who wishes further details on the subject. Chrystal then moves on to magnetic properties of matter in general, before coming to the relation of magnetism to other physical properties. In the section on forms, construction, and preservation of magnets, he comments that this subject used to occupy a large portion of most of the earliest treatises on magnetism. Much of the information given, however, has been superseded by the latest developments. He therefore, makes a few remarks, mainly historical, on the subject.

He then gives some theories of magnetic phenomena, like stream theory of Euler, Two-fluid theory by Poisson; molecular magnet theory (Weber's form and Maxwell's form); Ampère's hypothesis on molecular magnet theory; Weber's theory of diamagnetism and Maxwell's molecular vortex theory published in the Philosophical Magazine for 1861 and 1862 (4th series; vols.21 and 23).

Maxwell deduced without difficulty all the principal electrical and magnetic phenomena from this theory and he points out that its general conclusions have a value which does not depend upon the somewhat intricate kinematical arrangements supposed to exist in the magnetic medium. The theory certainly affords us a most instructive dynamical picture of the phenomena of electricity and magnetism and it is, so far as we know, the only successful attempt of this kind upto that time.

In fact his two articles "Electricity" and "Magnetism" are best considered together. The nature of the material they contain shows that Professor Chrystal must have worked hard to gather this material and then to organise all that in a beautiful form as they appear. The articles are not only a brief history of the two subjects, but they also contain all the necessary theory. Mathematics involved has been given in such a form that any person with a little knowledge of it could easily follow. In fact in these two articles theory and experiment go side by side and the two articles contain all the developments in these two branches up to the time of their publication. These were printed together with his article "Electrometer" and W. N. Shaw's article "Electrolysis" by A. and C. Black, London in 1894.

Professor H. L. Callendar while paying tribute to Professor Chrystal in his presidential address to Section "A" of British Association for Advancement of Science at its meeting in Dundee in 1912 had this to say about his two articles considered together:-

"I well remember as a student his admirable Article on 'Electricity and Magnetism' contributed to the 'Encyclopaedia Britannica,' which formed at that time the ground work of our studies at Cambridge under Sir J. J. Thomson. It would be difficult to find a more complete and concise statement of the mathematical theory at that time when that article was written".

In another letter of J. C. Maxwell to Chrystal on 3 June 1879[17] Chrystal was invited to make a book on experimental electricity in these words "....If you construct a book on experimental electricity you will have a good work before you. It will interfere with no plans of mine, but rather help them...."

As already mentioned earlier, in those days there used to be no classes in the Scottish Universities from April-October, thus Chrystal had ample time to quench his thirst for experimental physics.

During his occupation of the chair of mathematics in St. Andrews, in the summer he used to return to the Cavendish laboratory to carry on his experimental work and was guided by Maxwell through their regular correspondence and on returning to St. Andrews he used to prepare a report of his experimental work and present it to Maxwell. In his letter of 6th March 1879 to Professor Maxwell while presenting a report of his work of the summer of 1878 he writes[18]

" I feel I have done less than my duty in not sending you an account of my summer work sooner. I delayed hoping to find leisure to make it complete & my health this winter has not been so good as it usually is. I wanted also to be able to make of my mind whether I should come back to Cambridge next summer. On reckoning up last years' accounts and prospecting ways and means I see the thing to be impossible. There are also other reasons but those I mentioned are sufficient. I am very much disappointed but it can't be helped........"

As a result he did not go to Cambridge in the summer of 1879 but he continued to correspond with Professor J. C. Maxwell until the latter's death in November 1879. His contact with Cambridge was totally over after the death of Professor Maxwell and this he mentioned in a letter to Dr. Arthur Schuster[19] who was writing for the Cavendish volume. He writes:

"As perhaps you are aware, I was obliged owing to the lapse of my fellowship to leave Cambridge in the middle of my physical works; and take the first decent post that offered, being at that time entirely depending on what I would earn by work.

The result was that I conceived a disgust for Cambridge and everything connected with it; and I have revisited the place very rarely perhaps three times in over thirty years.

I did a great deal of preliminary work for the redetermination of electrical standards; But, owing to my departure and the death of Maxwell which followed soon after, I fancy no use, or very little was made of it.

In any case I would not now lay my hands on my notes or laboratory books, if they are in existence (which I doubt). So far as I recollect at present, I have no documents that would be of any use to you.

As I explained to the person who first wrote to me in the matter, the close of my connection with the Cavendish Laboratory and the end of my career in Cambridge forms a passage in my life which I cannot recall with pleasure and therefore have very naturally studied to forget.

I hope you are enjoying your new career of man of science at large".

His appointment as Professor of Mathematics in Edinburgh provided him once again the chance to continue his experimental work. Tait, his colleague in the chair of Natural Philosophy, invited him to work in the physical laboratory and to utilise to the full all its appliances.

C. G. Knott an assistant of Professor Tait, latter president of the Edinburgh Mathematical Society and General Secretary of the Royal Society of Edinburgh after 1911, describes Chrystal's eagerness of experimental work as follows[20]:

"It was my first year as Tait's assistant, and the incursion of this young professor of twenty-eight years into our midst gave all our minds a new orientation. His constant presence in the laboratory during the summer months and his ready accessibility at all times gave a great impetus to the experimental study of electricity and magnetism. Tait was himself at the time fully occupied with the corrections to be applied to the 'Challenger Thermometers' and with the related work on high pressure. This work was being done in the basement by a few of the senior students working directly under Tait's supervision; and Tait was rarely seen in the upper rooms where most of the other laboratory work was going on. Summer after summer Chrystal flitted through these laboratories, busy with his own researches, but not too busy to take a keen interest in all that was being done. Many a helpful suggestion he gave for new lines of work, and many an eager student did he encourage by inviting his co-operation in some special bit of investigation. The advanced students of these years came into more direct contact with him than with Tait, and much of their scientific progress was due to his sympathetic help. My own research work in magnetism, which has continued over many years, had its origin in a conversation over a passage in the article 'Magnetism'".

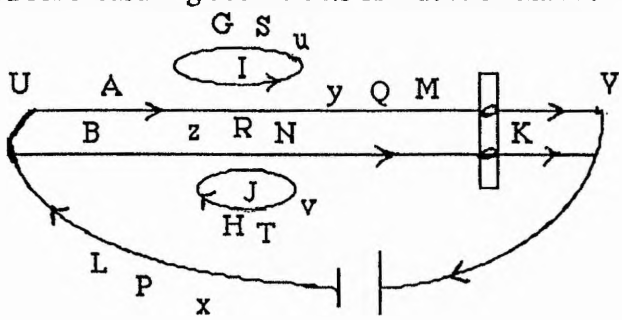
2.4 Differential Telephone And Other Related Works In Tait's Laboratory

Chrystal was working on the comparison of inductances and capacities using different methods as is mentioned in Chrystal/Maxwell correspondence. The paper containing the details of these was entitled,"On the Differential Telephone "[21]. Actually all the theoretical work was done by Chrystal in 1878 just after the invention of the telephone on the suggestion of Maxwell, but he did not get a chance to test it experimentally and now he got the chance to do it.

There were two leading null methods in use for measuring resistances, with a galvanometer as indicator,- the differential galvanometer method and Wheatstone's bridge. He considered the two corresponding methods for the telephone; but it was the former more especially to which he wished to direct attention. The instrument which takes the place of a differential galvanometer, he called a differential telephone. It is simply an ordinary telephone wound double like a differential galvanometer.

A multiple circuit of two branches A and B is inserted in a circuit containing a battery and an interrupter. A and B each contain one coil of the differential telephone, so that the currents pass in opposite direction round it. A and B have self-induction coefficients, M and N, which can be varied at will by altering the configuration of certain coils in the two circuits. If the resistances of A and B be Q and R then the conditions of equilibrium are shown to be M = N, and Q = R. There cannot be silence for all frequencies of the interrupter.

Chrystal pointed out that the instrument, and in fact the telephone generally, is better suited for measuring coefficients of induction than for measuring resistance.



The mathematical theory of the disturbance of the balance in the differential telephone by two independent circuits E and F neighbouring A and B is given. Let y and z be the current respectively in A and B; and u, S, G; v, T, H the currents, resistance, and coefficients of self-induction in E and F respectively. Let the coefficients of mutual induction of E, F and of the coils of the differential telephone be I, J, and K respectively. If U and V be the potentials at any time at the two points U, V and Asin nt the varying external force in the first circuit then,

$$(LD + P)x = A\sin nt + V - U \;. \;. \;. \;. \;. \;. \; (1)$$

$$(MD + Q)y - KDz + IDu = U - V \;. \;. \;. \;. \;. \;. \; (2)$$

$$-KDy + (ND + R)z + JDv = U - V \;. \;. \;. \;. \;. \;. \; (3)$$

$$(GD + S)u + IDy = 0 \;. \;. \;. \;. \;. \;. \; (4)$$

$$(HD + T)v + JDz = 0 \;. \;. \;. \;. \;. \;. \; (5)$$

$$x = y + z \;. \;. \;. \;. \;. \;. \; (6).$$

Here D stands for d/dt.

Now (1), (2), (3), and (6) can be replaced by

$$\{(L + M)D + P + Q\}y + \{(L - K)D + P\}z + IDu = A\sin nt \;. \;. \;. \;. \; (7),$$

$$\{(L - K)D + P\}y + \{(L + N)D + P + R\}z + JDv = A\sin nt \;. \;. \;. \;. \; (8).$$

Then for silence we must have y = z, where y and z can be found from (4), (5), (7) and (8) but the conditions to express the equality y = z(without solving the equations) are given by

$$\{(2L + M - K)D + 2P + Q - \frac{I^2D^2}{GD + S}\}y = A\sin nt,$$

$$\{(2L + N - K)D + 2P + R - \frac{J^2D^2}{HD + T}\}Y = A\sin nt.$$

These give

$$(\lambda + \mu D + \upsilon D^2 + \rho D^3)\sin nt = 0,$$

where $\lambda = (Q - R)ST$,

$\mu = (M - N)ST + (Q - R)(GT + HS)$,

$\upsilon = (Q - R)GH + (M - N)(GT + HS) + SJ^2 - TI^2$,

$\rho = (M - N)GH + GJ^2 - HI^2$.

Hence the conditions for silence are

$$\lambda - n^2\upsilon = 0, \quad \mu - n^2\rho = 0.$$

If there is to be silence for all frequencies, then we must have

$$\lambda = 0, \mu = 0, \upsilon = 0, \rho = 0,$$

which require

$$Q = R, M = N, SJ^2 = T I^2, GJ^2 = H I^2.$$

The mathematical theory of the measurement of capacities is then given. If the armatures of two condensers of capacities X and Y be attached, by wires whose resistances may be neglected, to the circuits A and B, so as to include between them all the self-induction of the circuits except that of the telephone coils, it is shown that there cannot be silence for all frequencies unless

$$Q = R, M = N, X = Y.$$

Another method is described for finding capacities in terms of resistances. In the circuit A of the differential telephone is inserted a multiple arc, in one branch of which is a condenser of capacity X, the resistance of this branch is Q'' and self-induction M''. In the other branch there is a resistance Q' and self-induction M'. The resistance and self-induction of rest of A are Q and M and the resistance and self-induction of B are R and N.

Suppose we take the ideal case where $M'' = 0$. The conditions of silence then reduce to

$M = N, R = Q + Q', Q'' = Q', M' = Q'^2 X.$

The last of these conditions means that the time constant of the coil (M', Q') and the condenser (X, Q') shall be equal. When this is the case, the multiple arc behaves like a resistance Q', having neither induction nor capacity.

The differential telephone was applied to the measurement of coefficients of induction and to the comparison of capacities and their evaluation in absolute measure. It was expected to prove useful in measuring specific inductive capacity, in investigating the properties of electrolytes and in examining the internal resistance and polarisation of batteries in action. The method last described was to give an improved determination of the ratio of the electrostatic to the electromagnetic unit.

The rest of the paper is occupied with a discussion of the use of the ordinary telephone in connection with Wheatstone's bridge.

The mathematical theory of various cases is examined and their application to the comparison and evaluation in absolute measure of electrical quantities is discussed.

This is the paper for which Chrystal was awarded the Keith Prize by the Royal Society of Edinburgh, for the biennial period 1879 - 81. The prize was presented to him on Monday, June 19th 1882 by the Right Honourable Lord Moncreiff, President of the Society.

Professor Tait, in explaining the grounds for the award, said that Professor Chrystal's paper was one of very high scientific interest and value and contained much that was wholly original. But, what would be more readily understood by the majority of the Society, it was one which contained the description and theory of a new instrument destined undoubtedly to improve in a marked manner the measurement of electric capacities. "Our means of measuring these have been, by this paper, raised from mere 'rule of thumb' to real experimental accuracy. And the subject for measurement is one whose scientific and whose practical importance are every day becoming greater".[22]

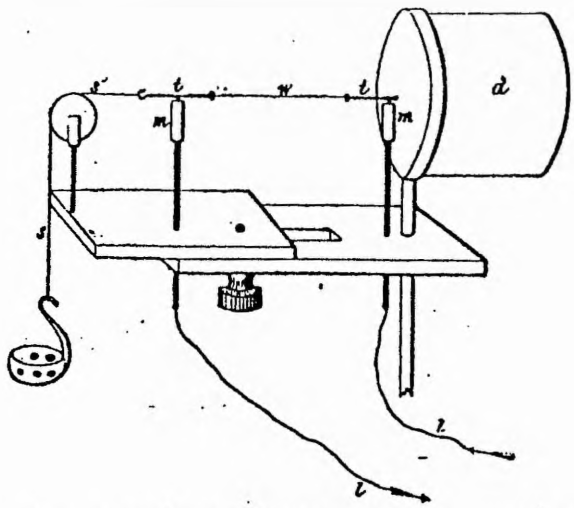
This paper, indeed, contains a complete theory up to the time, of comparison of inductances and capacities. During this period he installed a number of different apparatus for experiments of this sort. One of these, which he called the wire telephone, was used as a new form of telephone receiver. He published a paper on the subject entitled "On a New Telephone Receiver".[23] The experiment was originally devised as an illustration of the explanation of all kinds of microphone receivers, suggested by the beautiful experiments of Mr. Blyth, on loose contacts. Mr. James Blyth was then mathematical master in George Watson's Boys College. Professor Chrystal's idea was to replace Mr. Blyth's heated point of metal by a continuous portion of the circuit which should act in the same manner. In so doing he was led independently to construct an instrument, which was exhibited in action to the Royal Society of Edinburgh. Another instrument was constructed by William Henry Preece of the General Post Office, an account of which was communicated by Professor Stokes to the Royal Society of London titled "On Some Thermal Effects of Electric Currents". [24]

It happened, strangely, that the two instruments were nearly identical, although their constructors were unaware of each others' construction.

The experiments of Mr. Preece and Professor Chrystal had been to a considerable extent anticipated by some results given in a paper[25] by Dr. Ferguson of which Professor Chrystal was unaware when he made his experiments. It is true that Dr. Ferguson did not apply his apparatus to the transmission of music or of articulate sounds, as was done by Mr. Preece and Professor Chrystal, but he made the practically very important step of attaching a mechanical telephone to the wire which conveys the varying current and thus rendered the observation of De La Rive's sounds in iron and other metals both easy and certain.

Dr. Ferguson's paper contained an important result, which Professor Chrystal verified when his attention was drawn to it by Dr. Ferguson, that sounds can be produced in fine wires generally by induction currents of very feeble total heating effect.

For convenience in experimenting with different wires, Professor Chrystal constructed the apparatus which consisted of a fine palladium wire , 8 cm. long, soldered to two copper terminals, which are well amalgamated, and lie in the mercury of two cups forming the line terminals. One terminal was hooked to the membrane of a toy drum, the other end of which was removed and the other terminal was attached to a string, to which was hung a scale pan, with small weights for producing the requisite tension. With this apparatus he could produce the music of the violin. The apparatus is reproduced in the adjoining figure.



w, fine wire; t t, thicker copper terminals; m m, mercury cups over which t t pass; s s, string kept tight by small load; l l, line wires to microphone; d, drum head which distributes the sound.

Professor Chrystal did satisfy himself that the action of this instrument was not due to loose contacts or to earth's magnetism. He believed it to be due to the variations in the heating of the wire, which follow the variations of the current strength caused by the microphone. However Dr. Ferguson was of the opinion that these sounds are not due to heating effects but to some other molecular cause, which he did not elaborate much.

The other paper on the subject is entitled "On The Wire Telephone and its Applications to the Study of the Properties of Strongly Magnetic Metals".[26] In this paper by 'wire telephone' he means the instrument which has been described in the above paper. With this he conducted many experiments in Tait's laboratory and noticed four distinct sources of sound.

1. Variation of the longitudinal tension due to the variation of the heating effect of the current looks like the most probable cause for producing sound in the wire telephone for fine wires of ordinary weakly magnetic metals. He thought the lengthening of the wire due to heat as another cause, but as this was not established experimentally he rejected it.

He then made experiments using induction coils of different sizes, connected the violin and microphone with four Bunsen's cells and placed them in a circuit with primary of a small induction coil, and kept the wire telephone in circuit with the secondary. The music was reproduced but with "wiry" notes. Next he used a powerful induction coil, everything else kept unaltered. The sound in this case could just be heard. He tried with a large and very powerful induction coil under similar conditions but this gave no result of any kind at all.

2. *Electrostatic Action.*- However with the secondary circuit closed in the last case above, a loud hissing rattling noise was heard at the mercury pools of the break. This according to Prof. Chrystal was due to electrostatic action and the sounds in the Thomson's singing condenser were also due to this. He also believed that Edison and others made telephones using this principle.

3. *External Magnetic Action.*- According to Prof. Chrystal if the wire telephone be placed across the lines of force of a magnetic field of an external magnet and an interrupted current passed, loud sounds could be heard without any hissing or buzzing sounds. This sound could be heard with a thick wire of any metal. However, for a thin wire the amplitude of the transverse vibrations becomes large compared with the thick one.

4. *Effects of Internal Magnetism.*- Experiments were made by Prof. Chrystal to find the cause of the exceptional behaviour of iron wires in the experiments of De la Rive and Dr Ferguson.

He carried out his first experiment with an iron wire, and here unlike other ordinary metals the change in elasticity of the wire was not the cause of change in the sound. The sound instead depended on the temperature of the wire. At air temperature the sound was very feeble, at 200°c a high note was distinctly heard and was increasing, at 250°c this was accompanied by a fizzing sound, it became loudest at about dull red hot but with further rise in temperature it started falling and at bright red hot it totally stopped.

The sound in the above experiment depended on the temperature in the same way as the magnetic behaviour of the iron. Thus he concluded that the sound was due to magnetism of iron.

He then carried out experiments with a steel wire by putting it in the wire telephone. There was no sound at first. However, on magnetising it, a sound was heard quite distinctly which increased on magnetising the wire further. On gently heating the sound decreased but on cooling it again became as loud as before.

He then strongly heated the wire to white heat in order to deprive it of its permanent magnetism and tempered it by dropping it into cold water when dull red. By putting it back in the wire telephone there was no sound. The sound, however, reappeared when it was magnetised. He then gradually raised the temperature of the wire till it was bright red and allowed it to cool.

The sound changed with the temperature and was loudest at about dull red heat, moreover on cooling the phenomena recurred in the corresponding order.

He also carried out experiments with cobalt and nickle wires and in all cases the results were in agreement with the magnetic suceptibility of the corresponding metals. He was thus satisfied that the whole action is due to magnetism of the wire and also that earth's magnetism had nothing to do with it.

Moreover, to further verify such effects Chrystal made use of Professor Tait's thermoelectric diagram[27] with the assistance of Dr. C. G. Knott who had great experience in work of this kind. The agreement was found to be very striking.

His last paper on experimental work done in Tait's laboratory in the summer of 1881 was "Remarks On Dielectric Strength".[28] According to this paper the simplest way of representing the facts is "to imagine with Faraday that the non-conducting medium, or dielectric, between two charged conductors is the seat of mechanical stress, consisting of tension along, and pressure perpendicular to, the lines of force". The rupture of the dielectric may then be thought of as a phenomenon analogous to the rupture of an elastic body under stress. This leads to the conclusion that the rupture starts at the point where the tension first reaches a certain value, called the breaking tension or dielectric strength, which depends only on the material of the dielectric and on its physical condition at the time. The main thing in any experiment on dielectric strength is to know the tension at the point where the rupture begins. The beginning of disruptive discharge is conditioned solely by the nature of the dielectric in the immediate neighbourhood of a certain point on the surface of the discharging body and by the electric surface density at that point. This is the simplest and most scientific analysis of the phenomena but not one that accounts for all the observed facts. He conducted this experiment with the valuable assistance of Dr. Alexander Macfarlane, who was at the time an assistant to Tait, and was also a candidate for the Chair of Mathematics in 1879 with Professor Chrystal.

Dr. Alexander Macfarlane, himself a great mathematician and natural philosopher, was born in April, 1851 at Blairgowrie, Scotland, took his M. A. with honours from Edinburgh University in 1874, and obtained D. Sc. from the same University in 1878, the same year he was elected Fellow of the Royal Society of Edinburgh. He published principles of the Algebra of Logic in 1879, was interim Professor of Physics in 1880, occupied the Chair of Physics, University of Texas in 1885 and received the Honorary degree of LL. D. from the University of Michigan in 1887. In 1894 he resigned from the Chair of Physics in Texas and accepted the chair of lecturer in electrical engineering in Lehigh University[29].

Up to 1883 Chrystal was almost as strong an influence in Tait's laboratory as Tait himself but after a year Chrystal found himself forced to give up experimental work, in large measure almost certainly on account of increasing demands on his time by the duties of his own chair, and the fact that he found himself to be appropriating more and more of the really serviceable apparatus for his own experiments.

2.5 Hygrometry on Ben Nevis

Although Chrystal spent many years without doing any experimental work, he was always ready when occasion offered to advise and help others engaged in such work. For example, when he along with Professor Tait was named as representative of the Royal Society of Edinburgh to suggest a suitable site for the proposed observatory on the top of Ben Nevis[30], he accepted the responsibility. During the construction of the building he made a visit to Ben Nevis at Christmas in 1884, an account of which is given in Nature[31], and devised special forms of hygrometer and anemometer for use at these altitudes, specially during the winter season. Chrystal's hygrometer was designed on the principle of Dine's hygrometer, the nickel plated copper box, into which the thermometer bulb was inserted, being supplied by means of a double tap arrangement with warm or cold water at will. The temperature was adjusted until a film began to form on the box. It worked well but snow and fog-crystals proved too much for the anemometer designed by him.

The Ben Nevis observatory was formally opened on 17th October, 1883. His membership in this committee continued and in 1904 when the Royal Meteorological Council decided to discontinue some of the grant Sir Joseph Larmor F. R. S. a member of the Council asked for some help with the Ben Nevis work. Professor Chrystal in his reply to Larmor writes in his letter of 8th February, 1904[32] "I fear I cannot help you in the Ben Nevis matter. I took a good deal of interest in the practical details of the building of the observatory; but for a good many years I have not taken any part in the management. I wished to retire some years ago, because I have too many other things on hand, and know little at first hand about meteorology. For certain technical and formal reasons my name was retained on the board; but I became a obeying director.

I think the R. S. L. would be well advised to keep clear of the matter. This interference would certainly be resented here rightly or wrongly. Because, for a great many years back, a tide of indignation has been rising against the treatment accorded by the London officials - the Treasury in particular - to Scottish enterprises, scientific, artistic and indeed all round.

The breaking of this tide into surf is only a question of time. Some will have it that the R. S .L., in times past at least, have given bad advice in this matter; or, at least that some of its prominent members have formerly done so. Whether that be true or no, I need not now stop to discuss; but the mere fact that such things are said shows the drift of opinion. The Ben Nevis business is only one thing; there are many others as may appear anon. As to the Ben Nevis thing more particularly, I gather that the opinion here is that the constitution of the committee was not considered satisfactory from a Scottish point of view; and that an unfavourable verdict is discounted; but I did not know, what you say, that the evidence heard had been decisive of the merits of the case. What I did hear was that no sufficient evidence regarding the high level stations had been taken by the committee. Not having seen the evidence, I have no means of judging whether this is true. All I know of the matter is that I was asked to give evidence myself; and refused, because I have, as aforesaid, no first hand knowledge of meteorology. At the same time, I suggested that the committee should examine Mr. Michie Smith of the Lodai Kanal Observatory, Madras, who is a director of high level observatory, and certainly is a sound man and master of his business. Smith told me afterwards that his evidence was at first refused; and that he was afterwards asked to come after he had gone abroad and could not come!

You will of course know exactly how much of truth there is in this. If, it is true, it is pity; for the result will be embitterment of a feeling which is bitter enough already.

I could not say how the R. S. E. might look at the matter, the thing has been managed so that a cool view of it will no longer be easy to obtain. I could convene the council; but would take no such step without some kind of commission. If my head is to be broken in this Donny-brook fair I have a mind to get it broken in some business that I thoroughly understand; and there are others in which I shall certainly be involved before long".

As a postscript at the end he writes : "Is it the fact, as I have heard asserted, that the names of a number of high level experts were suggested; and that the committee called none of them ?"

In his next letter to Sir Joseph Larmor written on 13th February, 1904[33] he rejects the idea of using the Ben Nevis observatory just as an experimental station and not as an observatory. He questions the opinion of the Royal Meteorological Council and says that as the members of the Council are nominees of the R. S .L. they are acting at the instance of the Treasury. He therefore considered the R. S. L. and Treasury on trial in this case.

Chrystal acted as reviewer and critic of many scientific works chiefly in the columns of Nature. It may be said emphatically that Chrystal never wrote for the mere sake of writing. His desire always was to bring out what he believed to be truth, and this he did in many cases by exposing the errors. In many of such writings Chrystal wrote at length on the true way and the false in the teaching of science. Of these his review, which appeared in Nature, 12 January, 1882, of the second edition of Clerk Maxwell's great work "A Treatise on Electricity and Magnetism" is of great importance. In 1878 Maxwell decided to publish a revised edition of his important treatise on electricity and magnetism. The Treatise was incomplete at the untimely death of the great genius of nineteenth century in 1879 but was latter completed by William Garnett and published in two volumes at the Clarendon press in 1881.While preparing this Prof. Maxwell asked Chrystal in his letter of 9 July 1878 " Have you any errata or improvements in electrostatics? I am writing the chapter on systems of conductors". Professor Chrystal starts his review as follows :-

"These volumes have a melancholy interest for the student of electrical science, in as much as they are they are the unfinished works of one of its great masters. The printing of the second edition of the larger work had reached the second half of the first volume when it was interrupted by the premature death of the author. Up to this point considerable modifications have been introduced into the work; but the rest is merely a reprint under the superintendence of Mr. W. D. Niven, of Trinity College, Cambridge; who deserves praise for completing the work as he could".

Chrystal was such an influence in the laboratory that Dr. C. G. Knott states :-

"I have heard Tait to express the hope that when he retired from the chair of Natural Philosophy Chrystal would be his successor; but when the time of retirement came the situation had altered. Had it been Chrystal's fortune early in his professorial career to have had official control of a physical laboratory, he would certainly have founded a strong experimental school".

Taking a general view of Chrystal's experimental work we find it characterised by a true physical insight into the essential nature of each problem. Superfine accuracy was never his aim and perhaps from this point of view some of his investigations lack finish. His methods were in many cases rough and ready but they were always under complete mathematical control. Having laid down the broad lines of attack on any question he put together his apparatus with full attention to detail and his intuitions generally led him to right. For all his experimental work he owes much to his great master in the field Professor James Clerk Maxwell and also to his colleague Professor Tait. He never failed to give full credit to those who helped him carry his ideas to fruition.

References Chapter 2

1. Letter from George Chrystal to Professor Maxwell, 7 July 1874, Tübingen;

   Cambridge University Library; Add MSS. 7655/82.

2. *British Association Report*, Glasgow (1876); pp. 13-19.

3. *A History Of Cavendish Laboratory 1871 - 1910*, Longman. London. 1910. p.24.

4. Kohlraush and Nippoldt, "On The Validity Of Ohm's Law for Electrolytes And a Numerical Determination of the Conductivity of Sulphuric Acid";

   Poggendorff's Annalen, Vol. cxxxviii, (1869) p. 379.

5. Dr. A. Schuster, "On Unilateral Conductivity " and " Experiments On Electric Vibrations"

   Philosophical Magazine (1874); Series iv, Vol. xlviii , pp. 251, 340.

6. Report of the Committee for Testing Experimentally Ohm's Law;

   British Association Reports (1876) Glasgow; pp. 36 - 63.

7. *A History Of Cavendish Laboratory 1871 - 1910*; Longman. London (1910) p. 21.

8. Dr. J. Sutherland Black and Professor C. G. Knott; Obituary Notice of Professor George Chrystal;

   Proceedings Royal Society Of Edinburgh (1911-12); Vol.xxxii, p. 489.

9. Lewis Campbell and William Garnett *"The Life of J. C. Maxwell"*;

   London (1882), p. 300.

10. Dr.J. Sutherland Black and Professor C. G. Knott; Obituary Notice of Professor George Chrystal;

    Proceedings Royal Society of Edinburgh (1911 -12); Vol. xxxii, p. 489.

11. Oliver J. Lodge, "Verification Of Ohm's Law;"

    The Electrician, 25 July 1890, P. 309.

12. Dr. A. Schuster, "On Unilateral Conductivity" and "Experiments on Electric Vibrations"

Philosophical Magazine (1874); Series iv, Vol. xlviii, pp. 251, 340.

13. *Philosophical Magazine* (1876); Series v, Vol. ii, pp. 401-414.

14. Poggendorff, "Doppelsinnige Ablenkung", *Poggendorff's Annalen* (1838) Vol. xlv, p. 353.

15. Dr. Schuster, "On Unilateral Conductivity" *Philosophical Magazine* (1874),Series iv, Vol. xlviii, p. 257.

16. J. C. Maxwell, "Atom", *Encyclopaedia Britannica*, Vol.iii, Ninth Edition, pp. 36-49.

17. Letter from Professor J. C. Maxwell to Professor G. Chrystal, 3 June 1879; *Cambridge University Library*. Add. MSS. 8375.

18. Letter from Professor G. Chrystal to Professor J. C. Maxwell, 6 March 1879; *Cambridge University Library*. Add. MSS. 7655/ II / 183.

19. Letter from Professor G. Chrystal to Dr. A. Schuster, 19 June 1909; *Royal Society Library London*, MSS.

20. Dr. J. S. Black and Professor C. G. Knott; Obituary Notice Professor G. Chrystal, *Proceedings Royal Society of Edinburgh* (1911- 12); Vol. xxxii, p. 492.

21. *Transaction Royal Society of Edinburgh* (1880), Vol. xxix, pp. 609- 636.

22. *Proceedings Royal Society of Edinburgh* (1879 - 80), Vol. xi, p. 797.

23. *Proceedings Royal Society of Edinburgh* (1880), Vol. x, pp. 682-85.

24. Henry Preece, "On Some Thermal Effects of Electric Current", *Proceedings Royal Society of London* (1879-80), Vol. xxx, pp. 408-10.

25. Dr. Ferguson, "On the Indication of Molecular Action in the Telephone", *Proceedings Royal Society of Edinburgh* (1877-78), Vol. ix, pp. 615- 37.

26. *Nature*, Vol. xxii, 29 July 1880, pp. 303-07; and *Proceedings Royal Society of Edinburgh* (1880); Vol. X, pp: 303-07.

27. Professor Tait, "Thermoelectric Diagram" *Transactions Royal Society of Edinburgh* (1872), Vol. xxvii, p.134.

28. *Proceedings Royal Society of Edinburgh* (1882), Vol. xi, pp.487-98.

29. *American Mathematical Monthly*, Vol. ii, January 1885.

30. *Nature*, Vol. xxvii, 21 December 1882.

31. "A Christmas Visit to Ben Nevis Observatory", *Nature*, Vol. xxix, (1884).

32.Letter from Professor G. Chrystal to Sir J. Larmor,

Royal Society Library London, MSS. Lm. 259.

33. Letter from Professor G. Chrystal to Sir J. Larmor,

Royal Society Library London, MSS. Lm. 260.

---

chapter 3

CHRYSTAL'S EDUCATIONAL REFORMS

3.1 Introduction

Chrystal's professorial period had been a period of momentous change and unprecedented advance. Far-reaching discoveries and their practical applications in various fields had largely altered the material setting of life, and had, as Lord Beaconsfield declared, "affected social conditions and modes of life more profoundly than all the laws and codes of centuries". Consequently the social, political and economic ideas of the people changed. No formula expressing the prevailing tendencies of thought in the eighteen seventies represented the ideas and ideals in 1911. The wheel had come round almost full circle from individualism to collectivism, from competition to cooperation, and from a doctrine of *laissez faire* to that of state control.

Amid the flux of these years it would be strange indeed if education were found to be standing still. Education is never independent and self-contained. It is always a reflection of the conditions in the wider world of national ideas, and derives its inspiration and energy from the great movements of the time. As described by an educationist "Education follows, it does not lead, national opinion, and too often it follows afar off, for, like religion, it is essentially conservative in its outlook, and responds but slowly to changing needs".

Progress in education during these forty years does not "leap to the eye" as it does in more material spheres, but it was none the less real and momentous.

3.2 Primary and secondary education in Scotland

The report of the Royal Commission on Education in Scotland of 1867 presents a vivid and detailed picture of the educational conditions of the whole country at that time. Scotland with a long tradition of education, had been in possession of a National system of Education for nearly two hundred years prior to 1867. The Education Act of 1872 gave effect to the recommendations of the Commission and the Act improved primary education. Its object was to provide education for "the whole people of Scotland" and not merely for the labouring classes as was implied in the English measure.

In Scotland there were basically two kinds of schools. The parish schools, which originally were purely elementary, were encouraged to provide at least the elements of secondary education. These schools played this role so well that the Argyle Commission in its report of 1868 reported that over fifty per cent of the students attending the four Scottish universities came direct from parish schools[1].

The burgh or grammar schools, which were the true secondary schools, owing to the competition of the parish schools, were compelled to open their doors to primary pupils who were prepared to pay increased fees for the privilege. It is in this way that both types of schools became universal education providers, and gave to Scotland an education system far removed from the highly specialised character of continental schools. The general effect of this policy was to depress secondary education in the higher class reaches, but greatly to raise the level for the whole country. Through it, indeed, Scotland possessed for more than two hundred years the most democratic education system in the world and to a considerable extent in consequence of this it has enjoyed an influence and importance in the world altogether out of proportion to its size and population[2].

Since 1872 repeated efforts were made to remedy the more glaring defects in the original Act, some of them successful. According to Professor Chrystal since 1872 a wave of educational progress had begun to sweep over the schools and has risen rapidly. As an example, the Education Act of 1878 empowered the Education Department to conduct inspection of all higher class schools, but state inspection was not carried out because of financial difficulties until after the reorganisation of the Scottish Education Department in 1885. The Scottish Education Department was originated in 1839, but for our purpose we consider its beginning in 1872, the year in which a separate committee of the privy council was set up to administer the Scottish Education Department.

In 1882 The Educational Endowments (Scotland) Act was passed, under which such inspection was extended to all endowed schools and a commission consisting of seven commissioners was established, with Lord Balfour of Burleigh being chairman. Mr Alexander Gibson was appointed to be the secretary to the commissioners. The commissioners were provided full powers necessary for their job[3].

According to Dr J. S. Black[4]:

"Professor Chrystal was not a commissioner, but was on terms of intimate friendship with the secretary, and there can now be no impropriety in saying that the many questions the commission had to deal with were frequently discussed by Mr Gibson with Professor Chrystal and their common friend Professor Robertson Smith famous theologist of the time, who about that time was much in Edinburgh. Mr Gibson always found their opinions helpful and generally such as might profitably be suggested for the consideration of his commission. These discussions served to deepen in Chrystal's mind the interest he had long felt in educational reform as it ought to be regarded by a statesman".

When Professor Chrystal became professor of mathematics at St. Andrews in 1877 he was much concerned about the falling standards of education in Scotland, more specifically of secondary education and the university education. In his promoter's address of 1908 he described this as follows:[5]

"Many of the secondary schools were in a dying condition and others, which were apparently prosperous, were in reality much under-staffed and far from efficient, and were engaged, moreover, in the pursuit of low educational ideals. The universities had been enjoying a period of wholesome prosperity; the number attending them had increased but the standard of university work had fallen below the level of cultured nations of Europe".

As most of the problems were due to lack of finances, so in all his three promoter's addresses he pleaded the extension of the policy of state aid to secondary schools.

The Lord President and the Vice-President of the council acted as the responsible chiefs of both the English and Scottish committees and the same permanent official, Sir Francis Richard Sanford, was secretary to both bodies[6].

The English connection thus established lasted until the reorganisation of the Scottish Education Department in 1885, when Mr (later Sir) Henry Craik became permanent secretary. This English connection resulted in a great loss to Scottish education but, with the establishment of an independent Education Department, Scottish Education once more resumed its onward course. As a result of this administrative change Professor Chrystal along with some other Scottish professors took part in the inspection of secondary schools over a period of several years. The inspectors appointed by the Education Department in 1886 to investigate the conditions in the higher class schools presented a somewhat depressing report. The staff were found to be inadequate and underpaid, the curricula far behind the times and the methods antiquated and ineffective.

Thus Chrystal's concern about declining standards of education, specially of higher education proved to be true. He being a reformer started to think about the possible solutions of the problem.

3.3 The Scottish Leaving Certificate

The outstanding feature in the history of secondary education during the twenty years immediately after the passing of the Education Act of 1872 was the institution of the Leaving Certificate Examination, by which secondary education in the higher class public schools, the higher class schools, and the state aided schools, aimed at a common examination. This also provided a link to the work in the universities, and was one of the most important factors in the advance of the methods and results in secondary schools. From the beginning the system differed fundamentally from the German conception of leaving certificates. Certificates were awarded on the result of a purely external examination, although an attempt was made, but not very successfully, to keep in touch with school programmes by means of visits of inspection. These certificates were granted, not for success in specified groups of subjects, nor for the satisfactory completion of an approved school course, but for passes in the separate subjects of higher instruction. In this way pupils could boast of being in possession of four, five, and six leaving certificates. All this was latter changed, and a single certificate marked the successful close of the Intermediate and Post-Intermediate stage respectively.

Henry Craik after his appointment as secretary to the Scottish Education Department in 1885 was determined to raise and improve the standards of education in Scotland. To get a clearer picture of standards of education in the country, in particular of the higher education, he appointed an inquiry committee in 1886 to examine the conditions of education in Scotland.

The report of the committee according to Thomas B. Dobie[7],"showed a serious lack of uniformity between schools in various parts of the country". At the same time on 15 January 1886, the Scottish Education Department issued Circular No. 74 conveying its decision for inspection of secondary schools.

According to the circular such inspection was already compulsory for the Endowed Schools under Sections 19 and 45 of the Educational Endowments Act, 1882, and the cost of the inspection had to be paid from the funds of the endowment. Whereas the cost of inspection of the higher class public schools, which were liable for such inspection under Section 20 of the above Act, would be made out of a temporary grant obtained for the purpose however for the higher class schools which did not fall in either of the above categories the cost had to be paid by the managers.

In this circular the Department for the first time mentioned issuing a certificate as follows:[8]

"In connection with the inspection, the suggestion has been made that their Lordships should issue a certificate, based on the result of the examination of the highest classes in these schools, which could serve as a measure of the attainment fairly to be expected in the case of pupils completing a course of secondary education.

My Lords attach much importance to this proposal, which they find to be favourably received by all those most interested in the state of higher education in Scotland, and which, if adopted, might serve to mark the line dividing the sphere of such higher schools from that of universities. It would, of course, be necessary to give considerable freedom of choice as to subjects; but the examination might be so arranged as to cover generally the subjects of the Arts' course in the Scotch Universities, with modern Languages, history, and science as additional subjects of choice.

Before the standard for a certificate and its relation to the curriculum of each school can be definitely fixed, it would be necessary that the matter be carefully considered by the Department in communication with those who have most experience in regard to such schools".

Professor Chrystal had earlier expressed his concern about higher education in his promoter's address of April 1885 in these words "I have for the last fifteen years been an ardent student of everything relating to our higher education. In the course of that time I have been more or less intimately connected as student or teacher with five different universities, home and foreign; and ever since I became a Scottish professor, i.e; for the last eight years or so, I have largely availed myself of opportunities offered to me for examining secondary schools in England and Scotland".

The inspection of thirty-one schools was carried out successfully in 1886, and more and more schools opted for the inspection in the years to come. Henry Craik in his annual report of the year 1886-87 reported as follows:[9]

"We have been able to carry out this work we have given special consideration to the means by which this inspection might give satisfactory evidence as to the state of the schools, and might at the same time without unduly limiting the independence of local management raise the standards of secondary education throughout the country. With this end we have carefully considered the extent to which a leaving examination might be established in connection with the inspection, and how it might best be arranged. We have invited the opinions on this subject of those who have taken part in the inspection, as well as others".

As an Appendix to the main report a valuable report by Professor Chrystal[10], who was one of the few Professors working for the Department, was given.

Professor Chrystal, after being asked by the Department, examined students of mathematics in twelve secondary schools in Scotland. In his report he said:

"The result of this experiment has proved both interesting and important. It has shown clearly, to me at least, that, so far as mathematics is concerned, the institution of a leaving examination will be attended with no great difficulty, and probably with much advantage to our secondary schools".

In order to make clear the reasons for the course which Chrystal ultimately followed, it is necessary to state his ideas regarding the object and advantages of a leaving examination. The most important object according to him of a leaving examination is to set a minimum standard for the highest work of the secondary schools, and to mark, to some extent, the boundary between their province and that of the universities. This standard must be so high as not to discourage progress in the best equipped schools, and yet not so high as to snuff out the too many schools in Scotland that were inadequately equipped to meet the requirements.

According to Chrystal some people have thought that a university entrance examination would better answer the above object. But his experience had taught him otherwise, and to his thinking circumstantial forces would speedily lower the standard in any university entrance examination, beyond the level of poorest secondary schools.

One of the main advantages of such a certificate, according to him would be a reduction in the number of examinations, and that in turn would lighten the unnecessary burden which at that time oppressed the secondary schools in preparing pupils for different examinations.

To decide the standard for the leaving examination, Professor Chrystal made use of the general consensus at the time in Scotland that pupils from the highest class of a well conducted secondary school should have sufficient knowledge and ability to enter a three years course at a university. Thus he fixed the highest standard for such an examination to be the university entrance examination, but when he examined the students, in his assessment of the papers he found that only a few pupils were able to reach the standard just mentioned, whereas a majority of them were able to reach the standard set for the medical preliminary examination. He therefore fixed a minimum standard for the examination equivalent to this. In his recommendations to the Department, he recommended that if a national leaving certificate examination was instituted, it should be in two grades, a higher grade equivalent to the university entrance examination, and a lower grade, equivalent to the medical preliminary examination and the entrance examinations of other professional bodies.

In his report Professor Chrystal made certain criticisms and said:

"I trust that those concerned, whether teachers, governing bodies, or others, will listen with patience to my criticisms, believing that I have given them my most careful consideration, and that I have spoken neither lightly nor without experience. The time has arrived for plain speaking regarding our system of secondary education ....".

In May 1887 through Circular 88 the Scottish Education Department requested the examiners of higher class schools to present a short report on their views with regard to expediency of issuing a certificate as a result of a leaving examination.

The examiners in their reports agreed with the Department to institute a leaving certificate examination in all the main subjects.

The idea of instituting a leaving certificate examination was further strengthened by the reports of an Inquiry Committee which has been described by T. B. Dobie as follows:[11]

"Further impetus to the creation of a leaving certificate came from the committee of inquiry which had been set up by the Department in 1886 to inquire into 'certain questions relating to education in Scotland'. The committee, under the chairmanship of C.S.Parker, MP, issued three reports during the period 1887-88. In the third report concerned with a comprehensive study of secondary education, reference was made to the creation of a national certificate examination as a means of improving the organisation of secondary education in Scotland".

The positive response from the examiners of higher class schools, the report of Parker's committee, the general desirability of the public and the academicians encouraged Henry Craik to take the bold step of instituting a leaving certificate examination (this was bold in the sense that it had no parliamentary backing itself, though it was the outcome of the inspection under Education Acts of 1878, 1882). This was announced through Circular 91 of 14 November 1887 in these words:[12]

"The proposals as to an outgoing certificate for the pupils of such schools, which was first put forward in their Lordships Circular 74, has been much discussed, and strong representations have been made to my Lords as to the expediency of carrying out this arrangement as soon as possible".

The experiment proved to be very successful, one of the main reasons of its success being Craik's policy of consultation. In Circular 91, he asked for the opinion of the school boards and managers on the following points:

"1. What subjects should be embraced in such an examination, as representing the ordinary curriculum of their school?

2. A general indication of the standard which should be aimed at in each subject.

3. To which classes in the school under their management such an examination should be open, and what approximate estimate of the number of probable candidates from these classes.

4. My Lords presume that the month of June would be most convenient for the examination; but they would be glad to have any remarks as to this.

5. What minimum of attendance at the higher class school should be required from candidates?"

At the end of the circular the Department asked for their opinion on any other matter which they might think suitable in the case.

Encouraged by the Department's proper consideration of their opinions, school authorities in addition to replying to the above five points, made some other suggestions which proved to be very useful to the Department.

In fact the final shape of the examination owed much to the suggestions made by the school authorities.

After receiving necessary information and suggestions from school management and others involved directly or indirectly in the whole operation, the Department sent a letter to all Scottish Universities on 23 December 1887[13], mentioning the desirability of such a certificate from different corners, specially from the management of some of the largest and most prosperous higher class schools in Scotland.

The letter also explained the main aims and advantages of such a certificate. In this letter for the first time the Department specifically mentioned the type and grades of certificate as desired by the schools and the Department.

The letter gives the following kinds of certificate:

"1. Classical(with such branches of mathematical or other studies as are ordinarily taught on the classical side of a higher class school);

2. Scientific and technical;

3. Commercial;

while it might be convenient to add a fourth class, adopted for girls".

As regards the grades the Department agreed with the recommendations of Professor Chrystal in his report mentioned earlier and said that certificates would be issued in at least two grades.

The Department on behalf of the Marquis of Lothian invited the suggestions of the Senates of the Universities on all these matters and also desired their opinion about recognition of such a certificate(if proper standards are maintained) as an alternative for their entrance examination.

The Department arranged a conference in Edinburgh on 25 February 1888, mainly to discuss the replies of the school management in response to Circular 91 and other suggestions in this regard. The Department invited representatives from the four Scottish universities for the conference, in order to convince them about the merits of such a certificate, but the universities did not send any official representatives.

The universities by doing so avoided themselves being bound by any decisions taken by the conference.

According to T. B. Dobie[14],"Professors Ramsay of Glasgow and Chrystal of Edinburgh attended as private individuals".

No official record of the conference was kept, except that it was mentioned in Circular 93 of the Department, but according to T. B. Dobie:[15]

" Craik kept brief minutes which show that the main question discussed was whether the certificate be issued on a group or subject basis. Professor Chrystal advised that a single-subject certificate would best suit the needs of Scottish education at that time. It was generally agreed that, in principle, leaving certificates should be issued for groups of subjects, but that, as a matter of practical expediency, it would be necessary to begin with certificates in single subjects and let the group certificate develop in time".

Although the schools welcomed the new examination, it was clear that its acceptance was conditional on its being recognised as equivalent to the entrance examination of universities and other professional bodies. Four Scottish universities and other professional bodies were contacted by the Education Department to gain their recognition of the certificate examination. Professional bodies indicated a ready acceptance of the certificate examination as equivalent to their own entrance examinations. It was however very difficult for the Department to persuade universities to accept this certificate.

The Scottish Education Department in its circular 94 of 27 April 1888, announced its decision to hold examinations for the Institution of Leaving Certificate in the week beginning Monday 18 June 1888. It was decided that examinations would take place simultaneously in all the schools which opted for examination, and in addition to two grades already agreed, the Department would also issue an Honours Certificate in each subject. The final programme for holding this examination was conveyed by the Department through Circular 95 of 28 May 1888. The subjects in which the candidates were to be examined were:-

Mathematics(including arithmetic); English(including questions on modern history and geography); Latin; Greek; French, and German.

As was to be expected, the Leaving Certificate Examination underwent a good many changes in examination details with the passage of time. In 1889 another subject of Book-keeping with Commercial Arithmetic was added. The Department continued its struggle to gain university recognition of the Certificate, and at last in 1889 it succeeded in gaining this recognition. The Department in its report of 1890-91 added a memorandum giving a list of institutions and other bodies who had by then recognised this certificate as an alternate to their entrance examinations. These were:-

The Lords of the Council and Session(for the purposes of the Law Agents Act);

The War Office and Civil Service Commissioners, for the Army Preliminary Examination;

The University of Oxford;

The University of Cambridge;

The Universities of Edinburgh, Glasgow and St. Andrews, for the preliminary examination for the three years' course;

The General Medical Council;

The Royal College of Surgeons of Edinburgh;

The Pharmaceutical Society of Great Britain;

The Society of Solicitors before the Supreme Courts;

The Institute of Accountants and Actuaries in Glasgow; and

The Society of Accountants in Edinburgh.

The evolution of the leaving certificate examination itself was discussed by Professor Chrystal in his promoter's address of 1908 in these words:

"A small sum available for the purposes of secondary school inspection in Scotland had been wrung from the Treasury, and it occurred to me that it might be utilised to institute a leaving certificate examination. I was examining twelve schools for the Department in the year 1886, and it was proposed that I should demonstrate how such an examination, at least in a single subject, could be carried out. When I came to write my report the idea of a general leaving certificate examination had developed in my mind, and I sketched a complete scheme, in most of its essentials the same as now exists. To my great surprise, and no small gratification, the proposal was immediately taken up by the Scottish Education Department. The labour of carrying out the scheme in detail was taken up by Sir Henry Craik, then beginning his successful administration of the new Department. In an account of the subject that recently appeared in Scotsman, it has been very justly said that the introduction of the leaving certificate examination was perhaps the most important event of Sir Henry Craik's tenure of office, and he certainly deserves the highest credit for the tact and energy with which he carried out what proved under his guidance to be a great educational reform".

In his report of 1899 Sir Henry Craik mentioned strong representations to be made for issuing a Group Certificate as a replacement of its being issued in single subjects, he writes:[16]

"Strong representations continue to be made in favour of the issue of Leaving Certificate, not in single subjects, but in groups. The argument against this is that it might appear to impose a certain fixity of curriculum upon the schools which it is the desire of your Lordships to prevent. On the other hand such grouping undoubtedly represents a more satisfactory scheme of a comprehensive secondary education".

Here he also mentioned that as a preliminary experiment, the Department might issue Group Certificates, in addition to any issued in single subjects. This was confirmed by the Department in its Circulars 270, 271 of 20 December 1899, where it was also announced that a Leaving Certificate in Science would be introduced.

In these circulars the conditions for issuing group certificates were made clear and was announced that these certificates would be issued to those candidates[17],"who have been receiving higher instruction for not less than four years in some recognised school, and who have obtained, during that period, certificates of higher grade, or in honours, in at least four subjects of which one must be English, one an Ancient or Modern Foreign Language and one Mathematics, or in the case of girls, Higher Arithmetic. Two certificates of the lower grade will, for the present, be accepted in lieu of the fourth certificate of the higher grade, and a Leaving Certificate in Science may replace a certificate of the higher grade in Ancient or Modern Foreign Languages.

The grade in which a candidate passes in each subject will be recorded on the certificate".

Both the certificates were implemented in 1900, and proved to be successful.

With the view that a group certificate would be of general interest to the development of secondary education, the Department decided that from 1902, the Leaving Certificate would be issued on a group basis only. This was conveyed to the schools concerned through circulars 337, 338[18] of the Department.

After getting proper recognition of this from the universities and other professional bodies the Department announced the details through its circular 340 of 16 January 1902, according to which, "There shall be two classes of certificate. One of these, the Leaving Certificate Proper, is intended to mark the completion of a full course of secondary education. The other, to be called the Intermediate Certificate, is primarily intended to meet the case of those schools which, although they may be doing valuable work in secondary subjects, are yet unable, from one cause or another, to retain their pupils long enough to enable them to reach the standard of the Leaving Certificate Proper. This latter certificate will, however, always be open to pupils of any school who may satisfy the prescribed conditions. I am to remind you that, while candidates will no longer be furnished with a Leaving Certificate for each subject in which they may be successful, they will receive instead a document certifying that they have passed in a specified subject and grade in the Leaving Certificate Examination. Applicants for Leaving Certificates must have been receiving higher instruction at some recognised school for not less than four years. In the case of applicants for Intermediate Certificates, the corresponding period shall be two years".

To prevent excessive pressure on pupils, the minimum age for the former certificate was fixed to be seventeen years, and that for the latter fifteen years.

As far as the written examination was concerned, the candidates for the Leaving

Certificate must have passed in four subjects at the Higher Grade Standard, or at three subjects in the Higher Grade and two in the Lower. The candidates for the Intermediate Certificate must have passed in four subjects, at least one of these subjects being at the Higher Grade Standard. In addition the candidates of both certificates must have had specific training in either Languages or Science. English and Mathematics were compulsory for both the certificates. The Leaving Certificate thus prepared students for entering a university three years' course and the Intermediate Certificate provided fitness to enter on a mercantile or a technical career.

In addition to the above two certificates, the Department in 1903 agreed to issue a special group certificate, in a group of subjects which, any school management thought necessary for a particular career. The minimum age fixed for this was the same as for the Leaving Certificate. This was announced by the Department through circular 375 of 28 January 1903.

The Leaving Certificate instituted in 1888, did evolve successfully through the years which followed and played a vital role in improving secondary education in Scotland.

After twenty years of service as Secretary to the Scottish Education Department, Sir Henry Craik resigned in 1904, and was replaced by Mr (later Sir) J. Struthers in December 1904. In his first annual report on secondary education to the Committee of Council on Education in Scotland, he praised the outstanding services of Sir Henry Craik to the cause of education in his native country.

Improvements in the Leaving Certificate continued, and the Department in its circular 389 of 24 March 1906 declared the Intermediate Certificate as a prerequisite for entry to a course leading to the Leaving Certificate. In 1908, the Department through its circular 413 announced the introduction of yet another certificate named as Curricular Intermediate Certificate.

The Leaving Certificate because of its various merits proved to be successful, but this did not mean that it had no defects. It did have many defects, but its main success was the fact that whenever these defects were pointed out, the Scottish Education Department through its successive secretaries, tried to find a remedy for such defects.

The Secondary School Journal of November 1906, commented on the Leaving Certificate as follows:[19]

" In retrospect, the leaving certificate examination played a vital role in the development of secondary education in Scotland during the period 1888 to 1908. It also did much to raise the standard of education in secondary schools, in fact, the Scottish secondary school in 1908 was doing work which a quarter of a century previously had been done in the Arts classes of the universities".

It does not matter whether Chrystal's educational ideas particularly of the leaving certificate were original or derived from Germany, as pointed out by some of his critics. What does concern us is that he sketched a scheme which for more than three quarters of a century remained an ideal in Scottish education, in fact, the leaving certificate instituted in 1888 was the supreme award in Scottish secondary education, until its replacement by the Scottish Certificate of Education in 1962.

