On the Communication Discussion of Two Distributed Population-game Approaches for Optimization Purposes ⋆

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Abstract: Population games have become a powerful tool for solving resource-allocation problems in a distributed manner, and for the design of non-centralized optimization-based controllers. The aim of this paper is to illustrate the advantages of two recently introduced population-game approaches in comparison to other classical optimization methods. More specifically, the discussion is mainly devoted to the communication requirements. Finally, an illustrative example shows with more detail the advantages highlighted throughout the comparative discussion, i.e., fewer communications links are required for resource allocation problems, and there is not need of additional computation stages to solve the problem in a distributed manner.

1. INTRODUCTION

The game-theoretical approach has been widely used in the design of control systems, and in the solution of constrained optimization problems in a distributed manner (Marden and Shamma, 2014). In this paper, the advantages of population-games-based methods to solve constrained optimization problems are discussed. In this regard, the discussion is particularly focused on two different population-game approaches. The first approach consists on the distributed population dynamics presented in Barreiro-Gomez et al. (2017), which has been used in the solution of distributed resource allocation problems as in Obando et al. (2014), Pantoja and Quijano (2011), and Pashaie et al. (2015). In Pashaie et al. (2015), the passivity properties of the population dynamics have been exploited (Fox and Shamma, 2013). Moreover, these distributed population dynamics may only consider a unique coupled constraint, and guarantee the positiveness of decision variables. On the other hand, the second approach under consideration consists in population dynamics by adding mass dynamics, which has been proposed in Barreiro-Gomez et al. (2016), and that solves constrained optimization problems in a distributed manner. In Barreiro-Gomez et al. (2016), some features have been pointed out that the distributed population dynamics do not have, e.g., attractiveness of the feasible region, and the inclusion of more coupled constraints. Generally, the optimization problems, which can be solved by using these aforementioned population-game approaches, can also be solved by using other algorithms. Furthermore, due to the fact that the population-game-based method presented in Barreiro-Gomez et al. (2016) might use the Lagrangian function depending on the optimization problem form, it is appropriate to establish a comparative discussion with respect to other methods that also use the Lagrangian function, e.g., dual decomposition or alternative direction method of multipliers.

The contribution of this paper is twofold. First, a comparative assessment is developed for different non-centralized optimization algorithms. Besides, a discussion regarding communication requirements and/or information dependency in population-game-based methods with respect to other classical optimization methods is made. In particular, it is considered the optimization form corresponding to a resource allocation problem since it has straightforward application in engineering problems. The aim is to highlight and discuss the advantages of the population-game approach with respect to other optimization techniques. Over the end, an illustrative example shows in detail the advantages and the different required communication structures for all the discussed methods applied to solve a resource allocation problem.

The remainder of this paper is organized as follows. Section 2 introduces the preliminaries for optimization methods, which are discussed throughout the paper. Section 3 presents the communication requirements discussion. Afterwards, Section 4 shows an illustrative example recalling all the mentioned advantages identified in the discussion for a specific optimization-problem form corresponding to a resource allocation problem. Finally, conclusions are drawn in Section 5.

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Notation: The column vectors are denoted by bold font, e.g., $\mathbf{x}$. Differently, scalar numbers are denoted by non-bold style, e.g., n. Calligraphy style is used to denote sets, e.g., $\mathcal{S}$. The set of all non-negative real numbers are denoted by $\mathbb{R}_{\geq 0}$, whereas $\mathbb{R}_{>0}$ denotes the set of all positive real numbers, and $\mathbb{Z}_{>0}$ represents the set of positive integer numbers. The derivative with respect to time is denoted by $\dot{x} = \frac{dx}{dt}$. Finally, the continuous time is omitted throughout the manuscript in order to simplify the notation, and $k \in \mathbb{Z}_{>0}$ denotes the discrete time.

