Bifurcation of finitely deformed thick-walled electroelastic cylindrical tubes subject to a radial electric field

Andrey Melnikov and Ray W. Ogden

Abstract. This paper is concerned with the bifurcation analysis of a pressurized electroelastic circular cylindrical tube with closed ends and compliant electrodes on its curved boundaries. The theory of small incremental electroelastic deformations superimposed on a finitely deformed electroelastic tube is used to determine those underlying configurations for which the superimposed deformations do not maintain the perfect cylindrical shape of the tube. First, prismatic bifurcations are examined and solutions are obtained which show that for a neo-Hookean electroelastic material prismatic modes of bifurcation become possible under inflation. This result contrasts with that for the purely elastic case for which prismatic bifurcation modes were found only for an externally pressurized tube. Second, axisymmetric bifurcations are analyzed, and results for both neo-Hookean and Mooney–Rivlin electroelastic energy functions are obtained. The solutions show that in the presence of a moderate electric field the electroelastic tube becomes more susceptible to bifurcation, i.e., for fixed values of the axial stretch axisymmetric bifurcations become possible at lower values of the circumferential stretches than in the corresponding problems in the absence of an electric field. As the magnitude of the electric field increases, however, the possibility of bifurcation under internal pressure becomes restricted to a limited range of values of the axial stretch and is phased out completely for sufficiently large electric fields. Then, axisymmetric bifurcation is only possible under external pressure.

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1. Introduction

Electroactive elastomeric materials have attracted much interest in the literature in recent years because their properties are suitable for many potential applications in engineering science, such as in the production of actuators, sensors and other devices. One of the geometries of relevance in this context is that of a circular cylindrical tube actuator made of a dielectric elastomer with flexible electrodes on its curved boundaries (see, for example, Pelrine et al. [1]). Aspects of the associated theoretical analysis of this problem have been examined in [2] and in our recent paper [3], where the analysis was based on the tube maintaining its circular cylindrical shape. An important consideration for the effective operation of such an actuator is its ability to maintain its perfect cylindrical shape. In this paper we predict the onset of bifurcation from the cylindrical shape on the basis of the nonlinear theory of electroelasticity for specific material models.

It is well known that a tube made of a rubberlike material inflated by internal pressure may develop a bulge beyond a critical deformation, giving rise to deviations from the perfect cylindrical shape under symmetric loads. Relevant experimental data can be found in [4] and [5], for example, while for a dielectric elastomer tube recent experiments are described in [6]. For thin- and thick-walled cylindrical shells of elastic material under internal and external pressure, extensive bifurcation analyses have been provided in the papers [7,8] by Haughton and Ogden on the basis of the theory of small deformations superimposed on a finite deformation. Corresponding analysis for an electroactive tube, however, is limited. For example,

for an electroactive material the bifurcation of a tube subject to just an axial mechanical loading and an axial electric field was examined recently by Su et al. [9], who found that the stability of the tube depends on the electric field, the material constants and the geometrical dimensions of the tube. Prismatic bifurcations have been examined recently by Bortot and Shmuel [10] for a tube with fixed axial extension but no internal pressure.

In order to determine when deviations from the perfect cylindrical configuration are possible we use the theory of incremental deformations and electric fields superimposed on an underlying finite deformation and electric field. We note that by including dependence on time the problem of wave propagation in an electroelastic tube can be analyzed, as has been done by Shmuel and deBotton [11], who considered axisymmetric waves for an axially loaded tube with a radial electric field, but no radial mechanical loads. When the wave speed vanishes, corresponding bifurcation results can be obtained.

This paper is structured as follows. In Sect. 2 we summarize the most important ingredients of the general theory of electroelasticity as formulated by Dorfmann and Ogden [12]. The incremental electroelastic constitutive laws are also provided in this section, along with explicit formulas for the electroelastic moduli tensors that are used in subsequent sections.

In Sect. 3, following [3], we consider the underlying problem of the inflation and extension of an electroelastic tube in the presence of a radial electric field as developed in [3]. The electric field is generated by a potential difference between compliant electrodes on the inner and outer curved surfaces of the tube. Expressions are given for the internal pressure in a tube with closed ends and the axial load on its ends in respect of a general form of electroelastic energy function.

Then, in Sect. 4, we present a bifurcation analysis for the electroelastic tube. First, prismatic bifurcations are analyzed, i.e., we seek incremental solutions of the governing equations for which the cross section is independent of the axial location along the tube. For a specific model of a neo-Hookean electroelastic material we find that prismatic modes of bifurcation are possible under inflation in the presence of an electric field, which is in contrast to the result for the purely mechanical situation obtained in [8]. Second, axisymmetric bifurcations are considered. These are the configurations of the tube for which cross sections remain circular, but with the radius depending on the axial location. The results are compared with those in the purely elastic case: depending on the magnitude of the electric field its presence may make the electroelastic tube more susceptible to bifurcation for each of the two energy functions adopted (neo-Hookean and Mooney–Rivlin-based electroelastic models), i.e., bifurcation becomes possible at lower circumferential stretches than is the case for purely elastic materials without electromechanical interactions. But, for sufficiently large fields bifurcation under inflation is not possible and bifurcation requires external pressure.

For each of the prismatic and axisymmetric bifurcations numerical results are provided based on use of the MATLAB code listed in Appendix B of the PhD thesis [13]. Some important aspects of the code, which employs a numerical scheme used in [8], are discussed and explained briefly in [13].

2. The equations of nonlinear electroelasticity

We consider a deformable electrosensitive body which occupies the reference configuration $\mathcal{B}_r$ with boundary $\partial\mathcal{B}_r$, in the absence of mechanical loads and electric fields. Application of an electric field and mechanical loads induces deformation which results in a new configuration $\mathcal{B}$, with boundary $\partial\mathcal{B}$, commonly referred to as the deformed configuration. We label material points in the reference configuration $\mathcal{B}_r$ by their position vectors $\mathbf{X}$ and their images in the current configuration $\mathcal{B}$ by their position vectors $\mathbf{x}$. The deformation is described by the vector field $\boldsymbol{\chi}$, which relates the position of a particle in the reference configuration to the position of the same particle in the current configuration: $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$. The deformation gradient tensor, denoted $\mathbf{F}$, is defined by

$$\mathbf{F} = \mathrm{Grad}\,\boldsymbol{\chi}, \tag{1}$$

where Grad is the gradient operator defined with respect to $\mathbf{X}$.

The quantity defined by

$$J = \det \mathbf{F} \tag{2}$$

represents the local ratio of the volume in the deformed configuration to that in the reference configuration, and for an incompressible material the constraint

$$\det \mathbf{F} = 1 \tag{3}$$

must be satisfied at each $\mathbf{X}$.

Along with the deformation gradient we shall use the right and left Cauchy–Green deformation tensors, respectively, defined by

$$\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}, \quad \mathbf{b} = \mathbf{F}\mathbf{F}^{\mathrm{T}}. \tag{4}$$

2.1. Governing equations and boundary conditions

In the purely static context considered here the reduced forms of Maxwell's equations for a dielectric material are

$$\operatorname{curl}\mathbf{E} = \mathbf{0}, \quad \operatorname{div}\mathbf{D} = 0, \tag{5}$$

where $\mathbf{E}$ denotes the electric field vector and $\mathbf{D}$ is the electric displacement vector in the configuration $\mathcal{B}$. The operators curl and div are defined with respect to $\mathbf{x}$. Exterior to a material body the corresponding fields are denoted $\mathbf{E}^{\star}$ and $\mathbf{D}^{\star}$, which satisfy the standard relation $\mathbf{D}^{\star} = \varepsilon_0\mathbf{E}^{\star}$ in a nonpolarizable material, where ε_0 is the vacuum permittivity. They also satisfy field equations (5).

Associated with (5) are the standard boundary conditions

$$(\mathbf{E}^{\star} - \mathbf{E}) \times \mathbf{n} = \mathbf{0}, \quad (\mathbf{D}^{\star} - \mathbf{D}) \cdot \mathbf{n} = \sigma_{\mathrm{f}} \quad \text{on} \quad \partial\mathcal{B}, \tag{6}$$

where $\mathbf{n}$ is the unit outward normal to $\partial\mathcal{B}$ and σ_{f} is the free surface charge on $\partial\mathcal{B}$ per unit area.

In the absence of *mechanical* body forces the mechanical equilibrium equation incorporating electric body forces can be written simply as

$$\operatorname{div}\boldsymbol{\tau} = \mathbf{0}, \tag{7}$$

where $\boldsymbol{\tau}$ is the total Cauchy stress tensor, which depends on the deformation and electric field via a constitutive law, which will be discussed in Sect. 2.3. When there are no intrinsic mechanical couples, which we assume to be the case here, then $\boldsymbol{\tau}$ is symmetric.

The boundary condition associated with (7) is

$$\boldsymbol{\tau}\mathbf{n} = \mathbf{t}_{\mathrm{a}} + \mathbf{t}_{\mathrm{m}}^{\star} \quad \text{on} \quad \partial\mathcal{B}_{\mathrm{t}}, \tag{8}$$

where $\partial\mathcal{B}_{\mathrm{t}}$ is the part of the boundary where the traction is prescribed, $\mathbf{t}_{\mathrm{a}}$ being the mechanical part of the traction and $\mathbf{t}_{\mathrm{m}}^{\star} = \boldsymbol{\tau}_{\mathrm{m}}^{\star}\mathbf{n}$, which is the additional traction due to the Maxwell stress $\boldsymbol{\tau}_{\mathrm{m}}^{\star}$, calculated from the fields outside the body $\mathcal{B}$. The Maxwell stress is defined by

$$\boldsymbol{\tau}_{\mathrm{m}}^{\star} = \varepsilon_0\mathbf{E}^{\star} \otimes \mathbf{E}^{\star} - \frac{1}{2}\varepsilon_0(\mathbf{E}^{\star} \cdot \mathbf{E}^{\star})\mathbf{I}, \tag{9}$$

where $\mathbf{I}$ is the identity tensor.

2.2. Lagrangian forms of the electric fields

Lagrangian forms of the electric field vectors are defined by

$$\mathbf{E}_\mathrm{L} = \mathbf{F}^\mathrm{T}\mathbf{E}, \quad \mathbf{D}_\mathrm{L} = J\mathbf{F}^{-1}\mathbf{D}, \tag{10}$$

where we recall that $J = \det \mathbf{F}$. They satisfy the counterparts of equations (5) in the reference configuration, i.e.,

$$\mathrm{Curl}\,\mathbf{E}_\mathrm{L} = \mathbf{0}, \quad \mathrm{Div}\,\mathbf{D}_\mathrm{L} = 0. \tag{11}$$

Here the operators Curl and Div are defined with respect to $\mathbf{X}$.

In order to obtain a Lagrangian form of the equilibrium equation (7) we introduce the total nominal stress tensor $\mathbf{T}$ defined by

$$\mathbf{T} = J\mathbf{F}^{-1}\boldsymbol{\tau}, \tag{12}$$

which is a generalization of the nominal stress tensor in nonlinear elasticity to the electromechanical context. By a standard identity, this yields the Lagrangian form of the electromechanical equilibrium equation (7), i.e.,

$$\mathrm{Div}\,\mathbf{T} = \mathbf{0}. \tag{13}$$

The boundary condition associated with (13) can be obtained with the help of the relation

$$\boldsymbol{\tau}\mathbf{n}\mathrm{d}s = \mathbf{T}^\mathrm{T}\mathbf{N}\mathrm{d}S, \tag{14}$$

which makes use of Nanson's formula $\mathbf{n}\mathrm{d}s = J\mathbf{F}^{-\mathrm{T}}\mathbf{N}\mathrm{d}S$ connecting infinitesimal areas $\mathrm{d}s$ and $\mathrm{d}S$ in the current and reference configurations, $\mathbf{n}$ and $\mathbf{N}$ being the respective normals to these areas. Then, (8) transforms to

$$\mathbf{T}^\mathrm{T}\mathbf{N} = \mathbf{t}_\mathrm{A} + \mathbf{t}_\mathrm{M}^\star \quad \text{on} \quad \partial\mathcal{B}_\mathrm{rt}, \tag{15}$$

where $\partial\mathcal{B}_\mathrm{rt}$ is the pre-image of $\mathcal{B}_\mathrm{t}$, $\mathbf{t}_\mathrm{A}$ and $\mathbf{t}_\mathrm{M}^\star = \mathbf{T}_\mathrm{M}^{\star\mathrm{T}}\mathbf{N}$ (with $\mathbf{T}_\mathrm{M}^\star = J\mathbf{F}^{-1}\boldsymbol{\tau}_\mathrm{m}^\star$) being the mechanical traction and the Maxwell traction per unit reference area, respectively.

With the help of Nanson's formula and (10) the boundary conditions (6) convert to Lagrangian form as

$$(\mathbf{F}^\mathrm{T}\mathbf{E}^* - \mathbf{E}_\mathrm{L}) \times \mathbf{N} = \mathbf{0}, \quad (J\mathbf{F}^{-1}\mathbf{D}^* - \mathbf{D}_\mathrm{L}) \cdot \mathbf{N} = \sigma_\mathrm{F} \quad \text{on} \quad \partial\mathcal{B}_\mathrm{r}, \tag{16}$$

where $\mathbf{N}$ is the unit outward normal to $\partial\mathcal{B}_\mathrm{r}$ and σ_F is the free surface charge density per unit area of $\partial\mathcal{B}_\mathrm{r}$.

2.3. Constitutive equations

For the problem discussed in this paper it is convenient to choose $\mathbf{D}_\mathrm{L}$ as the independent electric variable. For mechanically incompressible materials the total stress tensor and the electric field in Lagrangian form are, respectively,

$$\mathbf{T} = \frac{\partial\Omega^*}{\partial\mathbf{F}} - p\mathbf{F}^{-1}, \quad \mathbf{E}_\mathrm{L} = \frac{\partial\Omega^*}{\partial\mathbf{D}_\mathrm{L}}, \tag{17}$$

where $\Omega^*(\mathbf{F}, \mathbf{D}_\mathrm{L})$ is a total energy density function, as defined in [12], and p is a Lagrange multiplier that accommodates the constraint (3).

For an incompressible isotropic electroelastic material Ω^* depends on $\mathbf{F}$ and $\mathbf{D}_\mathrm{L}$ through invariants of the right Cauchy–Green deformation tensor $\mathbf{C}$ and $\mathbf{D}_\mathrm{L}$, i.e.,

$$I_1 = \mathrm{tr}\,\mathbf{C}, \quad I_2 = \frac{1}{2}[(\mathrm{tr}\,\mathbf{C})^2 - \mathrm{tr}(\mathbf{C}^2)], \tag{18}$$

$$I_4 = \mathbf{D}_\mathrm{L} \cdot \mathbf{D}_\mathrm{L}, \quad I_5 = \mathbf{D}_\mathrm{L} \cdot (\mathbf{C}\mathbf{D}_\mathrm{L}), \quad I_6 = \mathbf{D}_\mathrm{L} \cdot (\mathbf{C}^2\mathbf{D}_\mathrm{L}), \tag{19}$$

with $I_3 = \det \mathbf{C} = 1$ omitted since we are considering incompressible materials.

By expressing Ω^* in terms of the invariants, then expanding out (17) in terms of the derivatives of Ω^* with respect to the invariants, we obtain, on use of (12) with $J = 1$ and $(10)_1$, the total stress $\boldsymbol{\tau}$ and electric field $\mathbf{E}$, i.e.,

$$\boldsymbol{\tau} = 2\Omega_1^*\mathbf{b} + 2\Omega_2^*(I_1\mathbf{b} - \mathbf{b}^2) - p\mathbf{I} + 2\Omega_5^*\mathbf{D} \otimes \mathbf{D} + 2\Omega_6^*(\mathbf{D} \otimes \mathbf{bD} + \mathbf{bD} \otimes \mathbf{D}), \tag{20}$$

$$\mathbf{E} = 2(\Omega_4^*\mathbf{b}^{-1} + \Omega_5^*\mathbf{I} + \Omega_6^*\mathbf{b})\mathbf{D}, \tag{21}$$

where Ω_i^* stands for $\partial\Omega^*/\partial I_i$, $i = 1, 2, 4, 5, 6$, and $\mathbf{b}$ is the left Cauchy–Green tensor defined in $(4)_2$.

2.4. Incremental formulation

In this section we summarize the equations governing incremental deformations and electric displacements superimposed on a deformed configuration and an initial electric field. A more detailed discussion of this theory can be found in [14,15].

2.4.1. Incremental equations and boundary conditions. The increment of each variable is now identified by a superimposed dot. For example, $\dot{\mathbf{x}}$ is the increment in the displacement and $\dot{\mathbf{F}} = \mathrm{Grad}\,\dot{\mathbf{x}}$ is the corresponding increment in the deformation gradient. The increments $\dot{\mathbf{E}}_L$, $\dot{\mathbf{D}}_L$, $\dot{\mathbf{T}}$ must satisfy the incremental governing equations

$$\mathrm{Curl}\,\dot{\mathbf{E}}_L = \mathbf{0}, \quad \mathrm{Div}\,\dot{\mathbf{D}}_L = 0, \quad \mathrm{Div}\,\dot{\mathbf{T}} = \mathbf{0}. \tag{22}$$

Outside the material increments in the field vectors are connected by $\dot{\mathbf{D}}^\star = \varepsilon_0\dot{\mathbf{E}}^\star$ and satisfy the equations

$$\mathrm{curl}\,\dot{\mathbf{E}}^\star = \mathbf{0}, \quad \mathrm{div}\,\dot{\mathbf{D}}^\star = 0. \tag{23}$$

The incremental forms of the boundary conditions (16) and (15) with $J = 1$ are

$$(\dot{\mathbf{F}}^T\mathbf{E}^\star + \mathbf{F}^T\dot{\mathbf{E}}^\star - \dot{\mathbf{E}}_L) \times \mathbf{N} = \mathbf{0} \quad \text{on} \quad \partial\mathcal{B}_r, \tag{24}$$

$$(\mathbf{F}^{-1}\dot{\mathbf{D}}^\star - \mathbf{F}^{-1}\dot{\mathbf{F}}\mathbf{F}^{-1}\mathbf{D}^\star - \dot{\mathbf{D}}_L) \cdot \mathbf{N} = \dot{\sigma}_F \quad \text{on} \quad \partial\mathcal{B}_r, \tag{25}$$

$$\dot{\mathbf{T}}^T\mathbf{N} = \dot{\mathbf{t}}_A + \dot{\boldsymbol{\tau}}_m^\star\mathbf{F}^{-T}\mathbf{N} - \boldsymbol{\tau}_m^\star\mathbf{F}^{-T}\dot{\mathbf{F}}^T\mathbf{F}^{-T}\mathbf{N} \quad \text{on} \quad \partial\mathcal{B}_r, \tag{26}$$

where $\dot{\boldsymbol{\tau}}_m^\star$ is the incremental Maxwell stress obtained from (9) and expressed as

$$\dot{\boldsymbol{\tau}}_m^\star = \varepsilon_0[\dot{\mathbf{E}}^\star \otimes \mathbf{E}^\star + \mathbf{E}^\star \otimes \dot{\mathbf{E}}^\star - (\mathbf{E}^\star \cdot \dot{\mathbf{E}}^\star)\mathbf{I}]. \tag{27}$$

We now work in terms of the push-forward versions of the increments in $\dot{\mathbf{E}}_L$, $\dot{\mathbf{D}}_L$ and $\dot{\mathbf{T}}$ defined by (since $J = 1$)

$$\dot{\mathbf{E}}_{L0} = \mathbf{F}^{-T}\dot{\mathbf{E}}_L, \quad \dot{\mathbf{D}}_{L0} = \mathbf{F}\dot{\mathbf{D}}_L, \quad \dot{\mathbf{T}}_0 = \mathbf{F}\dot{\mathbf{T}}, \tag{28}$$

which are the incremental counterparts of (10) and (12) and also referred to as quantities updated from the reference configuration $\mathcal{B}_r$ to the deformed configuration $\mathcal{B}$, with updated quantities identified by a zero subscript. A detailed discussion of the concept of updating can be found in [16] for purely mechanical problems, and here we use this approach for updating variables to the deformed configuration for our electromechanical problem.

