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QUALITATIVE PROPERTIES IN STRAIN GRADIENT THERMOELASTICITY WITH MICROTEMPERATURES

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Abstract: This paper is devoted to the strain gradient theory of thermoelastic materials whose microelements possess microtemperatures. The work is motivated by increasing use of materials which possess thermal variation at a microstructure level. In the first part of the paper we deduce the system of basic equations of the linear theory and formulate the boundary-initial-value problem. We establish existence, uniqueness and continuous dependence results by means of the semigroup theory. Then, we study the one-dimensional problem and establish the analyticity of solutions. Exponential stability and impossibility of localization are consequences of this result. In the case of anti-plane problem we derive uniqueness and instability results without assuming the positivity of the mechanical energy. Finally, we study the equilibrium theory and investigate the effects of a concentrated heat source in an unbounded body.

Keywords: Strain gradient thermoelasticity; Microtemperatures; Existence and uniqueness results; Analyticity

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1 Introduction

The linear theories of thermoelastic bodies with inner structure have been intensively studied. The origin of the theories of continua with microstructure goes back to the papers of Ericksen and Truesdell (1958), Toupin (1962, 1964), Mindlin (1964), Eringen and Suhubi (1964) and Green and Rivlin (1964). Toupin (1962, 1964) and Mindlin (1964) have established the theory of nonsimple elastic media, which is characterized by the inclusion of higher gradients of displacement in the basic postulates. The equations and the boundary conditions of the nonlinear strain gradient theory of elastic solids were first established by Toupin (1962, 1964). The linear theory has been developed by Mindlin (1964) and Mindlin and Eshel (1968)). The strain gradient theory of elasticity is adequate to investigate important problems related to size effects and to describe the deformation of chiral elastic solids (Papanicolopulos, 2011 and references therein). Examples of chiral materials include auxetic materials, bones, carbon nanotubes, honeycomb structures, as well as some porous composites. The gradient theories of thermomechanics have been studied in various papers (Ahmadi and Firoozbakhsh, 1975; Ieşan, 1983; Ieşan and Quintanilla, 1992; Forest et al., 2000, 2002; Ieşan, 2004; Forest and Amestoy, 2008). Mindlin (1964) formulated a theory of a continuum which has some properties of a crystal lattice as a result of inclusion of the idea of a unit cell. Mindlin begins with the general concept of a continuum, each material particle of which is a deformable medium. In this theory, each microelement is constrained to deform homogeneously. The spatial coordinates x_i' of the point X' of the microelement Ω are represented in the form $x_i' = x_i + \psi_{ik}\xi_k$, where x_i are the spatial coordinates of the centroid X of Ω, X_k' and X_k are the material coordinates of X' and X, and $\xi_k = X_k' - X_k$. The functions ψ_{ik} are called microdeformations. The theory of continua with microstructure has been studied extensively and an account of the basic results can be found in the works of Ciarletta and Iesan (1993), Eringen (1999) and Mariano (2001). Grot (1969) established a linear theory of thermodynamics of elastic solids with microstructure whose microelements possess microtemperatures. The Clausius-Duhem inequality is modified to include microtemperatures, and the first-order moment of the energy equations are added to the usual balance laws. In this theory the absolute temperature T' at the point X' of the microelement is a linear function of the microcoordinates ξ_k, of the form $T' = T + \tau_k \xi_k$, where T is the temperature at the centroid X. The vector with the components T_k defined by $T_k = -\tau_k/T_0$ is called the microtemperature vector. Here, T_0 is the absolute temperature in the natural state. The theory of continua with microtemperatures has been studied in various papers (see, e.g., Riha, 1975, 1976; Verma et al., 1979; Svanadze, 2004; Casas and Quintanilla , 2005; Iesan and Quintanilla, 2009). Riha (1976) presented a study of heat conduction in materials with microtemperatures. Experimental data for the silicone rubber containing spherical aluminium particles and for human blood were found to conform closely to predicted theoretical thermal conductivity.

This paper is structured as follows. In Section 2 we establish the basic equations of the linear strain gradient theory of thermoelastic materials whose microelements possess microtemperatures. Section 3 is devoted to a uniqueness theorem in the dynamic theory of anisotropic solids. In Section 4 we present an existence theorem by means of the linear semigroup theory. The one-dimensional dynamic theory of homogeneous and isotropic solids is investigated in Section 5. In this case we establish the analyticity of solution. This result implies the exponential stability of solutions and the impossibility of localization of solutions. In Section 6 we consider the anti-plane problem and prove an uniqueness result by means of the logarithmic convexity argument. The last section is concerned with the equilibrium theory of thermoelastic materials with microtemperatures. We present a uniqueness result and investigate the effects of a concentrated heat source in an unbounded body.

2 Basic equations

In this section we use the results of Toupin (1964), Eringen and Suhubi (1964), Grot (1969) and Eringen (1999) to establish the basic equations of a strain gradient theory of thermoelasticity with microtemperatures. We restrict our attention to the linear theory of thermoelasticity.

We consider a body that at some instant occupies the properly regular region B of Euclidean three-dimensional space and is bounded by the surface ∂B. The motion of the body is referred to a fixed system of rectangular cartesian axes Ox_i $(i = 1; 2; 3)$. We denote by n_k the outward unit normal of ∂B. We shall employ the usual summation and differentiation conventions: Latin subscripts, unless otherwise specified, are understood to range over the integers $(1, 2, 3)$, summation over repeated subscripts is implied, and subscripts preceded by a comma denote partial differentiation with respect to the corresponding cartesian coordinate. We use a superposed dot to denote partial differentiation with respect to the time.

Let $\mathcal{P}$ be an arbitrary material volume in the continuum, bounded by a surface $\partial\mathcal{P}$ at time t. We suppose that P is the corresponding region in the reference configuration, bounded by a surface ∂P. Let u_k be a displacement vector field on B.

Following Toupin (1964) we postulate an energy balance in the form

$$\int_P (\rho \ddot{u}_j \dot{u}_j + \rho \dot{e}) dv = \int_P \rho (f_i \dot{u}_i + S) dv +$$
$$+ \int_{\partial P} (t_i \dot{u}_i + \mu_{ji} \dot{u}_{i,j} + q) da, \tag{1}$$

for all regions P of B and every time, where ρ is the mass density in the reference configuration, e is the internal energy per unit mass, f_i is the body force per unit mass, S is is the heat supply per unit mass, t_i is a part of the stress vector associated with the surface $\partial\mathcal{P}$ but measured per unit area of ∂P, μ_{ij} is the dipolar surface force associated with the surface $\partial\mathcal{P}$ and measured per unit area of ∂P and q is the heat flux associated with surface ∂P and measured per unit area of ∂P. Following Toupin (1964) we assume that the dipolar body force and the spin inertia per unit mass are absent. As in the paper of Green and Rivlin (1964), we consider a motion of the body which differs from the given motion only by a superposed translation, the body occupying the same position at time t. From (1) we obtain

$$\int_P \rho \ddot{u}_j dv = \int_P \rho f_j dv + \int_{\partial P} t_j da, \tag{2}$$

for all regions P of B. Using the well-known method, from (2) we get

$$t_i = t_{ji} n_j, \tag{3}$$

where t_{ij} is the stress tensor. The local form of the relation (2) is given by

$$t_{ji,j} + \rho f_i = \rho \ddot{u}_i. \tag{4}$$

In view of (3) and (4), the relation (1) reduces to

$$\int_P \rho \dot{e} dv = \int_P (t_{ji} \dot{u}_{i,j} + S) dv + \int_{\partial P} (\mu_{ji} \dot{u}_{i,j} + q) da, \tag{5}$$

for all regions P of B and every time. With an argument similar to that used to derive the relation (3), from (5) we obtain

$$(\mu_{ji} - \mu_{kji} n_k) \dot{u}_{i,j} + q - q_j n_j = 0, \tag{6}$$

where μ_{ijk} is the double stress tensor and q_j is the heat flux vector.

If we use the relation (6) and the divergence theorem, then we find that the equation (5) can be written in the following local form

$$\rho \dot{e} = (t_{ji} + \mu_{kji,k})\dot{u}_{i,j} + \mu_{kji}\dot{u}_{i,jk} + q_{j,j} + \rho S. \tag{7}$$

In this theory the balance of first stress moments presented by Grot (1969) reduces to

$$\mu_{kji,k} + t_{ji} - \tau_{ji} = 0, \tag{8}$$

where τ_{ji} is the microstress tensor. By using (8), the equation (7) can be written in the following local form

$$\rho \dot{e} = \tau_{ij}\dot{u}_{i,j} + \mu_{ijk}\dot{u}_{i,jk} + q_{j,j} + \rho S. \tag{9}$$

Following the method of Green and Rivlin (1964) we consider a motion of the body which differs from the given motion by a superposed uniform rigid body angular velocity, the body occupying the same position at time t, and let us assume that $\dot{e}$, τ_{ij}, μ_{kji}, q_j and S are unaltered by such motion. From (9) we find that

$$\tau_{ij} = \tau_{ji}. \tag{10}$$

If we use the relation (10) then the balance of energy (9) can be written in the form

$$\rho \dot{e} = \tau_{ij}\dot{e}_{ij} + \mu_{ijk}\dot{\kappa}_{ijk} + q_{j,j} + \rho S, \tag{11}$$

where we have used the notations

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \kappa_{ijk} = u_{k,ij}. \tag{12}$$

