

# Minimal set of generators of controllability space for singular linear dynamical systems

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*Abstract:* Due to the significant role played by singular systems in the form  $E\dot{x}(t) = Ax(t)$ , on mathematical modeling of science and engineering problems; in the last years recent years its interest in the descriptive analysis of its structural and dynamic properties. However, much less effort has been devoted to studying the exact controllability by measuring the minimum set of controls needed to direct the entire system  $E\dot{x}(t) = Ax(t)$  to any desired state. In this work, we focus the study on obtaining the set of all matrices  $B$  with a minimal number of columns, by making the singular system  $E\dot{x}(t) = Ax(t) + Bu(t)$  controllable.

*Key-Words:* Controllability, exact controllability, eigenvalues, eigenvectors, singular linear systems.

## 1 Introduction

In these recent years, the study of the control of complex networks with linear dynamics has gained importance in both science and engineering; because of this kind of systems appear in applications such us electrical networks, simulation of the dynamics of multi-body systems, modelling chemical reactions among others.

Controllability of a dynamical system has being largely studied by several authors and under many different points of view, (see [2], [4], [5], [6], [11], [13], [14], [19], [20] and [22] for example). Between different aspects in which we can study the controllability we have the notion of structural controllability that has been proposed by Lin [15] as a framework for studying the controllability properties of directed complex networks where the dynamics of the system is governed by a linear system. Recent studies over the structural controllability can be found on [16].

Another important aspect of control is the notion of output controllability that describes the ability of an external data to move the output from any initial condition to any final in a finite time. Some results about can be found in [11]. This concept has some interest in codes theory (see [9], for example).

In this article, we analyze the exact controllability concept as a generalization for singular linear dynamical systems of the concept given in [23] for standard linear dynamical systems and in [8] for  $\ell$ -order standard linear systems. This concept is based on the maximum multiplicity to identify the minimum set of

driver nodes required to achieve full control of networks with arbitrary structures and link-weight distributions. The notion of exact controllability has interest in different topics as for example is an adequate notion in hyperbolic problems (see [12], [17]), also, can be used to explore the effect of interconnections' correlation on the controllability of multiplex networks, S. Nie, X. Wang and B.Wang in [18] find that the minimal number of driver nodes decreases with correlation for lower density of interconnections.

We were focusing the study on the obtention of the set of all matrices  $B$  making the system  $E\dot{x}(t) = Ax(t) + Bu(t)$  exact controllable. These sets are obtained from the quasi Weierstraß reduced form and from getting the transformation matrices of the system to its reduced form.

We have included some examples to make the work easier readable.

Finally, we introduce exact controllability for second order singular linear systems, because they are interest in application to power systems and they are also used in conjunction with the analysis and modelling of flexible beams [1].

## 2 Equivalence relation

It is well known that many complex networks have linear dynamics and they have a state space representation for its description:

$$E\dot{x}(t) = Ax(t) + Bu(t) \} \quad (1)$$

where  $E, A \in M_n(\mathbb{C})$  and  $B \in M_{m \times n}(\mathbb{C})$ .

When  $B = 0$  the system is called homogeneous.

For simplicity, from now on we will write the system 1 as the triple of matrices  $(E, A, B)$  or as a pair  $(E, A)$  for the homogeneous case.

Trying to understand the properties of the system they use purely algebraic techniques. The central aspect of this focus is defining an equivalence relation preserving these properties.

The equivalence relation considered corresponds to standard transformation basis changes for the state space and pre multiplication for an invertible matrix.

**Definition 1** *The systems  $E\dot{x}(t) = Ax(t)$  and  $\bar{E}\dot{\bar{x}}(t) = \bar{A}\bar{x}(t)$  are equivalent if and only if, there exist a basis change in the state space  $\bar{x}(t) = Px(t)$  and an invertible matrix  $Q \in Gl(n; \mathbb{C})$  such that*

$$\bar{E}\dot{\bar{x}}(t) = QEP\dot{x}(t) = QAPx(t) = \bar{A}\bar{x}(t)$$

We can characterize equivalent systems, by associating matrix pencils to them in a natural way:

The matrix pencil  $\lambda E + A$  is naturally associated to the pair  $(E, A)$  representing a singular linear system  $E\dot{x}(t) = Ax(t)$

Equivalent pairs are those whose associated matrix pencils are "strictly equivalent". Remember that two pencils  $\lambda E + A$  and  $\lambda \bar{E} + \bar{A}$  are strictly equivalent, if and only if, there exist invertible matrices  $P, A \in Gl(n; \mathbb{C})$  such that

$$\lambda \bar{E} + \bar{A} = Q(\lambda E + A)P = \lambda QEP + QAP.$$

Observe that in the case where the system is standard (i.e.  $E = I$ ), the equivalence relation considered, corresponds to the similarity relation of square matrices.

We will consider the case of systems where the matrix pencil  $\lambda E + A$  is regular, as it is usual, in order to ensure that the system has a unique solution for any sufficiently differentiable input function  $u(t)$ . Under this regularity assumption, there exist invertible matrices  $Q, P \in Gl_n(\mathbb{C})$  such that  $\bar{E} = QEP = \text{diag}(I_r, N)$ ,  $\bar{A} = QAP = \text{diag}(J, I_{n-r})$ , where  $J$  a Jordan matrix and  $N$  a nilpotent matrix and we will say that the pencil is in its canonical reduced form.