The governing equations (22) are then updated to

$$\mathrm{curl}\,\dot{\mathbf{E}}_{L0} = \mathbf{0}, \quad \mathrm{div}\,\dot{\mathbf{D}}_{L0} = 0, \quad \mathrm{div}\,\dot{\mathbf{T}}_0 = \mathbf{0}, \tag{29}$$

and the corresponding boundary conditions are updated to

$$(\dot{\mathbf{E}}^\star + \mathbf{L}^T\mathbf{E}^\star - \dot{\mathbf{E}}_{L0}) \times \mathbf{n} = \mathbf{0} \quad \text{on} \quad \partial\mathcal{B}, \tag{30}$$

$$(\dot{\mathbf{D}}^\star - \mathbf{L}\mathbf{D}^\star - \dot{\mathbf{D}}_{L0}) \cdot \mathbf{n} = \dot{\sigma}_{F0} \quad \text{on} \quad \partial\mathcal{B}, \tag{31}$$

$$\dot{\mathbf{T}}_0^T\mathbf{n} = \dot{\mathbf{t}}_{A0} + \dot{\boldsymbol{\tau}}_m^\star\mathbf{n} - \boldsymbol{\tau}_m^\star\mathbf{L}^T\mathbf{n} \quad \text{on} \quad \partial\mathcal{B}, \tag{32}$$

where $\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1} = \mathrm{grad}\,\mathbf{u}$, $\mathbf{u}\,(= \dot{\mathbf{x}})$ being a function of $\mathbf{x}$.

The incremental form of (2) is then obtained as $\dot{J} = J\mathrm{tr}(\dot{\mathbf{F}}\mathbf{F}^{-1})$, and for an incompressible material $\dot{J} = 0$ and the incremental form of the incompressibility condition (3) becomes

$$\mathrm{tr}\,\mathbf{L} \equiv \mathrm{div}\,\mathbf{u} = 0. \tag{33}$$

Let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ be unit basis vectors in an orthogonal curvilinear coordinate system. Then, in component form, Eq. $(29)_3$ yields the three scalar equations

$$\dot{T}_{0ji,j} + \dot{T}_{0ji}\mathbf{e}_k \cdot \mathbf{e}_{j,k} + \dot{T}_{0kj}\mathbf{e}_i \cdot \mathbf{e}_{j,k} = 0, \quad i = 1, 2, 3, \tag{34}$$

in which summation over repeated indices j and k from 1 to 3 is implied and the notation $_{,j}$ represents the derivative associated with the jth curvilinear coordinate and is made explicit in Sect. 4 for cylindrical polar coordinates.

2.4.2. Incremental constitutive equations. Increments $\dot{\mathbf{F}}$ and $\dot{\mathbf{D}}_{\mathrm{L}}$ in the deformation gradient and Lagrange electric displacement induce increments $\dot{\mathbf{T}}$ and $\dot{\mathbf{E}}_{\mathrm{L}}$ in the nominal stress and Lagrange electric field, leading to incremental forms of the constitutive laws. From (17) we obtain

$$\dot{\mathbf{T}} = \boldsymbol{\mathcal{A}}^*\dot{\mathbf{F}} + \mathbb{A}^*\dot{\mathbf{D}}_{\mathrm{L}} + p\mathbf{F}^{-1}\dot{\mathbf{F}}\mathbf{F}^{-1} - \dot{p}\mathbf{F}^{-1}, \quad \dot{\mathbf{E}}_{\mathrm{L}} = \mathbb{A}^{*\mathrm{T}}\dot{\mathbf{F}} + \mathbf{A}^*\dot{\mathbf{D}}_{\mathrm{L}}, \tag{35}$$

where $\boldsymbol{\mathcal{A}}^*$, $\mathbb{A}^*$, $\mathbf{A}^*$, which are, respectively, fourth-, third- and second-order tensors, denote electroelastic moduli associated with the total energy Ω^*. Their component forms are written

$$\mathcal{A}^*_{\alpha i\beta j} = \frac{\partial^2\Omega^*}{\partial F_{i\alpha}\partial F_{j\beta}}, \quad \mathbb{A}^*_{\alpha i|\beta} = \frac{\partial^2\Omega^*}{\partial F_{i\alpha}\partial D_{\mathrm{L}\beta}}, \quad \mathsf{A}^*_{\alpha\beta} = \frac{\partial^2\Omega^*}{\partial D_{\mathrm{L}\alpha}\partial D_{\mathrm{L}\beta}}, \tag{36}$$

which enjoy the symmetries

$$\mathcal{A}^*_{\alpha i\beta j} = \mathcal{A}^*_{\beta j\alpha i}, \quad \mathsf{A}^*_{\alpha\beta} = \mathsf{A}^*_{\beta\alpha}. \tag{37}$$

The vertical bar in the component form of $\mathbb{A}^*$ separates the first two and the third indices, because the first two indices are associated with a second-order tensor, whereas the third one is associated with a vector. The tensor $\mathbb{A}^*$ maps a vector into a second-order tensor, whereas its transpose maps a second-order tensor into a vector. In component form this is expressed as $\mathbb{A}^*_{\alpha i|\beta} = \mathbb{A}^{*\mathrm{T}}_{\beta|\alpha i}$. The component forms of the equations in Eq. (35) are then

$$\dot{T}_{\alpha i} = \mathcal{A}^*_{\alpha i\beta j}\dot{F}_{j\beta} + \mathbb{A}^*_{\alpha i|\beta}\dot{D}_{\mathrm{L}\beta} + pF^{-1}_{\alpha k}\dot{F}_{k\beta}F^{-1}_{\beta i} - \dot{p}F^{-1}_{\alpha i}, \quad \dot{E}_{\mathrm{L}\alpha} = \mathbb{A}^*_{\beta i|\alpha}\dot{F}_{i\beta} + \mathsf{A}^*_{\alpha\beta}\dot{D}_{\mathrm{L}\beta}, \tag{38}$$

where $F^{-1}_{\alpha i}$ is defined as $(\mathbf{F}^{-1})_{\alpha i}$.

When expressed in terms of the invariants I_m, $m \in \{1, 2, 4, 5, 6\}$, the components in (36) can be expanded as

$$\mathcal{A}^*_{\alpha i\beta j} = \sum_{m\in\mathcal{I}}\sum_{n\in\mathcal{I}}\Omega^*_{mn}\frac{\partial I_m}{\partial F_{i\alpha}}\frac{\partial I_n}{\partial F_{j\beta}} + \sum_{n\in\mathcal{I}}\Omega^*_n\frac{\partial^2 I_n}{\partial F_{i\alpha}\partial F_{j\beta}}, \tag{39}$$

$$\mathbb{A}^*_{\alpha i|\beta} = \sum_{m=4}^{6}\sum_{n\in\mathcal{I}}\Omega^*_{mn}\frac{\partial I_m}{\partial D_{\mathrm{L}\beta}}\frac{\partial I_n}{\partial F_{i\alpha}} + \sum_{n=5}^{6}\Omega^*_n\frac{\partial^2 I_n}{\partial F_{i\alpha}\partial D_{\mathrm{L}\beta}}, \tag{40}$$

$$\mathsf{A}^*_{\alpha\beta} = \sum_{m=4}^{6}\sum_{n=4}^{6}\Omega^*_{mn}\frac{\partial I_m}{\partial D_{\mathrm{L}\alpha}}\frac{\partial I_n}{\partial D_{\mathrm{L}\beta}} + \sum_{n=4}^{6}\Omega^*_n\frac{\partial^2 I_n}{\partial D_{\mathrm{L}\alpha}\partial D_{\mathrm{L}\beta}}, \tag{41}$$

where $\mathcal{I}$ is the index set $\{1, 2, 5, 6\}$, $\Omega^*_n = \partial\Omega^*/\partial I_n$ and $\Omega^*_{mn} = \partial^2\Omega^*/\partial I_m\partial I_n$, $m, n \in \{1, 2, 4, 5, 6\}$. The required expressions for the derivatives of the invariants with respect to $\mathbf{F}$ and $\mathbf{D}_{\mathrm{L}}$ in (39)–(41) are given in "Appendix A.1," and the resulting expressions for the moduli are listed in "Appendix A.2."

The updated versions of (35) are

$$\dot{\mathbf{T}}_0 = \boldsymbol{\mathcal{A}}^*_0\mathbf{L} + \mathbb{A}^*_0\dot{\mathbf{D}}_{\mathrm{L}0} + p\mathbf{L} - \dot{p}\mathbf{I}, \quad \dot{\mathbf{E}}_{\mathrm{L}0} = \mathbb{A}^{*\mathrm{T}}_0\mathbf{L} + \mathbf{A}^*_0\dot{\mathbf{D}}_{\mathrm{L}0}, \tag{42}$$

and in component form the connections between the electroelastic moduli tensors (36) and their updated forms are

$$\mathcal{A}^*_{0jilk} = F_{j\alpha}F_{l\beta}\mathcal{A}^*_{\alpha i\beta k}, \quad \mathbb{A}^*_{0ji|k} = F_{j\alpha}F^{-1}_{\beta k}\mathbb{A}^*_{\alpha i|\beta}, \quad \mathsf{A}^*_{0ij} = F^{-1}_{\alpha i}F^{-1}_{\beta j}\mathsf{A}^*_{\alpha\beta}. \tag{43}$$

The symmetries (37) carry over to the updated versions of the moduli, while $\mathbb{A}^*_0$ gains the symmetry

$$\mathbb{A}^*_{0ij|k} = \mathbb{A}^*_{0ji|k}. \tag{44}$$

We also record here the useful connection

$$\mathcal{A}^*_{0jisk} - \mathcal{A}^*_{0ijsk} = (\tau_{js} + p\delta_{js})\delta_{ik} - (\tau_{is} + p\delta_{is})\delta_{jk}. \tag{45}$$

3. Finite deformations of an electroelastic tube

3.1. Extension and inflation of a tube

The geometry of a circular tube is described by cylindrical polar coordinates R, Θ, Z in its reference configuration according to

$$0 < A \le R \le B, \quad 0 \le \Theta \le 2\pi, \quad 0 \le Z \le L, \tag{46}$$

where A and B are the internal and external radii and L is the length of a tube. The deformed geometry of the tube with circular symmetry maintained is described in terms of cylindrical polar coordinates r, θ, z as

$$a \le r \le b, \quad 0 \le \theta \le 2\pi, \quad 0 \le z \le l, \tag{47}$$

where a, b and l are the radii and the length in the deformed configuration.

For an incompressible material, on which we focus here, the deformation is given by

$$r^2 = a^2 + \lambda_z^{-1}(R^2 - A^2), \quad \theta = \Theta, \quad z = \lambda_z Z, \tag{48}$$

where λ_z is the uniform axial stretch of the tube. The azimuthal stretch is denoted λ and given by $\lambda = r/R$, and the deformation gradient is diagonal with respect to the cylindrical polar axes. Then, by the incompressibility condition (3), the radial stretch is $\lambda_r = \lambda^{-1}\lambda_z^{-1}$, and in terms of the independent stretches λ and λ_z the invariants I_1 and I_2 are then

$$I_1 = \lambda^{-2}\lambda_z^{-2} + \lambda^2 + \lambda_z^2, \quad I_2 = \lambda^2\lambda_z^2 + \lambda^{-2} + \lambda_z^{-2}. \tag{49}$$

3.1.1. Field variables and boundary conditions. Flexible electrodes are attached to the boundaries $R = A$ and $R = B$ in the reference configuration. A potential difference is then applied between the electrodes along with an axial mechanical load and internal pressure, resulting in the deformed configuration described above and charges on the electrodes with equal magnitudes and opposite signs. Let the charges be denoted $Q(a)$ and $Q(b)$, so that $Q(a) + Q(b) = 0$. Because of the symmetry it can be assumed that the electric field is purely radial and therefore independent of the coordinates θ and z, i.e., the geometry of the tube is such that end effects can be neglected. Let $D_r(r)$ then denote the only component of the electric displacement $\mathbf{D}$. The equation $(5)_2$ satisfied by $\mathbf{D}$ yields the specialization

$$\frac{\mathrm{d}(rD_r)}{\mathrm{d}r} = 0, \tag{50}$$

so that rD_r is a constant, and hence, by evaluation at the boundaries $r = a$ and $r = b$, we have

$$rD_r(r) = aD_r(a) = bD_r(b). \tag{51}$$

By Gauss's theorem there is no field outside the tube so that $\mathbf{D}^\star = \mathbf{0}$. The boundary condition $(6)_2$ applied on $r = a$ and $r = b$ then yields

$$D_r(a) = \sigma_{\mathrm{f}a}, \quad D_r(b) = -\sigma_{\mathrm{f}b}, \tag{52}$$

where $\sigma_{\mathrm{f}a}$ and $\sigma_{\mathrm{f}b}$ are the free surface charge densities on the two boundaries, i.e.,

$$\sigma_{\mathrm{f}a} = \frac{Q(a)}{2\pi a l}, \quad \sigma_{\mathrm{f}b} = \frac{Q(b)}{2\pi b l}. \tag{53}$$

For this problem the Lagrange variable $\mathbf{D}_{\mathrm{L}}$ has just a single component, which we denote by $D_{\mathrm{L}R}$, related by the specialization of $(10)_2$ to D_r by $D_{\mathrm{L}R} = \lambda_r^{-1} D_r = \lambda \lambda_z D_r$. From the definitions (19) of the invariants I_4, I_5, I_6 we then obtain the specializations

$$I_4 = D_{\mathrm{L}R}^2, \quad I_5 = \lambda^{-2}\lambda_z^{-2} I_4, \quad I_6 = \lambda^{-4}\lambda_z^{-4} I_4. \tag{54}$$

For an incompressible isotropic electroelastic material the energy function Ω^* is expressed as a function of the invariants I_1, I_2, I_4, I_5, I_6 and written with these arguments: $\Omega^*(I_1, I_2, I_4, I_5, I_6)$. Then the only nonzero component of the electric field $\mathbf{E}$, denoted E_r, is obtained from the specialization of (21) as

$$E_r = 2(\Omega_4^* \lambda^2 \lambda_z^2 + \Omega_5^* + \Omega_6^* \lambda^{-2}\lambda_z^{-2}) D_r, \tag{55}$$

and Eq. $(5)_1$ is satisfied automatically.

3.1.2. Stress components and mechanical equilibrium. From (20) the only nonzero (total) Cauchy stress components are given by

$$\tau_{rr} = 2\Omega_1^* \lambda^{-2}\lambda_z^{-2} + 2\Omega_2^*(\lambda_z^{-2} + \lambda^{-2}) - p + 2\Omega_5^* D_r^2 + 4\Omega_6^* \lambda^{-2}\lambda_z^{-2} D_r^2, \tag{56}$$

$$\tau_{\theta\theta} = 2\Omega_1^* \lambda^2 + 2\Omega_2^* \left(\lambda_z^{-2} + \lambda_z^2 \lambda^2\right) - p, \tag{57}$$

$$\tau_{zz} = 2\Omega_1^* \lambda_z^2 + 2\Omega_2^* \left(\lambda^{-2} + \lambda^2 \lambda_z^2\right) - p. \tag{58}$$

Since, from (49) and (54), the invariants are functions of two independent stretches and I_4, we now define a reduced energy function in the form

$$\omega^*(\lambda, \lambda_z, I_4) = \Omega^*(I_1, I_2, I_4, I_5, I_6), \tag{59}$$

with the invariants on the right-hand side given by (49) and (54). Elimination of p then leads to simple expressions for the stress differences, namely

$$\tau_{\theta\theta} - \tau_{rr} = \lambda \frac{\partial \omega^*}{\partial \lambda}, \quad \tau_{zz} - \tau_{rr} = \lambda_z \frac{\partial \omega^*}{\partial \lambda_z}. \tag{60}$$

Similarly, the expression (55) is simplified to

$$E_r = 2\lambda^2 \lambda_z^2 \frac{\partial \omega^*}{\partial I_4} D_r. \tag{61}$$

The equilibrium equation (7) reduces to

$$r \frac{\mathrm{d}\tau_{rr}}{\mathrm{d}r} = \tau_{\theta\theta} - \tau_{rr}, \tag{62}$$

with which we associate boundary conditions on $r = a$ and $r = b$. Since there is no electric field outside the tube the Maxwell stress (9) vanishes. Thus, the only load on the surfaces $r = a$ and $r = b$ is the internal pressure P on $r = a$ with zero traction on $r = b$, i.e.,

$$\tau_{rr} = -P \quad \text{on} \quad r = a, \quad \tau_{rr} = 0 \quad \text{on} \quad r = b. \tag{63}$$

Integration of (62) with the use of $(60)_1$ and boundary conditions (63) yields

$$P = \int_a^b \lambda \omega_\lambda^* \frac{\mathrm{d}r}{r}, \tag{64}$$

where $\omega_\lambda^* = \partial \omega^* / \partial \lambda$.