The balance of first moment of energy can be expressed as (Grot, 1969)

$$\rho \dot{\epsilon}_i = q_{ji,j} + q_i - Q_i + \rho G_i, \tag{13}$$

where q_{ij} is the first heat flux moment tensor, Q_{ij} is the microheat flux average and G_i is the first heat supply moment tensor. The local form for the second law of thermodynamics is given by (Grot, 1969)

$$\rho \dot{\eta} - (\frac{1}{T}q_k + \frac{1}{T}q_{km}T_m)_{,k} - \frac{1}{T}(S + G_i T_i) \geq 0, \tag{14}$$

where η is the entropy density per unit mass, T is the absolute temperature and T_j is the microtemperature vector. In this theory the temperature T' at the point X' of a microelement is a linear function of the microcoordinates

$\xi_k = X'_k - X_k$ of the form $T' = T + \tau_k \xi_k$, where T is the temperature at the centroid X. The vector T_k defined by $T_k = -\tau_k/T$ is called the microtemperature vector. With the help of (11) and (13), the inequality (14) can be written in the form

$$\rho(T\dot{\eta} - \dot{e} - T_i\dot{\epsilon}_i) + \tau_{ij}\dot{e}_{ij} + \mu_{ijk}\dot{\kappa}_{ijk} + \frac{1}{T}q_iT_i + \frac{1}{T}q_{ij}T_{,i}T_{,j} - q_{ij}T_{j,i} + (q_j - Q_j)T_j \geq 0. \tag{15}$$

We introduce the function ψ by

$$\psi = e - \eta T + \epsilon_j T_j. \tag{16}$$

The inequality (15) can be expressed as

$$-\rho(\dot{\psi} + \dot{T}\eta - \dot{T}_i\epsilon_i) + \tau_{ij}\dot{e}_{ij} + \mu_{ijk}\dot{\kappa}_{ijk} + \frac{1}{T}q_iT_i + \frac{1}{T}q_{ij}T_{,i}T_{,j} - q_{ij}T_{j,i} + (q_j - Q_j)T_j \geq 0. \tag{17}$$

Let us introduce the notation

$$\theta = T - T_0, \tag{18}$$

where T_0 is the absolute temperature in the reference configuration. In what follows we assume that T_0 is a positive constant. In the linear theory we assume that $u_i = \epsilon u'_i, \theta = \epsilon\theta'$ and $T_i = \epsilon T'_i$ where ϵ is a constant small enough for squares and higher powers to be neglected and u'_i, θ' and T'_i are independent of ϵ. We suppose that in the reference configuration the functions $\tau_{ij}, \mu_{ijk}, \eta, \epsilon_i, q_i, q_{ij}$ and Q_j all vanish. In the context of the linear theory the inequality (17) becomes

$$-\rho(\dot{\psi} + \dot{\theta}\eta - \dot{T}_i\epsilon_i) + \tau_{ij}\dot{e}_{ij} + \mu_{ijk}\dot{\kappa}_{ijk} + \frac{1}{T_0}q_i\theta_{,i} - q_{ij}T_{j,i} + (q_j - Q_j)T_j \geq 0. \tag{19}$$

We require constitutive equations for $\psi, \tau_{ij}, \mu_{ijk}, \eta, \epsilon_i, q_i, q_{ij}$ and Q_j and assume that these are functions of the set of variables $\Pi = (e_{ij}, \kappa_{ijk}, \theta, \theta_{,i}, T_i, T_{i,j})$. We assume that there is no kinematical constraint. Introduction of the constitutive equations of the form

$$\psi = \widetilde{\psi}(\Pi), \quad \tau_{ij} = \widetilde{\tau}_{ij}(\Pi), \ldots, Q_i = \widetilde{Q}_i(\Pi), \tag{20}$$

into the equation (19), yields

$$\begin{aligned} &(\tau_{ij} - \frac{\partial\sigma}{\partial e_{ij}})\dot{e}_{ij} + (\mu_{ijk} - \frac{\partial\sigma}{\partial\kappa_{ijk}})\dot{\kappa}_{ijk} - \dot{\theta}(\rho\eta + \frac{\partial\sigma}{\partial\theta}) \\ &+ \dot{T}_i(\rho\epsilon_i - \frac{\partial\sigma}{\partial T_i}) - \frac{\partial\sigma}{\partial T_{i,j}}\dot{T}_{i,j} + \frac{1}{T}q_i\theta_{,i} - q_{ij}T_{j,i} + (q_i - Q_i)T_i \geq 0. \end{aligned} \tag{21}$$

Here we have used the notation $\sigma = \rho\psi$. For simplicity we regard the material to be homogeneous. From (21) we obtain

$$\sigma = \tilde{\sigma}(e_{ij}, \kappa_{ijk}, \theta, T_i),$$
$$\tau_{ij} = \frac{\partial \sigma}{\partial e_{ij}}, \quad \mu_{ijk} = \frac{\partial \sigma}{\partial \kappa_{ijk}}, \quad \rho\eta = -\frac{\partial \sigma}{\partial \theta}, \quad \rho\epsilon_i = \frac{\partial \sigma}{\partial T_i}, \tag{22}$$

and

$$q_i\theta_{,i} - T_0 q_{ij} T_{j,i} + T_0(q_i - Q_i)T_i \geq 0. \tag{23}$$

In the context of the linear theory we have

$$\sigma = \frac{1}{2}A_{ijrs}e_{ij}e_{rs} + B_{ijpqr}e_{ij}\kappa_{pqr} + \frac{1}{2}C_{ijkpqr}\kappa_{ijk}\kappa_{pqr} - a_{ij}e_{ij}\theta + L_{ijk}e_{ij}T_k$$
$$- c_{ijk}\kappa_{ijk}\theta + N_{ijrs}\kappa_{ijr}T_s - \frac{1}{2}a\theta^2 - b_i\theta T_i - \frac{1}{2}D_{ij}T_iT_j, \tag{24}$$

where the constitutive coefficients have the following symmetries

$$A_{ijmn} = A_{jimn} = A_{mnij}, \quad B_{ijpqr} = B_{jipqr} = B_{ijqpr}, \quad C_{ijkpqr} = C_{pqrijk} = C_{jikpqr},$$
$$a_{ij} = a_{ji}, \quad L_{ijk} = L_{jik}, \quad c_{ijk} = c_{jik}, \quad N_{ijks} = N_{jiks}, \quad D_{ij} = D_{ji}. \tag{25}$$

In view of (22), (24) and (25) we get

$$\tau_{ij} = A_{ijmn}e_{mn} + B_{ijpqr}\kappa_{pqr} + L_{ijk}T_k - a_{ij}\theta,$$
$$\mu_{ijk} = B_{rsijk}e_{rs} + C_{ijkpqr}\kappa_{pqr} + N_{ijks}T_s - c_{ijk}\theta,$$
$$\rho\eta = a_{ij}e_{ij} + c_{ijk}\kappa_{ijk} + a\theta + b_iT_i,$$
$$\rho\epsilon_i = L_{rsi}e_{rs} + N_{pqri}\kappa_{pqr} - D_{ij}T_j - b_i\theta. \tag{26}$$

On the basis of (23), the linear approximations for q_j, Q_j and q_{ij} are given by

$$q_i = k_{ij}\theta_{,j} + H_{ij}T_j, \quad q_{ij} = -P_{ijrs}T_{s,r}, \quad Q_i = (k_{ij} - K_{ij})\theta_{,j} + (H_{ij} - \Lambda_{ij})T_j. \tag{27}$$

It follows from (23) that the constitutive coefficients $k_{ij}, K_{ij}, H_{ij}, \Lambda_{ij}$ and P_{ijrs} satisfy the inequality

$$k_{ij}\theta_{,i}\theta_{,j} + (H_{ji} + T_0K_{ij})\theta_{,j}T_i + T_0\Lambda_{ij}T_iT_j + T_0P_{ijrs}T_{s,r}T_{j,i} \geq 0. \tag{28}$$

With the help of (8), the equations of motion (4) become

$$\tau_{ji,j} - \mu_{kji,kj} + \rho f_i = \rho\ddot{u}_i. \tag{29}$$

By (16), (18), (22) and (26) we find that in the linear theory the balance of energy (9) becomes

$$\rho T_0 \dot{\eta} = q_{j,j} + \rho S. \tag{30}$$

We conclude that the basic equations of the linear theory are the equations of motion (29), the energy equation (30), the balance of the first moment of energy (13), the constitutive equations (26) and (27), and the geometrical equations (12).

For a given deformation, $\dot{u}_{i,j}$ in (6) may be chosen in an arbitrary way so that, on the basis of the constitutive equations (20) we get

$$\mu_{ji} = \mu_{kji} n_k, \ \ q = q_j n_j. \tag{31}$$

It is known (Grot, 1969) that the heat flux moment vector Λ_j at regular points of the boundary are given by

$$\Lambda_i = q_{ji} n_j. \tag{32}$$

Let us assume that the boundary of B consists in the union of a finite number of smooth surfaces, smooth curves (edges) and points (corners). Let C be the union of the edges. Following Toupin (1964) and Mindlin (1964) we obtain

$$\int_{\partial B} (t_i \dot{u}_i + \mu_{ji} \dot{u}_{i,j}) da = \int_{\partial B} (P_i \dot{u}_i + R_i D \dot{u}_i) da + \int_C \Omega_i \dot{u}_i dl, \tag{33}$$

where

$$\begin{aligned} &P_i = (\tau_{ki} - \mu_{rki,r}) n_k - D_j (n_s \mu_{sji}) + (D_j n_j) n_s n_p \mu_{spi}, \ \ R_i = \mu_{rsi} n_r n_s, \\ &\Omega_i = < \mu_{pji} n_p n_q > \epsilon_{jrq} s_r, \ \ Df = f_{,j} n_j. \end{aligned} \tag{34}$$

In (34), D_i are the components of the surface gradients, $D_i = (\delta_{ij} - n_i n_j)\partial/\partial x_j$, s_k are the components of the unit vector tangent to C, and $< f >$ denotes the difference of limits of f from the both sides of C.