So, considering  $x(t) = P\bar{x}(t)$  and premultiplying the system 1 by  $Q$  and calling  $\bar{B} = QB$ , the system can be written as  $\bar{E}\dot{\bar{x}} = \bar{A}\bar{x}(t) + \bar{B}u(t)$ , that is to say:

$$\begin{pmatrix} I_r & 0 \\ 0 & N \end{pmatrix} \begin{pmatrix} \dot{\bar{x}}_1(t) \\ \dot{\bar{x}}_2(t) \end{pmatrix} = \begin{pmatrix} J & 0 \\ 0 & I_{n-r} \end{pmatrix} \begin{pmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \end{pmatrix} + \begin{pmatrix} \bar{B}_1 \\ \bar{B}_2 \end{pmatrix} u(t) \quad (2)$$

In general we say that two systems  $(E, A, B)$  and  $(\bar{E}, \bar{A}, \bar{B})$  are equivalent if and only if, there exist invertible matrices  $P$  and  $Q$  such that  $(\bar{E}, \bar{A}, \bar{B}) = (QEP, QAP, QB)$ . This equivalence relation corresponds with strict equivalence of the pencil  $\begin{pmatrix} sE - A & B \end{pmatrix}$ . So, the collection of invariants of the pencil are the invariants for the system.

In particular, for the systems  $E\dot{x}(t) = Ax(t)$  the generalized eigenvalues of the system are the generalized eigenvalues of the pencil.

**Definition 2**  $\lambda_0$  is a generalized eigenvalue of the system, if and only if  $\text{rank}(\lambda_0 E - A) < n$ .

It is easy to observe that the generalized eigenvalues of  $sE - A$  are the eigenvalues of the matrix  $J$  in the reduced form 2:

$$\begin{aligned} \text{rank}(\lambda_0 E - A) &= \\ \text{rank}(\lambda_0 Q^{-1} \bar{E} P - Q^{-1} \bar{A} P) &= \\ \text{rank} Q^{-1}(\lambda_0 \bar{E} - \bar{A}) P &= \\ \text{rank}(\lambda_0 \bar{E} - \bar{A}) & \end{aligned}$$

and

$$\text{rank} \begin{pmatrix} \lambda_0 I_r - J & \\ & \lambda_0 N - I_{n-r} \end{pmatrix} < n$$

if and only if

$$\text{rank}(\lambda_0 I_r - J) < r.$$

In a more general form we have the following proposition.

**Proposition 3** *Let  $(E, A)$  and  $(\bar{E}, \bar{A})$  be two equivalent systems,  $\lambda_0$  is an eigenvalue of  $(E, A)$  if and only if it is an eigenvalue of  $(\bar{E}, \bar{A}) = (QEP, QAP)$ .*

If  $\lambda_0$  is a generalized eigenvalue of  $(E, A)$ , then there exists a vector  $0 \neq w_0$  such that  $(\lambda_0 E - A)w_0 = 0$ .

**Definition 4** *This vector is called the generalized eigenvector associated to  $\lambda_0$ .*

**Proposition 5** *Let  $(E, A)$  and  $(\bar{E}, \bar{A})$  be two equivalent systems,  $w_0$  is an eigenvector of  $(E, A)$  if and only if  $P^{-1}w_0$  is an eigenvector of  $(\bar{E}, \bar{A}) = (QEP, QAP)$ .*

**Proof:** Let  $(E, A)$  and  $(\bar{E}, \bar{A}) = (QEP, QAP)$  two equivalent systems.

$(\lambda_0 E - A)w_0 = 0$  if and only if  $(\lambda_0 Q^{-1} \bar{E} P - Q^{-1} \bar{A} P)w_0 = 0$ , equivalently if and only if  $Q^{-1}(\lambda_0 \bar{E} - \bar{A})P w_0 = 0$ , that is to say, if and only if  $(\lambda_0 \bar{E} - \bar{A})P^{-1}w_0 = 0$ .  $\square$

the particular case where the equivalent system is in the reduced form  $P^{-1}w_0 = (v_0^1, 0) \in \mathbb{C}^r \times \mathbb{C}^{n-r}$  and  $v_0^1$  is an eigenvector of  $J$ :

$$\begin{pmatrix} \lambda_0 I_r & \\ & \lambda_0 N \end{pmatrix} \begin{pmatrix} v_0^1 \\ v_0^2 \end{pmatrix} = \begin{pmatrix} J & \\ & I_{n-r} \end{pmatrix} \begin{pmatrix} v_0^1 \\ v_0^2 \end{pmatrix}$$

Clearly  $v_0^2 = 0$  and  $Jv_0^1 = \lambda_0 v_0^1$ .

It is important the following result.