From (48) we have $b = \sqrt{a^2 + \lambda_z^{-1}(B^2 - A^2)}$ and from $(54)_1$ with $D_{\mathrm{L}R} = \lambda \lambda_z D_r$, it follows from (51), (52), (53) and (48) that $I_4 = Q(a)^2/4\pi^2 L^2[A^2 + \lambda_z(r^2 - a^2)]$. Thus, for any given λ_z and initial geometry

of the tube, (64) provides an expression for the pressure as a function of the internal radius a and the charge on the electrodes.

The axial load N on any cross section of the cylinder is given by

$$N = 2\pi \int_a^b \tau_{zz} r \mathrm{d}r, \tag{65}$$

and a standard manipulation enables this to be re-cast in the form

$$F = N - \pi a^2 P = \pi \int_a^b (2\lambda_z \omega_{\lambda_z}^* - \lambda \omega_\lambda^*) r \mathrm{d}r, \tag{66}$$

where F denotes the so-called reduced axial load, which removes from N the contribution of the pressure on the ends of a closed-ended tube.

4. Bifurcation of an electroelastic circular cylinder

In this section, for consistency with the corresponding analysis in the purely elastic situation in [8], we re-order the coordinates r, θ, z as θ, z, r, associated with the stretches $\lambda, \lambda_z, \lambda_r$, respectively. We also use the notation $\lambda_1, \lambda_2, \lambda_3$ for these stretches in the same order, and for the associated components of vectors and tensors we adopt the indices $1, 2, 3$. The reduced energy function $\omega^*(\lambda, \lambda_z, I_4)$ is then replaced by $\omega^*(\lambda_1, \lambda_2, I_4)$, and stress differences (60) are rewritten as

$$\tau_{11} - \tau_{33} = \lambda \omega_\lambda^*, \quad \tau_{22} - \tau_{33} = \lambda_z \omega_{\lambda_z}^*, \tag{67}$$

where ω_λ^*, $\omega_{\lambda_z}^*$ denote the derivatives $\partial \omega^* / \partial \lambda$, $\partial \omega^* / \partial \lambda_z$.

The unit basis vectors associated with the cylindrical polar coordinates θ, z, r are denoted as $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, and the derivatives $(\cdot)_{,k}$ in (34) denoted by subscripts with commas become $\partial / r \partial \theta$, $\partial / \partial z$, $\partial / \partial r$ for $k = 1, 2, 3$, respectively. For the cylindrical polar coordinates the only nonzero scalar products $\mathbf{e}_i \cdot \mathbf{e}_{j,k}$ in (34) are

$$\mathbf{e}_1 \cdot \mathbf{e}_{3,1} = -\mathbf{e}_3 \cdot \mathbf{e}_{1,1} = \frac{1}{r}. \tag{68}$$

The incremental displacement $\dot{\mathbf{x}} = \mathbf{u}$ is written

$$\mathbf{u} = v\mathbf{e}_1 + w\mathbf{e}_2 + u\mathbf{e}_3, \tag{69}$$

and the matrix of components of $\mathbf{L} = \mathrm{grad}\, \mathbf{u}$ with respect to the basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is

$$[L_{ij}] = \begin{bmatrix} (u + v_\theta)/r & v_z & v_r \\ w_\theta/r & w_z & w_r \\ (u_\theta - v)/r & u_z & u_r \end{bmatrix}, \tag{70}$$

where the subscripts θ, z, r without a preceding comma indicate the corresponding partial derivatives.

The incremental incompressibility condition (33) specializes to

$$(u + v_\theta)/r + w_z + u_r = 0. \tag{71}$$

4.1. Prismatic bifurcations

For prismatic bifurcations u, v and w are independent of z. Furthermore, the equation for w decouples from those for u and v and any solution for w does not affect the cross-sectional shape of the tube. It is therefore convenient to set $w = 0$. The matrix of components of $\mathbf{L}$ then specializes to

$$[L_{ij}] = \begin{bmatrix} (u + v_\theta)/r & 0 & v_r \\ 0 & 0 & 0 \\ (u_\theta - v)/r & 0 & u_r \end{bmatrix}, \tag{72}$$

and the incompressibility condition reduces to

$$u + v_\theta + ru_r = 0. \tag{73}$$

Equation (73) is satisfied by taking a function $\phi(\theta, r)$ such that

$$u = \frac{\phi_\theta}{r}, \quad v = -\phi_r. \tag{74}$$

For the updated increment $\dot{\mathbf{D}}_{\mathrm{L}0}$ in the independent variable $\mathbf{D}_{\mathrm{L}}$ Eq. $(29)_2$ reduces to

$$\frac{\partial(r\dot{D}_{\mathrm{L}03})}{\partial r} + \frac{\partial \dot{D}_{\mathrm{L}01}}{\partial \theta} = 0, \tag{75}$$

which, similarly to (73), is satisfied by defining a function $\psi(\theta, r)$ such that

$$\dot{D}_{\mathrm{L}03} = \frac{\psi_\theta}{r}, \quad \dot{D}_{\mathrm{L}01} = -\psi_r. \tag{76}$$

The remaining governing equations are now used to obtain two coupled equations for ϕ and ψ. First, we note that the equilibrium equation for $i = 2$ in (34) is satisfied identically, while the equations for $i = 1, 3$ yield

$$\dot{T}_{011,1} + \dot{T}_{031,3} + \frac{1}{r}(\dot{T}_{031} + \dot{T}_{013}) = 0, \tag{77}$$

$$\dot{T}_{013,1} + \dot{T}_{033,3} + \frac{1}{r}(\dot{T}_{033} - \dot{T}_{011}) = 0. \tag{78}$$

For the considered underlying cylindrical configuration the components of $\dot{\mathbf{T}}_0$ in the above two equations are given by

$$\dot{T}_{011} = \mathcal{A}^*_{01111}L_{11} + \mathcal{A}^*_{01133}L_{33} + pL_{11} - \dot{p} + \mathbb{A}^*_{011|3}\dot{D}_{\mathrm{L}03}, \tag{79}$$

$$\dot{T}_{013} = \mathcal{A}^*_{01313}L_{31} + \mathcal{A}^*_{01331}L_{13} + pL_{13} + \mathbb{A}^*_{013|1}\dot{D}_{\mathrm{L}01}, \tag{80}$$

$$\dot{T}_{031} = \mathcal{A}^*_{03131}L_{13} + \mathcal{A}^*_{03113}L_{31} + pL_{31}, \tag{81}$$

$$\dot{T}_{033} = \mathcal{A}^*_{03311}L_{11} + \mathcal{A}^*_{03333}L_{33} + pL_{33} - \dot{p} + \mathbb{A}^*_{033|3}\dot{D}_{\mathrm{L}03}, \tag{82}$$

where the components of the electroelastic moduli tensors $\mathcal{A}^*_0$ and $\mathbb{A}^*_0$ are obtained by specializing the general expressions given in "Appendix A.2."

On substitution of these expressions into (77) and (78) and use of incompressibility condition (73), we obtain

$$\begin{aligned} \dot{p}_\theta = {}& [r(\mathcal{A}^{*\prime}_{03113} + p') + \mathcal{A}^*_{01313}](u_\theta - v)/r + (r\mathcal{A}^{*\prime}_{03131} + \mathcal{A}^*_{03131})v_r + \mathcal{A}^*_{03131}rv_{rr} \\ & + (\mathcal{A}^*_{01331} + \mathcal{A}^*_{01133} - \mathcal{A}^*_{01111})u_{r\theta} + \mathbb{A}^*_{011|3}\dot{D}_{\mathrm{L}03,\theta} + \mathbb{A}^*_{013|1}\dot{D}_{\mathrm{L}01}, \end{aligned} \tag{83}$$

$$\begin{aligned} \dot{p}_r = {}& [r(\mathcal{A}^{*\prime}_{03333} + p' - \mathcal{A}^*_{01133}) + \mathcal{A}^*_{03333} + \mathcal{A}^*_{01111} - 2\mathcal{A}^*_{01133}]u_r/r \\ & + (\mathcal{A}^*_{03333} - \mathcal{A}^*_{01133})u_{rr} + \mathcal{A}^*_{01313}(u_{\theta\theta} - v_\theta)/r^2 + \mathcal{A}^*_{01331}v_{r\theta}/r + \mathbb{A}^*_{013|1}\dot{D}_{\mathrm{L}01,\theta}/r \\ & + \mathbb{A}^{*\prime}_{033|3}\dot{D}_{\mathrm{L}03} + \mathbb{A}^*_{033|3}(\dot{D}_{\mathrm{L}03,r} + \dot{D}_{\mathrm{L}03}/r) - \mathbb{A}^*_{011|3}\dot{D}_{\mathrm{L}03}/r, \end{aligned} \tag{84}$$

respectively, where a prime denotes differentiation with respect to r and the partial derivatives of the components $\dot{D}_{\mathrm{L}01}$ and $\dot{D}_{\mathrm{L}03}$ with respect to r and θ are indicated by a prior comma.

Second, Eq. $(29)_1$ reduces to

$$\frac{\partial(r\dot{E}_{\mathrm{L}01})}{\partial r} - \frac{\partial \dot{E}_{\mathrm{L}03}}{\partial \theta} = 0. \tag{85}$$

From $(42)_2$ appropriately specialized we obtain

$$\dot{E}_{\mathrm{L}01} = \mathbb{A}^*_{013|1}L_{31} + \mathsf{A}^*_{011}\dot{D}_{\mathrm{L}01}, \quad \dot{E}_{\mathrm{L}03} = \mathbb{A}^*_{011|3}L_{11} + \mathbb{A}^*_{033|3}L_{33} + \mathsf{A}^*_{033}\dot{D}_{\mathrm{L}03}, \tag{86}$$

with the components of $\mathbb{A}^*_0$ and $\mathbf{A}^*_0$ given in "Appendix A.2." After substitution of (86) into (85) the latter gives

$$\mathbb{A}^*_{013|1}(u_\theta - v)/r + \mathsf{A}^*_{011}\dot{D}_{\mathrm{L}01} + \mathbb{A}^{*\prime}_{013|1}(u_\theta - v) + \mathbb{A}^*_{013|1}[(u_{\theta r} - v_r)r - u_\theta + v]/r$$
$$+ \mathsf{A}^{*\prime}_{011}r\dot{D}_{\mathrm{L}01} + \mathsf{A}^*_{011}r\dot{D}_{\mathrm{L}01,r} - \mathbb{A}^*_{011|3}(u_\theta + v_{\theta\theta})/r - \mathbb{A}^*_{033|3}u_{r\theta} - \mathsf{A}^*_{033}\dot{D}_{\mathrm{L}03,\theta} = 0. \tag{87}$$

By eliminating $\dot{p}_r$ and $\dot{p}_\theta$ from (83) and (84) by cross-differentiation and then substituting for expressions (74) and (76) in the result and in (87), followed by some rearrangements, we obtain two coupled equations for ϕ and ψ, namely

$$r^4\mathcal{A}^*_{03131}\phi_{rrrr} + \mathcal{A}^*_{01313}\phi_{\theta\theta\theta\theta} - (2\mathcal{A}^*_{01331} + 2\mathcal{A}^*_{01133} - \mathcal{A}^*_{01111} - \mathcal{A}^*_{03333})r^2\phi_{rr\theta\theta}$$
$$+ (2r^4\mathcal{A}^{*\prime}_{03131} + 2r^3\mathcal{A}^*_{03131})\phi_{rrr} - [(\mathcal{A}^*_{03333} + \mathcal{A}^*_{01111} - 2\mathcal{A}^*_{01331} - 2\mathcal{A}^*_{01133})r$$
$$+ (2\mathcal{A}^{*\prime}_{01331} + 2\mathcal{A}^{*\prime}_{01133} - \mathcal{A}^{*\prime}_{01111} - \mathcal{A}^{*\prime}_{03333})r^2]\,\phi_{r\theta\theta} - [2\mathcal{A}^*_{01331} - \mathcal{A}^*_{01111} - \mathcal{A}^*_{03333} + 2\mathcal{A}^*_{01133} - 2\mathcal{A}^*_{01313}$$
$$+ (\mathcal{A}^{*\prime}_{01313} - 2\mathcal{A}^{*\prime}_{01331} - 2\mathcal{A}^{*\prime}_{01133} + \mathcal{A}^{*\prime}_{01111} + \mathcal{A}^{*\prime}_{03333})r + (\mathcal{A}^{*\prime\prime}_{01331} + p^{\prime\prime})r^2]\phi_{\theta\theta}$$
$$- (\mathcal{A}^{*\prime}_{01331} + p^\prime + \mathcal{A}^*_{01313}/r - 2\mathcal{A}^{*\prime}_{03131} - r\mathcal{A}^{*\prime\prime}_{03131})r^3\phi_{rr} - (\mathcal{A}^{*\prime\prime}_{01331} + p^{\prime\prime} + \mathcal{A}^{*\prime}_{01313}/r - \mathcal{A}^*_{01313}/r^2)r^3\phi_r$$
$$- (\mathbb{A}^*_{011|3} + \mathbb{A}^*_{013|1} - \mathbb{A}^*_{033|3})r^2\psi_{r\theta\theta} - (\mathbb{A}^{*\prime}_{011|3} - \mathbb{A}^{*\prime}_{033|3})r^2\psi_{\theta\theta} + \mathbb{A}^*_{013|1}r^3\psi_{rr} + \mathbb{A}^{*\prime}_{013|1}r^3\psi_r = 0, \tag{88}$$

$$(\mathbb{A}^*_{013|1} + \mathbb{A}^*_{011|3} - \mathbb{A}^*_{033|3})r\phi_{r\theta\theta} + (\mathbb{A}^*_{033|3} - \mathbb{A}^*_{013|1} - \mathbb{A}^*_{011|3} + \mathbb{A}^{*\prime}_{013|1}r)\phi_{\theta\theta}$$
$$+ \mathbb{A}^*_{013|1}r^2\phi_{rr} + \mathbb{A}^{*\prime}_{013|1}r^2\phi_r - r^3\mathsf{A}^*_{011}\psi_{rr} - \mathsf{A}^*_{033}r\psi_{\theta\theta} - (\mathsf{A}^*_{011} + r\mathsf{A}^{*\prime}_{011})r^2\psi_r = 0. \tag{89}$$

We now turn to the boundary conditions associated with the latter two equations. First, the boundary condition (32) is specialized to

$$\dot{\mathbf{T}}_0^{\mathrm{T}}\mathbf{n} = \dot{\mathbf{t}}_{\mathrm{A}0} = \begin{cases} P\mathbf{L}^{\mathrm{T}}\mathbf{n} - \dot{P}\mathbf{n} & \text{on } r = a \\ 0 & \text{on } r = b \end{cases} \tag{90}$$

since $\mathbf{E}^\star = \mathbf{0}$, where $\dot{P}$ is a prescribed constant increment in P. Hence, by (9) and (27), both the Maxwell stress and its increment vanish. In terms of components this gives

$$\dot{T}_{031} = \begin{cases} PL_{31} & \text{on } r = a \\ 0 & \text{on } r = b, \end{cases} \quad \dot{T}_{033} = \begin{cases} PL_{33} - \dot{P} & \text{on } r = a \\ 0 & \text{on } r = b. \end{cases} \tag{91}$$

By using (81), (45) and the boundary conditions (63), the first of these can be written as

$$\mathcal{A}^*_{03131}(L_{13} + L_{31}) = 0 \quad \text{on} \quad r = a, b, \tag{92}$$

while on use of (82) and the incompressibility condition $L_{11} + L_{33} = 0$ the second can be rewritten as

$$(\mathcal{A}^*_{03333} - \mathcal{A}^*_{03311} + p)L_{33} - \dot{p} + \mathbb{A}^*_{033|3}\dot{D}_{\mathrm{L}03} = \begin{cases} PL_{33} - \dot{P} & \text{on } r = a \\ 0 & \text{on } r = b. \end{cases} \tag{93}$$

The term in $\dot{p}$ is eliminated by differentiating (93) with respect to θ (which removes the constant $\dot{P}$) and using (83). Then by using the expressions for the components L_{ij}, $\dot{D}_{\mathrm{L}01}$ and $\dot{D}_{\mathrm{L}03}$ in terms of the functions ϕ and ψ this yields the boundary conditions

$$
\begin{aligned}
&r(\mathcal{A}_{03333}^* - 2\mathcal{A}_{01133}^* - 2\mathcal{A}_{01331}^* + \mathcal{A}_{03131}^* + \mathcal{A}_{01111}^*)\phi_{r\theta\theta} + r^3 \mathcal{A}_{03131}^* \phi_{rrr} \\
&\quad - (r\mathcal{A}_{03131}^{*\prime} + \mathcal{A}_{03333}^* - 2\mathcal{A}_{01133}^* - 2\mathcal{A}_{01331}^* + 2\mathcal{A}_{03131}^* + \mathcal{A}_{01111}^*)\phi_{\theta\theta} \\
&\quad + r^2(r\mathcal{A}_{03131}^{*\prime} + \mathcal{A}_{03131}^*)\phi_{rr} - r(r\mathcal{A}_{03131}^{*\prime} + \mathcal{A}_{03131}^*)\phi_r \\
&\quad + r(\mathbb{A}_{033|3}^* - \mathbb{A}_{011|3}^*)\psi_{\theta\theta} + r^2 \mathbb{A}_{013|1}^* \psi_r = 0 \quad \text{on} \quad r = a, b,
\end{aligned}
\tag{94}
$$

in which we have also used the boundary condition $(63)_1$ in the form $\tau_{33} = -P$ on $r = a$. And from (92), assuming $\mathcal{A}_{03131}^* \neq 0$, we obtain similarly

$$
r^2 \phi_{rr} - r\phi_r - \phi_{\theta\theta} = 0 \quad \text{on} \quad r = a, b.
\tag{95}
$$

The incremental electric field has to satisfy Eq. $(29)_1$, which becomes (87) for the problem considered here, and it also has to satisfy the associated boundary condition (30). Since there is no exterior field, the latter reduces to

$$
\dot{E}_{\mathrm{L}01} = 0 \quad \text{on} \quad r = a, b,
\tag{96}
$$

which, from the constitutive component $(86)_1$ and the expression for the component L_{13} with (74) and $(76)_2$, can be written in terms of functions ϕ and ψ as

$$
\mathbb{A}_{013|1}^*(\phi_{\theta\theta} + r\phi_r)/r^2 - \mathsf{A}_{011}^* \psi_r = 0 \quad \text{on} \quad r = a, b.
\tag{97}
$$

We now have two coupled partial differential equations (88) and (89) for ϕ and ψ as functions of r and θ and two boundary conditions in each of (94), (95) and (97). No progress is possible theoretically, and in general, the system of equations and boundary conditions needs to be solved numerically. Toward this we consider solutions for ϕ and ψ of the form

$$
\phi = r f_n(r) \sin n\theta \quad \text{and} \quad \psi = g_n(r) \sin n\theta
\tag{98}
$$

for consistency and comparison with the corresponding purely elastic situation considered in [8], n being a nonnegative integer.