3.4 University Reforms

A year before George Chrystal occupied the chair of mathematics in St Andrews, the Royal Commission with Lord President Inglis as chairman was appointed to inquire into the Universities of Scotland. The commission had issued their report in 1878, which was introduced into parliament in 1883, 1884, 1885, 1887, 1888, and 1889, but it was not until the last mentioned year that it secured itself as legislation, and became the Universities(Scotland) Act, 1889.

As a result of the Universities(Scotland)Act, 1858, the curriculum for the M. A. degree was somewhat modified, but modifications were not enough and the degree was still very rigid. The report of the above commission contained the recommendation of replacing the uniform curriculum of seven subjects with new options. These recommendations included the requirement from every candidate for the degree of M. A. to pass a First Examination.

After passing this, the candidate was allowed either to follow the existing curriculum of seven subjects, or to take any one of the five areas of study, viz; Literature and Philology; Philosophy; Law and History; Mathematical Science; Natural Science. These areas included further branches. The contents of these reports were naturally discussed in gatherings of the staff of the Universities; and for Professor Chrystal(who always favoured sensible reforms) these provided a major subject for thinking.

He spoke on the topic in detail in his first promoter's address of 1885, where he says: "Although I have never hitherto taken any part in the public discussion of this matter, I have by no means been an indifferent spectator".

He further added that "I read every publication, good and bad, bearing on the subject which has come within my notice during the last ten years, the last of these being the Italian university bill".

Talking about higher education he says:

"Higher education is an expensive commodity, the furnishing of which involves most important practical questions regarding men and money. Who are the men that are to receive it ? Where are the men to come from who are to give it ? How is the money to be provided to maintain the givers of it, and to equip them with the necessary but costly apparatus ?"

As an answer to these questions he continues:

"The higher education in the strictest sense of the word must always be the possession of a very few, and yet the proposition that the avenues to it should be open to every one, however poor, who has shown special fitness to receive it, is to my mind so obvious, and is moreover so universally accepted in Scotland, that it would be idle to discuss it here. This proposition carries with it of course the admission that higher education must be supported to a large extent by the community at large, and can never be treated as a merely commercial article, subject to ordinary laws of supply and demand".

The Universities (Scotland) Act, 1889 greatly modified the structure of the Scottish Universities. It appointed the Scottish Universities Committee of the Privy Council as an administrative body, to take care of all the new ordinances and of all the petitions from or concerning the Universities. Moreover, under Section 14 of the Universities (Scotland) Act, 1889 an Executive Commission under the chairmanship of Lord Kinnear was appointed with enough powers to revise the the courses of study for all the degrees, to look after teaching methodology and granting degrees, to determine the length of the session, and the institution of an entrance examination in a university or a preliminary examination for entering a degree in any faculty, or both such examinations. It also considered the admission of women to instruction and graduation in any faculty.

Professor Chrystal not satisfied with the appointment of such a commission says in his address:

"No one in his senses expects that an executive commission will be able to sit down and draw up a scheme that will at once meet all our difficulties for all time coming. Such an idea belongs to the childhood of an educational reformer. What the commission will in all probability, do, _ what they certainly ought to do,_ is to put elasticity and, if need be, joints into the cast-iron framework of our university constitution, which will enable us gradually, as men and money can be found, to adapt ourselves to the existing want of our time".

He then quotes the views of Signor Coppino, the new Italian Minister of Public Education from a bill presented to the Italian Senate by his friend Professor Cremona(1830-1903), Professor of Mathematics and Director of the Engineering School at Rome and which according to him are the views as his own:

"The state should concede the most ample scientific-didactic freedom to the universities, meaning thereby the totality of university professors, who could be called to propose in new regulations or statutes of the faculties compiled by a commission elected by and common to all universities those parts of the scholastic regime which are not purely administrative, but are founded on scientific and technical criteria. Thus that part of the matter which by its nature ought to follow the progress of science and the movement of ideas would be determined by statutes made by experts and subjected to periodical revision at shorter intervals; while those parts should be determined by law which do not depend on scientific opinion, and which may without detriment remain unchanged for such a longer period of time as the life of an organic law regarding public instruction is wont to be".

To fulfil the general desire of widening the curriculum, the commissioners in their recommendations adopted a moderate policy of retaining seven as the number of subjects for the M. A. in Arts, together with the introduction of a large variety of options in the selection of those subjects. Much emphasis was laid on gradual improvement of science courses in the Arts degree.

The commissioners changed the regulations of degrees in Arts in the Scottish Universities by introducing Ordinance No. 11, which was passed in June 1892. It instituted the University Preliminary Examination in a specified form. For the ordinary M. A. degree it fixed four departments of study, viz; Language and Literature; Mental Philosophy; Science; History and Law; each with many fixed branches. It also increased the number of departments for an honours degree in Arts to eight.

From an academic point of view many of these curricula were of doubtful value and had little or no connection at all with each other.

Professor Chrystal expresses his views about mathematics and the new Arts ordinance in detail in his promoter's address of 1892:

"Regarding the general principle of the ordinance it would hardly be profitable to speak at length, as it has been tacitly agreed to give it a trial. I cannot, however, refrain from saying that after mature consideration I have come to think that it is of doubtful educational soundness".

He then goes on saying that the evil was so obvious and so likely to cause trouble both inside and outside the university, in departments of mathematics and natural philosophy, that it called for general condemnation.

Describing representations made to the commissioners and their effect on the ordinance he says:

"The commissioners are treating the representations made to them in a conciliatory spirit, and I hope a remedy will be provided which, if it does not effect all that some of us would desire, will yet prevent immediate disaster, and gives us time to devise a better plan after some years' experience of the new conditions".

He describes the school of mathematics and natural philosophy in Edinburgh as follows:

"The University of Edinburgh has been famous as a school of mathematics and natural philosophy ever since the Gregorys, in the latter part of seventeenth century brought into its teaching the spirit and methods of Newton. David Gregory, afterwards Savilion Professor in Oxford, was indeed a favourite follower, distinguished by Newton himself; and it was in his lecture room in the university of Edinburgh that the doctrines of the 'Principia' were first publicly taught in Great Britain. Ever since then the position of natural philosophy as an advanced subject, to which pure mathematics is in part ancillary, has been fixed in the Scottish universities".

He then says that in the draft ordinance for Arts degrees, while higher standards had been imposed on Latin and Greek as graduation subjects, nothing of the kind had been for the mathematical department.

This omission was thought to be mere accident but everyone was surprised when it was found in the final ordinance that mathematics and natural philosophy were placed as compulsory alternatives, with the higher standard of entrance for mathematics and the lower for natural philosophy. It was clear that natural philosophy would be used as the outlet for those who could not reach the higher standard in mathematics on entering the university.

Professor Chrystal explains his own position in these words:

"Ever since I became convinced that a majority of educated Scotsmen desired to break down the old curriculum of seven subjects, my watchword has been "Greater freedom and higher standards". It is obvious that in any subject which is generally compulsory the standards cannot be high. I was never very anxious that all Arts students, should take either mathematics or natural philosophy; but I have all along striven to secure, so far as possible, that those who do take these subjects should be well prepared to receive them. To meet the difficulty of those who desired to have no mathematics, I proposed that an alternative should be given of a physical or natural science with practical or laboratory work; that mathematics should be entered on the higher standard, and that natural philosophy should remain as Newton made it and Gregory expounded it. The Commissioners adopted the part of my proposal relating to entrance on mathematics; but made their action nugatory by ignoring the rest of it, although they had fully carried out the principle in the science ordinance".

According to Professor Chrystal all this was a result of lack of proper consultations and representation on the part of commissioners. Departments of mathematics were unfortunate in the evolution of the ordinances: first mathematics was practically dropped from the science degree, second honours mathematicians did not get full justice as compared with those studying classics. Mathematicians were obliged to take classics yet no classical honours student was required to take a mathematical subject.

George Chrystal also welcomed women into Arts classes and said "several women were distinguished for humanistic culture during the early days of the revival of classical learning and from Hypatia down to Madame Sophie Kovalevski, who died recently, women have from time to time distinguished themselves as mathematicians".

Professor Chrystal being Dean Faculty of Arts, was responsible to get the curricula modified as far as possible. It was the Dean who was also responsible to bring all the changes into practice. Professor Chrystal with his experience and hard work accomplished the job successfully. In view of the rapid developments that were taking place and many other reasons already pointed out by him, some of the fundamental changes made by the commissioners were subject to revision.

Under Section 21 of the Universities (Scotland) Act, 1889 the University Court of each university after expiration of the powers of the commissioners under the Act were given powers to make ordinances as they thought fit, subject to approval of the Scottish Universities Committee of the Privy Council. Thus in 1907 after the powers conferred on the commissioners expired, each of the Courts of the four Scottish Universities decided to use the powers conferred upon it of framing its own regulations for the M. A. degree. Edinburgh University Court prepared a new Arts ordinance called Edinburgh Ordinance No. 11, which was approved by the Privy Council on 5 May 1908.

The salient features of this ordinance were that there would be four departments each having many subjects. It made provision for specialised study and concentration on fewer subjects.

The curriculum, according to it consisted of five separate subjects; two of the five had to be studied for two sessions each, and the examinations in these had to be passed on a higher standard than in the other three.

Under the ordinance the Senatus was given power to reckon courses in two cognate subjects as two courses in one subject. The Senatus was also given permission to modify the list at any time, to define, group and regulate the order in which the subjects were to be studied, all of this subject to approval of the University Court.

The ordinance extended the Academic year to a minimum of twenty-five weeks, divided into three sessions between October and July.

The ordinance was to come into force at the beginning of the first Academic year after its approval which was the session 1909-10.

In the capacity as Dean of the Faculty of Arts, Professor Chrystal had to carry out the hard work of framing the new regulations and steering them through the Senatus and other committees. This was not an easy assignment, it required patience and administrative experience, which he fortunately had. How hard he worked to achieve the target, those who worked with him as heads of other departments know better than anyone else. Professor(later Sir) Richard Lodge, who was Professor of History at the time, and later succeeded Professor Chrystal as Dean of the Faculty of Arts, in his own reminiscences says:[20]

"The ordinance was mainly due to the initiative of an able and efficient Dean of the Faculty, the late Professor Chrystal....."

According to J. S. Black[21], " the minute of the Senatus drawn up at the end of his twenty years' term of office records: 'To his knowledge of public opinion, to his mastery of educational problems of the day, and to his unwearing zeal and administrative capacity, it was mainly due that these changes were successfully accomplished'".

In his promoter's address of 10 April 1908, Chrystal refers to these changes in the following words:

"The realisation of their consequences will be a matter of time and no little labour for the university staff, and will ultimately make heavy demands on university resources. I am keenly interested in the developments that lie before us, but I must confess that I shrink from the labour that they will involve. Yet the whole of my career has been a turmoil of university reform, beginning in Cambridge, and it may as well end as it began, if it be decreed that it is to continue any longer".

3.5 Edinburgh Provincial Committee for training of teachers

To carry out all the educational reforms as suggested under various education Acts after 1872, a great need for trained teachers was felt by the Department, the teacher being the pivot upon which the whole educational system moves. It was felt that failure here in quality or quantity would paralyse all and frustrate the brightest hopes. Thus it was decided in January 1905[22] to constitute provincial committees in each of the four Scottish universities for the training of teachers. Up to the time of institution of Provincial Committees the main work of teacher training was almost in the control of the churches.

Professor Chrystal had by this time been recognised as a national educational reformer, so it was natural, perhaps almost inevitable, that he should be called upon to take part as one of the five representative of the University in the Edinburgh provincial committee. It was entirely in character for him to respond positively when asked and he was eventually elected chairman of the committee[23]. Despite very limited powers given to the committee, he took a leading part in this process of framing a new system for the training of teachers.

A provincial committee was in reality responsible to no body, and the Scottish Education Department had much control in financial matters.

The success of the new system was commented on by the Educational News[24], which attributed this success to the chairman, conveners of different sub-committees, and other officials of the committee.

Professor Chrystal had these remarks to make on the new committees:[25]

"That scheme would attain a magnitude which was not anticipated by many, and would permeate the whole of their educational system. That new institution would have far-reaching effects, not only in connection with schools, but also, he hoped, in connection with the universities".

Under the Minute of the Committee of the Council on Education in Scotland for the establishment of provincial committees there were three immediate tasks confronted by the provincial committees. Firstly, framing rules for teacher training in Scotland and for this purpose a draft of the Department's new regulations was handed over to them for their consideration and discussion. Secondly, negotiations with the church authorities on transferring the existing training colleges in their control to the provincial committees.Thirdly, the committees were responsible for the organisation of all business arrangements for the new body.

Professor Chrystal with all his experience , managed all the three tasks very smoothly. He was elected chairman of the joint committee of the four provincial committees.

In view of their limited executive powers, the committees achieved much more than was expected from them. Some critics considered the committees to be the mouthpiece of the Department, but this policy proved to be very successful for the smooth running of the affairs of the committees.

In addition to the tasks already mentioned above, each committee was given powers to acquire or purchase suitable premises for training colleges and to appoint officers for instruction or discipline. All of this was subject to the approval of the Department. It was an enormous task and involved, in the case of the Edinburgh provincial committee alone, an expenditure of more than thirty thousand pounds of public money, as reported by the Educational News in its report of the committee on 23 April 1909.

Under the Minute, there was no fixed term of office for the committees and the Scottish Education Department was given authority to announce a date for appointing a new committee, and when the Department set 30 October 1909 as the date for appointment and election of a new provincial committees, Chrystal wrote:[26]

"I have been medically advised that for some time to come I must diminish the amount of business for which I am responsible, if I am not to court final unfitness for all business whatsoever. As the work of your committee was the last faggot added to the bundle, it must be the first removed".

The Scottish Education Department appreciated his services to the Department in this capacity. According to Dr J. S. Black, the then Assistant Secretary to the Scottish Education Department, Dr George Macdonald praised his services in the following words:[27]

"His work in this and many other ways met with comparatively scant recognition from the public. I dare say the average man might have thought more of him if he had accepted the knighthood which the government is understood to have offered him. But he cared for none of these things and was content with consciousness of having done his duty. If, however, the circle of those who learned, through his connection with the provincial committee, to appreciate his worth and to care for his personality was small, the measure of that appreciation and of the liking that was engendered, was large indeed. I do not know any instance of a public man whose labours and whose personality have been spoken of with great or more uniform cordiality by all those who were privileged to be in touch with what he was doing".

The members of the Edinburgh provincial committee, while recording their gratitude of his services, mentioned his general reforms in education for his native country, which according to them had opened a new era for education in Scotland. To them he deserved and had received,"the warmest appreciation of all who are conversant with the subject". Moreover in their view, " He has been an ideal chairman, and his resignation has been received by all with profound sense of personal loss......"[28]

As to his personal relations with his colleagues he wrote in his letter to the Department:[29]

"These relations, I am happy to say, are not clouded by a single unpleasant recollection.

The name of the local administrator is 'writ in water'.

He must look for his reward in the approbation of his own conscience, and in the keen sense of friendly comradeship which is generated by sharing a common enterprise for what is believed to be the public good. Such reward is enough, in my opinion for any one, certainly enough for me. During the four years that I have worked with you I have learned to know and like many men with whom I should otherwise never have become intimate. I hope the friends I have thus made will remember me as long as I shall remember them, and with equal pleasure".

At the end he thanked his colleagues for the uniform kindness and courtesy with which they had treated him during his term of office, and wished the same kind of treatment from the committee for his successor.

His great business capacity and the thoroughness and devotion with which he carried out the duties of his responsible office were much appreciated by the committee. They appreciated also his courtesy, his fairness of mind, his personal kindness, and his unselfish readiness to credit to others the success of work which he had himself inspired.

The opportunity is not always given to an academic to prove his competence in the world of business. Yet to Professor Chrystal the opportunity came and did not find him wanting. In the opinion, not only of his immediate colleagues on the Edinburgh provincial committee but of those who had occasion to meet under his chairmanship he was a valuable asset in starting and keeping on proper lines the new scheme for training of teachers.

There was another committee on which Professor Chrystal had also served. In 1901 he was appointed as a member of a committee which, as a result of the South African war, had been appointed by the War Office to advise on the education of officers. This was another opportunity for him to utilise his long experience as a teacher and a master of educational methods. Although the result of his services on this committee was not seen in his lifetime, his advice provided a sound base for the training of officers, which was recognised and appreciated by successive Secretaries of State[30].

References Chapter 3

1.*The Report Of the Royal Commission on Education in Scotland* 1868, p.ix.

2.*Problems of National Education* edited by John Clarke;
MacMillan and Co.London.(1919); P.11.

3.*Educational Endowments*(Scotland) Act, 1882, section 5.

4.Dr J. S. Black and Prof. C. G. Knott, Obituary Notice of Prof. Chrystal,
Proceedings Royal Society of Edinburgh, session 1911-12.Vol.xxxii, p.481.

5.Promoter's Address by Professor Chrystal to Arts Graduates;
University of Edinburgh (1908).

6.*Education (Scotland) Report 1873-74* , p.xviii.

7. T.R.Bone, *Studies in The History of Scottish Education 1872-1939* ;
University of London Press(1967), p.136.

8. *Education (Scotland) Report 1885-86* , p.116.

9. *Education (Scotland) Report 1886-87* , p.xxviii.

10. *Ibid* ; pp:119-22.

11. T.R.Bone, *Studies in The History of Scottish Education 1872-1939* ;
University of London Press(1967), pp:139-40.

12. *Education (Scotland) Report 1887-88* , p.109.

13. *Ibid* ; pp:110-11.

14. T.R.Bone, *Studies in The History of Scottish Education 1872-1939* ;
University of London Press(1967), p.149.

15. *Ibid* .

16. *Secondary Education Report (1899)* ; p.8.

17. *Secondary Education report (1900) ; Circulars 270, 271 ;* pp:30-31.

18. *Secondary Education report (1902)* ; pp:83-88.

19. *The Secondary School Journal,* November 1906; p. 66.

20. Margaret Lodge, *Sir Richard Lodge: A Biography* (1946); p.150.

21. Dr J. S. Black and Prof. C. G. Knott, Obituary Notice of Prof. Chrystal, *Proceedings Royal Society of Edinburgh,* 1911-12.Vol.xxxii, p.482.

22. *Minute Of The Committee Of Council On Education in Scotland,* 30 Jan.1905.

23.*The Educational News,* 25 Nov.1905,p.869.

24.*Ibid* ; 13 Dec.1907,p.1175.

25.*Ibid* ; 2 Dec. 1905,p.889.

26.Dr. J. S. Black and Prof. C. G. Knott, Obituary Notice of Prof. G. Chrystal; *Proceedings Royal Society of Edinburgh.* 1911-12.Vol.xxxii,p.486.

27. *Ibid* ; p.484.

28. *Ibid* ; p.486.

29. *Ibid* .

30. *Ibid* ; p.485.

Chapter 4

Mathematical Work

4.1 Introduction

Professor Chrystal's period as professor of mathematics in the Universities of St. Andrews and Edinburgh, was in fact the era in which mathematics had grown into an enormous and unwieldy structure. The subject matter in the existing fields was greatly extended and many new fields came into existence. This expansion became possible because a small aristocracy of mathematicians was supplanted by a far broader group. Mathematicians who were masters of the whole subject ceased to exist, instead, mathematicians began working in small corners of the subject often considering their area more important than others. Publications were no longer for a large public but for particular colleagues. The progress of mathematics during this period was greatly accelerated by the setting up of several new mathematical societies with their regularly published proceedings. The number of such periodicals rose from 210 in the eighteenth century to about 950 in the nineteenth century. As a result the total rate of mathematical publications doubled during the period, and the number of mathematicians also increased, a trend which was to continue through to the 1970's levelling off in the 1980's.

By 1900 mathematicians did recognise that they could no longer rely upon the physical truth of mathematics to be sure of its consistency. Instead the view that mathematics can introduce and deal with rather arbitrary concepts and theories that do not have immediate physical interpretation but may nevertheless be useful, as in the case of quaternions, or satisfy the desire of generality, as in the case of n-dimensional geometry, gained acceptance. Mathematicians for the first time carried their work far beyond being a tool for science and technology by posing and answering questions that had no bearing on real-world problems.

This was the period of rigorization of mathematics, something which had been neglected by many mathematicians for the previous two thousand years despite much progress being made in the subject matter. One of the most conspicuous and distinctive features of mathematical thought in this period was its critical spirit. In fact it is a period in which the foundations of mathematics were reexamined and the fundamental principles were worked out afresh. Mathematics became universal and its progress was not confined to any particular region or country, as was the case previously.

Great Britain, after the great set back due to the controversy between Newton and Leibnitz, once again proved to be a major centre of progress in the subject and was no longer secluded as in the preceding centuries. In fact British mathematicians were pioneers in many aspects with contributors such as W.C. Clifford, A. Cayley, J.J. Sylvester and many more. Mathematical physics developed rapidly in Great Britain, this development being led by Maxwell, Tait, Stokes, Kelvin, Rayleigh with many others also making major contributions.

The group concept was developed throughout the nineteenth century and, at its close, was seen to be one of the fundamental and most fruitful notions in the whole of science.

4.2 Non- Euclidean Geometry

When the time is ripe for a new idea to emerge, the idea often occurs to several people at different places in the world simultaneously. This was the case in the seventeenth century with the discovery of calculus by Newton in England and by Leibnitz in Germany and in the eighteenth century this happened with the discovery of non-euclidean geometry by Carl Friedrich Gauss in Germany, Janos Bolyai in Hungary and Nicolai Ivanovitsch Lobatschewsky in Russia. Twenty centuries of useless effort and in particular the last unsuccessful investigations for the proof of the fifth postulate, convinced many of the geometers, who thrived about the beginning of the nineteenth century, that the solution of the theory of the parallels problem was impossible.

Nevertheless keen interest was taken in the subject. Janos Bolyai (1802 - 1860) published his discoveries as a twenty-six page appendix to a book [Tentamen Juventutem Studiosam in Elementa Matheseos] by his father Wolfgang Bolyai. Wolfgang Bolyai sent a copy of this book to his old friend C. F. Gauss, who was the most eminent mathematician of the time. Janos must have expected appreciation from Gauss for his marvellous work, but in fact was very disappointed when he read the reply to his father from Gauss, in which he said "to praise it, would be to praise myself". A deep depression brought on by this reply caused Janos never again to publish his research.

The story of non-euclidean geometry did not end here, for also involved was a Russian mathematician N. I. Lobachevsky. He was actually the first to publish his work on non-euclidean geometry in 1829. This work, which was published in Russian, attracted little attention at the time and the Russians who read it were critical. In 1840 he published a treatise in German which attracted the attention of Gauss who praised it in a letter to Schumacher. Finally, in 1855, a year before his death, when blind, he dictated and published in Russian and French a complete exposition of his system of geometry, which he first called "Imaginary Geometry " and then " Pangeometry ".

It was not until the death of Gauss in 1855, when his correspondence was published, that the mathematical world began to take non-euclidean geometry seriously. Some of the best mathematicians of the time took up the subject, extended it, and applied it to other branches of mathematics, notably complex function theory. In 1868 the Italian mathematician Euginio Beltrami settled once and for all the question of the proof for the parallel postulate. He proved that no proof was possible. He did this by proving that non-euclidean geometry is just as consistent as euclidean geometry and is independent of it.

The first "Non-Euclidean boom" occurred after the publication of Gauss's correspondence in the late 1860's, and the first lecture in its favour in Germany was given only in 1870 by Hermann Von Helmholtz, delivered in the Dozenten Verein in Heidelberg.[1]

Professor Chrystal in association with Professor Tait wrote the obituary notice of his predecessor Professor Kelland in which he particularly praised Professor Kelland's memoir on non-euclidean geometry and investigations in wave motion presented to the Royal Society of Edinburgh. The same year he himself presented his first paper to the Royal Society of Edinburgh on non-euclidean geometry even before he was formally elected a Fellow of the Society, and interestingly the last work of his life was on the theory of seiches, a subject related to wave motion.

Professor Chrystal when asked by the Royal Society of Edinburgh to give an address, chose the subject "Non-Euclidean Geometry"[2] partly because his predecessor Professor Kelland had brought this to the notice of the Society, and partly because in his memoir Professor Kelland treated the subject from only one point of view and the other side of it was left out.

Although the paper does not contain any genuinely new results it does show Chrystal's potential as a mathematician. In the introduction he points out that as a result of discussions with school masters, he had reached the conclusion that geometrical insight and geometrical ideas, either natural or acquired, are essential to a teacher. In this address he had simply tried in a synthetic way to give a general idea of what was known. In so doing he had used the methods of Euclid as much as he could, sacrificing elegance for practical advantage. He adopted Euclid's propositions concerning angles at a point, viz ; I. 13, 14, 15 ; also the propositions as to congruency I. 4, 5, 6, 8, and first part of 26, saying that in many cases his proofs are unnecessarily indirect and difficult. All that is necessary for the proof of these propositions is the defining property of the straight line and the axioms and definitions of equality. He did not give any bibliographical details on the subject but referred to the bibliography given by Mr. Halsted in the first volume of the American Journal of Mathematics.

According to Professor Chrystal the question of the origin of the axioms of geometry is the most interesting for a thinker and most important for a mathematician but he does not take up the question directly. He divides the contributors to this subject into two groups, one consisting of mathematicians headed by Gauss, Lobatschewsky, Bolyai, and Riemann, the other consisting of physiologists represented by Helmholtz.

The mathematical investigators represent the subjective side of the subject, the physiologists represent the objective, although Helmholtz, the personal representative of the latter, is a union of both types of philosopher. Helmholtz's investigations were carefully examined by Marius Sophus Lie (1842 - 1899).

He considers the Non-Euclidean geometries in three dimensional space, and states that a point has no extension, a line is once extended, a surface twice, and a solid thrice. He says that two points ***in general*** determine a straight line.

Professor Chrystal distinguishing different cases which arise when two straight lines intersect, defines three different kinds of spaces :-

1. Each line is non-re-entrant. Space which has this characteristic he calls *hyperbolic space* . Later another case is distinguished under this head, namely homaloidal or *Euclidean space* .

2. The lines may intersect again. Space having this characteristic is called *elliptic space* . If two straight lines intersect in only one point it is single elliptic space, and if they intersect in two distinct points the space is called double Elliptic space.

According to Professor Chrystal a Euclidean space is a hyperbolic space (as he defined it), yet may be regarded as a limiting case of elliptic space. It is therefore the transition case lying between the other two.

In hyperbolic space, he proves that, either the *defect* (i.e. the amount by which the sum of the three angles of a triangle falls short of two right angles) of a triangle is always positive or it is always zero. The latter alternative gives Euclidean space while the former gives hyperbolic space.

He further proves that the area of a triangle in hyperbolic space is proportional to its defect. If A is the area of a rectilinear triangle of defect δ, we have $A = \rho^2\delta$ or $\delta = A/\rho^2$ where ρ is a linear constant characteristic of a hyperbolic space, this is also called the space constant in modern terminology. If $\rho \to \infty$, then $\delta = 0$ for every triangle of finite area. In other words, Euclidean space is simply a hyperbolic space whose linear constant is infinite.

Alternatively consider a hyperbolic space of given linear constant ρ. Take a region in this space whose greatest linear dimension is an infinitely small fraction of ρ, then the defect of every triangle within that region will be infinitely small, and its geometry will not differ greatly from that of a Euclidean space. Thus hyperbolic space is Euclidean in its smallest parts.

Hence in hyperbolic space Euclid's planimetry will apply to infinitely small figures. For example the ratio of the circumference of a circle to its diameter will be $\pi = 3.14159$....when the diameter is made infinitely small. Chrystal notes therefore that he can use all the formulæ of Euclidean trigonometry, if proper restrictions be observed.

According to Professor Chrystal the existence of this length ρ related to the space, but not directionally related, suggests the possibility of explaining the properties of 3-dimensional space by embedding it in a space of four or more dimensions.

He then remarks that he is not entering into speculations of this type because the conceivability of hyperspace of three dimensions rests on different grounds from that which must be assumed when we add another dimension. He further continues that in this he might be one of those whom Gauss playfully called Bœtians.

Professor Chrystal further proceeds to demonstrate the following propositions in the hyperbolic space geometry:

I. The exterior angle of any triangle is not less than the sum of the two interior opposite angles.

II.The greatest side of every triangle has the greatest angle opposite, and conversely.

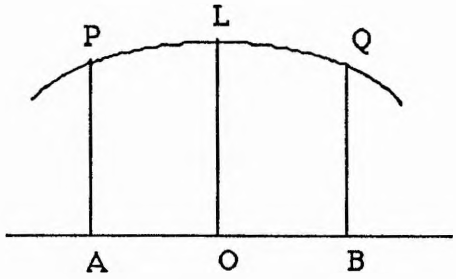
III. Any two sides of a triangle are together greater than the third side.

IV. If the defect of any triangle whose sides are finite be zero, then the defect of every finite triangle must be zero.

Definition:- Equidistants.- Professor Chrystal defines equidistants as follows:

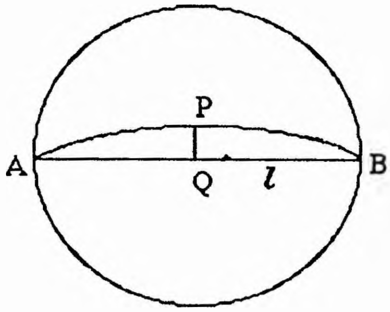
"If in any plane, perpendiculars of constant length be erected upon a given straight line, their extremities generate two curves, which are called the equidistants".

According to him an equidistant is a self congruent line, is at every point at right angles to the generating perpendicular, is curved and is concave towards the given line as given in the figure.

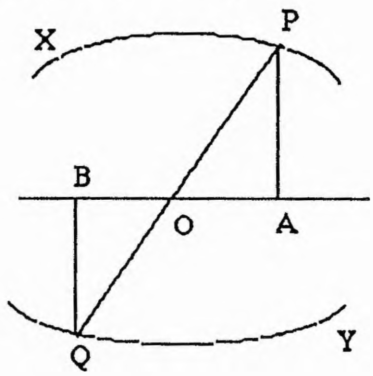


We can consider this in the light of later developments, by using Poincaré's model which was given by H. Poincaré about three years later in 1882 to establish the consistency of hyperbolic geometry. The form in which this model is commonly used for hyperbolic plane geometry is the one in which absolute is taken as a circle. This form seems to have first been given by Joseph Wellestein and H. Weber, in *Enzyklopädie der Elementar-Mathematik* , in 1905, and is attributed to Poincaré, because of its closeness to the model he gave in *Bull. Soc. Math. de France*, in 1887. By the use of this model a hyperbolic space can be represented in the Euclidean plane. This model represents angles accurately for the hyperbolic space but distorts distances.

Consider A and B to be the ideal end points of the given line *l*. Then the equidistant curve through a point P not on *l* is represented by the arc of the Euclidean circle passing through A, B and P and is orthogonal to all Poincaré lines perpendicular to the line *l*. It is also known the hypercircle through P.



Chrystal then defines conjugate equidistants, a term which is no longer used, and says that two equidistants corresponding to a given length of the perpendicular may be called conjugate equidistants and are given by XP, YQ in the figure.

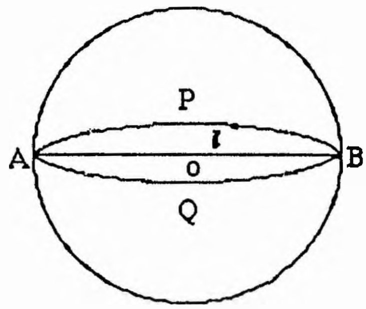


Next he establishes the following two propositions concerning conjugate equidistants:

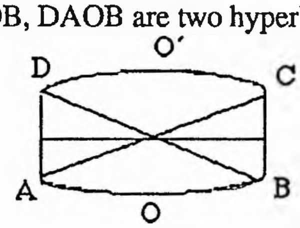
V. Every straight line terminated by a pair of conjugate equidistants to a given straight line is bisected by the given straight line, and makes equal alternate angles with the equidistant.

VI. The common perpendicular to two conjugate equidistants is the least distance between them, the oblique distances are greater according as the angle they make with the perpendicular is greater, and the length of an oblique can be increased without limit.

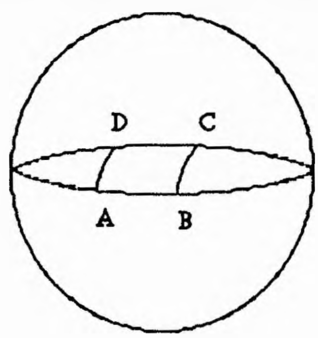
In the Poincaré disk model conjugate equidistants can be represented by the arcs of the Euclidean circles passing through A, P, B and A, Q, B as in the figure.



In hyperbolic geometry rectangles do not exist, but one can consider skew quadrilaterals. Interestingly Professor Chrystal defines a hyperbolic parallelogram by using his conjugate equidistants as the figure formed by joining equal arcs of two conjugate equidistants towards the same parts by two straight lines. He also defines a hyperbolic triangle to be a mixed triangle whose base is the arc of an equidistant, whose remaining sides are straight lines, and whose vertex lies on the conjugate equidistant. In the adjoining figure AOBCO′D is a hyperbolic parallelogram and CAOB, DAOB are two hyperbolic triangles.



In the Poincaré model these can be represented as in the figure.



Chrystal proves the following propositions for rectangles and triangles:

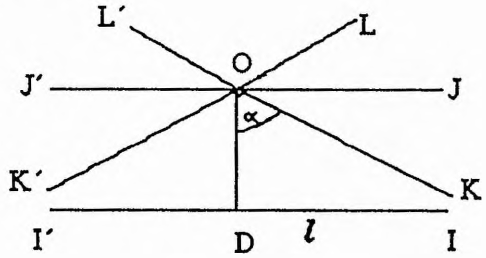
VII. Two hyperbolic triangles which have for common base the arc of an equidistant (and consequently have their vertices on the conjugate equidistant) are equal in area.

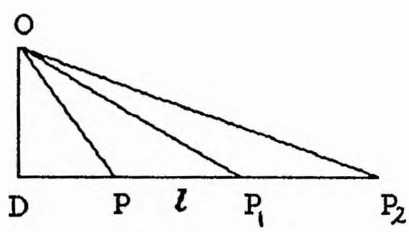
VIII. The rectilinear triangles on the same chord of an equidistant, whose vertices lie on the conjugate equidistant are equal in area and defect.

IX. We can always construct an isoceles triangle whose base is equal to one side of a given triangle, and whose area and defect are the same as those of the given triangle. Hence two triangles that have the same area must have the same defect, and conversely.

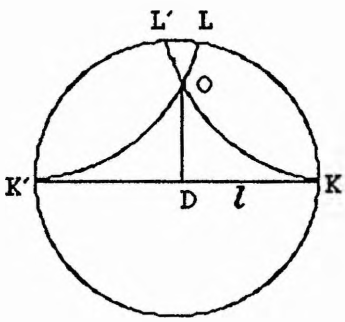
Definition: Given any line *l*, if O be a point not on *l*, P a point on *l*, to the right of the foot of the perpendicular from O on *l*, then the limiting position of OP, when P is moved in the direction DI to the right, without limit is called the parallel through O to DI. The corresponding limiting line on the other side of OD is called the parallel through O to DI′. These are exhibited in the following figures given by Professor Chrystal. He classifies the lines through O in three categories: (i) Intersectors; (ii) Non-Intersectors; (iii) Parallels.

According to him all lines in the angles KOL, K′OL′ are non-intersectors, all those lying in KOK′, LOL′ are intersectors, KOL′, K′OL are two parallels, and α is the angle of parallelism.





In the Poincaré model Chrystal's hyperbolic parallels as defined above can be represented as in the attached figure.



In the theory of the parallels for hyperbolic space Professor Chrystal establishes the following propositions:-

X. If a line is parallel to another at any point, it is so at every point of itself.

XI. Parallelism is mutual.

XII. Lines which are parallel to the same line are parallel to one another.

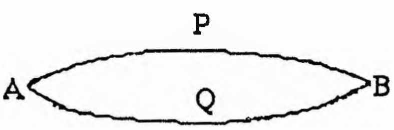
XIII. Lines that are parallel continually approach one another on the side towards which they are parallel.

XIV. Non-intersectors in the same plane have a minimum distance, which is the common perpendicular.

XV. The parallel angle is a function of the length of the perpendicular, increasing when the perpendicular diminishes.

Next Professor Chrystal demonstrates some propositions in the geometry of elliptic space. He establishes them in the case of single elliptic space but remarks that these can be modified to make them apply to double elliptic space. The term double elliptic geometry is no more in use; it is in fact now called spherical geometry. In single elliptic space every straight line returns into itself and two straight lines intersect in only one point. Thus, starting from any point P and proceeding in any direction continuously, we at last return to the point P; the length L travelled during this process is called the *complete straight line.*

In single (as well as in double) elliptic space two intersecting complete straight lines enclose a plane figure. Such a figure he called a *biangle* . In the figure here APBQA is a biangle.



The propositions proved by Chrystal are as follows:-

XVI. Two biangles are congruent when their angles are equal. All complete straight lines are of the same length, and all the straight lines emanating from the same point intersect in the same second point.

XVII. In single elliptic space the least distance between two points can never be greater than (1/2)L, and the greatest distance can never be greater than L.

XVIII. If we consider the plane determined by two intersecting straight lines AOA, BOB, and if we pass from O along OA through a length L, we return to O, but find ourselves on the opposite side of the plane to that from which we started, and only arrive at the same point O on the same side as before by travelling once more through a length L.

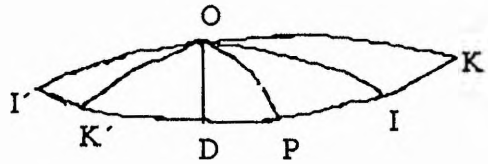
In order to establish the fundamental proposition concerning the sum of the angles of a plane triangle in elliptic space, Professor Chrystal makes use of his biangles, each of the biangles departs from the vertex on the upper half side of the plane and returns to the vertex on the lower side.He deduces that the excess (i.e; the amount by which the sum of the three angles of a triangle exceeds of two right angles) of every triangle is positive, and is proportional to its area.

In Poincaré's model a line in an elliptic space means a great circle on the sphere, and a point means a pair of antipodal points. Equidistants in this space can be represented by the circles of latitude, which are also called "small circles". All lines perpendicular to a line l are not parallel to each other but are concurrent, i.e., they meet in a single point, called the pole of l with l its polar. Professor Chrystal states that if we rotate a given plane about its polar the pole describes a straight line which may be called the conjugate of the given polar.The relation of these two lines is mutual, every point on one being at a distance (1/2)L from every point of the other.

According to Professor Chrystal, in elliptic space there is no such thing as a parallel, because there are no infinitely distant points on a straight line.

If O is a point not on a given line IDI′, then the two segments of the perpendicular from O are respectively the least and greatest distances from the given line. If OD is the least distance then as OP, starting from OD, revolves about O, it continually decreases until it has rotated through 180°, and then it is at its maximum after which it decreases again. Also as OP revolves from OD, the angle OPD decreases until OP is perpendicular to OD, and then

OPD is at its minimum value. By producing the line backwards through O, the angle increases again.



According to Professor Chrystal the line OI, perpendicular to OD, is all that there is in elliptic space to represent a parallel through O to the line I'DI. This in modern terminology is the same as the Clifford parallel.

He concludes that elliptic space is finite but unbounded- finite because all lines have finite length and look like circles, and unbounded because there is no boundary.

He concludes that if he had succeeded in explaining the results of research concerning the axioms of geometry, it will be clear that it is possible to develop three self-consistent kinds of geometry - the hyperbolic, the homaloidal, and the elliptic. If a priori grounds are to be sought for the axioms of geometry, such tests of its firmness "as the conceivability of the opposite" and others like it are not to be relied upon. They are merely an appeal to ignorance.

At the end he gives a brief Appendix on trigonometry where he remarks that the trigonometry of a single elliptic space is identical with the geodetic trigonometry of a sphere, although it would not be correct to say that the planimetry of single elliptic space is identical with the geodesy of a sphere.

For hyperbolic space the analogue is the pseudo-spherical surface of Beltrami. This representation was discovered by Beltrami by using previous work by F. Minding.

Professor Chrystal in addition to his major articles which have already been taken up in Chapter 2, wrote many short articles for the ninth edition of Encyclopædia Britannica. The majority of them were biographical notes of mathematicians and physicists. Some of these are discussed precisely as below:

"Mathematics" gives a precise account of the historical development of basic ideas of the subject. "Parallel" discusses in detail the historic background of the parallel axiom in

Euclid's geometry. "Perpetual Motion" contains the definition and history along with some experiments given in support of its proof. "Montucla" is a short biography of the famous historian of mathematics, whose book on the history of mathematics was completed after his death by Lalande and was published in Paris in four volumes during the period (1799-1802). According to Professor Chrystal it was the first book on the history of mathematics worthy of the name, and was characterised alike by the elegance of style and breadth of treatment. Only a part of the article "Pascal" was written by Chrystal, in which he describes Pascal as one of the greatest mathematicians of the time, specially for his contributions to geometry, probability and many other branches along with his calculating machine. He also describes his work as a natural philosopher. The most important contribution Pascal makes to this area is his treatise on Equilibrium of Fluids published in 1662. According to Chrystal this entitles Pascal to rank with Galileo and Stevinus as one of the founders of the science of hydrodynamics.

"Peacock" is a detailed biography of George Peacock giving his contributions to mathematics at large, particularly his enthusiasm to bring in Leibnitz's notations in Cambridge and undertaking other university reforms.

"Poisson" is a detailed biography of the celebrated French mathematician, who despite his many official duties found time to publish more than three hundred works in mathematics and related topics. Several of these works are extensive treatises and many are memoirs dealing with the most abstruse branches of pure and applied mathematics of the time. The article includes the list of Poisson's major works.

To decide about his rank amongst mathematicians of all ages, Chrystal compares him with his contemporaries, chief among whom were Lagrange and Laplace and says "..... although we can not see him alongside of these mighty sovereigns, yet it is impossible to deny him the nearest rank to them in the temple of mathematical fame".

In justification of his judgment he mentions a story given by Arago

" "I am told" said Lagrange to Poisson one day; " during my long intervals of sleepiness I divert myself by making numerical approximations. Keep this one; it may interest you. Hygens was thirteen years older than Newton, I am thirteen years older than Laplace. D'Alembert was thirty-two years older than Laplace, Laplace is thirty-two years older than you".

Professor Chrystal says, " Arago who gave this story, justly remarks that no more delicate way could be conceived of intimating to Poisson his admission into the inner circle of the fraternity of mathematical genius".

In addition to these Chrystal wrote another sixteen biographical notes for the ninth edition of Encyclopædia Britannica.

Professor Chrystal's second paper on "Minding's System of Forces"[3] was another of his contributions to geometry. Its importance is indicated by the fact that the paper was published in the Transactions of the Royal Society of Edinburgh.

F. Minding has proved a remarkable theorem concerning a variable system of forces. The points of application of the different forces, and their magnitudes are given, while the directions are such that a pencil of rays through any given point parallel to them moves as a rigid body. This theorem was proved by Minding in an excessively elaborate process in vols. xiv; xv of Crelle's Journal in 1835 and 1836. After Minding's original investigations several others were given. The last of these before Chrystal's work, due to Professor Tait[4], rests typically on a quaternion method and is so elegant and concise that it inspired Professor Chrystal to reinvestigate the whole subject by ordinary methods in the hope that the analysis might lead to further insight. He presented two methods of arriving at Minding's results, and reached a variety of other conclusions by means of the second method sufficient to suggest a full investigation of the complex formed by the single resultants of Minding's system.

First Method

The components of force and couple are found in terms of the Rodrigues coordinates

λ, μ, υ, which determine the position of the rigid pencil representing the direction of forces. The equations for the single resultant are then found in terms of two constants g and h, and the parameters λ, μ, υ. Equations are then deduced for the values of λ, μ, υ corresponding to a ray passing through a point (x, y, z). Eliminating μ and υ a biquadratic is found for λ. The system of resultant rays therefore forms a congruency of the fourth order. This biquadratic becomes wholly indeterminate for points on the real focal conics of the ellipsoid

$$\frac{x^2}{g^2+h^2} + \frac{y^2}{h^2} + \frac{z^2}{g^2} = 1 \text{ - - - - - - - - - (I)}$$

Some further discussion leads to the conclusion that the single resultant in Minding's system consists solely of the congruency of lines intersecting the two focal conics of (I).

Second Method

If ξ, η, ζ are coordinates of the feet of the perpendicular from the origin on any ray whose direction is (λ, μ, υ), and ρ the length of that perpendicular, it is shown that

$$\rho^2 = g^2\mu_1^2 + h^2 \upsilon_1^2 \quad \text{..................(II)}$$

$$\rho^4 + g^2\eta^2 + h^2 \zeta^2 = g^4\mu_1^2 + h^4 \upsilon_1^2 \quad \text{.................(III).}$$

(II) is true for central axes generally, and determines a complex of the second order which they form. Both (II) and (III) are true for the rays of a single resultant, and are twofold relations which determine a congruency with which they are identical. A discussion is given of the complex determined by the relation

$$\rho^4 = f^2 \lambda_1^2 + g^2 \mu_1^2 + h^2 \upsilon_1^2 \quad \text{..............(IV).}$$

of which (II) is a particular case.

The equations to Plücker's complex cone and equatorial and meridian surfaces are given, and various loci connected with the complex are discussed. A method of exploring the complex by means of central radii is then given.

It is found that the stretch on any radius that is intersected by rays of the complex perpendicular to that radius is in general finite. An equation for the distances of the ends of this stretch from the origin is found, and expressions for the direction cosines are given for the extreme rays which are at right angles to one another.

Various results concerning the lengths of perpendiculars are given, among them that the sum of the squares of the perpendiculars on three rays mutually at right angles to each other is constant. The solid locus of the feet of the perpendiculars on the central axis generally is found to be the space between the sheets of the surface

$$\frac{x^2}{r^2-f^2}+\frac{y^2}{r^2-g^2}+\frac{z^2}{r^2-h^2} = 0 \quad \text{..................(V)}$$

which is the reciprocal of the wave surface. Lastly, the congruency of the rays determined by (V), and the additional relation

$$\rho^4 + f^2\xi^2 + g^2\eta^2 + h^2\zeta^2 = f^4\lambda_1^2 + g^4\mu_1^2 + h^4\upsilon_1^2 \quad \text{..............(VI)}$$

is discussed, and shown to be of the fourth order. Minding's theorem is shown to hold when f=0. It is not true when f is not equal to zero. The equation of the single locus of the feet of the perpendiculars on the rays of the resultant is found and appears to be of the twelfth degree. In conclusion, the equations of various other loci connected with the congruence are given which show the power of the methods employed.

The importance of the paper lies in the methods employed to treat the particular problem in Plücker's "Line Geometry" and not in the results obtained. In the development of the results Chrystal followed Plücker's "Neue Geometrie ......" a posthumous work published in 1868 and edited by Felix Klein.

Professor George Chrystal was elected Fellow of the Royal Society of Edinburgh at its meeting held on Monday, 2nd February 1880, when Professor Douglas MacLagen, vice-president was presiding. In February 1881 he presented another paper to the Society "Note on Muir's Transformation of a Determinant into a Continuant"[5] in which he gave a simple way of arriving at some of Muir's elegant theorems, first taking a particular case and then generalising it.

On 20 June 1881, his paper "On a Special Class of Sturmians" [6] was published in which he started by defining a set of Sturmians of a rational integral function s_n of x of the nth degree. If $s_{n-1}, s_{n-2}, \ldots\ldots\ldots, s_2, s_1, s_0$ is a series of such functions of the (n-1)th, (n-2)th, ... degrees, so related to s_n that, when any one of the whole series $s_0, s_1, \ldots\ldots, s_n$ vanishes, the two on opposite sides have opposite signs, and further s_{n-1} and s_n have always opposite signs when x is just less than any real root of $s_n = 0$, then $s_0, s_1, \ldots\ldots\ldots, s_{n-1}$ may be called a set of Sturmians to s_n. The problem of finding such a set of functions has an infinite number of solutions.

The first discovery of such a set was made by Jacques Charles François Sturm, and the researches of J. J. Sylvester, C. Hermite and others showed how other solutions of the problem may be obtained. It occurred to Professor Chrystal while working on some physical problems that the properties of symmetrical determinants would provide the means of constructing a particular class of Sturmians. At first he thought that he has found a new result, but then he learnt that F. Joachimsthal has given the same series[7], however the method used by Professor Chrystal was different and much simpler than that used by Joachimsthal. Chrystal started with a symmetric determinant and proved a well-known proposition that if any principal minor vanishes then the next higher and the next lower have opposite signs.

In the second step by multiplication of matrices, he proved that, if r=n

$$S_n(x) = (-)^n \begin{vmatrix} 1 & x & x^2 & \cdots & x^n \\ s_p & s_{p+1} & s_{p+2} & \cdots & s_{p+n} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ s_{p+n-1} & s_{p+n} & s_{p+n+1} & \cdots & s_{p+2n-1} \end{vmatrix}$$

$$= \sum (a_1a_2......a_n)^p\zeta(a_1a_2........a_n)(x-a_1)(x-a_2).......(x-a_n)$$

where $\zeta(a_1a_2.....a_n)$ denotes, using Sylvester's notation, the product of the squares of all possible differences of $a_1a_2,...a_n$ and Σ denotes summation over all sets of n quantities.

If $a_1,a_2,......a_n$ are roots of

$x^n+p_1x^{n-1}+p_2x^{n-2}+..........+p_n = 0$ with a_i's being real or imaginary numbers, then

$$S_n(x) = \sum(a_1a_2......a_n)\zeta(a_1a_2.....a_n)(x^n+p_1x^{n-1}+......+p_n).$$

Moreover, if r > n, then $S_n(x) \equiv 0$.

In the third step he considered

$$\frac{dS_n(x)}{dx} \div S_{n-1}(x) \text{ when } x = a_1$$

and it came out to be equal to

$a_1\{ (a_1- a_2)(a_1 - a_3).......(a_1 - a_n) \}^2.$

Now, if a_1 is real, and p is an even non-zero integer, this ratio is real and positive. For, $a_1, a_2, ...,a_n$ being the roots of an equation with real coefficients, then for every imaginary in the series $a_1 - a_2$, $a_1 - a_3$, ..., $a_1 - a_n$, there will occur a corresponding imaginary conjugate so that the product of them all is real.

It follows that $S_{n-1}(x)$ and $S_n(x)$ have opposite signs when x is just less than any real root of $S_n(x) = 0$, which is the second characteristic of the first two functions of a Sturmian series.

The restriction of p being even may be removed if positive and negative roots are considered separately; but Chrystal supposes p to be always even.

In the fourth step, take the determinantal expression for S_n, multiply each column by x, and subtract the following column, leaving the last column unchanged, we get, denoting $s_p x - s_{p+1}$ by (p), $s_{p+1} x - s_{p+2}$ by (p + 1), etc;

$$S_n(x) = \begin{vmatrix} (p) & (p+1)(p+2) & \cdots & (p+n-1) \\ (p+1) & (p+2)(p+3) & \cdots & (p+n) \\ \cdot & \cdot & \cdot & \cdot \\ (p+n-1) & (p+n)(p+n+1) & \cdots & (p+2n-2) \end{vmatrix}$$

which is a symmetric determinant. $S_{n-1}(x)$, similarly transformed, becomes the first principal minor of this by deleting the last row and the last column, and so on. Hence by the first step, $S_n(x)$, $S_{n-1}(x)$, ..., $S_1(x)$, $S_0(x)$, the last being any positive constant, have the property that, when any one of the series vanishes, the next higher and next lower have opposite signs.

Then he applies his theory to determine Sturmian's for the cubic $x^3 + px + q = 0$.

4.3 Edinburgh Mathematical Society

In the last thirty five years of the nineteenth century a new medium was used for the promotion of mathematical activity namely the formation of professional societies in several countries. These societies held regular meetings in which papers were presented and each society sponsored one or more journals. The leading societies of the period were the London Mathematical Society founded in 1865, La Société Mathématique de France founded in 1872. In 1883 the Edinburgh Mathematical Society was founded by the initiative of Mr. A. Y. Fraser of Aberdeen, Mr. A. J. G. Barclay of Edinburgh and Mr. C.

G. Knott of Edinburgh. But it was the interest and support given by Professors Tait and Chrystal which strengthened the Society. The preliminary meeting was held on 2 February 1883 at which office bearers were elected and a committee was formed to draft a constitution. The first regular meeting of the Society was on 12 March 1883 and Professor Chrystal had the honour of giving the first introductory address, the subject of which was "Present Fields of Mathematical Research". In these early years he contributed a number of notes on mathematical subjects and greatly encouraged the members by his presence at the meetings. The Society thus fostered by Professor Chrystal, Thomas Muir, and J. S. Mackay and many others became an active centre for mathematical discussion and publication.

Professors Tait and Chrystal were the first two Honorary members elected by the Society and as Honorary members they were not, of course, eligible as officers of the Society. The Society now meets regularly in all eight Scottish universities, and it first began to meet annually in St. Andrews in 1922. Except for meetings in St. Andrews (and until recently in Aberdeen), which are held on a Saturday, ordinary meetings have, since the foundation of the society, been normally held on Friday evenings[8].

In 1884 Professor Chrystal presented three small notes to the Society, the first being "Application of the Multiplication of Matrices to Prove a Theorem in Spherical Geometry"[9]. The theorem in question was that if two of the diagonals of a spherical quadrilateral are quadrantal arcs, the third diagonal is also a quadrantal arc.

The second of these notes was "On the Discrimination of Conics Enveloped by the Rays Joining the corresponding points of two Projective Ranges"[10]. In this note he says that Steiner pointed out that the conic in question will be a parabola if, and cannot be a parabola unless, the point at infinity on one range corresponds to the point at infinity on the other. This according to Professor Chrystal is the converse of the well-known proposition [A movable tangent to a parabola divides two fixed tangents similarly]. As far as he knew neither J. Steiner nor T. Reye or any other writer on the projective geometry of conics took up the other cases.

The construction which he gave came from a course on the projective theory of conics, and was not supposed to assume any proposition regarding conics except the fundamental fact of their projective generation by the lines joining the corresponding points of two projective ranges.

A letter from Professor Cremona which was in reply to Professor Chrystal's enquiry "what construction he [Professor Cremona] used for asymptotes to a conic generated by means of its tangents" contained a construction which was more elegant than that followed by Chrystal in this note, but it proceeded on much the same lines.

The third note is "On a Problem in Partition of Numbers"[11]. The problem is finding $_nP_r$ (the number of partitions of r obtained by taking any of the numbers less than or equal to n). Professor Tait gave a method for $_nP_r$, but Professor Chrystal did not want to find an analytical expression for $_nP_r$, instead he wanted a process for quickly calculating a table of double entries of it. He gave a method to tabulate the values of $_nP_r$ on a piece of paper ruled into squares, and called the table Euler's Table because Euler dealt with such tables in a considerably extended form in his "Introductio in Analysin Infinitorum", Lib. I; chap. xvi (1748).

The theory of partitions had risen into great importance at that time in connection with the researches of Sylvester, Cayley and their followers on the theory of Invariants. It is also connected with the theory of series, as can be seen from Euler's enumeration of certain species of partitions by means of Euler's table.

