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ON THE EXISTENCE AND UNIQUENESS IN PHASE-LAG THERMOELASTICITY

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Abstract: This paper is devoted to analyze the phase-lag thermoelasticity problem. We study two different cases and we prove, for each one of them, that the solutions of the problem are determined by a quasi-contractive semigroup. As a consequence, existence, uniqueness and continuous dependence of the solutions are obtained.

Keywords: phase-lag thermoelasticity, quasi-contractive semigroups, existence, uniqueness, continuous dependence.

1. Introduction

In thermoelasticity, it is widely known that the models which use the Fourier constitutive equation to describe the behavior of heat lead to some paradoxes. For example, it has been proved that the thermal perturbations at one point of a solid can be observed instantly at another point of it, anywhere, independently of however distant both points are. This means that the thermal waves propagate with infinite speed. To overcome this drawback of the model and to satisfy the principle of causality, several alternative heat conduction theories have been suggested recently. Each one of them gives rise also to a new thermoelastic theory (without trying to be exhaustive, see, for instance, [5, 12, 13]). The applicability of these new proposed thermoelastic models has been the aim of study of several books [14, 32, 34].

One of these alternative heat conduction theories was suggested by Tzou [33] in 1995. There, the author proposed that the heat flux and the gradient of the temperature have a delay in the constitutive equation. When this consideration is taken into account, it is usual to speak about phase-lag theories. In the aforementioned case, the constitutive equation is given by:

$$q(\mathbf{x}, t+\tau_q) = -k\nabla\theta(\mathbf{x}, t+\tau_\theta), \qquad k>0. \tag{1.1}$$

Here q is the heat flux vector, θ is the temperature and τ_q and τ_θ are the delay parameters which are assumed to be positive. This equation suggests that the temperature gradient established across a material volume at the position $\mathbf{x}$ at time $t+\tau_\theta$ results in a heat flux to flow at a different instant of time $t+\tau_q$. These delays can be understood in terms of the microstructure of the medium. An extension of this theory was proposed in 2007 by Choudhuri [6] using the following constitutive equation

$$q(\mathbf{x}, t+\tau_q) = -\left(k\nabla\theta(\mathbf{x}, t+\tau_\theta) + k^*\nabla\nu(\mathbf{x}, t+\tau_\nu)\right). \tag{1.2}$$

The new variable ν is the thermal displacement and satisfies $\nu_t=\theta$. Constant k^*, that some authors call the rate of thermal conductivity of the medium, is a new parameter which is typical

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of the thermoelastic theories proposed by Green and Naghdi [10, 11], and τ_ν is another delay parameter which is also assumed to be positive.

As many other heat conduction theories, these two new ones are proposed from an intuitive viewpoint. Nevertheless, there is no any thermomechanical foundation for any of them. In fact, it can be proved that if we combine the proposed constitutive laws with the classical energy equation

$$-\operatorname{div} q(x,t) = c\theta_t(x,t), \qquad c>0, \tag{1.3}$$

there exists a sequence of solutions of the form

$$\theta_n(x,t) = \exp(\omega_n t)\Phi_n(x)$$

where the real part of ω_n tends to infinity [7].

This fact implies that there is no continuous dependence of the solutions with respect to the initial conditions and that the associated mathematical problem is *ill-posed* in the sense of Hadamard. Unfortunately, this disagrees with the *a priori* expectation and this theory has a very explosive behavior.

For this reason a big interest has grown to understand the formal Taylor approximations to the phase-lag constitutive equations [3, 16, 17, 19, 20, 22, 25, 26, 27, 28, 29]. It is worth recalling the contribution by Serdyukov *et al.* [30] where the authors discuss the postulates of extended irreversible thermodynamics for several Taylor approximations of the dual-phase-lag heat conducting models. Other recent contributions [2, 8, 9] try to give a thermodynamical basis to these theories.

These alternative propositions allow to obtain the well-posedness of the problems and the stability of solutions provided that certain conditions on the parameters hold. Another way to overcome the ill-posedness is combining the delays with the two temperatures theory [23, 24].

In this paper we also consider Taylor approximations to the general phase-lag theories. Plugging these into the energy equation (1.3), we obtain the heat equation[1]:

$$a_0\theta + a_1\theta^{(1)} + a_2\theta^{(2)} + \cdots + a_n\theta^{(n)} = b_0\Delta\theta + b_1\Delta\theta^{(1)} + \cdots + b_m\Delta\theta^{(m)}. \tag{1.4}$$

where $a_0,\ldots,a_n,b_0,\ldots,b_m$ are constants.

It has been shown [7] that the mathematical problem associated to this equation is again ill-posed in the sense of Hadamard whenever $n-m>2$. Here we analyze two cases for the thermoelastic system associated with this equation. In the next section we propose the problem for $n=m+1$ and we obtain the existence and uniqueness of solutions in Section 3. In Section 4 we prove the existence and uniqueness for $n=m+2$.

It is worth noting that the thermoelastic models that make use of this heat conduction formulation are being studied for its interesting mathematical properties but also for its possible applicability in engineering [15, 31, 21]. In fact, a deep mathematical and physical work are needed to clarify the applicability of these theories. Our contribution is addressed in this line.

In this paper we consider the thermoelastic theory based on the former heat equation (see (2.1)). The aim of the work is to prove the existence and uniqueness of solutions and, to this end, we use the semigroup theory of linear operators. In fact, we prove that the solutions of the problem are

[1] Here and from now on, $g^{(k)}$ denotes the k-th derivative of the function g with respect to the time and, in particular, $g^{(0)}=g$.

generated by a quasi-contractive semigroup. Therefore, continuous dependence of the solutions with respect to the initial data and supply terms is also obtained.

2. The system: case $n = m + 1$

The problem we analyze here is defined on a three-dimensional domain Ω with boundary smooth enough to allow the use of the divergence theorem.

The system of partial differential equations that models the phase-lag thermoelaticity can be obtained following the arguments used by Chandrasekharaiah [5]:

(2.1)
$$\begin{cases} \rho \ddot{u}_i = \left(C_{ijkl}u_{k,l} - \beta_{ij}\theta\right)_{,j} + l_i \\ c\frac{d}{dt}\left(a_0\theta + \cdots + a_m\theta^{(m)}\right) + \beta_{ij}\left(a_0 v_{i,j} + \cdots + a_m v_{i,j}^{(m)}\right) = \left(b_{ij}^0\theta_{,i} + b_{ij}^1\theta_{,i}^{(1)} + \cdots + b_{ij}^m\theta_{,i}^{(m)}\right)_{,j} + S \end{cases}$$

Here C_{ijkl} is the elasticity tensor satisfying the major symmetry $C_{ijkl} = C_{klij}$, $\rho > 0$ is the mass density, $c > 0$ is the thermal capacity constant, β_{ij} is the coupling tensor and the tensors b_{ij}^l are symmetric, that is $b_{ij}^l = b_{ji}^l$ for $l = 0, ..., m$. Finally, l_i and S are the supply terms. As usual, $\dot{u}_i = v_i$, and we will write $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{l} = (l_1, l_2, l_3)$.