Let S_r, $(r = 1, 2, ..., 8)$, be subsets of ∂B such that $\bar{S}_1 \cup S_2 = \bar{S}_3 \cup S_4 = \bar{S}_5 \cup S_6 = \bar{S}_7 \cup S_8 = \partial B$, and $S_1 \cap S_2 = S_3 \cap S_4 = S_5 \cap S_6 = S_7 \cap S_8 = \emptyset$. We consider the mixed problem characterized by the following boundary conditions

$$\begin{aligned} &u_i = \tilde{u}_i \text{ on } \bar{S}_1 \times I, \ \ P_i = \tilde{P}_i \text{ on } S_2 \times I, \ \ Du_i = \tilde{d}_i \text{ on } \bar{S}_3 \times I, \\ &R_i = \tilde{R}_i \text{ on } S_4 \times I, \ \ \theta = \tilde{\theta} \text{ on } \bar{S}_5 \times I, \ \ q_j n_j = \tilde{q} \text{ on } S_6 \times I, \\ &T_i = \tilde{T}_i \text{ on } \bar{S}_7 \times I, \ \ q_{ji} n_j = \tilde{\Lambda}_i \text{ on } S_8 \times I, \Omega_i = \widetilde{\Omega}_i \text{ on } C \times I, \end{aligned} \tag{35}$$

where $\tilde{u}_i, \tilde{P}_i, \tilde{d}_i, \tilde{R}_i, \tilde{\theta}, \tilde{q}, \tilde{T}_i$, $\widetilde{\Lambda}_i$ and $\widetilde{\Omega}_i$ are prescribed functions and $I = (0, \infty)$. The initial conditions have the form

$$u_i(x,0) = u_i^0(x),\ \ \dot{u}_i(x,0) = v_i^0(x),\ \ \theta(x,0) = \theta^0(x), T_i(x,0) = T_i^0(x),\ \ x \in B, \tag{36}$$

where u_i^0, v_i^0, θ^0 and T_i^0 are given.

We note that in the case of isotropic and homogeneous bodies, the constitutive equations become

$$\begin{aligned}
&\tau_{ij} = \lambda e_{rr}\delta_{ij} + 2\mu e_{ij} - \beta\theta\delta_{ij},\\
&\mu_{ijk} = \frac{1}{2}\alpha_1(\kappa_{rri}\delta_{jk} + 2\kappa_{krr}\delta_{ij} + \kappa_{rrj}\delta_{ik}) + \alpha_2(\kappa_{irr}\delta_{jk} + \kappa_{jrr}\delta_{ik})\\
&\ + 2\alpha_3\kappa_{rrk}\delta_{ij} + 2\alpha_4\kappa_{ijk} + \alpha_5(\kappa_{kji} + \kappa_{kij}) + \xi_1\delta_{ij}T_k + \xi_2(\delta_{ik}T_j + \delta_{jk}T_i),\\
&\rho\eta = \beta e_{rr} + a\theta,\\
&\rho\epsilon_i = \xi_1\kappa_{mmi} + 2\xi_2\kappa_{irr} - bT_i,\ \ q_i = k\theta_{,i} + k_1T_i,\\
&Q_i = (k_1 - k_2)T_i + (k - k_3)\theta_{,i},\ \ \ q_{ij} = -k_4T_{r,r}\delta_{ij} - k_5T_{i,j} - k_6T_{j,i},
\end{aligned} \tag{37}$$

where δ_{ij} is the Kronecker delta and $\lambda, \mu, \alpha_s, (s = 1, 2, ..., 5), \beta, \xi_1, \xi_2, a, b, k$ and k_r, $(r = 1, 2, ..., 6)$, are prescribed constants.

The basic equations can be expressed in terms of the functions u_j, θ and T_k. Thus in the case of isotropic and homogeneous bodies we obtain

$$\begin{aligned}
&(\mu - \nu_1\triangle)\triangle u_i + (\lambda + \mu - \nu_2\triangle)u_{j,ji} - \xi_1\triangle T_i - 2\xi_2T_{j,ji} - \beta\theta_{,i} + \rho f_i = \rho\ddot{u}_i,\\
&k\triangle\theta + k_1T_{j,j} - \beta T_0\dot{u}_{r,r} - aT_0\dot{\theta} = -\rho S,\\
&k_6\triangle T_i + (k_4 + k_5)T_{j,ji} + \xi_1\triangle\dot{u}_i + 2\xi_2\dot{u}_{j,ji} - b\dot{T}_i - k_2T_i - k_3\theta_{,i} = \rho G_i,
\end{aligned} \tag{38}$$

where $\triangle$ is the Laplacian operator and we have introduced the notations

$$\nu_1 = 2(\alpha_3 + \alpha_4),\ \ \ \nu_2 = 2(\alpha_1 + \alpha_2 + \alpha_5). \tag{39}$$

The inequality (28) implies the following relations (Grot,1969)

$$k \geq 0,\ \ 3k_4 + k_5 + k_6 \geq 0,\ \ k_5 + k_6 \geq 0,\ \ k_5 - k_6 \geq 0,\ \ (k_1 + T_0k_3)^2 \leq 4T_0kk_2. \tag{40}$$

We assume that: (i) f_i, S and G_i are continuous on $\overline{B} \times [0, \infty)$; (ii) $\rho, u_i^0, v_i^0, \theta^0$ and T_i^0 are continuous on $\overline{B}$; (iii) the constitutive coefficients are continuous differentiable on $\overline{B}$; (iv) $\widetilde{u}_i, \widetilde{\theta}$ and $\widetilde{T}_i$ are continuous on $S_1 \times I$, $S_5 \times I$ and $S_7 \times I$, respectively, and $\widetilde{d}_i$ are continuous in time and piecewise regular on $S_3 \times I$; (v) $\widetilde{P}_i, \widetilde{R}_i, \widetilde{q}$ and $\widetilde{\Lambda}_i$ are continuous in time and piecewise regular on $S_2 \times I$, $S_4 \times I$, $S_6 \times I$, and $S_8 \times I$, respectively; (vi) $\widetilde{\Omega}_i$ are continuous in time and piecewise regular on $C \times I$. Let P and Q be non-negative integers. We say that f is of class $C^{P,Q}$ on $B \times I$ if f is continuous on $B \times I$ and the functions

$$\frac{\partial^m}{\partial x_i \partial x_j \ldots \partial x_s}\left(\frac{\partial^n f}{\partial t^n}\right), m \in \{0, 1, 2, \ldots, P\}, n \in \{0, 1, 2, \ldots, Q\},$$

$m+n \leq \max\{P, Q\}$, exist and are continuous on $B \times I$. We write C^L for $C^{L,L}$. By an admissible process $\zeta = \{u_i, \theta, T_i, e_{ij}, \kappa_{ijk}, \tau_{ij}, \mu_{ijk}, \eta, \varepsilon_j, q_i, q_{ij}, Q_i\}$ we mean an ordered array of functions $u_i, \theta, T_i, e_{ij}, \kappa_{ijk}, \tau_{ij}, \mu_{ijk}, \eta, \varepsilon_j, q_i, q_{ij}$ and Q_i defined on $\overline{B} \times [0, \infty)$ with the following properties: (i) $u_i \in C^{4,2}$; $\theta, T_i \in C^{2,1}$; $e_{ij}, \kappa_{ijk} \in C^{2,0}$; $\tau_{ij}, q_j, q_{ij} \in C^{1,0}$; $\mu_{ijk} \in C^{2,0}$; $Q_i \in C^0$; $\eta, \varepsilon_i \in C^{0,1}$ on $B \times I$; (ii) $u_i, \dot{u}_i, \ddot{u}_i, u_{i,j}, u_{i,jk}, \theta, \theta_{,i}, T_i, T_{i,j}, e_{ij}, \kappa_{ijk}, \tau_{ij}, \mu_{ijk}, \eta, \dot{\eta}, \varepsilon_i, \dot{\varepsilon}_i, q_i, q_{ji}, Q_i$, $q_{j,j}$ and $q_{ij,i}$ are continuous on $\overline{B} \times [0, \infty)$. By a solution of the mixed problem we mean an admissible process which satisfies the equations (12), (13), (26), (27), (29) and (30) on $B \times I$, the boundary conditions (35) and the initial conditions (36).

3 Uniqueness

In this section we establish a uniqueness result in the strain gradient theory of thermoelasticity with microtemperatures. We consider the admissible process $\zeta = \{u_i, \theta, T_i, e_{ij}, \kappa_{ijk}, \tau_{ij}, \mu_{ijk}, \eta, \varepsilon_j, q_i, q_{ij}, Q_i\}$ and introduce the functions W_ζ, K_ζ and $\mathcal{D}_\zeta$ on $B \times I$, defined by

$$\begin{aligned} 2W_\zeta &= A_{ijrs} e_{ij} e_{rs} + 2B_{ijpqr} e_{ij} \kappa_{pqr} + C_{ijkpqr} \kappa_{ijk} \kappa_{pqr}, \\ 2K_\zeta &= a\theta^2 + 2b_i T_i \theta + D_{ij} T_i T_j, \\ \Pi_\zeta &= k_{ij} \theta_{,i} \theta_{,j} + (H_{ji} + T_0 K_{ij}) \theta_{,j} T_i + T_0 \Lambda_{ij} T_i T_j + T_0 P_{ijrs} T_{j,i} T_{s,r}. \end{aligned} \tag{41}$$

The entropy inequality implies that Π_ζ is positive semidefinite,

$$\Pi_\zeta \geq 0, \tag{42}$$

for any admissible process ζ.