Professor Chrystal gives his own view point in his Algebra as follows:

"Instead of making the theory of partitions depend on series, we might contemplate the various partitions directly, and develop their properties from their inherent character. Sylvester in 1882 considered the subject from this point of view, and had given what he calls a **constructive theory of partitions**[12], which throws a new light on many parts of the subject, and greatly simplifies some of the fundamental demonstrations".

1884 was the Tercentenary year of the University of Edinburgh, so Professor

Chrystal was busy entertaining mathematicians who were invited by the University on his recommendation to take part in the celebrations. The celebrations brought a new awakening in the University, which caused rapid development both academically and otherwise.

In January 1885 he presented a short paper "On the Problem to Construct the Minimum Circle Enclosing n Given Points in the Plane"[13]. He was inspired to work on this problem by the interest shown by Professor Hermite because of its use in the theory of functions of a complex variable and in the allied subject of the theory of potential. Chrystal sent the solution to Professor Hermite. However, before communicating it to the Edinburgh Mathematical Society, he discovered that the problem had originally been proposed by Professor Sylvester[14] and the solution had been given by him more than twenty years before[15] based on work of Professor Peirce, and was still in print at that time.

The method given by Professor Chrystal is as follows:-

Construct, by taking m of the points, a convex polygon enclosing them all. Take any side of this m-gon, and find the vertex at which it subtends the least angle. If this least angle is right or obtuse, the minimum circle is the circle on the chosen side as diameter. If the triangle formed by the three vertices is acute-angled its circumscribed circle is the minimum circle. If not, take the side opposite the obtuse angle, find the vertex at which it subtends the least angle and continue as before. The number of steps in the construction will be at most (1/2) m(m-1).

In a particular case it might be better to test whether a circle on the greatest side or diagonal encloses all the points. If this is not so, then the minimum circle is obtained by finding that triangle formed by three of the m points which is acute angled and has the greatest circumscribing circle. The theory given for the case of n points in the plane could be extended to n points in space. The method given by Professor Peirce is almost identical to the one given by Chrystal, but there is no mention that it can be extended to a three dimension problem.

However the connection between the minimum circle problem and the main part of

Sylvester's paper, which deals with the approximate representation of $\sqrt{(x^2+y^2)}$ and $\sqrt{(x^2+y^2+z^2)}$ by means of linear functions of x, y, z is very interesting.

Chrystal also submitted a small note "On Certain Formulae for Repeated Differentiation"[16] which was followed by another note "On a Method for Obtaining the Differential Equation to an Algebraical Curve"[17].

In his paper "On the Hessian"[18] read before the Royal Society of Edinburgh on 18 May 1885, he takes a problem of great geometrical interest i.e. "to calculate the number of intersections of an algebraic curve U of nth degree and the equation to its Hessian H of 3(n-2)th degree which are absorbed at a multiple point on the former". This problem had never been solved before directly except in a few simple cases. The Hessian plays a leading role in the theory of Invariants, a subject largely developed by A. Cayley and J. J. Sylvester. Cayley had given the theory that every singularity of an algebraic curve can be regarded as equivalent to a certain number δ of ordinary double points, and a certain number κ of ordinary cusps. However the proofs which were given of his theory by Max Noether (1844-1921), H. G. Zeuthen (1839-1920), Otto Stolz (1842-1905), Henry Smith (1826-1883), and other methods given for finding the indices δ and κ were of an indirect nature. At the time it had been doubted whether any proof of this theory could be given by the methods of coordinate geometry.

Before embarking on a general solution Professor Chrystal worked out a number of cases, some quite special, others of a more general character. Eventually the problem reduces to determining the number of intersections of two algebraical curves U = 0, V = 0, which coincide with a common point which is a multiple point on one or both. If the common point is a multiple point of order k on U and of order κ on V then if none of the k tangents of U coincide with any one of κ tangents on V, the number of intersections absorbed at the common point is $k\kappa$. But if l of the k tangents and λ of the κ tangents coincide with x=0, then we have

$$U \equiv x^l u_{k-l} + u_{k+1} + \dots\dots\dots$$

$$V \equiv x^\lambda v_{\kappa-\lambda} + v_{\kappa+1} + \dots\dots$$

or that x=0 is a multiple inflexional or undulatory tangent, so that

$$U \equiv x^l u_{k-l} + x^m u_{k+1-m} + x^n u_{k+2-n} + \dots\dots$$

$$V \equiv x^\lambda v_{\kappa-\lambda} + x^\mu v_{\kappa+1-\mu} + x^\upsilon v_{\kappa+2-\upsilon} + \dots\dots$$

and the problem becomes one of some difficulty, the difficulty being that of elimination which was faced by almost all the mathematicians of the time. Chrystal suggested that the problem would be simplified by substituting for U and V the approximations to their branches at the origin determined by the rule of Newton and Cramer. In this way the problem in general could be reduced to a series of others. He finds out in a simple theorem that the number of intersections of $U \equiv x^m - y^n = 0$ and $V \equiv x^\mu - y^\upsilon = 0$ at the point x=0,y=0 is $l(\mu n, m\upsilon)$, where $l(\mu n, m\upsilon)$ is the least of the two numbers μn, $m\upsilon$. Furthermore as far as points near x=0, y=0 are concerned we can replace U and V by

$$U' \equiv (x^{m_1} - A_1 y^{n_1})(x^{m_2} - A_2 y^{n_2}) \dots\dots\dots$$

$$V' \equiv (x^{\mu_1} - B_1 y^{\upsilon_1})(x^{\mu_2} - B_2 y^{\upsilon_2}) \dots\dots$$

where all the factors in U′ and V′ will in general be different and there is no factor common to U′ and V′. In this case the number of intersections of U′ and V′ will be the same as those of U and V and these will be given by

$$l(m_1\upsilon_1, n_1\mu_1) + l(m_1\upsilon_2, n_1\mu_2) + \dots$$

$+ l(m_2\upsilon_1, n_2\mu_1) + l(m_2\upsilon_2, n_2\mu_2) + \ldots$

$+\ldots\ldots$

The result still holds when factors are repeated in U′ or in V′ with some modification.

4.4 Text Book On Algebra

Chrystal had long felt the need for a text-book on algebra in which he could bring to a wider audience his distinctive approach to the subject. He described this desire in very powerful words in his famous presidential address as follows[19]:

"There are few things where the want of an enlightened scientific public strikes an expert more than the matter of scientific textbooks. It would naturally be expected that we should look carefully to the scientific education of our youth, to see that the best men and the best means that could be had were devoted to it; that we should endeavour to make for them a broad straight road to the newest and best of our scientific ideas; that we should exercise them when young on the best work of the greatest masters; familiarise them early with the great men and the great feats of science, both of the past and of the present.

For our teaching of algebra, I am afraid we can claim neither the sanction of antiquity nor the light of modern times. Whether we look at the elementary, or what is called the higher teaching of this subject the result is unsatisfactory. In the higher teaching which interests me most, I have to complain of the utter neglect of all important notion of algebraic form.

The logic of the subject, which, both educationally and scientifically speaking, is the most important part of it, is wholly neglected. The whole training consists in example grinding. What should have been the help to attain the end has become the end itself. The result is that algebra, as we teach it, is neither an art nor a science, but an ill-farrago of rules; whose object is the solution of examination problems. The end of all education nowadays is to fit the student to be examined; the end of every examination not to be an educational instrument, but to be an examination which a creditable number of men, however badly taught, shall pass. We reap, but we omit to sow. The problem for the writer of a text-book has come now, in fact, to be this_ to write a booklet so neatly trimmed and compacted that no coach on looking through it, can mark a single passage which the candidate for a minimum pass can safely omit. Some of these textbooks I have seen, where the scientific matter has been, like the lady's waist in the nursery song, compressed 'so gent and sma', that the thickness of it barely, if at all, surpasses that is devoted to the publisher's advertisements. The cure for all this evil is simply to give effect to a higher ideal of education in general, and of scientific education in particular. Science cannot live among the people, and scientific education cannot be more than a wordy rehearsal of dead textbooks, unless we have living contact with the working minds of living men. It takes the hand of God to make a great mind, but contact with a great mind will make a little mind greater".

Accordingly, he decided to devote his time to the writing of such a book and, as the work progressed, it inspired him to include all the relevant material. The first volume of the book, entitled "Algebra : An Elementary Textbook for the Higher Classes of Secondary Schools and for Colleges", was published in 1886. This is an elementary volume because "it begins at the beginning of the subject"; it is not written, however, "for babes". It will have been noticed how the address quoted above insisted upon the "all important notion of algebraic form". At the commencement Professor Chrystal lays down generally the three fundamental laws, and then proceeds deductively. This he does because this idea of algebraic form is "the foundation of all the modern developments of algebra, and the secret of analytical geometry, the most beautiful of all its applications".

The preface indicates the writer's aim and the prerequisites required of the reader. Outside algebra proper the reader was expected to be familiar with the definition of the trigonometrical functions, and to have a knowledge of their fundamental addition theorem.

The object was "to develop algebra as a science, and thereby to increase its usefulness as an educational discipline". Sources are indicated where relevant, and a most admirable feature is the introduction of numerous historical notes.

The subject is broken up into twenty-two chapters, and the arrangement "the result of some ten years experience as university teacher" deviates somewhat from what was usual at the time.

He continued his project producing the second volume in November 1889[20]. In this splendid volume we have the fulfilment of the author's promise, and the completion of his work on elementary algebra. There was no shirking difficulties, and Chrystal's mastery of his subject, and his acquaintance with the literature of his time and of earlier works, enabled him to produce ideas new and old from his wide knowledge. Still he did not write with the desire to show off, but with an eye as to what would be useful to the student.

The prominent features of the exposition as given by Professor Chrystal were its "singular ability and freshness of treatment" which were as conspicuous in the second volume as the first. The preface sets out Chrystal's aims:

"The main object of Part II is to deal as thoroughly as possible with those parts of algebra which form, to use Euler's title, 'Introductio In Analysin Infinitorum'.

A practice has sprung of late (encouraged by demands for premature knowledge in certain examinations) of hurrying young students into the manipulation of the machinery of the differential and integral calculus before they have grasped the preliminary notions of a limit and of infinite series, on which all the meaning and all the uses of the infinitesimal calculus are based. Besides being to a large extent an educational sham, this course is a sin against the spirit of mathematical progress. The methods of the differential and integral calculus which were once an outwork in the progress of pure mathematics, threatened for a time to become its grave. Mathematicians had fallen in to a habit of covering their inability to solve many particular problems by a vague wave of the hand towards some generality, like Taylor's Theorem, which was supposed to give "an account of all such things", subject only to the awkwardness of practical inapplicability. Much has happened to remove this danger and to reduce d/dx and $\int$ dx to their place as servants of the pure mathematician. In particular, the brilliant progress on the continent of function theory in the hands of Cauchy, Riemann, Weierstrass, and their followers has opened for us a prospect in which the symbolism of the differential and integral calculus is but a minor object. For the proper understanding of this important branch of modern mathematics a firm grasp of doctrine of limits and of the convergence and continuity of an infinite series is of much greater moment than familiarity with the symbols in which these ideas may be clothed. It is hoped that the chapters on Inequalities, Limits, and Convergence of Series will help to give the student all that is required both for entering on the study of the Theory Of Functions and for rapidly acquiring intelligent command of the infinitesimal calculus. In the chapters in question, I have avoided trenching on the ground already occupied by standard treatises: the subjects taken up, although they are all important, are either not treated at all or else treated very perfunctorily in other English textbooks".

What Chrystal aims at, and succeeds in achieving, is thoroughness. He draws on his skills as a brilliant teacher with a clear and critical mind caring more for the quality than the quantity of his work. Everything that flowed from his pen was of the highest standard.

The two opening chapters are on permutations and combinations, then the chapters mentioned above, followed by chapters applying the results proved to binomial and multinomial series of any index, and to exponential and logarithmic series. This forms a good introduction to the theory of functions. It enabled the student who assimilated the results to get a firm grasp of the calculus. The theory of functions is further illustrated in three other chapters in which Chrystal treats of graphs of the circular functions, Riemann surfaces, Hyperbolic functions, Gudermannian functions, the numbers of Bernoulli and Euler, and a host of other topics.There follow explanations of the method of finite differences, recurring series, and a collection of miscellaneous methods. The concluding five chapters were devoted to continued fractions, properties of integral numbers, and probability.

In the last chapter Professor Chrystal omitted certain material which he considered doubtful and of questionable usefulness one of these topics being the theory of inverse probability, the term used to denote cases in which the a priori probability of a cause is modified by the observation of some effect due to the cause. Objections to the notion of inverse probability were raised by many including Professor Chrystal who expressed his distaste in this last chapter as follows:

"The very meaning of some of the propositions usually stated in parts of these theories seem to us to be doubtful. Notwithstanding the weighty support of Laplace, Poisson, De Morgan, and others, we think that many of the criticisms of Mr Venn on this part of the doctrine of chances are unanswerable. The mildest judgment we could pronounce would be the following words of De Morgan himself, who seems, after all, to have "doubted":- " My own impression, derived from this and many other circumstances connected with the analysis of probabilities, is, that mathematical results have outrun their interpretation".[21]

It is also mentioned by Florian Cajori in his famous book History of Mathematics and by E. Seneta in the Mathematical Scientist[22].

In fact any one who wants to read the book today will find it still most readable, provided he approaches it with the requisite knowledge and ability, and when he has got to the end of the course he will have an excellent mathematical foundation. Chrystal gives good advice on reading his book in the preface with the following words:

"Every mathematical book that is worth anything must be read 'backwards and forwards', if I may use the expression. I would modify Lagrange's advice a little and say, ' Go on, but often return to strengthen your faith'. When you come on a hard or dreary passage, pass it over; and come back to it after you have seen its importance or found the need for it further on".

To facilitate this skimming process, he has given, after the table of contents, a suggestion for the course of a first reading.

There are numerous historical notes which form a conspicuous and useful feature of the whole work, any reader who wants to trace the history of the subject can benefit from these historical notes and numerous footnotes which are given whenever they are seen as necessary. Another prominent feature of the book is the use of proper notation. Chrystal was indeed able to utilise to the full extent his mastery of mathematical symbolism. Some of these were mentioned by Florian Cajori in the second volume of his famous 'History of Mathematical Notations'. The first of these is the factorial notation n! which was first used in 1808 by Christian Kramp (1760-1826) in his "Éléments d'arithmétique Universelle", although the notation was less frequently used in England in the nineteenth century but Chrystal chose this notation for his book[23]. This shows his insight about symbolism, because the same notation was recommended some twenty-six years later in 1915 by the Council of the London Mathematical Society for all practical purposes in this country.

"Subfactorial n" which chiefly occurs in connection with permutations was introduced in 1878 by W. A. Whitworth who represented it by the sign $\underline{||}$, but Chrystal[24] used the more convenient notation n¡ , which was more logical, convenient for printing, so was adopted quickly, and remains a common notation for subfactorial. Another distinct notation used by Professor Chrystal[25] is ${}_nH_r$, which he uses for the number of r combinations of n letters when each letter may be repeated any number of times up to r. The reason for using H here seems to be that the word " Habitué " represents the same meaning as he meant.

Chrystal always used notations free of any sort of controversies which were easy to use. He represents the product $(1+u_1)(1+u_2).....(1+u_n)$ by $\Pi^n(1+u_n)$ or simply by P_n, when n is increased without limit[26], the notation which is now commonly in use. The **arcsin tanh u** called the **Gudermannian of u,** the name invented by Arthur Cayley in honour of the German mathematician Gudermann (1798-1852), is represented by Chrystal using the notation gd u[27]. The numbers E_{2m} in the series

$$\sec x = \sum_0^\infty E_{2m} \frac{x^{2m}}{2m!}$$

were named by H. F. Scherk (1798-1885) as Euler's numbers and Chrystal in his book[28] denotes these numbers by $E_1, E_2, ...$ Thus $E_1= 1, E_2= 5, E_3= 61, E_4= 1385,$ ....

Chrystal uses a quadripartite symbol to denote the number of partitions:

"Thus, P(n | p | q) means the number of partitions of n into p parts, the greatest of which is q ; P (n | * | ⊁q) means the number of partitions of n into any number of parts, no one of which is to exceed q; P_u(n | >p | *), the number of partitions of n into p or any less number of unequal parts unrestricted in magnitude ... P(n | * |1,2,2^2,2^3,......) the number of partitions of n into any number of parts being a number in the series 1, 2, 2^2, 2^3, ..."[29].

H. Burmann and Sir J. W. Herschel gave the notation

$$a_1 + \frac{b_2}{a_2+}\ \frac{b_3}{a_3+}\ \frac{b_4}{a_4+}\ \ \ ....$$

for continued fractions in 1820, taken on latter by Professor Chrystal along with many other mathematicians. Chrystal also used symbolism, such as[30] :

$$\sqrt{13} = 3 + \frac{1}{*1+}\ \frac{1}{1+}\ \frac{1}{1+}\ \frac{1}{1+}\ \frac{1}{*6+}\ \frac{1}{1+}\ \ ......,$$

where * * indicate the beginning and end of the cycle of partial quotients, this notation was never before used, was quite simple and easy for printing.

Professor Chrystal, has mentioned in his Algebra[31] that the notation

$$b_0 + \frac{a_1}{b_1} \dot{+} \frac{a_2}{b_2} \dot{+} \ .\ .\ .\ .\ . \dot{+} \frac{a_n}{b_n},$$

devised by J. H. T. Müller in 1838 (where each + sign may be replaced by a - sign when parts are negative) found quite wide acceptance on the European continent.

In an Historical account of the subject Professor Chrystal[32] claims that "One of the earliest appearances of continued fractions in mathematics was the value of 4/π given by Lord Brouncker (about 1655). While discussing Brouncker's Fraction in his *Arithmetica*

Infinitorum (1656), Wallis gives a good many of the elementary properties of the convergents to a general continued fraction, including the rule for their formation. Saunderson, Euler, and Lambert all helped in developing the theory of the subject".

He has further given some important references for the interested reader in this note. According to Professor Chrystal, the first comprehensive treatment of the subject was given by Lagrange in his addendum to the French edition of Euler's Algebra (1795). This appears to be a standard work until Chrystal wrote his beautiful book, in which he treated the subject extensively.

It will be seen that many, if not most, complicated algorithms and processes in mathematical analysis are, as Chrystal has said, merely ' the creations of a common place'[33]. But, though this is so, it is usually because of powerful methods already in existence created by previous mathematical workers. Thus even Euler could say that he often felt his pen was, in some mysterious way, more powerful than himself during his mathematical investigations.

The results of Chrystal's book on students was clear as remarked by B. Branford in 1908:[34]

"Happily our best schools have made remarkable progress in the last ten years in the improvement of algebraic teaching, thanks largely to the arguments of Perry and the masterly work of Chrystal and of his great predecessors in this branch, namely Boole, Peacock, and De Morgan".

It is worth quoting excerpts from press opinions of the time about Chrystal's book:

" It is the completest work on Algebra that has yet come before us, and in lucidity of exposition it is second to none. The author views his subject from the high ground of the educationist, without reference to the exigencies of established examinations; yet neither the candidates who are training for such nor the teachers who prepare them will act wisely if they neglect his lessons"._Athenœum.

" The explanations are admirably clear, and the arrangement appears to be a very good one. No teacher of the high classes in our schools or of students preparing for the university examinations should be without this book. There is nothing like it in English, and it forms an excellent introduction to the various applications of Algebra to the higher analysis"._Academy.

This opinion considered even today applies to the book with some minor changes. Another prominent feature of Chrystal's Algebra which was mentioned in S. Abhyanker's review[35] is that it is very well suited for self-study and so students of mathematics could usefully study the book now exactly 100 years after its first publication.

The Chelsea Publishing Company is to be thanked for reprinting this precious book. It is still in print and even today used by students, which entitles Chrystal to rank with De Morgan and Clifford as one of the acknowledged masters of science, and has earned for him among mathematicians a world-wide reputation. It is, in fact, the first English textbook of mathematical analysis.

In the Mathematical Gazette of February 1895[36], the need for a textbook on Algebra in schools was put forcibly in these words:

"A teacher who makes it his endeavour to treat the subject in a more satisfactory way has, fortunately, the advantage of the guidance of eminent authorities. Chrystal's Algebra and Clifford's Common Sense of The Exact Sciences will help him to lay out the best road......

Nevertheless, these high authorities have not as yet had much influence on school books. I do not know of any English book suitable for elementary teaching that is constructed according to their recommendations. A 'Chrystal for beginners' is sadly wanted".

Partly in response to such demands, and partly due to the changes in the university system at the time and his long-standing promise to his publishers to provide an introduction to his larger textbook on Algebra, in 1898 Chrystal further enriched the literature of the subject by bringing out his **Introduction To Algebra: For the use of secondary schools and colleges**[37].

In the preface he puts forward his aim as follows:

"In the first place, I have kept the fundamental principles of the subject well to the front from the beginning. At the same time I have not forgotten, what every mathematical (and other) teacher should have perpetually in mind, that a general proposition is a property of no value to one that has not mastered the particulars. The utmost rigour of accurate logical deduction has therefore been less my aim than a gradual development of algebraic ideas.....

A mathematical truth is not made part of the mental furniture of a pupil merely by furnishing him with an irrefragable demonstration; it is not until he has tried it in particular cases, and seen not only where it succeeds, but where it fails to apply, that it becomes a sword loose in the scabbard and ready for emergencies.The rigorous demonstration is but the last polish given to the blade. It is better now and then to lead a learner to feel the need of a weapon before we place it in his hands...."

He further continues:

"In particular, I have excluded the treatment of subjects that depend on the theory of limits and convergency. The premature introduction of such subjects with loose and even misleading or false demonstrations has been one of the most glaring defects of our elementary mathematical textbooks. In this respect it is scarcely too much to say that many of them are half a century behind the age. Not only is teaching of this kind a waste of time, but it is an absolute obstruction to further progress".

Another prominent feature of this book is the constant use of graphical illustration; it is introduced in a simple form very early; and altogether about fifty pages are devoted to it exclusively. This proportion may have surprised some at the time but it is now standard practice in modern elementary teaching texts.

Regarding the first steps in teaching algebra Chrystal says:

"I hold, in common, I believe, with most teachers of mathematics who have deeply considered their business, that the teaching of Algebra- that is, of the science of arithmetical operations- should commence with the teaching of Arithmetic itself".

4.5 Some Further Contributions To Edinburgh Mathematical Society

It is significant once again to understand the close connection in Chrystal's mind between pure mathematics and applied science. His next paper "On Certain Inverse Roulette Problems"[38] is an example of this, in which he gives elegant and some new solutions of the problem earlier taken up by Professor J. C. Maxwell in the Transactions of the Royal Society of Edinburgh, vol. xvi; 1849. He finds solutions of particular cases of the problem taken in three different ways.

Firstly given the space centrode and the Roulette for one point of a plane figure moving in the plane, to find the body centrode.

In such cases if the space centrode is a straight line, take it as the x-axis, and then find an equation between the normal and the ordinate to the given curve. This equation gives the p-r Equation or pedal Equation to the body centrode.

Secondly given the body centrode and the Roulette for one point of a plane figure moving in the plane, to find the space centrode.

Finally he considers the problem when the body centrodes and the space centrodes are congruent curves and has to generate the Roulette for one point of a plane figure moving in its plane.

The problem of designing cams or centrodes to produce any motion in one plane was of some practical importance. The methods used by Professor Chrystal are founded, for the most part, on the use of the so-called Pedal Equation (or p-r Equation), which has the advantage in this case that it depends on the form but not on the position of the curve which it represents.

Chrystal's next publication is "On the Inequality $mx^{m-1}(x-1) \gtrless x^m-1 \gtrless m(x-1)$ and its Consequences"[39]. The object of the note is to establish the above inequality in its most general form, and to use it to deduce two of the principal propositions in the theory of inequalities, one of which is usually proved by means of infinite series. The logical advantage in making the theory of inequalities independent of that of infinite series is obvious to Chrystal who remarks that the discussion of the convergency of infinite series is strictly speaking a part of inequalities. The inequality in question was by no means new, nor had its importance been overlooked, as may be seen from the elegant use of it by Schlömilch in the second chapter of his 'Handbuch der Algebraischen Analysis'.

The inequality had not, however, usually been stated in quite so general a form as the one that Chrystal gave and, possibly in consequence, its application to the proof of the following theorem seems not to have been noticed before:

"If a, b, ..., k, are n positive quantities and p, q, ..., t, are n commensurable quantities, then

$$\frac{pa^m+qb^m+.....+tk^m}{p+q+.....+t} \lessgtr \left(\frac{pa+qb+.......+tk}{p+q+........+t}\right)^m$$

according as m does not or does lie between 0 and 1". This theorem was usually proved by a somewhat awkward combination of induction and the use of infinite series.

The history of the theorem is a little obscure. At first Chrystal suspected that it was due to Cauchy, but it does not appear in his 'Analyse Algebrique' (Paris, 1821). The earliest reference which he discovered was given to him by A. Y. Fraser, and occurs in 'Problèmes et Dévelopments sur Diverses Parties des Mathématiques', by M. Reynaud and M. Duhamel (1823), P.155. It is there deduced while finding the maximum and minimum values of $x^m + y^m$ subject to the condition $ax+by = c$.

The inequality $mx^{m-1}(x-1) \gtrless x^m-1 \gtrless m(x-1)$ has the merit of binding together a great variety of algebraical theorems which were usually presented without any connection whatever. Its power is cleary given by its close connection to the theorem

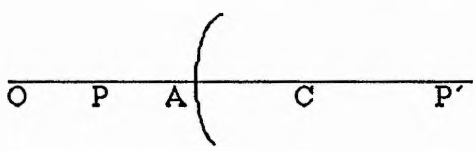
$\lim_{x\to 1}(x^m - 1) / (x-1) = m$, which is the fundamental result in the differentiation of algebraic functions.

This, along with many of his other notes contributed to the Proceedings of the

Edinburgh Mathematical Society, are incorporated in his Algebra textbook. These notes were always interesting in themselves and frequently presented old results from new points of view.

The most elaborate paper which Professor Chrystal communicated to the Edinburgh Mathematical Society was entitled "A Summary of the Theory of Refraction of Thin Approximately Axial Pencils Through a Series of Media Bounded by Coaxial Spherical Surfaces, with Application to a Photographic Triplet, etc."[40] The optical theory referred to in the title of this communication was about half a century old and had been well expounded in the famous English Treatises of the period. Despite its elegance and simplicity, and its practical importance to the theory of the optical instruments in ordinary use, its spread into general knowledge had been remarkably slow. Chrystal therefore decided to give a brief summary of the main principles and methods to find the constants of an optical system, together with their experimental verification. The whole theory depends on two elementary results concerning the refraction of a thin pencil at a single spherical surface, viz; the Law of Conjugate Focal Planes and Helmholtz's Law of Magnification.

Let P be any point and C the centre of the refracting surface considered.



The Law of Conjugate Focal Planes can be stated as follows. "Every pencil, all of whose rays diverge from (or converge to) a point lying in a small area through P perpendicular to the axis PC, will after refraction diverge from(or converge to) a point lying in a small area also perpendicular to the axis PC, provided we consider only rays whose inclinations to PC are small. The point P´ where the second plane meets the axis, is called the conjugate focus to P, and is in direct projective correspondence with P; that is to say, if x and x´ denote the distances of P and P´, both measured from a point O in the axis in the same direction (which we take to be that in which the light is proceeding, here we consider from left to right), then

$$Axx' + Bx + Cx' + D = 0$$

where A, B, C, D, are constants depending on the radius of the refracting surface, and on the indices of refraction of the media of which it is the boundary". The law is accurate to a first approximation only.

Helmholtz's Law of Magnification can be stated as follows. "If β and β' be the linear dimensions of an object at P and its image at P′, P and P′ being axial points, and α and α' the inclinations to the axis of an incident ray through P and the corresponding emergent ray through P′, then

$$\mu\beta\tan\alpha = \mu'\beta'\tan\alpha'$$

where μ and μ' are the refractive indices of the media in the order of the passage of rays".

These laws can be extended to any number of co-axial spherical surfaces.

Chrystal considered two main optical systems, telescopic and non-telescopic, where telescopic system is a system satisfying the equation $Axx'+Bx+Cx'+D=0$, with $A=0$, $B\neq0$, $C\neq0$ and a non-telescopic system is the one for which $A\neq0$. He prefers the use of a doublet of thin lenses over the use of thick lenses of special kind for optical experiments by the beginners because of economic and other reasons and remarks as follows: "a thin lens is not representative of optical systems in general; and it is not easy by means of experiments with lenses of moderate thickness to bring home to the beginner in optics the characteristic properties of a general system, because in such lenses the distance between the principal points barely exceeds the errors of such measurements as can be made without special refinements which are out of place in elementary instruction. The construction of thick lenses of special kinds would meet the difficulty, but would be troublesome and somewhat costly. On the other hand a doublet of thin lenses can always be constructed so as to have the same fundamental points as any given system".

According to him as far as landscape photography is concerned, it is essential to have a series of lenses of widely different focal length, but for general photographic purpose the handiest lens is a symmetrical doublet of the rapid rectilinear or euryscope type, and it had long been known that the focal length of such a combination can be varied within wide limits without destroying its efficiency as a photographic instrument by inserting between its elements a thin lens of positive or negative focal length.

Chrystal was led to take up the subject in connection with his own photographic work, and had worked out the theory in a very simple form. He had given helpful instruction to his students.

As far as the focal length of a triplet is concerned he came out with the remarkable conclusion that the position of the internal adjuster makes no practical difference, and the principal focal length F_3 of the triplet is given by

$$F_3 = \frac{F\phi}{\phi + \frac{1}{2}m - \frac{d^2}{2m}}$$

where F and ϕ are focal lengths of the doublet itself and of the adjuster respectively, m is the distance from the central point of the doublet to the inner principal focal point of either of its elements, while d is a constant positive or negative depending on the nature of the system. At the end of the paper Chrystal gave methods for determining experimentally the characteristic points of a symmetrical optical system.

4.6 Differential Equations

The introduction of the theory of functions had completely revolutionised one of the major fields of mathematical activity, the theory of differential equations. At the beginning of the nineteenth century many important results had indeed been established, particularly by Euler and Lagrange but the methods employed were artificial and broad comprehensive principles were lacking.

A new epoch began with Cauchy, who by means of his new theory of functions first rigorously established the existence of the solution of certain classes of equations in the vicinity of regular points. Sophus Lie said "The theory of differential equations is the most important branch of modern mathematics". The subject may be considered as occupying a central position from which different lines of development extend in many directions. The pure analytical direction leads to discussing infinite series, existence theorems and the theory of functions. Another direction leads to the differential geometry of curves and surfaces. Between the two lies the path first discovered by Lie, leading to continuous groups of transformation and their geometrical interpretation.

In the period 1891-96 Professor Chrystal presented to the Royal Society of Edinburgh three detailed and interesting papers on Differential Equations. According to Prof. C. G. Knott[41],"There was a rumour at one time that he purposed writing a book on this subject; whether this was so or not, the character of these papers shows that he had given careful consideration to the logical foundations of the theory of certain parts". In the first paper "A Demonstration of Lagrange's Rule for the Solution of a Linear Partial Differential Equation, with some Historical Remarks on Defective Demonstrations Hitherto Current"[42] Professor Chrystal remarks that he noticed an inadequate proof of Lagrange's Rule for solving a linear differential equation in the textbooks of the time, but as the method by which he treated Linear Partial Differential Equations in his lectures did not use them, it did not occur to him to investigate the exact nature of the defect. The consideration of certain special cases led him to the fact that most of the general proofs contain an obvious fallacy.

The special proof given in this paper was described more than a year before its publication, in correspondence with Mr A. R. Forsyth. The publication was delayed because Chrystal had to devote much time to administering the changes recommended as a result of the Universities (Scotland) Act, 1889 and he had to give priority to the changes in his own subject. In the mean time, M. Goursat's Leçons sur l'Intégration des Equations aux Derivées Partielles du Premier Ordre, Paris (1891), had been published, which contained the first rigorous proof of Lagrange's Rule with which Professor Chrystal was acquainted. Goursat's proof however depended on a thorough discussion of the passage from the ordinary linear equation to Jacobi's homogeneous linear equation, so that it was totally different from Chrystal's proof which was carried out independently. Before giving a summary of Chrystal's proof we consider the Historical Remarks made at the end of his proof.

The first germ of the rule itself was given by Lagrange in §52 of his memoir "Sur les Intégrales des Equations Différentiélles"[43]. It is given in complete and general form in Art.V.- Sur l'intégration des équations aux différences partielles du premier ordre, of his memoir "Sur Differentes Questions d'Analyse Relative a la Théorie des Intégrales Particulières"[44], and is there accompanied by a verification that the Lagrangian integral satisfies the differential equation. He returns to the subject in a memoir entitled " Méthode Générale Pour Intégrer les Equations aux Differences Partielles"[45].

Lagrange's proof, which seems intended to prove that every possible solution is of the Lagrangian form, may be summarised as follows. It is necessary to determine p and q as functions of (x, y, z), so that the two equations

$$Pp + Qq = R, \quad Pdx + Qdy = dz, \text{ with } p = \partial z/\partial x,\ q = \partial z/\partial y,$$

may be satisfied simultaneously. As a consequence of these

$$p(Pdz - Rdx) + q(Qdz - Rdy) = 0 \qquad \text{(I)}.$$

Let $u(x,y,z)=a$, $v(x,y,z)=b$ be the integrals of the system

$$Pdz - Rdx=0, \quad Qdz - Rdy=0.$$

Then, replacing the variables(x,y,z) by u, v, and any one of the others, say z, gives

$$Pdz - Rdx = \frac{R(v_y du - u_y dv)}{T},$$

$$Qdz - Rdy = \frac{R(u_x dv - v_x du)}{T},$$

where $T = u_y v_x - u_x v_y$.

Hence (I) becomes

$$\frac{R}{T}\left\{(pv_y - qv_x)du + (qu_x - pu_y)dv\right\} = 0 \ldots\ldots \quad (II).$$

Lagrange then writes (II) in the form

$$du + \frac{qu_x - pu_y}{pv_y - qv_x} dv = 0 \ldots\ldots \quad (III).$$

Then he reasons thus, since (III) contains only two differentials du and dv, it can on subsist if the coefficient of dv is a function of u and v alone. Consequently, on substitutin for x and y their values in terms of (u,v,z) as determined by the equations

$u(x,y,z) = u$, $v(x,y,z) = v$,

the variable z must disappear, and we must have

$$\frac{qu_x - pu_y}{pv_y - qv_x} = f(u,v) \ldots\ldots\ldots \quad (IV).$$

Then (III) becomes $du + f(u,v) = 0$, the integration of which gives a relation of the form $F(u,v)=0$.

Professor Chrystal claims that this proof is completely unsound. Not only doe to demonstrate that all solutions have the Lagrangian form, but it does not even apply to solutions that actually have the form. The fallacy seems to have arisen from not realising that (III) does not need to be satisfied identically, but only as a derivative of the relation which is the solution. Lagrange's proof is, therefore, completely fallacious.

Lagrange gives two other proofs of his rule, one in his "Leçons sur le Calcul des Fonctions (Leçon 20)", the other in his "Théorie des Fonctions (chap.xvi.)". These proofs contains the same confusion between identical and conditional equality .

Jacobi on page 2 of his "Dilucidationes de Æquationum Differentialium Vulgarum Systematis Earumque Connexione cum Æquationibus Differentialibus Partialibus Linearibus Primi Ordinis"[46] refers to Lagrange's work in these words,"**III. Lagrange(acad. Ber. a. 1779, pp.152-160)** aequationum differentialium partialium primi ordinis linearium solutionem, hoc est reductionem ad aequationes differentiales vulgares primum obiter et adumbrata tantum demonstratione dedit. De illa demonstratione pretiosa alio loco mihi agendum erit. Alium postea dedit demonstrationem in commentatione,' Méthode Générale pour intégrer les équations aux différences partielles et du premier ordre, lorsque ces différences ne sont que linéaires'(**Acad. Ber. a. 1785, pp.174-190)**". According to Professor Chrystal from this reference it does not appear that Jacobi had read Lagrange very closely, for the "Demonstratio Adumbrata" is a perfectly clear and good proof so far as it goes, while the "Alia Demonstratio" is fallacious. Jacobi's own method is very beautiful.

"He first considers the homogeneous equation

$$Z\frac{\partial f}{\partial z}+X_1\frac{\partial f}{\partial x_1}+\ .\ .\ .\ +X_n\frac{\partial f}{\partial x_n}=0\ .\ .\ .\ .\ .\ \text{(A)},$$

where Z, X_1, X_2,........ , X_n are functions of z, x_1,x_2, ..., x_n ; and shows that every integral of (A) is of the form

$H(f_1, f_2, ..., f_n) = 0$ ....(B),

where $f_1=a_1$, $f_2=a_2$, ... , $f_n = a_n$ ....(C),

is the integral system of

$$\frac{dx_1}{X_1}=\frac{dx_2}{X_2}=\ .\ .\ .\ =\frac{dx_n}{X_n}=\frac{dz}{Z}\ .\ .\ .\ .\ .\ \text{(D)}.$$

He then shows that the equation

$$X_1\frac{\partial z}{\partial x_1}+X_2\frac{\partial z}{\partial x_2}+\ .\ .\ .\ +X_n\frac{\partial z}{\partial x_n}=Z\ .\ .\ .\ .\ .\ \text{(E)},$$

can be transformed into (A) by supposing

$f(z, x_1, ..., x_n) = a$ ....(F),

where a is an arbitrary constant, to be a solution of (E), and replacing $\partial z/\partial x_1$, ... by their values

$$-\frac{\partial f}{\partial x_1}/\frac{\partial f}{\partial z},\ .\ .\ .\ .\ .$$

So far as proving that every possible solution of (E) is contained in (B) is concerned, this process is no better than Lagrange's 'Demonstratio Adumbrata'".

The question of how the equation (A) derived in this way from (E) is really equivalent to (E) is not discussed at all. In fact (A) is not equivalent to (E). Jacobi himself seems to have been aware of this defect.

In Boole's "Differential Equations" (2nd ed; 1865, p.332) a proof is given in which the confusion between a conditional equation and an identity occurs twice over. J. A. Serret (Calcul Intégral, 1868,p. 601) comes nearer a correct proof, but is also unsuccessful. W. G. Imschenetsky (Archive der Mathematik und Physik,Th.1; 1869, p. 295) reproduces the proof given by Lagrange in the "Calcul de Fonctions" without alluding to the fallacy it contains.

The proof given by Graindorge (Mémoire sur l'Intégration des Equations Derivées Partielles, 1872, p.12) contains the usual fallacy of confusion between an identical and a conditional equation. So also does the proof given by Paul Mansion of the University of Gand in his book (Théorie des Equations aux Derivées Partielles du Premier Ordre, 1875, p. 35), which is essentially the same as that given by Forsyth in his "Treatise on Differential Equations, 1885".

According to Professor Chrystal this mathematical history shows that even the best mathematicians make errors in mathematical reasoning. Chrystal advises readers to distrust general proofs which are not built on elementary foundations. Such proofs should constantly be tested on particular examples. The difficulty in proving Lagrange's rule was to remain until the nature of singular solutions had been determined, and the cases separated in which it was not true. However Chrystal does not excuse the bad logic of the claimed proofs.

Despite these comments Professor Chrystal was, in fact, a great admirer of Lagrange as mentioned in one of his letters to Sir Joseph Larmor on 30th March 1909[47], where he writes:

"...Please do not set me down as a denigrator of Lagrange who is one of my Lares & Penates. I am looking now at a shelf full of his works, some volumes of which I have actually read. I was only amusing myself, as little men are prone to do, by thinking of spots in the sun.

Lagrange, Laplace and Fourier had something rarer than Logic, namely Genius, a thing that never came out of a syllogism. The amusement consists in discovering the persistence with which their smaller successes will persuade themselves that they have reasoned out the conclusions of great men - even when they were erroneous".

In fact it was Lagrange who first considered in detail the relation of singular solutions to envelopes. However, he did not succeed in removing all the mystery about this delicate but important subject. The inconsistency noted in his theorems by the mathematicians in the last quarter of the nineteenth century, led them to reconsider the entire theory of singular solutions. Chrystal's efforts were also a result of this trend.

Professor Chrystal's proof owes something to the private criticism of A. R. Forsyth and his own perusal of M. Goursat's Leçons.

He begins with the linear equation

$Pp + Qq = R$..............(1),

with the same notation as above. A solution of (1) can be taken as

$\Psi(x,y,z) = 0$..............(2),

leading to one or more determinations of z as a finite, continuous, single-valued function of x, y, such that the resulting values of P, Q, R, p, q make (1) an identity. Then he defines a non-singular solution to be any solution having either of the following two characteristics:-

I. That the related values of (x,y,z) do not constitute a critical point of any one of the functions P, Q, R.

II. That the related values of (x,y,z) do not cause all the three functions P, Q, R to vanish.

Any values of (x,y,z)which violate the above two conditions he names as Critical Values. Excluding critical values of x, y, z Lagrange's auxiliary system has the integral system

$u = a$, $v = b$ ....(3),

where u and v are independent functions of x, y, z which may be taken as single-valued, continuous and finite for all values of x, y, z which constitute an ordinary solution of (1).

This gives the relations

$$u_xP + u_yQ + u_zR = 0, \qquad v_xP + v_yQ + v_zR = 0 \ \ldots\ldots\ldots\ldots\ldots (4)$$

which cannot be merely conditional, but must be identities. Thus neglecting any factor which contains (x, y, z) alone, equation (1) is equivalent to

$$\begin{vmatrix} p & q & -1 \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} = 0 \ . \ . \ . \ . \ (5).$$

Chrystal then proves a proposition showing that if any solution of (1) can be put into one of the forms

$x - g(u,v) = 0$, $y - h(u,v) = 0$, $z - k(u,v) = 0$,

then the relation between (x,y,z) involved in this solution must make $P = 0$, $Q = 0$, $R = 0$, respectively. Lastly he proves the main proposition :

Every non-singular solution of (1) can be expressed in the form

$$f(u,v) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (6).$$

Since the functions u and v are independent, not more than one of the determinants in the matrix

$$\left\| \begin{matrix} u_x & u_y & u_z \\ v_x & v_y & v_z \end{matrix} \right\| \ \ldots\ldots\ldots (7),$$

can vanish identically.

First, suppose that none of the determinants in (7) vanish. Then by means of the equations

$$u = u(x, y, z), \qquad v = v(x, y, z)$$

any function $\psi(x, y, z)$ can be expressed in each of the three forms

$$\chi_1(x, u, v), \qquad \chi_2(y, u, v), \qquad \chi_3(z, u, v).$$

Hence the solution (2) of (1) will be a derivative of (but not necessarily equivalent to) each of the three equations

$\chi_1(x,u,v) = 0$, $\chi_2(y,u,v) = 0$, $\chi_3(z,u,v) = 0$ ....(8).

Forgetting the point made in the parenthesis seems to have been the cause of the many defective proofs of Lagrange's rule, including his own.

It follows, therefore, that any relation which is part of a solution of (1) is either a derivative of an equation of the form $f(u,v) = 0$, or else it can be expressed in each of the three forms

$x - g(u,v) = 0$, $y - h(u,v) = 0$, $z - k(u,v) = 0$,

and is therefore such that it makes $P = 0$, $Q = 0$, $R = 0$, that is to say it is a singular solution.

Second, let us suppose that one of the determinants (7) vanishes identically, then we must have

$$P \equiv 0, \text{ or } Q \equiv 0, \text{ or } R \equiv 0,$$

that is to say, we have to prove Lagrange's Rule for the special equations

$Pp = R$, $Qq = R$, $Pp + Qq = 0$.

Considering each of these separately we reach at the same conclusion as before, that any relation which is part of a solution of (1) is either a derivative of an equation of the form $f(u,v)= 0$, or else it can be expressed in each of the three forms

$x - g(u,v) = 0$, $y - h(u,v) = 0$, $z - k(u,v) = 0$,

and therefore makes $P = 0$, $Q = 0$, $R = 0$, that is to say it is a singular solution.

At the end Chrystal states that the above reasoning may be applied to the case of n independent variables. He also gives some examples. In the last example, he considers the differential equation

$$\{1+\sqrt{(z - x - y)}\}p + q = 2,$$

which was named in his honour as Chrystal's equation. We consider its solution

$z - x - y = 0$. We can take $\Psi(x,y,z) \equiv z - x - y$, $u \equiv 2y - z$, $v \equiv y + 2\sqrt{(z - x - y)}$, and

$\chi_1(x,u,v) \equiv 2 + v - u - x - 2\sqrt{(1 + v - u - x)}$; $\chi_2(y,u,v) \equiv (1/4)(v - y)^2$;

$$\chi_3(z,u,v) \equiv (1/4)(2v - u - z)^2.$$

The solution z - x - y = 0, in this case can not be put into Lagrangian form at all. This agrees with the proof given by Professor Chrystal. For z - x - y = 0 is a locus of branch points of the coefficient of p, so this solution is a singular solution of the linear equation.

According to Professor Chrystal it sometimes might happen that the solutions classified as singular for the proof given by him are particular or limiting cases of a Lagrangian solution but this is not the case in the above example.

Professor Tait communicated this paper to the Royal Society of Edinburgh and remarked that the problem could be thought of in terms of quaternion principles.

Professor Chrystal's second paper on differential equations "A Fundamental Theorem Regarding The Equivalence of Systems of Ordinary Linear Differential Equations, and its Application to the Determination of the Order and the Systematic Solution of a Determinate System of such Equations"[48] aims at establishing rigorously a fundamental theorem regarding the equivalence of systems of ordinary linear differential equations. This leads to a systematic way for solving systems of this kind without the introduction of superfluous arbitrary constants. According to Chrystal systems of ordinary linear differential equations are of great importance, both from a practical and from a theoretical point of view. They figure largely in dynamical problems and Jacobi had shown that the general problem of determining the order of any system of ordinary differential equations can be reduced to the problem of determining the order of a linear system with constant coefficients. Nevertheless, at that time the state of the theory of such a system still left something to be desired. A logical and systematic process for the solution was given by Cauchy, which consists of first replacing the system by another in which only first order differential coefficients occur, by introducing as auxiliary variables the successive differential coefficients of the various dependent variables to the highest but one, and then saying that the order of the system is always the same as the order of the characteristic differential equation; i.e. the degree of the characteristic determinant.The terms "characteristic equation" and "characteristic determinant" are defined as follows:

Let x, y, z be dependent variables and t an independent variable, and let the system be

$$f_1(D)x + g_1(D)y + h_1(D)z = 0$$

$$f_2(D)x + g_2(D)y + h_2(D)z = 0$$

$$f_3(D)x + g_3(D)y + h_3(D)z = 0$$

where D stands for d/dt and f_1, f_2, etc; are integral functions of D with constant coefficients.

Then the determinant $|f_1(D), g_2(D), h_3(D)|$ is called the characteristic determinant and is denoted by K. Each of the variables satisfy the differential equation $K\xi = 0$, which is called the characteristic equation of the system.

Jacobi in his memoir "De Investigando Ordine Systematis Equationum"[49] made this theorem his starting point but the proof which he gave is unsatisfactory. The theorem in question is true but the difficulty in proving it lies in the fact that although each of the variables satisfies the characteristic equation, all of them may not be general integrals of the equation. In fact it may happen that no one of them is a general integral.

Both Jacobi and Cauchy had tried the problem in detail, but it still was impossible to predict the order of the system without going through the work of reduction.

Another method was the use of the characteristic equation of the system. However the assumptions made are not always true, and this process is not a very satisfactory proof of the order of the system.

Professor Chrystal continues, that there are various ways in which the solution may be amended in particular cases, but generally as far as he was aware there was no proof to the theorem. The proof given by Professor Chrystal in this paper is not subject to failure in practical cases.

He first wants to find necessary and sufficient conditions for the equivalence of two systems of linear equations with constant coefficients.

To do so he assumes $x_1, x_2, \ldots, x_n$ are n dependent variables, and t the independent variable and takes

$$U_r \equiv (r,1)x_1 + (r,2)x_2 + \ldots + (r,n)x_n + S_r,$$

$$V_r \equiv [r,1]x_1 + [r,2]x_2 + \ldots + [r,n]x_n + T_r,$$

with $(r,1), \ldots, (r,n), [r,1], \ldots, [r,n]$ as integral functions of D with constant coefficients; S_r, T_r as functions of t alone and r can take values $1, 2, \ldots, n$.

He considers two systems of m independent equations

$U_1 = 0, U_2 = 0, \ldots, U_m = 0$, and $V_1 = 0, V_2 = 0, \ldots, V_m = 0$, with m not greater than n, and calls them original and derived systems respectively. Derived in the sense that every solution of the first system is a solution of the second.

According to Chrystal since both the systems are linear with constant coefficients any process of derivation must consist in operating on the equations of the original system with integral powers of D, multiplying the resulting equations thus obtained by constants and adding.

Thus one gets

$$V_1 \equiv \xi_1 U_1 + \ldots + \xi_m U_m$$

$$V_2 \equiv \eta_1 U_1 + \ldots + \eta_m U_m$$

$$\cdots\cdots\cdots\cdots$$

$$V_m \equiv \kappa_1 U_1 + \ldots + \kappa_m U_m$$

where $\xi_1, \ldots, \xi_m, \ldots, \kappa_1, \ldots, \kappa_m$ are integral functions of D with constant coefficients; these he calls the multiplier system which derives the second system from the original.

He calls the determinant

$$\Delta = | \xi_1, \eta_2, \ldots, \kappa_m |$$

the modulus of the derived system with respect to the original.

The desired conditions he states in the form of an equivalence theorem which is as follows:

"When two systems of linear equations with constant coefficients are equivalent, the modulus of either with respect to the other must be constant, and the converse is also true".

This equivalence theorem he also expressed in another form, which to him is convenient for some purposes, "the necessary and sufficient condition that the two systems of linear equations with constant coefficients be equivalent, is that every determinant in the matrix of one system differs by the same constant multiplier from the corresponding determinant in the matrix of the other".

In particular, "for two determinate systems (of n equations in n dependent variables) to be equivalent, it is necessary and sufficient that the determinants of the two systems differ merely by a constant factor",i.e. $| (11), (22), \ldots, (nn)| = | [11], \ldots, [nn] | \times$ const. Where (11), (22), . . . , (nn) and [11], . . . , [nn] are integral functions of D with constant coefficients as defined earlier.

This second form of the equivalence theorem enables the equivalence of the systems to be tested directly, without calculating the system of multipliers.

Chrystal then gives a method of reduction of any determinate system of linear equations with constant coefficients to an equivalent diagonal system. By a diagonal system is meant a system of the form

$$[11]x_1 + [12]x_2 + [13]x_3 + \ldots\ldots + [1n]x_n + T_1 = 0$$

$$[22]x_2 + [23]x_3 + \ldots\ldots + [2n]x_n + T_2 = 0$$

$$[33]x_3 + \ldots\ldots + [3n]x_n + T_3 = 0$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$[nn]x_n + T_n = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(I),$$

A diagonal system is characterised by the order of the variables on the diagonal.

In order to do so Chrystal first shows that "every determinate system of linear equations with constant coefficients can be reduced to an equivalent diagonal system in which dependent variables have any assigned diagonal order".

Firstly he proves that for any two equations

$U_1 \equiv (11)x_1 + (12)x_2 + (13)x_3 + ... + (1n)x_n + S_1 = 0,$

$U_2 \equiv (21)x_1 + (22)x_2 + (23)x_3 + ... + (2n)x_n + S_2 = 0;$

we can always deduce an equivalent pair, one of which does not contain any assigned dependent variable, say, x_1.

Then he proves that "a determinate system of linear equations with constant coefficients can always be replaced by an equivalent system in which any given variable, say, x_1, occurs in only one of the equations".

The possibility of reducing any given system

$$U_1 \equiv (11)x_1 + (12)x_2 + (13)x_3 + ... + (1n)x_n + S_1 = 0$$

$$U_2 \equiv (21)x_1 + (22)x_2 + (23)x_3 + ... + (2n)x_n + S_2 = 0 \qquad ....(II),$$

....

$$U_n \equiv (n1)x_1 + (n2)x_2 + (n3)x_3 + ... + (nn)x_n + S_n = 0$$

to a diagonal system is now easy.

Arrange the dependent variables in any order, say $x_1, x_2, ..., x_n$, then deduce from (II) an equivalent system in which only one equation, say the first, contains x_1. The remaining (n-1) equations form a determinate system for $x_2, x_3, ..., x_n$. Now deduce an equivalent system, the first equation of which alone contains x_2 and so on. Eventually a diagonal system, such as (I), is reached.

Professor Chrystal who had a habit of doing each job systematically had by now matured this habit, so he then moves on to take the properties of the diagonal system in detail.

According to him, as the determinant of the diagonal system reduces to the product of the diagonal coefficients hence by the second form of the equivalence theorem it follows that "the product of the diagonal coefficients of any diagonal system is, to a constant factor, equal to the determinant of any system to which it is equivalent, i.e;

$$\left| (11)\ (22)\ \ldots\ldots\ldots\ (nn) \right| = [11]\ [22]\ \ldots\ldots\ [nn] \quad \ldots\ldots\ldots\ (III).$$

Every diagonal system may be solved by means of a series of linear equations with constant coefficients each involving only a single dependent variable". Hence from (III) we have a rigorous proof of the main theorem of this paper, viz; "the order of any determinate system of ordinary linear differential equations with constant coefficients is equal to the degree in D(=d/dt) of its characteristic determinant K. The equivalence of a diagonal system is not affected by adding to any equation $U_r = 0$ of the system any linear combination

$L_{r+1}U_{r+1} + \ldots\ldots + L_nU_n = 0$ (where L_{r+1} etc; are integral functions of D with constant coefficients) of all the equations that follow it".

It appears from this remark that the coefficients of a diagonal system, other than the diagonal coefficients, are not uniquely determined when the diagonal order of the dependent variables is assigned. However, the diagonal coefficient of any variable is determined when the aggregate of the variables that follow it in the diagonal order is given.

Professor Chrystal then invesigates rules for calculating the first and the last of the diagonal coefficients for any given order of the dependent variables, say $x_1, x_2, \ldots, x_n$.

Let $\begin{vmatrix} \xi_1 \ldots \xi_n \\ \ldots\ldots \\ \kappa_1 \ldots \kappa_n \end{vmatrix}$, $\begin{vmatrix} \xi_1' \ldots \xi_n' \\ \ldots\ldots \\ \kappa_1' \ldots \kappa_n' \end{vmatrix}$

be the systems of multipliers of (I) with respect to (II), and of (II) with respect to (I). Since the two systems are equivalent, all the multipliers must be integral functions of D. Then among other things one has

$$[11]\xi_1' = (11),\ [11]\eta_1' = (21), \ldots, [11]\kappa_1' = (n1) \quad \ldots\ldots \text{(IV)}$$

$$(11)\xi_1 + (21)\xi_2 + \ldots + (n1)\xi_n = [11] \quad \ldots\ldots\ldots\ldots\ldots \text{(V).}$$

From (IV), [11] must be a common divisor of (11), (21), . . . , (n1); and from (V), the greatest common multiple of (11), (21), . . . , (n1) must divide [11] exactly. Hence [11] must, to a constant factor, be simply the greatest common multiple of (11), (21), . . . , (n1), g_1 say.