The equations (88) and (89) are now expressed in terms of the functions $f_n(r)$ and $g_n(r)$ and their derivatives as

$$
\begin{aligned}
&r\{\mathcal{A}_{03131}^* r^3 f_n''' + (r\mathcal{A}_{03131}^{*\prime} + 3\mathcal{A}_{03131}^*)r^2 f_n'' \\
&\quad + [r\mathcal{A}_{03131}^{*\prime} - \mathcal{A}_{03131}^* + n^2(2\mathcal{A}_{01331}^* + 2\mathcal{A}_{01133}^* - \mathcal{A}_{01111}^* - \mathcal{A}_{03333}^*)]rf_n'\}' \\
&\quad + (n^2 - 1)[r^2 \mathcal{A}_{03131}^{*\prime\prime} + r\mathcal{A}_{03131}^{*\prime} + (n^2 - 1)\mathcal{A}_{03131}^* + n^2(\tau_{11} - \tau_{33})]f_n + \mathbb{A}_{013|1}^* r^2 g_n'' \\
&\quad + \mathbb{A}_{013|1}^{*\prime} r^2 g_n' + (\mathbb{A}_{011|3}^* + \mathbb{A}_{013|1}^* - \mathbb{A}_{033|3}^*)rn^2 g_n' + (\mathbb{A}_{011|3}^{*\prime} - \mathbb{A}_{033|3}^{*\prime})rn^2 g_n = 0,
\end{aligned}
\tag{99}
$$

$$
\begin{aligned}
&r^2 \mathbb{A}_{013|1}^* f_n'' + [r^2 \mathbb{A}_{013|1}^{*\prime} + 2r\mathbb{A}_{013|1}^* - rn^2(\mathbb{A}_{013|1}^* + \mathbb{A}_{011|3}^* - \mathbb{A}_{033|3}^*)]f_n' \\
&\quad + r(\mathbb{A}_{013|1}^{*\prime} - n^2 \mathbb{A}_{013|1}^{*\prime})f_n - r^2 \mathsf{A}_{011}^* g_n'' - (\mathsf{A}_{011}^* + r\mathsf{A}_{011}^{*\prime})rg_n' + \mathsf{A}_{033}^* n^2 g_n = 0.
\end{aligned}
\tag{100}
$$

To obtain (99) we have used the expression

$$
p' = \mathcal{A}_{03131}^{*\prime} - \mathcal{A}_{01331}^{*\prime} + (\mathcal{A}_{03131}^* - \mathcal{A}_{01313}^*)/r,
\tag{101}
$$

which can be obtained from (62) and (45), and its derivative

$$
p'' = \mathcal{A}_{03131}^{*\prime\prime} - \mathcal{A}_{01331}^{*\prime\prime} - (\mathcal{A}_{03131}^* - \mathcal{A}_{01313}^*)/r^2 + (\mathcal{A}_{03131}^{*\prime} - \mathcal{A}_{01313}^{*\prime})/r.
\tag{102}
$$

The boundary conditions (94), (95) and (97) are rewritten as

$$\mathcal{A}^*_{03131} r^3 f'''_n + (r\mathcal{A}^{*'}_{03131} + 4\mathcal{A}^*_{03131})r^2 f''_n$$
$$+ [r\mathcal{A}^{*'}_{03131} - (n^2-1)\mathcal{A}^*_{03131} + n^2(2\mathcal{A}^*_{01331} + 2\mathcal{A}^*_{01133} - \mathcal{A}^*_{01111} - \mathcal{A}^*_{03333})]rf'_n$$
$$+ (n^2-1)(r\mathcal{A}^{*'}_{03131} + \mathcal{A}^*_{03131})f_n - n^2(\mathbb{A}^*_{033|3} - \mathbb{A}^*_{011|3})g_n + \mathbb{A}^*_{013|1}rg'_n = 0 \quad \text{on} \quad r = a, b. \tag{103}$$

$$r^2 f''_n + rf'_n + (n^2-1)f_n = 0 \quad \text{on} \quad r = a, b, \tag{104}$$

$$\mathbb{A}^*_{013|1}[rf'_n + (1-n^2)f_n] - \mathsf{A}^*_{011}rg'_n = 0 \quad \text{on} \quad r = a, b. \tag{105}$$

4.1.1. Numerical solution. For purposes of the numerical solution we introduce the dimensionless quantities

$$\hat{r} = \frac{r}{A}, \quad \hat{f}_n(\hat{r}) = \frac{f_n(r)}{A}, \quad \hat{g}_n(\hat{r}) = \frac{g_n(r)}{\sigma_{\mathrm{fa}}A}, \tag{106}$$

$$\hat{\boldsymbol{\tau}} = \frac{\boldsymbol{\tau}}{\mu}, \quad \hat{\boldsymbol{\mathcal{A}}}^*_0 = \frac{\boldsymbol{\mathcal{A}}^*_0}{\mu}, \quad \hat{\mathbb{A}}^*_0 = \mathbb{A}^*_0\frac{\varepsilon}{\sigma_{\mathrm{fa}}}, \quad \hat{\mathbf{A}}^*_0 = \mathbf{A}^*_0\varepsilon, \tag{107}$$

where the constants μ and ε have dimensions of elastic modulus and permittivity, respectively, and we recall that σ_{fa} is the charge per unit area on the boundary $r = a$. We then rearrange the governing equations as a nondimensional system of six ordinary differential equations with the new variables

$$\hat{y}_1(\hat{r}) = \hat{f}_n(\hat{r}), \quad \hat{y}_2(\hat{r}) = \hat{f}'_n(\hat{r}), \quad \hat{y}_3(\hat{r}) = \hat{f}''_n(\hat{r}),$$
$$\hat{y}_4(\hat{r}) = \hat{f}'''_n(\hat{r}), \quad \hat{y}_5(\hat{r}) = \hat{g}_n(\hat{r}), \quad \hat{y}_6(\hat{r}) = \hat{g}'_n(\hat{r}), \tag{108}$$

suitable for use in MATLAB [17]. The resulting first-order system of equations may be written in the standard form as $\hat{y}'_i = M_{ij}\hat{y}_j$, $i = 1, \ldots, 6$, where a prime now indicates differentiation with respect to $\hat{r}$ and M_{ij} is a 6×6 matrix whose entries can be extracted from the connections

$$\hat{y}'_1 = \hat{y}_2, \quad \hat{y}'_2 = \hat{y}_3, \quad \hat{y}'_3 = \hat{y}_4, \quad \hat{y}'_5 = \hat{y}_6,$$
$$\hat{\mathcal{A}}^*_{03131}\hat{r}^4\hat{y}'_4 + (n^2-1)[\hat{r}^2\hat{\mathcal{A}}^{*''}_{03131} + \hat{r}\hat{\mathcal{A}}^{*'}_{03131} + (n^2-1)\hat{\mathcal{A}}^*_{03131} + n^2(\hat{\tau}_{11} - \hat{\tau}_{33})]\hat{y}_1$$
$$+ [\hat{r}^3\hat{\mathcal{A}}^{*''}_{03131} + n^2\hat{r}^2\hat{Q}'(\hat{r}) + \hat{r}^2\hat{\mathcal{A}}^{*'}_{03131} - \hat{r}\hat{\mathcal{A}}^*_{03131} + n^2\hat{r}\hat{Q}(\hat{r})]\hat{y}_2$$
$$+ [7\hat{r}^3\hat{\mathcal{A}}^{*'}_{03131} + \hat{r}^4\hat{\mathcal{A}}^{*''}_{03131} + 5\hat{r}^2\hat{\mathcal{A}}^*_{03131} + n^2\hat{r}^2\hat{Q}(\hat{r})]\hat{y}_3 + (6\hat{r}^3\hat{\mathcal{A}}^*_{03131} + 2\hat{r}^4\hat{\mathcal{A}}^{*'}_{03131})\hat{y}_4$$
$$+ \hat{\sigma}^2_{\mathrm{fa}}n^2\hat{r}(\hat{\mathbb{A}}^{*'}_{011|3} - \hat{\mathbb{A}}^{*'}_{033|3})\hat{y}_5 + \hat{\sigma}^2_{\mathrm{fa}}[\hat{r}^2\hat{\mathbb{A}}^{*'}_{013|1} + n^2\hat{r}(\hat{\mathbb{A}}^*_{011|3} + \hat{\mathbb{A}}^*_{013|1} - \hat{\mathbb{A}}^*_{033|3})]\hat{y}_6$$
$$+ \hat{\sigma}^2_{\mathrm{fa}}\hat{r}^2\hat{\mathbb{A}}^*_{013|1}\hat{y}'_6 = 0,$$
$$\hat{r}^2\hat{\mathsf{A}}^*_{011}\hat{y}'_6 = \hat{r}(\hat{\mathbb{A}}^{*'}_{013|1} - n^2\hat{\mathbb{A}}^*_{013|1})\hat{y}_1 + [\hat{r}^2\hat{\mathbb{A}}^{*'}_{013|1} + 2\hat{r}\hat{\mathbb{A}}^*_{013|1} - n^2\hat{r}(\hat{\mathbb{A}}^*_{013|1} + \hat{\mathbb{A}}^*_{011|3} - \hat{\mathbb{A}}^*_{033|3})]\hat{y}_2$$
$$+ \hat{r}^2\hat{\mathbb{A}}^*_{013|1}\hat{y}_3 + n^2\hat{\mathbb{A}}^*_{033}\hat{y}_5 - \hat{r}(\hat{\mathsf{A}}^*_{011} + \hat{r}\hat{\mathsf{A}}^{*'}_{011})\hat{y}_6, \tag{109}$$

where for brevity we have defined the notations

$$\hat{Q}(\hat{r}) = 2\hat{\mathcal{A}}^*_{01331} + 2\hat{\mathcal{A}}^*_{01133} - \hat{\mathcal{A}}^*_{01111} - \hat{\mathcal{A}}^*_{03333}, \tag{110}$$

and

$$\hat{\sigma}^2_{\mathrm{fa}} = \frac{\sigma^2_{\mathrm{fa}}}{\mu\varepsilon}. \tag{111}$$

Note that the equation for $\hat{y}'_4$ in $(109)_5$ involves $\hat{y}'_6$, which can be obtained from $(109)_6$ for substitution into $(109)_5$ in order to obtain the matrix M_{ij} if required.

Also, we nondimensionalize the boundary conditions (103), (104) and (105) and express them in terms of the new variables as

$$\hat{r}^3\hat{\mathcal{A}}^*_{03131}\hat{y}_4 + \hat{r}^2(\hat{r}\hat{\mathcal{A}}^{*'}_{03131} + 4\hat{\mathcal{A}}^*_{03131})\hat{y}_3 + \hat{r}[\hat{r}\hat{\mathcal{A}}^{*'}_{03131} - (n^2-1)\hat{\mathcal{A}}^*_{03131} + n^2\hat{Q}(\hat{r})]\hat{y}_2$$
$$+ (n^2-1)(\hat{r}\hat{\mathcal{A}}^{*'}_{03131} + \hat{\mathcal{A}}^*_{03131})\hat{y}_1 - \hat{\sigma}^2_{\mathrm{fa}}n^2(\hat{\mathbb{A}}^*_{033|3} - \hat{\mathbb{A}}^*_{011|3})\hat{y}_5 + \hat{\sigma}^2_{\mathrm{fa}}\hat{r}\hat{\mathbb{A}}^*_{013|1}\hat{y}_6 = 0 \quad \text{on} \quad \hat{r} = \hat{a}, \hat{b}, \tag{112}$$

$$\hat{r}^2\hat{y}_3 + \hat{r}\hat{y}_2 + (n^2-1)\hat{y}_1 = 0 \quad \text{on} \quad \hat{r} = \hat{a}, \hat{b}, \tag{113}$$

TABLE 1. *Bifurcation values λ_a, λ_b and nondimensional pressure P/μ for neo-Hookean electroelastic material for different values of parameter $\hat{\sigma}_{fa}$ and different values of A/B; $\lambda_z = 1$, mode number $n = 2$ in all cases. Also, for comparison bifurcation values of λ_b from [8] for the three-term strain-energy function [18] are given in column H*

| | H | neo-Hookean electroelastic material (115) | | | | | | | | | | | |
| | | $\hat{\sigma}_{fa} = 0$ | | | $\hat{\sigma}_{fa} = 0.25$ | | | $\hat{\sigma}_{fa} = 0.5$ | | | $\hat{\sigma}_{fa} = 0.75$ | | |
A/B	λ_b	λ_a	λ_b	P/μ	λ_a	λ_b	P/μ	λ_a	λ_b	P/μ	λ_a	λ_b	P/μ
0.95	0.999	0.9993	0.999	-0.000	1.0651	1.058	0.008	1.3013	1.275	0.025	1.7789	1.718	0.036
0.9	0.997	0.9969	0.997	-0.001	1.0628	1.051	0.155	1.3032	1.251	0.050	1.8014	1.678	0.074
0.85	0.994	0.9923	0.994	-0.004	1.0581	1.042	0.020	1.3029	1.226	0.075	1.8248	1.638	0.115
0.8	0.990	0.9848	0.990	-0.011	1.0503	1.032	0.022	1.2992	1.200	0.099	1.8488	1.596	0.157
0.7	0.979	0.9576	0.979	-0.046	1.0211	1.010	0.005	1.2761	1.143	0.136	1.8963	1.507	0.247
0.6	0.968	0.9066	0.967	-0.139	0.9647	0.987	-0.068	1.2148	1.082	0.135	1.9339	1.409	0.343

$$\hat{\mathbb{A}}^*_{013|1}[\hat{r}\hat{y}_2 + (1 - n^2)\hat{y}_1] - \hat{r}\hat{\mathbb{A}}^*_{011}\hat{y}_6 = 0 \quad \text{on} \quad \hat{r} = \hat{a}, \hat{b}, \tag{114}$$

where $\hat{a} = a/A, \hat{b} = b/A$.

For the numerical calculations we use a simple energy function for which the purely elastic part corresponds to the neo-Hookean model and the electric part is quadratic in the electric displacement, so that

$$\Omega^*(I_1, I_5) = \frac{1}{2}\mu(I_1 - 3) + \frac{1}{2}\varepsilon^{-1}I_5, \tag{115}$$

where the constant μ is the shear modulus of the neo-Hookean material in the absence of an electric field and the constant ε is the electric permittivity of the electroelastic material. The required components of the electroelastic moduli are listed in "Appendix B.1," and the numerical scheme used for the calculations is described in Sect. 4.2.

For different values of the parameter $\hat{\sigma}_{fa}$ and for $\lambda_z = 1$ the values of $\lambda_a = a/A$ (the azimuthal stretch at the inner boundary) at which prismatic bifurcations become possible for mode number $n = 2$ are given in Table 1, along with the corresponding value of $\lambda_b = b/B$ (the azimuthal stretch at the outer boundary). The results for the neo-Hookean electroelastic material with $\hat{\sigma}_{fa} = 0$ are almost identical to those reported in [8] for a different strain-energy function. This is consistent with the findings in [8] that under *external* pressure the values of λ_b (equivalently λ_a) are essentially independent of the form of strain-energy function.

For the considered electroelastic material an important difference from the purely elastic case is that the tube may bifurcate into a prismatic configuration under internal pressure ($P > 0$). In [8] it was reported that prismatic modes are not possible under *internal* pressure for the material models considered, but only under external pressure ($P < 0$). This can also be observed here in Table 1 for the case $\hat{\sigma}_{fa} = 0$: all nondimensional pressures at which prismatic bifurcations are possible are negative. The values of P/μ in Table 1 were calculated using formula (64) and the specialization of (115) in the form

$$\omega^*(\lambda, \lambda_z, I_4) = \frac{1}{2}\mu(\lambda^2 + \lambda_z^2 + \lambda^{-2}\lambda_z^{-2} - 3) + \frac{1}{2}\varepsilon^{-1}\lambda^{-2}\lambda_z^{-2}I_4. \tag{116}$$

A closed-form expression for P was given in [3] for this constitutive model.

In [10], prismatic bifurcations in the absence of an internal pressure were analyzed and solutions showing the shapes of different bifurcation modes were obtained for both the Gent and neo-Hookean models.

Note that for any given underlying boundary-value problem we may prescribe either the charge or potential on the boundary, and the potential difference between $r = b$ and $r = a$, V say, or charge density σ_{fa}, adjusts according to the formula

$$V = -\frac{a\sigma_{fa}}{\varepsilon}\log(b/a). \tag{117}$$

This was given for the material model (115) in [3] (in a slightly different notation). Thus, results for a fixed potential difference can be obtained from Table 1.

4.2. Axisymmetric bifurcations

For axisymmetric bifurcations we take $v = 0$ and u and w to be independent of θ, so that the components of the displacement gradient specialize to

$$[L_{ij}] = \begin{bmatrix} u/r & 0 & 0 \\ 0 & w_z & w_r \\ 0 & u_z & u_r \end{bmatrix}, \tag{118}$$

and the incompressibility condition then has the form

$$u/r + w_z + u_r = 0. \tag{119}$$

A function $\phi(z, r)$ is then introduced such that

$$u = \frac{\phi_z}{r}, \quad w = -\frac{\phi_r}{r} \tag{120}$$

and Eq. (119) is satisfied.