To simplify the analysis, through the paper, we assume that all the constitutive coefficients are constant and, in consequence, they do not depend on the material points. However, the extension to the non-homogeneous case does not seem difficult.

We will assume the following three conditions:

I. Coefficient a_m is strictly positive.
II. The matrix b_{ij}^m is positive definite, that is, there exists a positive constant M such that
$$b_{ij}^m\xi_i\xi_j \geq M\xi_i\xi_i$$
for every vector (ξ_i).
III. The elasticity tensor C_{ijkl} is positive definite, that is, there exists a positive constant C such that
$$\int_\Omega C_{ijkl}\xi_{ij}\xi_{kl}\,dV \geq C\int_\Omega \xi_{ij}\xi_{ij}\,dV$$
for every tensor (ξ_{ij}).

To have a well-posed problem we need to impose initial and boundary conditions. As initial conditions we take

$$u_i(\mathbf{x},0) = u_i^0(\mathbf{x}), \quad \dot{u}_i(\mathbf{x},0) = v_i^0(\mathbf{x}) \text{ for } i = 1,2,3 \text{ and}$$
$$\theta^{(k)}(\mathbf{x},0) = \theta_0^{(k)}(\mathbf{x}) \text{ for } k = 0,1,...,m-1. \tag{2.2}$$

And we consider null Dirichlet boundary conditions:

$$u_i(\mathbf{x},t) = \theta(\mathbf{x},t) = 0 \text{ for } i = 1,2,3 \text{ and } \mathbf{x} \in \partial\Omega \text{ with } t \geq 0. \tag{2.3}$$

In view of system (2.1) we introduce the following notation: $\tilde{g} = a_0 g + a_1 g^{(1)} + \cdots + a_m g^{(m)}$.

Notice that the first equation of (2.1) implies

$$\rho\ddot{\tilde{u}}_i = \left(C_{ijkl}\tilde{u}_{k,l} - \beta_{ij}\tilde{\theta}\right)_{,j} + \tilde{l}_i.$$

We consider, therefore, the new system

$$\left\{\begin{array}{l} \rho\ddot{\tilde{u}}_i=\left(C_{ijkl}\tilde{u}_{k,l}-\beta_{ij}\tilde{\theta}\right)_{,j}+\tilde{l}_i \\ c\frac{d}{dt}\left(a_0\theta+\cdots+a_m\theta^{(m)}\right)+\beta_{ij}\tilde{v}_{i,j}=\left(b^0_{ij}\theta_{,i}+b^1_{ij}\theta^{(1)}_{,i}+\cdots+b^m_{ij}\theta^{(m)}_{,i}\right)_{,j}+S \end{array}\right. \tag{2.4}$$

Notice that if system (2.4) can be solved, then system (2.1) can be solved too, because

$$\tilde{u}_i=a_0u_i+a_1u_i^{(1)}+\cdots+a_mu_i^{(m)}$$

defines a linear ordinary differential equation.

We will transform system (2.4) in an abstract problem involving a convenient Hilbert space and matrix operators. In fact, we will work in the Hilbert space $\mathcal{H}$ defined by

$$\mathcal{H}=\mathbf{W}_0^{1,2}(\Omega)\times\mathbf{L}_2(\Omega)\times\left(W_0^{1,2}(\Omega)\right)^m\times L_2(\Omega),$$

where $\mathbf{W}_0^{1,2}(\Omega)$ denotes the cartesian product $\left(W_0^{1,2}(\Omega)\right)^3$, $\mathbf{L}_2(\Omega)$ represents also the cartesian product $(L_2(\Omega))^3$, and $W_0^{1,2}$ and L_2 are the usual Sobolev spaces [1].

To be consistent, we need to introduce a suitable notation for the variables in $\mathcal{H}$. We will use the new variables $\theta,\theta^{\{1\}},\theta^{\{2\}},...,\theta^{\{m-1\}}$ and $\theta^{\{m\}}$. Therefore, our working variables will be

$$\tilde{u}_i,\tilde{v}_i,\theta,\theta^{\{1\}},\theta^{\{2\}},...,\theta^{\{m-1\}},\theta^{\{m\}}.$$

Following the same idea that we used before, we write now $\hat{\theta}=a_0\theta+a_1\theta^{\{1\}}+\cdots+a_m\theta^{\{m\}}$.

From here on, to ease the notation we remove the tilde from the variables u_i and v_i.

We define an inner product in $\mathcal{H}$ by

$$\langle U,U^*\rangle_{\mathcal{H}}=\int_\Omega\left(C_{ijkl}u_{i,j}u^*_{k,l}+\rho v_iv_i^*+\lambda_0\theta_{,i}\theta^*_{,i}+\lambda_1\theta^{\{1\}}_{,i}\theta^{\{1\}*}_{,i}+\cdots+\lambda_{m-1}\theta^{\{m-1\}}_{,i}\theta^{\{m-1\}*}_{,i}+c\,\hat{\theta}\hat{\theta}^*\right)dV \tag{2.5}$$

In this inner product, constants λ_i are positive real numbers as greater as necessary. Notice that, for every choice of the positive numbers λ_i, this inner product defines a norm in $\mathcal{H}$ which is equivalent to the usual one.

3. Existence and uniqueness of solutions

We will rewrite system (2.4) in terms of matrix operators and, afterwards, we will use the technique of quasi-contractive semigroups to prove the existence of solutions.

In order to obtain a written synthetic expression to the above problem, we define the following operators.