Theorem 3.1. *Assume that*
(i) W_ζ is a positive semidefinite quadratic form;
(ii) ρ is strictly positive;
(iii) K_ζ is a positive definite quadratic form for any admissible process ζ;
(iv) the symmetry relations (25) hold.

Then, the mixed problem of thermoelasticity has at most one solution.

Proof. By using the constitutive equations (26) and the notations (41) we get

$$\tau_{ij}\dot{e}_{ij} + \mu_{ijk}\dot{\kappa}_{ijk} + \rho\dot{\eta}\theta - \rho\dot{\varepsilon}_i T_i = \dot{W}_\zeta + \dot{K}_\zeta. \tag{43}$$

On the other hand, with the help of relations (12), (13) and (29) we find that

$$\begin{aligned}
&\tau_{ij}\dot{e}_{ij} + \mu_{ijk}\dot{\kappa}_{ijk} + \rho\dot{\eta}\theta - \rho\dot{\varepsilon}_i T_i = \\
&= [(\tau_{ji} - \mu_{kji,k})\dot{u}_i + \mu_{jik}\dot{u}_{k,i} + \frac{1}{T_0}q_j\theta - T_i q_{ji}]_{,j} - \\
&- (\tau_{ji,j} - \mu_{kji,kj})\dot{u}_i + \frac{1}{T_0}\rho S\theta - \rho G_i T_i - \frac{1}{T_0}q_j\theta_{,j} + +T_{i,j}q_{ji} - T_i(q_i - Q_i).
\end{aligned}$$

In view of (27), (30) and (41) we obtain

$$\begin{aligned}
&\tau_{ij}\dot{e}_{ij} + \mu_{ijk}\dot{\kappa}_{ijk} + \rho\dot{\eta}\theta - \rho\dot{\varepsilon}_i T_i = \\
&= (t_{ji}\dot{u}_i + \mu_{jik}\dot{u}_{k,i} + \frac{1}{T_0}q_j\theta - T_i q_{ji})_{,j} - \rho\ddot{u}_i\dot{u}_i + \rho f_i\dot{u}_i + \\
&+ \frac{1}{T_0}\rho S\theta - \rho G_i T_i - \frac{1}{T_0}\Pi_\zeta.
\end{aligned} \tag{44}$$

From (43) and (44) we get

$$\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}(\rho\dot{u}_i\dot{u}_i + 2W_\zeta + 2K_\zeta) = (t_{ji}\dot{u}_i + \mu_{jik}\dot{u}_{k,i} + \frac{1}{T_0}q_j\theta - T_i q_{ji})_{,j} + \\
&+ \rho(f_i\dot{u}_i + \frac{1}{T_0}S\theta - \rho G_i T_i) - \frac{1}{T_0}\Pi_\zeta.
\end{aligned} \tag{45}$$

Let us introduce the function U_ζ defined on $[0, \infty)$ by

$$U_\zeta = \int_B (\frac{1}{2}\rho\dot{u}_i\dot{u}_i + W_\zeta + K_\zeta)dv. \tag{46}$$

If we integrate the relation (45) over B and use the divergence theorem and relations (3), (31)-(33), then we obtain

$$\begin{aligned}
\dot{U}_\zeta = &\int_{\partial B}(P_i\dot{u}_i + R_i D\dot{u}_i + \frac{1}{T_0}q\theta - \Lambda_i T_i)da \\
&+ \int_C \Omega_i\dot{u}_i dl + \int_B \rho(f_i\dot{u}_i + \frac{1}{T_0}S\theta - G_i T_i)dv - \frac{1}{T_0}\int_B \Pi_\zeta dv.
\end{aligned} \tag{47}$$

Suppose that there are two solutions of the mixed problem, $\zeta^{(\alpha)} = \{u_i^{(\alpha)}, \theta^{(\alpha)}, T_i^{(\alpha)}, e_{ij}^{(\alpha)}, \kappa_{ijk}^{(\alpha)}, \tau_{ij}^{(\alpha)}, \mu_{ijk}^{(\alpha)}, \eta^{(\alpha)}, \varepsilon_j^{(\alpha)}, q_i^{(\alpha)}, q_{ij}^{(\alpha)}, Q_i^{(\alpha)}\}$, $(\alpha = 1, 2)$. We denote $u_i^* = u_i^{(1)} - u_i^{(2)}$, $\theta^* = \theta^{(1)} - \theta^{(2)}$, $T_i^* = T_i^{(1)} - T_i^{(2)}$, $e_{ij}^* = e_{ij}^{(1)} - e_{ij}^{(2)}$, $\kappa_{ijk}^* = \kappa_{ijk}^{(1)} - \kappa_{ijk}^{(2)}$, $\tau_{ij}^* = \tau_{ij}^{(1)} - \tau_{ij}^{(2)}$, $\mu_{ijk}^* = \mu_{ijk}^{(1)} - \mu_{ijk}^{(2)}$, $\eta^* = \eta^{(1)} - \eta^{(2)}$, $\varepsilon_{ij}^* = \varepsilon_{ij}^{(1)} - \varepsilon_{ij}^{(2)}$, $q_i^* = q_i^{(1)} - q_i^{(2)}$, $q_{ij}^* = q_{ij}^{(1)} - q_{ij}^{(2)}$, $Q_i^* = Q_i^{(1)} - Q_i^{(2)}$. Then, the process $\vartheta = \{u_i^*, \theta^*, T_i^*, e_{ij}^*, \kappa_{ijk}^*, \tau_{ij}^*, \mu_{ijk}^*, \eta^*, \varepsilon_j^*, q_i^*, q_{ij}^*, Q_i^*\}$ corresponds to null data. From (42) and (47) we conclude that

$$\dot{U}_\zeta \leq 0 \quad \text{on} \quad [0, \infty). \tag{48}$$

With the help of initial data we find that $U_\vartheta \leq 0$ on I. In view of hypotheses of the theorem, from (48), we obtain $\dot{u}_i^* = 0$, $\theta^* = 0$ and $T_i^* = 0$ on I. By using the initial data we get $u_i^* = 0$, $\theta^* = 0$ and $T_i^* = 0$ on I, and the proof is complete.□

Uniqueness results in thermoelastodynamics have been established in various works (see, e.g., Ieşan, 2004).

4 An existence theorem

In this section we consider the case of isotropic and homogeneous bodies and use a semigroup approach (see Goldstein, 1985) to obtain an existence result in the dynamical theory.

We assume that the boundary ∂B is smooth and consider the following homogeneous boundary conditions

$$u_i = Du_i = T_i = \theta = 0 \text{ on } \partial B. \tag{49}$$

We assume that the initial conditions (36) hold.

We introduce the mechanical internal energy by

$$\begin{aligned} \widehat{\sigma} = & \frac{1}{2}\lambda e_{rr} e_{ss} + \mu e_{ij} e_{ij} + \alpha_1 \kappa_{iik}\kappa_{kjj} + \alpha_2 \kappa_{ijj}\kappa_{ikk} \\ & + \alpha_3 \kappa_{iik}\kappa_{jjk} + \alpha_4 \kappa_{ijk}\kappa_{ijk} + \alpha_5 \kappa_{ijk}\kappa_{kji}. \end{aligned} \tag{50}$$

In this section we assume that:

(i) ρ, a and b are positive constants;

(ii) the following inequalities

$$k > 0,\ 3k_4 + k_5 + k_6 > 0,\ k_5 + k_6 > 0,\ k_5 - k_6 > 0,\ (k_1 + T_0 k_3)^2 < 4T_0 k k_2, \tag{51}$$

are satisfied.

(iii) the function $\widehat{\sigma}$ is a positive definite quadratic form.

Let $W_0^{2,2}$, $W^{4,2}$ and L^2 be the usual Hilbert spaces and denote

$$\mathcal{Z}=\{(\mathbf{u},\mathbf{v},\theta,\mathbf{T}),\mathbf{u}\in\mathbf{W}_0^{2,2}(B),\ \mathbf{v},\mathbf{T}\in\mathbf{L^2(B)},\ \theta\in\mathbf{L^2(B)}\},$$

where $\mathbf{W}_0^{2,2}=[W_0^{2,2}]^3$ and $\mathbf{L}^2=[L^2]^3$.

We introduce the operators

$$\begin{aligned}
&M_i\mathbf{u}=\rho^{-1}[(\mu-\nu_1\triangle)\triangle u_i+(\lambda+\mu-\nu_2\triangle)u_{j,ji}],\\
&N_i\mathbf{T}=\rho^{-1}(-\xi_1\triangle T_i-2\xi_2T_{j,ji}),\ \widehat{P}_i\theta=\rho^{-1}\beta\theta_{,i},\\
&R\mathbf{v}=-a^{-1}\beta v_{r,r},\ X\theta=(aT_0)^{-1}k\triangle\theta,\\
&U\mathbf{T}=(aT_0)^{-1}k_1T_{j,j},\ V_i\mathbf{v}=b^{-1}(\xi_1\triangle v_i+2\xi_2v_{j,ji},\\
&W_i\theta=-b^{-1}k_3\theta_{,i},\\
&Z_i\mathbf{T}=b^{-1}(k_6\triangle T_i+(k_4+k_5)T_{j,ji}-k_2T_i).
\end{aligned}$$

Let us consider the matrix operator $\mathcal{A}$ defined on $\mathcal{Z}$ by

$$\begin{pmatrix}\mathbf{0}&\mathbf{Id}&\mathbf{0}&\mathbf{0}\\ \mathbf{M}&\mathbf{0}&\widehat{\mathbf{P}}&\mathbf{N}\\ 0&R&X&U\\ \mathbf{0}&\mathbf{V}&\mathbf{W}&\mathbf{Z}\end{pmatrix},\tag{52}$$

where $\mathbf{Id}$ is the identity operator , $\mathbf{M}=(M_i),\mathbf{N}=(N_i),\widehat{\mathbf{P}}=(\widehat{P}_i)$, $\mathbf{V}=(V_i)$, $\mathbf{W}=(W_i)$ and $\mathbf{Z}=(Z_i)$.