The governing equation $(29)_2$ reduces to

$$\frac{\partial(r\dot{D}_{\mathrm{L}03})}{\partial r} + \frac{\partial(r\dot{D}_{\mathrm{L}02})}{\partial z} = 0, \tag{121}$$

which can be satisfied by defining a function $\psi(z, r)$ such that

$$\dot{D}_{\mathrm{L}03} = \frac{\psi_z}{r}, \quad \dot{D}_{\mathrm{L}02} = -\frac{\psi_r}{r}. \tag{122}$$

For axisymmetric incremental deformations with no dependence on θ and $v = 0$ the component of equilibrium equation (34) for $i = 1$ is satisfied identically, while the components for $i = 3, 2$ specialize, respectively, to

$$\dot{T}_{023,2} + \dot{T}_{033,3} + \frac{1}{r}(\dot{T}_{033} - \dot{T}_{011}) = 0, \tag{123}$$

$$\dot{T}_{022,2} + \dot{T}_{032,3} + \frac{1}{r}\dot{T}_{032} = 0, \tag{124}$$

wherein the components

$$\dot{T}_{023} = \mathcal{A}^*_{02323}L_{32} + \mathcal{A}^*_{02332}L_{23} + pL_{23} + \mathbb{A}^*_{023|2}\dot{D}_{\mathrm{L}02}, \tag{125}$$

$$\dot{T}_{032} = \mathcal{A}^*_{03232}L_{23} + \mathcal{A}^*_{03223}L_{32} + pL_{32}, \tag{126}$$

$$\dot{T}_{011} = \mathcal{A}^*_{01111}L_{11} + \mathcal{A}^*_{01122}L_{22} + \mathcal{A}^*_{01133}L_{33} + pL_{11} - \dot{p} + \mathbb{A}^*_{011|3}\dot{D}_{\mathrm{L}03}, \tag{127}$$

$$\dot{T}_{022} = \mathcal{A}^*_{02211}L_{11} + \mathcal{A}^*_{02222}L_{22} + \mathcal{A}^*_{02233}L_{33} + pL_{22} - \dot{p} + \mathbb{A}^*_{022|3}\dot{D}_{\mathrm{L}03}, \tag{128}$$

$$\dot{T}_{033} = \mathcal{A}^*_{03311}L_{11} + \mathcal{A}^*_{03322}L_{22} + \mathcal{A}^*_{03333}L_{33} + pL_{33} - \dot{p} + \mathbb{A}^*_{033|3}\dot{D}_{\mathrm{L}03} \tag{129}$$

are obtained by specializing $(42)_1$. Substitution of these into (123) and (124) and use of (119) yields

$$\begin{aligned} \dot{p}_r = {}&(r\mathcal{A}^{*\prime}_{01133} - \mathcal{A}^*_{01111})u/r^2 + (r\mathcal{A}^{*\prime}_{03333} + rp' + \mathcal{A}^*_{03333})u_r/r + \mathcal{A}^*_{03333}u_{rr} \\ &+ \mathcal{A}^*_{02323}u_{zz} + (r\mathcal{A}^{*\prime}_{02233} + \mathcal{A}^*_{02233} - \mathcal{A}^*_{01122})w_z/r + (\mathcal{A}^*_{02233} + \mathcal{A}^*_{03223})w_{rz} \\ &+ \mathbb{A}^*_{023|2}\dot{D}_{\mathrm{L}02,z} + \mathbb{A}^{*\prime}_{033|3}\dot{D}_{\mathrm{L}03} + \mathbb{A}^*_{033|3}\dot{D}_{\mathrm{L}03,r} + (\mathbb{A}^*_{033|3} - \mathbb{A}^*_{011|3})\dot{D}_{\mathrm{L}03}/r, \end{aligned} \tag{130}$$

$$\begin{aligned} \dot{p}_z = {}&\mathcal{A}^*_{03232}w_{rr} + (r\mathcal{A}^{*\prime}_{03232} + \mathcal{A}^*_{03232})w_r/r + \mathcal{A}^*_{02222}w_{zz} + (\mathcal{A}^*_{02233} + \mathcal{A}^*_{03223})u_{rz} \\ &+ (r\mathcal{A}^{*\prime}_{03223} + rp' + \mathcal{A}^*_{03223} + \mathcal{A}^*_{01122})u_z/r + \mathbb{A}^*_{022|3}\dot{D}_{\mathrm{L}03,z}. \end{aligned} \tag{131}$$

The governing equation $(29)_1$ reduces to

$$\frac{\partial \dot{E}_{\text{L}03}}{\partial z} - \frac{\partial \dot{E}_{\text{L}02}}{\partial r} = 0. \tag{132}$$

From $(42)_2$ we have

$$\dot{E}_{\text{L}02} = \mathbb{A}^*_{023|2} u_z + \mathsf{A}^*_{022} \dot{D}_{\text{L}02}, \quad \dot{E}_{\text{L}03} = \mathbb{A}^*_{011|3} u/r + \mathbb{A}^*_{022|3} w_z + \mathbb{A}^*_{033|3} u_r + \mathsf{A}^*_{033} \dot{D}_{\text{L}03}, \tag{133}$$

and when these are substituted into (132) we obtain

$$\mathbb{A}^*_{011|3} u_z/r + \mathbb{A}^*_{022|3} w_{zz} + \mathbb{A}^*_{033|3} u_{rz} + \mathsf{A}^*_{033} \dot{D}_{\text{L}03,z}$$
$$- \mathbb{A}^{*\prime}_{023|2} u_z - \mathbb{A}^*_{023|2} u_{zr} - \mathsf{A}^{*\prime}_{022} \dot{D}_{\text{L}02} - \mathsf{A}^*_{022} \dot{D}_{\text{L}02,r} = 0. \tag{134}$$

On use of (120) and (122) this is then expressed in terms of ϕ and ψ as

$$r(\mathbb{A}^*_{033|3} - \mathbb{A}^*_{022|3} - \mathbb{A}^*_{023|2})\phi_{rzz} + (\mathbb{A}^*_{011|3} - \mathbb{A}^*_{033|3} + \mathbb{A}^*_{023|2} - r\mathbb{A}^{*\prime}_{023|2})\phi_{zz}$$
$$+ \mathsf{A}^*_{033} r\psi_{zz} + \mathsf{A}^*_{022} r\psi_{rr} + (r\mathsf{A}^{*\prime}_{022} - \mathsf{A}^*_{022})\psi_r = 0. \tag{135}$$

To express (130) and (131) in terms of ϕ and ψ we eliminate the terms in $\dot{p}$ by cross-differentiation to arrive at

$$\mathcal{A}^*_{03232} r^3 \phi_{rrrr} + \mathcal{A}^*_{02323} r^3 \phi_{zzzz} - (2\mathcal{A}^*_{02233} + 2\mathcal{A}^*_{03223} - \mathcal{A}^*_{02222} - \mathcal{A}^*_{03333}) r^3 \phi_{rrzz}$$
$$- 2(\mathcal{A}^*_{03232} - r\mathcal{A}^{*\prime}_{03232}) r^2 \phi_{rrr} - \big[\mathcal{A}^*_{02222} - 2\mathcal{A}^*_{02233} - 2\mathcal{A}^*_{03223} + \mathcal{A}^*_{03333}$$
$$+ r(2\mathcal{A}^{*\prime}_{02233} + 2\mathcal{A}^{*\prime}_{03223} - \mathcal{A}^{*\prime}_{02222} - \mathcal{A}^{*\prime}_{03333})\big] r^2 \phi_{rzz} + (3\mathcal{A}^*_{03232} - 3\mathcal{A}^{*\prime}_{03232} r + \mathcal{A}^{*\prime\prime}_{03232} r^2) r\phi_{rr}$$
$$- \big[2\mathcal{A}^*_{02233} - 2\mathcal{A}^*_{01122} - r\mathcal{A}^{*\prime}_{01133} + \mathcal{A}^*_{01111} - \mathcal{A}^*_{03333}$$
$$+ r(\mathcal{A}^{*\prime}_{01122} - \mathcal{A}^{*\prime}_{02233} - \mathcal{A}^{*\prime}_{03223} + \mathcal{A}^{*\prime}_{03333}) + r^2(\mathcal{A}^{*\prime\prime}_{03223} + p'')\big] r\phi_{zz}$$
$$- (3\mathcal{A}^*_{03232} - 3r\mathcal{A}^{*\prime}_{03232} + r^2 \mathcal{A}^{*\prime\prime}_{03232})\phi_r - (\mathbb{A}^*_{022|3} + \mathbb{A}^*_{023|2} - \mathbb{A}^*_{033|3}) r^3 \psi_{rzz}$$
$$- \big[\mathbb{A}^*_{011|3} - \mathbb{A}^*_{022|3} + r(\mathbb{A}^{*\prime}_{022|3} - \mathbb{A}^{*\prime}_{033|3})\big] r^2 \psi_{zz} = 0, \tag{136}$$

p'' being given by (102).

Thus, we have two coupled equations (135) and (136), to which we now append the boundary conditions. Similarly to Sect. 4.1, the electric field boundary condition (30) reduces to

$$\dot{E}_{\text{L}02} = 0 \quad \text{on} \quad r = a, b, \tag{137}$$

which, on application of the specialization of the incremental constitutive law $(42)_2$, yields

$$\mathbb{A}^*_{023|2} \phi_{zz} - \mathsf{A}^*_{022} \psi_r = 0 \quad \text{on} \quad r = a, b \tag{138}$$

in terms of functions ϕ and ψ.

The traction boundary condition again has form (90) but now specializes to

$$\dot{T}_{032} = \begin{cases} PL_{32} & \text{on } r = a \\ 0 & \text{on } r = b, \end{cases} \qquad \dot{T}_{033} = \begin{cases} PL_{33} - \dot{P} & \text{on } r = a \\ 0 & \text{on } r = b, \end{cases} \tag{139}$$

and hence, on using the components (126) and (129), we obtain

$$\mathcal{A}^*_{03232}(L_{23} + L_{32}) = 0 \quad \text{on} \quad r = a, b, \tag{140}$$

and, on use of (129) and the incompressibility condition $L_{11} + L_{22} + L_{33} = 0$,

$$(\mathcal{A}^*_{03333} - \mathcal{A}^*_{02233} + p)L_{33} + (\mathcal{A}^*_{01133} - \mathcal{A}^*_{02233})L_{11} - \dot{p} + \mathbb{A}^*_{033|3} \dot{D}_{\text{L}03} = \begin{cases} PL_{33} - \dot{P} & \text{on } r = a \\ 0 & \text{on } r = b. \end{cases} \tag{141}$$

With the help of (120) and the assumption $\mathcal{A}^*_{03232} \neq 0$ Eq. (140) can be expressed in terms of ϕ as

$$r\phi_{zz} - r\phi_{rr} + \phi_r = 0 \quad \text{on} \quad r = a, b. \tag{142}$$

For the boundary conditions (141) the terms in $\dot{p}$ are eliminated by differentiating the equations with respect to z and using the expression for $\dot{p}_z$ in (131) to obtain, after use of (120) and (122), and some rearrangement with the help of (45) and (62),

$$r^2 \mathcal{A}^*_{03232}\phi_{rrr} + r^2(\mathcal{A}^*_{02222} + \mathcal{A}^*_{03333} - 2\mathcal{A}^*_{02233} + \mathcal{A}^*_{03232} - 2\mathcal{A}^*_{03223})\phi_{rzz} + r(r\mathcal{A}^{*\prime}_{03232} - \mathcal{A}^*_{03232})\phi_{rr}$$
$$- r(\mathcal{A}^*_{03333} + \mathcal{A}^*_{01122} - \mathcal{A}^*_{01133} - \mathcal{A}^*_{02233} + r\mathcal{A}^{*\prime}_{03232} + \mathcal{A}^*_{03232} - \mathcal{A}^*_{03223} + \mathcal{A}^*_{03131} - \mathcal{A}^*_{01313})\phi_{zz}$$
$$- (r\mathcal{A}^{*\prime}_{03232} - \mathcal{A}^*_{03232})\phi_r + r^2(\mathbb{A}^*_{033|3} - \mathbb{A}^*_{022|3})\psi_{zz} = 0 \quad \text{on} \quad r = a, b \tag{143}$$

in terms of functions ϕ and ψ.

Analogously to (98) we now write

$$\phi(z, r) = r f_\alpha(r)\cos\alpha z \quad \text{and} \quad \psi(z, r) = g_\alpha(r)\cos\alpha z, \tag{144}$$

where α is a constant to be determined by the boundary conditions on the ends of the tube.

The equations (135) and (136) can then be expressed, respectively, as

$$\alpha^2 r^2(\mathbb{A}^*_{033|3} - \mathbb{A}^*_{022|3} - \mathbb{A}^*_{023|2})f'_\alpha + \alpha^2 r(\mathbb{A}^*_{011|3} - \mathbb{A}^*_{022|3} - \mathbb{A}^{*\prime}_{023|2}r)f_\alpha$$
$$- \mathsf{A}^*_{022}rg''_\alpha - (\mathsf{A}^{*\prime}_{022}r - \mathsf{A}^*_{022})g'_\alpha + \alpha^2 r\mathsf{A}^*_{033}g_\alpha = 0 \tag{145}$$

and

$$r^4[\mathcal{A}^*_{03232}f'''_\alpha + (r\mathcal{A}^{*\prime}_{03232} + 2\mathcal{A}^*_{03232})f''_\alpha/r + (r\mathcal{A}^{*\prime}_{03232} - \mathcal{A}^*_{03232})f'_\alpha/r^2 - (r\mathcal{A}^{*\prime}_{03232} - \mathcal{A}^*_{03232})f_\alpha/r^3]'$$
$$+ \alpha^2 r^2[(2\mathcal{A}^*_{02233} + 2\mathcal{A}^*_{03223} - \mathcal{A}^*_{03333} - \mathcal{A}^*_{02222})r^2 f''_\alpha$$
$$+ (2r\mathcal{A}^{*\prime}_{03223} + 2r\mathcal{A}^{*\prime}_{02233} - r\mathcal{A}^{*\prime}_{03333} - r\mathcal{A}^{*\prime}_{02222} - \mathcal{A}^*_{03333} - \mathcal{A}^*_{02222} + 2\mathcal{A}^*_{02233} + 2\mathcal{A}^*_{03223})r f'_\alpha$$
$$+ (r^2\mathcal{A}^{*\prime\prime}_{03223} + r^2 p'' + r\mathcal{A}^{*\prime}_{03223} + r\mathcal{A}^{*\prime}_{01122} - r\mathcal{A}^{*\prime}_{01133} - r\mathcal{A}^{*\prime}_{02222} + r\mathcal{A}^{*\prime}_{02233}$$
$$+ \mathcal{A}^*_{01111} + \mathcal{A}^*_{02222} - 2\mathcal{A}^*_{01122} - 2\mathcal{A}^*_{03223})f_\alpha] + \alpha^4 r^4 \mathcal{A}^*_{02323}f_\alpha + \alpha^2 r^3(\mathbb{A}^*_{022|3} + \mathbb{A}^*_{023|2} - \mathbb{A}^*_{033|3})g'_\alpha$$
$$+ \alpha^2[(\mathbb{A}^*_{011|3} - \mathbb{A}^*_{022|3})r^2 + (\mathbb{A}^{*\prime}_{022|3} - \mathbb{A}^{*\prime}_{033|3})r^3]g_\alpha = 0, \tag{146}$$

and the corresponding boundary conditions (138), (142) and (143) as

$$\mathbb{A}^*_{023|2}\alpha^2 r f_\alpha + \mathsf{A}^*_{022}g'_\alpha = 0 \quad \text{on} \quad r = a, b, \tag{147}$$

$$r^2 f''_\alpha + r f'_\alpha + (\alpha^2 r^2 - 1)f_\alpha = 0 \quad \text{on} \quad r = a, b \tag{148}$$

and

$$\mathcal{A}^*_{03232}r^3 f'''_\alpha + (r\mathcal{A}^{*\prime}_{03232} + 2\mathcal{A}^*_{03232})r^2 f''_\alpha + (r\mathcal{A}^{*\prime}_{03232} - \mathcal{A}^*_{03232})r f'_\alpha$$
$$- (r\mathcal{A}^{*\prime}_{03232} - \mathcal{A}^*_{03232})f_\alpha - \alpha^2 r^2[(\mathcal{A}^*_{03333} + \mathcal{A}^*_{02222} - 2\mathcal{A}^*_{02233} - 2\mathcal{A}^*_{03223} + \mathcal{A}^*_{03232})r f'_\alpha$$
$$- (r\mathcal{A}^{*\prime}_{03232} + \mathcal{A}^*_{01122} - \mathcal{A}^*_{02222} + \mathcal{A}^*_{02233} - \mathcal{A}^*_{01133} + \mathcal{A}^*_{03223} + \mathcal{A}^*_{03131} - \mathcal{A}^*_{01313})f_\alpha]$$
$$- \alpha^2 r^2(\mathbb{A}^*_{033|3} - \mathbb{A}^*_{022|3})g_\alpha = 0 \quad \text{on} \quad r = a, b. \tag{149}$$

4.2.1. Numerical solution. In order to proceed further we use the incremental boundary condition

$$u = 0 \quad \text{on} \quad z = 0, l, \tag{150}$$

so that radial displacements at the ends of the cylinder are not allowed (and the increment $\dot{T}_{022}$ in the axial load also vanishes). Therefore, from the previous relation, $(120)_1$ and $(144)_1$, we obtain

$$\alpha = \frac{\pi n}{l} = \frac{\pi n}{\lambda_z L}, \tag{151}$$

where $n = 1, 2, 3, \ldots$ is the axisymmetric mode number. We see from (151) that α may be changed either by mode number n or the length of the cylinder L. We fix $n = 1$ and we perform our analysis for different lengths of the cylinder, recognizing that an increase in the mode number n is captured by a decrease in the value of L.