$A_i(\mathbf{u}) = \frac{1}{\rho} C_{ijkl} u_{k,lj}$

$B_i^0(\theta) = -\frac{a_0 \beta_{ij}}{\rho} \theta_{,j}$

$B_i^1(\theta^{\{1\}}) = -\frac{a_1 \beta_{ij}}{\rho} \theta_{,j}^{\{1\}}$

$\ldots$

$B_i^m(\theta^{\{m\}}) = -\frac{a_m \beta_{ij}}{\rho} \theta_{,j}^{\{m\}}$

$D(\mathbf{v}) = -\frac{\beta_{ij}}{c\, a_m} v_{i,j}$

$M^{\{0\}}(\theta) = \frac{1}{c\, a_m} b_{ij}^0 \theta_{,ij}$

$M^{\{1\}}(\theta^{\{1\}}) = \frac{1}{c a_m} \left(b_{ij}^1 \theta_{,ij}^{\{1\}} - c a_0 \theta^{\{1\}} \right)$

$\ldots$

$M^{\{m\}}(\theta^{\{m\}}) = \frac{1}{c a_m} \left(b_{ij}^m \theta_{,ij}^{\{m\}} - c a_{m-1} \theta^{\{m\}} \right)$

Let us write $\mathbf{U} = (\mathbf{u}, \mathbf{v}, \theta, \theta^{\{1\}}, \theta^{\{2\}}, \ldots, \theta^{\{m\}})$, $\mathbf{A} = (A_i)$ and $\mathbf{B^k} = (B_i^k)$ for $k = 0, ..., m$. Therefore, system (2.4) can be written as

$$\frac{d\mathbf{U}}{dt} = \mathcal{A}\mathbf{U} + \mathcal{F}, \text{ with } \mathbf{U}(0) = (\mathbf{u}^0, \mathbf{v}^0, \theta_0, \theta_0^{\{1\}}, ..., \theta_0^{\{m\}}) \text{ for } t \in [0, t_1], \tag{3.1}$$

where $\mathcal{A}$ is the following matrix operator

$$\mathcal{A} = \begin{pmatrix} 0 & \mathbf{I} & 0 & 0 & \cdots & 0 \\ \mathbf{A} & 0 & \mathbf{B^0} & \mathbf{B^1} & \cdots & \mathbf{B^m} \\ 0 & 0 & 0 & I & \cdots & 0 \\ 0 & 0 & 0 & & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & I \\ 0 & D & M^{\{0\}} & M^{\{1\}} & \cdots & M^{\{m\}} \end{pmatrix} \text{ and } \mathcal{F} = \begin{pmatrix} 0 \\ \mathbf{l} \\ 0 \\ 0 \\ \vdots \\ 0 \\ S \end{pmatrix}. \tag{3.2}$$

Following the Lumer-Philips Theorem, an useful special corollary of the Hille-Yosida Theorem (see reference [18], page 340), the operator $\mathcal{A}$ will be the generator of a quasi-contractive semi-group whenever the three following conditions are satisfied:

- the domain of $\mathcal{A}$ is dense in $\mathcal{H}$.
- $\langle \mathcal{A}U, U \rangle_{\mathcal{H}} \leq C \|U\|_{\mathcal{H}}^2$.
- $\delta I - \mathcal{A}$ is exhaustive for $\delta \in \mathbb{R}$ large enough.

The domain of the operator is the set $\{\mathbf{U} \in \mathcal{H} : \mathcal{A}\mathbf{U} \in \mathcal{H}\}$.

We have that

$$\begin{aligned}
&\mathbf{v} \in \mathbf{W}_0^{1,2},\\
&\theta, \theta^{\{1\}}, ..., \theta^{\{m\}} \in W_0^{1,2}\\
&\mathbf{Au} + \textstyle\sum_{k=0}^m \mathbf{B^k}\theta^{\{k\}} \in \mathbf{L}^2\\
&D\mathbf{v} + \textstyle\sum_{k=0}^m \mathbf{M^k}\theta^{\{k\}} \in L^2
\end{aligned}$$

In view of the definitions of the operators, we see that the set

$$\{U \in \mathcal{H} : \ \mathbf{u} \in \mathbf{W}_0^{1,2} \cap \mathbf{W}^{2,2}, \mathbf{v} \in \mathbf{W}_0^{1,2}, \theta, \theta^{\{1\}}, ..., \theta^{\{m\}} \in W_0^{1,2} \cap W^{2,2}\}$$

is contained in the domain of $\mathcal{A}$. Therefore, the domain of $\mathcal{A}$ is dense in $\mathcal{H}$.

Lemma 3.1. $\langle \mathcal{A}U, U\rangle_{\mathcal{H}} \leq C\|U\|^2_{\mathcal{H}}$.

Proof. On the one hand, direct calculations and the use of the divergence theorem show that

$$\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = \int_\Omega \left(\sum_{k=0}^{m-1} \lambda_k \theta^{\{k\}}_{,i} \theta^{\{k+1\}}_{,i} - \sum_{k,l=0}^{m} b^k_{ij} a_l \theta^{\{k\}}_{,i} \theta^{\{l\}}_{,j} \right) dV. \tag{3.3}$$

On the other hand,

$$\|U\|^2_{\mathcal{H}} = \int_\Omega \left(C_{ijkl} u_{i,j} u_{k,l} + \rho v_i^2 + \sum_{k=0}^{m-1} \lambda_k \theta^{\{k\}}_{,i} \theta^{\{k\}}_{,i} + c\hat{\theta}^2 \right) dV.$$

Applying the Poincaré inequality, it can be shown that

$$\int_\Omega \left(\sum_{k=0}^{m-1} \lambda_k \theta^{\{k\}}_{,i} \theta^{\{k\}}_{,i} + c\hat{\theta}^2 \right) dV$$

is, basically, equivalent to

$$\int_\Omega \left(\sum_{k=0}^{m-1} \Lambda_k \theta^{\{k\}}_{,i} \theta^{\{k\}}_{,i} + c\theta^{\{m\}}\theta^{\{m\}} \right) dV,$$

where Λ_k are positive real numbers.

We will use this last expression to compare with (3.3).