The complete system for determining $\kappa_1, \kappa_2, \ldots, \kappa_n$ is

$$(11)\kappa_1 + (21)\kappa_2 + \ldots + (n1)\kappa_n = 0$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$(1, n\text{-}1)\kappa_1 + (2, n\text{-}1)\kappa_2 + \ldots + (n, n\text{-}1)\kappa_n = 0$$

$$(1n)\kappa_1 + (2n)\kappa_2 + \ldots + (nn)\kappa_n = [nn] \quad \ldots\ldots \text{(VI).}$$

If therefore $\left| \{11\}\ \{22\} \ldots\ldots \{nn\} \right|$ . . . . (VII)

denotes the reciprocal matrix of $\left| (11)\ (22) \ldots\ldots (nn) \right|$, then from the first (n-1) equations of (VI), $\kappa_1 : \kappa_2 : \ldots : \kappa_n = \{1n\}' : \{2n\}' : \ldots : \{nn\}'$, where $\{11\}', \ldots, \{nn\}'$ are the set of relatively prime integral functions of D which arise by dividing $\{1n\}, \ldots, \{nn\}$ by their greatest common multiple, G_n say. Therefore

$\kappa_1 = \lambda\{1n\}'$, $\kappa_2 = \lambda\{2n\}'$, . . . , $\kappa_n = \lambda\{nn\}'$, with λ some integral function of D, or a constant.

Now, since $|\xi_1, \eta_2, \ldots, \kappa_n|$ must be a constant, (I) and (II) being equivalent, it follows that λ must be a constant. For $|\xi_1, \eta_2, \ldots, \kappa_n|$ obviously contains the factor λ. Therefore

$$[nn] = \lambda[(1n)\{1n\}' + (2n)\{2n\}' + \ldots + (nn)\{nn\}'] = \lambda|(11), (22), \ldots, (nn)| \div G_n,$$

i.e. to a constant factor, $[nn] = K/G_n$, where K is the characteristic determinant of the original system.

This leads to the following general rule:-

Form the schemes

$$\begin{array}{c|cccc|} & x_1 & x_2 & \ldots & x_n \\ U_1 & (11) & (12) & \ldots & (1n) \\ U_2 & (21) & (22) & \ldots & (2n) \\ \vdots & \cdot & \cdot & \ldots & \cdot \\ U_n & (n1) & (n2) & \ldots & (nn) \\ & g_1 & g_2 & \ldots & g_n \end{array} \qquad \text{(VIII)},$$

and

$$\begin{array}{|cccc|} x_1 & x_2 & \ldots & x_n \\ \{11\} & \{12\} & \ldots & \{1n\} \\ \{21\} & \{22\} & \ldots & \{2n\} \\ \cdot & \cdot & \ldots & \cdot \\ \{n1\} & \{n2\} & \ldots & \{nn\} \\ G_1 & G_2 & \ldots & G_n \end{array} \qquad \text{(IX)}.$$

by means of the matrix of the given system and its reciprocal matrix, $g_1, g_2, \ldots\ldots, g_n$, and $G_1, G_2, \ldots\ldots, G_n$, being the greatest common multiples of the constituents of the respective columns, then

$$g_1, \quad g_2, \quad \ldots\ldots, \quad g_n ;$$

$$K/G_1, K/G_2, \ldots\ldots, K/G_n \qquad \ldots\ldots\ldots\ldots(X)$$

are the diagonal coefficients of the variables $x_1, x_2, \ldots\ldots, x_n$, when these are first and last in the diagonal order respectively.

Further, the differential equations for determining the variables separately are

(since $V_n \equiv K_1U_1 + \ldots + K_nU_n$)

$$\frac{K}{G_1}x_1 + \frac{\{11\}}{G_1}S_1 + \frac{\{21\}}{G_1}S_2 + \ldots + \frac{\{n1\}}{G_1}S_n = 0$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$\frac{K}{G_n}x_n + \frac{\{1n\}}{G_n}S_1 + \frac{\{2n\}}{G_n}S_2 + \ldots + \frac{\{nn\}}{G_n}S_n = 0 \quad \ldots \quad \text{(XI).}$$

In other words, these are the last equations in diagonal systems when $x_1, x_2, \ldots, x_n$ are last in the diagonal order respectively.

Chrystal noted that (XI), considered as a system, is not equivalent to the given system, although it gives correctly all the variables separately - that is to say, it gives correctly a value for each variable which, along with a corresponding set of properly-determined values, will constitute a solution of the system.

He then defines what he calls a simple diagonal system, as a diagonal system which contains only one differential equation, this of course being the last. In this case the order of the system is the order of this last differential equation. All the diagonal coefficients reduce to constants, for none can vanish in a determinate system, and the first (n-1) are non-differential equations, by means of which one can calculate successively the variables in terms of those previously found, and of their differential coefficients.

Professor Chrystal then mentioned two possible important criteria for reduction of a given system to a simple diagonal system.

The first of these is that " If K be irreducible, then every equivalent diagonal system to which a given system can be reduced is simple". Here the irreducibility of K is in the sense that it has no integral factors whose coefficients are integral functions of the coefficients in the operator coefficients in the original system.

The second criteria is that " For every prime column in the reciprocal matrix of a given system a series of equivalent simple diagonal systems can be formed in which the corresponding variable is the last variable.

In particular, if every column of the reciprocal matrix be prime, then every equivalent diagonal system will be simple, and the expression for each of the dependent variables will contain all the arbitrary constants of the system".

If all the columns of the characteristic determinant of a system be prime, such a system he calls a prime system.

To Professor Chrystal the most natural method for finding practically the solutions of linear systems with constant coefficients is to transform the system into a prime system, and then reduce this prime system to an equivalent diagonal system, simlify this as much as possible, solve it, and then pass back to the original system.

At the end of the paper he gives three examples which illustrate the use of the methods described in the theory. I give here one of these examples. Let the system of equations be

$$(D^3 - D + 2)x + (2D - 2)y = e^{-t},$$

$$(-3D^2 - 4D + 1)x + (D^3 + 3D^2 + 5D - 1)y = 0.$$

Here the characteristic determinant of the system

$$\begin{vmatrix} D^3 - D + 2, & 2D - 2 \\ -3D^2 - 4D + 1, & D^3 + 3D^2 + 5D - 1 \end{vmatrix}$$

$$\equiv D(D + 1)^3(D^2 + 1).$$

Since it is not constant, so we consider the reciprocal matrix

$$\begin{vmatrix} D^3 + 3D^2 + 5D - 1, & 3D^2 + 4D - 1 \\ -2D + 2, & D^3 - D + 2 \end{vmatrix}$$

which has all its columns prime. Hence any equivalent system must be simple.

If we consider x to be the last variable, the matrix of the system of multipliers will be

$$\begin{vmatrix} L & M \\ D^3 + 3D^2 + 5D - 1, & -2D + 2 \end{vmatrix}$$

where L and M satisfy the equation $L(2D-2) + M(D^3 + 3D^2 + 5D - 1) = \text{const.}$

He finds $L = D^2 + 4D + 9$, $M = -2$.

By using these multipliers the original system reduces to

$$-16y + (D^5 + 4D^4 + 8D^3 + 4D^2 + 7D + 16)x = 6e^{-t},$$

$$D(D+1)^3(D^2+1)x = -4e^{-t}.$$

From this we get by the common methods used

$x = A+(B+Ct+Et^2)e^{-t}+F\cos t+G\sin t+(1/3)t^3e^{-t}$, and so

$y = A+(1/4)\{(2B+3C-3E)+(2C+6E)t+2Et^2\}e^{-t}+$

$$F\cos t+G\sin t+(1/8)\{-1-6t+6t^2+(4/3)t^3\}e^{-t}$$

Professor Chrystal says nothing about the interesting question as to how far the general principles laid down in the paper could be extended to systems of ordinary linear equations whose coefficients are not constant.

This paper was followed by another entitled "On the p-discriminant of a Differential Equation of the First Order, and on certain points in the General Theory of Envelopes Connected Therewith"[50]. Although criteria for distinguishing between singular solutions and particular solutions of differential equations of the first order were established at the end of the eighteenth century and the first half of the nineteenth century, however the entire theory of singular solutions was reinvestigated in the last quarter of the nineteenth century along new paths by J. G. Darboux, A. Cayley, E. C. Catalan, F. Casorati and others including Professor Chrystal.

In the beginning of his paper he comments that the theory of the singular solutions of the first order, even in the interesting form due to Professor A. Cayley[51], as given in Messenger of Mathematics is defective, because it gives no indication of what are abnormal phenomena. Moreover, Cayley continued his theory[52] investigating the circumstances under which a singular solution exists, which Chrystal claims is misleading so far as the theory of differential equations is concerned, if not altogether incorrect.

The main purpose of this paper is to clarify this last point by means of a number of examples. Moreover, he took the opportunity to give simple proofs of several well-known theorems regarding the p-discriminant which did not appear in text-books of the time. This shows his instinct for helping students of mathematics to improve their understanding of the subject.

In the paper he investigates the nature of the integral of the equation

$A_0 + A_1p + A_2p^2 + \ \dots\ + A_np^n = 0$, at a point on the p-discriminant locus, in the most general case. Here $A_0, A_1, \dots, A_n$ are supposed to be continuous functions, so that we have

$$A_0 = a_0 + b_0x + c_0y + d_0x^2 + \dots$$

$$A_1 = a_1 + b_1x + c_1y + d_1x^2 + \dots$$

$$A_2 = a_2 + b_2x + c_2y + d_2x^2 + \dots$$

$$\dots\dots\dots\dots\dots\dots\dots\dots \quad \dots$$

$$A_n = a_n + b_nx + c_ny + d_nx^2 + \dots$$

He concludes that the p-discriminant locus is in general the locus of cusps on the integral curves of the differential equation.

He then takes the nature of the integral at a point on the p-discriminant locus where the primitive in question touches that locus. He finds that in general, when the primitive touches the p-discriminant two integral curves touch each other, and we have a tac-point. If any branch of the p-discriminant is a solution of the differential equation that branch is an envelope singular solution, that is, at every point there is a primitive distinct from the branch of the p-discriminant in question that touches that branch.

He investigates the conditions in the most general case that the p-discriminant gives a tac-locus, and uses Newton's diagram to estimate the relative orders of the terms in the differential equation,

$$\phi(x, y, p) = 0.$$

These conditions are that four equations in x, y, p

$$\phi = 0,\ \phi_p = 0,\ \phi_x = 0,\ \phi_y = 0$$

have infinitely many simple common solutions, the corresponding points being the tac-locus.

In special cases, the four equations may have a finite number of common solutions and the corresponding points are tac-points, in which the touching integral branches do not in general touch the p-discriminant, but at which the p-discriminant itself has double points.

Considering into the locus of the points of inflexion on the integral curves of the differential equation (1), he proves an important result that at a tac-point where the tangent does not touch the p-discriminant locus, this locus must have a point in common with the inflexion-loci of the original family and of the orthogonal trajectories. The conditions for the existence of a tac-locus, where there is no contact with the p-discriminant locus is that the p-discriminant locus, the locus of inflexions, and the locus of inflexions on the orthogonal trajectories must have a branch in common.

The general existence of the cusp-locus of the integral family of a differential equation of the first order was shown as early as 1851 by De Morgan[53]. The earliest absolutely explicit statement of the above theorem seems to have been made by Darboux[54]. In an extremely interesting paper "Sur les Solutions Singulières des Equations aux Derivées Ordinaires du Premier Ordre"[55]. Darboux established most of the propositions given by Professor Chrystal in this paper. It is surprising that Darboux's work did not attract the notice of Professor A. Cayley. Reference may also be made to the work of R. F. A. Clebsch[56].

Professor Chrystal used the approximative method first used by C. A. A. Briot and J. C. Bouquet to deduce these results, because this method is a general one, applicable to the discrimination of special cases, such as arise when an envelope is also a cusp-locus or a tac-locus, etc. Moreover, the method was little used by English mathematicians.

He has then showed that if the general equation considered has an envelope singular solution its integral is a family of algebraic curves, and finds the condition that the equation considered may have an algebraic integral.

Finally he gives an example of a differential equation which has an algebraic primitive but has no singular solution. After a thorough discussion he concludes that when a differential equation has a singular solution, its primitive is algebraic yet, on the other hand, it is the exception, and not the rule, that it has a singular solution when its primitive is algebraic.

In Cayley's second paper on the subject published again in Messenger of Mathematics (vol.vi.P.23,1877) the following passage occurs:-" Consider now a system of algebraic curves U= 0, where U is, as regards (x,y), a rational integral function of order m, and depends in any manner on an arbitrary parameter C, I say that there is always a proper envelope, which envelope is the singular solution of the differential equation in regard to (x,y). It follows that the differential equation $(L,M,N)(p,1)^2 = 0$, which has no singular solution, does not admit of an integral of the form in question, U = 0, viz., an integral representing a system of algebraic curves".

According to Chrystal, Professor Cayley rests his conclusion, on a proof which amounts to the following:-

"Consider an algebraic curve U = 0 of order m, having singularities equivalent to δ double points and κ cusps. Of the intersections of U = 0, with its consecutive $U + \delta_c U = 0$, two will coincide with each double point and three with each cusp, leaving $m^2 - 2\delta - 3\kappa$ other points of intersection. If n be the class of the curve $m^2 - 2\delta - 3\kappa = m + n$[which is Plücker's formula given by him in his "System der analytischen Geometrie (der Ebene)" 1834], a number which can not vanish; hence there is always an envelope, viz; the locus of these $m^2 - 2\delta - 3\kappa$ points".

The fallacy in this reasoning Chrystal points out "consists in assuming that $m^2 - 2\delta - 3\kappa$ points are necessarily spread out into a locus. It is, in fact, an inference that may be drawn from the above investigation that, in general, when a differential equation of the first order has an algebraic integral, this is not the case. So that only particular kinds of algebraic families can be integrals of equations of this description".

Chrystal expresses surprise that J. W. L. Glaisher[57] seems to endorse the statement of

Cayley just referred to, seeing that Glaisher's examples (iii), (xiv), (xv), (xx) are instances to the contrary.

Professor Chrystal also mentions that Professor A. R. Forsyth has been kind enough to inform him that he is not to be understood as endorsing the proposition in dispute.

According to Professor C. G. Knott, "It is interesting to note that there had been a steady demand for copies of this particular paper on the p-discriminant and the connected theory of envelopes".

The idea of his papers seems to show that the theory of differential equations, was far from being exhausted, and was full of possible developments even from its very foundations. These developments rested on the equal use of constructive power and of critical faculty. He wrote all his papers keeping in view the help to teachers and students alike, and explained most of his theory by giving illustrative examples. His knowledge in mathematics was very deep, and the way he wrote his papers showed that he used to read all the literature available on the subject, without any reservations.

His continued interest in differential equations, in particular their singular solutions, shows that he was beginning to approach mathematics from a more algebraic point of view. However, his instinct for applied science forced him to abandon this course when Sir John Murray asked for his advice on observations of seiches in Scottish lakes.

References Chapter 4

1. H. V. Helmholtz," *Popular Lectures On Scientific Subjects*," Vol.ii;
Translated by E. Atkinson(1900); Longmans: London. pp. 27-71.

2. *Proceedings Of The Royal Society Of Edinburgh* (1880); Vol.x, pp.638-664.

3. *Transactions Royal Society Of Edinburgh* (1880); Vol.xxix, part II, pp.519-530.

4. Professor Tait, "Quaternion Proof Of Minding's Theorem";
Proceedings London Mathematical Society (1877-78); Vol.ix-x,pp.101-103.

5. *Transactions Royal Society Of Edinburgh* (1883); Vol. xxx, Part I, pp.13-14.

6. *Ibid* ; pp. 161-165.

7. Ferdinand Joachimsthal, "Bemerkungen Uber Den Sturm'schen Satz";
Crelle's Journal (1854); Vol. xlviii, p.386.

8. R. A. Rankin, "The First Hundred Years(1883-1983)";
Proceedings Edinburgh Mathematical Society (1983); Vol.26, pp.135-150.

9. *Proceedings Edinburgh Mathematical Society* (1884); Vol. II, pp. 45-47.

10. *Ibid* ; pp.47-49.

11. *Ibid* ; pp. 49-50.

12. J. J. Sylvester, "Constructive Theory Of Partitions";
American Journal Of Mathematics (1882); Vol. v, pp.151-330.

13. *Proceedings Edinburgh Mathematical Society* (1885); Vol. iii, pp.30-33.

14. J. J. Sylvester, "A Question In The Geometry Of Situation";
Quarterly Journal Of Mathematics (1857); Vol. i, p. 79.

15. J. J. Sylvester, "On Poncelet's Approximate Linear Valuation Of Surd Forms";
Philosophical Magazine (1860); Fourth Series, Vol.xx, pp. 206-214.

16. *Proceedings Edinburgh Mathematical Society* (1885); Vol. iii, p.94.

17. *Ibid* ; pp. 95-100.

18. *Transactions Royal Society Of Edinburgh* (1885);
Vol. xxxii, part. ii, pp.645-650.

19. Address Section "A" *British Association For Advancement Of Science* (1885), Aberdeen.

20. *Algebra: An Elementary Textbook* , Part.II;
Adam and Charles Black, Edinburgh(1889).

21. *Ibid*; p. 576.

22. Florian Cajori, *A History Of Mathematics* (1985) 4th ed;
Chelsea Publishing Co. p.378.

23. *Algebra: An Elementary Textbook* , Part. II(1889);
Adam And Charles Black, Edinburgh. p.4.

24. *Ibid* ; p. 25.

25. *Ibid* ; p. 10.

26. *Ibid* ; p.135.

27. *Ibid* ; p. 288.

28. *Ibid* ; p. 318.

29. Florian Cajori, *A History Of Mathematical Notations* (1929); Vol.ii, p.37.

30. *Algebra: An Elementary Textbook*, part. II(1889); p.402.
Adam And Charles Black, Edinburgh.

31. *Ibid* ; p. 397.

32. *Ibid* ; p. 421.

33. Banchara Branford, *A Study Of Mathematical Education* (1908);
The Clarendon Press, Oxford.p.216.

34. *Ibid* ; p.253.

35. S. Abhyanker," Review Of Chrystal's Algebra";
The Mathematical Intelligencer (1978). Vol. i, no.I, p.37.

36. G. Heppel, "Algebra In Schools";

The Mathematical Gazette (1895); No.4, p. 25.

37. *Introduction To Algebra* ; Adam And Charles Black, London(1898).

38. *Proceedings Edinburgh Mathematical Society* (1886-87); Vol.v, pp.38-50.

39. *Proceedings Edinburgh MathematicalSociety* (1888); Vol. vi, pp.29-33.

40. *Proceedings Edinburgh Mathematical Society* (1895); Vol. xiv, pp.2-25.

41.Dr. J. S. Black and Prof. C. G. Knott, Obituary Notice Of Prof. Chrystal; *Proceedings Royal Society Of Edinburgh* (1911-12); Vol.xxxii, p.489.

42. *Transactions Royal Society Of Edinburgh* (1891); Vol.xxxvi,Part.II, pp.551-562.

43. J. L. Lagrange, "Sur Les Intégrales Des Equations Differentielles"; *Nouv. Mem.D. L'Academy Des Berlin* (1774); Œuvres; t. iv, p.82.

44. *Ibid* ; (1779), Œuvres, t. iv, p. 624-34.

45. *Ibid* ; (1785), Œuvres, t.v, p. 543.

46. C. G. J. Jacobi, " Dilucidationes De Æquationum Differentialium Vulgarum Systematis" *Crelle's Journal* (1842); Bd.23, pp:1-104; Werke,Bd.iv,p.150.

47. Letter From Professor G. Chrystal To Sir Joseph Larmor FRS, 30th March 1909; *Royal Society Of London, Library* , MSS. Lm.269.

48. *Transactions Royal Society Of Edinburgh* (1895); Vol.xxxviii,Part.I, pp. 163-178.

49. C. G. J. Jacobi, " De Investigando Ordine Systematis Æquationum Differentialium Vulgarium Cujus_cunque"; *Werke* , Bd.v. pp:191-216.

50. *Transactions Royal Society Of Edinburgh* (1896); Vol. xxxviii, pp. 803-824.

51. A. Cayley, " On The Theory Of Singular Solutions Of Differential Equations Of The First Order"; *Messenger Of Mathematics* (1873), Vol. ii, pp. 6-12.

52. A. Cayley, " Equations Of The First Order"; *Messanger Of Mathematics* (1877); Vol. vi, pp. 23-27.

53. De Morgan, *Cambridge Philosophical Transactions* (1851); Vol. ix, Part. 2, p. 113.

54. Darboux, *Comptes Rendus* (1870); t. lxx, p. 1331; also t. lxxi,p. 267.

55. Darboux, *Bulletin Des Sciences Mathématiques* (1873); t. iv, p.158.

56. R. F. A. Clebsch, *Mathematische Annalen* (1873); Bd. vi, p. 211.

57. J. W. L. Glaisher, *Messenger Of Mathematics* (1883); Vol.xii, p. 2.

Chapter 5

Theory Of Seiches

5.1 Introduction

In lakes and along sea coasts, oscillations of water level occur, under certain circumstances, which are called *SEICHES* .

The term seiche, supposedly derived from the Latin word Siccus, meaning dry, hence exposed, had centuries of usage for describing the occasional rise and fall of water at the narrow end of Lake Geneva. At different places different terms are used for this kind of oscillation of water,e.g., in Norway it is called *floing* , in Sweden *lunken* , but the term seiche is internationally recognised. According to an old chronicle by Shulthaiss, oscillations of this kind were noted at the Lake of Constance(Switzerland) as early as 1549; J. P. E. Vaucher in 1804, nevertheless, appears to have been first to note that it was a feature of many lakes and that it was in some way associated with atmospheric conditions. In Scotland the first seiche was noted in November, 1755 in Lochs Lomond, Ness, Lung, and Katrine as reported in the Scots Magazine[November 1755, p. 593], caused by the great earthquake which destroyed Lisbon on the morning of 1 November , 1755. In two important respects this seiche is, however, exceptional. Its amplitude, is much out of the common, and, so far as we know, ordinary seiches, which are plentiful enough, are very rarely caused by earthquakes.

The history of investigation of seiches is a long one, Merian in 1828 studied oscillations in long rectangular basins and deduced a formula for the period when the rectangle was long and narrow, and the depth constant.

The modern history of seiche begins with the researches of Professor F. A. Forel at

Morges, in 1869. From that date for nearly fifty years seiche investigations received much attention. Professor Forel made his early observations with a small portable apparatus called Plemyrameter. This was a small tank fixed on the shore, connected with the water in the lake by means of a syphon having a horizontal glass tube connected at each end with the tank and with the lake by rubber tubing. In the glass a weighted cork float was placed, this float indicated the rise and fall in the lake by its position in the tube. The instrument was supplemented by a very delicate tide gauge, which recorded the amplitude of the seiche. But in 1876 he reportedly was using a self-registering limnograph and another portable limnograph. He appears to have been first to apply Merian's formula. Forel along with his friend Plantamour(1877), Sarasin(1879), and others made a thorough investigation of seiches of the Swiss lakes. In 1877 Plantamour developed another self-registering limnograph which worked nicely near his villa at Sechron, on lake Geneva and remained in action for a long time. Two years later in 1879 Sarasin made further improvements and invented a portable limnograph, which was used at many stations on lake Geneva as well as on other Swiss lakes. In 1880 the French Government showed its interest in the study. As a result, Government Engineers installed a fixed limnograph at Thonon, and appointed Delebecque, Du Boys, and Lauriol to carry on the observations.

Ebert(1901), Halbfass(1902), and Endrös(1903) have done a similar service for many of the German Lakes; the results of Endrös in particular being of great interest and variety. P. Du Boys(1891) made serious attempt to submit seiche phenomena in detail to mathematical calculation. Marinelli(1891), Bettoni(1900), Palazzo(1904), and Magrini(1905) have worked in Italy; Von Cholnoky(1897) in Hungary; W. B. Dawson(1894-1920), Gaillard(1904), N. Denison(1897), Henry(1902), and others in North America; Burton(1892), Nakamura and Yoshida(1903) in Japan; Airy(1845), Chrystal, Murray, White, Watson, and Wedderburn in Britain. But the great central authority who inspired all others to work is Forel, whom Chrystal named the *Faraday of seiches* . To Chrystal, more than any one else, belongs the credit for having evolved the main elements of the hydrodynamical theory of seiches, though important contributions have been added subsequently by R. Harris(1908)[1], Proudman(1914)[2], Doodson(1920)[3], Sterneck(1916)[4], Defant(1917)[5], Goldsbrough(1930), Lamb(1932)[6], Hidaka(1932, 1936), Ertel(1933), Goldberg(1947) Caloi, Bottomley(1956), Zore(1955) and others.

Systematic observations of seiches in the lakes of Scotland are of the first decade of the twentieth century. Dr Johnston and Mr J. Parson, working for the British Lake Survey, observed a slight periodic variation in the level of water on Loch Treig in 1902. It was a uninodal seiche with amplitude 0.56 inch and period 9.50 minutes. The same year Mr James Murray and Mr J. Hewitt made similar determinations for many lochs in Scotland. Mr James Murray also observed a binodal seiche on one of these lochs. In 1903 Mr T. R. H. Garret made observations on loch Ness and had an approximation of uninodal period. Even though all of these observations were very rough, yet they proved to be a good start for a long and detailed survey of these lochs.

Professor Chrystal's connection with the matter began in February 1903, when, at the request of Sir John Murray, he gave a brief account of the hydrodynamical principles of the subject, with suggestions to his surveyors regarding the observations they might make on Scottish lochs in the Scottish Lake Survey.

This survey was mainly organised by Sir John Murray for a systematic study of Scottish fresh-water lochs, in order to fulfil the needs of geologists, fishermen, engineers and other scientists. He brought the matter before the councils of the Royal Societies of London and Edinburgh. After much consideration both the councils made very strong representations to the Government during 1883-84, urging that a bathymetrical survey of the Scottish fresh-water lochs should at once be undertaken in the interest of scientific progress. But the reply of the Treasury soon made it clear that a survey of the kind did not come within the functions of the Admiralty.

Sir John Murray who was depressed by the reply , but was determined to get the job done for the benefit of scientific investigators, commenced the task in a systematic way in 1897, with the assistance and encouragement of his young friend Mr Fred Pullar.

In February 1901, the work came to a standstill, because of the untimely death of Mr F.

Pullar, who lost his life on 15 February 1901, while gallantly attempting to rescue a number of people who, through an ice accident, had been immersed in Airthrey Loch. This untoward event obliged Sir John Murray to think of abandoning the survey altogether.

However, Mr Laurence Pullar, the father of F. P. Pullar, wished, to see the work in which his son had taken so deep an interest brought to a satisfactory conclusion. He expressed his willingness to take his son's place so far as possible, and, at all events, to set apart a sum of money to carry out the work and to publish the results of the investigations. Mr Laurence Pullar desired to be assured on two points:

first, that there was no likelihood of the Government undertaking such a survey in the near future; and *second*, that this survey was considered by competent scientific authorities to be desirable and important from a national point of view. In these circumstances the question of the renewal of the survey work was brought before the councils of the Royal Societies of London and Edinburgh, as well as before the British Association at its meeting in Glasgow in 1901. All these organisations passed resolutions that they learned with great satisfaction that arrangements were under consideration for the completion of the survey commenced by Sir John Murray and the late Mr F. P. Pullar, and confirmed the opinion as to the great scientific importance of the investigation.

Mr Laurence Pullar at once handed over to a small trust a sum of £10,000 to provide the means for carrying on the work on the lines that had been indicated.

Although His Majesty's Government could not see its way to undertake a bathymetrical survey of Scottish fresh-water lochs, still several Public Departments did take a deep interest in the work and provided important assistance.

Up to the time of Mr Pullar's death, 15 lochs had been surveyed; during 1902-06 another 547 lochs were surveyed, thus making a total of 562 lochs in all.

The above is a brief account of the history of the Survey taken from the report given by Sir John Murray in 1910.

5.2 The Hydrodynamical Theory Of Seiches

The seiches problem was just the kind to awake Chrystal's keenest interest. He caught the seiche madness and devoted most of his leisure to the subject in the last years of his life. It involved hydrodynamical problems of great difficulty, most of them never attempted before, which could be overcome only by the use of mathematics of a higher order. It demanded experimentation of an alluring nature, full of difficulties. Chrystal showed the same interest for investigating seiches in Scotland as Forel did in Switzerland, and was able to gather a large group of volunteers to work on the subject, these including ordinary observers as well as experts of different fields.

A year after his first address to surveyors of the Scottish Lake Survey, he presented his first paper on the subject entitled " Some Results In The Mathematical Theory Of Seiches,"[7] and a year later his second paper " Some Further Results In The Theory Of Seiches,"[8] which was a continuation of his first paper. In fact both of these papers, published in the Proceedings of The Royal Society of Edinburgh, were the abstract of his most important and valuable paper entitled " On The Hydrodynamical Theory Of Seiches,"[9]which was published in the Transactions of The Royal Society of Edinburgh. It consisted of three parts:

Part I(pp:599-612)._ Giving the general summary of the work, and was presented to encourage and guide the enthusiastic observers, who were engaged in procuring accurate data regarding seiches.

Part II(pp:612-643)._ Dealing in detail with the mathematical theory concerning seiche phenomena; a major part of which was already covered in his first two papers.

Part III(pp:644-649)._ A sketch of the bibliography of seiches, which consisted of books and memoirs on the subject from the very beginning of its investigation till his own time. He tried his best to make it complete, but still he did not claim it to be exhaustive. To any one interested in the history of the subject this list is a most useful reference.

F. A. Forel originally established that a seiche is a synchronised standing oscillation of lakes, estuaries and other inland waters. Usually the seiches are longitudinal but transverse seiches do exist.

In order to explain some of the fundamental terms used in the study of seiche, Professor Chrystal considered a longitudinal seiche in any lake of rectangular section and uniform breadth but of varying depth. Then the horizontal and vertical displacements of any particle originally at a distance x from a fixed point of reference would be given by

$$\xi = \phi_n(x) \sin nt, \quad \zeta = \chi_n(x) \sin nt,$$

where t is the time measured from any instant, and $T=2\pi/n$ is the period of oscillation.

The periods and position of nodes of a seiche always depend on the lake configuration, so the functions $\phi_n(x)$ and $\chi_n(x)$ are determined for admissible values of n.

For any value of n, say n_ν, the function $\chi_{n_\nu}(x)$ vanishes for n_ν different values of x, so that at these points there is no vertical motion these points are called *nodes* of the seiche. The seiche corresponding to $n = 1, 2, 3$, etc., are termed as *uninodal, binodal, trinodal,* etc. Any number of these may exist together; and the total seiche displacement is obtained by adding these. When only one of these harmonic components is involved the seiche is called a *pure seiche* . A compound seiche being a superposition of two pure seiches is termed a *dicrote seiche* ; and so on.

Intermediate between these nodal points, there are values of x for which $\phi_n(x)$ vanishes, i.e., there are points where there is no horizontal motion of the surface particles. These points are called *ventral* or *antinodal* points. Four times the distance between a node and next ventral(antinodal) point is called the *wave length* . When the wave length is large compared with the depth of the water basin, the wave is called a *long wave* , and a seiche is always a long wave.

The theory of long waves was first described by Lagrange(1781)[10]; and the first detailed memoir on the subject was given by J. R. Merian(1828)[11] which dealt fully and effectively with the problem of stationary waves in a canal of uniform depth h and length l. In Merian's paper the formula

$$T = \sqrt{\{\pi l/g \tanh(\pi h/l)\}}$$

appeared for the first time in connection with stationary waves. There is no evidence that Merian's theory was immediately applied to explain the seiches of the lakes. In fact it was forgotten until it was reproduced by the author's grandnephew, Von Der Mühl[12]. Meanwhile Merian's results had been rediscovered by other mathematicians. Merian's formula was applied by F. A. Forel in 1876[13].

In 1845 G. B. Airy published his article " Tides and Waves,"[14] one of the classic treatises on the subject. He gave a general explanation of the semi-diurnal tides of the Adriatic Sea and the Bay of Fundy.

In his paper " On The Hydrodynamical Theory Of Seiches", Professor Chrystal discussed the theory of seiches in an elongated lake on the assumption that a seiche may be treated as a "long" stationary wave. So far as seiches of lower nodalities were concerned, this amounted to assuming that the square of the ratio of the wavelength of the seiche at the end of the lake to the length of the lake was negligible. With this assumption applied Professor Chrystal, achieved the following results:-

1. When the lake was of uniform breadth and depth, the periods were proportional to the harmonic sequence 1, 1/2, 1/3, 1/4, ... and the quarter wave length was the same all over.

2. When the depth or breadth, or both, varied, the periods were in general no longer commensurable. Thus, for a *complete parabolic lake* the ν-nodal period was given by

$$T_\nu = \pi l/\sqrt{\{\nu(\nu+1)gh\}},$$

where l is the length and h the maximum depth; i.e. the periods were proportional to

$$\frac{1}{\sqrt{(1\text{x}2)}}, \frac{1}{\sqrt{(2\text{x}3)}}, \frac{1}{\sqrt{(3\text{x}4)}}, \frac{1}{\sqrt{(4\text{x}5)}}, \ldots$$

Also for a lake whose longitudinal section was a certain quartic curve,

$T_v = \rho/\sqrt{(v^2+\varepsilon)}$, where ρ and ε depend on the dimensions of the lake and ε may be positive or negative, according to circumstances.

3. Hence it followed that the ratio of the binodal to the uninodal period might be less than, equal to, or greater than 1/2, according to circumstances_ a fact which seemed to have puzzled seiche observers considerably. Indeed Professor Chrystal showed that quartic lakes could be imagined in which periods T_1, T_2, T_3, . . . might be as nearly all equal as one likes.

4. A shallow or other obstruction, or deep near a node, greatly affected the corresponding period, a deep decreasing it. Also a shallow attracted the node towards itself, and a deep repelled it. Thus, for example, the binodes in a parabolic lake were nearer the ends than in a rectangular one.

If the obstruction at a node was very great, it might have rendered the corresponding seiche unstable, or prevented its occurrence altogether. This clarified the absence in certain particular lakes of certain seiches of theoretically possible series.

5. When the breadth and the form of the transverse section of an elongated lake varied as well as the depth, provided these variations were not too abrupt, it was possible to submit it to calculation by the introduction of two new variables, viz., σ, which was the product of the area of the transverse section by the breadth of this section at the surface; and v, which was the area of the surface of the lake between the trace on the surface of the transverse section corresponding to σ, and any other similar line chosen for reference.

In order to submit the lake to calculation, its line of maximum depth was taken and laid out straight, and practically the lake was treated as if it were a lake of uniform breadth and rectangular cross section, whose longitudinal section was the curve at any point of which the abscissa is v and ordinate is σ . This curve was called by Professor Chrystal the *normal curve* of the lake.

Though under the assumptions made by Professor Chrystal, it was difficult to have a normal curve for lakes of every possible shape, yet it provided solution for many lakes.Judged by the results for Lochs Treig and Earn, these assumptions were sufficiently

correct for ordinary concave lakes at least.

6. It became obvious that a seiche, properly so called, differed from an ocean tide. The origin of a seiche, and the absolute and relative magnitudes of the pure seiches of which it was composed, no doubt depended on external circumstances; but the periods and the positions of the nodes of the component seiches depended for the most part on the configuration of the lake-basin, and on the surface-level of the water at that time.

In a tide, on the other hand, the periods were dependent on external disturbing agencies, chiefly the sun and the moon. In the language of the physicists, a seiche is a *free oscillation* ; a tide a *forced oscillation* .

About Part II of this memoir which is on mathematical theory, Professor Chrystal mentions that his purpose is twofold:-

" First, to establish generally, and also by means of special instances, the leading principles stated above;

secondly, to furnish formulæ and methods which can be applied in the investigation *a priori* of the periods and nodes of a lake whose length is considerably greater than its breadth, and which does not present excessive or abrupt variations in the configuration of its basin. I believe that in many such cases a sufficiently close approximation can be obtained by finding the normal curve of the lake, and replacing this curve by a combination of parabolas, straight lines, or simple quartic curves".

He identifies the seiche problem with the theory of a vibrating string as follows:-

"Consider a stretched string, fixed at both ends, whose length in conveniently chosen units represents the median line of the lake, which we take to be of uniform breadth and rectangular cross section, but of variable depth, h(x), at any distance x along the median line. Suppose the string is so constructed that its density at x is 1/h(x); and let u=P(x)sinnt be the transverse displacement in any fundamental (normal) mode of vibration at time t of the point which was originally at x. Then, if the tension of the string be properly adjusted, we shall have

$$\xi d(x) = u, \quad \zeta = - du/dx,$$

ξ and ζ denoting the horizontal and vertical displacements of the seiche. The motion of the string, therefore, exactly represents the seiche movement, the transverse displacement of the string corresponding to the horizontal displacement of the seiche; and the gradient of the curve formed by the string at any moment to the vertical displacement of a seiche.

It will be noticed that nodes of the string correspond to ventral(antinodal) points of the lake, and the ventral(antinodal) points on the string to nodes on the lake".

According to him identification of the seiche problem with the theory of vibrating strings is very helpful from both physical and mathematical points of view. As an example he mentions that "when we have worked out the periods and nodes of a seiche for any simple configuration approximately fitting a given lake, we can correct for the divergence of the actual lake from the assumed mathematical form by means of the beautiful method described by Lord Rayleigh in his *Theory of Sound* , Vol.i.§ 90".

The beautiful method referred to in the above is an approximation for determining the periods and types of vibration of a given system of vibrating strings.

Another important conclusion suggested by this analogy is that, "while the boundary conditions at the end of the lake may seriously affect seiche of high nodality, they have comparatively little influence on the seiche of lower nodality, in particular on the uninodal and binodal seiche".

5.2.1 Du Boys' Theory

The only serious attempt before Chrystal to submit seiche phenomena in detail to mathematical calculation had been made by Du Boys and is given in his paper entitled " Essai Théoretique Sur les Seiches".[15] In this paper, Du Boys gives an approximate method for calculating periods of a seiche. He treats the seiche as the interference of two waves travelling backwards and forwards in the lake, the velocity of propagation being at each section that due to the greatest depth there. He thus arrives at the formula

$T_v = \frac{2}{v}\int_0^l \frac{dx}{\sqrt{(gh)}}$, where $\int_0^l \frac{dx}{\sqrt{(gh)}}$ simply means the time that a man would take

to travel from one end of the lake to the other along the line of greatest depth, his speed at each point being that which a stone would have after it has fallen from rest through a distance equal to half the depth at that point.

According to Professor Chrystal "This formula is exact for a lake of uniform breadth and depth, but its results vary in excess for a lake having a concave, and in defect for a lake having a convex, bottom. But approximation becomes better as the nodality rises; and, for that and other reasons, his rule is very useful in limnographic calculations". Moreover, in Du Boys' Theory according to him "no good reason can be given for the fact that, in order to get a good approximation for T_1 we must integrate along the line of greatest depth, instead of using Kelland's Formula[16] for the wave velocity, and putting

$$T_1 = 2\int\sqrt{\left(\frac{b}{ag}\right)}\,dl,$$

where b is the surface breadth, and a the area of cross section".

However, his work was of great use, and after Merian was the first to give a definite formula for calculating the period of the first node. It also gives best results for the lakes whose normal curve is neither purely concave nor purely convex, because in these cases the errors of the formula compensate one another and the results approximately come near the true values.

Nevertheless, it is valuable as an emperical formula, easy of application and giving in many cases a good first approximation.

5.2.2 Introduction Of Seiche Functions

Before introducing the seiche functions, let us consider precisely the equations of motion and continuity for seiches as derived by Professor Chrystal.

The equations for the long waves and so for seiches can be derived from the general equations of motion in wave theory, if we bear in mind the following characteristics of long waves:-

As the depth is small compared with the wave length, so horizontal motion becomes more important than the vertical motion and vertical acceleration of the water particles can be neglected altogether. Hence it follows that in a vertical plane perpendicular to the direction of propagation of the wave, the horizontal motion is always virtually the same for all water particles; i.e., horizontal velocity is a function of the direction and time only. Moreover the pressure in each case at any point under consideration is equal to the statical pressure exerted by a column of water extending from the free surface to the depth at that particular point.

We consider the x- axis parallel to the length of the lake, the z-axis vertical and upwards, the y-axis horizontal and perpendicular to x-axis.

According to Professor Chrystal consider a section of the lake at a distance x from the origin, with area $A(x)$ and breadth $b(x)$. Let us consider another section parallel to this and at a distance dx from this. Then the volume of the water column in between two sections, to the first order of small quantities will be $A(x)dx$.

Suppose, after a time, t, this water column has moved to a new position, with the distance of its posterior face from the origin being $x+\xi$. Then its breadth in this new position will be $dx(1+\partial\xi/\partial x)$; and part of its volume below the normal level of the lake will be $A(x+\xi)dx(1+\partial\xi/\partial x)$.

If we suppose the rise in level of the water column under consideration to be the same throughout, say ζ. Then under the assumptions made above, the equation of motion for this water column, regarded as a whole, will be

$$dx(1+\partial\xi/\partial x)\rho\partial^2\xi/\partial t^2 = -g(\partial\xi/\partial x)dx$$

i.e., $\partial^2\xi/\partial t^2 = -(g\partial\xi/\partial x) \div (1+\partial\xi/\partial x)$, ρ being the density of the liquid;

and the equation of continuity is

$$A(x)dx = \{A(x+\xi)+b(x)\zeta\}dx(1+\partial\xi/\partial x)$$

$$\text{i.e., } \zeta b(x) = A(x) \div (1+\partial\xi/\partial x) - A(x+\xi).$$

The amplitude being small, the quantities of the order ξ^2, $(\partial\xi/\partial x)^2$ shall be neglected, thus equation of motion for long waves becomes

$$\partial^2\xi/\partial t^2 = -g(\partial\zeta/\partial x);$$

and the equation of continuity takes the form

$$\zeta b(x) = A(x)(1-\partial\xi/\partial x) - A(x) - \xi\{\partial[A(x)]/\partial x\}$$

$$= -A(x)\partial\xi/\partial x - \xi\partial[A(x)]/\partial x$$

$$= -(\partial/\partial x)\{A(x)\xi\} \text{ or } \zeta = -\{1/b(x)\}\partial/\partial x\{A(x)\xi\}.$$

He then introduces two new variables by the equations

$$u = A(x)\xi, \quad v = \int b(x)dx$$

and by the use of the equation of motion and the equation of continuity he obtains

$$\partial^2 u/\partial t^2 = g\sigma(v)\partial^2 u/\partial v^2, \ \zeta = -\partial u/\partial v,$$

where v is determined as a function of x by the equation

$$v = \int b(x)dx, \text{ and } \sigma(v) = A(x)b(x).$$

According to him since a seiche is a standing oscillation, so ξ and therefore u is a periodic function of the time. Supposing this periodic function analysed into simple harmonic terms one can write $u = P\sin n(t - \tau)$,

where P is a function of v alone and τ is a constant.

He further remarks that the values of n admissible depend on the circumstances of each case.

But in order that the last equation for u satisfy $\partial^2u/\partial t^2= g\sigma(v)\partial^2u/\partial v^2$, we must have

$$-n^2P= g\sigma(v)\partial^2P/\partial v^2.$$

Thus the mathematical theory of a seiche of small amplitude depends on the differential equation

$$d^2P/dv^2+(n^2/ g\sigma(v))P = 0.$$

So professor Chrystal reduced the theoretical determination of the longitudinal seiche of a lake to the finding of the solutions of the differential equation

$$d^2P/dv^2+(n^2/ g\sigma(v))P = 0$$

for which $P=0$ at $v=0$ and $v=a$.

Here v denotes the area of the surface of the lake from one end up to any transverse section and ranges from 0 to a, a being the total area of the surface of the lake;

$\sigma(v)=A(x)b(x)$; n being the frequency of the periodic motion and g the acceleration due to gravity; P denotes the total volume of water which has passed the section up to the time t.

In order to find the general solution for a parabolic longitudinal section, Professor Chrystal considers the differential equation

$$d^2P/dv^2+[c/(1 - \lambda v^2)]\, P = 0.$$

Assuming $P = a_0+ a_1v+ a_2v^2+ \;.\;.\;.\;.\;,$ he gets

$$c\, a_0 + 1.2\, a_2 = 0,\ \ c\, a_1 + 2.3\, a_3 = 0,$$

$$(c - 1.2\,\lambda)\, a_2 + 3.4\, a_4 = 0,\ \ (c - 2.3\,\lambda)\, a_3 + 4.5\, a_5 = 0,$$

$$\cdots\cdots \qquad \cdots\cdots$$

$$\{c - (n - 3)(n - 2)\,\lambda\}\, a_{n-2} + (n - 1)\, n\, a_n = 0.$$

Thus getting

$$P = A\{1 - \frac{c}{(1x2)} v^2 + \frac{c[c - (1x2)\lambda]}{(1x2)(3x4)} v^4 - \ldots\} + B\sqrt{c}\{v - \frac{c}{(2x3)} v^3 + \frac{c[c - (2x3)\lambda]}{(2x3)(4x5)} v^5 - \ldots\},$$

where A and B are arbitrary constants. The series in brackets are obviously convergent if $|v| \leq 1/\sqrt{|\lambda|}$.

He then introduces new notations to represent the two convergent series in the brackets and writes

$$C(c, \lambda, v) = 1 - \frac{c}{1.2} v^2 + \frac{c(c - 1.2\lambda)}{1.2 \text{ x } 3.4} v^4 - \ldots\ldots,$$

$$S(c, \lambda, v) = \sqrt{c}\, v\, \{1 - \frac{c}{2.3} v^2 + \frac{c(c - 2.3\lambda)}{2.3 \text{ x } 4.5} v^4 - \ldots\ldots\}.$$

He purposely represents them with C and S, because they have some properties in common with circular functions; e.g.,

$$C(c, \lambda, -v) = C(c, \lambda, v);$$

$$S(c, \lambda, -v) = -S(c, \lambda, v);$$

$$C(c, \lambda, 0) = 1, S(c, \lambda, 0) = 0.$$

Moreover $C(c, 0, v) = \cos(\sqrt{c}v)$, $S(c, 0, v) = \sin(\sqrt{c}v)$; thus Cosine and Sine are particular forms of the functions defined above..

He next by the use of Euler's identity shows that the functions C and S can be represented as infinite products i.e.,

$$C(c, \lambda, \frac{1}{\sqrt{\lambda}}) = (1 - \frac{c/\lambda}{1.2})(1 - \frac{c/\lambda}{3.4}) \ldots\ldots,$$
$$S(c, \lambda, \frac{1}{\sqrt{\lambda}}) = \frac{\sqrt{c}}{\sqrt{\lambda}}(1 - \frac{c/\lambda}{2.3})(1 - \frac{c/\lambda}{4.5}) \ldots\ldots$$

He remarks that for applications of these in seiches, one can without any loss of generality put $\lambda = \pm 1$, and can omit the parameter λ, thus giving

$$C(c,v) = 1 - \frac{c}{(1x2)} v^2 + \frac{c(c - 1x2)}{(1x2)(3x4)} v^4 - \ldots\ldots;$$

$$\text{and } S(c,v) = v - \frac{c}{(2x3)} v^3 + \frac{c(c - 2x3)}{(2x3)(4x5)} v^5 - \ldots\ldots$$

He calls them *seiche-cosine* and *seiche-sine* respectively.

They are connected by the relation

$$C(c,v)S'(c,v) - C'(c,v)S(c,v) = 1,$$

an analogue of the relation $\cos^2\theta + \sin^2\theta = 1$, for the circular functions.

He also defined _the hyperbolic seiche-cosine_ and _the hyperbolic seiche-sine_ by the equations

$$\mathfrak{C}(c,v) = 1 - \frac{c}{(1x2)} v^2 + \frac{c(c + 1x2)}{(1x2)(3x4)} v^4 - \ldots\ldots;$$

$$\text{and } \mathfrak{S}(c,v) = v - \frac{c}{(2x3)} v^3 + \frac{c(c + 2x3)}{(2x3)(4x5)} v^5 - \ldots\ldots$$

as integrals of the differential equation

$$(1 + v^2)\frac{d^2P}{dv^2} + cP = 0.$$

They are connected by the relation

$$\mathfrak{C}(c,v)\mathfrak{S}'(c,v) - \mathfrak{C}'(c,v)\mathfrak{S}(c,v) = 1,$$

which is the analogue of the relation $\cosh^2\theta - \sinh^2\theta = 1$, for hyperbolic functions.

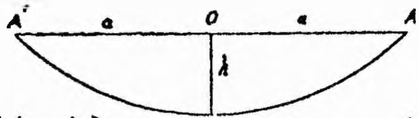
The Seiche Functions introduced by Professor Chrystal did not play as important a role in applied mathematics as he expected them to play. Yet they proved to be very useful in the calculation of periods and nodes for different lake configurations, which he considered in detail in the rest of this mathematical theory.

Professor Chrystal has given the solution for various longitudinal profiles d(x), of which some are given in the following section.

5.3 Main Lake Configurations Considered by Prof. Chrystal and his Introduction of the Lake Function

a. Seiches In A Concave Symmetric Parabolic Lake

For such a lake $d(x) = h(1 - x^2/a^2)$. The equation determining P in this case is $\partial^2P/\partial x^2+[n^2/gd(x)]P=0$; or, if $w=x/a$, this becomes $(1-w^2)d^2P/dw^2+[n^2a^2/gh]P=0$ or simply $(1-w^2)d^2P/dw^2+cP=0$, where $c= n^2a^2/gh$.



In this case $\xi h(1-w^2) = u = \{A\mathrm{C}(c,w)+ B\mathrm{S}(c,w)\}\sin nt$, where A,B are arbitrary constants. Also $\zeta = - \partial u/\partial x = - (1/a)(du/dw)$. Since ξ is finite, we have u=0, when $w=\pm1$, using these boundary conditions, for seiches with odd and even numbers of nodes, seiche displacement equations are given by

$$\xi = \frac{A}{h}\,\frac{\mathrm{C}(c_{2s-1}, w)}{1 - w^2}\sin n_{2s-1}t\,,\ \zeta = -\frac{A}{a}\mathrm{C}'(c_{2s-1}, w)\sin n_{2s-1}t\ ;$$

$$\text{and}\ \ \xi = \frac{B}{h}\,\frac{\mathrm{S}(c_{2s}, w)}{1 - w^2}\sin n_{2s}t\,,\ \zeta = -\frac{B}{a}\mathrm{S}'(c_{2s}, w)\sin n_{2s}t\,.$$

The roots of the equation $\mathrm{C}(c,1)=0$ are $c_1=1\times2$, $c_3=3\times4$, . . . , $c_{2s-1}=(2s-1)2s$, ..., and those of $\mathrm{S}(c,1)=0$ are $c_2=2\times3$, $c_4=4\times5$, . . . , $c_{2s}=2s(2s+1)$, ... We have in general $c_\nu = \nu(\nu + 1)$, and so T_ν the period of the ν-nodal seiche is given by

$$T_\upsilon = \frac{2\pi}{n_\upsilon} = \frac{2\pi a}{\sqrt{(c_\upsilon gh)}} = \frac{\pi l}{\sqrt{\{\upsilon(\upsilon+1)gh\}}},$$

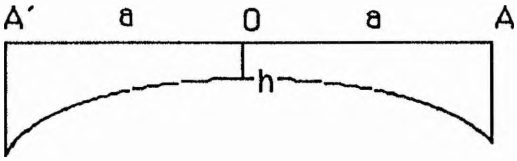
where $l = 2a$ is the total length of the lake.

The Nodes in this case lie at x = 0(uninodal); x= ± 0.5774a(binodal); x = 0, ±0.7746a(trinodal); x=±0.3400a,±0.8621a(quadrinodal); x=0,±0.5384a, ±0.9062a(quibodal), and

$T_1: T_2: T_3: T_4 = 100: 57.7: 40.8: 31.7.$

b. Seiche In a Convex Symmetric Parabolic Lake

In this case $d(x) = h(1+x^2/a^2)$.



According to Professor Chrystal if $c_1, c_2, c_3, \ldots, c_v, \ldots$ be the real positive roots taken in order of magnitude of the equations $\mathfrak{C}(c,1) = 0$, and $\mathfrak{S}(c,1) = 0$, so that c_1 is the smallest positive root of $\mathfrak{C}(c,1) = 0$, c_2 is the smallest positive root of $\mathfrak{S}(c,1) = 0$, and so on, then, for seiches with an odd number of nodes,

$$\xi = \frac{A}{h}\frac{\mathfrak{C}(c_{2s-1}, w)}{1+w^2}\sin n_{2s-1}t, \quad \zeta = -\frac{A}{a}\mathfrak{C}'(c_{2s-1}, w)\sin n_{2s-1}t,$$

and for seiches with an even number of nodes,

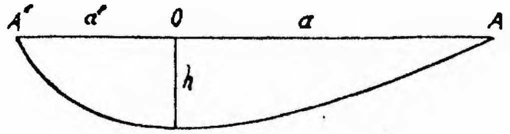
$$\xi = \frac{B}{h}\frac{\mathfrak{S}(c_{2s}, w)}{1+w^2}\sin n_{2s}t, \quad \zeta = -\frac{B}{a}\mathfrak{S}'(c_{2s}, w)\sin n_{2s}t.$$

By using the equation $T_\upsilon/T_1 = \sqrt{(c_1/c_\upsilon)}$, we get

$T_1 : T_2 : T_3 : T_4 = 100 : 47.2 : 31.2 : 23.4$, as given by Dr Halm in his paper,"On a Group of Linear Differential Equations......", p.666, considered later in section 5.4.

The binode lies at about 0.472a.

c. Seiches In a Concave Asymmetric Biparabolic Lake



O is considered as the origin for the two parabolas taken separately with OA=a, OA′=a′. Moving from the deepest point O along OA, the depth function is given by

$d(x) = h\,(1 - x^2/a^2)$, and along OA′ it is given by $d(x) = h\,(1 - x^2/a'^2)$.

If $w = x/a$, $w' = x/a'$; $c = n^2a^2/gh$, $c' = n^2a'^2/gh$, then for the two portions OA and OA′

$$\xi h(1 - w^2) = \frac{A}{S(c,1)}\{S(c,1)C(c,w) - C(c,1)S(c,w)\}\sin nt,$$

$$\zeta = -\frac{A}{aS(c,1)}\{S(c,1)C'(c,w) - C(c,1)S'(c,w)\}\sin nt;$$

and

$$\xi' h(1-w'^2) = \frac{A}{S(c',1)}\{S(c',1)C'(c',w')+C(c',1)S(c',w')\}\sin nt,$$

$$\zeta' = -\frac{A}{a'S(c',1)}\{S(c',1)C'(c',w')+C(c',1)S'(c',w')\}\sin nt,$$

respectively.

The boundary conditions at A and A′ give

$$AC(c,1)+BS(c,1)=0;$$

$$A'C(c',1) - B'S(c',1)=0.$$

The conditions $\xi=\xi'$, $\zeta=\zeta'$ at O give A=A′; B/a =B′/a′.

Thus we have

$$a' C(c,1)\, S(c',1) + a\, C(c',1) S(c,1) = 0,$$

which is the equation by the use of which we can calculate c or c′ as $c/c' = a^2/a'^2$.