The domain $\mathcal{D}$ of the operator $\mathcal{A}$ is the set

$$\{(\mathbf{u},\mathbf{v},\theta,\boldsymbol{T}),\ \text{such that}\ \mathcal{A}\begin{pmatrix}\mathbf{u}\\ \mathbf{v}\\ \theta\\ \mathbf{T}\end{pmatrix}\in\mathcal{Z},\ \theta=T_i=0\ \text{on}\ \partial B\}.$$

We note that

$$\mathbf{W}_0^{2,2}\cap\mathbf{W}^{4,2}\times\mathbf{W}_0^{2,2}\times W_0^{1,2}\cap W^{2,2}\times\mathbf{W}_0^{1,2}\cap\mathbf{W}^{2,2}$$

is a dense subspace of the Hilbert space $\mathcal{Z}$ which is contained in $\mathcal{D}$. Therefore the domain of the operator is dense.

The boundary-initial-value problem can be transformed into the following abstract equation in the space $\mathcal{Z}$

$$\frac{d\omega}{dt}=\mathcal{A}\omega+F(t),\quad \omega(0)=\omega_0,\tag{53}$$

where $F(t) = (\mathbf{0}, \mathbf{f}, (aT_0)^{-1}\rho S, -b^{-1}\rho \mathbf{G})$, $\omega_0 = (\mathbf{u}^0, \mathbf{v}^0, \theta^0, \boldsymbol{T}^0)$.

We introduce in the Hilbert space the following inner product

$$< (\mathbf{u}, \mathbf{v}, \theta, \boldsymbol{T}), (\mathbf{u}^*, \mathbf{v}^*, \theta^*, \boldsymbol{T}^*) >= \int_B (\rho v_i v_i^* + a\theta\theta^* + bT_iT_i^* + 2W^*)\, dv, \tag{54}$$

where

$$\begin{aligned} 2W^* =& \lambda u_{r,r}u^*_{j,j} + 2\mu u_{(i,j)}u^*_{(i,j)} + \alpha_1(u_{j,rr}u^*_{k,jk} + u_{k,jk}u^*_{j,rr}) \\ &+ 2\alpha_2 u_{j,ij}u^*_{k,ik} + 2\alpha_3 u_{k,ii}u^*_{k,jj} + 2\alpha_4 u_{k,ij}u^*_{k,ij} + \\ &+ 2\alpha_5 u_{j,ik}u^*_{k,jk}, \end{aligned}$$

$u_{(i,j)} = (u_{i,j} + u_{j,i})/2$.

It is worth noting that this inner product is equivalent to the usual one in the Hilbert space. It defines the norm

$$||\omega||^2 = \int_B (\rho v_i v_i + a\theta\theta + bT_iT_i + 2\widehat{\sigma})\, dv. \tag{55}$$

We also note that for every $\omega \in \mathcal{D}$, we have

$$< \mathcal{A}\omega, \omega >= -\int_B D^* dv, \tag{56}$$

where

$$D^* = \frac{k}{T_0}\theta_{,i}\theta_{,i} + (\frac{k_1}{T_0} + k_3)T_i\theta_{,i} + k_5T_{i,j}T_{j,i} + k_4T_{i,i}T_{j,j} + k_6T_{i,j}T_{i,j} + k_2T_iT_i. \tag{57}$$

In view of the conditions (ii) we see that

$$< \mathcal{A}\omega, \omega > \leq 0, \tag{58}$$

for every $\omega \in \mathcal{D}$.

Lemma 4.1. *Suppose that hypotheses (i)-(iii) hold. Let $\rho(\mathcal{A})$ be the resolvent of $\mathcal{A}$. Then, $0 \in \rho(\mathcal{A})$.*

Proof. Let us show that we can find $\omega = (\mathbf{u}, \mathbf{v}, \theta, \boldsymbol{T}) \in \mathcal{D}$ such that

$$\mathcal{A}\omega = \mathcal{F}, \tag{59}$$

for any $\mathcal{F} = (\mathbf{f}_1, \mathbf{f}_2, f_3, \mathbf{f}_4) \in \mathcal{Z}$. In terms of the components we get

$$\mathbf{v} = \mathbf{f}_1, \;\; \mathbf{M}\mathbf{u} + \widehat{\mathbf{P}}\theta + \mathbf{N}\mathbf{T} = \mathbf{f}_2, R\mathbf{v} + X\theta + U\mathbf{T} = f_3 \tag{60}$$

$$\mathbf{V}\mathbf{v} + \mathbf{W}\theta + \mathbf{Z}\boldsymbol{T} = f_4. \tag{61}$$

From these equation we see that $\mathbf{v} \in \mathbf{W}_0^{2,2}$ and we can write the system

$$\mathbf{M}\mathbf{u} + \widehat{\mathbf{P}}\theta + \mathbf{N}\mathbf{T} = \mathbf{f}_2, X\theta + U\mathbf{T} = f_3 - R\mathbf{f}_1, \mathbf{W}\theta + \mathbf{Z}\boldsymbol{T} = f_4 - \mathbf{V}\mathbf{v}. \tag{62}$$

To study last two equations of this system we define the bilinear form:

$$\mathcal{B}[(\theta, \boldsymbol{T}), (\theta^*, \boldsymbol{T}^*)] = \int_B I dv,$$

where

$$I = a(X\theta + U\mathbf{T})\theta^* + b(W_i\theta + Z_i\boldsymbol{T})T_i^*.$$

After the use of the divergence theorem we see that this is a bounded bilinear form defined in $W_0^{1,2} \times \mathbf{W}_0^{1,2}$ which is coercive. The right-hand side belongs to $W^{-1,2} \times \mathbf{W}^{-1,2}$. The solution of this system is guarantee on the basis of the Lax-Milgram theorem (see Gilbarg and Trudinger, 1983). Consequently, there exists $(\theta, \mathbf{T}) \in W_0^{1,2} \times \mathbf{W}_0^{1,2}$ satisfying the last two equations of the system (62). Then, we can solve the first equation

$$\mathbf{M}\mathbf{u} = \mathbf{f}_2 - (\widehat{\mathbf{P}}\theta + \mathbf{N}\mathbf{T}). \tag{63}$$

Thus, we conclude that the equation (59) has a solution in the domain of the operator and the lemma is proved. □

Theorem 4.1. *Suppose that hypotheses (i)-(iii) hold. Then the operator $\mathcal{A}$ is the generator of a C^0-semigroup of contractions in the Hilbert space $\mathcal{Z}$.*

Proof. The proof is a direct consequence of the Lumer-Phillips theorem, since the operator $\mathcal{A}$ is dissipative, with a dense domain and $0 \in \rho(\mathcal{A})$ (see Liu and Zheng, 1999).

Now, we can state the main result of this section.

Theorem 4.2. *Suppose that hypotheses (i)-(iii) hold. Let $F(t) \in C^1(\mathbb{R}^+, \mathcal{Z}) \cap C^0(\mathbb{R}^+, \mathcal{D})$ and $\omega_0 \in \mathcal{D}$. Then, there exists a unique solution $\omega(t) \in C^1(\mathbb{R}^+, \mathcal{Z}) \cap C^0(\mathbb{R}^+, \mathcal{D})$ to the problem (53).*

5 Analyticity of solutions

In this section we consider an isotropic and homogeneous body which occupies the layer defined by

$$B = \{(x_1, x_2, x_3) : 0 < x_1 < L, |x_2| < \infty,\ |x_3| < \infty\}.$$

We suppose that the body loads are absent, so that $f_i = 0$, $S = 0$ and $G_i = 0$. Moreover, we assume that the initial data are independent of coordinates x_2 and x_3. We consider the following boundary conditions

$$u_i = 0, \;\; R_i = 0, \;\; \theta = 0, T_i = 0 \;\text{ for }\; x_1 = 0 \;\text{ and }\; x_1 = L. \tag{64}$$

We say that the layer is subjected to a one-dimensional deformation if the functions u_i, θ and T_i are independent of x_2 and x_3, and $u_2 = u_3 = 0$, $T_2 = T_3 = 0$. Thus, we have

$$u_1 = u(x_1, t), \;\; \theta = \theta(x_1, t), \;\; T_1 = \widehat{T}(x_1, t). \tag{65}$$

In the one-dimensional theory the equations (38) become

$$\begin{aligned} &-\widehat{\nu} u_{,1111} + \widehat{\mu} u_{,11} - \widehat{\xi}\widehat{T}_{,11} - \beta\theta_{,1} = \rho\ddot{u}, \\ &k\theta_{,11} + k_1\widehat{T}_{,1} - \beta T_0 \dot{u}_{,1} = aT_0\dot{\theta}, \\ &k^*\widehat{T}_{,11} + \widehat{\xi}\dot{u}_{,11} - k_2\widehat{T} - k_3\theta_{,1} = b\partial\widehat{T}/\partial t, \end{aligned} \tag{66}$$

where

$$\widehat{\nu} = \nu_1 + \nu_2, \;\; \widehat{\mu} = \lambda + 2\mu, \;\; k^* = k_4 + k_5 + k_6, \;\; \widehat{\xi} = \xi_1 + 2\xi_2.$$

In this section we assume that $\widehat{\nu}, \widehat{\mu}$ and k^* are positive and $\widehat{\xi}$ is different from zero. The conditions (64) reduce to

$$u = 0, \;\; u_{,11} = 0, \;\; \theta = 0, \;\; \widehat{T} = 0 \;\text{ for } x_1 = 0 \text{ and } x_1 = L. \tag{67}$$

The aim of this section is to prove the analyticity of the solutions for the one-dimensional homogeneous version of the system (38). To do that we will use the result (see Liu and Zheng, 1999).