We concentrate first in the terms $\lambda_k \theta^{\{k\}}_{,i} \theta^{\{k+1\}}_{,i}$ of (3.3) and we distinguish two cases:

- $k < m-1$: for each term of the sum we have the following inequality:
$$\int_\Omega \lambda_k \theta^{\{k\}}_{,i} \theta^{\{k+1\}}_{,i}\, dV \leq C_k \int_\Omega \left(\theta^{\{k\}}_{,i} \theta^{\{k\}}_{,i} + \theta^{\{k+1\}}_{,i} \theta^{\{k+1\}}_{,i} \right) dV$$
and then, it is clear that each term is bounded by $\|U\|^2_{\mathcal{H}}$ multiplied by an appropriate constant.
- $k = m-1$: for this term we have
$$\int_\Omega \lambda_{m-1} \theta^{\{m-1\}}_{,i} \theta^{\{m\}}_{,i}\, dV \leq C_{m-1} \int_\Omega \theta^{\{m-1\}}_{,i} \theta^{\{m-1\}}_{,i}\, dV + \epsilon \int_\Omega \theta^{\{m\}}_{,i} \theta^{\{m\}}_{,i}\, dV,$$
where ϵ is a positive real number as small as necessary and C_{m-1} is also a positive real number. Again, $C_{m-1} \int_\Omega \theta^{\{m-1\}}_{,i} \theta^{\{m-1\}}_{,i}\, dV$ is bounded by $\|U\|^2_{\mathcal{H}}$ multiplied by an appropriate constant.

Now, we focus on the terms $b^k_{ij} a_l \theta^{\{k\}}_{,i} \theta^{\{l\}}_{,j}$ and, as before, we distinguish different cases (four, now):

- $k, l < m$: for each term we have
$$\int_\Omega \left| b^k_{ij} a_l \theta^{\{k\}}_{,i} \theta^{\{l\}}_{,j} \right| \, dV \leq C_{kl} \int_\Omega \left(\theta^{\{k\}}_{,i} \theta^{\{k\}}_{,i} + \theta^{\{l\}}_{,j} \theta^{\{l\}}_{,j} \right) \, dV.$$
- $k = m, l < m$: for each $l < m$ we have
$$\int_\Omega \left| b^m_{ij} a_l \theta^{\{m\}}_{,i} \theta^{\{l\}}_{,j} \right| \, dV \leq C_{ml} \int_\Omega \theta^{\{l\}}_{,j} \theta^{\{l\}}_{,j} \, dV + \epsilon_{ml} \int_\Omega \theta^{\{m\}}_{,i} \theta^{\{m\}}_{,i} \, dV$$
- $l = m, k < m$: for each $k < m$ we have
$$\int_\Omega \left| b^k_{ij} a_m \theta^{\{k\}}_{,i} \theta^{\{m\}}_{,j} \right| \leq C_{km} \int_\Omega \theta^{\{k\}}_{,j} \theta^{\{k\}}_{,j} \, dV + \epsilon_{km} \int_\Omega \theta^{\{m\}}_{,i} \theta^{\{m\}}_{,i} \, dV$$
- $k = l = m$: by hypothesis $a_m > 0$ and the tensor b^m_{ij} is positive definite, hence there exists a positive constant M such that
$$-\int_\Omega a_m b^m_{ij} \theta^{\{m\}}_{,i} \theta^{\{m\}}_{,j} \, dV \leq -M \, a_m \int_\Omega \theta^{\{m\}}_{,i} \theta^{\{m\}}_{,i} \, dV.$$

As before, the terms of the right hand side of the three first inequalities are bounded by $\|U\|^2_{\mathcal{H}}$ multiplied by an appropriate constant. Therefore, if we take the epsilons such that $\epsilon + \sum_{k=0}^{m-1} \epsilon_{km} + \sum_{l=0}^{m-1} \epsilon_{ml} \leq M \, a_m$, then the desired inequality will be satisfied and the proof is completed. □

Lemma 3.2. *$\delta I - \mathcal{A}$ is exhaustive for $\delta \in \mathbb{R}$ large enough.*

Proof. We consider $(\mathbf{f_1}, \mathbf{f_2}, f_3, \dots, f_{m+2}, f_{m+3}) \in \mathcal{H}$. We have to prove that system

$$\left\{ \begin{array}{l} \delta \mathbf{u} - \mathbf{v} = \mathbf{f_1} \\ \delta \mathbf{v} - \mathbf{A}\mathbf{u} - \sum_{k=0}^m \mathbf{B^k} \theta^{\{k\}} = \mathbf{f_2} \\ \delta\theta - \theta^{\{1\}} = f_3 \\ \delta\theta^{\{1\}} - \theta^{\{2\}} = f_4 \\ \vdots \\ \delta\theta^{\{m-1\}} - \theta^{\{m\}} = f_{m+2} \\ \delta\theta^{\{m\}} - D\mathbf{v} - \sum_{k=0}^m M^{\{k\}} \theta^{\{k\}} = f_{m+3} \end{array} \right. \tag{3.4}$$

has a solution in the domain of $\mathcal{A}$.

From the first equation of the above system we get $\mathbf{v} = \delta\mathbf{u} - \mathbf{f_1}$. Then, the second equation becomes

$$\delta^2 \mathbf{u} - \mathbf{A}\mathbf{u} - \sum_{k=0}^m \mathbf{B^k} \theta^{\{k\}} = \delta \mathbf{f_1} + \mathbf{f_2}. \tag{3.5}$$

From the third to the $m+2$ equations we obtain the following relations:

$$\begin{array}{l} \theta^{\{1\}} = \delta\theta - f_3 \\ \theta^{\{2\}} = \delta^2\theta - \delta f_3 - f_4 \\ \vdots \\ \theta^{\{m\}} = \delta^m\theta - \delta^{m-1} f_3 - \delta^{m-2} f_4 - \cdots - \delta f_{m+1} - f_{m+2}. \end{array} \tag{3.6}$$

Taking into account the above relations and the operators $\mathbf{B^k}$, equation (3.5) can be written as

$$\delta^2 u_i - A_i(\mathbf{u}) + \sum_{k=0}^{m} a_k \delta^k \frac{\beta_{ij}}{\rho}\theta_{,j} = F_{1i}, \tag{3.7}$$

where $\mathbf{F_1}$ is a linear combination of $\mathbf{f_1}$, $\mathbf{f_2}$ and the first derivatives of f_k for $k = 3, ..., m + 3$.

The last equation of (3.4) can also be written in terms of $\mathbf{u}$ and θ:

$$\frac{\delta}{ca_m}\beta_{ij}u_{i,j} + \left(\delta^{m+1} + \frac{1}{a_m}\sum_{k=1}^{m} a_{k-1}\delta^k\right)\theta - \frac{1}{ca_m}\sum_{k=0}^{m} b_{ij}^k \delta^k \theta_{,ij} = F_2, \tag{3.8}$$

where F_2 is another linear combination of f_k, for $k = 3, ..., m + 3$.

We have now a new system of equations, (3.7) and (3.8). We will prove that this system has an unique solution by using the Lax-Milgram lemma. In order to do so, we define an inner product equivalent to the usual one and we have to prove that it is coercive and bounded.