If $a'^2 c = a^2 c' = n^2 a^2 a'^2/gh = z$, then the last equation can be written as

$$a\left(1-\frac{z}{1\times2a^2}\right)\left(1-\frac{z}{3\times4a^2}\right)\ldots\left(1-\frac{z}{2\times3a'^2}\right)\left(1-\frac{z}{4\times5a'^2}\right)\ldots$$

$$\ldots + a'\left(1-\frac{z}{1\times2a'^2}\right)\left(1-\frac{z}{3\times4a'^2}\right)\ldots\left(1-\frac{z}{2\times3a^2}\right)\left(1-\frac{z}{4\times5a^2}\right)\ldots = 0$$

The period of ν-nodal seiche is given by

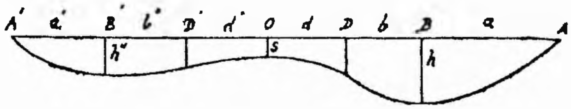
$$T_\nu = 2\pi/n_\nu = 2\pi a a'/\sqrt{(z_\nu g h)}\ ;$$

where z_ν is the corresponding root of the above equation.

d. Seiches In an Unsymmetrical Lake with one Shallow and Two Maximal Depths

In order to have a good approximation to the form of a lake section in many cases that occur in nature, Professor Chrystal pieced together six parabolæ, so as to form a continuous curve.

If s be the minimum, h, h′ the two maximum depths; D, D′ the points of inflexion(the depths at which cannot be arbitrarily assigned); AB = a_1, A′B′ = a'_1, BD = b , B′D′ = b′, OD = d , OD′= d′, then the portions AB, BD, DO, OD′, D′B′, B′A′ were represented by Professor Chrystal by the six parabolæ:-



$d(x) = h(1 - x^2/a_1^2)$; $d(x) = h(1 - x^2/a_2^2)$; $d(x) = s(1 - x^2/a_3^2)$;

$d(x) = s(1 - x^2/a'_3{}^2)$; $d(x) = h'(1 - x^2/a'_2{}^2)$; $d(x) = h'(1 - x^2/a'_1{}^2)$.

The conditions of continuity led him to

$a_2^2 = hb(d+b)/(h-s)$, $a_3^2 = sd(d+b)/(h-s)$;

$a'_2{}^2 = h'b'(d'+b')/(h'-s)$, $a'_3{}^2 = sd'(d'+b')/(h'-s)$.

With the exception of a_2, a_3, a_2', a_3', and the depths at D and D′, the other quanties may be chosen arbitrarily.

The origin for x is in each case the vertex of the corresponding parabola.

The formulæ for ξ, ζ, and the period equation had been worked out by him in this case, which involved all the four seiche functions defined so far.

Introduction Of The Lake Function

In order to give an alternative solution for parabolic lakes, he shifted the origin to the positive end and substituted $w = 1 - 2z$. Thus the equation

$$(1 - w^2)\frac{d^2P}{dw^2} + cP = 0,$$

became

$$(z - z^2)\frac{d^2P}{dz^2} + cP = 0.$$

As a solution to this equation he found

$$P = A_1\left(z - \frac{c}{1^2}\frac{z^2}{2} + \frac{c(c - 1x2)}{1^2.2^2}\frac{z^3}{3} - \frac{c(c - 1x2)(c - 2x3)}{1^2.2^2.3^2}\frac{z^4}{4} + \ldots \right)$$

for $-1 \leq z \leq +1$.

Dropping the multiplicative constants,

$$P = z - c\frac{z^2}{1x2} - c\left(1 - \frac{c}{1x2}\right)\frac{z^3}{2x3} - c\left(1 - \frac{c}{1x2}\right)\left(1 - \frac{c}{2x3}\right)\frac{z^4}{3x4} - \ldots$$

$$= L(c,z).$$

The function so defined was called by Professor Chrystal the *Lake Function;* it is related to the seiche-cosine and the seiche-sine by the relation

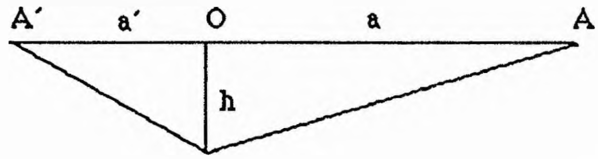
$$2L(c,z) = S(c,1)C(c,1-2z) - C(c,1)S(c,1-2z).$$

In particular

$L(c,1/2) = 1/2\ S(c,1)$, $L'(c,1/2) = C(c,1)$, where dash on L denotes differentiation with respect to z.

The practical advantage of the introduction of the Lake Function, as described by Professor Chrystal, was that it resulted in highly convergent series at points where the series for C(c,w) and S(c,w) converged slowly. However, it was not convenient to use it for truncated parabolic lakes and convex parabolic lakes.

e. Seiches In Rectilinear Lakes, Unsymmetric, Shelving At Both Ends



The laws of depth for the two parts being given by $d(x) = h(1 - x/a)$ and $d(x) = h(1+ x/a')$, if the origin be considered at the junction of the two slopes, where the bottom slopes upwards on both sides of the junction.

In this case $\xi d(x) = u = P \sin nt$, $\zeta = - \partial u/\partial x$, where P is determined by $d^2P/dx^2+ [n^2/gd(x)]P = 0$. If we put $P = Rw$, where $w = 2na\sqrt{(1 - x/a)} \div \sqrt{(gh)}$, the above equation becomes $d^2R/dw^2+(1/w)dR/dw+(1 -1/w^2)R = 0$, which is a particular case of the Bessel equation. If

$$w' = 2na'\sqrt{(1+ x/a')} \div \sqrt{(gh)};\quad \alpha = 2a/\sqrt{(gh)},\ \alpha' = 2a'/ \sqrt{(gh)}.$$

Then for the two portions OA and OA'

$$w\xi = \alpha A\ [J_1(w)/ J_1(n\alpha)] \sin nt,\ \ \zeta = (hA/2)[J_0(w)/J_1(n\alpha)]\sin nt;$$

and $w'\xi' = \alpha' A[J_1(w')/J_1(n\alpha')]\sin nt$, $\zeta' = -(hA/2)[J_0(w')/J_1(n\alpha')]\sin nt$, respectively,

$J_n(w)$ being the Bessel Function and A an arbitrary constant.

The period equation is $J_0(n\alpha)J_1(n\alpha') + J_0(n\alpha')J_1(n\alpha) = 0$.

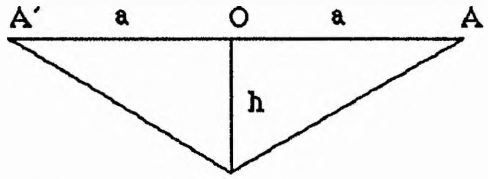
The nodes are given by

$J_0(w) = 0$ in OA; $J_0(w') = 0$ in OA′: where for the ν-nodal seiche $w = n_\nu\alpha$, $w' = n_\nu\alpha'$.

The positive roots of the equation $J_0(z) = 0$ are denoted by j_{2k-1} and those of $J_1(z) = 0$ by j_{2k}, with the exception of j_0 which is equal to zero. So that we have approximately $j_1=2.405$, $j_2=3.832$, $j_3=5.520$, $j_4=7.016$, $j_5=8.654$, $j_6=10.173$ and so on. For large values of n, $j_n \simeq (2n+1)\pi/4$.

f. Seiches In Rectilinear Lakes, Symmetric, Shelving At Both Ends

In this case since $a = a'$; so $\alpha = \alpha'$.



Thus

$w\xi = AJ_1(w)\sin nt$, $\zeta = (hA/2a)\,J_0(w)\sin nt$;

and $w'\xi' = A\,J_1(w')\sin nt$, $\zeta' = -(hA/2a)J_0(w')\sin nt$.

The period equation breaks up into

$J_0(n\alpha) = 0$, $J_1(n\alpha) = 0$.

Hence

$T_\nu = 4\pi a/j_\nu\surd(gh)$,

where j_ν are the positive roots of the equations $J_0(z) = 0$ and $J_1(z) = 0$ as in the previous case. For large values of ν, $T_\nu= 16a/(2\nu+1)\surd(gh)$. When $(1/2\nu)$ is negligible,

$T_\nu= 8a/\nu\surd(gh)= 4l/\nu\surd(gh)$.

Moreover the ratios of the periods for a complete symmetric rectilinear lake is

$T_1{:}T_2{:}T_3{:}T_4{:}T_5{:}T_6{:}T_7{:}T_8$= 100: 62.76: 43.57: 34.28: 27.79: 23.65: 20.40: 18.05.

The nodes are at $x=0$ (uninodal); $x=\pm0.6057a$ (binodal); $x=0, \pm0.8102a$ (trinodal); $x=\pm0.3809a, \pm0.8825a$ (quadrinodal); $x=0, \pm0.5930a, \pm0.9228a$ (quinquinodal).

Professor Chrystal then took the case of other lakes, for which the normal or σ–v-curve was a simple quartic curve; viz: $\sigma=h(1\pm v^2/a^2)^2$. In order to simplify the case without any loss of generality, he assumed the lake to have uniform breadth and rectangular cross section, so that the expression for depth at a distance x from origin became $d(x)=h(a^2\pm x^2)^2$.

He started his investigations by considering a slightly generalised form of an equation used by Professor G. G. Stokes in 1849, in his well known paper on 'Breaking Of Railway Bridges'[17]; viz:-

$(z-a)^2(z-b)^2d^2y/dz^2+cy=0$,

which had the general solution

$y=A(z-a)^m(z-b)^n+B(z-a)^n(z-b)^m$,

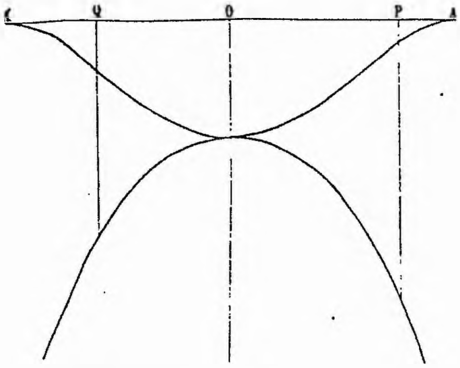
where A, B were arbitrary constants and m, n were the roots of the quadratic

$\rho^2-\rho+c/(a-b)^2= 0$.

He considered the following two cases:-

g. Concave Truncated Quartic Lake

Professor Chrystal in this case took origin at O the deepest point. The length PQ as l, and P, Q corresponding to $x=p$, $x=q$.



The depths at P, Q, and O were r, s, d respectively; so that

$p=a\sqrt{(1-\sqrt{r/d})}$, $q=\mp a\sqrt{(1-\sqrt{s/d})}$; $l=a\{\sqrt{(1-\sqrt{r/d})}\pm\sqrt{(1-\sqrt{s/d})}\}=a\gamma$, say.

The upper signs correspond to the case figured, where P and Q were considered to be on the opposite sides of the origin.

Then considering ξ and ζ as the horizontal and vertical displacements respectively, of the seiche under investigation, he obtained

$$\xi h(a^2-x^2)^2=u=A(a^2-x^2)^{1/2}\sin\{(\nu\pi/k)[\log(a+x/a-x)-\log(a+p/a-p)]\}\sin n_\nu t,\ \zeta=-\partial u/\partial x;$$

and for the period T_ν of the ν-nodal seiche he got

$$T_\nu=2\pi l\div\gamma\sqrt{\{gd[(4\nu^2\pi^2/k^2)+1]\}},$$

where $\gamma=\sqrt{(1-\sqrt{r/d})}\pm\sqrt{(1-\sqrt{s/d})}$;

$$k=\log\{\frac{1+\sqrt{(1-\sqrt{\frac{r}{d}})}}{1-\sqrt{(1-\sqrt{\frac{r}{d}})}}\cdot\frac{1\mp\sqrt{(1-\sqrt{\frac{s}{d}})}}{1\pm\sqrt{(1-\sqrt{\frac{s}{d}})}}\}.$$

This case is extremely interesting theoretically. It corresponds to a lake with very gently shelving ends and a deep concave central depth d. Such a shape approximates to many actual lakes.

When he considered the end barriers P and Q approaching to the infinitely shallow theoretical ends at A and A´, the periods of all seiches became more and more nearly equal to each other and to $\pi l/\sqrt{(gd)}$, which Professor Chrystal called the period of anomalous seiche.

Although the 'anomalous seiche' as he called it can not in general exist because in practice it is not possible, but it might happen that the lakes were connected at the ends P and Q with canals or other lakes.

According to Professor Chrystal

"Apart from the anomalous seiche, the possibility of which arises from the fact that a concave quartic lake approaches the shores of its theoretic end with an infinitely small gradient, the results obtained for concave quartic lakes repeat the main features of the seiche phenomena as calculated for parabolic lakes".

h. Convex Quartic Lakes

The symbols used in this case are the same as defined in the case of concave quartic lakes, the only difference being that

$p = a\sqrt{[\sqrt{(r/d)} - 1]}$, $q = \mp a\sqrt{[\sqrt{(s/d)} - 1]}$;

$l = a\{\sqrt{[\sqrt{(r/d)} - 1]} \pm \sqrt{[\sqrt{(s/d)} - 1]}\} = a\gamma$, say;

and ξ and ζ are given by

$\xi h(a^2+x^2)^2 = u = (a^2+x^2)^{1/2}\sin\{(2\nu\pi/k)[\tan^{-1}(x/a) - \tan^{-1}(p/a)]\}\sin n_\nu t$, $\zeta = -\partial u/\partial x$.

If T_ν be the period of the ν-nodal seiche in this case then

$T_\nu = 2\pi l/\gamma\sqrt{\{gd[(4\nu^2\pi^2/k^2) - 1]\}}$;

where $\gamma = \{\sqrt{[\sqrt{(r/d)} - 1]} \pm \sqrt{[\sqrt{(s/d)} - 1]}\}$;

$k = 2\tan^{-1}\{\sqrt{[\sqrt{(r/d)} - 1]} - 2\tan^{-1}\sqrt{[\sqrt{(s/d)} - 1]}\}$.

The great value of Chrystal's investigations is that they permit certain general principles to be deduced. A shallow near a node may render a seiche unstable, however, or prevent its occurrence entirely. Concave profiles tend to reduce the difference between the periods of seiches of different nodalities, convex profiles tend to increase them. The particular combination of concavity and convexities discussed here, in which the periods approach each other, gives of course the least difference. Since convex basins are rarer than concave, and since many lakes are likely to have littoral shelves at least at their upper ends, roughly simulating the quartic form of truncated concave quartic lake, it is not surprising that $T_2 \geq (1/2)T_1$ in the majority of lakes.

It is worth noting that earlier investigations of seiches often recorded a seiche of shorter period than that of the uninodal, but of longer period than half that of the latter. Wrongly supposing that in all cases $T_2 = (1/2)T_1$. A. F. Forel considered such seiches to be analogous to the fifth in musical Theory and termed them seiche à la quinte. Professor Chrystal has this to say:-

"The *seiche à la quinte* of which Forel speaks in the cases of Constance, Garda, and Starnberg is in all probability simply the binodal seiche; and the seiche whose period is approximately half the longest period is a trinodal".

Professor Chrystal considered in detail a variety of lake configurations, in order to construct a theoretical curve to represent the normal curve of any lake deduced from bathymetrical data; This could be done by combining pieces of parabolas, straight lines or quartics at will; and the variety of formulæ given by him were sufficient for most practical purposes. Even though the computation involved, even in simple cases, was not small. His method has not been applied often in practice; yet whenever it has been used it has given very satisfactory results. For example, F. Bergsten in 1926 in his work on the seiches of Lake Vättern(Sweden)[18]which is the most careful and thorough analysis of the seiches of a great lake, divided the lake into four parts, and was able to approximate the normal curve very well, by means of parabolic and quartic curves and then apply Chrystal's method for the determinations.

F. Defant in 1953 computed the seiches of Lake Michigan[19], applying methods by G. Chrystal(1905), A. Defant(1918), and Ertel(1933). The results obtained were in good agreement, so that all these methods seem to be nearly equivalent. The results can be found in the following table.

Table. Observed and computed values of the periods of Lake Vättern(taken in minutes), and Lake Michigan (taken in hours).

	Lake Vättern		Lake Michigan		
	observed	computed	computed		
		(1)	(1)	(2)	(3)
T_1	178.99	177.94	9.04	9.08	9.02
T_2	97.52	95.95	4.87	4.92	4.69
T_3	80.74	79.17	–	–	–
T_4	57.89	59.83	–	–	–
T_5	48.10	49.77	–	–	–
T_6	42.59	42.46	–	–	–

(1) According to Chrystal's Method.
(2) According to Defant's Method.
(3) According to Ertel's Method.

The values in the above table have been taken from the book by A. Defant [Physical Oceanography, Vol. II, p.183, Pergamon (London) 1961], where he has wrongly given the unit for the periods of Lake Michigan in minutes instead of in hours.

5.4 Mathematical Analysis Of Professor Chrystal's Seiche Equations Carried out By Dr J. Halm

Dr J. Halm who was lecturer in Astronomy in the University of Edinburgh had taken the task of analysing Professor Chrystal's seiche equations in his paper " A Group Of Linear Differential Equations Of Second Order,"[20] published in the Transactions of the Royal Society of Edinburgh. In this paper he had obtained a number of interesting results relating Chrystal's seiche equations with Hypergeometric and Legendre Functions and had justified the introduction of Chrystal's Seiche-Functions for different purposes. He had also given tables for calculating values of Seiche-Functions, and graphs of these functions.

In the first place Dr Halm had shown that the two differential equations

$(1 - w^2)d^2y/dw^2 + n(n - 1)y = 0$ . . . . . (1),

which is Professor Chrystal's seiche equation, and

$(1+x^2)^2 d^2y/dx^2 + n(n+2)y = 0$ . . . . (2),

which is Stokes equation both are special cases of a more general equation

$(1+w^2)d^2y/dw^2 - (2a+1)dy/dw + n(n+2a)y = 0$ . . . . (3).

This in turn could be obtained by a number of different substitutions from the hypergeometric differential equation

$v(1 - v)d^2y/dv^2 + [\gamma - (\alpha+\beta+1)]dy/dv - \alpha\beta y = 0$ . . . . (4),

whose solution is given by the contour integral

$$y = \text{const.} \int_c u^{\alpha-\gamma}(1 - u)^{\gamma-\beta-1}(u - v)^{-\alpha}du \quad . \; . \; . \; . \quad (5).$$

where c is any closed contour in the u-plane such that the integrand resumes its initial value after u has described it.

Halm then deduced that by the introduction of $u = (1 - t)/2$, under the integral sign in (5), equation (3) is satisfied by the integral

$$y = \text{const.} \int_c \frac{(1 - t^2)^{n+a-\frac{1}{2}}}{(t - w)^{n+2a}} dt \quad . \; . \; . \; . \quad (6).$$

Next he wrote

$y = (1 - w^2)^{(1/2)-a}Y$, and found that Y is a solution of

$(1 - w^2)d^2Y/dw^2 + (2a - 3)dY/dw + (n+1)(n+2a - 1)Y = 0.$

He then substituted $\alpha = n+1$, $\beta = 1 - n - 2a$, $\gamma = (3/2) - a$ in (4) and (5) and obtained

$$Y = \text{const.} \int_c \frac{(1 - t^2)^{n+a-1/2}}{(t - w)^{n+1}} dt \ ,$$

which gave him

$$y = \text{const.}\ (1 - w^2)^{1/2 - a} \int_c \frac{(1 - t^2)^{n+a-1/2}}{(t - w)^{n+1}} dt \ . \ . \ . \ . \quad (7),$$

as another solution of equation (3).

Dr Halm noticed that for a = 1/2, the integrals (6) and (7) become identical, viz.,

$$y = \text{const.} \int_c \frac{(1 - t^2)^n}{(t - w)^{n+1}} dt,$$

which in fact is the Schläfli's contour integral of the Legendre function, and where c is a contour which encircles the point w once counter-clockwise, and also encloses the points $t = \pm 1$.So he concluded that equation (3) also involves Legendre functions.

Dr Halm, after making substitution $v = w^2$, $\alpha = n/2+a$, $\beta = -n/2$, $\gamma = 1/2$, and using Jacobi's Schematic Table[21] of particular solutions for hypergeometric differential equation, obtained twenty-four particular integrals of (3), of which

$Cos_a(w) = F(n/2+a, -n/2, 1/2; w^2)$

corresponded to Chrystal's seiche-cosine for a = -1/2; and

$Sin_a(w) = wF(1/2+n/2+a, 1/2 - n/2, 3/2; w^2)$

corresponded to Chrystal's seiche-sine for a = -1/2. From these he was able to obtain corresponding expressions for hyperbolic seiche-functions by replacing n by $n\iota - a$ and w by $w\iota$, where ι is the imaginary unit.

He had then suggested an easy representation for the solution of (3) as $C_n^a(w)$, so that seiche-functions could be represented by $C_n^{-1/2}(w)$, the Legendre functions by $C_n^{1/2}(w)$, and the Stokes functions by $C_n^1(w)$.

He further added that if n was a positive integer, the functions $C_n^a(w)$ were the coefficients of the powers h^n in the series

$$(1 - 2hw + h^2)^{-a} = \sum_0^\infty h^n C_n^a(w).$$

As from (6) and (7)

$$C_n^a(w) = \text{const.}\int_c \frac{(1 - t^2)^{n+a-1/2}}{(t - w)^{n+2a}}\,dt = \text{const.}(1 - w^2)^{1/2-a}\int_c \frac{(1 - t^2)^{n+a-1/2}}{(t - w)^{n+1}}\,dt,$$

according to Cauchy's integral formula

$$\frac{d^n f(w)}{dw^n} = \frac{n!}{2\pi\iota}\int_C \frac{f(t)dt}{(t - w)^{n+1}},$$

the function $C_n^a(w)$ becomes proportional to

$$\frac{d^{n+2a-1}}{dw^{n+2a-1}}\left\{(1 - w^2)^{n+a-1/2}\right\}$$

and $$(1 - w^2)^{1/2-a}\frac{d^n}{dw^n}\left\{(1 - w^2)^{n+a-1/2}\right\}.$$

This gives the following relations :

$$C_n^a(w) = (-2)^n \frac{a(a+1)(a+2)\ .\ .\ .\ .\ (a+n-1)}{1.2.3\ .\ .\ .\ .\ (2a+2n-1)}(-1)^{a-1/2}\frac{d^{n+2a-1}}{dw^{n+2a-1}}\left\{(1-w^2)^{n+a-1/2}\right\}$$

$$= (-2)^n \frac{a(a+1)(a+2)\ .\ .\ .\ .\ (a+n-1)}{n!(2n+2a-1)(2n+2a-2)\ .\ .\ .\ .\ (n+2a)}(1-w^2)^{1/2-a}\frac{d^n}{dw^n}\left\{(1-w^2)^{n+a-1/2}\right\}.$$

. . . . . (8).

Dr Halm then had concluded that relations (8) were of particular interest in the seiche theory because they led to elegant expressions for ξ and ζ.

For example, in the case of parabolic concave lakes, using Professor Chrystal's notations, these become

$(-)^{n-1}2.4.\ .\ .\ .\ (2n-2)(\xi/\sin n_\nu t) = -\{2A/h(1-w^2)\}d^{n-2}\{(1-w^2)\}/dw^{n-2}$

$= \{2A/hn(n-1)\}d^n\{(1-w^2)^{n-1}\}/dw^n,$

$(-)^{n-1}\, 2.4. \;.\;.\;.\; (2n-2)(\zeta/\sin n_\nu t) = (2A/a)d^{n-1}\{(1-w^2)^{n-1}\}/dw^{n-1},$

where for the uninodal seiche n =2, $\nu = 1$; for the binodal seiche n = 3, $\nu = 2$; and so on.

Dr J. Halm showed that the positions of the nodes could be represented in a convenient graphical form, which not only showed clearly their dependence on the curvature of the lake, but at the same time enabled him to find the nodes for the curves lying between those which were considered by Professor Chrystal and those, which, according to him were not amenable to direct analytical treatment. In fig.1. Dr Halm showed the halves of the vertical longitudinal sections of symmetric lakes. In this OB represented a , the half-length, and OA the central depth, h , of the lake, where AB, AC, AD, and AE signified the intersections of the vertical plane with the concave-parabolic, the plane-horizontal, the convex-parabolic, and the convex-quartic floors. Then, on each of those curves the nodes had been marked by the points B_2, B_3, B_4, etc., in such a way that for instance the distance of B_2 from AO agreed with the value of w referring to the binodal seiche in a lake with concave parabolic floor,i.e. w =x/a= 0.577.

In the same way E_2 was drawn at a distance 0.447 from AO, thus representing the position of the binode in a convex-quartic lake. Having secured the corresponding four points on each of the curves AB, AC, AD, and AE, he drew the curved lines B_2E_2, B_3E_3, and B_4E_4, and those lines were obviously the loci of the nodes. He realised in all cases the displacements of the nodal points towards the shallow water, a phenomenon specially marked in concave lakes. He then considered, as an illustration, the case of a convex lake with depth represented by $h\sqrt{(1+w^2)}$, it being represented in the diagram by the dotted curve AF. He then asserted that without knowing the solutions of the corresponding differential equation, and without computing the nodes and periods of the particular seiches by analytical methods, the nodes could be approximated directly from the diagram, being represented by the points of intersection between the loci BE and the curve AF.

Dr Halm at the request of Professor Chrystal subjoined tables from which the numerical values of $C(c,1)$ and $S(c,1)$ could be taken for any values of c. He had written

$a = 1/4 + (1/2)\sqrt{(c+1/4)}$, $\Theta(a) = \{\Gamma(a)/\Gamma(a+1/2)\} \sin(a\pi)/\sqrt{\pi}$,

$\Sigma(a) = -\{\Gamma(a+1/2)/\Gamma(a)\} \cos(a\pi)/\sqrt{\pi}$,

and found from the above formulæ the following relations:

$0<a<1$: $C(c,1) = -\{(a+1/2)\div a\}\Theta(a+1)$; $c/2\ S(c,1) = -\{a\div(a+1/2)\}\Sigma(a+1)$;

$1<a<2$: $C(c,1) = \Theta(a)$; $c/2\ S(c,1) = \Sigma(a)$;

$2<a<3$: $C(c,1) = -\{(a-1)\div(a-1/2)\}\Theta(a-1)$; $c/2\ S(c,1) = -\{(a-1/2)\div(a-1)\}\Sigma(a-1)$;

$3<a<4$: $C(c,1) = \{(a-2)(a-1)\div(a-3/2)(a-1/2)\}\Theta(a-2)$;

$c/2\ S(c,1) = \{(a-3/2)(a-1/2)\div(a-2)(a-1)\}\Sigma(a-2)$;

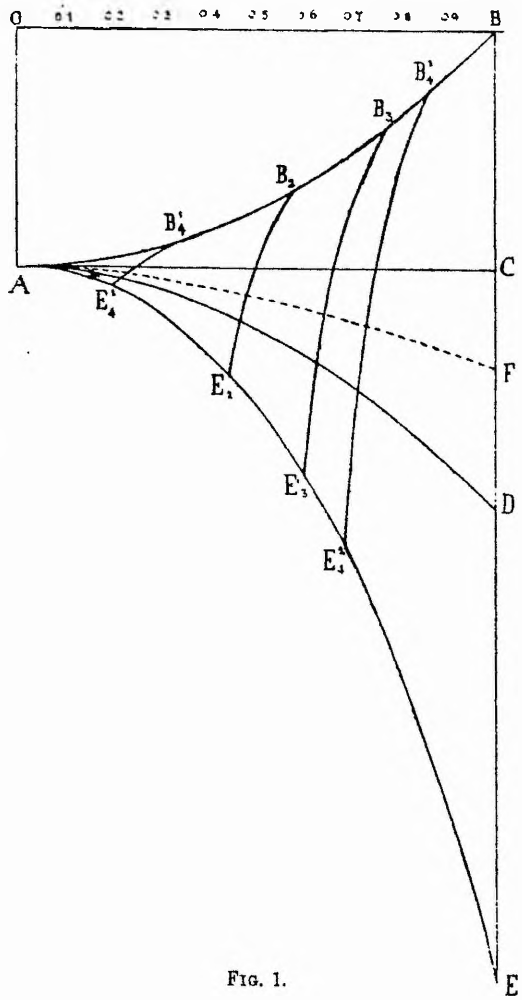
. . . .

where the values of Θ and Σ were to be taken from Table I.

He had then given a table containing numerical values of the two seiche functions directly obtained for values of c between 0 and 30.0 as in Table II. The intervals chosen were sufficiently close to permit an easy interpolation of these functions for the intermediate values of c. The table could be extended by the use of Table I, and the preceding formulæ. Dr Halm also managed to prepare a table containing numerical values of the two Hyperbolic seiche-functions for values of c between 0 and 30.0 as given in Table III.

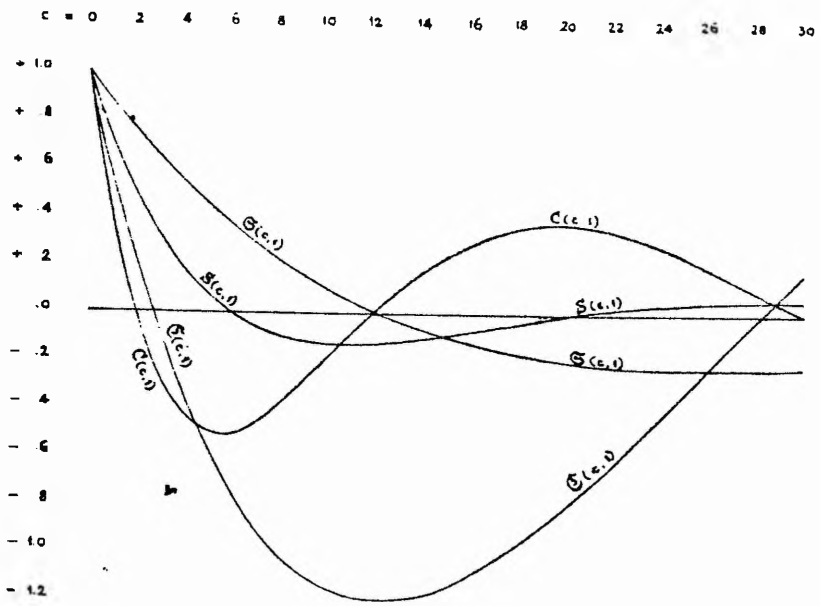
To exhibit, more correctly, the character of the four seiche functions, Dr Halm had given their graphs as given in Fig.2.

The work done by Dr. Halm in this paper made the calculation of the roots of the



O
B
B'4
B3
B2
B'4
A
C
E'4
F
E1
E3
D
E'3
E

Fig. 1.



c = 0
2
4
6
8
10
12
14
16
18
20
22
24
26
28
30
+ 1.0
- 1.0
- 1.2
C(c,1)
S(c,1)

Fig. 2.

Table I

a	Θ(a)	a	Θ(a)	a	Θ(a)	a	Θ(a)	a	Θ(a)
1.00	0.00000	1.20	-0.33511	1.40	-0.49501	1.60	-0.45815	1.80	-0.26474
1.01	-0.01988	1.21	-0.34769	1.41	-0.49775	1.61	-0.45163	1.81	-0.25237
1.02	-0.03949	1.22	-0.35984	1.42	-0.49999	1.62	-0.44471	1.82	-0.23983
1.03	-0.05884	1.23	-0.37152	1.43	-0.50171	1.63	-0.43741	1.83	-0.22712
1.04	-0.07790	1.24	-0.38271	1.44	-0.50294	1.64	-0.42975	1.84	-0.21430
1.05	-0.09666	1.25	-0.39346	1.45	-0.50366	1.65	-0.42171	1.85	-0.20133
1.06	-0.11512	1.26	-0.40370	1.46	-0.50391	1.66	-0.41332	1.86	-0.18824
1.07	-0.13326	1.27	-0.41347	1.47	-0.50365	1.67	-0.40457	1.87	-0.17505
1.08	-0.15106	1.28	-0.42275	1.48	-0.50292	1.68	-0.39552	1.88	-0.16177
1.09	-0.16852	1.29	-0.43153	1.49	-0.50171	1.69	-0.38612	1.89	-0.14842
1.10	-0.18563	1.30	-0.43982	1.50	-0.50000	1.70	-0.37643	1.90	-0.13499
1.11	-0.20237	1.31	-0.44762	1.51	-0.49785	1.71	-0.36641	1.91	-0.12151
1.12	-0.21874	1.32	-0.45490	1.52	-0.49520	1.72	-0.35612	1.92	-0.10800
1.13	-0.23473	1.33	-0.46170	1.53	-0.49211	1.73	-0.34555	1.93	-0.09445
1.14	-0.25032	1.34	-0.46798	1.54	-0.48858	1.74	-0.33472	1.94	-0.08090
1.15	-0.26551	1.35	-0.47375	1.55	-0.48458	1.75	-0.32362	1.95	-0.06734
1.16	-0.28029	1.36	-0.47902	1.56	-0.48014	1.76	-0.31228	1.96	-0.05380
1.17	-0.29464	1.37	-0.48378	1.57	-0.47527	1.77	-0.30071	1.97	-0.04028
1.18	-0.30855	1.38	-0.48804	1.58	-0.46998	1.78	-0.28892	1.98	-0.02680
1.19	-0.32206	1.39	-0.49177	1.59	-0.46426	1.79	-0.27692	1.99	-0.01337

a	Σ(a)	a	Σ(a)	a	Σ(a)	a	Σ(a)	a	Σ(a)
1.00	+0.50000	1.20	+0.45171	1.40	+0.18899	1.60	-0.20419	1.80	-0.57178
1.01	+0.50281	1.21+44336		1.41	+0.17133	1.61	-0.22463	1.81	-0.58638
1.02	+0.50510	1.22	+0.43447	1.42	+0.15336	1.62	-0.24499	1.82	-0.60047
1.03	+0.50687	1.23	+0.42503	1.43	+0.13507	1.63	-0.26525	1.83	-0.61406
1.04	+0.50811	1.24	+0.41505	1.44	+0.11649	1.64	-0.28536	1.84	-0.62708
1.05	+0.50880	1.25	+0.40453	1.45	+0.09765	1.65	-0.30533	1.85	-0.63956
1.06	+0.50894	1.26	+0.39347	1.46	+0.07855	1.66	-0.32513	1.86	-0.65148
1.07	+0.50853	1.27	+0.38190	1.47	+0.05921	1.67	-0.34473	1.87	-0.66279
1.08	+0.50758	1.28	+0.36982	1.48	+0.03966	1.68	-0.36410	1.88	-0.67348
1.09	+0.50604	1.29	+0.35723	1.49	+0.01992	1.69	-0.38325	1.89	-0.68355
1.10	+0.50396	1.30	+0.34416	1.50	0.00000	1.70	-0.40213	1.90	-0.69303
1.11	+0.50130	1.31	+0.33060	1.51	-0.02007	1.71	-0.42072	1.91	-0.70183
1.12	+0.49807	1.32	+0.31657	1.52	-0.04028	1.72	-0.43901	1.92	-0.70998
1.13	+0.49428	1.33	+0.30207	1.53	-0.06060	1.73	-0.45697	1.93	-0.71744
1.14	+0.48991	1.34	+0.28716	1.54	-0.08101	1.74	-0.47456	1.94	-0.72423
1.15	+0.48496	1.35	+0.27179	1.55	-0.10149	1.75	-0.49181	1.95	-0.73032
1.16	+0.47946	1.36	+0.25601	1.56	-0.12203	1.76	-0.50866	1.96	-0.73570
1.17	+0.47335	1.37	+0.23982	1.57	-0.14259	1.77	-0.52511	1.97	-0.74037
1.18	+0.46671	1.38	+0.22324	1.58	-0.16315	1.78	-0.54111	1.98	-0.74432
1.19	+0.45948	1.39	+0.20630	1.59	-0.18369	1.79	-0.55667	1.99	-0.74755

Table II

c	C(c,1)	S(c,1)	c	C(c,1)	S(c,1)
0.0	+1.00000	+1.00000	5.6	-0.50392	+0.02851
0.2	+0.86568	+0.94030	5.8	-0.50292	+0.01375
0.4	+0.74008	+0.88216	6.0	-0.50000	0.00000
0.6	+0.62247	+0.82647	7.0	-0.46052	-0.05615
0.8	+0.51257	+0.77390	8.0	-0.38976	-0.09397
1.0	+0.41034	+0.72302	9.0	-0.29986	-0.11695
1.2	+0.31593	+0.67440	10.0	-0.20033	-0.12809
1.4	+0.22694	+0.62778	11.0	-0.09837	-0.13006
1.6	+0.14511	+0.58331	12.0	-0.00000	-0.12500
1.8	+0.06956	+0.54071	13.0	+0.09082	-0.11479
2.0	0.00000	+0.50000	14.0	+0.17106	-0.10101
2.2	-0.06385	+0.46111	15.0	+0.23802	-0.08524
2.4	-0.12221	+0.42406	16.0	+0.29323	-0.06753
2.6	-0.17561	+0.38864	17.0	+0.33377	-0.04971
2.8	-0.22395	+0.35492	18.0	+0.36035	-0.03222
3.0	-0.26758	+0.32282	19.0	+0.37382	-0.01550
3.2	-0.30673	+0.29227	20.0	+0.37500	0.00000
3.4	-0.34177	+0.26317	21.0	+0.36508	+0.01404
3.6	-0.37273	+0.23555	22.0	+0.34531	+0.02640
3.8	-0.39987	+0.20933	23.0	+0.31710	+0.03698
4.0	-0.42343	+0.18442	24.0	+0.28192	+0.04572
4.2	-0.44360	+0.16083	25.0	+0.24103	+0.05266
4.4	-0.46051	+0.13853	26.0	+0.19611	+0.05779
4.6	-0.47438	+0.11741	27.0	+0.14830	+0.06123
4.8	-0.48542	+0.09744	28.0	+0.09889	+0.06308
5.0	-0.49366	+0.07865	29.0	+0.04919	+0.06345
5.2	-0.49941	+0.06088	30.0	0.00000	+0.06250
5.4	-0.50277	+0.04422			

Table III

c	$\mathfrak{C}$(c,1)	$\mathfrak{S}$(c,1)	c	$\mathfrak{C}$(c,1)	$\mathfrak{S}$(c,1)
0.0	+1.0000	+1.0000	13.0	-1.1946	-0.0259
0.5	+0.7877	+0.9353	14.0	-1.1741	-0.0619
1.0	+0.5894	+0.8731	15.0	-1.1389	-0.0932
1.5	+0.4047	+0.8133	16.0	-1.0908	-0.1208
2.0	+0.2328	+0.7557	17.0	-1.0314	-0.1446
2.5	+0.0730	+0.7008	18.0	-0.9626	-0.1648
3.0	-0.0748	+0.6480	19.0	-0.8854	-0.1817
3.5	-0.2111	+0.5974	20.0	-0.8019	-0.1956
4.0	-0.3368	+0.5489	21.0	-0.7126	-0.2067
4.5	-0.4519	+0.5024	22.0	-0.6192	-0.2153
5.0	-0.5568	+0.4580	23.0	-0.5224	-0.2214
6.0	-0.7388	+0.3747	24.0	-0.4235	-0.2255
7.0	-0.8856	+0.2989	25.0	-0.3229	-0.2275
8.0	-1.0007	+0.2299	26.0	-0.2225	-0.2277
9.0	-1.0868	+0.1673	27.0	-0.1219	-0.2264
10.0	-1.1468	+0.1108	28.0	-0.0226	-0.2234
11.0	-1.1831	+0.0600	29.0	+0.0755	-0.2192
12.0	-1.1984	+0.0146	30.0	+0.1707	-0.2137

equations $\mathrm{C}(c,1)=0$, $\mathrm{S}(c,1)=0$, $\mathfrak{C}(c,1)=0$, $\mathfrak{S}(c,1)=0$ much easier; these are the main ingredient in calculation of nodes and periods for seiches in lakes by Chrystal's method. It also to some extent fulfilled the desire of Professor Chrystal, who wanted his seiche functions to play a vital role in mathematics. Though it did not happen, yet this paper provided some significance of these functions.

5.5 Calculation Of The Periods And Nodes Of Lochs Earn And Treig

In a memoir " On The Calculation Of Periods And Nodes Of Lochs Earn And Treig, From The Bathymetrical Data Of The Scottish Lake Survey,"[22] Mr E. M. Wedderburn and Professor Chrystal applied the hydrodynamical theory to calculate the seiche constants for the three seiches of lowest nodality in Lochs Earn and Treig. This being the first attempt to solve completely a problem of the kind, the lakes selected for the purpose had as simple a configuration as possible.

For the purpose Professor Chrystal had made use of the Bathymetric Data of the Lake Survey, which was the result of a series of observations made on Treig in October, November 1904 under the superintendence of Mr E. MacLagan-Wedderburn, and on Earn in June 1905, by Mr James Murray under the superintendence of Professor Chrystal. The observations on Loch Treig had been brought to an untimely conclusion by the partial destruction of the instrument during a storm.

As a first step they determined the normal curves of the two lakes as given in figures 3 and 4.

The calculated values for the lowest three periods of Lochs Earn and Treig were found to be $T_1=14.50'$, $T_2=8.14'$, $T_3=5.74'$; and $T_1=9.14'$, $T_2=5.10'$, $T_3=3.59'$ respectively.

To calculate the the position of the nodes for these periods, they found it more convenient for nodes near the deepest parts of the lakes to use the formulæ:

Fig. 3. NORMAL CURVE FOR EARN.

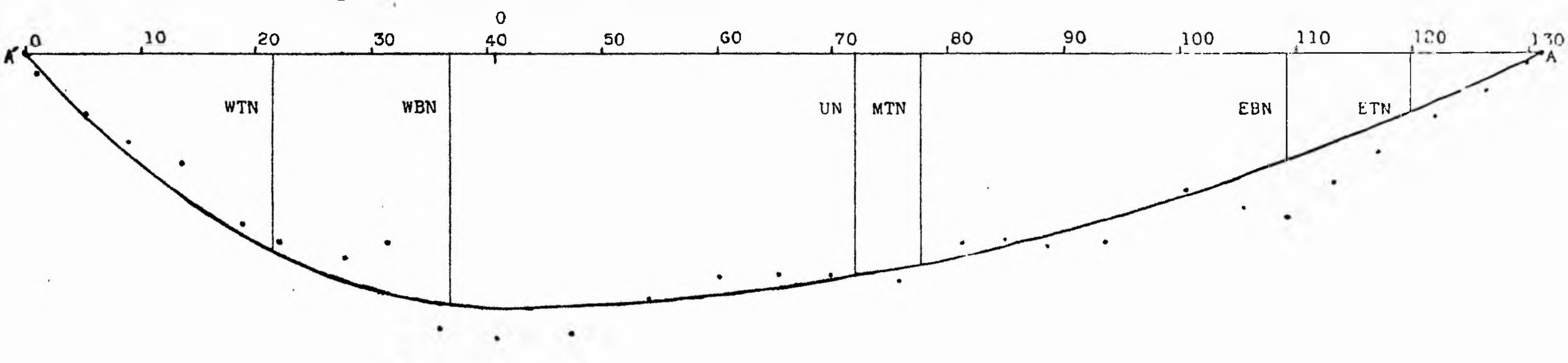
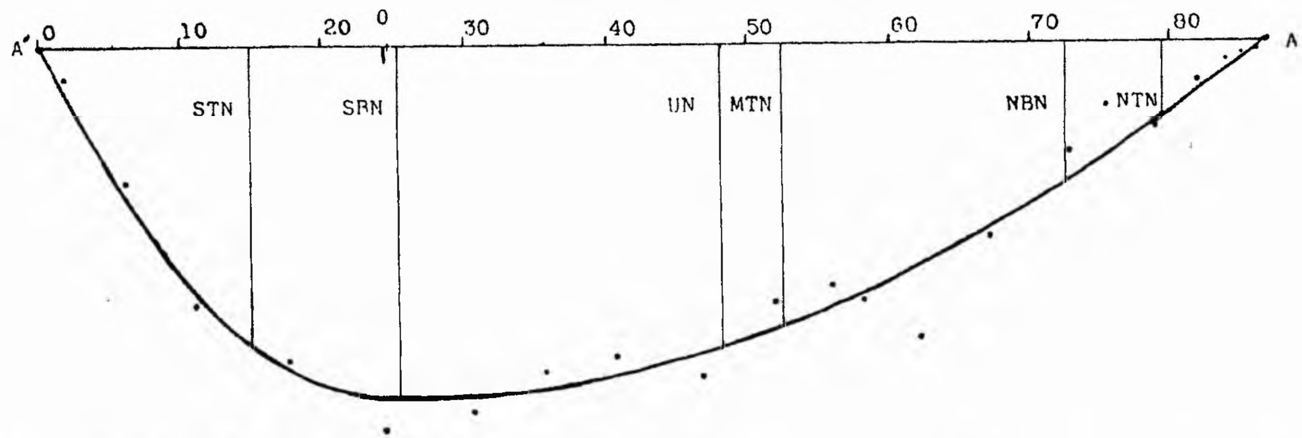


Fig. 4. NORMAL CURVE FOR TREIG.



$\phi(w) = S'(c,w)\,K(c,1) - C'(c,w) = 0,$

$\phi(w) = S'(c',w)\,K(c',1) - C'(c',w) = 0\,,$

where w denoted the scalar values of v/a or v/a′ in all cases; $K(c,1) = C(c,1)+S(c,1)$; c, c′ were calculated from the equation

$\chi(c)= K(c,1)+\rho K(c',1) = 0\,,\ \rho=a/a'.$

And for nodes near the ends the formulæ:

$L'(c,z) = 0,\ L(c',z) = 0$, where $z = (1-w)/2$, and c,c′ are as before.

5.6 Investigation Of Seiches Of Loch Earn

In 1905 Professor Chrystal organised an investigation of the seiches of Loch Earn, being accompanied by observations for comparison on Lochs Tay and Lubnaig. He himself supervised the entire work and had the valuable assistance of Messrs P. White and W. Watson, two students of Professor Macgregor, successor of Prof. P. G. Tait, in the Chair of Natural Philosophy, Edinburgh University, during 1901-1913. To finance this work, he had in addition to the funds of the Lake Survey, a small grant from the Government Fund for Scientific Research. He also had the advice and assistance of his old friend from Cambridge, Mr W. N. Shaw, Director of the Meteorological Office, London, which were of great help on the meteorological side of this enterprise.

The next paper of Professor Chrystal, " An Investigation Of The Seiches Of Loch Earn By The Scottish Lake Survey,"[23] divided into five parts, contains the detailed report of that work, the first two parts of which were published together and the other three together.

5.6.1 Limnographic Apparatus Used In The Survey Of Lochs Earn, Tay, And Lubnaig

The first two parts of this paper dealt with the detail of the instruments used for different observation. According to Professor Chrystal one of the simplest and most effective of the instruments for measuring denivellation of a lake was the index limnograph, originally devised by Endrös. According to Professor Chrystal with this an observation was possible every half-minute, and a corresponding dot was made on the recording paper. Through those dots was drawn a curve called a " limnogram".

For many purposes it was desirable to have a continuous record, extending over a considerable time, for both day and night. For this purpose Professor Chrystal designed a special instrument , which, was called the "Waggon Recorder" as reproduced in fig.5.

According to the description given by Professor Chrystal it was a combination of the essential principles of the older limnographs of Plantamour and Sarasin. The string of the index limnograph was replaced by a steel tape, which passed horizontally over two pulleys, between which it dragged backwards and forwards a little waggon carefully mounted by means of three wheels, which ran two on one and one on another of two parallel rails. The waggon carried an ordinary stylographic pen, so mounted as to write on a long strip of paper which was moved horizontally by rollers driven by clockwork. As the motion of the paper was perpendicular to the motion of the pen, caused by the rise and fall of water, the result was the same as before, only the work and the patience were transferred from living observer to the waggon and the clock, and the record was absolutely continuous.

Figs. 6 and 7 give an idea how the instrument was mounted by the lake side, in order to resist the combined efforts of rain, wind, and waves to make an end of the observations of the limnographer. These are reproductions from the original photos given in this paper by Professor Chrystal.

According to Professor Chrystal, the precautions taken were by no means unnecessary,

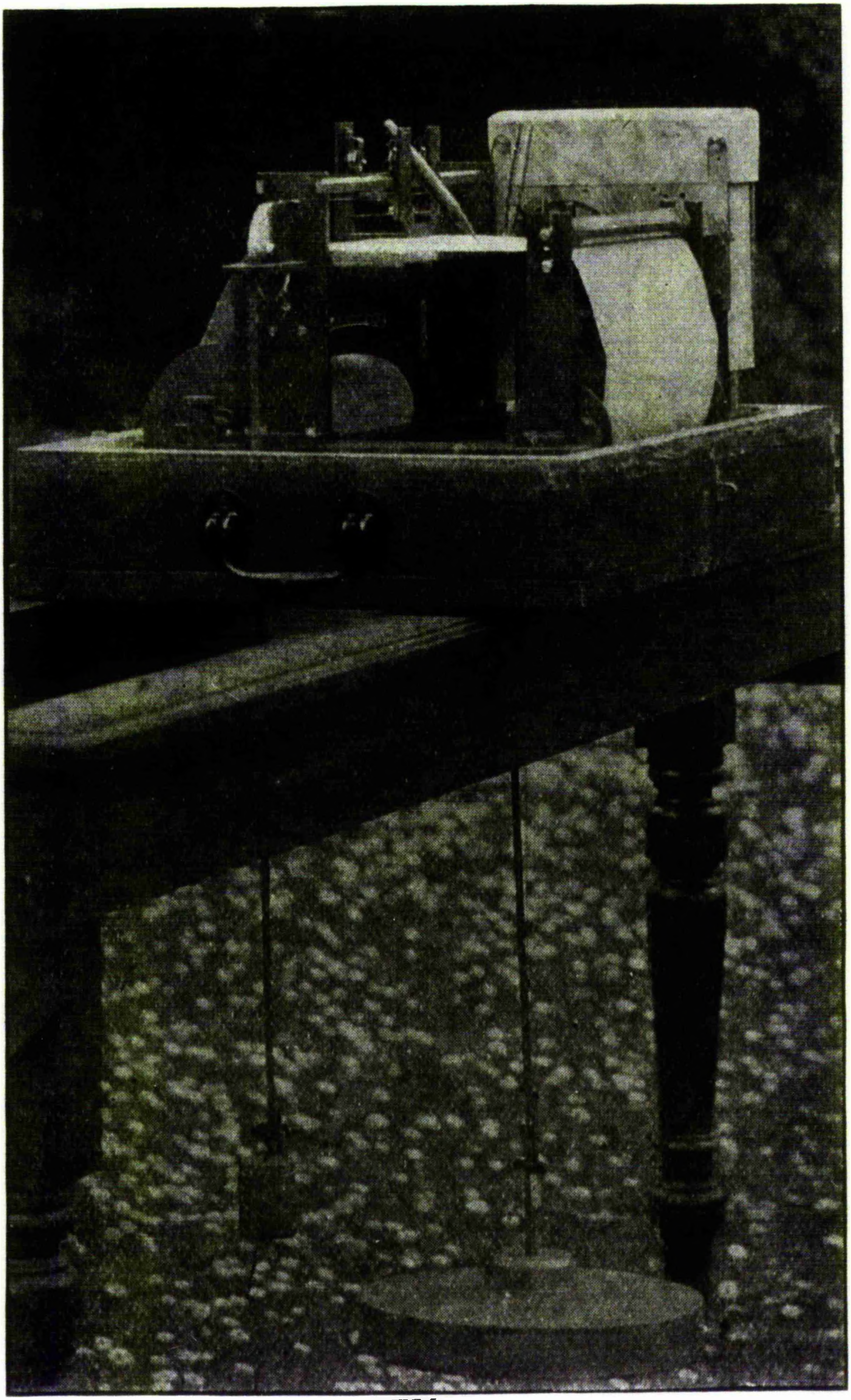


FIG.5.

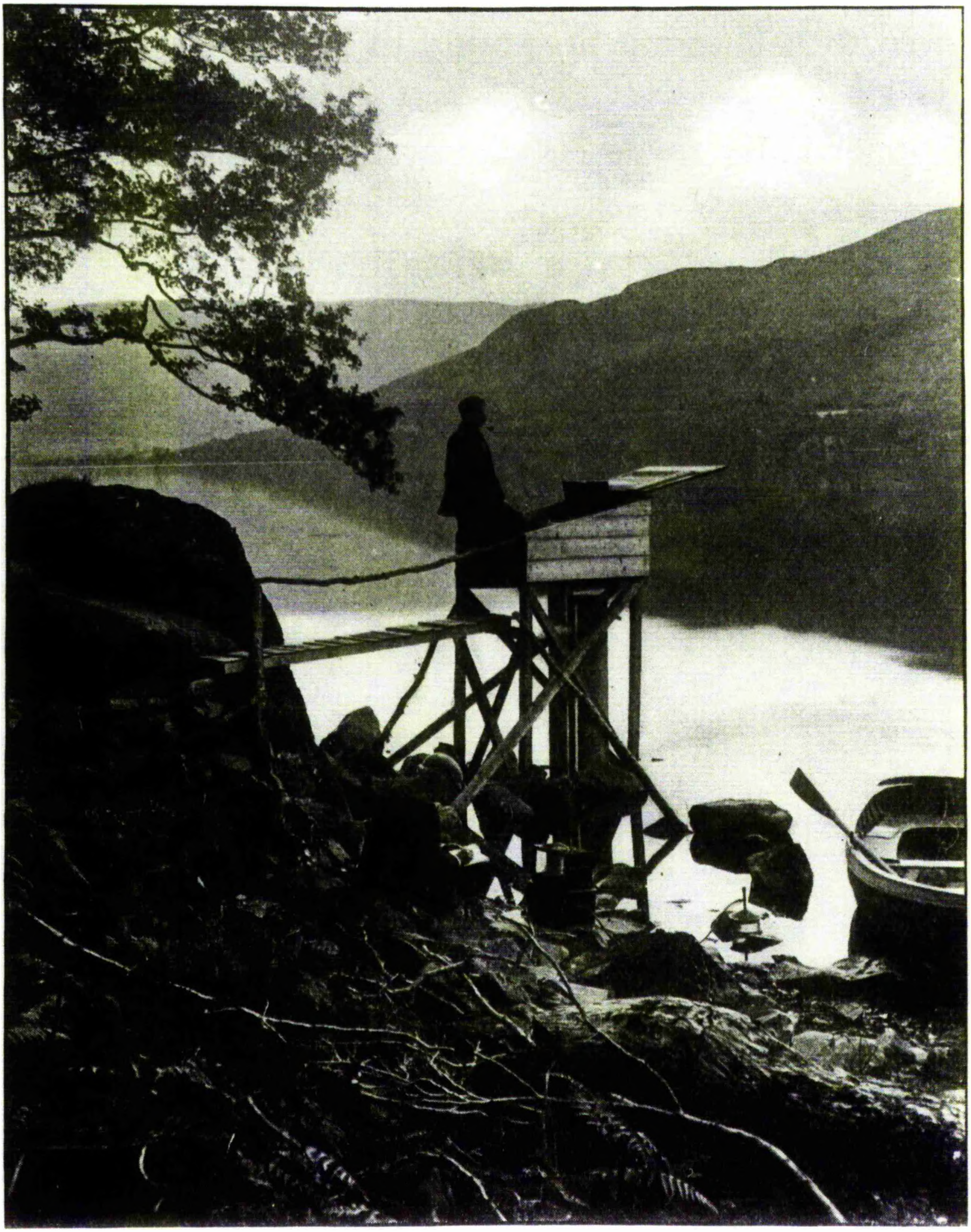


FIG.6.