2. SOME OPTIMIZATION METHODS

In order to compare different optimization methods, first the preliminaries corresponding to the two distributed population-games-based methods under discussion are presented. Then, the preliminary concepts for the dual decomposition (DD) method and the alternative direction method of multipliers (ADMM) are introduced.

2.1 Distributed population dynamics (DPD)

This population-game approach is an extension of the population dynamics presented in Sandholm (2010) for the distributed case. Moreover, this approach has been widely discussed in Barreiro-Gomez et al. (2017), where different distributed population dynamics have been studied. Also, in Pantoja and Quijano (2011), and (Pantoja and Quijano, 2012) the distributed replicator dynamics have been used for a smart-grid application. Consider a large and finite population of agents represented by a positive mass denoted by $m \in \mathbb{R}_{>0}$. It is assumed that the agents are rational in the sense that they are able to make decisions in order to improve their benefits. The mentioned decisions consist in selecting among a set of n available strategies denoted by $\mathcal{S} = \{1, \ldots, n\}$. Let $x_i \in \mathbb{R}_{\geq 0}$ denote a portion of agents choosing the strategy $i \in \mathcal{S}$. The collection of all these portions represent the population strategic distribution denoted by $\mathbf{x} \in \mathbb{R}^n_{\geq 0}$. Therefore, the evolution of the population only admits population states belonging to the simplex set denoted by $\Delta = \{\mathbf{x} \in \mathbb{R}^n_{\geq 0} : \sum_{i \in \mathcal{S}} x_i = m\}$. The benefits that agents receive for selecting the strategy $i \in \mathcal{S}$ are given by the mapping function $f_i : \Delta \mapsto \mathbb{R}$. The collection of fitness functions represent the population function given by $\mathbf{F}(\mathbf{x}) = [f_1(\mathbf{x}) \quad \ldots \quad f_n(\mathbf{x})]^\top \in \mathbb{R}^n$. Under the framework of full-potential and stable games, the fitness functions are computed as $\nabla V(\mathbf{x}) = \mathbf{F}(\mathbf{x})$, where $V(\mathbf{x})$ is a strict concave function known as *potential function*. These distributed dynamics consider that the evolution is made over a non-well-mixed population whose structure is represented by a graph $\mathcal{G} = (\mathcal{S}, \mathcal{E})$, where $\mathcal{S}$ is the set of nodes (strategies), and $\mathcal{E} \subset \{(i,j) : i,j \in \mathcal{S}\}$ is the set of links representing the possible strategic interaction. Moreover, the set of neighbors of a node $i \in \mathcal{S}$ is given by $\mathcal{N}_i = \{j : (i,j) \in \mathcal{E}\}$. One of the distributed population dynamics presented in Barreiro-Gomez et al. (2017) are the distributed replicator dynamics given by

$$\dot{x}_i = x_i \left(f_i \sum_{j \in \mathcal{N}_i} x_j - \sum_{j \in \mathcal{N}_i} f_j x_j \right), \quad \forall i \in \mathcal{S}. \tag{1}$$

Additionally, there is a close relationship between the introduced population-game framework and constrained optimization problems as presented in Theorem 1.

Theorem 1. (Adapted from (Sandholm, 2010)) If $\mathbf{F}$ is a full-potential game, i.e., there exists a full potential function V such that $\nabla V(\mathbf{x}) = \mathbf{F}(\mathbf{x})$, then the Nash equilibria of the game $\mathbf{F}$ satisfy the Karush-Kuhn-Tucker first-order necessary conditions of the following optimization problem:

$$\max_{\mathbf{x}} \; V(\mathbf{x}) \tag{2a}$$

$$\text{s. t.} \; \sum_{i \in \mathcal{S}} x_i = m, \tag{2b}$$

$$x_i \geq 0, \; \forall \, i \in \mathcal{S}. \tag{2c}$$

Moreover, if V is concave, then $\mathbf{F}$ is a stable game. ♢

Proof. This proof may be found in Sandholm (2010). ■

The optimization problem presented in (2) corresponds to a resource allocation problem, i.e., the case in which it is necessary to maximize a cost function by distributing optimally a resource denoted by m throughout n different agents. Finally, it should be pointed out that the initial conditions of this method should belong to the simplex, i.e., $\mathbf{x}(0) \in \Delta$.