To simplify the notation, we write $p_1 = \frac{1}{\rho}\sum_{k=0}^{m} a_k\delta^k$, and $p_2 = \frac{\delta}{ca_m}$, and we define the operators $\mathbf{B}\theta = \beta_{ij}\theta_{,j}$, $B^*\mathbf{u} = \beta_{ij}u_{i,j}$ and $P_3\theta = \left(\delta^{m+1} + \frac{1}{a_m}\sum_{k=1}^{m} a_{k-1}\delta^k\right)\theta - \frac{1}{ca_m}\sum_{k=0}^{m} b_{ij}^k\delta^k\theta_{,ij}$. Therefore, the system of equations we want to study reduces to

$$\begin{cases} (\delta^2\mathbf{I} - \mathbf{A})\mathbf{u} + p_1\mathbf{B}\theta = \mathbf{F_1} \\ p_2 B^*\mathbf{u} + P_3\theta = F_2 \end{cases} \tag{3.9}$$

The inner product that we define is given by

$$\begin{aligned} \mathbf{C}[(\mathbf{u},\theta),(\mathbf{u}^*,\theta^*)] &= \langle \big(p_2\big((\delta^2\mathbf{I} - \mathbf{A})\mathbf{u} + p_1\mathbf{B}\theta\big), p_1(p_2\mathbf{B}^*\mathbf{u} + P_3\theta)\big), (\mathbf{u}^*,\theta^*)\rangle_{L^2\times L^2} \\ &= p_2\delta^2\langle\mathbf{u},\mathbf{u}^*\rangle - p_2\langle\mathbf{A}\mathbf{u},\mathbf{u}^*\rangle + p_1p_2\langle\mathbf{B}\theta,\mathbf{u}^*\rangle + p_1p_2\langle B^*u,\theta^*\rangle + p_1P_3\langle\theta,\theta^*\rangle. \end{aligned}$$

It is clear that $\mathbf{C}$ is bounded in $\mathbf{W}_0^{1,2} \times W_0^{1,2}$.

In particular,

$$\mathbf{C}[(\mathbf{u},\theta),(\mathbf{u},\theta)] = p_2\delta^2\langle\mathbf{u},\mathbf{u}\rangle - p_2\langle\mathbf{A}\mathbf{u},\mathbf{u}\rangle + p_1\langle P_3\theta,\theta\rangle.$$

Notice that, from the definition of $\mathcal{A}$, we get

$$p_2\delta^2\int_\Omega u_iu_i\,dV + p_2\int_\Omega C_{ijkl}u_{i,j}u_{k,l}\,dV \geq K\|\mathbf{u}\|^2_{\mathbf{W}_0^{1,2}}.$$

On the other hand,

$$p_1\langle P_3\theta,\theta\rangle = \int_\Omega p_1\Big(\delta^{m+1}\theta^2 + \frac{1}{a_m}\sum_{k=1}^{m} a_{k-1}\delta^k\theta^2\Big)\,dV + \int_\Omega \frac{1}{ca_m}\sum_{k=0}^{m} b_{ij}^k\delta^k\theta_{,i}\theta_{,j}\,dV.$$

Notice also that, from the assumptions over the b_{ij}^k tensors, we have

$$\begin{aligned}
&\left|\int_\Omega b_{ij}^0\theta_{,i}\theta_{,j}\,dV\right| \leq K_0\int_\Omega \theta_{,i}\theta_{,i}\,dV\\
&\left|\int_\Omega b_{ij}^1\delta\theta_{,i}\theta_{,j}\,dV\right| \leq K_1\delta\int_\Omega \theta_{,i}\theta_{,i}\,dV\\
&\vdots\\
&\left|\int_\Omega b_{ij}^{m-1}\delta^{m-1}\theta_{,i}\theta_{,j}\,dV\right| \leq K_{m-1}\delta^{m-1}\int_\Omega \theta_{,i}\theta_{,i}\,dV\\
&\int_\Omega b_{ij}^m\delta^m\theta_{,i}\theta_{,j}\,dV \geq M\delta^m\int_\Omega \theta_{,i}\theta_{,i}\,dV
\end{aligned}$$

It is worth noting that constants K_0, K_1,..., K_{m-1} are positive and calculable real numbers. Adding all these inequalities we get

$$\sum_{k=0}^m\int_\Omega b_{ij}^k\delta^k\theta_{,i}\theta_{,j}\,dV \geq \left(M\delta^m - K_{m-1}\delta^{m-1} - \cdots C_1\delta - C_0\right)\int_\Omega \theta_{,i}\theta_{,i}\,dV.$$

And, therefore, for δ great enough,

$$\sum_{k=0}^m\int_\Omega b_{ij}^k\delta^k\theta_{,i}\theta_{,j}\,dV \geq K'\int_\Omega \theta_{,i}\theta_{,i}\,dV \geq K''\|\theta\|^2_{W_0^{1,2}}.$$

Finally,

$$\int_\Omega p_1\Big(\delta^{m+1}\theta^2 + \frac{1}{a_m}\sum_{k=1}^m a_{k-1}\delta^k\theta^2\Big)\,dV \geq K^{'''}\|\theta\|^2_{L_2}$$

because the first term of the sum is a polynomial in δ with positive principal coefficient and degree greater than the polynomial in δ contained in the second term of the sum. $\square$

Theorem 3.3. *The operator $\mathcal{A}$ defined at (3.2) is the generator of a quasi-contractive semigroup.*

As a consequence, we have the following results.

Theorem 3.4. *Assume that conditions I, II and III are satisfied. Assume also that the supply terms verify*

$$\begin{aligned}
&\mathbf{l}\in\mathcal{C}^1\left([0,t_1],\mathbf{L}^2\right)\cap\mathcal{C}^0\left([0,t_1],\mathbf{W}_0^{1,2}\cap\mathbf{W}_0^{2,2}\right)\\
&S\in\mathcal{C}^1\left([0,t_1],L^2\right)\cap\mathcal{C}^0\left([0,t_1],W_0^{2,2}\right).
\end{aligned}$$

Then, for any $U(0)=\left(\mathbf{u}^0,\mathbf{v}^0,\theta_0,\theta_0^{\{1\}},...,\theta_0^{\{m\}}\right)\in\mathcal{D}$ *there exists a unique solution*

$$U(t)=\left(\mathbf{u}(t),\mathbf{v}(t),\theta(t),\theta^{\{1\}}(t),...,\theta^{\{m\}}(t)\right)\in\mathcal{C}^1\left([0,t_1],\mathcal{H}\right)\cap\mathcal{C}^0\left([0,t_1],\mathcal{D}\right)$$

which satisfies equation (3.1) with the aforementioned initial conditions.