**Proposition 6** *Eigenvectors corresponding to different eigenvalues are independent.*

**Proof:** Let  $w_1, \dots, w_\ell$   $\ell$  eigenvectors corresponding to  $\lambda_1, \dots, \lambda_\ell$  with  $\lambda_i \neq \lambda_j$  for all  $i \neq j$  and consider  $\sum_{i=1}^\ell \alpha_i w_i$ .

Then  $\sum_{i=1}^\ell \alpha_i P^{-1} w_i = 0$  and  $\bar{A}^k \sum_{i=1}^\ell \alpha_i P^{-1} w_i = \sum_{i=1}^\ell \alpha_i \lambda_i^k P^{-1} w_i = 0$

Solving the system

$$\left. \begin{array}{l} \sum_{i=1}^\ell \alpha_i P^{-1} w_i = 0 \\ \sum_{i=1}^\ell \alpha_i \lambda_i P^{-1} w_i = 0 \\ \vdots \\ \sum_{i=1}^\ell \alpha_i \lambda_i^{\ell-1} P^{-1} w_i = 0 \end{array} \right\},$$

we obtain  $\alpha_i = 0$ , for  $i = 1, \dots, \ell$ . Consequently, the vectors are linearly independent.  $\square$

It is important to remark the following proposition corresponding to the eigenvectors of infinity.

**Proposition 7** *Let  $(E, A)$  and  $(\bar{E}, \bar{A})$  be two equivalent systems and  $0 \neq w \in \mathbb{C}^n$ .  $w \in \text{Ker } E$  if and only if  $P^{-1}w \in \text{Ker } \bar{E}$  with  $(\bar{E}, \bar{A}) = (QEP, QAP)$ .*

**Proof:** Let  $(E, A)$  and  $(\bar{E}, \bar{A}) = (QEP, QAP)$  two equivalent systems.

$Ew = 0$  if and only if  $Q^{-1}\bar{E}P^{-1}w = 0$ , equivalently if and only if  $\bar{E}P^{-1}w = 0$ .  $\square$

## 2.1 Quasi-Weierstraß form

The pair of matrices  $(E, A)$  corresponding to a regular pencil, can be reduced to a weaker form called "Quasi-Weierstraß form" (see [3]) in the following manner:

Let  $P = \begin{pmatrix} V & W \end{pmatrix}$  and  $Q = \begin{pmatrix} EV & AW \end{pmatrix}^{-1}$ . Matrices  $V \in M_{n \times r}(C)$  and  $W \in M_{n \times (n-r)}(C)$  are in such a way that  $\begin{pmatrix} V & W \end{pmatrix}$  and  $\begin{pmatrix} EV & AW \end{pmatrix}$  are invertible.

$$(QEP, QAP) = \left( \begin{pmatrix} I_r & \\ & N \end{pmatrix}, \begin{pmatrix} A_r & \\ & I_{n-r} \end{pmatrix} \right) = (\tilde{E}, \tilde{A}),$$

where  $A_r$  is some matrix and  $N$  is nilpotent.

The vector spaces  $\text{Im } V$  and  $\text{Im } W$  are spanned by the generalized eigenvector at the finite and infinite eigenvalues respectively, and they are derived by the following recursive subspace iteration with a limited number of steps called Wong sequences [21].

$$\begin{array}{ll} V_0 = C^n, & V_{i+1} = \{v \in C^n \mid Av \in E(V_i)\} \\ W_0 = \{0\}, & W_{i+1} = \{v \in C^n \mid Ev \in A(W_i)\} \end{array}$$

verifying

$$V_0 \supseteq V_1 \supseteq \dots \supseteq V_\ell = V_{\ell+1} = \dots V_{\ell+q} = V^* \supseteq \text{Ker } A \\ W_0 \subseteq W_1 \subseteq \dots \subseteq W_m = W_{m+1} = \dots W_{m+q} = W^*$$

It is easy to prove that  $\ell = m$  and satisfy  $AV^* \subseteq EV^*$  and  $EW^* \subseteq AW^*$ .

Matrices  $V$  and  $W$  are defined in such away that  $V^* = \text{Im } V$  and  $W^* = \text{Im } W$ .

**Example 8** *Let  $(E, A)$  a system with*

$$E = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

$$W_0 = \{0\}$$

$$W_1 = \text{Ker } E = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = W_2 = W$$

$$V_0 = \mathbb{R}^3$$

$$V_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = V_2 = V$$

then,

$$\begin{pmatrix} 3 & 1 & 2 \\ 4 & 2 & 2 \\ 3 & 1 & -4 \end{pmatrix}^{-1} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \tilde{E}$$

$$\begin{pmatrix} 3 & 1 & 2 \\ 4 & 2 & 2 \\ 3 & 1 & -4 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 2 & -2 & 0 \\ -5 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \tilde{A}$$

Similarly to propositions 3, 5 and 7 we can prove the following results.