We again use the dimensionless variable $\hat{r} = r/A$, and as in the case considered for prismatic bifurcations we use the nondimensional quantity $\hat{\sigma}_{fa}$ defined in (111) as an electric parameter that is a measure of the charge on the inner boundary of a tube. We also introduce new nondimensional variables

$$\hat{\alpha} = \alpha A, \quad \hat{f}_\alpha(\hat{r}) = \frac{f_\alpha(r)}{A^2}, \quad \hat{g}_\alpha(\hat{r}) = \frac{g_\alpha(r)}{\sigma_{fa}A^2}, \quad \hat{p}(\hat{r}) = \frac{p(r)}{\mu}, \tag{152}$$

and analogously to (108) we define the variables

$$\hat{y}_1(\hat{r}) = \hat{f}_\alpha(\hat{r}), \quad \hat{y}_2(\hat{r}) = \hat{f}'_\alpha(\hat{r}), \quad \hat{y}_3(\hat{r}) = \hat{f}''_\alpha(\hat{r}),$$
$$\hat{y}_4(\hat{r}) = \hat{f}'''_\alpha(\hat{r}), \quad \hat{y}_5(\hat{r}) = \hat{g}_\alpha(\hat{r}), \quad \hat{y}_6(\hat{r}) = \hat{g}'_\alpha(\hat{r}). \tag{153}$$

Then we again have a first-order system of equations written in the form $\hat{y}'_i = M_{ij}\hat{y}_j$, with the coefficients M_{ij} obtained from

$$\hat{y}'_1 = \hat{y}_2, \quad \hat{y}'_2 = \hat{y}_3, \quad \hat{y}'_3 = \hat{y}_4, \quad \hat{y}'_5 = \hat{y}_6,$$

$$\hat{r}^4 \hat{\mathcal{A}}^*_{03232}\hat{y}'_4 = -[3(\hat{r}\hat{\mathcal{A}}^{*\prime}_{03232} - \hat{\mathcal{A}}^*_{03232}) - \hat{r}^2\hat{\mathcal{A}}^{*\prime\prime}_{03232} + \hat{\alpha}^2\hat{r}^2(\hat{r}^2\hat{\mathcal{A}}^{*\prime\prime}_{03223} + \hat{r}^2\hat{p}'' + \hat{r}\hat{\mathcal{A}}^{*\prime}_{03223} + \hat{r}\hat{\mathcal{A}}^{*\prime}_{01122}$$
$$- \hat{r}\hat{\mathcal{A}}^{*\prime}_{01133} - \hat{r}\hat{\mathcal{A}}^{*\prime}_{02222} + \hat{r}\hat{\mathcal{A}}^{*\prime}_{02233} + \hat{\mathcal{A}}^*_{01111} + \hat{\mathcal{A}}^*_{02222} - 2\hat{\mathcal{A}}^*_{01122} - 2\hat{\mathcal{A}}^*_{03223}) + \hat{\alpha}^4\hat{r}^4\hat{\mathcal{A}}^*_{02323}]\hat{y}_1$$
$$- [\hat{r}^3\hat{\mathcal{A}}^{*\prime\prime}_{03232} - 3\hat{r}^2\hat{\mathcal{A}}^{*\prime}_{03232} + 3\hat{r}\hat{\mathcal{A}}^*_{03232} + \hat{\alpha}^2\hat{r}^3(2\hat{r}\hat{\mathcal{A}}^{*\prime}_{03223} + 2\hat{r}\hat{\mathcal{A}}^{*\prime}_{02233} - \hat{r}\hat{\mathcal{A}}^{*\prime}_{03333} - \hat{r}\hat{\mathcal{A}}^{*\prime}_{02222}$$
$$- \hat{\mathcal{A}}^*_{03333} - \hat{\mathcal{A}}^*_{02222} + 2\hat{\mathcal{A}}^*_{02233} + 2\hat{\mathcal{A}}^*_{03223})]\hat{y}_2$$
$$- [\hat{r}^3(3\hat{\mathcal{A}}^{*\prime}_{03232} + \hat{r}\hat{\mathcal{A}}^{*\prime\prime}_{03232}) - 3\hat{r}^2\mathcal{A}^*_{03232} + \hat{\alpha}^2\hat{r}^4(2\hat{\mathcal{A}}^*_{02233} + 2\hat{\mathcal{A}}^*_{03223} - \hat{\mathcal{A}}^*_{03333} - \hat{\mathcal{A}}^*_{02222})]\hat{y}_3$$
$$- (2\hat{r}^4\hat{\mathcal{A}}^{*\prime}_{03232} + 2\hat{r}^3\hat{\mathcal{A}}^*_{03232})\hat{y}_4 - \hat{\alpha}^2[(\hat{\mathbb{A}}^*_{011|3} - \hat{\mathbb{A}}^*_{022|3})\hat{r}^2 + (\hat{\mathbb{A}}^{*\prime}_{022|3} - \hat{\mathbb{A}}^{*\prime}_{033|3})\hat{r}^3]\hat{\sigma}^2_{fa}\hat{y}_5$$
$$- \hat{\alpha}^2\hat{r}^3(\hat{\mathbb{A}}^*_{022|3} + \hat{\mathbb{A}}^*_{023|2} - \hat{\mathbb{A}}^*_{033|3})\hat{\sigma}^2_{fa}\hat{y}_6,$$

$$\hat{\mathbb{A}}^*_{022}\hat{r}\hat{y}'_6 = \hat{\alpha}^2(\hat{\mathbb{A}}^*_{011|3}\hat{r} - \hat{\mathbb{A}}^*_{022|3}\hat{r} - \hat{\mathbb{A}}^{*\prime}_{023|2}\hat{r}^2)\hat{y}_1 + \hat{\alpha}^2\hat{r}^2(\hat{\mathbb{A}}^*_{033|3} - \hat{\mathbb{A}}^*_{022|3} - \hat{\mathbb{A}}^*_{023|2})\hat{y}_2$$
$$+ \hat{\alpha}^2\hat{r}\hat{\mathbb{A}}^*_{033}\hat{y}_5 - (\hat{\mathbb{A}}^{*\prime}_{022}\hat{r} - \hat{\mathbb{A}}^*_{022})\hat{y}_6, \tag{154}$$

and boundary conditions

$$\hat{\mathbb{A}}^*_{023|2}\hat{\alpha}^2\hat{r}\hat{y}_1 + \hat{\mathbb{A}}^*_{022}\hat{y}_6 = 0, \quad \text{on} \quad r = \hat{a}, \hat{b}, \tag{155}$$

$$\hat{r}^2\hat{y}_3 + \hat{r}\hat{y}_2 + (\hat{\alpha}^2\hat{r}^2 - 1)\hat{y}_1 = 0, \quad \text{on} \quad r = \hat{a}, \hat{b}, \tag{156}$$

$$\hat{\mathcal{A}}^*_{03232}\hat{r}^3\hat{y}_4 + (\hat{r}\hat{\mathcal{A}}^{*\prime}_{03232} + 2\hat{\mathcal{A}}^*_{03232})\hat{r}^2\hat{y}_3 + (\hat{r}\hat{\mathcal{A}}^{*\prime}_{03232} - \hat{\mathcal{A}}^*_{03232})\hat{r}\hat{y}_2 - (\hat{r}\hat{\mathcal{A}}^{*\prime}_{03232} - \hat{\mathcal{A}}^*_{03232})\hat{y}_1$$
$$- \hat{\alpha}^2\hat{r}^2[(\hat{\mathcal{A}}^*_{03333} + \hat{\mathcal{A}}^*_{02222} - 2\hat{\mathcal{A}}^*_{02233} - 2\hat{\mathcal{A}}^*_{03223} + \hat{\mathcal{A}}^*_{03232})\hat{r}\hat{y}_2 - (\hat{r}\hat{\mathcal{A}}^{*\prime}_{03232} + \hat{\mathcal{A}}^*_{03223} + \hat{\mathcal{A}}^*_{03131} - \hat{\mathcal{A}}^*_{01313}$$
$$+ \hat{\mathcal{A}}^*_{01122} - \hat{\mathcal{A}}^*_{02222} + \hat{\mathcal{A}}^*_{02233} - \hat{\mathcal{A}}^*_{01133})\hat{y}_1] - \hat{\alpha}^2\hat{r}^2(\hat{\mathbb{A}}^*_{033|3} - \hat{\mathbb{A}}^*_{022|3})\hat{\sigma}^2_{fa}\hat{y}_5 = 0, \quad \text{on} \quad r = \hat{a}, \hat{b}. \tag{157}$$

Expressions for the electroelastic moduli in respect of energy function (115) are given in "Appendix B.2," in both dimensional and dimensionless forms.

We define initial values for the system (154) in the form

$$\hat{y}_i(\hat{a}) = \delta_{ik}, \quad i = 1, \dots, 6, \tag{158}$$

where δ_{ik} is the Kronecker delta. Each $k (= 1, \dots, 6)$ in (158) corresponds to the solution $\mathbf{y}^k$ of the system (154), the general solution of which can be written in the form

$$\hat{\mathbf{y}} = \sum_{k=1}^{6} c_k \hat{\mathbf{y}}^k, \tag{159}$$

where c_k are constants.

We require the solution (159) to satisfy boundary conditions (155)–(157), and we are interested in solutions (159) for which at least one constant c_k is nonzero. Substitution of (159) into (155)–(157) leads
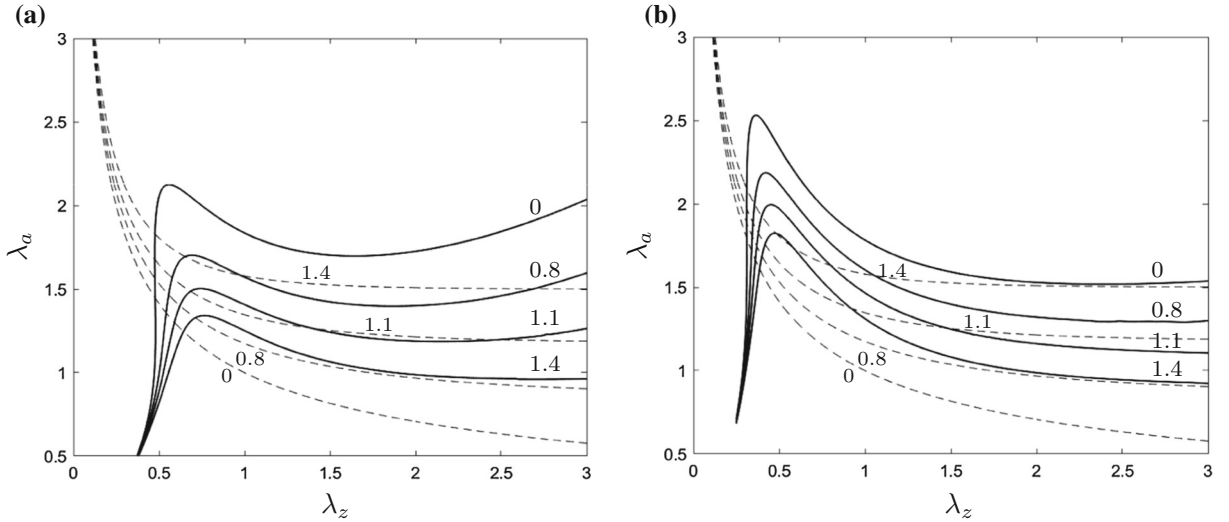
FIG. 1. Plots of the axisymmetric bifurcation curves (continuous curves) and the curves of zero pressure (dashed curves) for the neo-Hookean electroelastic material with $A/B = 0.85$, $\hat{\sigma}_{\mathrm{f}a} = 0, 0.8, 1.1, 1.4$: **a** $L/B = 5$; **b** $L/B = 10$. The values of $\hat{\sigma}_{\mathrm{f}a}$ are shown next to the relevant curves, the smaller fonts being associated with the zero pressure curves

to a 6×6 determinant of coefficients of c_k, vanishing of which represents the bifurcation criterion for this problem. This method was also used for the prismatic bifurcations.

In addition to the neo-Hookean electroelastic model (115) we consider its Mooney–Rivlin generalization in the form

$$\Omega^* = \frac{1}{2}\mu_1(I_1 - 3) - \frac{1}{2}\mu_2(I_2 - 3) + \frac{1}{2}\varepsilon^{-1}I_5, \tag{160}$$

where $\mu_1 \geq 0$ and $\mu_2 \leq 0$ are material constants satisfying $\mu_1 - \mu_2 = \mu$. In what follows we set $\mu_1 = 0.8\mu$ and $\mu_2 = -0.2\mu$. The associated dimensional and nondimensional electroelastic moduli and their required derivatives are listed in "Appendix B.2."

In Fig. 1, for a relatively thin-walled tube with $A/B = 0.85$, curves of λ_a versus λ_z for which the bifurcation criterion is satisfied are plotted (continuous curves) together with the curves corresponding to $P = 0$ (dashed curves) for several values of $\hat{\sigma}_{\mathrm{f}a}$, separately for two values of L/B. These include results for $\hat{\sigma}_{\mathrm{f}a} = 0$, which corresponds to the result obtained in [8] for the neo-Hookean purely elastic material (subject to the factor 2 correction in the values of L/B in figure 3 of [8], which was also noted in [19]). Note that, for each value of $\hat{\sigma}_{\mathrm{f}a}$, P is positive above the associated dashed curve and negative below it. For the smaller values of $\hat{\sigma}_{\mathrm{f}a}$ the bifurcation criterion is met under inflation for any fixed value of λ_z above a minimum value, but as $\hat{\sigma}_{\mathrm{f}a}$ increases the range of λ_z for which bifurcation is possible decreases and eventually disappears for large enough $\hat{\sigma}_{\mathrm{f}a}$. This can be seen, for example, for the value $\hat{\sigma}_{\mathrm{f}a} = 1.4$ in Fig. 1a, for which the dashed curve is entirely above the bifurcation curve, in which case bifurcation under inflation is not possible. Bifurcation is possible for negative P, which amounts to bifurcation under external pressure. In this and the subsequent figures the uppermost curve corresponds to the purely elastic case (or, equivalently, to the case when there is no potential between the electrodes and thus no electric field). As the electric field is applied and increased, the corresponding curves drop to lower values of λ_a.

For the two values of L/B used in Fig. 1, the corresponding results are shown for a thicker-walled tube with $A/B = 0.5$ in Fig. 2, and with the same values of $\hat{\sigma}_{\mathrm{f}a}$. This shows that the qualitative features are the same as in Fig. 1. The quantitative differences are that, for a given value of λ_z, larger values of the azimuthal stretch are required to achieve bifurcation under inflation corresponding to higher pressure,

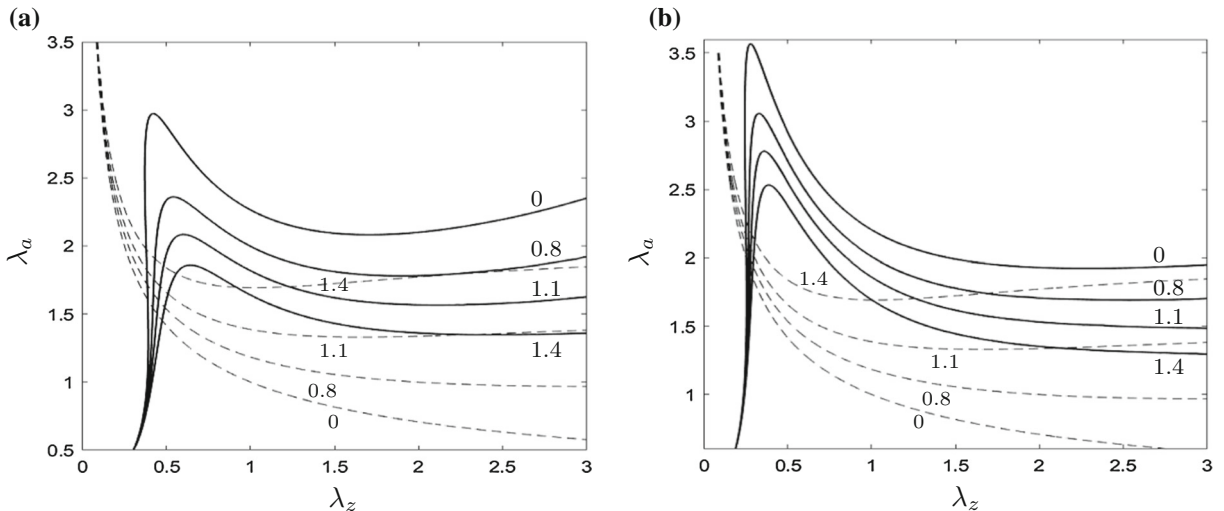**(a)**   **(b)**

FIG. 2. Plots of the axisymmetric bifurcation curves (continuous curves) and the curves of zero pressure (dashed curves) for the neo-Hookean electroelastic material with $A/B = 0.5$, $\hat{\sigma}_{fa} = 0, 0.8, 1.1, 1.4$: **a** $L/B = 5$; **b** $L/B = 10$. The values of $\hat{\sigma}_{fa}$ are shown next to the relevant curves, the smaller fonts being associated with the zero pressure curves

and that larger values of $\hat{\sigma}_{fa}$ are required to completely phase out the possibility of bifurcation under inflation. Results for shorter tubes with $L/B = 2.5$ and the same values of A/B as in Figs. 1 and 2 are illustrated in Fig. 3. Again, the main qualitative features are unchanged, but as λ_z increases, larger values of λ_a are required for bifurcation with correspondingly larger pressures, and there is no phasing out of bifurcation under inflation for the considered range of values of $\hat{\sigma}_{fa}$. The nonmonotonicity of the λ_a versus λ_z curves after the maxima evident in Figs. 1a and 2a is more pronounced in Fig. 3b. However, this does not correspond to any form of snap-through instability since the values of λ_z (when there are more than one) associated with a given λ_a correspond here to significantly different values of P, whereas snap through between different values of λ_z occurs at fixed P.

Figure 4 shows representative plots for the Mooney–Rivlin material (160) for $A/B = 0.85$ analogous to those in Fig. 1. For the purely elastic case considered in [8], the values of μ_1 and μ_2 used were not specified so a direct comparison of our results with those in [8] is not possible. However, our results are qualitatively very similar to those reported in Figure 3 of [8]. For this material there is no phasing out of the bifurcation under internal pressure unless the electric field becomes very large (not shown), but as for the neo-Hookean material increasing electric field strength reduces the values of λ_a for which bifurcation can occur for those fixed λ_z for which bifurcation is possible under inflation. Moreover, the range of values of λ_z for which bifurcation is possible is more limited than for the neo-Hookean model.

The results described herein are for fixed values of $\hat{\sigma}_{fa}$. Equation (117) gives the corresponding values of the potential difference and can be used to translate the results to those for a fixed potential difference.