Moreover, we now know that there is continuous dependence of the solutions with respect to the initial data.

Remark 3.5. Since $\mathcal{A}$ is the generator of a quasi-contractive semigroup, the following estimate for the solution is satisfied

$$\|U(t)\|_{\mathcal{H}} \leq \left(\|U(0)\|_{\mathcal{H}} + K_2 \int_0^t h(s)ds \right) e^{K_1 t},$$

where

$$h^2(t) = \int_\Omega (\rho_0 l_i l_i + a_m S^2)\, dV$$

and K_1 and K_2 are calculable positive constants.

Remark 3.6. The last theorem and the above remark allow to state that, whenever the assumptions on the constitutive coefficients are satisfied, the problem of the phase-lag thermoelasticity defined by (2.1) with initial conditions (2.2) and boundary conditions (2.3) is a well-posed problem in the sense of Hadamard.

Remark 3.7. Another interesting question related with this problem is the analysis of the time stability of the solutions in the homogeneous situation. In this sense, it is suitable to recall here that Borgmeyer [4] obtained exponential stability in the one-dimensional case for some particular models. However, there are some other cases where the time behavior of the solutions is still an open question.

4. Case $n = m + 2$

In this section we study the case where $n = m + 2$. To save repetitive analysis we only sketch several steps. We think that the main contribution of this section is the definition of the inner product (energy function) in the corresponding Hilbert space. This is the point that we will emphasized.

Following the arguments of Chandrasekharaiah [5] the system of equations becomes
(4.1)

$$\begin{cases} \rho \ddot{u}_i = \left(C_{ijkl} u_{k,l} - \beta_{ij}\theta \right)_{,j} + l_i \\ c\frac{d}{dt}\left(a_0\theta + \cdots + a_{m+1}\theta^{(m+1)} \right) + \beta_{ij}\left(a_0 v_{i,j} + \cdots + a_{m+1} v_{i,j}^{(m+1)} \right) = \left(b_{ij}^0 \theta_{,i} + \cdots + b_{ij}^m \theta_{,i}^{(m)} \right)_{,j} + S \end{cases}$$

We assume the same notation and assumptions as in Section 2 for system (2.1). We suppose again that the coefficients and tensors do not depend on the material point (again, the extension to the non-homogeneous case does not seem difficult). In this section we impose that a_{m+1} is strictly positive as well as conditions II and III of the Setion 2.

To define a well-posed problem we impose the initial conditions (2.2) plus

$$\theta^{(m)}(\mathbf{x}, 0) = \theta_0^{(m)}(\mathbf{x}).$$

We impose also boundary conditions (2.3).

We abuse a little bit the notation and write $\tilde{g} = a_0 g + a_1 g^{(1)} + \cdots + a_{m+1} g^{(m+1)}$. Therefore, system (4.1) can be written as

$$\begin{cases} \rho \ddot{\tilde{u}}_i = \left(C_{ijkl} \tilde{u}_{k,l} - \beta_{ij}\tilde{\theta} \right)_{,j} + \tilde{l}_i \\ c\frac{d}{dt}\left(a_0\theta + \cdots + a_{m+1}\theta^{(m+1)} \right) + \beta_{ij}\tilde{v}_{i,j} = \left(b_{ij}^0 \theta_{,i} + b_{ij}^1 \theta_{,i}^{(1)} + \cdots + b_{ij}^m \theta_{,i}^{(m)} \right)_{,j} + S \end{cases} \tag{4.2}$$

As we did in Section 2, we transform our system into an abstract ordinary differential equation in a suitable Hilbert space $\mathcal{H}^*$:

$$\mathcal{H}^* = \mathbf{W}_0^{1,2}(\Omega) \times \mathbf{L}_2(\Omega) \times \left(W_0^{1,2}(\Omega)\right)^{m+1} \times L_2(\Omega).$$

To ease the notation, we remove the tilde again.

We now consider the following inner product in $\mathcal{H}^*$:

(4.3)

$$\langle U, U^* \rangle_{\mathcal{H}^*} = \int_\Omega \Big(C_{ijkl} u_{i,j} u^*_{k,l} + \rho v_i v_i^* + \lambda_0 \theta_{,i} \theta^*_{,i} + \lambda_1 \theta^{\{1\}}_{,i} \theta^{\{1\}*}_{,i} + \cdots + \lambda_{m-1} \theta^{\{m-1\}}_{,i} \theta^{\{m-1\}*}_{,i} + c\,\hat\theta \hat\theta^*$$
$$+ a_{m+1} b^m_{ij} \theta^{\{m\}}_{,i} \theta^{\{m\}*}_{,j} + \sum_{k=0}^{m-1} a_{m+1} b^k_{ij} (\theta^{\{k\}}_{,i} \theta^{\{m\}*}_{,j} + \theta^{\{k\}*}_{,i} \theta^{\{m\}}_{,j}) \Big)\, dV.$$

In this product, the constants λ_i are positive real numbers as greater as needed to guarantee that the inner product is equivalent to the usual one in $\mathcal{H}^*$. We use $\hat\theta = a_0\theta + a_1\theta^{\{1\}} + \cdots + a_{m+1}\theta^{\{m+1\}}$.

Notice that $\|U\|^2_{\mathcal{H}^*}$ is equivalent to

$$\int_\Omega \Big(C_{ijkl} u_{i,j} u_{k,l} + \rho v_i, v_i + \sum_{j=0}^{m} \theta^{\{j\}}_{,i} \theta^{\{j\}}_{,i} + |\theta^{\{m+1\}}|^2 \Big) dV.$$