Theorem 5.1. *Let $S(t) = e^{\mathcal{A}(t)}$ be a C_0-semigroup of contraction in a Hilbert space. Suppose that*

$$i\mathbb{R} \subseteq \rho(\mathcal{A}), \tag{68}$$

where $\rho(\mathcal{A})$ is the resolvent of $\mathcal{A}$. Then, $S(t)$ is analytic if and only if

$$\limsup_{|\widetilde{\beta}|\to\infty} ||\widetilde{\beta}(i\widetilde{\beta}\mathcal{I} - \mathcal{A})|| < \infty \tag{69}$$

holds.

It is worth noting that the existence of semigroup obtained in the previous section can be adapted directly to this system with initial conditions

$$u(x,0) = u^0(x), \;\; \dot{u}(x,0) = v^0(x), \;\; \theta(x,0) = \theta^0(x), \widehat{T}(x,0) = T^0(x), \;\; x \in (0, L).$$

In fact, the existence theorem can be established for an associated system formulated in terms of complex valued functions. This is similar to the arguments proposed by Liu and Zheng (1999) for several thermoelastic problems. In this context, we will study now the spectrum of the operator.

We note that in this case the corresponding Hilbert space is

$$\mathcal{Z} = W^{2,2} \cap W_0^{1,2} \times L^2 \times L^2 \times L^2$$

and the domain is

$$\mathcal{D}(\mathcal{A}) = \{(u, v, \theta, \widehat{T}) \in \mathcal{Z},\ v, \theta, \widehat{T} \in W^{2,2} \cap W_0^{1,2}, u_{,11} = 0 \text{ for } x_1 = 0 \text{ and } x_1 = L\}.$$

Before to state our theorem we need a couple of lemmas

Lemma 5.1. *The one-dimensional version of the operator $\mathcal{A}$ defined by (52) satisfies the condition (68).*

Proof. The proof consists in three steps

(a) We now suppose that 0 is in the resolvent of $\mathcal{A}$. The contraction mapping theorem shows that for any real number γ with $\gamma < ||\mathcal{A}^{-1}||^{-1}$, the operator $i\gamma\mathcal{I} - \mathcal{A} = \mathcal{A}(i\gamma\mathcal{A}^{-1} - \mathcal{I})$ is invertible. Moreover, $||(i\gamma\mathcal{I} - \mathcal{A})^{-1}||$ is a continuous function of γ in the interval $(-||\mathcal{A}^{-1}||^{-1}, ||\mathcal{A}^{-1}||^{-1})$.

(b) If $\sup\{||(i\gamma\mathcal{I} - \mathcal{A})^{-1}||, |\gamma| < ||\mathcal{A}^{-1}||^{-1}\} = M < \infty$, then by the contraction theorem, the operator

$$i\gamma\mathcal{I} - \mathcal{A} = (i\gamma_0\mathcal{I} - \mathcal{A})(\mathcal{I} + i(\gamma - \gamma_0)(i\gamma_0\mathcal{I} - \mathcal{A})^{-1}),$$

with $|\gamma_0| < ||\mathcal{A}^{-1}||^{-1}$ is invertible for $|\gamma - \gamma_0| < M^{-1}$. It turns out that by choosing $|\gamma_0|$ as close to $||\mathcal{A}^{-1}||^{-1}$ as we can, the set $\{\gamma, |\gamma| < ||\mathcal{A}^{-1}||^{-1} + M^{-1}\}$ is contained in the resolvent of $\mathcal{A}$ and $||(i\gamma\mathcal{I} - \mathcal{A})^{-1}||$ is a continuous function of γ in the interval $(-||\mathcal{A}^{-1}||^{-1} - M^{-1}, ||\mathcal{A}^{-1}||^{-1} + M^{-1})$.

(c) Thus, it follows from the argument in (b) that if the imaginary axis is not contained in the resolvent, then there is a real number τ with $||\mathcal{A}^{-1}||^{-1} \leq |\tau| < \infty$ such that the set $\{i\gamma, |\gamma| < |\tau|\ \}$ is in the resolvent of $\mathcal{A}$ and $\sup\{||(i\gamma\mathcal{I} - \mathcal{A})^{-1}||, |\gamma| < |\tau|\} = \infty$. Therefore, there exists a sequence of real numbers γ_n with $\gamma_n \to \tau$, $|\gamma_n| < |\tau|$ and a sequence of vectors $\omega_n = (u_n, v_n, \theta_n, \widehat{T}_n)$ in the domain of the operator $\mathcal{A}$ and with unit norm such that

$$||(i\gamma_n\mathcal{I} - \mathcal{A})\omega_n||_{\mathcal{Z}} \to 0 \text{ as } n \to \infty. \tag{70}$$

This is

$$i\gamma_n u_n - v_n \to 0 \text{ in } W^{2,2}, \tag{71}$$

$$i\gamma_n v_n - Mu_n - P\theta_n - N\widehat{T}_n \to 0 \text{ in } L^2, \tag{72}$$

$$i\gamma_n\theta_n - Rv_n - X\theta_n - U\widehat{T}_n \to 0 \text{ in } L^2, \tag{73}$$

$$i\gamma_n\widehat{T}_n - Vv_n - W\theta_n - Z\widehat{T}_n \to 0 \text{ in } L^2, \tag{74}$$

By (56) we see that $\theta_{n,1}$ and $\widehat{T}_{n,1}$ tend to zero in L^2. From (70) we see that

$$Vv_n + W\theta_n + Z\widehat{T}_n \to 0 \text{ in } L^2, \tag{75}$$

As the spatial derivative of u_n is bounded in L^2 we obtain that

$$\widehat{\xi} < u_{n,1}, v_{n,1} >_{L^2} \to 0. \tag{76}$$

In view of (71) we also have that $u_{n,1}$ tends to zero and then $v_{n,1}$ also tends to zero. The multiplication of (72) by u_n implies that u_n tends to zero in $W^{2,2}$ which contradicts the assumption that ω_n has unit norm.□

Lemma 5.2. *The one-dimensional version of the operator $\mathcal{A}$ defined by (52) satisfies the condition (69).*

Proof. Let us to assume that (59) does not hold. Then there exists a sequence $\gamma_n > 0$ and $\gamma_n \to \infty$; and a sequence of vectors $\omega_n = (u_n, v_n, \theta_n, T_n)$ in the domain of the operator $\mathcal{A}$ and with unit norm such that

$$\lim_{n\to\infty} \gamma_n^{-1}||(i\gamma_n\mathcal{I} - \mathcal{A})\omega_n||_{\mathcal{Z}} = 0. \tag{77}$$

This is

$$\gamma_n^{-1}(i\gamma_n u_n - v_n) \to 0 \text{ in } W^{2,2}, \tag{78}$$

$$\gamma_n^{-1}(i\gamma_n v_n - Mu_n - P\theta_n - N\widehat{T}_n) \to 0 \text{ in } L^2, \tag{79}$$

$$\gamma_n^{-1}(i\gamma_n\theta_n - Rv_n - X\theta_n - U\widehat{T}_n) \to 0 \text{ in } L^2, \tag{80}$$

$$\gamma_n^{-1}(i\gamma_n T_n - Vv_n - W\theta_n - Z\widehat{T}_n) \to 0 \text{ in } L^2. \tag{81}$$

From the dissipation properties of the operator $\mathcal{A}$ and the condition (67) we have that

$$\gamma_n^{-1/2}(||\theta_{n,1}|| + ||\widehat{T}_n|| + ||\widehat{T}_{n,1}||) \to 0. \tag{82}$$

We can write (81) as

$$\gamma_n^{-1}(i\gamma_n\widehat{T}_n - \widehat{\xi}v_{n,11} - k^*\widehat{T}_{n,11}) \to 0 \text{ in } L^2. \tag{83}$$

If we take the inner product by T_n in L^2, after integration by parts, we obtain that

$$i||\widehat{T}_n||^2 + \widehat{\xi} < \gamma_n^{-1/2}v_{n,1}, \gamma_n^{-1/2}\widehat{T}_{n,1} > \to 0. \tag{84}$$

We now that $\gamma_n^{-1}||v_n||_{W^{2,2}}$ is uniformly bounded. On the other hand $||v_n|| \leq 1$. Then by the Gagliardo-Niremberg interpolation inequality we see that ${\gamma_n}^{-1/2}||v_n||_{W^{1,2}}$ is also bounded. Following (82) we obtain that $||\widehat{T}_n|| \to 0$. Working in a similar way with the equation (80) we see that $||\theta_n|| \to 0$.