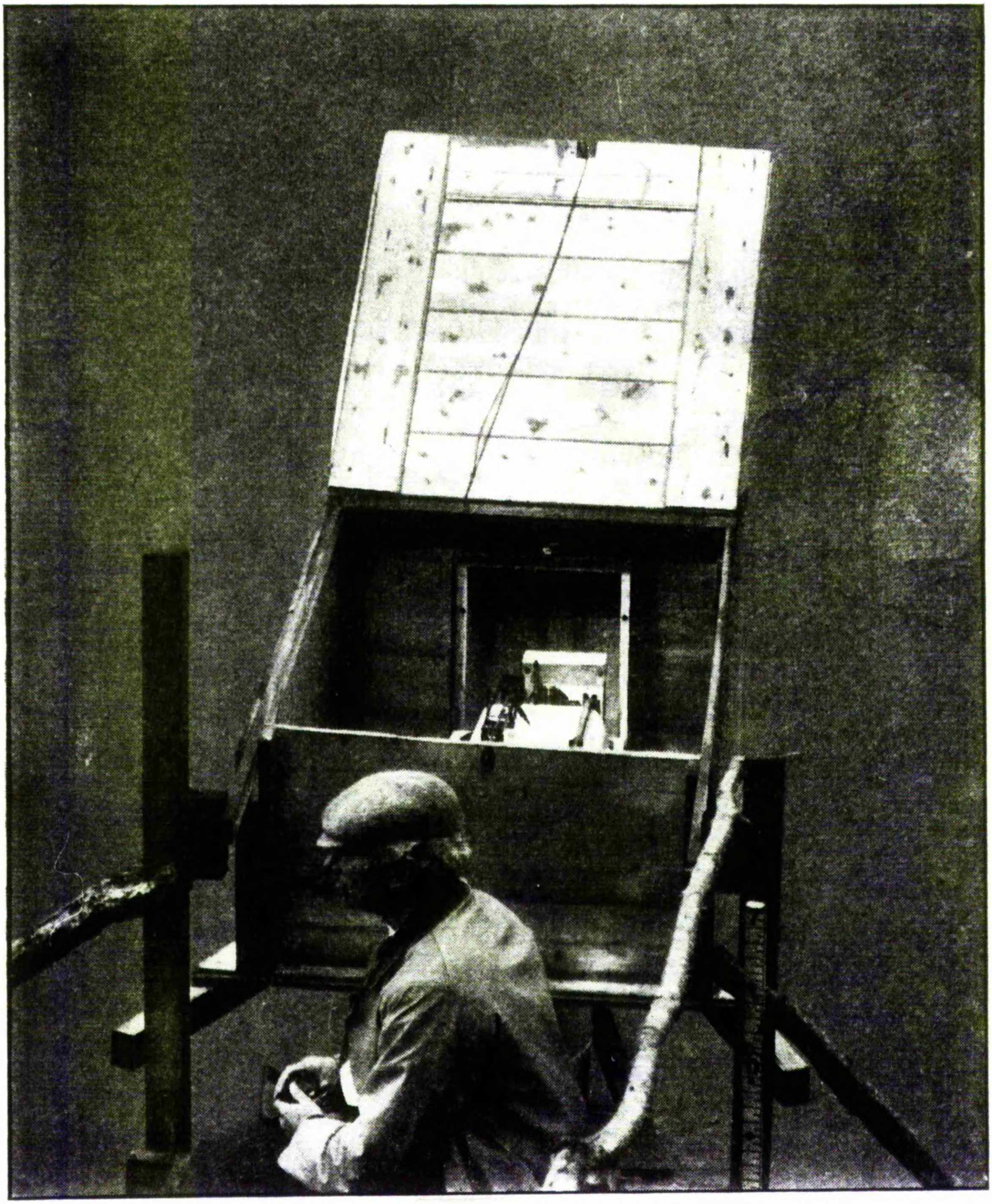


FIG.7.
Professor Chrystal and his "waggon recorder" limnograph, as mounted in position by the side of a lake.

for, in November 1904 part of the Sarasin Limnograph under Mr Wedderburn's charge on Loch Treig was destroyed during a storm, and there were times during the months of August and September 1905 when he was worried for the security of his installations.

Professor Chrystal also devised a form of limnograph for investigation of the nature of the embroidery on the limnograms which is shown in fig.8. According to the description given by him, this consisted essentially of a large and very sensitive barograph (Richard Statoscope), DSS, which was connected with a closed well, W, placed partly in, partly out of, the lake. The rise and fall of the lake level caused a corresponding rise and fall of the water level inside this closed well, thereby increasing and diminishing the air-pressure in the cylinder, SS, into which were fixed the barograph capsules, which were thus compressed and extended like the bellows of a concertina. This compression and extension were transferred by a system of multiplying levers, working the pen which wrote on the recording drum, D. The inertia of the working parts was very small, and the sensitiveness to alteration of pressure was fifteen or twenty times that of an ordinary mercury barometer.

According to Professor Chrystal this instrument was able to show quite plainly extremely small denivellations of the lake, and could be made more or less sensitive by increasing or diminishing the diameter of the well. By merely turning the stopcock, and shuttering off the communication with the well, the instrument could be turned into a very sensitive barometer. The curve which it traced was thus changed, at a moment's notice, from a limnogram into a barograph, so that one could alternately record the denivellation of the lake and the variation of the atmospheric pressure.

Professor Chrystal further said that if, instead of closing the stopcock C, it was left open, and the other end of the tube CC allowed to communicate with the open air through a capillary tube of properly chosen length and bore, the statoscope would act exactly like the Dines-Shaw Microbarograph, with the advantage of a larger time-scale.

Professor Chrystal called it a Statolimnograph.

In addition to the Statolimnograph, and four index limnographs, which worked constantly at varying points of the lakes, he had three fixed Limnographs_ one near St.

Fillans (the Waggon Recorder), one near the binode (a Sarasin), and one near the uninode (a Sarasin).

As the gears of the Sarasin Limnograph proved too crude to deal with the delicate seiches of Loch Earn; Professor Chrystal was able to remodel it on the plan of the Waggon Recorder, which proved very successful at St. Fillans.

Besides the limnographic apparatus, they had a number of other meteorological instruments_three Microbarographs of the Dines-Shaw pattern, and a Dines Pressure Anemograph.

5.6.2 Various Causes Of The Denivellation Registered In A Limnogram

The ordinate of the limnogram taken at any particular station on a lake shows the height at different times of the lake surface at that station above a certain arbitrarily chosen level. Retardation and damping due to the instrument being allowed for, the limnogram gives the total denivellation at the station due to all causes whatsoever.

The examples reproduced in fig.9 were from Loch Earn, all taken by the Waggon Recorder near St. Fillans. They give a good idea of the great variety in the form of the limnographic record, and of the complexity of the phenomena to be explained. The two upper curves were very smooth, and furnished excellent examples of what was called the configuration period of a dicrote seiche. The third curve was an example of the strongly marked embroidery, which appeared on the limnogram during stormy weather. The fourth curve was an example of a seiche in moderately calm weather broken by varying weather conditions. The fifth curve, except for the slight wind embroidery, gave an example of an almost pure sinusoidal curve; it was taken near the Eastern Binode.

In Part I of the paper Professor Chrystal also gave various causes which might affect the level of a lake, which had been enumerated by him as follows:-

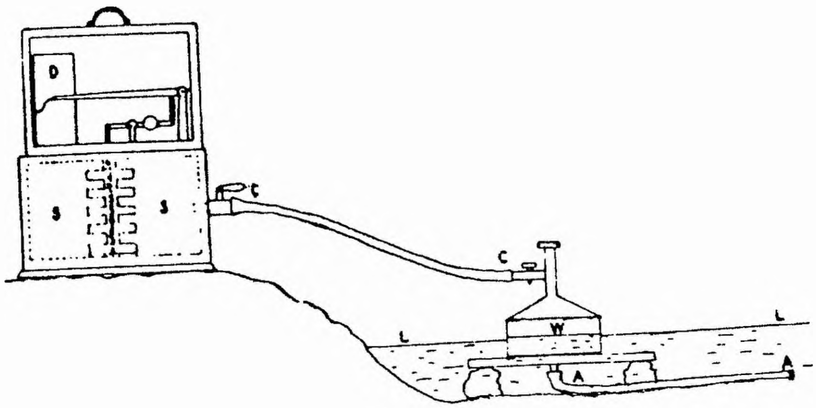


FIG. 8.

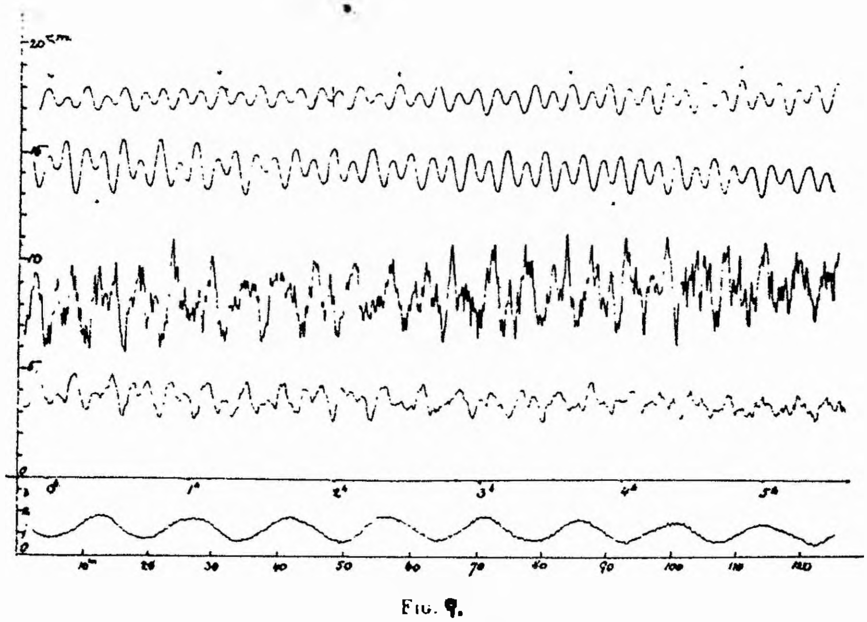


FIG. 9.

1. Volume denivellations ._ Caused by precipitation or evaporation. These were usually of slow variation, easily traced to their causes, and not directly concerned with seiche phenomena.

2. Persistent wind denivellation ._ Due to the heaping up of water at one end of a lake, or in shallow places, where the bottom friction prevents the development of an under return current to counteract the surface wind current. These denivellations were slow and irregular in their variation, and again easily traced to their cause.

3. Fluctuating wind and other denivellations ._ Due to the propagation of trains of waves on the surface of the lake by the passage of wind- squalls, and associated with the rapid variation of wind pressure shown by a self-registering anemograph. Such wave trains may also be started by passing steamers or other accidental causes.

4. Swell denivellations ._ After a persistent wind had blown for some time over a stretch of water of a certain length, a kind of dynamical equilibrium was established between the wind and the water, and the surface became covered with more or less regular trains of progressive waves. Owing to reflection at banks and retardation at shores and shallows, and also to unsteadiness of the wind, there was an interference of superposed trains, which spoiled the wave pattern, and prevented absolutely regular periodicity in the denivellation at any given point. The general effect was, however, a fairly regular pattern of small progressive waves of apparently constant length, usually diversified by wave maxima at approximately equal intervals. This system persisted for some time after the wind fell, and at this stage it was usually spoken of as "Swell".

5. Seiche denivellations ._ These were stationary oscillations of the whole lake, having nodes(i.e. places of no vertical motion), ventral points(i.e. places of no horizontal motion), and periods depending only on the configuration of the lake-basin.

5.6.3 Composition Of Seiches, And The Analysis Of A Limnogram By Residuation

Another important coverage in Part I of the paper was the *method of Residuation* , as it was called by Professor Chrystal. In essence the method consists of moving the record initially about half a uninodal wave length to the right and averaging with the initial curve. This will remove the uninodal seiche, so that the mean amplitude of the binodal becomes in most cases the conspicuous element in the record. The process can be repeated and it is possible to work backwards to a more accurate determination of the uninodal period by subtracting the first approximations to the binodal and other plurinodal seiches. The method is a semi-empirical mathematical analysis and can be applied when an approximate uninodal period is available.

The residuation method as explained by Professor Chrystal can be described precisely as follows:

He assumes that if the period of two components approximate to a simple numerical proportion, say 9:5, as in the case of the uninodal and binodal seiches of Loch Earn, the result is a limnogram with a periodically recurring configuration like a wall-paper, the individual waves of which approximate to the waves of one of the two components if the amplitude of that component predominates, but which fluctuates if the two amplitudes are not very different. In fig.10 the thin and dotted lines represent the component seiches, and the thick line the resultant component seiche, the ordinate of which is the algebraic sum of the ordinates of the components. The thick curves in Nos.i., ii., and iii. imitate very closely the smooth dicrote seiches reproduced in Nos.1 and 2 of fig.9.

Conversely, these principles may be used in the difficult process of analysing an actual limnogram, so as to discover the periods of the components of the seiche which it records. At the bottom of fig.11 is reproduced part of a fine limnogram obtained by Mr James Murray from Loch Earn by a series of half-minute observations with an index limnograph,

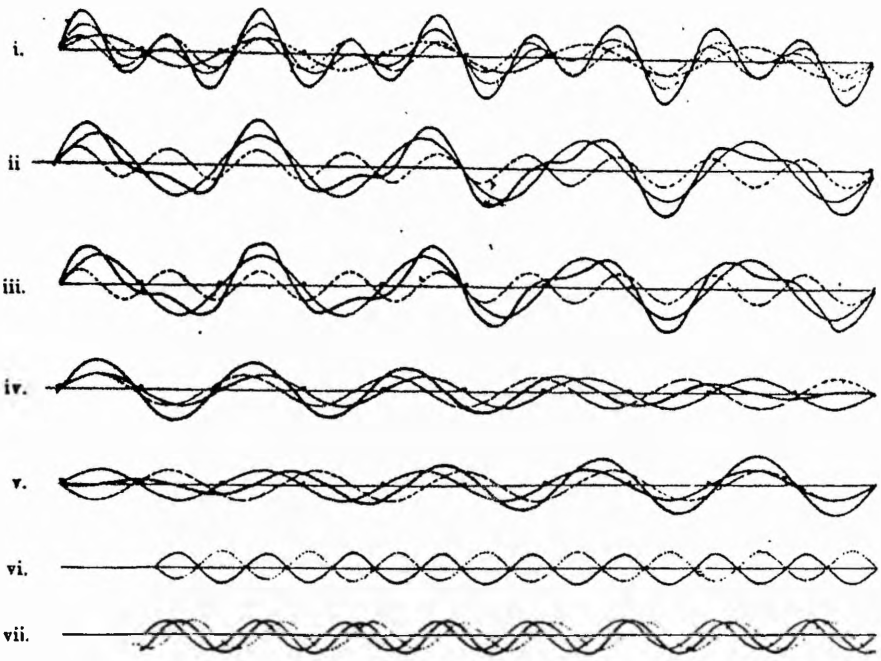

i.
ii
iii.
iv.
v.
vi.
vii.

FIG. 10.

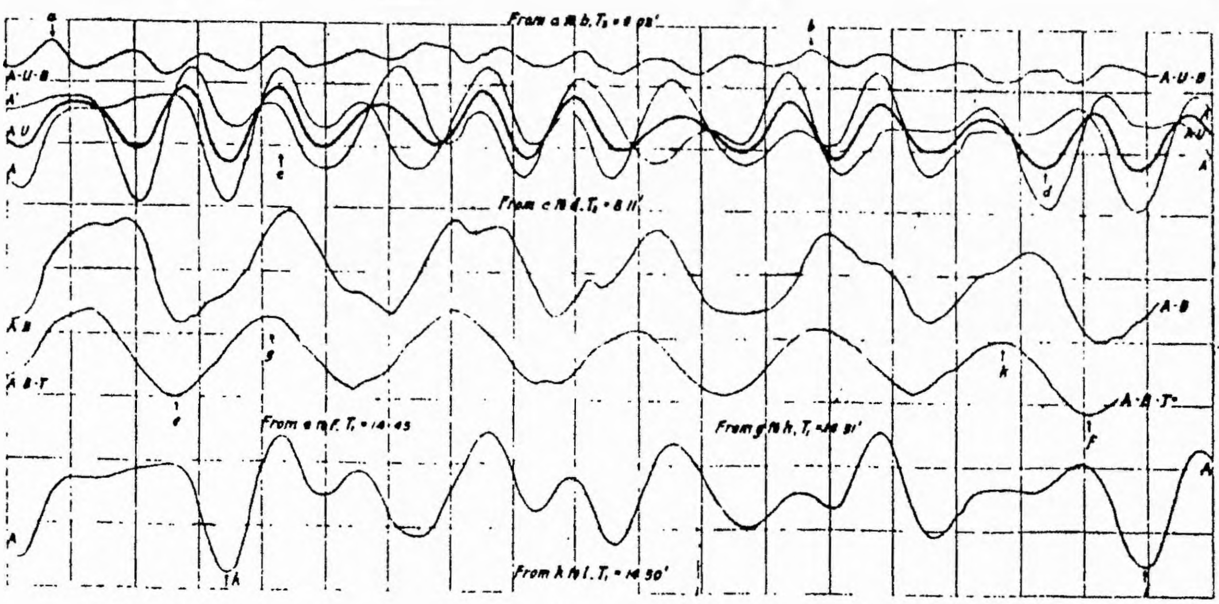

A·U·B
A·B
A·B·T
From e to f. T_1 = 14·45
From k to l. T_1 = 14·50'

FIG. 11.

which extended over eight hours by counting and measuring between two nearly symmetric minima; it is readily found that the longest period is about T_1=14.5 min. On the limnogram is now superposed a tracing of itself, displaced to the left through a distance 14.50/2=7.25′, and the two curves are compounded by taking at each point half the sum of their ordinates. In the resulting curve, A-U in fig.11, the uninodal seiche is destroyed, or at least greatly reduced. It would be quite accurate. The other component seiches are altered in a known way as regards phase and amplitude, but the periods are unaltered. The result is a curve still impure, but with a well-marked period of T_2=8.11 min. Eliminating this component as before, we get the uppermost curve, which gives a period of T_3=6.02 min. These are good approximations to the first three calculated periods of Loch Earn, which are T_1=14.50′, T_2=8.14′, T_3=5.74′. The approximation may be refined by now residuating out the binodal and trinodal, and redetermining T_1 from the purified curve; then residuating out T_1 and improving the value of T_2; and so on.

The method has also been carried successfully later by A. Defant(1908)[24] and Nakamura and Honda(1911)[25].

5.6.4 Periods And Nodes Of Loch Earn

The memoir containing Parts III, IV, and V of Professor Chrystal's paper, " An Investigation of the Seiches of Loch Earn by the Scottish Lake Survey"[26] was communicated to the Royal Society of Edinburgh in 1908. In Part II of this paper Professor Chrystal presented an account of the observations to determine the periods and nodes of Loch Earn.

In order to give an idea of the accuracy that was possible under the most favourable circumstances in determining the periods of a lake, Professor Chrystal subjoined some of the tables which were used in determining the first three periods of Loch Earn and are reproduced here as Tables IV--VIII.The station, the instrument, and instrumental adjustments were the same in each table, but not the same in any two tables for the same period. In the second column of each table was given the height of the surface of the lake above a certain arbitrarily fixed level.

According to Professor Chrystal,"the weight in taking the mean is assumed to be inversely proportional to the estimated possible error, or directly proportional to the number of oscillations counted".

Naturally the period most accurately determined was the uninodal. It would be seen that the mean of the results from Tables IV, V, and VI was 14.524, which differed by less than 0.005 from any one of the three.

According to Professor Chrystal, the extreme accuracy of determination of the uninodal period of Loch Earn, and incidentally also of the binodal period, was due to the great regularity and persistence of the seiches of this lake, and to the fact that the ratio $T_1 : T_2$ was very nearly equal to 9: 5.

Professor Chrystal further noticed that during the observations the mean level of Loch Earn varied through a range of nearly 20 inches, but a careful examination of the tables of values of the periods under different circumstances did not appear to show any correlation between the depth of the lake and the various periods. He therefore concluded that in the case of Loch Earn within the range of observations, the periods were independent of the depth.

From the theoretical point of view, there was nothing surprising in the results arrived at. For if one considers elongated lakes of uniform breadth, and assumes that the same normal curve continues to represent the lake-basin when the mean level rises or falls. Then a lake whose longitudinal section is parabolic has $T_\nu = \pi l \div \sqrt{\{\nu(\nu+1)gh\}}$ [Hydrodynamical Theory of Seiches, P.622]. In this case l is proportional to $\sqrt{h}$; hence all the periods are independent of the depth of the lake. From the analysis in "Hydrodynamical Theory of Seiches; P.628", the same is true of a biparabolic lake.

Table IV. Observations with the Waggon Recorder at St. Fillans(Picnic Point)

Date 1905.	staff. Feet.	T_1. Minutes.	Number of Oscillations.
Aug. 11	2.07	14.64	41
13	1 95	14.30	19
14	1.88	14.54	15
14	1.88	14.60	39
15	1.82	14.67	10
15	1 82	14 56	46
16	1.78	14.57	57
17	1.72	14.47	49
18	1.80	14.50	76
20	2.27	14.55	65
21	2.20	14.64	15
22	2 30	14.57	45
25	2.25	14.56	72
Sept. 2	1 90	14.58	38
3	1.85	14.63	40
5	1.83	14.54	71
6	1.80	14.45	40
17	2.80	14.49	30
18	2.48	14.45	36
18	2.48	14.47	66
18	2.16	14.47	14
18	2.16	14.55	30
18	2.16	14.53	44
20	2.00	14.57	27 1/2
23	1.82	14.52	24
23-27	1.82-1.65	14.52	375
24	1.80	14.53	79
25	1.76	14.50	95
26	1.70	41.53	100
27	1.66	14.52	101

Weighted mean T_1=14.529

Table V. Observations with the Sarasin(at low speed) near the E. Binode.

Date. 1905.	Staff. Feet.	T_1. Minutes.	Number of Oscillations.
Aug. 6	2.25	14.56	41
7	2.25	14.47	42
7	2.25	14.51	83
8	2.15	14.52	75
9	2.10	14.53	75
20	2.30	14.62	48
22	2.40	14.51	56
22	2.40	14.45	74
24	2.32	14.54	118
Sep 1	2.07	14.55	30
3	1.85	14.52	50
4	1.87	14.53	95
24	1.80	14.55	64
25	1.76	14.49	106
26	1.70	14.54	100
23 - 26	1.82-1.68	14.52	270
27	1.66	14.51	93

Weighted mean T_1=14.524.

Table VI. Observations with the Sarasin(at higher speed) near the E. Binode.

Date. 1905	Staff. Feet.	T_1. Minutes.	Number of Oscillations.
Sept. 9	2.60	14.64	15
14	2.60	14.54	10
15	2.45	14.52	45
16	2.30	14.53	10
20	2 00	14.44	25
22	1.87	14.51	50

Weighted mean T_1=14.521.

Table VII. Observations with the Waggon Recorder near St. Fillans(Picnic Point).

Date. 1905.	Staff. Feet.	T_2. Minutes.	Number of Oscillations.
Aug. 11	2.07	8.12	74
14	1.88	8.08	27
14	1.88	8.13	70
15	1.82	8.15	18
15	1.82	8.07	83
16	1.78	8.065	103
17	1.72	8.055	88
18	1.80	8.06	138
20	2.27	8.08	117
21	2.20	8.13	27
22	2.30	8.10	81
Sept. 2	1.90	8.15	68
3	1.85	8.12	72
5	1.83	8.06	128
6	1.80	8.05	72
18	2.16	8.10	25
18	2.16	8.08	54
18	2.16	8.09	79
24	1.80	8.085	142
27	1.66	8.086	169

Weighted mean T_2=8.086

Table VIII. Observations with the Waggon Recorder near St Fillans(Picnic Point).

Date. 1905.	Staff. Feet.	T_3. Minutes.	Number of Oscillations.
Aug. 12	2.07	6.14	36
13	1.95	5.89	29
21	2.20	5.91	45
29	2.22	5.98	36
29	2.22	6.01	48
30	2.15	6.07	65
31	2.10	6.00	60
Sept. 3	1.85	5.98	36
23	1.82	6.008	58

Weighted mean T_3=6.005.

According to Professor Chrystal, "Loch Earn occupies an intermediate position; the constancy of its periods is therefore an indication that the assumption of a biparabolic normal curve is a good first approximation".

To me, Professor Chrystal talks of an intermediate position of Loch Earn above in terms of its dimensions.

Professor Chrystal further remarked that the difficulties in determining the nodes by direct observation were more than realised in practice. When the range of the seiche was large, there was nearly always a great deal of wind-embroidery of an irregular character, which it was impossible to eliminate entirely either by damping or by residuating the limnogram. Also, where the amplitude was small, there was almost always a sensible disturbance arising from an aperiodic variation of the lake level, probably due to the heaping up of the water on the shallow shore, an effect which would vary with the slope of the beach. The varying slope also affected the range of the seiche at the margin of the lake to an extent which it would be difficult to calculate with any degree of accuracy. Both these causes introduced uncertainty in the method of observing with index limnographs on two sides of the node where the seiche was found in opposite phases, and then deducing its position by interpolation. A mere null method would have scarcely led to a satisfactory result, unless under exceptional circumstances. Of the many attempts made by Professor Chrystal and his co-workers, only a few led to limnograms which could be utilised; and in every case the curves had to be purified by residuation.

The actual observed position for the *uninode of Loch Earn* was 105 yards west of the position determined in the normal curve by calculation.

The *Eastern Binode and* the southern end of the *Western Binode* were respectively 117, 305 yards west of their respective positions determined in the normal curve.

Similarly, the southern end of the *Eastern Trinodal* was 88 yards west of its position fixed in the normal curve.

The observations made for the *Middle Trinode* by Professor Chrystal were rendered useless by casual wind-disturbances, and according to him no observations of sufficient accuracy were available for fixing the *Western Trinode* .

5.6.5 Effect Of meteorological Conditions Upon The Denivellation Of Lakes

In Part IV of his paper under consideration Professor Chrystal first of all describes the general character of the seiches of Loch Earn as observed by him and his co-workers. He also presents here a comparison of seiches of Loch Earn with Lochs Tay and Lubnaig and finally discusses in detail the origin of seiches and gives the detailed effect of these causes as observed by him during the observations made on Loch Earn from August-December 1905.

General character of the seiches on Loch Earn ._

Professor Chrystal gives the general character of the Loch Earn as follows:-

"Owing to the comparatively regular shape of its basin, and the fact that the depth is considerable compared with the length, the seiches on Loch Earn are very regular and very persistent. Also, probably because its longest axis is more or less parallel to the path of the major and minor atmospheric disturbances, Loch Earn is very rarely free from seiches. During 1070 hours, from 10th August to 28th September, the waggon recorder at Picnic Point was almost constantly in action, yet only two-and-a-half hours of calm were recorded. During 1350 hours, from 12th October to 7th December, while the waggon recorder was in action at Lochearnhead, there were in all 90 hours of calm. Of these, 81 hours were made up by continuous stretches of 21^h, 37^h, and 23^h on 4th, 16th, and 20th November".

He next gives the greatest ranges observed on Loch Earn in August and September as below:

"The greatest ranges observed in August and September were 79 mm., 66 mm., 73 mm., 55 mm., 55 mm., 63 mm., on 19th and 21st August and 3rd, 7th, 8th, and 9th September respectively. Only one very exceptional range was observed between 12th October and 7th December, viz. 55mm. on 7th December".

Professor Chrystal further comments about the seiche behaviour of Loch Earn in different times of the year as follows:

"The range of seiche at St. Fillans is usually over 10 mm. A rough estimate showed that during the 1070 hours of observations at Picnic Point the range of the seiche was over 30 mm. during 214 hours; and during the 1350 hours at Lochearnhead it was over 30 mm. during 57 hours only. It follows that, whether we test by hours of calm, by hours of excess over 30 mm., or by occurrence of exceptional ranges, the period from 12th October to 7th December showed much less seiche activity than the period from 10th August to 28th September".

Describing further the seiches of Loch Earn, Prof. Chrystal said that in more or less settled weather, by far the commonest seiche configuration on Loch Earn was a uninodal and binodal dicrote, shortly written as UB-dicrote, that varied between the two extremes, where the binodal on the one hand and the uninodal on the other were scarcely noticeable.

In fact he and his observers during their observation never noticed any purely uninodal or purely binodal seiche. In these seiches according to him 5-9-period combination by the interference of the uninodal and binodal components was usually reproduced with the most beautiful regularity, sometimes for a whole day or longer. As an example, Prof. Chrystal presented a seiche observed at Lochearnhead between 16th and 22nd October 1905, which lasted about six-and-a -half days, say 127 configuration periods, only six of these periods were found too short by one uninodal, and three too long by the same amount.

He remarked, it was probable that the gradual change of phase accompanying the rise and fall of the amplitudes of the components more than compensated for the fact that 9/5 was not so close an approximation to T_1/T_2 as was the sixth convergent 70/39, in the continued fraction for the ratio.

In order to find that a particular shape considered for the lake is correct, Prof. Chrystal considered in fig.12 three simultaneous limnograms, taken from top to bottom near the uninode, near the binode, and 480 yards from the eastern end of the lake respectively. All of which were according to him somewhat embroidered by the wind, but the lowest one was a

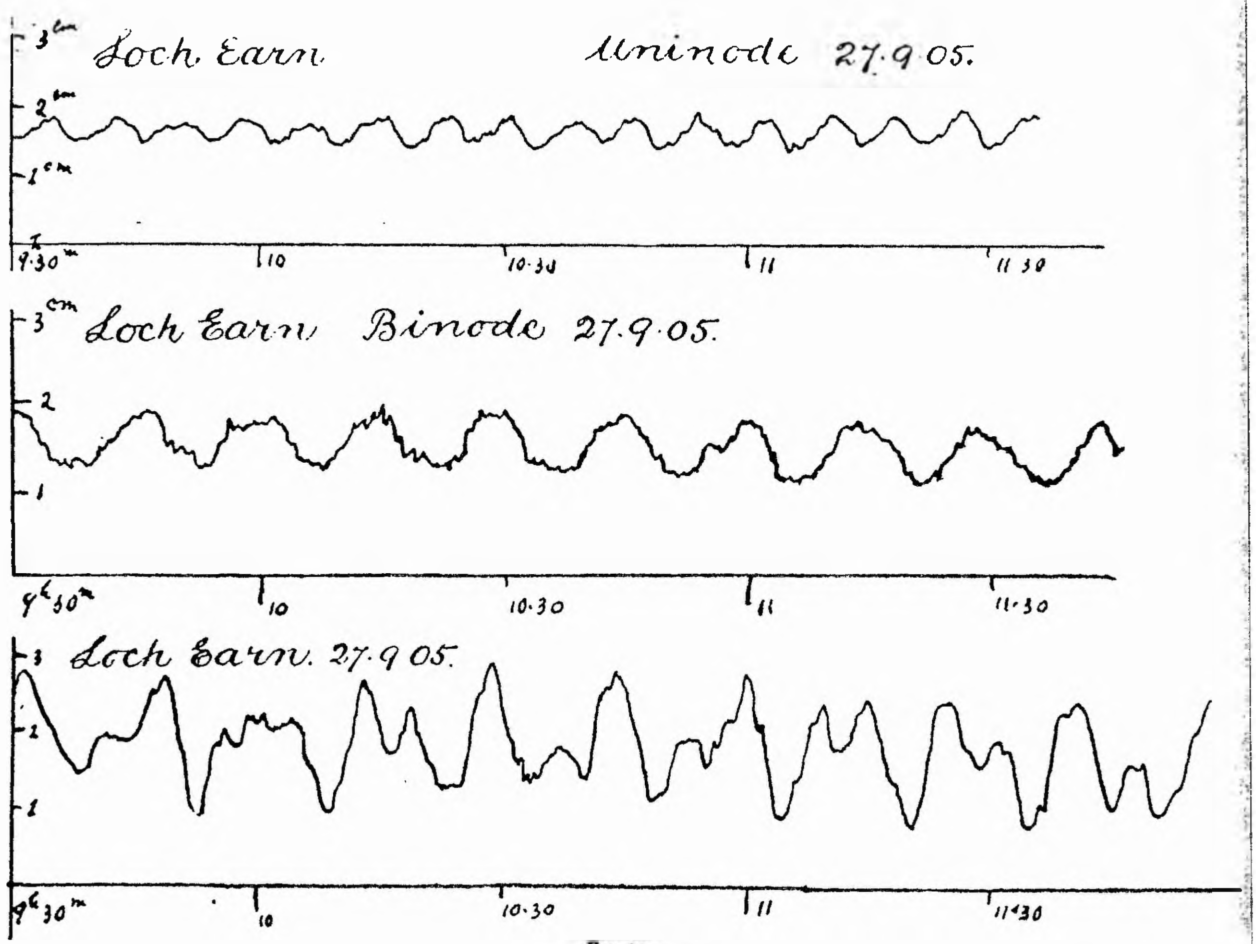

3cm
Loch Earn
Uninode 27.9.05.
2cm
1cm
9.30m
10
10.30
11
11.30
3cm
Loch Earn Binode 27.9.05.
2
1
9h 30m
10
10.30
11
11.30
3
Loch Earn. 27.9.05.
2
1
9h 30m
10
10.30
11
11.30

Fig. 12.

UB-dicrote, the middle one a nearly pure uninodal, and the uppermost one a nearly pure binodal.

He concluded that the figure was a verification of the approximate accuracy of the mathematical theory of Loch Earn regarded as a biparabolic lake.

In comparing the seiche observations of Lochs Tay and Lubnaig with Loch Earn, he said that the seiche behaviour of Lochs Tay and Lubnaig was a matter of disappointment. According to him," Loch Tay is relatively shallower, is more crooked, and the relation of its axial line to the path of the minor atmospheric disturbance is different".

Giving his opinion about Loch Lubnaig he said,"it is very shallow, has a very irregular basin, and lies across the path of the atmospheric disturbances".

But after all the disappointment he found consolation in the beautiful seiche behaviour of Loch Earn, which according to him, " can be regarded as a small but elegant daughter of the Lake of Geneva, the great mother of seiches".

In the next part of the paper he describes different causes of seiches, discussed in the following section.

5.7 Origin Of Seiches

Forel "Faraday of seiches" discussed the causes of seiches in detail in his treatise, "Le Léman, Monographie Limnologique(1895);"[27] many of his followers also considered the matter of whom Du Boys(1891)[28]; and Von Cholnoky(1897)[29]; are of great importance. In 1903, just the time when Prof. Chrystal started taking interest in the subject, Endrös in his important memoir, "Seeschwankungen beobachtet am Chiemsee,"[30] confirmed the conclusions of his predecessors, and added some fresh details of great importance. In fact, according to Prof. Chrystal this memoir was one of the most complete examples of the exact observations of seiches known to him.

Though a number of possible processes clearly may start a lake oscillating, it is reasonably certain that pressure denivellations and wind denivellations are the commonest.

The theoretical adequacy of these and other causes has been demonstrated by Prof. Chrystal in this part. Earlier Forel (1895) believed barometric pressure changes to be most effective on Lake Geneva. Prof. Chrystal concluded here that small travelling changes in pressure were more important than wind denivellations on Loch Earn, which like Lake Geneva lies on a west-east axis. There is often some difficulty in distinguishing wind effects from pressure effects, because a large wind denivellation is likely to be accompanied by disturbances in pressure distribution. It has been noticed that very small variations in pressure may have a quite remarkable effect if they happen to come in an irregular quasi-periodic series, because when the pressure changes happen to occur at intervals corresponding to the period of the seiche, they can force a persistent oscillation.

The results of Prof. Chrystal in this paper suggest the likelihood of this being the major method of origin of seiches of Loch Earn.

Like Endrös(1903) for Chiemsee, Prof. Chrystal here gave examples for Loch Earn, of seiches apparently generated by local showers of rain. It can be shown that the impact of such a shower is, in certain cases, quite adequate to account for the effect. Local denivellations due to the sudden inflow of water were early believed to cause the seiches of the Lake of Geneva; this is certainly incorrect, but Prof. Chrystal has recorded a case from Loch Earn reasonably explained in such a way.

Besides the seiches generated by the Lisbon earthquake of 1755 which have already been mentioned in the introduction, there are other instances of seiches caused by earthquake mentioned by R. C. H. Russel and D. H. MacMillan in their book [Waves and Tides, Hutchinson's Scientific Publications (1952)]. At San Francisco in August 1883, a seiche of range 6 inches and period 40 minutes was recorded 18 hours after the Krakatoa earthquake some 10,000 miles away in Pacific. In November 1922, an earthquake on the east coast of Chile, 5,000 miles distant, caused a seiche of the same period and amplitude.But Forel's critical studies showed that, in general, seiches are not due to seismic phenomena, and Prof. Chrystal agrees with him. Chrystal has also pointed out that even in the case of Lisbon earthquake of 1755, Loch Lomond responded, with a trinodal or quadrinodal rather than a uninodal seiche.

After the examination of the limnograms he showed that seiches may be generated "suddenly," i.e. attain their full range in one or two oscillations, or may be generated "gradually," i.e. the full range may be attained only after a considerable number of oscillations.

Among the causes that might generate seiches suddenly, Prof. Chrystal considered the following:-

1. The sudden release of a static denivellation of the whole lake-surface, due to the progression of the general system of the atmospheric isobars.

2. Sudden release of a denivellation caused by the transport of water from one end of the lake to other by a wind which has blown in one direction for a time and then fallen calm or reversed its direction.

3. A sudden denivellation in one part of the lake due to very heavy flooding.

4. A sudden denivellation due to heavy fall of rain, snow, or hail over a part of the lake. This might be partly static, i.e. due merely to the generation of the precipitated water; or it might be partly dynamic, i.e. due to the impact of the precipitated water.

5. Sudden alteration of the atmospheric pressure, due to the passage over parts of the lake of a local atmospheric disturbance(squall), such as is indicated by a disturbance on the microbarogram.

6. The impact of wind-gusts on the lake-surface.

According to Prof. Chrystal among the causes that might generate seiches gradually the following may be mentioned:-

7. The action over portions of the lake-surface of small fluctuations of the barometric pressure, which, happen to synchronise more or less nearly with some of the seiche periods of the lake.

8. Action similar to last of fluctuations in the velocity and pressure of the wind, as shown in the anemogram.

In a lecture[31] delivered at the Royal Institution on Friday 17 May 1907, Prof. Chrystal had demonstrated some experiments with a miniature parabolic lake, to illustrate the origin of seiches. He had followed the method used by Messrs White and Watson in their interesting experiments on the subject[32].

However the method of generating seiches this way did not correspond to anything observable under ordinary circumstances in a lake; but according to Prof. Chrystal that was interesting in view of the important discovery then made by the Japanese observers[33], that the secondary oscillations in many of the bays on the coasts of Japan were seiches, having a node at the mouth and a loop at the bottom of the bay. These oscillations, which were sometimes of considerable range, were apparently due to resonance with comparatively inconspicuous undulations in the external oceanic swell, the periods of which were equal to some of the natural periods of the bay.

5.8 The Vibrations Which Cause The Embroidery On The Limnogram

A very rapid transitory oscillation on the limnograph, usually of small amplitude, is often observed. Such a disturbance of the lake-surface was termed by Forel a *vibration* . On Lake Geneva at Morges, Forel found that such vibrations may be started by the wind or by steamers passing at some distance from the coast. He supposed that they were analogous to the so-called sea seiches that have frequently been observed in marine coastal waters. Prof. Chrystal also considered the matter. He examined the simultaneous limnograms taken on Loch Earn and Tay during October and November 1905, to see whether there was any connection between the vibrations on the two lakes pointing to a common atmospheric cause. It was found by him that the average of the maximum ranges and of the periods was much the same for both lakes; but there seemed to be no connection between the occurrence of a particular period in the two. He concluded that the range might be high in both lakes and the periods different; or vice versa; or there might be vibrations of considerable range on one of the lakes, and none, or only the merest tremor, on the other.

Prof. Chrystal made some suggestions regarding the nature of these lake vibrations which are summarised as follows:-

1. They might be longitudinal seiches of very high nodality.

2. The vibrations might be transversal seiches of the lake.

3. Another cause of the embroidery of the limnogram may possibly be found in progressive surface waves and wave groups.

4. According to Prof. Chrystal, Prof. Börgen in his paper,"On the Relation Between the Velocity and the Dimension of Oceanic Waves, with an Explanation of the Waves of Longer Period on Open Coasts,"[34] had suggested that the secondary tidal oscillations and waves of unusually long periods occasionally observed on open coasts, where the circumstances do not seem to justify the assumption of a seiche, may be due to difference and summation waves(whose theoretical existence arises from the non-applicability of the theory of the linear superposition of small motions), after the analogy of the difference and summation tones of Helmholtz. Prof. Chrystal's opinion about this was that it was quite possible that some such explanation might apply in part to lake vibrations; but he himself found no evidence for or against such a hypothesis.

5. In cases where the lake vibrationsare followed by the wind. To Prof. Chrystal the best possible explanation of this seemed to be that the squall exerts a horizontal traction on the water and causes a drift current. By and by this current becomes greater than the compensating return current underneath. Thus a hump (or a group of waves) is raised on the surface, which is propagated in the water with a speed usually exceeding the velocity of the wind in a moderate breeze. This phenomenon had been confirmed by Prof. Chrystal in his observations on Loch Earn.

5.9 Mathematical Theory For The Effect Of Pressure Disturbance Upon The Seiches In a Uniform Parabolic Lake

In Part V of the paper Prof. Chrystal has given considerable attention to the influence of different atmospheric pressure disturbances in developing seiches in a lake with parabolic normal curve, neglecting friction and the rotation of the earth; and to make the results applicable to Loch Earn, he supposed the length, 2a, of the symmetrical parabolic lake as 6 miles or 10^6 cm., and the maximum depth 270 feet say 8000 cm. approximately. He made use of the method of Normal Co-ordinates introduced by Lord Rayleigh[35], but was originally given in Thomson and Tait's "Natural Philosophy (1867), §337".

This method used here by Prof. Chrystal furnished the change in the extreme amplitudes of the various seiche components due to a given disturbance of pressure, to the same degree of approximation as Rayleigh's Method.

As the total energy K of a seiche is at any moment partly kinetic $\mathfrak{L}$ and partly potential $\mathfrak{B}$, except of course that the kinetic energy is zero at the moment of maximum displacement and the potential energy is zero at the moment of zero displacement from the level. If p be the pressure at any point x of the water surface, and V_v the velocity of the water at that point in the direction of the normal to the surface drawn towards the water, then the following equation holds[36]:

$$\frac{dK}{dt} = \int_{-a}^{+a} pV_v dx = a\int_{-1}^{+1} pV_v dw, \text{ where } w = \frac{x}{a}.$$

Prof. Chrystal rewrote this equation as follows:

$$\frac{dK}{dt} = -a\int_{-1}^{+1} p\zeta^{\bullet} dw,$$

where ζ is the vertical displacement of a particle at the surface at time t and so $V_v = -\zeta^{\bullet}$.

The advantage of so doing was according to him that one neglects quantities of the order of k_v^2/ha or k_v^2/a^2, already negligible if one is to apply the theory of long waves, where k_v is the extreme amplitude of the ν-nodal seiches corresponding to x=+a, and h the maximum depth.

According to Prof. Chrystal, let us consider the seiche motion to be analysed into ν-nodal components with amplitudes at the ends of the lake as k_v. Since these components are normal modes of motion for the parabolic lake, the total energies K_v for each of these seiches can be calculated separately and independently, so that $K=\Sigma K_v$. For a ν-nodal seiche, the above equation gives

$$\frac{dK_v}{dt} = -a\int_{-1}^{+1} p\dot{\zeta}_v dw.$$

Since the integral of a constant pressure all over the surface of the lake must be zero, so any constant addition to p requires no attention. Hence one needs only to consider the disturbing pressure $\partial p = f(w,t)$, which may be expressed in centimetres of water.

If ∂K_v denote the total increment of the energy of the ν-nodal seiche by this disturbing pressure from t=0, to t=T, Prof. Chrystal got the result

$$\partial K_v = ag\int_{-1}^{+1} Q_v'(w)dw\int_0^T k_v n_v \sin n_v(t-\tau_v)f(w,t)dt \quad . \; . \; . \; . \; . \quad (1),$$

where $Q_v(w)$ is the solution of the original seiche equation and k_v the amplitudes of the ν-nodal at the ends of the lake. Also the energy of the ν-nodal seiche equals its potential energy in the configuration of maximum potential and zero kinetic energy. Hence, as the co-ordinates are normal

$$K_v = \mathfrak{B} = \{ ga/(2v+1)\}k_v^2 \; . \; . \; . \; . \; . \; . \; (2),$$

and so $$\partial K_v = \{2ga/(2v+1)\}k_v \partial k_v \; . \; . \; . \; . \; . \; . \; . \; (3).$$

Prof. Chrystal further said that strictly regarded, k_v is a function of the time; for the energy of the seiche is being continually altered by the disturbing surface pressure, so that the extreme amplitude k_v of the seiche at each moment, which would be left if the disturbing pressure were suddenly to cease, varied with time. However, as the variation of k_v was small, and f(w,t) was also small, the squares and products of derivatives may be neglected, and k_v can be regarded as constant in the integral on the right of (1).

From (1) and (3) he got

$$\partial k_v = \frac{1}{2}(2v+1)\int_{-1}^{+1} Q_v'(w)dw \int_0^T n_v \sin n_v(t-\tau_v) f(w,t)dt \; . \; . \; . \; . \; . \; (4),$$

a formula which summarised his whole theory so far as disturbance of the extreme amplitudes of the various seiches was concerned. So it followed that

$$\partial k_v = A'_v \cos n_v\tau_v + B'_v \sin n_v\tau_v;$$

where $$A'_v = \frac{1}{2}(2v+1)\int_{-1}^{+1} Q_v'(w)dw \int_0^T n_v \sin n_v t \, . \, f(w,t)dt,$$

$$B'_v = -\frac{1}{2}(2v+1)\int_{-1}^{+1} Q_v'(w)dw \int_0^T n_v \cos n_v t \, . \, f(w,t)dt \; . \; . \; . \; . \; .(5).$$

He considered many special cases by giving several examples.

Prof. Chrystal concluded that the methods of calculation which he used for a symmetric lake were, of course, applicable to any lake for which the normal modes of motion could be found. All one has to do is to use, instead of the Legendre functions, the general seiche functions, Bessel's functions, or other functions appropriate to the special form of lake basin in question.

For the uninodal seiche of Loch Earn, if the maximal displacement at the ends is 10cm., the total energy is about 2.4×10^{15} ergs or 2.4×10^{5} ergs $cm.^{-2}$

5.10 Theory Of Leaking Microbarograph

Professor Chrystal's last published paper was a continuation of his seiche investigations. In this paper entitled "On The Theory of the Leaking Microbarograph; and on some observations made with a Triad of Dines-Shaw instruments,"[37] he discussed how the minor fluctuations of the atmospheric pressure had an effect on the seiche. According to him many meteorologists of that time were also closely engaged in investigating the phenomena, and his own attention had been drawn to them on account of their connection with the seiche in general.

He explained that as these fluctuations often did not exceed a millimetre or two of water they were not shown by ordinary self-registering apparatus. In order to record them a specially sensitive form of barograph, such as Richard Statoscope was needed, which was delicate enough to show those small fluctuations, and yet could be brought back to a momentary zero whenever the indicator threatened to go off the scale or beyond the limits of safety. As previously mentioned a Statoscope could be used as a Dines-Shaw Microbarograph.

Professor Chrystal, however, raises two objections to the measurements taken by the Richard Statoscope. The first objection according to him is that the instrument must either be under constant watch, or else provided with automatic arrangement for altering the zero.

The second objection is due to the "secondary oscillations" and according to Professor Chrystal, this is more serious and radical. He explains these oscillations as "certain oscillations of much smaller range than the tidal oscillation proper, and of much shorter period, say fifteen to twenty minutes". Fig.13 is a reproduction of this kind of a curve.

FIG 13

Professor Chrystal discusses the actual appearance of these oscillations in these words "at first sight it would appear that the secondary oscillations occur only near high and low water. As a matter of fact, however, they occur throughout the whole day, and are nearly masked in the mareogram at the zero of the tide, because the rapid rise of the curve due to the proper tide wipes out turning- points and even the inflexions due to the minor oscillation".

He felt the need to devise an instrument able to neglect the barometric variations of larger range, longer period and to record only the minor fluctuations. The method which he used for the purpose was more or less the converse of the method of damping out the seiche of higher nodality in a limnogram which he described in his memoir "on the investigation of seiches on Loch Earn".

The method was briefly described by Professor Chrystal as follows:

"This method consisted essentially in measuring the difference between atmospheric pressure and the pressure in a vessel which communicates with the atmosphere by means of a small leak, say through a capillary tube of sufficient length or fineness of bore. If the bore were infinitely fine, the instrument would simply register the atmospheric pressure with all its variations; if the bore was very wide, it would register nothing at all; and by properly adjusting the tube we can arrange so that it only registers the fluctuations of short duration, which pass away before the small flow through the capillary has had time to establish equality of pressure inside and outside".

He concluded that the method could be applied by the use of any form of microbarograph, and he himself used three Dines-Shaw microbarographs for the microbarometric observations at Loch Earn in the autumn of 1905. He placed the three microbarographs for his observation at three different stations of Loch Earn which if joined formed a rough triangle. He named the three vertices of this triangle as A, B, C representing the three stations Killan, Lochearnhead and Ardrostan respectively. He denoted the sides and angles of this triangle by a, b, c; A, B, C; so that roughly a=5.60, b=8.75, c=6.10 (miles); A=39°, B=98°, C=43°.The direction BC being about 4° north of east.

In order to interpret a set of observations from three stations only, it was necessary to make some assumptions regarding the wave-front to be a straight line, and the propagation to be rectilinear, and uniform with velocity V.

Then t_1, t_2, t_3 being the time passage(in minutes) of the same phase of a disturbance at A, B, and C; θ the inclination (northward) of the wave-front to BC, and $\phi=90^\circ-\theta$ the inclination of the direction of propagation (southward) to BC; he obtained the formula

$$\cot\theta=\{(t_3-t_1)\cot B+(t_2-t_1)\cot C\}\div(t_3-t_2),$$

and, since $\phi=90^\circ-\theta$, cot B=-0.14, cot C=1.07, he finally got the formula

$$\tan\phi=\{1.07(t_2-t_1)-0.14(t_3-t_1)\}\div(t_3-t_2).$$

Taking V the velocity in miles per hour, he had $V=(60a \sin\theta)\div(t_3-t_2)=336\sin\theta/(t_3-t_2)$. These two formulæ enabled him to calculate the direction and velocity of propagation of any disturbance, the same phase of which had been observed at A, B, and C.

He then gave a brief account of the Earn observations,which, he thought would be useful for the future investigators, investigating the minor fluctuations of the atmospheric pressure, and would have more perfect means for such investigations.

As regards the nature of the phenomena under observation, Professor Chrystal noted the following points:-

1. It was unlikely that the assumption of a rectilinear wave-front could be more than a very rough approximation.
2. Because of the absolute similarity in the instruments used on the three stations, there seemed to be no doubt whatsoever that the distribution of pressure disturbance varied as it progressed. This introduced uncertainty in identifying the points on the microbarogram which corresponded to the same phase of disturbance.

He then presented specimen of the results obtained at Loch Earn without claiming the work to be anything more than a mere preliminary survey in a very interesting but still an almost wholly unknown region of meteorology.

These results are reproduced in the following table, where the first column gives the day of the month on which the disturbance occurred, the second, the time, reckoned from mid-night, when the maximum or minimum passed Ardrostan; the third, the direction from which the disturbance came; the fourth, the velocity of propagation in miles per hour. The letters α prefixed to the date means that the phase that was timed was a maximum, β a minimum.

Day.	Hour.	Direction.	V.	Day.	Hour.	Direction.	V.
Aug.	h.m.			Sept.	h.m.		
α 18	1 53	W.	6.7	α 2	15 11	E.25°.N.	75
β 21	12 17	W.19°.S.	27	α "	15 37	E.44°.N	19
α "	13 53	W.41°.S	47	β "	21 18	W.59°.S.	21
β "	15 2	W.52°.S.	19	β 3	14 12	W.25°.N.	26
α "	18 17	W.26°.N.	26	α "	16 8	W.30°.S.	48
α "	19 39	W.36°.N.	22	α "	26 18	E.55°.N	30
β "	21 47	W.46°.S.	13	α "	8 28	E.56°.N.	19
α "	23 44	W.49°.S.	36	α "	16 34	W.39°.S.	9
α 23	14 7	W.62°.N.	15	α 8	3 39	W.15°.S.	34
β 31	18 36	W.42°.S	41	α "	17 13	W.6°.N	27
β "	26 48	W.62°.S	68	β 9	12 53	W.63°.S	17
sept.				α "	13 58	W.68°.S.	22
α 1	20 31	W.4°.S.	21	α 13	3 31	W.33°.S.	45
α 2	10 42	E.71°.N.	46	β "	8 8	W.44°.S.	20

Out of the twenty-seven cases tabulated, the disturbance came from an easterly direction in only five, which, according to Prof.Chrystal was of considerable surprise, not to say skepticism, to one of the referees of this paper, himself a competent meteorologist.

His last illness and short life span prevented him from doing more in the field, but despite his engagements as Professor of mathematics, Dean of the Faculty of Arts of the University of Edinburgh and chairman of the Edinburgh Provincial Committee for Training of Teachers, he best utilised his leisure in investigating the seiches in different fresh-water lochs of Scotland. The work provided a sound base for the future investigators in the subject. In fact, most of the latter investigators did make use of his theory and observations.

For example, J. Proudman[38] in 1915 presented a general solution of the theoretical determination of the longitudinal seiches in a lake; which was, in fact, an improved version of Chrystal's method, and did not involve approximations similar to those of Chrystal; moreover, it had taken account of all the natural irregularities of the lake pointed out by Prof. Chrystal at different stages. The other famous methods based on his theory are those of A. Defant(1918)[39] and Ertel(1933)[40].

It is because of the importance of his work that even in the latest texts and papers carrying details of seiches, his work is frequently quoted.

In July 1905 in a letter to J. Larmor[41], he indicated his wish to extend his investigations to study the phenomena in the sea provided he lived so long, which unfortunately he did not, and his life came to an end just at the time when he completed his seiche investigations on some selected fresh-water lochs of Scotland.

His seiche investigations brought him in close contact with F. A. Forel and on his advice the Council of the Royal Society of Edinburgh invited F. A. Forel for a lecture, which he accepted and delivered a lecture on *fata morgana* [42]. This is a phenomenon of meteorological optics, which can precisely be explained as follows:-

During the process of transition between inverse and direct thermal gradients, it is possible for the rays from a single point to be refracted along both concave and convex paths, so that they appear to reach the eye from a vertical line. As a result the waves at the horizon, and distant objects in general, are distorted into rectangular blocks of varying colour and intensity, which may look like buildings bearing turrets, the so called *castles in the air* . This appearance is characteristically observed in the vicinity of the straits of Messina. The fata morgana has been extensively studied on the Lake of Geneva by F. A. Forel.