**Proposition 9** *Let  $(E, A)$  be a system and  $(\tilde{E}, \tilde{A})$  its quasi-Weierstraß form.  $\lambda_0$  is an eigenvalue of  $(E, A)$  if and only if it is an eigenvalue of  $(\tilde{E}, \tilde{A})$ .*

**Proposition 10** *Let  $(E, A)$  be a system and  $(\tilde{E}, \tilde{A})$  its quasi-Weierstraß form.  $w_0$  is an eigenvector of  $(E, A)$  if and only if  $P^{-1}w_0$  is an eigenvector of  $(\tilde{E}, \tilde{A})$  and  $P^{-1}w_0 = (v_0^1, 0) \in \mathbb{C}^r \times \mathbb{C}^{n-r}$  and  $v_0^1$  is an eigenvector of  $A_r$ .*

It is important to remark the following proposition corresponding to the eigenvectors at the infinity.

**Proposition 11** *Let  $(E, A)$  be a system and  $(\tilde{E}, \tilde{A})$  its quasi-Weierstraß form. Let us consider a non-zero vector  $w \in \mathbb{C}^n$ . Then,  $w \in \text{Ker } E$  if and only if  $P^{-1}w \in \text{Ker } \tilde{E}$  with  $(\tilde{E}, \tilde{A}) = (QEP, QAP)$ .*

## 2.2 Controllability

An important concept concerning structural properties is the controllability that is defined as follows

**Definition 12** *The system 1 is called controllable if, for any  $t_1 > 0$ ,  $x(0) \in \mathbb{C}^n$  and  $w \in \mathbb{C}^n$ , there exists a control input  $u(t)$  sufficiently smooth such that  $x(t_1) = w$ .*

The controllability character can be computed by means the generalized Hautus test for controllability of singular systems.

**Proposition 13 ([10])** *The system 1 is controllable if and only if:*

$$\left. \begin{aligned} \text{rank} \begin{pmatrix} E & B \end{pmatrix} &= n \\ \text{rank} \begin{pmatrix} sE - A & B \end{pmatrix} &= n, \forall s \in \mathbb{C} \end{aligned} \right\}. \quad (3)$$

**Remark 14** *The first condition of the proposition implies that the system is standardizable under derivative feedback.*

*Remember that, system is standardizable under derivative feedback if and only if, there exists a matrix  $F \in M_{m \times n}(\mathbb{C})$  such that  $E + BF$  is invertible and the system obtained by derivative feedback  $(E + BF)\dot{x}(t) = Ax(t) + Bu(t)$  can be standardized premultiplying it by  $(E + BF)^{-1}$ .*

**Remark 15** *Controllability character can be computed by means the rank of a certain numerical matrix constructed gluing matrix blocks  $\begin{pmatrix} E & B & 0 \\ A & 0 & B \end{pmatrix}$  in the lower right corner (see [10]).*

## 3 Exact controllability

There are many possible control matrices  $B$  in the system 1 that satisfy the controllability condition. The goal is to find the set of all possible matrices  $B$ , having the minimum number of columns corresponding to the minimum number  $n_B(E, A)$  of independent controllers required to control the whole network.

**Definition 16** *Let  $(E, A)$  be a pair of matrices. The exact controllability  $n_B(E, A)$  is the minimum of the rank of all possible matrices  $B$  making the system 1 controllable.*

$$n_B(E, A) = \min \{ \text{rank } B, \forall B \in M_{n \times i} \mid 1 \leq i \leq n, (E, A, B) \text{ controllable} \}. \quad (4)$$

If confusion is not possible we will write simply  $n_B$ .

Taking into account the generalized Hautus condition 3, it is straightforward the following proposition.

**Proposition 17** *The exact controllability  $n_B$  is invariant under equivalence relation considered, that is to say: for any couple of invertible matrices  $(Q, P)$ ,*

$$n_B(E, A) = n_B(QEP, QAP).$$

**Proof:**

$$\begin{aligned} \text{rank} \begin{pmatrix} QEP & QB \end{pmatrix} &= \\ \text{rank } Q \begin{pmatrix} E & B \end{pmatrix} \begin{pmatrix} P \\ I \end{pmatrix} &= \text{rank} \begin{pmatrix} E & B \end{pmatrix} \\ \text{rank} \begin{pmatrix} sQEP - QAP & QB \end{pmatrix} &= \\ \text{rank } Q \begin{pmatrix} sE - A & B \end{pmatrix} \begin{pmatrix} P \\ I \end{pmatrix} &= \\ \text{rank} \begin{pmatrix} sE - A & B \end{pmatrix} & \end{aligned}$$

□

As a consequence, if necessary we can consider  $(E, A)$  in its canonical reduced form.

**Example 18** 1) *If  $E = A = 0$ ,  $n_B = n$*

2) *If  $E = I$  and  $A = \text{diag}(\lambda_1, \dots, \lambda_n)$  with  $\lambda_i \neq \lambda_j$  for all  $i \neq j$ , then  $n_B = 1$ , (it suffices to take  $B = (1 \dots 1)^t$ ).*

3) *If  $E = \begin{pmatrix} 1 & \\ & 0 \end{pmatrix}$  and  $A = \begin{pmatrix} 0 & \\ & 1 \end{pmatrix}$ ,  $n_B = 1$ . It suffices to consider  $B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ .*

**Remark 19** Not every matrix  $B$  having  $n_B$  columns is valid to make the system controllable. For example if  $E = I$ ,  $A = \text{diag}(1, 2, 3)$  and  $B = (1, 0, 0)^t$ , the system  $(A, B)$  is not controllable, ( $\text{rank} \begin{pmatrix} B & AB & A^2B \end{pmatrix} = 1 < 3$ , or equivalently  $\text{rank} \begin{pmatrix} A - \lambda I & B \end{pmatrix} = 2$  for  $\lambda = 2, 3$ ).