If we replace $\gamma_n^{-1}\widehat{\xi}v_{n,11}$ by $i\widehat{\xi}u_{n,11}$, we can rewrite (71) as

$$\gamma_n^{-1}(k^*\widehat{T}_{n,11} + i\widehat{\xi}u_{n,11}) \to 0 \text{ in } L^2. \quad (85)$$

Taking the inner product by $u_{n,11}$ and after integration by parts we obtain

$$k^* < \gamma_n^{-1/2}\widehat{T}_{n,1}, \gamma_n^{-1/2}u_{n,111} > +i\widehat{\xi}||u_{,11}||^2 \to 0.$$

To prove that $||u_{,11}||$ tends to zero it is sufficient to show that ${\gamma_n}^{-1/2}u_{n,111}$ is bounded in L^2. To this end we multiply (79) by $u_{n,11}$, to get

$$< iv_n, u_{n,11} > -\widehat{\nu}\gamma_n^{-1}||u_{n,111}||^2 - \widehat{\xi} < \gamma_n^{-1/2}\widehat{T}_{n,1}, \gamma_n^{-1/2}u_{n,111} > \to 0. \quad (86)$$

As $v_n, u_{n,11}$ and ${\gamma_n}^{-1/2}T_{n,1}$ are bounded, it follows that ${\gamma_n}^{-1/2}u_{n,111}$ is bounded in L^2. We see that $u_{n,11}$ tends to zero in L^2. Finally , from (71) we also obtain that v_n tends to zero in L^2 which contradicts that the sequence has unit norm.

Remark. The proposed arguments can be adapted to the case that we assume that the boundary conditions are

$$u = u_{,1} = \theta = \widehat{T} = 0 \text{ on } x_1 = 0, \text{and } \ x_1 = L.$$

The interested reader can find the main ideas in the paper by Liu and Quintanilla(2010). Therefore we can obtain the following result:

Theorem 5.2. *The semigroup is analytic.*

We note that a consequence of this result is the exponential stability of solutions. That is, there exists two positive constant C and τ such that

$$E(t) \leq CE(0)\exp(-\tau t), \quad (87)$$

for every $t \geq 0$, where

$$E(t) = ||(u, v, \theta, \widehat{T})(t)||^2.$$

As the solutions are analytic functions, we have that the only solution which can be identically zero after a finite time is the null solution.

Corollary 5.1. *Let $(u, \theta, \widehat{T})$ be a solution to the boundary-initial value problem that vanishes for every $t \geq t_0$ where $t_0 < \infty$. Then $(u, \theta, \widehat{T})$ is the null solution.*

The importance of the three-dimensional counterpart of this result is clear, but this question remains open.

6 Anti-plane shear deformations

In this section we consider another particular class of solutions for the system (38). These are solutions of the form $u_1 = u_2 = \theta = T_1 = T_2 = 0$ and $u_3 = u(x_1, x_2, t)$, $T_3 = \mathcal{T}(x_1, x_2, t)$. Assuming that the supply terms vanish, the equations (38) reduce to

$$(\mu - \nu_1 \triangle)\triangle u - \xi_1 \triangle \mathcal{T} = \rho \ddot{u}, \quad k_6 \triangle \mathcal{T} + \xi_1 \triangle \dot{u} - b\dot{\mathcal{T}} - k_2 \mathcal{T} = 0. \tag{88}$$

The equations (88) are defined on a two dimensional domain P^* smooth enough to apply the divergence theorem. To the equations (88) we add initial and boundary conditions. We assume that

$$u(x, 0) = u^0(x), \quad \dot{u}(x, 0) = v^0(x), \;\; \mathcal{T}(x, 0) = T^0(x), \;\; x \in P^*, \tag{89}$$

and that

$$u = Du = \mathcal{T} = 0 \text{ on } \partial P^*. \tag{90}$$

We now present a uniqueness result. We assume that: (α) the mass density ρ and the parameter b are strictly positive; (β) the parameters k_2 and k_6 are strictly positive. It is worth remarking that we do not impose any assumption on the parameters μ and ν_1.

First, we note that the energy

$$E(t) = \int_{P^*} (\rho|\dot{u}|^2 + \mu|\nabla u|^2 + \nu_1|\triangle u|^2 + b\mathcal{T}^2)da \tag{91}$$

$$+2\int_0^t \int_{P^*} (k_2\mathcal{T}^2 + k_6|\nabla \mathcal{T}|^2)dads = E(0),$$

is conserved.

Logarithmic convexity argument is based on the choice of a *good* function which satisfies some requirements. We consider several relations to define this function. First we integrate with respect to time the second equation of (88). We have

$$\int_0^t (k_6 \triangle \mathcal{T} - k_2 \mathcal{T})ds + \xi_1 \triangle u = b\mathcal{T} + \xi_1 \triangle u^0 - bT^0. \tag{92}$$

Now, we define Q as the solution to the Poisson equation

$$k_6 \triangle Q - k_2 Q = bT^0 - \xi_1 \triangle u^0, \tag{93}$$

subject to the boundary condition $Q = 0$ on ∂P^*. We note that the existence of such function is guaranteed by the classical results for the elliptic equations (see Gilbarg and Trudinger 1983). If we denote

$$\widehat{\beta} = \psi + Q, \text{ where } \psi = \int_0^t \mathcal{T}(s)ds, \tag{94}$$

then we obtain that the second equation on (88) can be written as

$$k_6 \triangle \widehat{\beta} - k_2 \widehat{\beta} + \xi_1 \triangle u = b\mathcal{T}. \tag{95}$$

We now define the function

$$F_{\omega,t_1} = \int_P \left(\rho |u|^2 + \int_0^t (k_6 |\nabla \widehat{\beta}|^2 + k_2 \widehat{\beta}^2) ds \right) da + \omega(t + t_1)^2, \tag{96}$$

where ω and t_1 are two constants to select later. In what follows, for the sake of simplicity, we shall use the notation $F_{\omega,t_1} = f$. We proceed to compute the first two derivatives of $f(t)$. Thus

$$\dot{f} = 2 \int_{P^*} \rho u \dot{u} \, da + \int_P^* (k_6 |\nabla \widehat{\beta}|^2 + k_2 \widehat{\beta}^2) dv + 2\omega(t + t_1). \tag{97}$$

$$\ddot{f} = 2 \int_{P^*} (\rho |\dot{u}|^2 + \rho u \ddot{u}) da + 2 \int_{P^*} (k_6 \widehat{\beta}_{,i} \mathcal{T}_{,i} + k_2 \widehat{\beta} \mathcal{T}) da + 2\omega. \tag{98}$$

If we substitute into the first equation of (88) and (95) and apply the divergence theorem we obtain that

$$\ddot{f} = 2 \int_{P^*} (\rho |\dot{u}|^2 - \mu |\nabla u|^2 - \nu_1 |\triangle u|^2 - b\mathcal{T}^2) da + 2\omega \tag{99}$$

In view of the energy equation (91), we get

$$\ddot{f} = 4 \int_{P^*} (\rho |\dot{u}|^2 + \int_0^t (k_6 |\nabla \mathcal{T}|^2 + k_2 \mathcal{T}^2) ds da + 2(\omega - E(0)). \tag{100}$$

It is also worth noting that (97) can be also written as

$$\dot{f} = 2 \int_{P^*} \rho u \dot{u} da + 2 \int_0^t \int_{P^*} (k_6 \widehat{\beta}_{,i} \mathcal{T}_{,i} + k_2 \widehat{\beta} \mathcal{T}) da ds \tag{101}$$

$$+ \int_P (k_6 Q_{,i} Q_{,i} + k_2 |Q|^2) da + 2\omega(t + t_1).$$

From (96), (101) and (100) we obtain that

$$f\ddot{f} - (\dot{f} - \frac{\Gamma}{2})^2 \geq -2(E(0) + \omega)f, \tag{102}$$

where

$$\Gamma = 2\int_{P^*} (k_6 Q_{,i} Q_{,i} + k_2 |Q|^2) da. \tag{103}$$

Inequality (102) is fundamental in our analysis. In the case of null initial data we get $E(0) = 0$ and $\Gamma = 0$. If we takes $\omega = t_1 = 0$ the inequality (102) becomes

$$F\ddot{F} - (\dot{F})^2 \geq 0, \tag{104}$$

where we have used the notation and we use $F(t)$ for the function $F_{0,0}(t)$. From (104) we obtain the estimate

$$F(t) \leq F(0)^{1-\frac{t}{t^*}} F(T)^{\frac{t}{t^*}}, \ \ 0 \leq t \leq t^*. \tag{105}$$

We then conclude that $F(t)$ vanishes for $0 \leq t \leq t^*$ and we get a uniqueness result. In the general case we obtain that

$$f\ddot{f} - (\dot{f})^2 \geq -\Gamma \dot{f}, \tag{106}$$

where we have selected $\omega = -E(0)$. The inequality (106) implies that

$$\frac{d}{dt}\left(\frac{\dot{f}}{f}\right) \geq -\Gamma \frac{\dot{f}}{f^2}.$$

Therefore, we see that the function

$$\frac{\dot{f} - \Gamma}{f}$$

is increasing in time. In particular we see that

$$\frac{\dot{f}(t) - \Gamma}{f(t)} \geq \frac{\dot{f}(0) - \Gamma}{f(0)}.$$

Now, we select the arbitrary positive constant t_1 to be large enough to satisfy $\dot{f}(0) - \Gamma > 0$. After a quadrature we obtain

$$f(t) \geq \frac{f(0)\dot{f}(0)}{\dot{f}(0) - \Gamma} \exp\left(\frac{\dot{f}(0) - \Gamma}{f(0)}\right) t - \frac{\Gamma f(0)}{\dot{f}(0) - \Gamma}.$$

Thus, we have established that $F(t)$ grows exponentially for large time. We have proved the following result:

Theorem 6.1. *Under the assumptions made at begining of this section, we have*:

(i) *The first boundary value problem has at most one solution;*

(ii) *If $E(0) < 0$, then the solutions becomes unbounded in an exponential way.*