2.2 Population dynamics with mass dynamics (PD-MD)

This population-game approach has been introduced in Barreiro-Gomez et al. (2016), where it is proposed that the graph $\mathcal{G} = (\mathcal{S}, \mathcal{E})$ exhibits the topology of a society, where $\mathcal{S}$ is the set of n available strategies in a social game given by $\mathcal{S} = \{1, \ldots, n\}$, and $\mathcal{E} \subset \{(i,j) : i,j \in \mathcal{S}\}$ is the set of edges of $\mathcal{G}$ that determines the possible interactions among social strategies. The graph $\mathcal{G}$ is divided into $\pi \in \mathbb{Z}_{>0}$ sub-complete graphs known as *cliques* (see Bomze et al. (2000), and Johnston (1976)) representing a population within the society. The set of populations is denoted by $\mathcal{P} = \{1, \ldots, \pi\}$, and the set of cliques is denoted by $\mathcal{C} = \{\mathcal{C}^p : p \in \mathcal{P}\}$. The clique of the population $p \in \mathcal{P}$ is a complete graph given by $\mathcal{C}^p = (\mathcal{S}^p, \mathcal{E}^p)$, where the set $\mathcal{S}^p$ represents the set of n^p available strategies in a population game, and $\mathcal{E}^p = \{(i,j) : i,j \in \mathcal{S}^p\}$ is the set of all the possible links in $\mathcal{C}^p$ determining full interaction among the population strategies. The number of cliques that contain a node $i \in \mathcal{S}$ is given by $G(i) = \sum_{p \in \mathcal{P}} g(i,p)$, where $g(i,p) = 1$ if $i \in \mathcal{S}^p$, and $g(i,p) = 0$ otherwise. The set of intersection nodes in a population $p \in \mathcal{P}$ is denoted by $\mathcal{I}^p = \{i \in \mathcal{S}^p : G(i) > 1\}$, and the set of intersection nodes in the graph $\mathcal{G}$ is denoted by $\mathcal{I} = \bigcup_{p \in \mathcal{P}} \mathcal{I}^p$. The set of all the populations that include a certain node $i \in \mathcal{S}$ is denoted by $\mathcal{P}_i = \{p : i \in \mathcal{S}^p\}$, where $\mathcal{P}_i \subseteq \mathcal{P}$. The scalar $x_i \in \mathbb{R}_{\geq 0}$ ($x_i^p \in \mathbb{R}_{\geq 0}$) corresponds to the proportion of agents in the society (population) selecting the strategy $i \in \mathcal{S}$ ($i \in \mathcal{S}^p$ in the population $p \in \mathcal{P}$). Moreover, the distribution of agents throughout the available strategies in the society or populations is known as the social state and the population state denoted by $\mathbf{x} \in \mathbb{R}^n_{\geq 0}$, and $\mathbf{x}^p \in \mathbb{R}^{n^p}_{\geq 0}$, respectively. The set of possible social (population) states is given by a simplex denoted by $\Delta = \left\{\mathbf{x} \in \mathbb{R}^n_{\geq 0} : \sum_{i \in \mathcal{S}} x_i = m\right\}$ $\left(\Delta^p = \left\{\mathbf{x}^p \in \mathbb{R}^{n^p}_{\geq 0} : \sum_{i \in \mathcal{S}^p} x_i = m^p\right\}\right)$, where $m \in \mathbb{R}_{>0}$ ($m^p \in \mathbb{R}_{>0}$) is the mass of agents in the society (population $p \in \mathcal{P}$). Furthermore, there is a relationship between the social states and the population states given by $x_i = G(i)^{-1}\sum_{p\in\mathcal{P}_i} x_i^p$. Fitness functions for society are defined as in Section 2.1. Furthermore, let $f_i^p : \Delta^p \mapsto \mathbb{R}$ be the mapping of the fitness function for the proportion of agents playing the strategy $i \in \mathcal{S}^p$ in the population $p \in \mathcal{P}$. The fitness corresponding to a strategy $i \in \mathcal{S}$ is the same as the fitness for a strategy $j \in \mathcal{S}^p$ for all $p \in \mathcal{P}$ if $i = j$. Consequently, for all $i \in \mathcal{I}$ and for all $p \in \mathcal{P}_i$, $f_i(\mathbf{x}) = f_i^p(\mathbf{x}^p)$, if $x_i = x_i^p$. Besides, there is a relationship between the population and social mass given by $m = \sum_{p\in\mathcal{P}} m^p - \sum_{i\in\mathcal{S}} (G(i)-1)x_i$. Then, a game is solved for each population with constraints given by the population mass m^p, for all $p \in \mathcal{P}$, which varies dynamically. Dynamics associated to each population are shown in (3a). There are π different dynamics of this form, i.e., one for each clique $\mathcal{C}^p$ for all $p \in \mathcal{P}$ as follows:

$$\dot{x}_i^p = x_i^p \left(f_i^p - \frac{1}{m^p} \sum_{j\in\mathcal{S}^p} x_j^p f_j^p - \varphi^p \right), \ \forall\, i \in \mathcal{S}^p, \tag{3a}$$

$$\varphi^p = \beta \left(\frac{1}{m^p} \sum_{j\in\mathcal{S}^p} x_j^p - 1 \right), \tag{3b}$$

where β is the convergence factor for the whole system that takes a positive and finite value. On the other hand, there are as many mass dynamics as intersection nodes in the graph, i.e., one for each $i \in \mathcal{I}$. The dynamics for population masses m^p are given by

$$\dot{m}_i^p = m_i^p \left(x_i - x_i^p - \psi_i \right), \ \forall\, p \in \mathcal{P}_i, \tag{4a}$$

$$\psi_i = \beta \left(\frac{1}{\kappa_i + (G(i)-1)\, x_i} \sum_{q\in\mathcal{P}_i} \frac{m_i^q}{|\mathcal{I}^q|} - 1 \right), \tag{4b}$$

where $\kappa_i \in \mathbb{R}_{>0}$ is a distribution of the social mass m. Then, it should be satisfied that $\sum_{i\in\mathcal{I}} \kappa_i = m$. There is a relationship between m_i^p, for all $i \in \mathcal{I}^p$, and the population masses m^p given by $m^p = |\mathcal{I}^p|^{-1} \sum_{i\in\mathcal{I}^p} m_i^p$, for all $p \in \mathcal{P}$. The population-game approach presented in (3), and (4) may also be used to solve, in a distributed manner, constrained optimization problems of the form in (2) applying Theorem 1. The difference with respect to the DPD is that the initial conditions for this method should only satisfy the fact that they belong to the positive orthant.

2.3 Dual decomposition (DD)

This optimization approach mainly uses the Lagrangian function associated to the optimization problem (2) as in Boyd and Vandenberghe (2004). This method is presented as a minimization problem. Then, in order to treat the optimization problem as a minimization, let $\min_{\mathbf{x}} -V(\mathbf{x}) = \tilde{V}(\mathbf{x})$ where $V(\mathbf{x})$ is a concave function, and for simplicity it is assumed that the argument, which maximizes it, belongs to the positive orthant. For the specific constrained optimization problem presented in (2), and omitting the positiveness constraint (2c) (since it is assumed the optimal point corresponds to an argument that belongs to the positive orthant), the Lagrangian function with mapping $l : \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}$ is defined as

$$l(\mathbf{x}, \mu) = \tilde{V}(x) + \mu \left(\sum_{i=1}^{n} x_i - m \right),$$