Professor Chrystal while recommending to invite F. A. Forel never anticipated that he himself would not be able to receive or to listen to his lecture.

This work of Professor Chrystal was recognised as a major break through in the investigations initiated by F. A. Forel in 1869. This recognition came by awarding the Gunning Victoria Award to him by the Royal Society of Edinburgh, one of the most prestigious awards of the society, and the Royal Medal by the Royal Society of London, the confirmation of which came from the King just two hours after his death.

References Chapter 5

1. R. A. Harris, Manual Of Tides, Part IV(1904);

   B. U. S. A. Coast Geod. Surv.(Washington, D. C.).

2. J. Proudman, " Free and Forced Longitudinal Tidal Motion In a Lake,"

   Proceedings of the London Mathematical Society (1915);

   Series 2, Vol. 14.pp.240-50.

3. A. T. Doodson; R. M. Carey; and R. Baldwin, " Theoretical Determination of the Longitudinal Seiches of Lake Geneva,"

   Transactions of the Royal Society of Edinburgh (1920);

   Vol. lii, Part III, pp.629-42.

4. R. Sterneck, " Hydrodynamische Theorie der Halbtägigen Gezeiten des Mittelmeeres," *Sitz. K. Akad. Wiss. Wien, Math. Nature. Kl* (1915);

   Vol. v. No. 124, P.905.

5. A. Defant, " Neue Methode zur Ermittlung der Eigenschwingungen(seiches) von abgeschlossenen Wassermassen,"

   Annalen der Hydrographie und Maritimen Meteorologie (1918); Vol. 46, p.78.

6. Horace Lamb, *Hydrodynamics*, 3rd ed. Cambridge University Press (1932).

7. *Proceedings of the Royal Society of Edinburgh* (1904);

   Vol. xxv, Part I, pp.328-37.

8. *Proceedings of the Royal Society of Edinburgh* (1905);

   Vol. xxv, Part II, pp.637-47.

9. *Transactions of the Royal Society of Edinburgh* (1906); Vol. xli. pp.599-649.

10. J. Lagrange, "Sur la Théorie du Mouvement des Fluides,"

    Mémoirs Acad. Berlin (1781).

11. J. R. Merian, *Über die Bewegung tropfbarer Flüssigkeiten in Gefässen* (1828);

    Basle.

12. Von Der Mühl, *Math. Ann* .(1885); Vol. xxviii. p.575.

13.F. A. Forel, "La Formule des Seiches,"

Archives des Sciences Physiques et Naturelles (1876), Genève. p.278.

14. G. B. Airy, "Tides and Waves,"

Encyclopaedia Metropolitana (1845), London. Vol. V, pp. 241-396.

15. P. Du Boys, " Essai Théorique sur les Seiches, avec Appendice par F. A. Forel,"

Archives des Sciences Physiques et Naturelles (1891), Genève.

Vol. xxv. p.628.

16. Phillip Kelland, "On the Theory of Waves, "

Transactions of the Royal Society of Edinburgh (1839); Vol. xiv. p.524.

17. G. G. Stokes, "Breaking of Railway Bridges(1849),"

Collected Works ; Vol. ii. §7, p.186.

18. F. Bergsten, "The Seiches of Lake Vättern,"

Geogr. Ann. Stockholm (1926); Vol. 8. pp.1-73.

19. Fr. Defant, "Theorie der Seiches des Michigansees und ihre Abwandlung durch wirkung der Carioliskraft,"

Arch. Met., Wien .(1953); Series A, Vol. 6, No. 2.

20. J. Halm,"On a Group of Linear Differential Equations of the 2nd order, including Professor Chrystal's Seiche-equations,"

Transactions of the Royal Society of Edinburgh (1906); Vol. xli. pp.651-76.

21. C. G. Jacobi,"Untersuchungen über die Differentialgleichung der Hypergeometrischen Reihe,"

Crelle's Journal (1859); Vol. lvi. pp.149-65.

22. *Transactions of the Royal Society of Edinburgh* (1905);

Vol. xli, Part III, pp.823-50.

23. *Transactions of the Royal Society of Edinburgh* (1906);

Vol. xlv, Part II, pp.361-96.

24. A. Defant, "Über die Stehenden Seespiegelschwankungen in Riva am Gardasee,"

S. B. Akad. Wiss. Wien, Math.-Nat. Kl. (1908); Vol. 117, Abt. IIa, pp.697-780.

25. S. Nakamura; and K. Honda, "Seiches in Some Lakes of Japan,"
Journal of the College of Science, Imperial University, Tokyo(1911);
Vol. xxviii, No. 5, 95pp.

26. *Transactions of the Royal Society of Edinburgh* (1908);
Vol.xlvi, Part III, pp.455-517.

27. F. A. Forel, *Le Léman, Monographie Limnologique*;
Sixième Partie, pp.1-288.

28. P. Du Boys, "Essai Théorique sur les Seiches, avec Appendice par F. A. Forel,"
Archives des Sciences Physiques et Naturelles (1891), Genéve; Vol. xxv, p.628.

29. E. Von Cholnoky, *Limnologie des Plattensees*. Wien.(1897).

30. A. Endrös, Seeschwankungen beobachtet am Chiemsee.(dissertation.)
K. Technischen Hochschule zu München, Traunstein(1903); 117 pp. Also
Zeitschrift für Instrumentenkunde, June 1904; p.180.

31. *Proceedings of the Royal Institute*; Friday, 17 May 1907. pp.657-76.

32. Peter White and William Watson, " Some Experimental Results in Connection with the Hydrodynamical Theory of Seiches,"
Proceedings of the Royal Society of Edinburgh (1905-06); Vol. xxvi, pp.142-56.

33. K. Honda; T. Terada; Y. Yoshida,"Secondary Undulations of Oceanic Tides,"
Journal of the College of Science, Imperial University, Tokyo(1908);
Vol. xxiv. pp.1-113.

34. Professor Börgen,"On the Relation between the Velocity of the Wind and the Dimension of Oceanic Waves, with an Explanation of the Waves of Longer Period on Open Coasts,"
Annalen der Hydrographie und Maritimen Meteorologie (1890); Heft I.

35. Lord Rayleigh, *Theory of Sound*; Vol. I(1877); §87.

36. Horace Lamb, *Hydrodynamics*, 3rd ed. Cambridge University Press(1906); p.9.

37. *Proceedings of the Royal Society of Edinburgh* (1907-08);

Vol. xxviii. pp.437-60.

38. J. Proudman, "Free and Forced Longitudinal Tidal Motion in a Lake," *Proceedings of the London Mathematical Society* (1915); Series 2, Vol. 14, pp.240-50.

39. A. Defant, "Neue Methode zur Ermittlung der Eigenschwingungen(seiches) von abgeschlossenen wassermassen," *Ann. Hydro. Met* .(1918); Vol. 46, p. 78.

40. Ertel, "Eine neue Methode zur Berechnung der Eigenschwingungen von wassermassen in Seen unregelmässiger Gestalt," *S. B. Preuss. Akad. Wiss.* (1938); Vol. 24.

41. Letter from Professor G. Chrystal to Sir Joseph Larmor FRS; *Royal Society of London, Library* , MSS. No. Lm 261.

42. F. A. Forel, " The Fata Morgana," *Proceedings of the Royal Society of Edinburgh* (1912); Vol. xxxii. pp.175-82.

Chapter 6

Professor Chrystal as a Mathematician, Scientist, and Educationist.

6.1 Introduction

One particular aspect of Chrystal's mathematics makes any view of his mathematical thinking necessarily incomplete. This was his deeply rooted reluctance to discuss the nature of mathematics in his addresses and writings. He did present a number of historical notes in his writings, but he nowhere offered his views on foundational questions. He did not generally channel his abilities into foundational questions and mathematical philosophy. It seems that mathematical philosophy was the only mathematical subject which did not attract his attention.

He was much interested in the theory of functions, the new developments in geometry, and algebra with applications to geometry, that is reflected in his book on algebra of which 1989 was incidentally the centenary year.

So he was a pure mathematician in the sense it was taken generally in Great Britain in the nineteenth century. In spite of his great contributions to the technical developments of mathematics, Chrystal must not be regarded as an unreflecting mathematician.

Ever since the foundation of the Scottish universities, mathematics had been studied independently in Scotland. The universities of Scotland, unlike those of England, instead of nursing an inclusive spirit and encouraging only scanty intercourse between teachers and students of different centres, lived in constant exchange of Professors and ideas_ much in the same way as has always been the custom on a larger scale among German and other

Continental universities. Though this is destructive of that individual character of the university or the college which was so highly prized by many English Fellows, it is certainly more conducive to the progress of studies and of research, and it is the reason why in the early history of science the universities of Scotland played so much more important a part than those of England. Whilst in England modern science was cultivated outside the universities by Priestley, Davy, Wollaston, Young, Dalton, Faraday, and Joule, to whom we may add Green and Boole, all eminent Scottish men of science, such as Gregory, Simson, MacLaurin, Playfair, Black, Thomson, Leslie, Brewster, and Forbes were university professors, many of whom did not confine their labours to one centre, but spread the light of their ideas and research all over the country.

6.2 Chrystal as a Mathematician

After an analysis of Professor Chrystal's mathematical work one concludes that he was not an original mathematician in the sense of carrying out original research, and with the exception of a few, in almost all his papers he studied a subject already dealt with by many previous mathematicians. The outstanding characteristic of these papers is that in them he has attempted to present the proofs very clearly, free of any ambiguity, and has further clarified them by giving numerous examples. To many he might seem to be a person finding fallacies in others' work instead of undertaking his own research, but this was not the case. Indeed, he pointed out all these fallacies in the service of mathematics to fulfil his main object of the diffusion of knowledge in a right way.

Nonetheless his knowledge of mathematics was very deep, and he had a firm grasp of all parts which interested him most; and was well acquainted with the history of the subject. This is evident from all his papers and also from his famous text-book on Algebra. To him history was the most important part in learning the background and significance of one's work, as well as in understanding the general nature of mathematics.

He had in mind the kind of knowledge about mathematics that will enable one to detect gaps where new concepts are needed; spot broad areas where new structures would provide unification and consolidation of seemingly diverse concepts; and recognise when a field has borne nearly all the mathematical fruit of which it is capable, so that it needs either to be given new vigour by fertilisation with ideas from other branches of mathematics, or possibly abandoned if its benefits to other fields are nil. So Professor Chrystal's achievement lies not in finding new mathematical ideas but in extending the domains of already existing ones to new areas. He, in contrast to others of his time, showed what extraordinary accomplishments mathematical thought can perform in the service of general questions.

Some people adopt a profession which is developed in them as a natural gift, Professor Chrystal was one of them and he had all the virtues which a good teacher should have.

In mathematics he was well abreast of the new developments, irrespective of their place of origin. He never kept anything for himself and tried to convey all the knowledge he had to the students who were eager to acquire it.

The following sentences drafted by his eldest son Mr(later Sir) George Chrystal for his obituary note, give what may be regarded as Chrystal's own estimate of himself in the rank of mathematicians[1]:-

"My father in familiar conversation with me always declined the title of a great original mathematician. How far this was justified, I have no means of judging; but his real bent seemed to be towards physical science_ towards the concrete rather than the abstract. With this, however, he had a keen appreciation, a great knowledge, and a thorough understanding of what had been achieved by the giants of the mathematical world_the Cayleys and Riemanns, whose results, as he used to tell me, were sometimes reached by stages and processes which even these great men themselves could not always thoroughly explain or account for.

What he regarded (I believe, though he never told me so in terms) as his special service to mathematics was that, by study and diligence and the exercise of intellectual power which he possessed, he had been able to consolidate some of the conquests made by the great mathematicians, his predecessors and contemporaries, and had evolved and excogitated a method by which the intelligent student of average ability could retread the path which had conducted the man of genius to his discoveries.

This method requires two things: *in the first place,* the abandonment of traditional practice of occupying, as it were, isolated points in the terrain to be conquered by science, from which isolated forays or raids were conducted under the guise of problem-solving and other virtuosities. Henceforth the pupil was to be conducted by an orderly series of reasonings up a sort of inclined plane from one well-defined conception to another, to the higher levels of science_ morphology in the words of Sylvester, was to be introduced into algebra and mathematical analysis in general.

Secondly: even in its elementary stages science of algebra required setting in order, and the morphological method required a new, a precise, and to some minds a 'forbidding' terminology. This was the 'reasoning of standard' playfully and ruefully described by J. M. Barrie in *An Edinburgh Eleven*" .

Looking carefully at students' accounts given of him, his lecture notes and papers, one can conclude that he had a thorough knowledge of the subject and was quite influential amongst mathematicians of the time in Great Britain. He was very close to his colleague Professor P. G. Tait, who was one of the examiners at Cambridge in 1875, when Chrystal was in his final year at Cambridge. According to Tait's remarks he was the best amongst candidates of the Mathematical Tripos of that year and his mind was highly original as compared with others.

After Tait's death in 1901, when Professor Chrystal was busy in writing an obituary notice of him for *Nature* , Lord Kelvin wrote to George Chrystal:[2]

"We[Kelvin and Tait] have had a thirty-eight years' war over quaternions. He had been captivated by the originality and extraordinary beauty of Hamilton's genius in this respect, and had accepted, I believe, definitely from Hamilton to take charge of quaternions after his death, which he has most loyally executed. Times without number I offered to let quaternions into Thomson and Tait[the treatise], if he could only show that in any case our work would be helped by their use. You will see that from beginning to end they were never introduced".

Professor Tait was one of the great promoter's of quaternions but was unable to convince his two close friends Lord Kelvin and Chrystal about the power of quaternions. Professors Chrystal and Tait had many virtues in common, both of them besides performing original work in mathematics did a great deal of experimental work. They were also fond of the right use of mathematical symbolism and both used this gift properly by introducing some new symbols in mathematics.

Since Professor Chrystal did not involve himself in foundational questions, his article "Mathematics" in the ninth edition of the Encyclopaedia Britannica carries special importance as being the only source of his views about mathematics. This article gives a brief exposition of the historical developments of different branches of mathematics.

Professor Chrystal defines a mathematical conception as," any conception which is definitely and completely determined by means of a finite number of specifications, say by assigning a finite number of elements". He further remarks that, " a mathematical conception is, from its very nature, abstract; indeed its abstractness is usually of a higher order than the abstractness of a logician".

He claims that the most convenient word at that time to draw attention at once to the fundamental idea involved in mathematical conception was "manifoldness," which he divided into two categories discrete and continuous. To him *Arithmetic* and *Number theory* were the results of discrete manifoldness, and that continuous manifoldness could be dealt with by the use of two methods, which may be called the *synoptic* and the *analytic* methods. In the synoptic method one could deduce the properties of a manifoldness by considering it as a whole, and wherever possible one could understand it better by a diagram, a model, or any other concrete device more or less refined according to circumstances. In the analytic method the properties of an element could be examined in the most general manner, and from that the properties of the manifoldness as a whole could be deduced.

The best and most familiar examples of the synoptic treatment of manifoldness as given by Chrystal are different kinds of pure geometry (e.g. Greek Geometry; Descriptive Geometry of Monge ; the Projective Geometry of Poncelet, Steiner, and Von Staudt; the Geometry of transformation in general, of which, Projective Geometry, is a special case). Those of the analytic treatment are Infinitesimal Calculus; Algebra; Cartesian geometry; The Géométrie de position of Carnot; The Line Geometry of Plücker and its use in various branches of applied mathematics, of which geometry is merely one of the simplest. The analytic method is far more common than the synoptic method, although most branches of applied mathematics are mixtures using one or the other, as happens to be convenient.

In his opinion " the two great methods employed in the investigation of manifoldness must of course be, at bottom, identical; and every conclusion arrived at by the one must be reachable by the other".

This indicates that Professor Chrystal considered that the conflict between the synthetic and analytic methods, which arose near the end of the eighteenth century and the beginning of the nineteenth century, had come to an end. Neither has come out as victorious. The two sides have realised that their greatest strength lies in the friendly competition and not in suppression of the other.

Another potential source for examining Chrystal's mathematical view point is his text-book on Algebra, the book which served as the first appropriate English book of the century on the subject and brought not only for him but for the whole of Scotland a long lasting fame.

This book is full of treasures of historic importance, which will remain useful for many generations to come. The following are few of numerous examples of this, which, also present the opinion of others about his work:-

William C. Waterhouse, a teacher of mathematics at Pennsylvania State University, New Park, in one of his articles[3]"Do Symmetric Problems Have Symmetric Solutions?" related to the history of mathematics has dealt with a general principle dealing with problems where the function under study is symmetric in several variables and its maximum or minimum occurs when the variables are equal.(e.g. of all rectangles with given perimeter, the square has the largest area). Waterhouse proposed that the principle be called the Purkiss Principle, to honour the Englishman H. J. Purkiss(1842-1865), a senior wrangler and Smith prizeman, who died at the early age of twenty-three by drowning in the river at Cambridge. The principle he stated as follows:

"Let $f(x_1, x_2, \ldots, x_n)$ and $g(x_1, x_2, \ldots, x_n)$ be two symmetric functions. On the set where g stays equal to $g(r, r, \ldots, r)$, the function f should have a local maximum or minimum at $(r, r, \ldots, r)$".

Professor Chrystal was the first to note the insufficiency of the proof presented by Purkiss in 1862[4] in the Messenger of Mathematics, of which, he was one of the founders, and Waterhouse acknowledges this as follows:-

"Purkiss did manage to show that the points (r, r, . . . , r) are critical points. But then he undeniably fell into error. When there is only one free variable, a critical point is 'in general' a local maximum or minimum _ i.e., it will be one or the other except for degenerate cases where the second derivative vanishes. But in several variables it is not true, and nondegenerate critical points can equally well be saddle points. Purkiss simply ignored this possibility, and thus he left a major gap in his argument. Yet it must be admitted that in our original examples we did not in fact encounter saddle points. Support for the principle can also be drawn from George Chrystal, who in 1889 included it in Part II of his famous Text-book on Algebra (PP:61-63). Recognising the inadequacy of Purkiss's proof, Chrystal treated only the case where f and g are symmetric polynomials; there he rewrote in terms of the principle (apart from some degenerate situations)".

Encouraged by the support, which he got from Chrystal, Waterhouse took the Purkiss principle where Purkiss had left it and tried to show that for some reason, one never gets saddle points.

He gives credit to Professor Chrystal as being the first one who gave the first proof of any significant form of the Purkiss Principle, and says that " the temptation to quote him was irresistible".

Another example is his omission in the last chapter of his book of a certain matter of doubtful soundness ; i.e., Inverse Probability. It is remarked by J. F. Box in his book " *The Life of a Scientist: a biography of R. A. Fisher* " [5] that " in the latter half of the nineteenth century, the criticisms of men like Boole, Venn, and Chrystal brought about a change of scientific opinion about inverse probability such that the subject fell into disrepute".

The possibility of posteriori probability distributions of parameters such as velocity of light was extremely attractive, and no one could see how to bridge the gap between probability theory and scientific inference without invoking the inverse probability argument. In its favour was the most persuasive fact that the method gave results that were consonant with reason; for example, on the assumption that errors were normally distributed, it yielded the method of least squares. Thus, despite recurrent doubt, it gained admission into statistical practice. Even today the logical status of Bayes' inference is uncertain; it has remained a ghost to haunt mathematical statisticians, being neither incorporated into the body of accepted theory nor finally laid to rest.

Laplace admitted the principle of inverse probability into the foundations of his exposition. Since he also developed mathematical analysis to a high state of sophistication, his free use of prior probabilities tended to obscure the logical issue. Gauss may well have had reservations on the subject, but he used the argument from inverse probability as one means by which to deduce and justify the least-squares procedure. His procedure was thus sometimes perceived as being dependent on inverse probability.

D. Higgs of the University of Waterloo, Ontario, presented a note[6] " On Products of Transpositions and Their Graphs," in which along with other theorems he proved the following:

Theorem 1. A product of transpositions of even length cannot be equivalent to a product of transpositions of odd length.

Theorem 4. A product of transpositions Π is minimal iff $|\Pi| = n - c(\Pi)$, where $|\Pi|$ is the length of Π, n the cardinality of the set on which all permutations are considered to act from the right, $c(\Pi)$ is the number of factors in the disjoint cycle decomposition of Π.

Professor Chrystal's involvement in these two particular theorems has been remarked by

D. Higgs as follows:-

" We have been unable to determine with certainty to whom Theorem 4. is due; our oldest reference is Chrystal [Algebra, Part II, Chap. 6, PP: 28-30], but he does not claim the result. It is worth remarking that his argument yields unified proof of Theorem 1 and Theorem 4 as given in the note".

These are only a few of such examples. There are many more, particularly concerning his dealing in detail with continued fractions.

His membership of the British Association for Advancement of Science was of great significance. He presided at Section"A" of the Association in its Aberdeen meeting in 1885. He was a member of many working committees of the Association, of which his membership in 1902 of the Committee on Teaching of Elementary Mathematics is of great importance. It shows recognition of his position as an educational reformer not only in Scotland but throughout Britain. Another significance of this committee is that a similar committee was appointed about thirty years earlier to consider a part of the matter, viz.,' the Possibility of Improving the Methods of Instruction in Elementary Geometry', of which Professor Cayley was a member. The report of the committee in 1902 was mainly influenced by the ideas of Professor Chrystal, the committee in its report recommended certain gradual improvements in both teaching and examinations. These included the rectification of the recommendations of the previous committee and the encouragement of the use of graphical methods. In particular, the members agreed that the general idea of co-ordinate geometry could be made familiar by the use of graphs and many of the notions underlying the methods of the infinitesimal calculus could similarly be given to comparatively youthful students long before the formal study of the calculus was begun.

This move was used by Professor Chrystal in his "Introduction to Algebra" in 1898 where he used a large section of the book for graphical methods. The committeee also recommended, having different levels of evaluation for different classes of students and that the teaching of demonstrative geometry be preceded by the teaching of experimental and practical geometry.

He also delivered a lecture to the section "A" of the Association in its Edinburgh meeting in 1892, " On a Curious Point Connected with the Parallel Axiom".

6.3 Chrystal as Scientist

As an experimentalist Chrystal secured a high position for himself by carrying out the experimental verification of Ohm's Law, which has already been dealt with in detail in chapter 2. Although he carried out only the experiments suggested by the committee, however setting up the apparatus and carrying out all the mathematics involved was not an easy job and this was acknowledged by Prof. Maxwell and other great scientists of the time. He did carry out some experiments devised by himself and the results were remarkable.

It is also fair to state that Alexander Graham Bell's invention of the telephone in 1876 encouraged many to investigate further usage of this important invention. These included Tait, George Forbes, Chrystal, and James Blyth. Prof. Chrystal's contributions in this respect played a significant role to promote research in this field.

Being an occupant of the chair of mathematics, he had to abandon all his experimental work. His position here was just like Plücker, but the other way round. Plücker occupied a chair of experimental physics, but was much interested in mathematical research, which he was later forced to abandon for sometime, and for nearly twenty years he devoted his energies to physics. But towards the close of his life he returned to his first love of mathematics, and enriched it with new discoveries.

Whenever Prof. Chrystal got a chance to satisfy his natural instinct he took advantage of the opportunity. He devised a special Hygrometer and Anemometer for higher altitudes, which were installed in the newly established Ben Nevis Observatory in 1884.

His membership of the Committee appointed for the purpose of constructing and issuing practical standards for use in Electrical Measurements is also of great importance. The Committee consisted of the following British scientists of great eminence:

Prof. G. Carey Foster, Sir William Thomson (later Lord Kelvin), Prof. Ayrton, Prof. J.

Perry, Prof. W. G. Adams, Lord Rayleigh, Dr. O. J. Lodge, Dr. John Hopkins, Dr. A. Muirhead, Mr W. H. Preece, Mr. Herbert Taylor, Prof. Everett, Prof. Schuster, Dr. J. H. Fleming, Prof. G. F. Fitzgerald, Mr. R. T. Glazebrook, Prof. Chrystal, Mr. H. Tomlinson, Prof. W. Garnett, Prof. J. J. Thomson, Mr. W. N. Shaw, Mr. J. T. Bottomley, and Mr. T. Gray. This committee was appointed to carry out the resolutions of the Paris Congress with regard to Electrical Standards of 1881 and later years. The Congress was essentially conducted by Helmholtz and Sir William Thomson. There, under the chairmanship of the French Minister of Commerce Cochery, the international standards_ Volt, Coulomb, Ampère, Farad, Joule, Watt, were established. The term "Gauss" for the unit of magnetic field strength gained acceptance only later, through a British proposal.

Although for many years he did not take part in any significant experimental work, still Chrystal remained in touch with science and the scientists.

He was always ready to advise and help others engaged in scientific investigation. Mr James Blyth mathematical teacher in George Watson's College was helped by Professor Chrystal in his experiments on the wire telephone by making exact observations as well as suggesting various changes to the experiment to bring about or eliminate particular effects. His involvement in seiche investigation is another example, for which he specially devised a Limnograph that proved to be very useful, and later converted many old type of limnographs into his form.

One test of how successful he was in these investigations is the manner in which it stimulated other investigators. Estimated in this way Prof. Chrystal's work is of great value, for we find a chain of numerous observers all over the world who have made elaborate seiche investigations, and in all cases the observations which have been made have fallen properly into line with Prof. Chrystal's mathematical theory.

Prof. Chrystal diligently read the literature of any subject in which he was specially interested and his knowledge and appreciation of the real significance of the far reaching work of the last century were probably unsurpassed.

His intimacy with Larmor, Maxwell, Stokes, Tait, Thomson, a few of the geniuses of the century, brought him into immediate contact with the springs of physical thought.

His great Articles " Electricity" and "Magnetism" achieved world wide recognition, and have been quoted by John Theodore Merz in his famous " History of European Thought in the Nineteenth Century," published in 1903.

6.4 Chrystal as an Educationist

During two years of teaching at Peter House and Corpus Christi College, Cambridge, he proved himself to be a successful teacher. The recognition of this came when, despite tough competition, he was appointed by the Crown as Regius Professor in the University of St. Andrews in 1877. In the next two years he further strengthened his role as a teacher as a result of which he was elected to the chair of mathematics at the University of Edinburgh.

He started his role as an educational reformer when he was a student at Cambridge where he took an active part in various university and college reforms, and continued playing this role during his two years occupancy of the St. Andrews chair. But his actual role as an educational reformer came during his stay at Edinburgh. All of these have been discussed in detail in Chapter 3. Here I want to make some conclusive remarks about his position as a reformer.

The turning point of his part in reforming secondary education in Scottish schools came with the reconstitution of the Scottish Education Department in 1885. As a result he, along with some other professors, was asked to inspect a number of secondary schools and present a report.

The idea of instituting a Leaving Certificate Examination was a result of this inspection. The idea was not originally due to him, in fact a Leaving Certificate Examination had been in operation in Prussia exactly hundred years before its institution by the Scottish Education Department. They were instituted in Prussian Gymnasia by the Royal Decree of 23 December 1788, and were intended to mark the completion of the course of secondary education.

But there is no evidence that he took the idea from the German Gymnasia, because if this would have been the case, Prof. Chrystal must have acknowledged it, as he did in many of his scientific and mathematical papers; but he gave no such acknowledgement. Instead, in his last promoter's Address of 1908, he made it clear that this was indeed his own idea. With this I agree and give him full credit for planning such an excellent idea which dominated Scottish education for almost a century; but now changes are desired from many quarters and seem to be imminent.

Sir Henry Craik the then Scottish Education Secretary, also deserves full credit for implementing all which was proposed by Prof. Chrystal, because it was an even more difficult job to convince universities and other professional bodies to recognise this certificate for entrance into these organisations, yet without this recognition all the effort was useless.

In 1886 Chrystal along with Prof. Laurie, Professor of Education, was elected by the Senatus of the University of Edinburgh to represent the university on the newly constituted governing body of the Heriot Trust and they continued to hold that office until 1902. Thus Chrystal played an influential part in laying the foundations of the Heriot-Watt Technical College, to commemorate the services of the famous Scots George Heriot and James Watt. The college which is now the Heriot-Watt University, was given its Charter on 31 January 1966.

Another important part of his reforms is the part which he played in the institution of a preliminary examination, greatly widening the Arts Curriculum, by introducing into it an elaborate system of options, setting up a new system for the Honours degree, the establishment of summer session in Arts, and for the first time admitting women to lectures and graduation. All these changes occurred during the period when he was Dean of the Faculty of Arts and he played a dominant role in working out the details of all these ordinances through the Senatus.

If one wanted to make a decision about the Arts curricula of Scottish universities in 1911-12 on sound educational lines, one would find that from Chrystal came the inspiration and guidance. The one influence making coherent and definite proposals towards change and advance within the precincts of the university came from him.

It is just a coincidence that most of his University reforms were exactly the same as those composed by Professor Cremona, himself a great Italian mathematician, in his report on Italian Educational Reforms presented to the Italian Senate. This Prof. Chrystal made clear in the very first of his Promoter's Addresses, where he quoted some parts of this report.

Last, but by no means least, was the work Prof. Chrystal did from 1905-09 as chairman of the Edinburgh Provincial Committee for the Training of Teachers.In the opinion, not only of his immediate colleagues on Edinburgh Provincial Committee but of those of other Provincial Committees who had the opportunity to meet under his chairmanship, in all matters pertaining to joint action affecting the whole of Scotland, Professor Chrystal was a valuable asset in all that pertained to starting and keeping on proper lines the new scheme for the training of teachers. One would require to ransack the dictionary for all the epithets necessary to describe a perfect chairman in order to give a fair estimate of the work done by Prof. Chrystal as chairman of Edinburgh Provincial Committee.

He made it possible once again to recognise that the schoolmaster was a powerful instrument for the furthering of the moral and spiritual as well as the intellectual welfare of the youth of the nation.Now is again the time, when the nation needs another Chrystal to get the position of a school teacher recognised in its proper form.Although there is no shortage of the talented young graduates who want to join the teaching profession, yet there are no good prospects in store for them.

So it is time for the Education Department to realise once again the fact that instead of importing foreign trained teachers and to implement this and other short term solutions, they should provide the talented young people in this country with sufficient means for maintaining themselves during a thorough course of University training. Not only this but there should be bright prospects held out for them after they complete their training, in the form of better pay, better chances of promotion, and the respect from the public at large. The training colleges and school administration should co-ordinate a well planned scheme, keeping in view the needs of the country.

As a teacher Chrystal was always kindly and sympathetic. Although at all times mathematics is thought to be very difficult, yet Professor Chrystal as a teacher dealt with it in a very interesting and illustrative way. According to a note in *The Student* magazine[7]:

"Professor Chrystal is never at a loss for an illustration. The other day he selected one from the region of domestic economy. The class listened with sympathetic cheers while he expatiated on the impossibility of getting a servant to do right, if she be bent on doing wrong. The ladies, no doubt on future house keeping thoughts intent, laughed merrily. And yet there are people that call mathematics dry!"

To the students who passed through the General class of mathematics on the way to an ordinary degree Chrystal was the superb lecturer and nothing more. Those who entered the advanced class to read for an Honours degree were better able to appreciate his varied gifts; but a full revelation of the great personality came only to the lucky few who acted as his assistants, or who worked with him.

He let nothing interfere with his official duties towards his class, declining on principle to make mention of anything but what had a direct connection with university regulations.

His Inaugural address as occupant of the chair of mathematics in Edinburgh, his address as president of the Section 'A' of the British Association meeting held at Aberdeen in 1885, and his Promoter's addresses of 1885, 1892, 1908, all reflect that he had firm ideas and with the passage of time and age his ideas remained mainly unaffected. In his addresses he was, for most, very lively and vivacious, and was always appreciated by his audience. In all these addresses he covered a wide range of subjects relevant to the educational needs of the time.

His inaugural address, " On History of Mathematics" with special reference to his predecessors in the Chair of Mathematics at Edinburgh, though not fully available anywhere, has been cited largely by Sir Alexander Grant, the then Principal, University of Edinburgh in Volume II of his famous book, "The Story of the University of Edinburgh:

During its first three hundred years" section VII, pp. 292-306.

The chair of mathematics in Edinburgh University was established in 1674 by the Town Council, and James Gregory was the first occupant of the chair, but his occupation of the chair was short lived, and he died of sudden blindness in October 1675. The chair had been occupied in succession by great mathematicians like David Gregory, Colin MacLaurin, John Playfair, John Leslie and many others.

An analysis of his addresses shows that he particularly stressed the need for diffusion of knowledge, and strived hard to achieve this. It was as a result of such attempts made by enthusiasts like him that made Scotland certainly a major contributor towards diffusion of modern scientific knowledge. He was indeed a great orator.

Professor Chrystal had a very strong personality, and fought bravely all the hardships he faced during his life. He was a man of business, a man of the world, and a diplomat.

References Chapter 6

1. Dr. J. Sutherland Black and Prof. C. G. Knott, " Obituary Notice of Prof. Chrystal," *Proceedings of The Royal Society of Edinburgh* (1911-12); Vol. XXXII, PP:496-97.

2. Obituary Notice of Prof. Tait, *Nature* (25 July 1901), Vol. LXIV, P. 306.

3. William C. Waterhouse, " Do Symmetric Problems have Symmetric Solutions?" *American Mathematical Monthly* (1983); VOL. 90, PP:378-87.

4. H. J. Purkiss, "Theorem in Maxima and Minima," *Messenger of Mathematics* (1862); VOL. I, PP: 180-83.

5. J. F. Box, *The Life of a Scientist: A Biography of R. A. Fisher* ; Wiley, New York(1920), P.68.

6. D. Higgs, "On Products of Transpositions and Their Graphs," *American Mathematical Monthly* (1979); Vol. 86, PP: 376-79.

7. *The Student* , 4 March 1897; Vol. XI, P.273.

PROFESSOR CHRYSTAL (1885)

PROFESSOR CHRYSTAL (1908)

Appendix A

Scientific Correspondence

Following is his correspondence with different men of science. Majority of this correspondence consists of letters after he became General Secretary of the Royal Society of Edinburgh in 1901, except the correspondence with Prof. J. C. Maxwell; Prof. G. G. Stokes; and Lord Rayleigh, which was in the beginning of his career as Professor of Mathematics and some even before that.

a. Royal Society Of London Library

1. One letter from G. Chrystal To Professor Arthur Schuster,
Mss Sch 48.

2. Correspondence with Sir Joseph Larmor,
Mss Lm 255-71.

b. University Of Cambridge Library

1. Correspondence with Sir William Thomson(later Lord Kelvin), consists of two letters from Lord Kelvin to Chrystal; Add Mss 7342 C86-88.
In addition there are drafts of twenty-eight letters from Lord Kelvin to Prof. G. Chrystal.

2. Chrystal/ Maxwell correspondence consisting of:

(i) Letters from George Chrystal to Prof. J. C. Maxwell,
Add Mss7655: II/82, 133, 141, 159, 161, 183.

(ii) Letters from Prof. J. C. Maxwell to George Chrystal,
Add Mss8375: 1, 4-11, 13-22.

3. Correspondence with Prof. G. G. Stokes consisting of six letters from Chrystal to Prof.Stokes, and two from Prof. Stokes to Chrystal(one being a testimonial), Add Mss7656 C529-535.

4. Correspondence with Sir J. Larmor,
Add Mss7656 L82-83.

5. Correspondence with Lord Rayleigh,

Add Mss8375 28-33.

c. **St. Andrews University Library**

1. Letter from George Chrystal to J. D. Forbes;

J. D. Forbes correspondence Mss 1880/47.

2.(i) Letters from D'Arcy Wentworth Thompson to Professor Chrystal;

D. W. T. Corr 517 Mss 13243-53.

(ii) Letters from Professor G. Chrystal to D'arcy Wentworth Thompson;

Mss 13254-56.

Appendix B

Scientific Communications, Books And Addresses

The following are the collected works of Professor Chrystal including published and unpublished work. these also include some of his class lecture notes available in the manuscript sections of the Libraries of St. Andrews and Edinburgh Universities.

1. Report Of The Committee For Testing Experimentally Ohm's Law.

   B. A. Reports, Glasgow Meeting (1876), pp. 36-63.

2. Results Of A Comparison Of The B. A. Units Of Electrical Resistance(With S. A. Saunder). *Ibid* ; pp. 13-19.

3. Bi- And Uni-Lateral Galvanometer Deflections.

   Philosophical Magazine , December 1876, 5th Series,

   Vol. II, No. 3, pp. 401-14.

4. Article," Electricity."

   Encyclopaedia Britannica , 9th Edition(1878), Vol. VIII, pp. 3-104.

5. Article," Electrometer."

   Ibid ; pp. 117-22.

6. Obituary Notice Of Professor P. Kelland(In Conjunction With Professor Tait).

   Proceedings Of The Royal Society Of Edinburgh (1879); Vol. X, pp. 321-29.

7. Article," Galvanometer."

   Encyclopaedia Britannica , 9th Edition(1879); Vol. X, pp. 49-53.

8. Article," Goniometer."

   Ibid ; pp. 771-72.

9. On Minding's System Of Forces.

   Proceedings Of The Royal Society Of Edinburgh (1880); Vol. X, pp. 397-400;

   And *Transactions Of The Royal Society Of Edinburgh* (1880);

   Vol. XXIX, pp. 519-30.

10. Address On Non-Euclidean Geometry.

Ibid ; pp. 638-64.

11. On A New Telephone Receiver.

Ibid ; pp. 682-84; And *Nature*, 24 June 1880; Vol. XXII, pp. 168-69.

12. On The Differential Telephone.

Proceedings Of The Royal Society Of Edinburgh (1880); Vol. X, pp. 685-89;

And *Transactions Of The Royal Society Of Edinburgh* (1880);

Vol. XXIX, pp. 609-36.

13. On The Wire Telephone And Its Application To The Study Of Strongly Magnetic Metals.

Proceedings Of The Royal Society Of Edinburgh (1880); Vol. X, pp. 707-10;

And *Nature* , 29 July 1880; Vol. XXII, pp. 303-07.

14. Note On Thomas Muir's Transformation Of A Determinant Into A Continuant.

Transactions Of The Royal Society Of Edinburgh (1881);

VOL. XXX, pp. 13-14.

15. On A Special Class Of Sturmians.

Ibid ; pp. 161-65.

16. A Review Of Clerk Maxwell's," A Treatise On Electricity And Magnetism," 2nd Ed. *Nature* , 12 January 1882; Vol. XXV, pp. 237-40.

17. Article," J. Von Lamont."

Encyclopaedia Britannica (1882); 9th Edition, Vol. XIV, p. 244.

18. Remarks On Dielectric Strength.

Proceedings Of The Royal Society Of Edinburgh (1882); Vol. XI, pp. 487-98.

19. Introductory Address To The First Meeting Of Edinburgh Mathematical Society On" Present Fields Of Mathematical Research." (Title Only).

Proceedings Of Edinburgh Mathematical Society (1883); Vol. I, P. 3.

20. Article," Magnetism."

Encyclopaedia Britannica (1883); 9th Edition, Vol. XV, pp. 219-76.

21. Article," Mascheroni."

Ibid ; p. 608.

22. Article," Mathematics."

Ibid ; pp. 629-30.

23. Article," Michell."

Encyclopaedia Britannica (1883); 9th Edition, Vol. XVI, p. 237.

24. Article," Montucla."

Ibid ; p. 798.

25. Article," R. Murphy."

Encyclopaedia Britannica (1884); 9th Edition, Vol. XVII, P. 57.

26. Article," Musschenbroek."

Ibid ; pp. 109-10.

27. Mathematical Models Chiefly Of The Surfaces Of The Second Degree.

Proceedings Of The Edinburgh Mathematical Society (1884); Vol. II, P. 5.

28. Application Of The Multiplication Of Matrices To Prove A Theorem In Spherical Trigonometry. *Ibid* ; pp. 45-47.

29. On The Discrimination Of Conics Developed By The Rays Joining The Corresponding Points Of Two Projective Ranges. *Ibid* ; pp. 47-49.

30. On A Problem In The Partition Of Numbers.

Ibid ; pp. 49-50.

31. A Christmas Visit To Ben Nevis Observatory.

Nature , 3 January 1884, Vol. XXIX, pp. 219-22.

32. Three Hundredth Anniversary Of The Edinburgh University.

Nature , 17 April 1884, Vol. XXIX, P. 577.

33. Promoter's Address To Graduates Of *Edinburgh University* . 22 April 1885.

34. Presidential Address Section 'A' Of The British Association For Advancement Of Science. 10 September 1885; *B. A. Reports, Aberdeen* (1885). pp. 889-96.

35. Article," Oughtred."

Encyclopaedia Britannica (1885); 9th Edition, Vol. XVIII, P. 74.

36. Article," Parallels."

Ibid ; pp. 254-55.

37. Article," Pascal(Part)."

Ibid ; p. 338.

38. Article," G. Peacock."

Ibid ; pp. 343-44.

39. Article," J. Pell."

Ibid ; pp. 475-76.

40. Article," Perpetual Motion."

Ibid ; pp. 553-55.

41. Article," J. F. Pfaff."

Ibid ; p. 725.

42. Article," J. Playfair."

Encyclopaedia Britannica (1885); 9th Edition, Vol. XIX, P. 217.

43. Article," J. Plücker."

Ibid ; pp. 228-30.

44. Article," J. C. Poggendorf."

Ibid ; pp. 273-74.

45. Article," S. D. Poisson."

Ibid ; pp. 279-81.

46. On The Problem To Construct The Minimum Circle Enclosing Given Points In A Plane.

Proceedings Of The Edinburgh Mathematical Society (1885);

Vol. III, pp. 30-38.

47. On Certain Formulæ For Repeated Differentiation.

Ibid ; p. 94.

48. On A Method For Obtaining The Differential Equation To An Algebraic Curve.

Ibid ; pp. 95-100.

49. On The Hessian.

Transactions Of The Royal Society Of Edinburgh (1885);

Vol. XXXII, pp. 645-50.

50. Article," Robert Recorde."

Encyclopaedia Britannica (1886); 9th Edition, Vol XX, P. 310.

51. Article," Rheticus."

Ibid ; p. 508.

52. Article," G. F. B. Riemann."

Ibid ; pp. 550-51.

53. Article," B. Robins."

Ibid ; p. 607.

54. Article," J. C. F. Sturm."

Encyclopaedia Britannica (1886); 9th Edition, Vol. XXII, P. 612.

55. *Algebra: An Elementary Text Book For The Higher Classes Of Secondary Schools And For Colleges* . Part I.

Adam And Charles Black: Edinburgh(1886).

56. Obituary Notice Of Joseph Liouville.

Proceedings Of The Royal Society Of Edinburgh (1886-87);

Vol. XIV, pp. 83-91.

57. On Certain Inverse Roulette Problems.

Proceedings Of The Edinburgh Mathematical Society (1887);

Vol. V, pp. 38-50.

58. On The Inequality $mx^{m-1}(x-1) \gtrless x^m-1 \gtrless m(x-1)$ And Its Consequences.

Proceedings Of The Edinburgh Mathematical Society (1888);

Vol VI, pp. 29-33.

59. An Electrical Method Of Reversing Deep Sea Thermometers.(Title Only).

Proceedings Of The Royal Society Of Edinburgh (1888); Vol. XV, P. 298.

60. An Elementary Discussion Of The Closeness Of The Approximation

In Stirling's Theorem. (Title Only).

Proceedings Of The Edinburgh Mathematical Society (1889);

Vol. VII, P. 26.

61. *Algebra: An Elementary Text Book* . Part II.

Adam And Charles Black: Edinburgh(1889).

62. On Some Fundamental Principles In The Theory Of Probability.

Transactions Actuarial Society Edinburgh (1891); Vol. II, P. 13.

63. A Demonstration Of Lagrange's Rule For The Solution Of A Linear Differential

Equation, With Some Historical Remarks On Defective Demonstrations.

Transactions Of The Royal Society Of Edinburgh (1891);

Vol. XXXVI, pp. 551-62.

64. Promoter's Address To Graduates Of Edinburgh University(1892).

The Scotsman ; Friday, 15 April 1892.

65. On a Curious Point Connected With The Parallel Axiom.

British Association Meeting , Edinburgh(1892).

66. A Fundamental Theorem Regarding The Equivalence Of Systems Of Ordinary Linear Differential Equations And Its Applications To The Determination Of The Order And The Systematic Solution Of A Determinate System Of Such Equations.

Transactions Of The Royal Society Of Edinburgh (1895);

Vol. XXXVIII. pp. 163-78.

67. A Summary Of The Theory Of The Refraction Of Thin Approximately Axial Pencils Through A Series Of Media Bounded By Spherical Surfaces, With Application To A Photographic Triplet, Etc.

Proceedings Of The Edinburgh Mathematical Society (1896);

Vol. XIV, pp. 2-25.

68. On The p-Discriminant Of A Differential Equation Of The First Order, And On Certain Points In The General Theory Of Envelopes Connected Therewith.

Transactions Of The Royal Society Of Edinburgh (1896);

Vol. XXXVIII, pp. 803-24.

69. *Introduction To Algebra For The Use Of Secondary Schools And Technical Colleges*

Adam And Charles Black: London(1898).

70. Some Elementary Theorems Regarding Surds.

Proceedings Of The Edinburgh Mathematical Society (1901);

VOL.XIX, pp. 46-49.

71. Note On The Mathematical Theory Of Miller's Trisector, And Its Relation To Other Solutions Of The Problem Of Trisection.

Proceedings Of The Royal Society Of Edinburgh (1901);

VOL. XXIV, pp. 9-16.

72. On The Relation Of Miller's Trisector To The Quartic Trisectrix, With A Description Of A Seven-Bar Limaçonograph.

Ibid ; pp. 17-20.

73. Obituary Notice Of Professor Tait.

Nature , 25 July 1901; Vol. LXIV, pp. 305-07.

74. with E. M. Wedderburn, "On The Theory Of Seiches. Experimental Illustrations" (Title Only).

Proceedings Of The Royal Society Of Edinburgh (1903); Vol. XXIV, P. 607.

75. Obituary Notice Of Luigi Cremona.

Ibid ; pp. 648-49.

76. Some Results In The Mathematical Theory Of Seiches.

Proceedings Of The Royal Society Of Edinburgh (1904); Vol. XXV. Part I. PP:328-37.

77. Some Further Results In The Mathematical Theory Of Seiches.

Ibid ; Part II. pp. 637-47.

78. On The Hydrodynamical Theory Of Seiches.

Transactions Of The Royal Society Of Edinburgh (1905); Vol. XLI, pp. 599-649.

79. Calculations Of The Periods And Nodes Of Loch Earn And Traig From The Bathymetric Data Of The Scottish Lake Survey. In Conjunction With E. M. Wedderburn. *Ibid* ; pp. 823-50.

80. An Investigation Of The Seiches Of Loch Earn By The Scottish Lake Survey.

Part I._ Limnographic Instruments And Methods Of Observation.

Part II.(By James Murray)_ Limnographic Observations.

Transactions Of The Royal Society Of Edinburgh (1906); Vol. XLV, pp. 361-96.

81. Seiches In The Lakes Of Scotland.

Proceedings Of The Royal Institution Of Great Britain (1905-07). pp. 657-76.

82. An Investigation Of The Seiches Of Loch Earn By The Scottish Lake Survey.

Part III._ Observations To Determine The Periods And Nodes.

Part IV._ Effect Of Meteorological Conditions Upon The Denivellation Of Lakes.

Part V._ Mathematical Appendix On The Effect Of Pressure Disturbances Upon The Seiches In A Symmetric Parabolic Lake.

Transactions Of The Royal Society Of Edinburgh (1908); Vol. XLVI, pp. 455-517.

83. On The Theory Of Leaking Microbarograph, And On Some Observations Made With A Triad Of Dines-Shaw Instruments.

Proceedings Of The Royal Society Of Edinburgh (1908); Vol. XXVIII, pp. 437-60.

84. Seiches And Other Oscillations Of Lake-Surfaces, Observed By The Scottish Lake Survey. *Bathymetrical Survey Of The Scottish Fresh Water Lochs* . Edited By Sir John Murray And James Pullar(1910); Vol. I, pp. 29-90.

85. (a)C. C. Easterbrook M. A. 1887,"Lectures on Senior Mathematics" *Edinburgh University Library*, MSS E72/26, Gen. 1903.

(b)Lectures on Mathematics (1894-95), Edinburgh University Library, MSS. E75/59, Gen. 1992/2.

(c)Robert Russel," 1887-88 Class Notes"; *St. Andrews University Library, Special Collections.*

Appendix C

PROMOTOR'S ADDRESSES

When Professor Chrystal took charge of the Chair of Mathematics in University of Edinburgh, every professor in the Faculty of Arts was required to deliver promoter's address to the new graduates of the University. In the begining of his professoriate, since there were only seven professors in the Faculty, so his first and second promoter's addresses were delivered in 1885, 1892, but then the number of professors in the Faculty increased and so his third and last promoter's address was delivered in 1908.

These addresses of Prof. Chrystal are full of valuable suggestions and cover many aspects of education at that time.

First address, 22nd APRIL 1885.

Gentlemen, -- In conformity with a custom of a good many years' standing, it is now my duty to address you, the newly promoted graduates of this university.

There is one topic that will doubtless occur to all of you as appropriate for this address, for there is one figure that used to be prominent among us on this occasion that will appear here no more. So suddenly did the death of Sir Alexander Grant overtake us in the hurry of the busy session that we scarcely had time to realise our loss before we were whirled away in the rapid current of college work. Now that leisure has come to us once more, that loss will be felt anew; and I greatly regret that the office of promoter is not occupied by one better fitted than I am to give expression to this common feeling. I have the double disadvantage of not having been personally intimate with Sir Alexander Grant, and of being little versed in the department of scholarship in which he made his reputation. My relations with him were solely in the way of university business. Few as even these were, they led me to form a very high opinion of the acute business-like character of his intellect. I was particularly struck with this while we were engaged in drawing up the financial statement criticising the Treasury clauses of the University Bill. Sir Alexander seized and retained the points of the financial character. During the whole course of discussions regarding

University reform I was much impressed with the width and liberality of our late Principal's views. Even when some of us happened to be unable to come to the same conclusion as he did, we never failed to recognise the manly straightforwardness of his course, and to perceive that he thought only of the highest interests of the University regarded as an institution for the advancement of sound learning. The principalship of Sir Alexander Grant will unquestionably be a marked era in the history of our University, for during his time it attained a prosperity such as it had never seen before. I can of course judge from what I have observed during the last five years of the period in question; but from that I should say that much of this prosperity must have been directly due to him. At all events I am certain of this, that during the last few years of his principalship, when the prosperity of the university had brought her difficulties naturally arising out of her increase, his tact and moderation did much to retain the many firm friends which she still happily possesses. To his tact also it was in a large measure due that an appeal to those friends resulted in the palatial buildings in which our brethren of the medical faculty are now established. The memory of the great Tercentenary Festival is still so fresh that I need not allude to it, except to mention its success as a striking testimony to the boldness in conceiving and skill in executing of our late Principal. A heavy share of the work connected with that celebration fell upon him, and I fear that the effects of it shortened his life. One of the most amiable characteristics of Sir Alexander Grant was his lively and continual interest in all that concerned the personal welfare of the students. The last great University scheme about which he busied himself was the founding of a Union Society and Club Room for the students and graduates of the University; and I venture to suggest that the best tribute you can pay to his memory would be to put your shoulders to the wheel, and carry into immediate execution this project which interested him so greatly.

The next topic which naturally suggests itself to a University Professor at the present moment is the burning question of university reform; and I propose to say a word or two on this all-important subject. Although I have never hitherto taken any part in the public discussion of this matter, I have by no means been an indifferent spectator. I have for the last fifteen years been an ardent student of everything relating to our higher education. In the course of that time I have been more or less intimately connected as student or teacher with five different universities, home and foreign; and ever since I became a Scottish Professor, i.e., for the last eight years or so, I have largely availed myself of opportunities offered me for examining secondary schools in England and Scotland. I may also add that I have read every publication, good or bad, bearing on the subject which has come within my notice during the last ten years. To the last of these, viz., the Italian University Bill, I shall make some allusions by and by.

It would be merest affection to say that I have not learned much from all this experience; but it is quite within the truth to say that there are many practical questions of high importance, on the solution of which my experience throws no light whatever. My observation regarding much of the writing on this subject has been that the more confident the proposer of a scheme, the more profoundly superficial, if I may use such an expression, has it appeared on examination. It is not unnatural therefore that I should be somewhat confident in setting forth my own views on the subject. Some of my conclusions are merely of a negative character, and for others, more positive in nature, I claim merely that they are the result of careful consideration, after the weighing of experience certainly wider than that of many of those who have come forward as physicians of our educational maladies. Above all, I wish it to be understood that I am a mere "vagrom man," claiming to represent no sect, party, or body of men whatsoever, not even my colleagues in the University.

Higher education is an expensive commodity, the furnishing of which involves most important practical questions regarding *men* and *money*. Who are the men that are to receive it? Where are the men to come from who are to give it? How is the money to be provided to maintain the givers of it, and to equip them with the necessary but costly apparatus? The higher education in the strictest sense of the word must always be the possession of a very few, and yet the proposition that avenues to it should be open to everyone, however poor, who has shown special fitness to receive it, is to my mind so obvious, and is moreover so universally accepted in Scotland, that it would be idle to discuss it here. This proposition carries with it of course the admission that the higher education must be supported to a large extent by the community at large, and can never be treated as a merely commercial article, subject to the ordinary laws of supply and demand. In most civilised modern nations a large part of this support for secondary education is drawn from the state; and in Scotland, as every one knows, the same practice has been contemplated ever since the great scheme for reorganising our national instruction was planned by John Knox. One of the most objectionable features of the last University Bill was the reactionary proposal that the rights of the Scottish universities should be bought off, and their connection with the state practically severed. Happily we are to hear no more of this idea, and with it disappears the greater part of my objection to the Government measure. On reading the report containing the Italian Bill represented to the Italian Senate by my friend Professor Cremona, I was greatly struck to find that a similar proposal had been made in the original draft of their measure as it came from the former Minister of Education, and that it had been met and overthrown by arguments almost identical with those used by the University of Edinburgh and many others in Scotland.

The idea of the original Italian measure was precisely the notion to which we were treated by some of the Treasury officials_viz., *autonomia* and *dotazione fissa*, i.e., autonomy and a fixed grant. It was said by many, in excuse of the Treasury, that the finality clause meant nothing; there never was a greater mistake. We found the authorities in London far more ready to bargain with us as to the sum than to give the principle. They wanted to be done with us, that was plain. It was a matter of so and so many thousand pounds. Now, as regards the interests and the vested rights of the University officials of the present generation, that might be so; but what about the birthright of the Scottish Universities? The finality clause was, to use the very words of the Italian report, " a reform in the wrong direction, which would inevitably have caused the decline instead of the progress of our studies, because our universities, even the best of them, are very far from having attained the full development demanded by the actual state of science." The finality clause gone, one of the greatest obstacles to a gradual reform of our higher education disappears. No one in his senses expects that an executive commission will be able to sit down and draw up a scheme that will at once meet all our difficulties for all time coming. Such an idea belongs to the childhood of an educational reformer. What the commission will, in all probability, do, --what they certainly ought to do,-- is to put elasticity and,if need be, joints into the cast-iron framework of our University Constitution, which will enable us gradually, as men and money can be found, to adopt ourselves to the existing want of our time. I will quote again, from the account of the views of Signor Coppino, the new Minister of Public Education, a sentence or two which express my views exactly. " I would prefer that the state, reserving to itself a high surveillance and the right of approval, should concede the most ample scientific-didactic freedom to the universities, meaning thereby the totality of university professors, who would be called to propose in new regulations or statutes of the faculties compiled by a commission elected by and common to all the universities those parts of the scholastic *régime* which are not purely administrative, but are founded on scientific and technical criteria. Thus that part of the matter which by its nature ought to follow the progress of science and the movement of ideas would be determined by statutes made by experts and subjected to periodical revision at shorter intervals; while those parts should be determined by law which do not depend on scientific opinion, and which may without detriment remain unchanged for such a longer period of time as the life of an organic law regarding public instruction is wont to be."