For standard systems we have the following result.

**Proposition 20 ([23])**

$$n_B = \max_i \{\mu(\lambda_i)\}$$

where  $\mu(\lambda_i) = \dim \text{Ker} (A - \lambda_i I)$  is the geometric multiplicity of the eigenvalue  $\lambda_i$ .

**Example 21 ([7])** 1) If  $A = 0$ ,  $n_D = n$

2) If  $A = \text{diag}(\lambda_1, \dots, \lambda_n)$  with  $\lambda_i \neq \lambda_j$  for all  $i \neq j$ , then  $n_D = 1$ , (it suffices to take  $B = (1 \dots 1)^t$ ).

3) Not every matrix  $B$  having  $n_D$  columns is valid to make the system controllable. For example if  $A = \text{diag}(1, 2, 3)$  and  $B = (1, 0, 0)^t$ , the system  $(A, B)$  is not controllable, ( $\text{rank} \begin{pmatrix} B & AB & A^2B \end{pmatrix} = 1 < 3$ , or equivalently  $\text{rank} \begin{pmatrix} A - \lambda I & B \end{pmatrix} = 2$  for  $\lambda = 2, 3$ ).

For singular systems it is obvious that  $n_B \geq n - \text{rank } E = n_E$ .

**Theorem 22** Let  $(E, A)$  be a singular system. The exact controllability  $n_B$  is computed in the following manner.

$$n_B = \max \{n_E, \mu(\lambda_i)\}$$

where  $\mu(\lambda_i) = \dim \text{Ker} (\lambda_i E - A)$  and  $\lambda_i$  (for each  $i$ ) is the eigenvalue of pencil  $sE - A$ .

**Proof:** Proposition 17 permit us to consider the system in its canonical reduced form

$$\text{rank} (E, B) = \text{rank} \left( \begin{pmatrix} I & \\ & N \end{pmatrix} \begin{pmatrix} \bar{B}_1 \\ \bar{B}_2 \end{pmatrix} \right) = n_1 + \text{rank} (N, \bar{B}_2)$$

$$N = \text{diag} (N_1, \dots, N_{n_E}) \text{ with } N_i = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}$$

Taking

$$\bar{B}_2 = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \\ \ddots \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = (w_1^\infty \dots w_{n_E}^\infty),$$

we have that  $\text{rank} (N, \bar{B}_2) = n_2$

$$\text{rank} \begin{pmatrix} \lambda_i E - A & B \end{pmatrix} = \text{rank} \left( \lambda_i \begin{pmatrix} I & \\ & N \end{pmatrix} - \begin{pmatrix} J & \\ & I \end{pmatrix} \begin{pmatrix} \bar{B}_1 \\ \bar{B}_2 \end{pmatrix} \right) = n_2 + \text{rank} \begin{pmatrix} \lambda_i I - J & \bar{B}_1 \end{pmatrix}.$$

$$J = \text{diag} (J_1(\lambda_1), \dots, J_r(\lambda_r)), \quad J_i(\lambda_i) = \text{diag} (J_{i1}(\lambda_i), \dots, J_{i r_i}(\lambda_i)) \quad \text{and} \quad J_{ij}(\lambda_i) =$$

$$\begin{pmatrix} \lambda_i & 1 & & \\ & \ddots & \ddots & \\ & & \lambda_i & 1 \\ & & & \lambda_i \end{pmatrix}$$

Taking

$$\bar{B}_{1_i} = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \\ \ddots \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = (w_1^{\lambda_i} \dots w_{\mu_i(\lambda_i)}^{\lambda_i}),$$

we have that  $\text{rank} (\lambda_i I - J, \bar{B}_{1_i}) = \mu(\lambda_i)$

Consider now the following collection of vectors

$$w_1^{\lambda_1}, \dots, w_\ell^{\lambda_1} \\ \vdots \\ w_1^{\lambda_r}, \dots, w_\ell^{\lambda_r} \\ w_1^\infty, \dots, w_\ell^\infty$$

where  $\ell = \max (\mu(\lambda_1), \dots, \mu(\lambda_r), n_E)$  and we complete each series of vectors with the zero vectors in case its length is less than  $\ell$ .