where $\mu \in \mathbb{R}$ corresponds to the Lagrange multiplier associated to the coupled constraint (2b). The dual decomposition method consists in the following algorithm:

$$x_i(k+1) = \arg\min_{x_i}\ l(\mathbf{x}(k), \mu(k)), \ \forall\, i \in \mathcal{S}, \tag{5a}$$

$$\mu(k+1) = \mu(k) + \alpha \left(\sum_{i=1}^{n} x_i(k+1) - m \right), \tag{5b}$$

where $k \in \mathbb{Z}_{\geq 0}$. For this method, notice that the Lagrange multiplier μ in (5b) requires information from all the decision variables x_i, for all $i \in \mathcal{S}$ due to the existence of a coupled constraint (2b). Furthermore, let $\mathcal{G}_2 = (\mathcal{S}_2, \mathcal{E}_2)$ be the graph representing the required communication network for this optimization method and the particular constrained problem (2), where $\mathcal{S}_2 = \{1, \ldots, n+1\}$, with $n+1 \in \mathcal{S}_2$ representing the Lagrange multiplier μ. Regarding the set of links, it is necessary to incorporate the information required by the Lagrange multiplier, then

$$\mathcal{E}_2 = \mathcal{E} \cup \{(n+1, j) : j \in \mathcal{S}\}.$$

Therefore, in comparison to the population-game approaches, one extra node and $\varepsilon = n$ extra links are required, i.e., $|\mathcal{S}_2| = |\mathcal{S}| + 1$, and $|\mathcal{E}_2| = |\mathcal{E}| + \varepsilon$.

Making distributed DD algorithm for a unique coupled constraint: The aspect that makes the DD algorithm centralized is the coupled constraint in (5b), i.e., information from all $x_i(k+1)$, for all $i \in \mathcal{S}$, is required in order to determine the evolution of the Lagrange multiplier μ, obtaining a communication graph $\mathcal{G}_2$. However, for the particular optimization problem (2), where there is only one coupled constraint, it is possible to solve the DD algorithm (5) in a distributed manner.

The main idea is to compute the sum $\sum_{i\in\mathcal{S}} x_i(k+1)$ in a distributed manner as an additional step within the DD algorithm. The execution order consists on computing (5a), then the sum $\sum_{i\in\mathcal{S}} x_i(k+1)$ is computed in a distributed manner, and it follows to compute (5b). In this regard, the DD algoritm can be performed by using the same graph $\mathcal{G}$. However, it is important to highlight that an additional process, that is explained next, should be considered.

Let $\boldsymbol{\xi} \in \mathbb{R}^n$ be a vector of auxiliary variables, i.e., a variable $\xi_i \in \mathbb{R}$ corresponding to a node $i \in \mathcal{S}$. The variables are initialized with the result obtained from (5a), i.e., $\xi_i(0) = x_i(k+1)$, for all $i \in \mathcal{S}$. Therefore, a continuous-time standard average consensus algorithm is computed, i.e.,

$$\dot{\xi}_i = \sum_{j\in\mathcal{N}_i} \xi_j - \xi_i. \tag{6}$$

According to Olfati-Saber et al. (2007), if the communication graph $\mathcal{G}$ is connected, then the dynamics in (6) converge to $\boldsymbol{\xi}^\star \in \mathbb{R}^n$, where $\xi_i^\star = \sum_{i\in\mathcal{S}} \xi_i(0)/n$, for all $i \in \mathcal{S}$. In this regard, it is obtained that the required value is

$$\frac{\xi_i^\star}{n} = \sum_{j\in\mathcal{S}} x_j(k+1), \quad \forall\, i \in \mathcal{S},$$

then, each node in the graph $\mathcal{G}$ has information about $\sum_{j\in\mathcal{S}} x_j(k+1)$ and each one can compute (5b) in a distributed manner.