Following the same idea of Section 2, we define the operators:

$$A_i(\mathbf{u}) = \tfrac{1}{\rho} C_{ijkl} u_{k,lj}$$

$$B_i^0(\theta) = -\tfrac{a_0 \beta_{ij}}{\rho} \theta_{,j}$$

$$B_i^1(\theta^{\{1\}}) = -\tfrac{a_1 \beta_{ij}}{\rho} \theta^{\{1\}}_{,j}$$

$$\ldots$$

$$B_i^{m+1}(\theta^{\{m+1\}}) = -\tfrac{a_{m+1} \beta_{ij}}{\rho} \theta^{\{m+1\}}_{,j}$$

$$D(\mathbf{v}) = -\tfrac{\beta_{ij}}{c\, a_{m+1}} v_{i,j}$$

$$M^{\{0\}}(\theta) = \tfrac{1}{c\, a_{m+1}} b^0_{ij} \theta_{,ij}$$

$$M^{\{1\}}(\theta^{\{1\}}) = \tfrac{1}{c a_{m+1}} \left(b^1_{ij} \theta^{\{1\}}_{,ij} - c a_0 \theta^{\{1\}} \right)$$

$$\ldots$$

$$M^{\{m\}}(\theta^{\{m\}}) = \tfrac{1}{c a_{m+1}} \left(b^m_{ij} \theta^{\{m\}}_{,ij} - c a_{m-1} \theta^{\{m\}} \right)$$

$$N(\theta^{\{m+1\}}) = -\tfrac{a_m}{a_{m+1}} \theta^{\{m+1\}}$$

Therefore, system (4.2) becomes

(4.4) $$\frac{d\mathbf{U}}{dt} = \mathcal{A}^*\mathbf{U} + \mathcal{F}, \text{ with } \mathbf{U}(0) = (\mathbf{u}^0, \mathbf{v}^0, \theta_0, \theta_0^{\{1\}}, ..., \theta_0^{\{m+1\}}) \text{ for } t \in [0, t_1],$$

where $\mathcal{A}^*$ is the following matrix operator

$$\mathcal{A}^* = \begin{pmatrix} 0 & \mathbf{I} & 0 & 0 & \cdots & 0 & 0 \\ \mathbf{A} & 0 & \mathbf{B^0} & \mathbf{B^1} & \cdots & \mathbf{B^m} & \mathbf{B^{m+1}} \\ 0 & 0 & 0 & I & \cdots & 0 & 0 \\ 0 & 0 & 0 & & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ 0 & 0 & 0 & 0 & \cdots & 0 & I \\ 0 & D & M^{\{0\}} & M^{\{1\}} & \cdots & M^{\{m\}} & N \end{pmatrix} \text{ and } \mathcal{F} = \begin{pmatrix} 0 \\ \mathbf{l} \\ 0 \\ 0 \\ \vdots \\ 0 \\ S \end{pmatrix}. \tag{4.5}$$

Theorem 4.1. *The operator $\mathcal{A}^*$ defined at (4.5) generates a quasi-contractive semigroup.*

Proof. We have to show the three points proposed in Section 3 for the operator $\mathcal{A}$.

First of all, the domain of $\mathcal{A}^*$ is the set $\{\mathbf{U} \in \mathcal{H}^* : \mathcal{A}^*\mathbf{U} \in \mathcal{H}^*\}$. Therefore, it is dense in $\mathcal{H}^*$. The proof is analogous to the one proposed in Section 3 for the operator $\mathcal{A}$.

To prove the second condition we note that

$$\begin{aligned} \langle \mathcal{A}^* U, U\rangle_{\mathcal{H}^*} = \int_\Omega \Big(\sum_{k=0}^{m-1} \lambda_k \theta^{\{k\}}_{,i} \theta^{\{k+1\}}_{,i} + b^m_{ij} a_{m+1} \theta^{\{m\}}_{,i} \theta^{\{m+1\}}_{,j} + \sum_{k=0}^{m-1} b^k_{ij} a_{m+1} \theta^{\{k\}}_{,i} \theta^{\{m+1\}}_{,j} \\ - \sum_{k=0}^{m+1} \sum_{l=0}^{m} a_k b^l_{ij} \theta^{\{k\}}_{,j} \theta^{\{l\}}_{,j} + \sum_{k=0}^{m-1} a_{m+1} b^k_{ij} \theta^{\{k+1\}}_{,i} \theta^{\{m\}}_{,j} \Big) dV \end{aligned} \tag{4.6}$$

Simplifying we obtain:

$$\langle \mathcal{A}^* U, U\rangle_{\mathcal{H}^*} = \int_\Omega \Big(\sum_{k=0}^{m-1} \lambda_k \theta^{\{k\}}_{,i} \theta^{\{k+1\}}_{,i} - \sum_{k,l=0}^{m} b^k_{ij} a_l \theta^{\{k\}}_{,i} \theta^{\{l\}}_{,j} + \sum_{k=0}^{m-1} a_{m+1} b^k_{ij} \theta^{\{k+1\}}_{,i} \theta^{\{m\}}_{,j} \Big) dV \tag{4.7}$$

As we did in the proof of Lemma 3.1, we can estimate each term of the above integral and we obtain

$$\langle \mathcal{A}^* U, U\rangle_{\mathcal{H}^*} \leq C^* \int_\Omega \sum_{k=0}^{m} \theta^{\{k\}}_{,i} \theta^{\{k\}}_{,i}\, dV,$$

for a calculable constant C^*. This implies the desired inequality.

To finish the proof we only need to show that $\delta\mathcal{I} - \mathcal{A}^*$ is exhaustive for δ large enough, but this can be done, mutatis mutandis, as we have done in the proof of Lemma 3.2. □

Hence, we have proved the following result.

Theorem 4.2. *Assume that $a_{m+1} > 0$ and that conditions II and III are satisfied. Assume also that the supply terms verify*

$$\begin{aligned} \mathbf{l} \in \mathcal{C}^1\left([0,t_1], \mathbf{L}^2\right) \cap \mathcal{C}^0\left([0,t_1], \mathbf{W}^{1,2}_0 \cap \mathbf{W}^{2,2}_0\right) \\ S \in \mathcal{C}^1\left([0,t_1], L^2\right) \cap \mathcal{C}^0\left([0,t_1], W^{2,2}_0\right). \end{aligned}$$

Then, for any $U(0) = \left(\mathbf{u}^0, \mathbf{v}^0, \theta_0, \theta^{\{1\}}_0, ..., \theta^{\{m+1\}}_0\right) \in \mathcal{D}$ there exists a unique solution

$$U(t) = \left(\mathbf{u}(t), \mathbf{v}(t), \theta(t), \theta^{\{1\}}(t), ..., \theta^{\{m+1\}}(t)\right) \in \mathcal{C}^1\left([0,t_1], \mathcal{H}^*\right) \cap \mathcal{C}^0\left([0,t_1], \mathcal{D}\right)$$

which satisfies equation (4.4) with the aforementioned initial conditions.

Remarks 3.5, 3.6 and 3.7 have a natural counterpart in this situation.