Finally we construct the family

$$w_1 = w_1^{\lambda_1} + \dots + w_1^{\lambda_r} + w_1^\infty, \dots, w_\ell = w_\ell^{\lambda_1} + \dots + w_\ell^{\lambda_r} + w_\ell^\infty$$

Clearly,

$$\begin{aligned} \text{rank} \begin{pmatrix} E & B \end{pmatrix} &= n \\ \text{rank} \begin{pmatrix} \lambda E - A & B \end{pmatrix} &= n, \text{ for all } \lambda \in \mathbb{C} \end{aligned}$$

Now, it suffices to remark that if we consider  $B = (b_{ij}) \in M_{n \times m}(\mathbb{C})$  with  $m < \ell$

- i) if  $\ell = n_E$  then  $\text{rank} \begin{pmatrix} E & B \end{pmatrix} < n$
- ii) if  $\ell = \mu(\lambda_i)$  then  $\text{rank} \begin{pmatrix} \lambda_i E - A & B \end{pmatrix} < n$

□

**Example 23** Let  $(E, A)$  be a singular system with

$$E = \left( \begin{array}{cccccccc|cccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 \end{array} \right),$$

and

$$A = \left( \begin{array}{cccccccc|cccccccc} 3 & 0 \\ 1 & 3 & 0 \\ 0 & 1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

We have:

$$\begin{aligned} \text{rank } E &= 10 \\ \text{rank} (sE - A) &= \begin{cases} 11 & \text{for } s = 3 \\ 12 & \text{for } s = 2 \\ 13 & \text{for all } s \neq 2, 3. \end{cases} \end{aligned}$$

So,

$$\begin{aligned} n_E &= 3, \\ \mu(3) &= 2, \\ \mu(2) &= 1, \end{aligned}$$

then  $n_B = \max(3, 2, 1) = 3$ .

In fact, taking

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\text{rank} \begin{pmatrix} E & B \end{pmatrix} = 13$$

$$\text{rank} \begin{pmatrix} sE - A & B \end{pmatrix} = 13 \text{ for all } s \in \mathbb{C}.$$

Obviously, for all matrix

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \\ b_{41} & b_{42} \\ b_{51} & b_{52} \\ b_{61} & b_{62} \\ b_{71} & b_{72} \\ b_{81} & b_{82} \\ b_{91} & b_{92} \\ b_{101} & b_{102} \\ b_{111} & b_{112} \\ b_{121} & b_{122} \\ b_{131} & b_{132} \end{pmatrix}$$

$$\text{rank} \begin{pmatrix} E & B \end{pmatrix} < 13.$$

## 4 Generators of control space

As we have discussed in the previous section, not every matrix  $B$  serves to make the system controllable. Of all possible, we want to find those with the least number of columns.

To make the paper more understandable, we begin showing some particular cases.

**Proposition 24** Let  $(E, A)$  be a system with  $E$  invertible. Then, the matrices  $B$  making the system  $(E, A, B)$  controllable are those that make the standard system  $\dot{z} = AE^{-1}z$ , controllable.

**Proof:**

$$\begin{aligned} \text{rank} \begin{pmatrix} E & B \end{pmatrix} &= n \text{ for all matrix } B \\ \text{rank} \begin{pmatrix} sE - A & B \end{pmatrix} &= \\ \text{rank} \begin{pmatrix} sI - AE^{-1} & B \end{pmatrix} \begin{pmatrix} E & \\ & I \end{pmatrix} &= \\ \text{rank} \begin{pmatrix} sI - AE^{-1} & B \end{pmatrix} & \end{aligned}$$

Then,

$$\text{rank} \begin{pmatrix} sE - A & B \end{pmatrix} = n$$

if and only if

$$\text{rank} \begin{pmatrix} sI - AE^{-1} & B \end{pmatrix} = n.$$

□

The minimal sets of matrices  $B$  making a standard systems controllable are described in [7].

We want to remark that the solution of the problem for standard systems is linked to the eigenstructure of the matrix  $AE^{-1}$ . In our particular setup, the eigenstructure of the  $AE^{-1}$  corresponds to the eigenstructure of the pair  $(E, A)$  because of  $\det(sI - AE^{-1}) \det E = \det((sI - AE^{-1})E) = \det(sE - A)$ .

**Proposition 25** Suppose that the pencil  $(\tilde{E}, \tilde{A})$  is in its quasi-Weierstraß form corresponding to a fast singular systems and let  $m_1 \geq \dots m_s$  the nilpotent indices of  $\tilde{E}$ . Consider  $0 \neq \tilde{w}_1 \in \text{Ker } \tilde{E}^{m_1} \setminus \text{Ker } \tilde{E}^{m_1-1}$ ,  $0 \neq \tilde{w}_2 \in \text{Ker } \tilde{E}^{m_2} \setminus \text{Ker } \tilde{E}^{m_2-1}$ , linearly independent with  $w_1, \dots, 0 \neq \tilde{w}_s \in \text{Ker } \tilde{E}^{m_s} \setminus \text{Ker } \tilde{E}^{m_s-1}$ , linearly independent with  $w_1, \dots, w_s$ .

If we consider  $\text{Im } B = [w_1, \dots, w_s]$ , then  $(\tilde{E}, \tilde{A}, \tilde{B})$  is controllable.

**Corollary 26** Let  $(E, A)$  be a singular fast system and  $(\tilde{E}, \tilde{A}) = (QEP, QAP)$  its quasi-Weierstraß form. Then, taking  $B = Q\tilde{B}$  with  $\tilde{B}$  as in proposition 25, the system  $(E, A, B)$  is controllable.