So much for the question of the *highest* education, and the duty of the state to provide it for those, be they rich or poor, who have shown themselves *fittest* to receive it. I shall return immediately to the consideration of the best means to be adopted for providing such education; but let me just mention that there is another part of the subject which the above consideration do not touch. How far is it the right of every one, not unfit, but not necessarily of the fittest, who is willing to pay for it to receive the highest education? This raises a variety of very difficult questions, on which I am not well prepared to give any definite opinion. How far, for instance, shall the higher education be self-supporting, i.e., be paid for by those who receive it ? What subjects shall be included in a course of general culture? What proportion of a course of general culture shall be given inside the universities? and how much relegated to the schools? How far shall secondary teachers and professors be paid by the fees of their pupils, and how far by fixed endowments? I have always felt myself, and now more than ever, since the not unfrequent attacks made of late upon University officials, that these questions to be decided by the cultured public, or by the state as representing them. After all, it is mainly for those who are to be educated to say what education shall be, and not for the teacher, who is their paid servant. The unsatisfactory part of this matter is that a close study of the opinion emitted, although it discovers a decided tendency to depart from the old course, shows no approach to unanimity regarding the path or paths we are to follow in the future. If I may be allowed, without pressing it on any one, to state my own private inclination, I should say that I have a preference for the old M. A. course of the Scottish Universities, especially in Aberdeen form, where it embraces a class of Natural Science. I went through it myself, and took full advantage of almost every part of it, and it has served me well; and if in the course of my wanderings I have often found men far my superior in special knowledge, I never saw any one of whom I could say that he seemed to me to have had better opportunities for laying the foundation of a good general education. Of course I do not mean that that course is not susceptible of great improvement, particularly in the earlier stages in and outside the University. There has been much talk of the Universities depressing the standard in the schools. In my department, at least, much of that is somewhat of an exaggeration, and what there is of truth in it was due to the illiberal treatment which my subject used to receive under the *régime* of those very classicists who are loudest in their complaints against the Universities. When I entered the University of Aberdeen some eighteen years ago I was a moderate classical scholar, but I had learned practically no mathematics. We used to read the first book of Euclid as far as the *pons asinorum* ; but regularly as we reached that dreadful pass we were turned back for a revisal. Algebra I had none, not to speak of other mathematical furniture. Yet large demands were made upon me during my second session under Professors Fuller, and I had to work hard during the spare time of my first year to be able to take his junior class with advantage. The fact that mathematical students from Aberdeen had been doing well in the world long before the time I allude to, was due to no exertions on the part of schools, but simply to the presence in the Faculty of Arts of two teachers, Professor Fuller and Thomson, of exceptional energy and ability, whose efforts were ably seconded by a private tutor, Mr Rennet, well known and much beloved by all Aberdeen graduates, who combined in a way more happy than common the power of dealing at once with the best and with the worst material that came up to the University. With regard to the rest of the teachers of my first *alma mater* , I have this to say, that I entered their lecture rooms a child intellectually, and that I emerged a man, and that during no other part of my mental life have I made so much intellectual progress as I did under their tuition.

Much improvement has doubtless been effected in the mathematical department of the schools of the north since my day. I had in fact occasion to remark it in a recent examination of my old school; and perhaps in the south, where there were secondary schools at all, the state of matters was never quite so bad; but I find no difficulty in my department of the University in providing for the better prepared students that come up to me. The fact is that one of my difficulties is that so few of those who pass the entrance examination in mathematics avail themselves of the privilege to which they are entitled. The number who pass is yearly on the increase; last year it was about fifty, but the fraction of these that actually enter my second class, as they ought to do, is very small; and this postponement of their promotion greatly aggravates their difficulties when they come to read for honours. All my representations on this subject to students and schoolmasters have hitherto been of no avail. The fact is that nothing prevents a well-prepared student from entering at once my third class. One once proposed to do so, and I went to the trouble of obtaining the consent of the faculty, which indeed was readily given, but the young man finally did not appear. Without enlarging more on this subject, I may be allowed to say that the difficulty I find in the meantime is not in giving a sufficient amount of the higher teaching, but in getting men to receive it.

This brings me to what to my mind is our most glaring defect in Scotland --the fact that we do not produce so many men with the highest special training as we ought to do. There can be no doubt in the mind of any one who has compared our secondary schools even with those to be met with in England--and that is not a high standard of measurement-- that our secondary schoolmasters are very much behind the age in the matter of special training. The reasons for this are not far to seek. It does not lie in the character of the teachers themselves, for a more intelligent and devoted body of men I have never met with; and in my discussions with them--which have been many -- as to the best means for raising the standards of education, I have met everywhere with the heartiest sympathy; and if we have accomplished anything of late, and I think we have perhaps done a very little, that is quite as much due to their co-operation as to any other cause that I am aware of. One of the best evidences of their readiness to avail themselves of every aid in their profession is the wonderful success of the Edinburgh Mathematical Society, which was founded with a view to the spreading, more particularly among our younger schoolmasters, of the latest and freshest ideas in mathematical science. It now numbers about 150 members drawn from all parts of the country.

No, the fault is not in the men, but in the poor prospect held out to them, and the scanty means at their disposal for maintaining themselves during a thorough course of University training. Time after time I lose honour men simply because they cannot afford to stay with me any longer; but must either go to some educational centre, such as Cambridge, where inducements are held out to them, or they must turn themselves to the making of their livelihood. The posts open to such men as teachers in Scotland are so few and so wretchedly paid that we cannot retain in the country the few good men we produce. They go away to India, to Australia, to New Zealand, and to the English Grammar Schools, because at home people will talk about the reform of our educational institutions, and every now and then abuse somebody in the newspapers, but will do nothing in the way of pressing upon the State the necessity for removing the disgrace we incur in the eyes of civilised nations by failing to provide an organised system of schools for the higher instruction. To institute entrance examinations and tinker University constitutions ever so cunningly, and expect thereby to cure this radical defect, is as reasonable as to hope by combing the hair of a patient to cure him of spinal disease.

However, in the meantime, let the patient's hair be combed by all means. If that be judiciously done, it will do no harm. In the suggestions I am now to make I have in view chiefly the special training of a higher class of secondary schoolmasters than we now possess, and the opening up of academic careers for men who are to take up this profession, and who may also aspire ultimately to become University teachers and cultivators of the higher branches of learning. After much careful consideration I have gradually come to the conclusion that this will be best done by adopting such a modification of the German University System as will suit our circumstances. I have been much strengthened in this conclusion by finding that the Italians, whose educational evils are very like our own, viz., a superabundance of ill-equipped Universities and a defective system of secondary schools, have adopted this course.

I think that in the future the system of paying our University Professors so largely by fees directly assigned to them should be departed from; and its place taken by an extension of the plan of endowment. There should no longer be the same rigidity as to the exact number and rank of professors; there should be more of them, and they should be graded as to pay and position. In the scientific didactic sense I would give them the utmost liberty;

that is to say, I would prevent no one from lecturing on a subject which he was specially fitted to treat, simply because that subject was touched upon in the course of a colleague.

In the future, professors should be appointed to a faculty and a scientific department more than to a particular chair, as at present. There ought to be various ranks, say ordinary professors, extra-ordinary professors, with less experience and lower pay, but with prospects of advancement, both of these should be fixed; then lecturers with temporary appointments, who might be either in probation or merely appointed for a time, on account of some special literary or scientific demand for instruction which they were peculiarly fitted to impart. Finally, I would concede to every man who has taken honours in a special subject, and who has passed a two years' probation in that subject, and during that time done such work as shall satisfy his faculty that he is fit for the trust, the liberty to give lectures to matriculated University students within the University, under the University discipline -- these lectures to be remunerated by fees from those attending them. I would also contemplate that pecuniary assistance of moderate amount should be given to the more deserving of these *privatim docentes* , regard being had in its distribution alike to service rendered to the University as a teaching body, and to original contributions made to literature and science.

Most important questions of detail arise in connection with this scheme which time will not permit me to discuss. There is the question as to the relative share of this body of teachers in the government of the University. I should contemplate a full share in the general government being given only to those of higher rank; but all the professors, ordinary and extraordinary, should have some voice in didactic matters, i.e., in that part of the routine work of the faculties which consists in arranging programmes of instruction; but in all matters, such as discipline, that require immediate action, the responsibility should be concentrated -- the determining voices weighty but few.

So far as examinations are to form part of the University instruction (and I may say, and every experienced teacher now-a-days will agree with me, that their number ought to consist of all the professors, ordinary and extra-ordinary, in the respective subjects, along with state appointed assessors, if that be thought necessary, to secure impartiality, and prevent the examination from becoming a sham, as they are apt to do, in spite even of the efforts of honest examiners to prevent it.

I do not contemplate that the number of ordinary professors should be fixed, only it ought to be kept below a certain maximum proportion of the whole staff. There might, under certain circumstances, be no ordinary professors at all in a particular department, but only energetic, promising extraordinaries, who had not yet attained the highest academic rank, and the corresponding pay. The great advantage of this plan is that it offers a continuous career for young men, perfectly open and unrestricted at the beginning (save by vocation), but giving in its course the fullest freedom for the selection of the fittest. The great defect of our present system is that it lifts a man at once to his maximum income, and puts in general no prospect before him whatsoever. Very possibly he may have the chagrin to see his pay diminishing year by year, from causes which he cannot control, and his work ever-increasing. Yet another advantage of the system I advocate, is that it would no longer necessitate the abrupt retirement of professors, which is at present so necessary, and so costly a feature of our system. Often a Professor retires at present simply because some trifling infirmity renders him unfit to lecture to a large class; but under the free system, all that would be necessary would be to promote him, with or without some trifling increase of pay, and indicate to him that he was better fitted to give lectures on some speciality to a select class. A younger man would then take his place with less pay, and would be well content to do so, knowing that if he worked hard in his vigorous youth, he would be promoted to lighter labour and not less pay when he grew old. Surely this is a better -- a more natural -- system than the one at present in vogue; moreover it works well in the German Universities, where a professor is allowed to die in honour among his academic colleagues.

But you will say this is a revolution. I grant it freely; but it will be a very gradual one.

The full carrying out of the plan in every department would take more men than now exist in Scotland fit for the work indicated, and more money than we could hope to get just at present, -- more even than it would be right to expend all at once on anything of the kind; but I want is that the University Commission should give us framework of such a system as I describe, so that we could fill it in gradually, as the means come to us. Benefactors of the University have never been wanting, and if we had room in our system for such posts as I describe, I believe that many would be found to perpetuate their memory by helping to endow them. No reform of anything like the requisite depth can take full effect in much less than ten or fifteen years; and how many obstacles will be swept away in that time? The vested rights, about which there has been so much needless bad language, would have well-nigh disappeared. When I look to the University of Aberdeen, I find in the Faculty of Arts only one Professor, who taught there in my day.

There is one thing that must be attended in the gradual introduction of a system of free teaching and free study, viz., that one or two years, according to circumstances, of the earlier part of the Scottish University students' course is virtually a gymnasial course, when judged by continental standards. To this part of the students' career, so long as it continues within the Universities, the ideas of *lehr* - and *lern-freiheit* may have to be applied with some caution. There may have, for instance, to be a somewhat more careful attention on the part of the Faculty to the division of this part of certain professors' work. But, in point of fact, there always is, even in Germany, except in the rarest cases, an understanding among the professors and *privat-docenten* as to the division of the work of each session. Here I would quote again, from the Italian report, words which express my views exactly, and which will, I believe, meet with the approval of every expert in the theory and practice of education. " We do not participate in the illusion of those who attach great importance to a competition on the ground of the same teaching between private and official teacher, as to who shall attract the most students, be it by the most honourable means. Such a competition is very rare, even in Germany, where the system of private teachers has been rooted for a long time. The competition usually takes place in quite another way. If there are vigorous youths qualified for free teaching, it is impossible for the professor to go to sleep; he is considered to aim high, and to go forward, in order to keep himself worthy of his office. And the height to which he rises, and at which he maintains himself, exerts in turn an action on the private teacher, and prevents him from slackening rein. Moreover, the free teaching fills the lacunæ of the official courses, and relieves the Government of the necessity for providing an excessive number of officials."

And now, this brings me to the last of my themes, how our Universities should *not* be altered.

In the scheme I sketched a little ago it is essential that all the teaching, whether by ordinary or extraordinary professors, by lecturers or by *privatim docentes* , shall be give under exactly the same conditions as to supervision and discipline. The private teaching, to be effective and free from abuses, must be given either in the lecture rooms of the University, or in places as fully under University supervision as these lecture rooms are. To adopt a phrase of my colleague, Dr Crum Brown, the University in this matter is or ought to be the educational arm of the State. Here again the words of the admirable report of Cremona:--" After the qualification has been given, the (free) teaching ought to be given in the buildings of the University, since it does not appear right that the State should have to grant a legal value to courses given beyond all discipline and vigilance. Just as the liberal professions are exercised under the discipline and guarantee of the State, so education ought to be disciplined and guaranteed."

Here of course I come into collision with extra-mural doctrine as it is understood in Scotland. I have studied this system anxiously among the other objects relating to University reform, and I see no harm at present juncture of indulging in a little plain speaking as to my conclusions. I am very much afraid, from what fell from some of our distinguished guests during the Tercentenary Festival, that the famous Edinburgh extra-mural system of medical teaching -- great as its claims to recognition have been in the past, and important as are the advantages it still to some degree presents -- is regarded by high authorities as little better than a relic of academic barbarism, bearing as little relation to the elaborate organisation of higher education in modern civilised States as do the bows and arrows of Crecy and Agincourt to the long range artillery of the battle of Sedan. The illusion mentioned in the Italian report has a good deal to do with the beliefs of the people who advocate the expansion of extra-academical teaching in its present form. It is a not unnatural one for those to entertain who have read of foreign institutions without seeing them at work. They constantly identify the continental system of *privatim docentes* with the home product, but in truth no two things could be more unlike in their actual working.

Let me begin by conceding the advantages of the extra-mural plan as we now have it. It affords a training school for professors, and it occasionally (of late not so frequently as of yore) supplies the lacuna in the official instruction caused by a useless professor, or by the want of a lectureship on some special subject. The most superficial examination of the two systems shows that the one I have sketched as preferable fulfils these requirements in a far more effectual way. It does that systematically which in the order is left to chance. In my system one great defect of our present one is remedied. Now a rising man is either included in or excluded from the University -- that is to say, he must adhere to one or other of two antagonistic cliques, the appointed or the disappointed professors. Once a man's chance of a chair is past, he remains forever outside. It will sometimes happen that a man justly thinks that he is better than his official rival; and it is no libel on human nature to say that the outsider will in general be more or less inclined to look at the territory from which he is excluded through a medium not quite free from special colour. Be it understood that I point to no individual; I merely lay finger on the inevitable consequences of human weakness. Can any one who values the academic reputation of his country read with complacency the recent discussions regarding educational questions-- volumes regarding contending interests, barely a mention of the requirements of sound learning? Then, look again at the miserable contentions that arise about examinations. A University student passes, it is well;

he fails, nothing amiss. An extra-academical student passes, it is also well; but he is plucked, and behold the examination was necessarily unfair. It is very natural--incidental, in fact, to the system--that he, and possibly also his teacher, should believe that. The tendency of this is, of course, to depress the educational standard of the University examination. We hear the most extraordinary arguments based on fallacies kindred to this. In the same speech you will hear a man advocating free competition outside and inside the Universities-- among other things, because it tends to the advancement of science-- and saying that it is monstrous that grants and laboratories and scientific apparatus should be given to the Universities, because that would be furnishing their professors with the implements of a lucrative trade, and giving them an unfair advantage over their outside rivals. This is cutting off the scientific tail to please the tailless scientific fox with vengeance. Within the hearing of such talk, is it altogether unintelligible that the permanent officials of the Treasury should stint the grant to our Universities, and the Government officials should draw bills aiming at the destruction of the influence of our schools of Medicine? If it were the case, as some say, that the teaching of the University is nothing but a lucrative trade, and the activity of the extra-mural schools nothing but a struggle for a share in it, then I say--Away with both of them, and substitute something better.

Then, again, is it the fact that any meritorious young man can start at once as an extra-academical teacher? Is it not the fact that to do so in many cases requires a certain amount of capital, small it may be, but more than is at the disposal of every young man who might wish to try the experiment; while, on the other hand, the possession of the capital will enable him to do so, quite irrespective of his fitness in other respects? No doubt a good many do manage to get a start, because there are various ways and means available to a man launched on a definite profession like Medicine, such as scanty professional earnings, hospital posts, and so on; but these facilities are limited, even in Medicine, and in the department of Arts there are none such. Most of my good men earn their living even while at college, and, as I have explained, often have to cut short their career for want of means to continue it. How one of them is to live by lecturing, say on definite integrals, is a mystery to the solution of which the advocates of pure and unadulterated extra-muralism have not addressed themselves. Is it not the fact that a considerable portion of the extra-mural medical teaching is in the hands of well-to-do practitioners, whose high standing and pecuniary independence puts them above the temptation to stoop to the trade of cramming for examinations, and that it is really the influence of these that keeps the system from degenerating? But where, may I ask, is the position to which an extra-mural teacher in Arts-- nay, even for the matter of that, a professor in Arts-- could aspire comparable with that of a successful Edinburgh surgeon or physician?

Finally, let me ask this pertinent question. The extra-academical system has been in force here for many a year; has it prevented overgrowth of classes in the Medical Faculty? is that evil not more remarkable there than in any other part of the University? Nay, does the present system not occasionally aggravate that serious difficulty, when, for instance, a successful extra-academical teacher is transferred to the University and this *clientèle* , as is not unnatural, to a large extent follow him? The truth is that, considering the enormous number of our students, there is work enough inside the University for all the competent teachers that Edinburgh at present can produce, were that work properly divided, and the teaching body properly organised, instead of being allowed or compelled, as they now are, to compete in paying subjects merely.

If I think that the extra-academical teaching at present arranged is not a good thing for the Faculty of Medicine, I need scarcely say that I think it would be worse thing for the Faculty of Arts. It would lead either to a mockery or to an abuse. As to providing for the real wants of our universities, in the way of higher instruction, it would be a mockery, because the subjects to be taught are not in such demand that men could live by teaching them. The extra-academical teacher would simply have to compete in the preparation for examinations--to become a private "coach," in fact. With the operation of this system, as seen in the English Universities, I am well acquainted; there (in mathematics, at least) it has invaded the higher subjects as well as the lower. The result is that the education of the University is to a large extent in the hands of a few private coaches; and the place is wholly given over to idolatry in the shape of reading for examinations. This is why the English Universities have, notwithstanding their princely endowments, done, comparatively speaking, so little for the advancement of science and learning. England, in the words of the Italian report, is the country of examinations. With an honours list in mathematics alone that passes a hundred, there are terms in which the lectures of one of the greatest mathematicians of Europe are not attended by more than three out of the whole University, and among these not a single undergraduate. The system of education in Cambridge owing to this abuse is, scholastically considered, certainly the most expensive, and perhaps the most ineffective in Europe. If you do not believe me, ask an English University reformer. Listen to one more story of the result of ill-regulated extra-academical teaching at Naples. It is Professor Villari, who writes in the *Nuova Antologia* as follows:-

"Owing to the want of a true and proper University, the higher teaching at Naples under the Bourbons was given by free teachers, some of them of the greatest merit and highest character. When a good University was founded by the Italian Government, in which many of these free teachers became professors, the remainder, who certainly were not the best, seeing their gains placed in danger, complained against the violation of liberty of teaching; and the more degenerate suddenly took to being preparers for the examinations, and invoked protection for this new species of trade. The Government soon began to yield, which multiplied the traders, increased their audacity and the number of their protectors, rendered the ministry less stern of brow toward them, and disgusted those who studied and taught in very deed. From concession to concession, these so-called (*professori pareggiaati*) professors co-ordinate of Naples obtained first one post on the examining commission, then two, then three, and finally four. There was conceded to them by law the right to get from the University a part of the scholastic fees according to the number of lectures beyond the official course, which left a new margin for the co-ordinates. And all this was done with the good intention of protecting and promoting liberty of teaching; but what was the real consequence? To-day there are at Naples a number of co-ordinates who, without giving a single lecture, gain three, or four, even six or seven thousand francs per annum, sometimes even more, and these odd thousands are paid by the state. The student who in November arrives at the railway station in Naples, immediately finds an agent who invites him to put down his name for a few free courses. "You will lose nothing," he says, "and will cause the professor to gain, who will then be among the examiners. You are under no obligation to go to his lectures; you can go if you like to those of the official professor." And sometimes for the readier persuasion he offers him a proportion of the fee, generally five francs for every inscription. If all this does not take place at the railway station, it happens in the house of the student, or in the University buildings, where another student, or even the professor co-ordinate himself, for economy, plays in person the part of agent . . . . The good and true free teachers, who work much and gain little, complain grievously of a state of affairs which discredits their office, and the faculty has often energetically protested. But no minister has the force to resist, because the traders have their *clientèle* , and the others think solely of working and holding their tongue."

You see I have spoken plainly, and why not ? I am a young man speaking to my fellow young men, and we are not identified with any effete system of education, be it extra- or intra-mural. Our aspirations are towards academic freedom of teaching and learning, with proper academic appliances, and under proper academic laws-- under arrangements, in short, such that the academic race shall be not to the longest purse, but to the longest head.

Speaking of young men suggests to me a reflection calculated to make us both grave and glad. It is our duty doubtless to listen respectfully to what our elders say in their speeches and write in the newspapers about the glorious deeds of the past, and the weapons with which they were accomplished, and about what ought to be done in the future. But although the past, and partly also the present is theirs, we the young men can say of the future in the words of an old northern proverb

" The Gordons hae the guidin' o't."

Second address, Friday April 15, 1892

Gentlemen - It is difficult for me to realise that seven years have passed away, and it has again become my duty to address the newly-promoted graduates in Arts and science in the university of Edinburgh. The last occasion on which I had this task seems like yesterday. Probably the flight of time has been so little noticed because so much has happened in the interval. Seven years ago the university had just held its tercentenary, and was lamentating the untimely death of Sir Alexander Grant, under whose reign as principal we had reached a hitherto unsurpassed degree of prosperity, if prosperity is to be measured by increase of wealth and number of students, we were then entering, or supposed we were entering, on the ordeal of a university commission which was to amend all our defects and to bring all our merits into greater perfection. I find, on looking at my old address, that, while I pointed out many of the difficulties that beset the way of reform, I drew a growing and hopeful picture of the future of the university of Edinburgh as I imagined it, and I gloried, with all the delightful confidence of a young man, in the prospect of taking part in the great development which seemed in store for the university. Seven years added to my age, seven years' hard work against adverse circumstances, and one years' experience of outcome of all the talk about university improvement, have done more to damp my youthful enthusiasm that I could have imagined seven years ago whether it be the additional years, the hard work, or the disillusion that depresses me I cannot say; but I feel today more inclined, like an old man, to dwell on the past than to look hopefully, like a young man, to the future. As this is probably the last occasion on which one of the Sacred Seven Professors - (laughter) - will address the graduates as Promoter, It would have been fitting that I should have given you a brief retrospect of the working of the Old Arts

Curriculum, and have pointed out what excellent service it has done for Scotland. But the time left me by imperative university duties for preparing such a retrospect has been so diminished by unforeseen circumstances that I have been obliged to give up the attempt.

CHANGE IN PROFESSORIATE

I shall therefore confine myself to a few desulatory remarks on subjects of interest suggested by the present position of the university, more particularly that department of it with which I am immediately connected. In the first place, I have to remind you of the losses we have sustained in the year that is past. My amiable and respected colleague Professor Campbell Fraser no longer holds the chair of Logic and Metaphysics, and no longer presides the Faculty of Arts as Dean. He served the university so ably and so long in both these capacities, besides giving part of his time to the business of the University Court, that he seemed to us to have become an essential part of our structure. Happily, his parting from us was not the final parting which severs all relations, but merely retirement to well-earned rest. Now and again we still have the pleasure of seeing his familiar figure in the old quadrangle, and always when that happens I feel the impulse to go up and consult him on some matter of Faculty administration, forgetting that with his Professorship he laid aside the cares and troubles of the Deanery of the Faculty of Arts. By the death of Professor Fraser Tytler we lost another colleague, of whom I cannot speak from such intimate knowledge. The little I saw of him gave me a high impression of genial courtesy combined with great practical good sense; and I am told by the Professors of Law that in him they have lost a valuable colleague. In reckoning our losses it is well that we should not forget to count our gains. In the Faculty of Arts we have welcomed the appointment of two able Professors. Professor Seth is, I believe, the man whom, of all others, Professor Campbell Fraser would have welcomed as his successor, and I have heard his appointment commended, so far from home as the Pacific Coast of America, as one bringing honour to the University. Professor Goodhart has already gained the respect and confidence of his students, and gives us good reason to hope that under him the recent reputation of this

University for Classical Scholarship will suffer no diminution. It is well that the Faculty of Arts should have thus secured the assistance of two young and able officers, for in the period of storm and stress which seems in store for it, it will need all the efforts of the crew to keep the ship afloat. A word of nonsense has been talked about the decline of the Universities. In the first place, the measure applied to their prosperity - viz, mere numbers - is utterly deceptive.

It is quite impossible that a University should be crowded to overflowing, and yet to be so far from prosperous as not even to deserve the name of a university at all. All kinds of explanations have been given of the decline in the number of students, most of them ignoring the fact that any explanation, so far as Arts is concerned, to be good must be general. I noticed, for instance, that the years in which the greatest drop occurred in the numbers of Arts students in Edinburgh a similar phenomenon occurred in University College, London; and the general decline is now beginning, I believe, to be felt even in the great English Universities, whose clientele depends so much on old tradition and caste prejudices that they are the last institutions in the country to be susceptible to general educational influences, so far at least their ordinary degrees are concerned.

THE UNIVERSITY AND SCHOOLS

The truth of the matter is that the Universities and University Colleges of Great Britain have now come to the parting of the ways. For years back they have been, so far as their teaching of ordinary pass men is concerned, doing work that under happier auspices would have been better done at school. A wave of educational progress began to sweep over the schools some twenty years ago; and of late it has been rapidly rising. So long as the schools did nothing but teach a little classics, and that for the most part in a lame and heartless fashion, the universities were a natural resort for those who wanted something better than mere wooden learning. Classics and Mathematics for the many could be had there at least as good, and under an able professor better than at school, and there were some other things to be had, viz., a little amount of academic freedom, and some little contact with what was newest and best of the time. But now schools are gradually awakening, and some are in full pursuit of a higher ideal. Some, it is true, formed on the old fashioned English Models, still profess merely classics and athletics - chiefly the latter - (laughter) - but in the case of the best boys, at least, the classics are better done. And some schools of the modern sort have now a curriculum which, so far as the older studies are concerned, is not inferior, if in something it be not superior, to what was to be found in many University Colleges some twenty years ago. The effect of this has been ever clearly marked of late years by increase in the number of those that avail themselves of the privilege of the three years' curriculum, which used to be almost a dead letter (and this is one of the causes of the apparent decrease in our numbers). Another effect, of course, is that many boys now stay longer at school, and many finish their education there without seeking to enter the university at all, unless it be to take up the technical studies connected with some profession. I think this tendency to press young men at once into the business of life is being overdone. It lowers the standard of the professional classes, and aggravates the block in these lines of life, of which so much is heard, and which has irritated many of our educational diseases to the stage of acuteness. Moreover, it is a mistake from the point of view of the individual, for I have noticed with satisfaction that of the few Arts graduates that now enter Medicine, for instance, a very large proportion are to be found among those that ultimately distinguish themselves by success in their profession. Be the tendency right or wrong, it has undoubtedly made itself felt in diminishing the number of students in Arts. Again, the Arts Faculty has now many rivals which it had not formerly. At one time it drew to itself all students that were not formally engaged in studying Divinity, Law or Medicine.

This led to the formation within the Faculty of Arts of the rudiments of various technical and scientific departments. Now, however, special institutions for the higher technical and scientific instruction are springing up on all sides, with which the universities, owing to their imperfect equipment have hitherto, at least, been competing at a serious disadvantage. It is perfectly true that many of the young men that frequent these newer institutions are not better fitted to profit by university instruction of the right kind. They are simply nonchalant young fellows, with a healthy disinclination for immediate mental exertion of any kind, gathering rosebuds while they may, only they gather no longer in the old fields of the Faculty of Arts. (laughter and applause): Then the number of institutions giving or pretending to give courses in Arts has been greatly multiplied, so that the students are divided more than they used to be. This would be a great advantage had the increase of new colleges been accompanied by a rise in the standard of the work done, both in the new and in the old. I greatly fear, however, that so far as past work is concerned, this cannot be said of either.

UNIVERSITY EXTENSION

There has been a great deal of university extensions; but very little university intention in this country of late. It is an excellent thing to interest the population of London, for example, by giving popular lectures on various branches of university culture, and by organising excursions to Oxford and Cambridge to hear a young university Don or two give dozen lectures on some tolerably digestible university subject; to take a walk along the banks of Isis or Cam, to see where Erasmus lived and Newton worked, and where their degenerate successors live and dine - (laughter and applause) - but, as the advocates of a teaching university for London very pertinently insisted lately, all this does nothing for the higher learning in London or elsewhere. Possibly one or two may be led to take a real university course in this way, but nothing is done to send up to the universities a supply of young men really fitted to receive the higher culture. All the fuss and restless activity of these movements is to my mind a morbid symptom. Much of the energy that is turned loose in these schemes aught to be concentrated upon higher objects. It would be amusing, if it were not so sad, to hear the magniloquent title of " University Extension Course" applied to six lectures, followed by indispensable examination to give an air of seriousness to the little plaything. I have no quarrel, however with university extension, although I cannot regard it with gravity of some of its promoters. It can do no harm, I merely mention it to emphasise by contrast the idea of the functions of a university as it presents itself to those who aim at doing something more than playing with the higher education of the country.

AT THE PARTING OF THE WAYS

As I have said, the universities of this country have now come to the parting of the ways. Either they are to go on competing with the secondary schools for the work which the latter can do well or better than they can, or else they are to specialise their functions, and aim at beginning where the secondary schools may be supposed to end. In my opinion, the latter is now the course to follow. It is opinion of so many others that I should scarcely have thought it worth while to insist upon it again, if it were not that it is clear that this opinion has not yet entirely entered the field of practice. The reason for this hesitation to put fully into action a widely accepted educational view are not far to seek, although they are not sufficient. The carrying out of the reform involved will of necessity largely diminish the number of students attending the universities. It would clear the poll men almost entirely out of the English Universities, and in the various provincial colleges of England that have been established on this model. I trust that those who have the direction of this matter will cherish no illusion on this head. Any considerable rise in standard sufficient to differentiate the functions of school and university must of necessity have the effect indicated. Any attempt to tinker the educational pan by taking a middle course will only make the whole larger, and may end in the ruin of the university reconstruction in a financial ordinance. All questions regarding degrees and courses, however important, are of secondary consequence. Institutions for the higher learning can no longer be conducted with profit to the state as quasi-private commercial enterprise; they can no longer be expected to pay their way by attracting large number of students. If it were necessary to argue this matter, a reference to the position of my own chair would, I think, be conclusive. The position of the Professor of Mathematics is this - he draws the main part of his income from the fees, the larger part of this comes from the Junior Class; for his higher work he receives practically nothing. Every step that he takes in improving the teaching of his subject, every schoolmaster that he helps to train to teach mathematics better in Scotland, aids in diminishing the number of students attending the junior class, the effect of which is to diminish the Professor's income, and to bring down upon him abuse in the newspapers regarding the fall of numbers in his department, and unpopularity with parents because the standard for the pass degree shows a tendency to rise in sympathy with improvement in the learning both in and outside the university. The unfortunate Professor may be accused in one and the same day of teaching too low in order to secure fees, and of examining too high for the same base purpose. (laughter and applause). As a matter of fact, the ordinances already issued, and in all probability about to become law, have gone so far that financial reconstruction cannot be further delayed without grave injury to the university. My own department will have to be wholly reorganised by separating honours from the pass teaching; and this cannot be done even temporarily without new financial arrangements. I am at present unable to tell those of my students who intend to specialise under the new ordinance what additional honours courses are to be provided for next session, because we cannot say what work can be done until we know how many workers we are to have, and how they are to be paid.

MATHEMATICS AND THE NEW ARTS ORDINANCE

This leads me to make a remark or two on the new Arts ordinance now before parliament. Regarding the general principle of that ordinance it would hardly be profitable to speak at length, as it has been tacitly agreed on all hands to give it a trial. I cannot, however, refrain from saying that after mature consideration I have come to think that it is of doubtful educational soundness, in so far it sets up a kind of competition by Dutch auction between the kindred subjects to the old departments. There will, I fear, be a tendency in the future for pass men to crowd, not to the better and more thorough teacher, but to the more popular subject and the easier going Professor. This will be seen in all compulsory departments, and to the optional subjects to some extent also. The evil was so obvious and so likely to be mischievous both inside and outside the university, in the departments of Mathematics and Natural Philosophy, that it called forth general condemnation. The commissioners are treating the representations made to them in a conciliatory spirit, and I hope a remedy will be provided which, if it do not effect all that some of us would desire, will yet prevent immediate disaster, and give us time to devise better plan after some years' experience of the new conditions. As I have taken some part in the discussion on this matter, I may be allowed to take this public opportunity of briefly stating my whole position. The university of Edinburgh has been famous as a school of Mathematics and Natural Philosophy ever since Gregorys, in the latter part of seventeenth century brought into its teaching the spirit and the methods of Newton. David Gregory, afterwards Savilian Professor in Oxford, was indeed a favourite follower, distinguished by Newton himself; and it was in his lecture room in the university of Edinburgh that the doctrines of "Principia" were first publicly taught in Great Britain. Ever since then the position of Natural Philosophy as an advanced subject, to which Pure Mathematics is in part ancillary, has been fixed in the Scottish Universities: It was taught when I was a student in Aberdeen, it was taught by my colleague, Professor Swan, in St. Andrews; it is so taught now in the University of Edinburgh. In the draft ordinance for Arts degrees, while higher standards had been imposed on Latin and Greek as graduation subjects, nothing of the kind had been done for the Mathematical Departments. Several oversights had been made in the Arts and in the Science ordinance regarding this department of university study, and it was assumed by some of us that omission was merely accidental. It occasioned , therefore, considerable surprise when it was found that in the final ordinance Mathematics and Natural Philosophy were placed as compulsory alternatives, with the higher standard of entrance for the former and lower for the latter.

It became evident that Natural Philosophy would be used as the outlet for those who were unable to reach the higher standard in Mathematics on entering the university. I need not point out again what would have been inevitable consequences. I wish, however, to point out what, as it appears to me, would have been the proper remedy for I believe has been difficulty of the commissioners. Ever since I became convinced that a majority of educated Scotsmen desired to break down the old curriculum of the seven subjects, my watchword has been "Greater freedom and higher standards". It is obvious that in any subject which is generally compulsory the standards cannot be high. I never was very anxious that all Arts students, should take either Mathematics or Natural Philosophy; but I have all along striven to secure, so far as possible, that those who do take these subjects should be well prepared to receive them. To meet the difficulty of those who desired to have no Mathematics, I proposed that an alternative should be given of a physical or natural science with practical or laboratory work; that Mathematics should be entered on the higher standard, and the Natural Philosophy should remain as Newton made it and Gregory expounded it.

The commissioners adopted the part of my proposal relating to entrance on Mathematics; but made their action nugatory by ignoring the rest of it, although they had fully carried out the principle in the science ordinance. I have reason to hope that they are now convinced that their action was against the opinion of the majority of those best qualified to judge. If they were in any doubt on the subject, evidence of public opinion would easily have been obtained by calling the proper representative men before them. This, I believe, was never done. Neither Professor Tait nor myself, nor, so far as I know, any of the Mathematical Professors or leading Schoolmasters were ever consulted on this point. What the evidence was on which commissioners did proceed we shall not know until their final report is published. Doubtless they have had difficulties over the matter, of which we know nothing; and, in any case, I consider it my duty to thank them publicly for their willingness to reconsider the matter at this last stage under no small inconvenience to themselves. Our department has been peculiarly unfortunate in the evolution of these ordinances. Besides the point just referred to, we were in the first instance dropped out of the science degree altogether, or practically altogether, and even now our honours men do not get full justice as compared with their classical brethren. No classical honours man need now take a mathematical subject as part of his degree course; why should a mathematical honours man be obliged to take classics? However, if only the mathematical men will cry out as loudly as the classical and the natural science people, no doubt that will be remedied hereafter. In any case, this is not a matter for which it is worth while delaying the course of public business. It is pleasant to turn from these vexed questions to a reform concerning which all men appear to be agreed. The admission of women to the universities was at one time the hottest of debated questions. It is surprising to see how quietly it has been settled at last. The commissioners have issued an ordinance, which has provoked little or no discussion, and which probably pass into law without any opposition worth mentioning.

UNIVERSITY EDUCATION OF WOMEN

Meanwhile the Faculty of Arts have unanimously recommended that the Arts classes shall be thrown open to women as soon as the ordinance has passed. The Senatus and the Court have approved, and the Heriot Trust are to offer entrance bursaries. Women will, therefore, enter upon Arts studies next session with full academic privileges. I am sure that I speak your mind as well as my own when I say that we give them a hearty welcome. I do not think that the university education of women is likely to be a large question for some time to come; but I never could see why any women who desired it should not be allowed to take up any Arts study that was likely to be either interesting or useful to her. It was due, I fancy, more to the monastic character of the early universities than to the exclusive spirit of learning that women were debarred so long from university privileges, for they have occasionally distinguished themselves in most branches of the old Arts curriculum. Several women were distinguished for humanistic culture during the early days of the revival of classical learnings and from Hypatia to Madame Sophie Kovalevsky, who died only the other day, women have from time to time distinguished themselves as mathematicians. It cannot be doubted, I think, that the exclusion of women from the universities was partly accountable for the disgraceful state in which the school education of women remained until very recently. I do not expect that any large number of women will enter my department, but if they work as enthusiastically as did pupils I used to have at Shandwick Place some years ago, they will be decided addition to the elite of the university.

VALEDICTORY

And now, gentlemen, I have to say a word of farewell to you - the promoted graduates of our university. The occasion is a peculiarly interesting one to me, because this is the last time that a Professor of Mathematics will be able to address the whole of the graduates in Arts and Science as men with whom he has been in personal contact. In the future, students will be more divided, and although it may be hoped that intimacy between student and Professor may be closer, it can no longer be so general. You are now at the close of what I am sure many of you will afterwards regard as the happiest period of your life. If you were asked what you have gained at the university, many of you would be puzzled to say exactly, but few, I am certain, would say that it had been little; and probably none would be willing to part with that portion of your life's experience which has been gathered there. The influences of university life are many and complicated and difficult to explain; but the effect that they produce for good on the better kind of character is a thing that even the extremes of good and bad fortune never afterwards efface. Someone once asked why he should send his sons to the university - would he learn there how to make money? The answer was, no; but if he be a lad of the right sort he may learn there how to spend money wisely if he gets it, and how to be contented if he never gets any. In the university you are in the world, but not of it. You have there the world's bustle and rivalry without the world's bitterness. It is a world whose inhabitants are all of your own age, and comparatively unrestrained by the barriers of caste and convention. Students can sympathise with each other, and fraternise in a way that is impossible beyond the pale of academic freedom. There you find the only true democracy. You will never again see so deep into the hearts of other men as you did if you used opportunities well at the university. Thus it is that the friendship there formed usually proves so lasting. The amount of positive learning that you have acquired during your student years will vary greatly from individual to individual. We do not flatter ourselves that we have made among you many classical scholars, expert mathematicians, profound philosophers, or men of science. The production of specialists is only one of the functions of a university for the majority of its graduates it has another office to fulfil. It is quite possible for a student to have acquired a head full of special learning at college and yet have missed the main object of a university training. The young scholar who can readily construct a piece of Crabbed Greek but who cannot write a sentence of decent English, who prides himself of his ignorance of the first principles of mathematics, and who knows nothing of the history of human thought, has not caught the " Spirit of the Place", as little has the solver of multitudious problems whose interest never wanders beyond the boards of a mathematical text book. The object of a university training is to familiarise a man with the thought of the past, and to bring in contact with the highest mental activity of the present in as many of its varied forms as possible. It aims at combining in a man that love and reverence for the past which is the characteristic of the true scholar, with the tolerance of what is new and strange if only true, the want of which is the ear-mark of the Philistine, and that willingness to put both new and old to the test of reason, which is the highest attribute of the man of science. In short, the business of the university is to help to make you men in the noble sense of Shakespear's definition - creatures " looking before and after". If you have fully taken the advantage of all your opportunities; if you have marshalled out of the past with Professor Masson the noble company of English authors; appreciated with Professors Goodhart and Butcher the graceful humour of Horace and the stately rhythm and lofty wisdom of the

Greek tragedians; in my own department followed for an hour the steps of Archimedes, and pondered, however superficially, the problems that engaged the mind of Newton, and finally, with Professors Seth and Calderwood, examined the manifold of its own experience - if any of you have done all this, and made no progress towards Shakespear's manliness, then I fear that you must be classed in the category so pithily described by Burns as those that " gang in strike and come out asses"? I am tempted before concluding to say a word or two of a personal character. The life of a Arts Professor has more disadvantages than meets the public eye. His teaching is all crowded into one-half of the year, during which, if he does his duty, he is overwhelmed with work of all kinds - much of it mere drudgery - to such an extent that he must suspend all independent work and gets no time even for bodily exercise. The sessions hurrying on and hurrying through mark time with dismal monotony that is at times saddening, but the Professor has one perennial pleasure - his students never grow old. He is met with a constant stream of young fresh faces, which lighten-up his classroom and greet him pleasantly in the streets and in the college quadrangle. This constant association with the young and enthusiastic is the best antidote against old age. If a university professor grows old in body, he must be singularly unfortunate if he grows old in mind; for he has constantly around him the best minds of the rising generation, and the faintest attempt to do his duty must draw him more or less into the current of intellectual progress. In the course of my work in Edinburgh I have had uphill stretches; but the companionship of sympathetic students has never failed me.

Although I am popularly supposed to represent the least attractive of the old seven subjects, I have never had much reason to complain of want of attention in my pupils, and never any to complain of their courtesy, wherever it may be my lot to teach in the future, I can wish nothing better than the audience of Edinburgh students. Every man who does his duty will meet with difficulties; but I am bound to say that most of mine have been with the old and not with the young. This leads me to the last words I have to say to you.

You are young; make the best of your youth; it is the season of enjoyment and pleasurable sensation; and the best of enjoyment is hard work done in the right spirit. Work while you are young, and be not too particular about taking up what comes to your hands to do; for men are often poor judges of the importance or value of what they are doing. The main thing is to see that the work you do is honest in its aim, and that you do it well. Do not waste the priceless years of youth in quarrelling with your tools and your environment, and in futile schemes for world generation on the grand scale as the manner now is. The progress of humanity is neither faster nor slower than the progress of its human atoms. It is perfectly true that some are called to play what to the vulgar eye seems a large part in the world's affairs; but the call is accepted for the most part unconsciously, and the part is rarely premeditated. A king may be a fool or a madman; and when a great movement for the regeneration of mankind is ready, a leader, as history tells us, may be found in the stable and not in the palace. Rejoice in your youth, and do not envy the position or dignity that comes with old age. To drift into an eddy, and to swirl round and round, and never be able to enter again the great stream of human life and action, is a poor end for any man's ambition; and that is too often the lot of old age. The real pleasure of life is the struggle. The victor's palm is nothing but a withered branch, which reminds him of of the glorious efforts by which the fight was won. (loud applause).

Third Address, April 11, 1908

At the beginning of this session, of the colleagues, forty or so in number, with whom I joined the senatus of this university twenty-eight years ago, there remained but six. The sudden death of our genial and distinguished colleague, Prof. Annendale, has taken one of these, and now only five remain. Of my sixteen original colleagues in the Faculty of Arts, only one is still there.This is now the third time that I have given the Promoter's Address. The first time I was young, and, to judge by a dusty copy which I came across and read with some amusement the other day, I was filled then with a young man's serenity and confidence. The second time I was, to use the sonorous phrase of Dante, Nel mezzo del camin di nostra vita(in the middle of life's way), and still not ill-pleased with the course of the journey. Now I am giving this address in all probability for the last time. When I entered the university of Aberdeen, a little over forty years ago, the demands made on the "Bajans", or freshmen, were very small. There was no entrance examination, unless a voluntary participation in the bursary competition be so called. This was a very restricted test; the main thing was to string together snippets from a Latin phrase book, without any very obvious violation of the rules of syntax; so as to produce what was called a "version" of an easy piece of English prose. The work in all the ordinary classes of the university was very elementary.

Life at Cambridge university

When I went to the university of Cambridge, four or five years later, I found that the course there for the ordinary degree in Arts was greatly inferior in educational quality to the Scottish one. On the other hand, the courses in honours were on a very much higher standard, although they suffered greatly from the chaotic organisation of the English universities which, about that time, were, to use a mathematical phrase, passing through a minimum turning-point in their history. I might liken the difference between the English and Scottish university courses at that time to the differences that then existed between their national styles of cookery. The Scottish cuisine was characterised by lightness and variety; the English cuisine was noted for plenty and excellence of material, but lacked variety, and the defective preparation of its dishes often left them heavy and indigestible. I have frequently been tempted to think that the three years I spent as an undergraduate at Cambridge were wasted years of my life; if they were to be valued merely by the amount of new knowledge acquired, no doubt they were largely wasted; but, on the other hand, they were of great advantage to me in other respects. I made the acquaintance of a large number of the ablest young men of my generation, and it was no small matter to come even within view of such men as Cayley, Adams, Stokes, and Maxwell; and to have lived for a time within the college walls which had sheltered Tait and Kelvin. Cambridge at that time presented strange contrasts. Although almost decadent as an educational institution, it numbered among its members, as the names I have just quoted proves, perhaps the greatest galaxy of intellectual stars that ever illustrated any period of the history of a university. It was doubtless these great men who sowed, it may be unconsciously the seeds of that great resurrection which has again raised my second alma mater, inspite of many picturesque absurdities, to her present high position she now holds, not because of these stars in her intellectual firmament, although such are not wanting, but because she possesses a great body of devoted teachers and investigators, all inspired in their various ways by the high ideals of the work of a university.

Secondary Education

During my absence from Scotland Lord Young's education Act of 1872 had revolutionised primary education, but, on my return to St. Andrews in 1877, I found that secondary education had not only not kept pace with primary education, but had, on the whole, perhaps retrograded. Many of the secondary schools were in a dying condition and others, which are apparently prosperous, were in reality much under-staffed and far from efficient, and were engaged, moreover, in the pursuit of low educational ideals. The universities had been enjoying a period of wholesome prosperity; the number attending them had increased, but the standard of university work had fallen below the level of cultured nations of Europe. They were, in fact, to a considerable extent engaged in performing the work which the secondary schools of the country were for the most part unable to do. The Educational Endowment Act of 1882 brought a partial remedy for this state of affairs, which, however, can only be finally and radically cured by an extension of the policy of state aid to secondary schools.

The beginnings of this extension are to be seen at present, and the consequences will be far reaching. A small sum available for the purpose of secondary school inspection in Scotland had been wrung from the treasury, and it occurred to me that it might be utilised to institute a leaving certificate examination. I was examining twelve schools for the department in the year 1886, and proposed that I should demonstrate how such an examination, at least in single subject, could be carried out. When I came to write my report the idea of a general leaving certificate examination had developed in my mind, and I sketched a complete scheme, in most of its essentials the same as now exists. To my great surprise, and no small gratification, the proposal was immediately taken up by the Scotch Education Department.The labour of carrying out the scheme was taken by Sir Henry Craik, then beginning his successful administration of the new department. In an account of the subject that recently appeared in Scotsman it has been very justly said that the introduction of the leaving certificate examination was perhaps the most important event of Sir Henry Craik's tenure of office, and he certainly deserves the highest credit for the tact and energy with which he carried out what proved to be a great educational reform.

The universities commission reorganised the financial administration of the universities, and profoundly modified their curricula by breaking down to a large extent the monoply of the old seven subjects, and admitting to more or less full academic enfranchisement the other twenty-two subjects of the present curriculum. At the same time a preliminary examination was instituted, which was to be managed by a joint board representing all the universities, in order to secure uniformity in the delimitation of the territories of school and university.

Here let me say that part of my original proposal to the Scotch Education Department regarding the leaving certificate examination was the creation of a National Board of Surveillance, on which the department, the schools, the universities, and certain other public bodies were to be represented. One of the many subjects I had in view was to forestall the necessity for the institution of a university preliminary examination. I foresaw that a generally accepted standard for entrance to the university was an inevitable element in university reform. But I held then, and after fifteen years' experience of the Joint Board I hold more firmly now that the administration of a general leaving examination for schools is not the proper business of the universities. No doubt one of the functions of the leaving certificate should be to qualify for an academic course, but it has many other functions besides, now all that the universities should claim is a share in the surveillance of the leaving certificate in so far as it concerns them.

[After quoting from the report of the universities commission Professor Chrystal went on to say that the advance of secondary education, in all over Scotland, is rapidly preparing way, if it has not already prepared it, for carrying out the ideal of the commissioners].

I turn therefore, with renewed hope and renewed insistence to the men of wisdom and influence, who hold in their hands our educational destiny, and ask them to consider once more my old proposal for a National Board, which shall regulate the school leaving certificate, so that it shall become the normal portal of admission to the universities, and render the present preliminary examination and the present joint board and all its works unnecessary. This reform must, of course, be taken up as a national affair. It is no matter of the autonomy of the universities. It concerns the welfare and good government of all the secondary schools of the country; also, I may say, the relation of our standards of secondary education to similar standards all over British Empire. For this reason it was wise in the recently promoted Arts ordinance to avoid touching the preliminary examination.

Over Pressure

The effect of the leaving certificate, combined with the pressure caused by the composition of younger rival institutions elsewhere, was to raise gradually but surely the standard in various subjects that were common to school and university, most of which were made, at least alternatively, compulsory in the reformed curriculum. A great variety of new subjects were opened out for the graduand, many of them making special demands on his time, either for practical work or for special preparation beyond a school training. Also, a demand has risen for teachers able to teach subjects of the school curriculum beyond former standards- a demand which for the moment exceeds the supply. Then, again the increased variety of university culture among classes of the community for whom the old exclusive had no interest and certainly no utility. As the universities of Scotland are the State institutions, and not merely resorts for the young people of the wealthy or leisured classes of society, it is a necessary condition of their existence that they should meet the general demands of the nation, and admit and provide for all who can advance reasonable claims for higher culture. If three sessions, each crushed into twenty- and- twenty practically consecutive weeks, were not more than enough for the old superficial curriculum of seven subjects, it will be clear that the same arrangement is no longer sufficient for a curriculum of seven subjects on the modern standard. The result of the attempt to put new wine into the old bottles has been over-pressure both for teachers and for the taught, more specially for the latter. This over-pressure shows itself in the frequent breakdown of our students, more particularly of the women students, and in the considerable percentage of those who, notwithstanding the selection by a somewhat stringent entrance test, are unable to complete their academic course in the three specified years. It is recognised that this over-pressure arises in three ways:-

(1) By compulsion to take too many subjects;

(2) By compulsion to take unnecessary or uncongenial subjects;

(3) By compression of the students' work into too short a period of the year. When, therefore, movements arose within the universities of Glasgow and Edinburgh five years ago in favour of an extension of university study over a longer portion of the year, and for a more concentrated curriculum of five degree subjects, free from more of the irksome compulsions of the curriculum with which you are familiar, it was speedily recognised by the more thoughtful in the academic world that these things were inevitable, because these are the natural consequences of what has gone before. The five years' deliberation and consultation with the other universities has resulted in this, that the universities of Glasgow and Edinburgh have recently presented to the Privy Council new Arts Ordinances, practically identical, which render it possible for each university to carry out, in the way

that best suits its own circumstances, one or all of the reforms that are now recognised as necessary.

The university of Aberdeen has rapidly followed suit, and that these ordinances will become law very speedily, I cannot doubt. The realisation of their consequences will be a matter of time and no little labour for the university staff, and will ultimately make heavy demands on university resources. I am keenly interested in the developments that lie before us, but I must confess that I shrink from the labour that they will involve. Yet the whole of my career has been a turmoil of university reform, beginning in Cambridge, and it may as well end as it began, if it be decreed that it is to continue any longer.

Advice To The Graduates

But now some of the audience might say, what interest lies for us in this retrospect and prospect of university politics? We have come through the existing ordeal, satisfied the last examiner, paid the last fee, and graduated; the matter concerns us no more. Yet surely this is not so. You are the newly admitted citizens of an ancient state, and it must, I think, interest you to hear something of its history from an old citizen, and to get some knowledge of its politics, especially at a time when that ancient state is on the eve of a revolution- a revolution in which many of you must participate hereafter, no longer as mere pieces on the board, but as players in the game. Every graduate in Arts here present can help in the right guiding of public opinion in matters concerning the higher education. If not, to what purpose was the course of university study? If your study here has not made you better citizens, able to advise and help in the higher education of our people, then the state has been a loser in the bargain by which it contributed part of the cost of your university education. Moreover, many of you are to be teachers, whose business it will be to train boys more thoroughly than before for the more varied and more concentrated curriculum which university is to offer in the future. From a purely personal and selfish point of view, be pleased to note that, although there is a great demand for your services at the present moment, in the future you will have competitors who have been trained under more favourable circumstances than you have been. It rests with you to see that you make good use of the start that time has given you. Take care, in short, that your education does not cease, as it ought not to cease, when you leave the university.

Academic Failures

There are three kinds of academic failures which always strike me as pitiably amusing. There is the academic person who treats his knowledge as a miser does his hoard, never spending, always carefully increasing it, concealing it from public view for the most part, and only exhibiting it now and then, when he wishes to rub into some less fortunate individual the fact that he is poor. Then there is the pragmatic university graduate who thinks that the scraps of knowledge gathered in a few university classes is sufficient stock-in-trade wherewithal to reconstruct the practice, laws, and customs of the nation. Also there is the offensively ostentatious academical who, whether rich or poor in knowledge, goes about in the world with his nose in the air, concerned mainly to impress upon his fellows that he has had the social advantage of spending, well or ill, a few years of his life at a university. Few of you, I trust, will fall into any of these blunders of tact or judgment. Once in the world you will soon find that the main lesson of a university training- I mean the ethical lesson- can, if less easily, be also learned elsewhere.