In considering the propagation of axisymmetric electroelastic waves in a tube in the absence of pressure Shmuel and deBotton [11] obtained results for vanishing wave speed, which corresponds to bifurcation. Such results can be read off as special cases of the results here corresponding to points in the above figures where the (dashed) zero pressure curves intersect with the corresponding bifurcation curves. Note, however, that in [11] an incremental electric field exterior to the body needed to be accounted for.
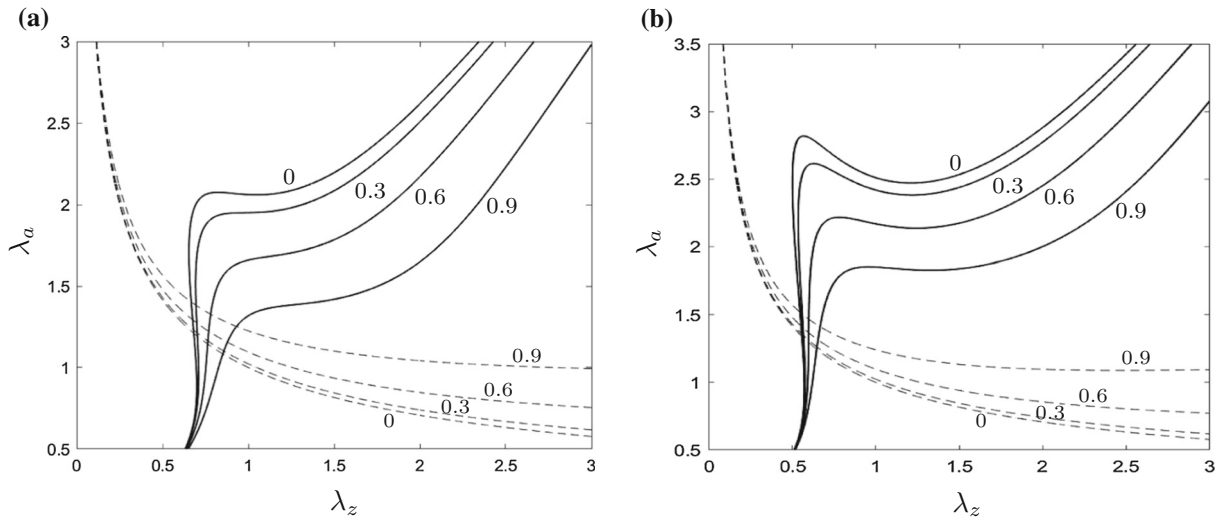
(a) **(b)**

FIG. 3. Plots of the axisymmetric bifurcation curves (continuous curves) and the curves of zero pressure (dashed curves) for the neo-Hookean electroelastic material with $L/B = 2.5$, $\hat{\sigma}_{\mathrm{f}a} = 0, 0.3, 0.6, 0.9$: **a** $A/B = 0.85$; **b** $A/B = 0.5$. The values of $\hat{\sigma}_{\mathrm{f}a}$ are shown next to the relevant curves, the smaller fonts being associated with the zero pressure curves
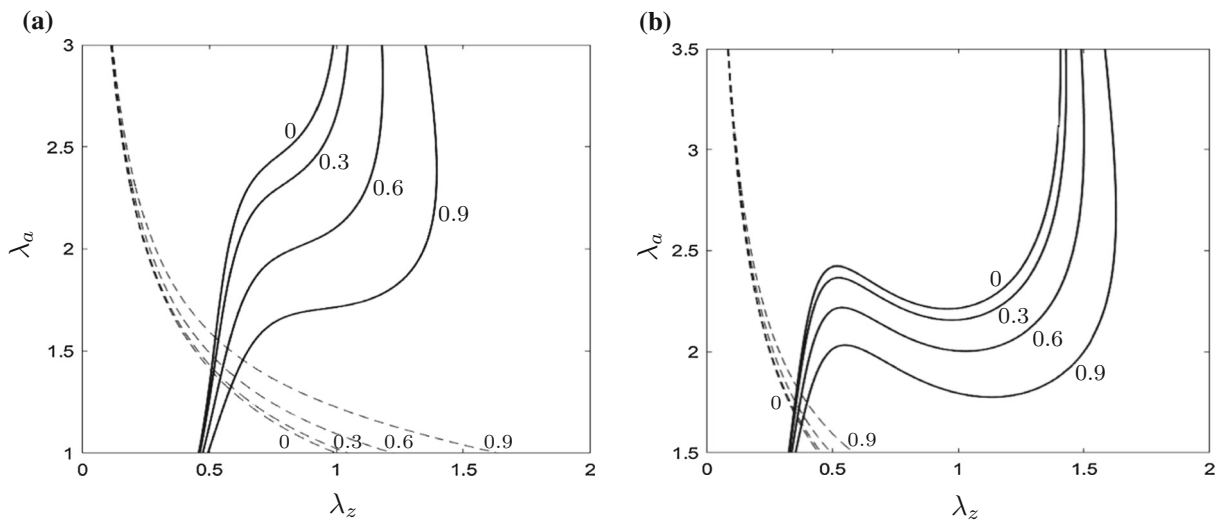
(a) **(b)**

FIG. 4. Plots of the axisymmetric bifurcation curves (continuous curves) and the curves of zero pressure (dashed curves) for the Mooney–Rivlin electroelastic material with $A/B = 0.85$, $\hat{\sigma}_{\mathrm{f}a} = 0, 0.3, 0.6, 0.9$: **a** $L/B = 5$; **b** $L/B = 10$. The values of $\hat{\sigma}_{\mathrm{f}a}$ are shown next to the relevant curves, the smaller fonts being associated with the zero pressure curves (to avoid congestion the smaller labels 0.3 and 0.6 are omitted in **b**)

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Appendix A. Electroelastic moduli

A.1. Derivatives of the invariants I_m, $m \in \{1, 2, 4, 5, 6\}$, with respect to $\mathbf{F}$ and $\mathbf{D_L}$

In component form the nonzero first derivatives have the following expressions

$$\frac{\partial I_1}{\partial F_{i\alpha}} = 2F_{i\alpha}, \quad \frac{\partial I_2}{\partial F_{i\alpha}} = 2(C_{\gamma\gamma}F_{i\alpha} - C_{\alpha\gamma}F_{i\gamma}), \quad \frac{\partial I_5}{\partial F_{i\alpha}} = 2D_{\mathrm{L}\alpha}(F_{i\gamma}D_{\mathrm{L}\gamma}),$$

$$\frac{\partial I_6}{\partial F_{i\alpha}} = 2(C_{\alpha\beta}D_{\mathrm{L}\beta}F_{i\gamma}D_{\mathrm{L}\gamma} + D_{\mathrm{L}\alpha}F_{i\gamma}C_{\gamma\beta}D_{\mathrm{L}\beta}),$$

$$\frac{\partial I_4}{\partial D_{\mathrm{L}\alpha}} = 2D_{\mathrm{L}\alpha}, \quad \frac{\partial I_5}{\partial D_{\mathrm{L}\alpha}} = 2C_{\alpha\beta}D_{\mathrm{L}\beta}, \quad \frac{\partial I_6}{\partial D_{\mathrm{L}\alpha}} = 2C_{\alpha\beta}^2 D_{\mathrm{L}\beta}.$$

The nonzero second derivatives with respect to $\mathbf{F}$ are

$$\frac{\partial^2 I_1}{\partial F_{i\alpha}\partial F_{j\beta}} = 2\delta_{ij}\delta_{\alpha\beta}, \quad \frac{\partial^2 I_2}{\partial F_{i\alpha}\partial F_{j\beta}} = 2(2F_{i\alpha}F_{j\beta} - F_{i\beta}F_{j\alpha} + C_{\gamma\gamma}\delta_{ij}\delta_{\alpha\beta} - b_{ij}\delta_{\alpha\beta} - C_{\alpha\beta}\delta_{ij}),$$

$$\frac{\partial^2 I_5}{\partial F_{i\alpha}\partial F_{j\beta}} = 2\delta_{ij}D_{\mathrm{L}\alpha}D_{\mathrm{L}\beta},$$

$$\frac{\partial^2 I_6}{\partial F_{i\alpha}\partial F_{j\beta}} = 2[\delta_{ij}(C_{\alpha\gamma}D_{\mathrm{L}\gamma}D_{\mathrm{L}\beta} + C_{\beta\gamma}D_{\mathrm{L}\gamma}D_{\mathrm{L}\alpha}) + \delta_{\alpha\beta}F_{i\gamma}D_{\mathrm{L}\gamma}F_{j\delta}D_{\mathrm{L}\delta}$$
$$+ F_{i\gamma}D_{\mathrm{L}\gamma}F_{j\alpha}D_{\mathrm{L}\beta} + F_{j\gamma}D_{\mathrm{L}\gamma}F_{i\beta}D_{\mathrm{L}\alpha} + b_{ij}D_{\mathrm{L}\alpha}D_{\mathrm{L}\beta}].$$

The second derivatives of I_4, I_5, I_6 with respect to $\mathbf{D_L}$ are

$$\frac{\partial^2 I_4}{\partial D_{\mathrm{L}\alpha}\partial D_{\mathrm{L}\beta}} = 2\delta_{\alpha\beta}, \quad \frac{\partial^2 I_5}{\partial D_{\mathrm{L}\alpha}\partial D_{\mathrm{L}\beta}} = 2C_{\alpha\beta}, \quad \frac{\partial^2 I_6}{\partial D_{\mathrm{L}\alpha}\partial D_{\mathrm{L}\beta}} = 2C_{\alpha\beta}^2,$$

where $C_{\alpha\beta}^2$ is defined as $(\mathbf{C}^2)_{\alpha\beta}$.

The mixed derivatives of I_1, I_2 and I_4 with respect to $\mathbf{F}$ and $\mathbf{D_L}$ are equal to zero. For I_5, I_6 they are

$$\frac{\partial^2 I_5}{\partial F_{i\alpha}\partial D_{\mathrm{L}\beta}} = 2\delta_{\alpha\beta}F_{i\gamma}D_{\mathrm{L}\gamma} + 2D_{\mathrm{L}\alpha}F_{i\beta},$$

$$\frac{\partial^2 I_6}{\partial F_{i\alpha}\partial D_{\mathrm{L}\beta}} = 2F_{i\beta}C_{\alpha\gamma}D_{\mathrm{L}\gamma} + 2F_{i\gamma}D_{\mathrm{L}\gamma}C_{\alpha\beta} + 2F_{i\gamma}C_{\gamma\beta}D_{\mathrm{L}\alpha} + 2\delta_{\alpha\beta}F_{i\gamma}C_{\gamma\delta}D_{\mathrm{L}\delta}.$$

A.2. Electroelastic moduli

We introduce the notation $\bar{\mathbf{b}} = I_1\mathbf{b} - \mathbf{b}^2$, $\mathbf{D}^{(1)} = \mathbf{b}\mathbf{D}$ and $\mathbf{D}^{(-1)} = \mathbf{b}^{-1}\mathbf{D}$ and obtain the following expressions for the components of the electroelastic moduli tensors:

$$\mathcal{A}_{0piqj}^* = 4\{\Omega_{11}^* b_{ip}b_{jq} + \Omega_{12}^*(b_{ip}\bar{b}_{jq} + b_{jq}\bar{b}_{ip}) + \Omega_{22}^*\bar{b}_{ip}\bar{b}_{jq} + \Omega_{55}^* D_i D_j D_p D_q$$
$$+ \Omega_{15}^*(b_{ip}D_j D_q + b_{jq}D_i D_p) + \Omega_{25}^*(D_i D_p\bar{b}_{jq} + D_j D_q\bar{b}_{ip})$$

$$+\Omega_{66}^*(D_i^{(1)}D_p + D_p^{(1)}D_i)(D_j^{(1)}D_q + D_q^{(1)}D_j)$$

$$+\Omega_{16}^*[b_{ip}(D_j^{(1)}D_q + D_q^{(1)}D_j) + b_{jq}(D_i^{(1)}D_p + D_p^{(1)}D_i)]$$

$$+\Omega_{26}^*[\bar{b}_{ip}(D_j^{(1)}D_q + D_q^{(1)}D_j) + \bar{b}_{jq}(D_i^{(1)}D_p + D_p^{(1)}D_i)]$$

$$+\Omega_{56}^*(D_p^{(1)}D_iD_qD_j + D_i^{(1)}D_pD_qD_j + D_q^{(1)}D_pD_iD_j + D_j^{(1)}D_pD_iD_q)\}$$

$$+2\{\Omega_1^*\delta_{ij}b_{pq} + \Omega_2^*[2b_{ip}b_{jq} - b_{iq}b_{jp} + \delta_{ij}\bar{b}_{pq} - b_{ij}b_{pq}] + \Omega_5^*\delta_{ij}D_pD_q$$

$$+\Omega_6^*[\delta_{ij}(D_p^{(1)}D_q + D_q^{(1)}D_p) + b_{pq}D_iD_j + b_{jp}D_iD_q + b_{iq}D_jD_p + b_{ij}D_pD_q]\},$$

$$\mathbb{A}_{0pi|q}^* = 4[\Omega_{14}^*b_{ip}D_q^{(-1)} + \Omega_{24}^*\bar{b}_{ip}D_q^{(-1)} + \Omega_{45}^*D_iD_pD_q^{(-1)} + \Omega_{46}^*(D_i^{(1)}D_p + D_p^{(1)}D_i)D_q^{(-1)}$$

$$+\Omega_{15}^*b_{ip}D_q + \Omega_{25}^*\bar{b}_{ip}D_q + \Omega_{55}^*D_iD_pD_q + \Omega_{56}^*(D_p^{(1)}D_iD_q + D_i^{(1)}D_pD_q + D_q^{(1)}D_iD_p)$$

$$+\Omega_{16}^*b_{ip}D_q^{(1)} + \Omega_{26}^*\bar{b}_{ip}D_q^{(1)} + +\Omega_{66}^*(D_i^{(1)}D_p + D_p^{(1)}D_i)D_q^{(1)}]$$

$$+2[\Omega_5^*(\delta_{pq}D_i + \delta_{iq}D_p) + \Omega_6^*(\delta_{iq}D_p^{(1)} + \delta_{pq}D_i^{(1)} + b_{pq}D_i + b_{iq}D_p)],$$

$$\mathsf{A}_{0ij}^* = 4[\Omega_{44}^*D_i^{(-1)}D_j^{(-1)} + \Omega_{55}^*D_iD_j + \Omega_{66}^*D_i^{(1)}D_j^{(1)} + \Omega_{45}^*(D_i^{(-1)}D_j + D_j^{(-1)}D_i)$$

$$+\Omega_{46}^*(D_i^{(-1)}D_j^{(1)} + D_j^{(-1)}D_i^{(1)}) + \Omega_{56}^*(D_i^{(1)}D_j + D_j^{(1)}D_i)] + 2(\Omega_4^*b_{ij}^{(-1)} + \Omega_5^*\delta_{ij} + \Omega_6^*b_{ij}).$$

With respect to the principal axes of **b** the previous expressions in terms of principal stretches λ_1, λ_2, λ_3 and components (D_1, D_2, D_3) of the electric displacement vector **D** are written as follows, noting that the indices are such that $i \neq j \neq k \neq i$:

$$\mathcal{A}_{0iiii}^* = 2\lambda_i^2[\Omega_1^* + (\lambda_j^2 + \lambda_k^2)\Omega_2^* + \lambda_j^2\lambda_k^2D_i^2(\Omega_5^* + 6\lambda_i^2\Omega_6^*)]$$

$$+4\lambda_i^4\{\Omega_{11}^* + 2(\lambda_j^2 + \lambda_k^2)\Omega_{12}^* + (\lambda_j^2 + \lambda_k^2)^2\Omega_{22}^*$$

$$+2\lambda_j^2\lambda_k^2D_i^2[\Omega_{15}^* + 2\lambda_i^2\Omega_{16}^* + (\lambda_j^2 + \lambda_k^2)\Omega_{25}^* + 2\lambda_i^2(\lambda_j^2 + \lambda_k^2)\Omega_{26}^*]$$

$$+\lambda_j^4\lambda_k^4D_i^4(\Omega_{55}^* + 4\lambda_i^2\Omega_{56}^* + 4\lambda_i^4\Omega_{66}^*)\},$$

$$\mathcal{A}_{0iiij}^* = 4D_iD_j\lambda_i^2\{\Omega_6^* + \Omega_{15}^* + (\lambda_j^2 + \lambda_k^2)\Omega_{25}^* + (\lambda_i^2 + \lambda_j^2)[\Omega_{16}^* + (\lambda_j^2 + \lambda_k^2)\Omega_{26}^*]$$

$$+\lambda_j^2\lambda_k^2D_i^2[\Omega_{55}^* + (3\lambda_i^2 + \lambda_j^2)\Omega_{56}^* + 2\lambda_i^2(\lambda_i^2 + \lambda_j^2)\Omega_{66}^*]\},$$

$$\mathcal{A}_{0iiji}^* = 2D_iD_j\{\Omega_5^* + (\lambda_j^2 + 3\lambda_i^2)\Omega_6^* + 2\lambda_i^2[\Omega_{15}^* + (\lambda_j^2 + \lambda_k^2)\Omega_{25}^*]$$

$$+2\lambda_i^2(\lambda_i^2 + \lambda_j^2)[\Omega_{16}^* + (\lambda_j^2 + \lambda_k^2)\Omega_{26}^*]$$

$$+2I_3D_i^2[\Omega_{55}^* + (3\lambda_i^2 + \lambda_j^2)\Omega_{56}^* + 2\lambda_i^2(\lambda_i^2 + \lambda_j^2)\Omega_{66}^*]\},$$

$$\mathcal{A}_{0iijj}^* = 4\lambda_i^2\lambda_j^2\{\Omega_2^* + \lambda_k^2\Omega_3^* + \Omega_{11}^* + (I_1 + \lambda_k^2)\Omega_{12}^* + (I_2 + \lambda_k^4)\Omega_{22}^*$$

$$+\lambda_k^2(\lambda_j^2D_i^2 + \lambda_i^2D_j^2)(\Omega_{15}^* + \lambda_k^2\Omega_{25}^*) + 2(\lambda_i^2D_i^2 + \lambda_j^2D_j^2)\Omega_{26}^*$$

$$+(D_i^2 + D_j^2)(2\Omega_{16}^* + \Omega_{25}^* + 2\lambda_k^2\Omega_{26}^*) + \lambda_k^2D_i^2D_j^2[\Omega_{55}^* + 2(\lambda_i^2 + \lambda_j^2)\Omega_{56}^* + 4\lambda_i^2\lambda_j^2\Omega_{66}^*]\},$$

$$\mathcal{A}_{0ijij}^* = 2\lambda_i^2\{\Omega_1^* + \lambda_k^2\Omega_2^* + D_i^2\lambda_j^2\lambda_k^2\Omega_5^* + \lambda_j^2\lambda_k^2(2D_i^2\lambda_i^2 + D_i^2\lambda_j^2 + D_j^2\lambda_i^2)\Omega_6^*$$

$$+2D_i^2D_j^2\lambda_j^2\lambda_k^2[\Omega_{55}^* + 2(\lambda_i^2 + \lambda_j^2)\Omega_{56}^* + (\lambda_i^2 + \lambda_j^2)^2\Omega_{66}^*]\},$$

$$\mathcal{A}^*_{0ijji} = 2\lambda_i^2\lambda_j^2\{-\Omega_2^* + \lambda_k^2(\lambda_j^2 D_i^2 + \lambda_i^2 D_j^2)\Omega_6^* \\ + 2D_i^2 D_j^2 \lambda_k^2 [\Omega_{55}^* + 2(\lambda_i^2 + \lambda_j^2)\Omega_{56}^* + (\lambda_i^2 + \lambda_j^2)^2 \Omega_{66}^*]\},$$

$$\mathcal{A}^*_{0iijk} = 4D_j D_k \lambda_i^2 \{\Omega_{15}^* + (\lambda_j^2 + \lambda_k^2)(\Omega_{25}^* + \Omega_{16}^*) + (\lambda_j^2 + \lambda_k^2)^2 \Omega_{26}^* \\ + D_i^2 \lambda_j^2 \lambda_k^2 [\Omega_{55}^* + (I_1 + \lambda_i^2)\Omega_{56}^* + 2\lambda_i^2(\lambda_j^2 + \lambda_k^2)\Omega_{66}^*]\},$$