**Proof:**

$$\begin{aligned} \text{rank} \begin{pmatrix} E & Q\tilde{B} \end{pmatrix} &= \\ \text{rank } Q^{-1} \begin{pmatrix} E & Q\tilde{B} \end{pmatrix} \begin{pmatrix} P^{-1} & \\ & I \end{pmatrix} &= \\ \text{rank} \begin{pmatrix} \tilde{E} & \tilde{B} \end{pmatrix} &= \\ \text{rank} \begin{pmatrix} sE - A & Q\tilde{B} \end{pmatrix} &= \\ \text{rank } Q^{-1} \begin{pmatrix} E & Q\tilde{B} \end{pmatrix} \begin{pmatrix} P^{-1} & \\ & I \end{pmatrix} &= \\ \text{rank} \begin{pmatrix} s\tilde{E} - \tilde{A} & \tilde{B} \end{pmatrix} & \end{aligned}$$

□

**Proposition 27** Let  $(\tilde{E}, \tilde{A})$  a singular system in its quasi-Weierstraß form. Then,  $(\tilde{E}, \tilde{A}, \tilde{B})$  is controllable, where  $\tilde{B} = \begin{pmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{pmatrix}$ , with  $\tilde{B}_1$  and  $\tilde{B}_2$  as in proposition and . (if both matrices do not have the same number of columns we complete the one that has less number of columns with columns of zeros).

**Proof:** Let  $\tilde{B}_1 \in M_{r \times m}(\mathbb{C})$  and  $\tilde{B}_2 \in M_{n-r \times \ell}(\mathbb{C})$  be the matrix such that  $(N, I_{n-r}, \tilde{B}_2)$  are controllable constructed as proposition 24 and proposition 25 respectively .

If  $m \neq \ell$  we complete with zero columns the matrix which the number of columns is smaller, matching in this way the size.

$$\text{rank} \left( \begin{pmatrix} I_r & \\ & N \end{pmatrix}, \begin{pmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{pmatrix} \right) = r + \text{rank} \left( \begin{pmatrix} N \\ \tilde{B}_2 \end{pmatrix} \right)$$

$$\text{rank} \left( s \begin{pmatrix} I_r & \\ & N \end{pmatrix} - \begin{pmatrix} A_r & \\ & I_{n-r} \end{pmatrix}, \begin{pmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{pmatrix} \right) = n - r + \text{rank} \left( \begin{pmatrix} sI_r - A_r \\ \tilde{B}_2 \end{pmatrix} \right)$$

□

**Theorem 28** Let  $(E, A)$  be a singular system and  $(\tilde{E}, \tilde{A}) = (QEP, QAP)$  its quasi-Weierstraß form. Then, taking  $B = Q\tilde{B}$  with  $\tilde{B}$  as in proposition 27, the system  $(E, A, B)$  is controllable.

**Example 29** Retaking example 8, we have that

$$\tilde{B}_1 = \begin{pmatrix} \alpha + 2\beta \\ \alpha - 5\beta \end{pmatrix}$$

with  $\alpha, \beta \neq 0$  and  $\tilde{B}_2 = (\gamma)$  with  $\gamma \neq 0$ . Then

$$B = Q\tilde{B} = \begin{pmatrix} 3 & 1 & 2 \\ 4 & 2 & 2 \\ 3 & 1 & -4 \end{pmatrix}^{-1} \begin{pmatrix} \alpha + 2\beta \\ \alpha - 5\beta \\ \gamma \end{pmatrix} = \begin{pmatrix} \alpha/3 + 25\beta/6 + \gamma/6 \\ -\alpha/3 - 67\beta/6 - \gamma/6 \\ \alpha/6 + \beta/3 - \gamma/6 \end{pmatrix}$$

And the system  $(E, A, B)$  is controllable.

## 5 Exact controllability of second order singular linear systems

Let us consider a homogeneous second order singular linear systems in the form

$$E\ddot{x}(t) = A_1\dot{x}(t) + A_0x(t) \quad (5)$$

And we ask for minimum of the rank of all possible matrices  $B$  making the system 5 controllable. Remember that:

**Definition 30** *The second-order linear system*

$$E\ddot{x}(t) = A_1\dot{x}(t) + A_0x(t) + Bu(t) \quad (6)$$

is controllable if and only if there exists a control  $u_1(t) = u - F_1\dot{x} - F_0x^{(0)}$ , with  $F_i \in M_{m \times n}(\mathbb{C})$  such that the equation

$$E\ddot{x}(t) = (A_1 + BF_1)\dot{x}(t) + (A_0 + BF_0)x(t) \quad (7)$$

has a stable solution.

For simplicity we will write the system as a quadruple of matrices  $(E, A_1, A_0, B)$  and as a triple  $(E, A_1, A_0)$  for the homogeneous case. The exact controllability  $n_B(E, A_1, A_0)$  is the minimum of the rank of all possible matrices  $B$  making the system 5 controllable.