7 Thermoelastostatics. Concentrated heat source

The fundamental system of field equations for the time-independent behaviour of a thermoelastic solid consists of the equations of equilibrium

$$\tau_{ji,j} - \mu_{kji,kj} + \rho f_i = 0, \tag{107}$$

the balance of energy

$$q_{j,j} + \rho S = 0, \tag{108}$$

the balance of the first moment of energy

$$q_{ji,j} + q_i - Q_i + \rho G_i = 0, \tag{109}$$

the constitutive equations (26) and (27), and the geometrical equations (12). To the basic equations we have to add boundary conditions. Let us assume that the boundary ∂B is smooth. The first boundary-value problem of thermoelastostatics is characterized by the following boundary conditions

$$u_i = \widetilde{u}_i,\ Du_i = \widetilde{d}_i,\ \theta = \widetilde{\theta},\ T_i = \widetilde{T}_i \ \text{ on } \partial B, \tag{110}$$

where $\widetilde{u}_i, \widetilde{d}_i, \widetilde{\theta}$ and $\widetilde{T}_i$ are prescribed functions. In the second boundary-value problem the boundary conditions are

$$P_i = \widetilde{P}_i,\ \ R_i = \widetilde{R}_i,\ \ q_i n_i = \widetilde{q},\ \ q_{ji} n_j = \widetilde{\Lambda}_i \ \text{ on } \ \partial B, \tag{111}$$

where the functions $\widetilde{P}_i, \widetilde{R}_i, \widetilde{q}$ and $\widetilde{\Lambda}_i$ are given. We note that the uniqueness theorems presented by Mindlin and Tiersten (1968) and Ieşan (2007) can be used to obtain uniqueness results in thermoelastostatics. By a rigid state we mean an ordered array of functions $\{u_i^*, \theta^*, T_i^*\}$ of the form

$$u_i^* = a_i^{(1)} + \varepsilon_{ijk} a_j^{(2)} x_k,\ \ \theta^* = a^{(3)},\ \ T_i^* = 0,$$

where $a_i^{(1)}, a_i^{(2)}$ and $a^{(3)}$ are arbitrary constants. As in Section 3 we can prove the following result.

Theorem 7.1. *Assume that W_ζ and Π_ζ are positive definite quadratic forms. Then,* *(i) the first boundary-value problem has at most one solution;*

(ii) any two solutions of the second boundary-value problem are equal modulo a rigid state.

In the equilibrium theory of homogeneous and isotropic solids the equations (38) become

$$\begin{aligned}&(\mu-\nu_1\Delta)\Delta u_i+(\lambda+\mu-\nu_2\Delta)u_{j,ji}-\xi_1\Delta T_i-2\xi_2 T_{j,ji}-\beta\theta_{,i}=-\rho f_i,\\&k\Delta\theta+k_1T_{j,j}=-\rho S,\\&k_6\Delta T_i+(k_4+k_5)T_{j,ji}-k_2T_i-k_3\theta_{,i}=\rho G_i.\end{aligned}\tag{112}$$

We note that in the equilibrium theory we can first study the problem of finding the functions θ and T_i, and then the problem of finding the displacements u_j.

Mindlin (1964) established a general solution of the displacement equations in gradient elastostatics and used it to derive the solution to the problem of a concentrated force acting in an infinite region.

In what follows we study a special problem of thermoelastostatics. We investigate the effects of a concentrated heat source acting in an isotropic and homogeneous body that occupies the entire three-dimensional space. First, we assume that

$$f_i=0,\ \ \rho S=\Lambda(r),\ \ G_i=0,\tag{113}$$

where Λ is a prescribed function, $r=[(x_i-y_i)(x_i-y_i)]^{1/2}$, and (y_1,y_2,y_3) is a fixed point. The conditions at infinity are

$$\begin{aligned}&u_i=O(1),\ \theta=O(r^{-1}),\ \ T_i=O(r^{-1}),\\&u_{i,j}=O(r^{-1}),\ u_{i,jk}=O(r^{-2}),\ \theta_{,i}=O(r^{-2}),\ T_{i,j}=O(r^{-2}).\end{aligned}\tag{114}$$

We seek the solution of the system (114), in the form

$$u_i=\Phi_{,i},\ \ \theta=\chi,\ \ T_i=\Psi_{,i},\tag{115}$$

where Φ,χ and Ψ are unknown functions which depend only on the variable r. The equations are satisfied if the functions Φ,χ and ψ satisfy the following equations

$$\begin{aligned}&[\lambda+2\mu-(\nu_1+\nu_2)\Delta]\Delta\Phi-(\xi_1+2\xi_2)\Delta\Psi-\beta\chi=0,\\&k\Delta\chi+k_1\Delta\Psi=-M,\\&(k_4+k_5+k_6)\Delta\psi-k_2\Psi-k_3\chi=0.\end{aligned}\tag{116}$$

We introduce the notations

$$S_1 = \lambda + 2\mu - (\nu + \nu)\Delta = -(\nu_1 + \nu_2)(\Delta - p^2),$$
$$S_2 = k(k_4 + k_5 + k_6)(\Delta - s^2),\ S_3 = (S_2 - k_1 k_3)/k, \tag{117}$$
$$p = \left(\frac{\lambda + 2\mu}{\nu_1 + \nu_2}\right)^{1/2},\quad s = \left[\frac{kk_2 - k_1 k_3}{k(k_4 + k_5 + k_6)}\right]^{1/2}.$$

Let

$$\Phi = -[\beta S_3 + k_3(\xi_1 + 2\xi_2)\Delta]\Omega,$$
$$\chi = -S_1 S_3 \Delta\Omega, \tag{118}$$
$$\Psi = -k_3 S_1 \Delta\Omega,$$

where Ω is a function of class C^8 which satisfies the equation

$$S_1 S_2 \Delta\Delta\Omega = M. \tag{119}$$

It is easy to see that the functions Φ, χ and Ψ satisfy the equations (116).

The equation (119) can be written in the form

$$(\Delta - p^2)(\Delta - s^2)\Delta\Delta\Omega = -M_0, \tag{120}$$

where $M_0 = M/[k(\nu_1 + \nu_2)(k_4 + k_5 + k_6)]$. Let us consider the functions Ω_k, $(k = 1, 2, 3, 4)$, that satisfy the equations

$$(\Delta - p^2)\Omega_1 = -M_0,\ (\Delta - s_2)\Omega_2 = -M_0,\ \Delta\Delta\Omega_3 = -M_0,\ \ \Delta\Omega_4 = -M_0. \tag{121}$$

We can see that the solution of the equation (120) can be expressed in the form

$$\Omega = \sum_{j=1}^{4} c_j \Omega_j, \tag{122}$$

where

$$c_1 = \frac{1}{p^2(^2 - s^2)},\ c_2 = \frac{1}{s(s^2 - p^2)},\ c_3 = \frac{1}{p^2 s^2},\ c_4 = \frac{p^2 + s^2}{p^4 s^4}. \tag{123}$$

We now investigate the effect of a concentrated heat source. Let us assume that $M = \delta(\boldsymbol{x} - \boldsymbol{y})$ where $\delta(\cdot)$ is the Dirac delta and $\boldsymbol{y}$ is a fixed point. In this case we get

$$\Omega_1 = \frac{q_0}{4\pi r}\exp(-pr),\ \Omega_2 = \frac{q_0}{4\pi r}\exp(-sr),\ \Omega_3 = \frac{q_0 r}{8\pi},\ \Omega_4 = \frac{q_0}{4\pi r}, \tag{124}$$

where $q_0 = [k(\nu_1+\nu_2)(k+k_5+k_6)]^{-1}$. Thus, from (122) we obtain

$$\Omega = \frac{q_0}{4\pi p^4 s^4 (p^2-s^2) r}[s^4\exp(-pr) - p^4\exp(-sr)] + \frac{q_0}{8\pi p^4 s^4 r}[p^2 s^2 r^2 + 2(p^2+s^2)]. \tag{125}$$

It follows from (118) and (125) that

$$\begin{aligned} \Phi &= \frac{q_0}{4\pi r}[c_{11}\exp(-pr) + c_{12}\exp(-sr) + c_{13}r + c_{14}],\\ \chi &= \frac{(\nu_1+\nu_2)q_0}{4\pi r}[c_{21}\exp(-sr) + c_{22}],\\ \Psi &= -\frac{(\nu_1+\nu_2)q_0 k_3}{4\pi s^2 r}[p^{-2}\exp(-sr) - 1], \end{aligned} \tag{126}$$

where

$$\begin{aligned} c_{11} &= \frac{A}{p^2 s^2 (s^2+p^2)} + \frac{k_2\beta}{p^4(p^2-s^2)}, A = \beta(k_4+k_5+k_6) + k_3(\xi_1+2\xi_2),\\ c_{12} &= \frac{A}{s^2(p^2-s^2)} - \frac{k^2\beta}{s^4(p^2-s^2)}, c_{13} = \frac{k_2\beta}{2p^2s^2}, c_{14} = \frac{k_2\beta(p^2+s^2)}{p^4s^4},\\ c_{21} &= k_4+k_5+k_6-k_2s^{-2},\ c_{22} = k_2 s^{-2}. \end{aligned}$$

The displacement vector, thermal field and the microtemperatures corresponding to the concentrated heat source are given by (115) and (126).

8 Conclusions

The original results established in this paper can be summarized as follows:

(a) We establish the basic equations of the strain gradient theory of thermoelastic materials whose microelements possess microtemperatures.

(b) In the dynamic theory we establish existence, uniqueness and continuous dependence results by means of the semigroup theory.

(c) We study the one-dimensional theory and establish the analyticity of solutions. Exponential stability and impossibility of localization are consequences of this result.

(d) We study the anti-plane problem and derive uniqueness and instability results without assuming the positivity of the mechanical energy.

(e) We study the equilibrium theory and investigate the effects of a concentrated heat source in an unbounded body.

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