$$\mathcal{A}^*_{0ijki} = \mathcal{A}_{0ijik} = 2D_j D_k \{\lambda_i^2 \Omega_6^* + 2D_i^2[\Omega_{55}^* + (I_1 + \lambda_i^2)\Omega_{56}^* + (I_2 + \lambda_i^4)\Omega_{66}^*]\},$$

$$\mathcal{A}^*_{0jiki} = 2D_j D_k \{\Omega_5^* + I_1 \Omega_6^* + 2D_i^2[\Omega_{55}^* + (I_1 + \lambda_i^2)\Omega_{56}^* + (I_2 + \lambda_i^4)\Omega_{66}^*]\},$$

$$\mathbb{A}^*_{0ii|i} = 4D_i\{\Omega_5^* + 2\lambda_i^2\Omega_6^* + \Omega_{14}^* + \lambda_i^2\Omega_{15}^* + \lambda_i^4\Omega_{16}^* + (\lambda_j^2 + \lambda_k^2)(\Omega_{24}^* + \lambda_i^2\Omega_{25}^* + \lambda_i^4\Omega_{26}^*) \\ + D_i^2\lambda_j^2\lambda_k^2[\Omega_{45}^* + \lambda_i^2\Omega_{55}^* + \lambda_i^4\Omega_{56}^* + 2\lambda_i^2(\Omega_{46}^* + \lambda_i^2\Omega_{56}^* + \lambda_i^4\Omega_{66}^*)]\},$$

$$\mathbb{A}^*_{0ii|j} = 4D_j\lambda_i^2\lambda_j^{-2}\{\Omega_{14}^* + \lambda_j^2\Omega_{15}^* + \lambda_j^4\Omega_{16}^* + (\lambda_j^2 + \lambda_k^2)(\Omega_{24}^* + \lambda_j^2\Omega_{25}^* + \lambda_j^4\Omega_{26}^*) \\ + D_i^2\lambda_j^2\lambda_k^2[\Omega_{45}^* + \lambda_j^2\Omega_{55}^* + \lambda_j^4\Omega_{56}^* + 2\lambda_i^2(\Omega_{46}^* + \lambda_j^2\Omega_{56}^* + \lambda_j^4\Omega_{66}^*)]\},$$

$$\mathbb{A}^*_{0ij|i} = 2D_j\{\Omega_5^* + (\lambda_i^2 + \lambda_j^2)\Omega_6^* + 2D_i^2\lambda_j^2\lambda_k^2[\Omega_{45}^* + \lambda_i^2\Omega_{55}^* + \lambda_i^4\Omega_{56}^* \\ + (\lambda_i^2 + \lambda_j^2)(\Omega_{46}^* + \lambda_i^2\Omega_{56}^* + \lambda_i^4\Omega_{66}^*)]\},$$

$$\mathbb{A}^*_{0ij|k} = 4D_i D_j D_k \lambda_i^2\lambda_j^2[\Omega_{45}^* + \lambda_k^2\Omega_{55}^* + \lambda_k^4\Omega_{56}^* + (\lambda_i^2 + \lambda_j^2)(\Omega_{46}^* + \lambda_k^2\Omega_{56}^* + \lambda_k^4\Omega_{66}^*)],$$

$$\mathsf{A}^*_{0ii} = 2\lambda_i^{-2}\{\Omega_4^* + \lambda_i^2\Omega_5^* + \lambda_i^4\Omega_6^* + 2D_i^2\lambda_j^2\lambda_k^2[\Omega_{44}^* + \lambda_i^2\Omega_{45}^* + \lambda_i^4\Omega_{46}^* \\ + \lambda_i^2(\Omega_{45}^* + \lambda_i^2\Omega_{55}^* + \lambda_i^4\Omega_{56}^*) + \lambda_i^4(\Omega_{46}^* + \lambda_i^2\Omega_{56}^* + \lambda_i^4\Omega_{66}^*)]\},$$

$$\mathsf{A}^*_{0ij} = 4D_i D_j \lambda_k^2[\Omega_{44}^* + \lambda_i^2\Omega_{45}^* + \lambda_i^4\Omega_{46}^* + \lambda_j^2(\Omega_{45}^* + \lambda_i^2\Omega_{55}^* + \lambda_i^4\Omega_{56}^*) \\ + \lambda_j^4(\Omega_{46}^* + \lambda_i^2\Omega_{56}^* + \lambda_i^4\Omega_{66}^*)].$$

Appendix B. Electroelastic moduli for the specific energy functions

B.1. Prismatic bifurcations

For prismatic bifurcations the electroelastic moduli specialized for neo-Hookean electroelastic material (115) are

$$\mathcal{A}^*_{03131} = \mathcal{A}^*_{03333} = \mu\lambda_3^2 + D_3^2/\varepsilon, \quad \mathcal{A}^*_{01313} = \mathcal{A}^*_{01111} = \mu\lambda_1^2, \quad \mathcal{A}^*_{01331} = \mathcal{A}^*_{01133} = 0,$$
$$2\mathbb{A}^*_{013|1} = \mathbb{A}^*_{033|3} = 2D_3/\varepsilon, \quad \mathbb{A}^*_{011|3} = 0, \quad \mathsf{A}^*_{011} = \mathsf{A}^*_{033} = 1/\varepsilon.$$

In nondimensional form the nonzero moduli can be expressed as functions of $\hat{r}$ as follows:

$$\hat{\mathcal{A}}^*_{03131} = \hat{\mathcal{A}}^*_{03333} = [\lambda_z^{-1}(\hat{r}^2 - \hat{a}^2) + \lambda_z^{-2} + \hat{\sigma}_{fa}^2\hat{a}^2]/\hat{r}^2, \quad \hat{\mathcal{A}}^*_{01313} = \hat{\mathcal{A}}^*_{01111} = \hat{r}^2/[\lambda_z(\hat{r}^2 - \hat{a}^2) + 1],$$
$$2\hat{\mathbb{A}}^*_{013|1} = \hat{\mathbb{A}}^*_{033|3} = 2\hat{a}/\hat{r}, \quad \hat{\mathsf{A}}^*_{011} = \hat{\mathsf{A}}^*_{033} = 1.$$

B.2. Axisymmetric bifurcations

B.2.1. Moduli for the neo-Hookean model. The electroelastic moduli take the following values for energy function (115):

$$\mathcal{A}^*_{03131} = \mathcal{A}^*_{03232} = \mathcal{A}^*_{03333} = \mu\lambda_3^2 + D_3^2/\varepsilon, \quad \mathcal{A}^*_{02222} = \mathcal{A}^*_{02323} = \mu\lambda_2^2,$$

$$\mathcal{A}^*_{01111} = \mathcal{A}_{01313} = \mu\lambda_1^2, \quad \mathcal{A}^*_{01122} = \mathcal{A}^*_{01133} = \mathcal{A}^*_{03223} = \mathcal{A}^*_{02233} = \mathcal{A}^*_{01331} = 0,$$

$$\mathbb{A}^*_{033|3} = 2\mathbb{A}^*_{023|2} = 2D_3/\varepsilon, \quad \mathbb{A}^*_{022|3} = \mathbb{A}^*_{011|3} = 0, \quad \mathsf{A}_{022} = \mathsf{A}^*_{033} = 1/\varepsilon,$$

and in nondimensional form the nonzero ones are

$$\hat{\mathcal{A}}^*_{03131}(\hat{r}) = \hat{\mathcal{A}}^*_{03232}(\hat{r}) = \hat{\mathcal{A}}^*_{03333}(\hat{r}) = [\lambda_z^{-1}(\hat{r}^2 - \hat{a}^2) + \lambda_z^{-2} + \hat{\sigma}_{\mathrm{f}a}^2\hat{a}^2]/\hat{r}^2,$$

$$\hat{\mathcal{A}}^*_{02222}(\hat{r}) = \hat{\mathcal{A}}^*_{02323}(\hat{r}) = \lambda_z^2, \quad \hat{\mathcal{A}}^*_{01111}(\hat{r}) = \hat{\mathcal{A}}^*_{01313}(\hat{r}) = \hat{r}^2/[\lambda_z(\hat{r}^2 - \hat{a}^2) + 1],$$

$$\hat{\mathbb{A}}^*_{033|3}(\hat{r}) = 2\hat{\mathbb{A}}^*_{023|2}(\hat{r}) = 2\hat{a}/\hat{r}, \quad \hat{\mathsf{A}}^*_{022}(\hat{r}) = \hat{\mathsf{A}}^*_{033}(\hat{r}) = 1.$$

B.2.2. Moduli for the Mooney–Rivlin model. For Mooney–Rivlin model (160) the electroelastic moduli specialize to

$$\mathcal{A}^*_{03131} = \mu_1\lambda_3^2 - \mu_2\lambda_1^{-2} + D_3^2/\varepsilon, \quad \mathcal{A}^*_{03232} = \mu_1\lambda_3^2 - \mu_2\lambda_2^{-2} + D_3^2/\varepsilon,$$

$$\mathcal{A}^*_{03333} = \mu_1\lambda_3^2 - \mu_2\lambda_3^2(\lambda_1^2 + \lambda_2^2) + D_3^2/\varepsilon, \quad \mathcal{A}^*_{01313} = \mu_1\lambda_1^2 - \mu_2\lambda_2^2, \quad \mathcal{A}^*_{01122} = -2\mu_2\lambda_1^2\lambda_2^2,$$

$$\mathcal{A}^*_{01133} = -2\mu_2\lambda_1^2\lambda_3^2, \quad \mathcal{A}^*_{03223} = \mu_2\lambda_3^2\lambda_2^2, \quad \mathcal{A}^*_{02233} = -2\mu_2\lambda_2^2\lambda_3^2, \quad \mathcal{A}^*_{01331} = \mu_2\lambda_1^2\lambda_3^2,$$

$$\mathcal{A}^*_{02222} = \mu_1\lambda_2^2 - \mu_2\lambda_2^2(\lambda_1^2 + \lambda_3^2), \quad \mathcal{A}^*_{02323} = \mu_1\lambda_2^2 - \mu_2\lambda_1^2\lambda_2^2, \quad \mathcal{A}^*_{01111} = \mu_1\lambda_1^2 - \mu_2\lambda_1^2(\lambda_2^2 + \lambda_3^2),$$

$$\mathbb{A}^*_{033|3} = 2D_3/\varepsilon, \quad \mathbb{A}^*_{023|2} = D_3/\varepsilon, \quad \mathbb{A}^*_{022|3} = 0, \quad \mathbb{A}^*_{011|3} = 0, \quad \mathsf{A}^*_{022} = \mathsf{A}^*_{033} = 1/\varepsilon.$$

With $\mu_1 = 0.8\mu$ and $\mu_2 = -0.2\mu$, the nonzero terms have nondimensional forms in terms of $\hat{r}$ as

$$\hat{\mathcal{A}}^*_{03131}(\hat{r}) = \{0.8[\lambda_z^{-1}(\hat{r}^2 - \hat{a}^2) + \lambda_z^{-2}] + 0.2[\lambda_z(\hat{r}^2 - \hat{a}^2) + 1] + \hat{\sigma}_{\mathrm{f}a}^2\hat{a}^2\}/\hat{r}^2,$$

$$\hat{\mathcal{A}}^*_{03232}(\hat{r}) = \{0.8[\lambda_z^{-1}(\hat{r}^2 - \hat{a}^2) + \lambda_z^{-2}] + \hat{\sigma}_{\mathrm{f}a}^2\hat{a}^2\}/\hat{r}^2 + 0.2\lambda_z^{-2},$$

$$\hat{\mathcal{A}}^*_{03333}(\hat{r}) = \{0.8[\lambda_z^{-1}(\hat{r}^2 - \hat{a}^2) + \lambda_z^{-2}] + 0.2[\lambda_z(\hat{r}^2 - \hat{a}^2) + 1] + \hat{\sigma}_{\mathrm{f}a}^2\hat{a}^2\}/\hat{r}^2 + 0.2\lambda_z^{-2},$$

$$\hat{\mathcal{A}}^*_{01313}(\hat{r}) = [0.8\lambda_z^{-2} + 0.2]\hat{r}^2/[\lambda_z^{-1}(\hat{r}^2 - \hat{a}^2) + \lambda_z^{-2}], \quad \hat{\mathcal{A}}^*_{01122}(\hat{r}) = 0.4\hat{r}^2/[\lambda_z^{-1}(\hat{r}^2 - \hat{a}^2) + \lambda_z^{-2}],$$

$$\hat{\mathcal{A}}^*_{01133}(\hat{r}) = 0.4\lambda_z^{-2}, \quad \hat{\mathcal{A}}^*_{03223}(\hat{r}) = -0.2[\lambda_z(\hat{r}^2 - \hat{a}^2) + 1]/\hat{r}^2,$$

$$\hat{\mathcal{A}}^*_{02233}(\hat{r}) = 0.4[\lambda_z(\hat{r}^2 - \hat{a}^2) + 1]/\hat{r}^2, \quad \hat{\mathcal{A}}^*_{01331}(\hat{r}) = -0.2\lambda_z^{-2}$$

$$\hat{\mathcal{A}}^*_{02222}(\hat{r}) = 0.8\lambda_z^2 + 0.2\{\hat{r}^2/[\lambda_z^{-1}(\hat{r}^2 - \hat{a}^2) + \lambda_z^{-2}] + [\lambda_z(\hat{r}^2 - \hat{a}^2) + 1]/\hat{r}^2\},$$

$$\hat{\mathcal{A}}^*_{02323}(\hat{r}) = 0.8\lambda_z^2 + 0.2\hat{r}^2/[\lambda_z^{-1}(\hat{r}^2 - \hat{a}^2) + \lambda_z^{-2}],$$

$$\hat{\mathcal{A}}^*_{01111}(\hat{r}) = 0.8\hat{r}^2/[\lambda_z(\hat{r}^2 - \hat{a}^2) + 1] + 0.2\hat{r}^2/[\lambda_z^{-1}(\hat{r}^2 - \hat{a}^2) + \lambda_z^{-2}] + 0.2\lambda_z^{-2},$$

$$\hat{\mathbb{A}}^*_{033|3}(\hat{r}) = 2\hat{a}/\hat{r}, \quad \hat{\mathbb{A}}^*_{023|2}(\hat{r}) = \hat{a}/\hat{r}, \quad \hat{\mathsf{A}}^*_{022}(\hat{r}) = 1, \quad \hat{\mathsf{A}}^*_{033}(\hat{r}) = 1.$$

The required derivatives with respect to $\hat{r}$ of the electroelastic moduli are

$$\hat{\mathcal{A}}^{*\prime}_{03131}(\hat{r}) = \frac{-1.6 + 1.6\lambda_z\hat{a}^2 - (0.4 + 2\hat{\sigma}_{\mathrm{f}a}^2\hat{a}^2)\lambda_z^2 + 0.4\hat{a}^2\lambda_z^3}{\hat{r}^3\lambda_z^2},$$

$$\hat{\mathcal{A}}^{*\prime\prime}_{03131}(\hat{r}) = \frac{4.8 - 4.8\hat{a}^2\lambda_z + (1.2 + 6\hat{a}^2\hat{\sigma}_{\mathrm{f}a}^2)\lambda_z^2 - 1.2\hat{a}^2\lambda_z^3}{\hat{r}^4\lambda_z^2},$$

$$\hat{\mathcal{A}}^{*\prime}_{03232}(\hat{r}) = \frac{-1.6 + 1.6\hat{a}^2\lambda_z - 2\hat{a}^2\hat{\sigma}_{\mathrm{f}a}^2\lambda_z^2}{\hat{r}^3\lambda_z^2}, \quad \hat{\mathcal{A}}^{*\prime\prime}_{03232}(\hat{r}) = \frac{4.8 + \hat{a}^2\lambda_z(-4.8 + 6\hat{\sigma}_{\mathrm{f}a}^2\lambda_z)}{\hat{r}^4\lambda_z^2},$$

$$\hat{\mathcal{A}}_{01313}^{*\prime}(\hat{r}) = \frac{\hat{r}(1.6 - 1.6\hat{a}^2\lambda_z + 0.4\lambda_z^2 - 0.4\hat{a}^2\lambda_z^3)}{[1 + (-\hat{a}^2 + \hat{r}^2)\lambda_z]^2},$$

$$\hat{\mathcal{A}}_{03223}^{*\prime}(\hat{r}) = \frac{0.4 - 0.4\hat{a}^2\lambda_z}{\hat{r}^3}, \quad \hat{\mathcal{A}}_{03223}^{*\prime\prime}(\hat{r}) = \frac{-1.2 + 1.2\hat{a}^2\lambda_z}{\hat{r}^4}, \quad \hat{\mathcal{A}}_{02233}^{*\prime}(\hat{r}) = \frac{-0.8 + 0.8\hat{a}^2\lambda_z}{\hat{r}^3},$$

$$\hat{\mathcal{A}}_{03333}^{*\prime}(\hat{r}) = \frac{(1.6 + 0.4\lambda_z^2)(\hat{a}^2\lambda_z - 1) - 2\hat{\sigma}_{\mathrm{fa}}^2\lambda_z^2\hat{a}^2}{\hat{r}^3\lambda_z^2},$$

$$\hat{\mathcal{A}}_{01122}^{*\prime}(\hat{r}) = \frac{0.8(1 - \lambda_z\hat{a}^2)\hat{r}\lambda_z^2}{[1 + \lambda_z(\hat{r}^2 - \hat{a}^2)]^2},$$

$$\hat{\mathcal{A}}_{02222}^{*\prime}(\hat{r}) = -0.4(\lambda_z\hat{a}^2 - 1)^2(2\lambda_z\hat{r}^2 - \lambda_z\hat{a}^2 + 1)/\{\hat{r}^3[1 + \lambda_z(\hat{r}^2 - \hat{a}^2)]^2\}$$

$$\hat{\mathbb{A}}_{033|3}^{*\prime}(\hat{r}) = -2\hat{a}/\hat{r}^2, \quad \hat{\mathbb{A}}_{023|2}^{*\prime}(\hat{r}) = -\hat{a}/\hat{r}^2.$$

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Andrey Melnikov
Department of Earth Sciences
Memorial University of Newfoundland
St. John's
Canada

Ray W. Ogden
School of Mathematics and Statistics
University of Glasgow
Glasgow
UK
e-mail: rwo@maths.gla.ac.uk