**Definition 31**

$$n_B(E, A_1, A_0) = \min \{ \text{rank } B, \forall B \in M_{n \times i} \ 1 \leq i \leq n, (E, A_1, A_0, B) \text{ controllable} \}. \quad (8)$$

A manner to study that is linearizing the system in the following manner:

$$X(t) = \begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix}, \quad \dot{X}(t) = \begin{pmatrix} \dot{x}(t) \\ \ddot{x}(t) \end{pmatrix},$$

$$\begin{pmatrix} I \\ E \end{pmatrix} \dot{X}(t) = \begin{pmatrix} I & I \\ A_0 & A_1 \end{pmatrix} X(t) + \begin{pmatrix} 0 \\ B \end{pmatrix} u(t).$$

that we can write in a simple way:

$$\mathbf{E}\dot{X}(t) = \mathbf{A}X(t) + \mathbf{B}u(t) \quad (9)$$

Taking into account that  $X_1(t) = \begin{pmatrix} x_1(t) \\ \dot{x}_1(t) \end{pmatrix}$  is a solution of the linear system associated  $\dot{X}(t) = \mathbf{A}X(t) + \mathbf{B}u(t)$ , if and only if  $x_1(t)$  is a solution of the equation 6, it is not difficult to prove the following proposition.

**Proposition 32** *The second order singular linear system is controllable if and only if the singular linear system associated 9 is controllable.*

**Proof:** The controllability of  $X^{(1)} = \mathbf{A}X + \mathbf{B}u$  ensures the existence of  $\mathbf{F} \in M_{m \times \ell n}(\mathbb{C})$  such that  $X^{(1)} = \mathbf{A}X(t) + \mathbf{B}u_1(t)$  with  $u_1(t) = u(t) - \mathbf{F}X(t)$  has a stable solution. Partitioning the matrix  $\mathbf{F}$  into two blocks  $\mathbf{F} = \begin{pmatrix} F_0 & F_1 \end{pmatrix}$  we have that the equation  $E\ddot{x}(t) = A_1\dot{x}(t) + A_0x(t) + Bu_1(t)$  with  $u_1(t) = u(t) - F_0x(t) - F_1\dot{x}(t)$  has a stable solution. Converse is analogous.  $\square$

So, controllability character of second order singular linear systems it is reflected as follows

$$\left. \begin{array}{l} \text{rank} \begin{pmatrix} \mathbf{E} & \mathbf{B} \end{pmatrix} = 2n \\ \text{rank} \begin{pmatrix} s\mathbf{E} - \mathbf{A} & \mathbf{B} \end{pmatrix} = 2n, \forall s \in \mathbb{C} \end{array} \right\}. \quad (10)$$

Now we present the main result that permit us to analyze the controllability character directly from the initial equation (1.1).

**Theorem 33** *The second order linear 6 is controllable if and only if,*

$$\left. \begin{array}{l} \text{rank} \begin{pmatrix} E & B \end{pmatrix} = n \\ \text{rank} \begin{pmatrix} s^2E - sA_1 - A_0 & B \end{pmatrix} = n. \end{array} \right\} \quad (11)$$

for all  $s \in \mathbb{C}$ .

**Proof:**

Making row and column elementary transformations we obtain

$$\text{rank} \begin{pmatrix} \mathbf{E} & \mathbf{B} \end{pmatrix} = \begin{pmatrix} I & & \\ & E & B \end{pmatrix} = n + \text{rank} \begin{pmatrix} E & B \end{pmatrix}$$

$$\begin{aligned} & \text{rank} \begin{pmatrix} s\mathbf{E} - \mathbf{A} & \mathbf{B} \end{pmatrix} \\ &= \text{rank} \begin{pmatrix} sI & -I & \\ -A_0 & sE - A_1 & B \end{pmatrix} = \\ & \text{rank} \begin{pmatrix} 0 & -I & \\ s^2E - sA_1 - A_0 & sE - A_1 & B \end{pmatrix} = \\ & \text{rank} \begin{pmatrix} 0 & -I & \\ s^2E - sA_1 - A_0 & 0 & B \end{pmatrix} = \\ & n + \text{rank} \begin{pmatrix} s^2E - sA_1 - A_0 & B \end{pmatrix} \end{aligned}$$

$\square$

So, as a consequence we can enunciate the following result:



### Theorem 34

$$n_B(E, A_1, A_0) = \max_i \{n_E, \mu(\lambda_i)\}$$

where  $\mu(\lambda_i) = \dim \text{Ker} (\lambda_i \mathbf{E} - \mathbf{A})$ .

### Corollary 35

$$n_B(E, A_1, A_0) = \max_i \{n_E, \nu(s_i)\}$$

where

$$\nu(s_i) = \dim \text{Ker} (s^2 E - sA_1 - A_0)$$

for all  $s_i \in \mathbb{C}$  such that

$$\det(s^2 E - sA_1 - A_0) = 0.$$

## 6 Conclusion

In this work, given two  $n$ -order square matrices  $E, A$  defining regular generalized systems  $E\dot{x} = Ax$ . We ask for minimal number of columns that must have a matrix  $B$  in order to make the system  $(E, A, B)$  controllable.

The sets of minimal generators of controllability spaces are obtained using the quasi Weierstraß reduced form. Examples have been included to make the work easier readable.

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