5. Conclusions

We have considered a very general phase-lag thermoelasticity theory based in Taylor approximations. Although its thermomechanical foundations still need to be well investigated, its mathematical setting is interesting enough. In fact, as we pointed out in the Introduction, there are a lot of recent contributions about the phase-lag theories. We have proved, by means of the semigroup theory of linear operators, the existence, the uniqueness and the continuous dependence of the solutions with respect to the initial data and supply terms when $n-m=1,2$. Or, in other words, we have proved that, under certain hypotheses over the coefficients of the system, the problem is well posed in the sense of Hadamard.

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References

[1] R. A. Adams, *Sobolev Spaces,* Academic Press, New York (1975).
[2] G. Amendola, M. Fabrizio, M. Golden and B. Lazzari, Second-order approximation for heat conduction: dissipation principle and free energies, Proceedings of the Royal Society A 472(2016), 1-17.
[3] K. Borgmeyer, R. Quintanilla and R. Racke, Phase-lag heat conduction: decay rates for limit problems and well-posedness, J. Evolution Equations 14 (2014), 863-884.
[4] K. Borgmeyer, Dual- und Three-Phase-lag-Modelle: Zeitliche Asymptotik von Lsungen. Dissertation (Ph.D. Thesis), University of Konstanz (2016).
[5] D. S. Chandrasekharaiah, Hyperbolic thermoelasticity: A review of recent literature, Appl. Mech. Rev., 51(1998), 705-729.
[6] S. K. R. Choudhuri, On a thermoelastic three-phase-lag model, J. Thermal Stresses, 30(2007), 231-238.
[7] M. Dreher, R. Quintanilla and R. Racke, Ill-posed problems in thermomechanics, Appl. Math. Letters 22(2009), 1374-1379.
[8] M. Fabrizio and B. Lazzari, Stability and second law of thermodynamics in dual-phase-lag heat conduction, Int. J. Heat Mass Transfer 74(2014), 484-489.
[9] M. Fabrizio, B. Lazzari and V. Tibullo, Stability and Thermodynamic Restrictions for a Dual-Phase-Lag Thermal Model, J. Non-Equilib. Thermodynamics, DOI: https://doi.org/10.1515/jnet-2016-0039.
[10] A. E. Green and P. M. Naghdi, On undamped heat waves in an elastic solid, J. Thermal Stresses 15(1992), 253-264.
[11] A. E. Green and P. M. Naghdi, Thermoelasticity without energy dissipation, J. Elasticity, 31(1993), 189-208.
[12] R. B. Hetnarski and J. Ignaczak, Generalized thermoelasticity, J. Thermal Stresses, 22(1999), 451-470.
[13] R. B. Hetnarski and J. Ignaczak, Nonclassical dynamical thermoelasticity, International J. Solids Structures, 37(2000), 215-224.
[14] J. Ignaczak and M. Ostoja-Starzewski, *Thermoelasticity with Finite Wave Speeds*, Oxford: Oxford Mathematical Monographs, (2010).
[15] S. Kothari and S. Mukhopadhyay, Some Theorems in Linear Thermoelasticity with Dual Phase-Lags for an Anisotropic Medium, J. Thermal Stresses, 36(2013), 985-1000.
[16] Z. Liu, R. Quintanilla and Y. Wang, On the phase-lag heat equation with spatial dependent lags, J. Math. Anal. and Appl., 455(2017), 422-438.

[17] Z. Liu and R. Quintanilla, Time decay in dual-phase-lag thermoelasticity: Critical case, Comm. Pure Appl. Anal. (in press).

[18] J.E. Marsden and T.J.R Hughes, *Mathematical Foundations of Elasticity*, Prentice-Hall, Inc., Englewood Cliffs, N. J. (1983).

[19] A. Miranville and R. Quintanilla, A phase-field model based on a three-phase-lag heat conduction, Applied Mathematics and Optimization, 63 (2011), 133-150.

[20] S. Mukhopadhyay, S. Kothari and R. Kumar, On the representation of solutions for the theory of generalized thermoelasticity with three phase-lags, Acta Mechanica, 214 (2010), 305-314.

[21] R. Prasad, R. Kumar and S. Mukhopadhyay, Propagation of harmonic waves under thermoelasticity with dual-phase-lags, Int. J. Eng. Science, 48(2010), 2028-2043.

[22] R. Quintanilla, Exponential stability in the dual-phase-lag heat conduction theory, J. Non-Equilibrium Thermodynamics, 27 (2002), 217-227.

[23] R. Quintanilla, A Well-Posed Problem for the Dual-Phase-Lag Heat Conduction, J. Thermal Stresses, 31 (2008), 260-269.

[24] R. Quintanilla, A Well-Posed Problem for the Three-Dual-Phase-Lag Heat Conduction, J. Thermal Stresses, 32 (2009), 1270-1278.

[25] R. Quintanilla and R. Racke, Qualitative aspects in dual-phase-lag thermoelasticity. SIAM J. Appl. Math., 66 (2006), 977-1001.

[26] R. Quintanilla and R. Racke, A note on stability of dual-phase-lag heat conduction, Int. J. Heat Mass Transfer, 49 (2006), 1209-1213.

[27] R. Quintanilla and R. Racke, Qualitative aspects in dual-phase-lag heat conduction, Proc. Royal Society London A., 463 (2007), 659-674.

[28] R. Quintanilla and R. Racke, A note on stability in three-phase-lag heat conduction, Int. J. Heat Mass Transfer, 51 (2008), 24-29.

[29] R. Quintanilla and R. Racke, Spatial behavior in phase-lag heat conduction, Differential and Integral Equations, 28 (2015), 291-308.

[30] S. I. Serdyukov, N. M. Voskresenskii, V. K. Bel'nov and I. I. Karpov, Extended irreversible thermodynamics and generalization of the dual-phase-lag model in heat transfer, J. Non-Equilib. Thermodynamics, 28(2003), 1-13.

[31] B. Singh, Wave Propagation in Dual-Phase-Lag Anisotropic Thermoelasticity, Continuum Mechanics and Thermodynamics, 25(2013), 675-683.

[32] B. Straughan, *Heat Waves*, Appl. Math. Sci. **177**, Springer-Verlag, Berlin, 2011.

[33] D. Y. Tzou, A unified approach for heat conduction from macro to micro-scales, ASME J. Heat Transfer, 117(1995), 8-16.

[34] L. Wang, X. Zhou, and X. Wei, *Heat Conduction, Mathematical Models and Analytical Solutions*. Springer-Verlag, Berlin Heidelberg, (2008).