2.4 Alternative direction method of multipliers (ADMM)

Inheriting the minimization problem from Section 2.3, i.e., $\min_{\mathbf{x}} \; \tilde{V}(\mathbf{x})$, an extended version of the Lagrangian function with mapping $l_\varrho : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$, and omitting the positiveness constraint (2c), is represented as in Boyd et al. (2011), i.e.,

$$l_\varrho(\mathbf{x},\mu,\varrho) = \tilde{V}(x) + \mu\left(\sum_{i=1}^{n} x_i - m\right) + \frac{\varrho}{2}\left\|\sum_{i=1}^{n} x_i - m\right\|_2^2, \quad (7)$$

where $\mu \in \mathbb{R}$ corresponds to the Lagrange multiplier associated to the coupled constraint (2b), and $\varrho \in \mathbb{R}$ penalizes decision variables that do not satisfy the coupled constraint (2b). The ADMM consists in the following algorithm:

$$x_i(k+1) = \arg\min_{x_i} \; l_\varrho(\mathbf{x}(k),\mu(k)), \quad (8a)$$

$$\mu(k+1) = \mu(k) + \varrho\left(\sum_{i=1}^{n} x_i(k+1) - m\right). \quad (8b)$$

Let $\mathcal{G}_3 = (\mathcal{S}_3, \mathcal{E}_3)$ be the graph representing the required communication network for the ADMM for the particular constrained problem (2), where $\mathcal{S}_3 = \{1,\dots,n+1\}$, with $n+1 \in \mathcal{S}_3$ representing the Lagrange multiplier μ as in the case of the DD method. Regarding the set of links, more connectivity is required for the evolution of decision variables x_i, for all $i \in \mathcal{S}$, and for the evolution of the Lagrange multiplier, i.e., $\mathcal{E}_3 = \mathcal{E} \cup \{(i,j) : i,j \in \mathcal{S}, \text{ and } (i,j) \notin \mathcal{E}\} \cup \{(n+1,a) : a \in \mathcal{S}\}$. In comparison to the population-game approaches, an extra node and $\varepsilon = [(n-1)n]/2 + n - |\mathcal{E}|$ extra links are required, i.e., $|\mathcal{S}_3| = |\mathcal{S}| + 1$, $|\mathcal{E}_3| = |\mathcal{E}| + \varepsilon$.

3. COMMUNICATION REQUIREMENTS DISCUSSION

The required communication links comparison among different optimization methods corresponding to the resource allocation problem in (2) is summarized in Table 1. According to this table, it is not necessary to add nodes neither links in order to solve the optimization problem in a distributed manner. This fact represents a solid advantage of the population-game approaches with respect to the other considered approaches.

Table 1. Additional requirements for the optimization problem in (2).

Optim. method	**Additional links ε**	**Additional nodes**
DPD	0	0
PD-MD	0	0
DD	n	1
ADMM	$\frac{(n-1)n}{2} + n - \lvert\mathcal{E}\rvert$	1

In order to illustrate a comparison of the number of required communication links among the different optimization methods, consider for instance the example given by the collection of serial triangles as presented in Figure 1. Suppose that an optimization problem of the form

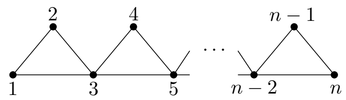

Fig. 1. Collection of serial triangle cliques composing the graph $\mathcal{G}$.

(2) is desired to be solved satisfying the communication established by the graph $\mathcal{G}$ (see Figure 1).

It is possible to solve the optimization problem with the population-game approaches (i.e., DPD, or PD-MD) without modifying the graph $\mathcal{G}$. However, in case it is desired to solve the problem by using DD or ADMM, it is necessary to add a node corresponding to the Lagrange multiplier associated to the coupled constraint, and it is also necessary to add information links. Figure 2 shows the number of additional links denoted by ε for the topology presented in Figure 1 and for different number of nodes (i.e., $n = 3,5,7,\dots$).

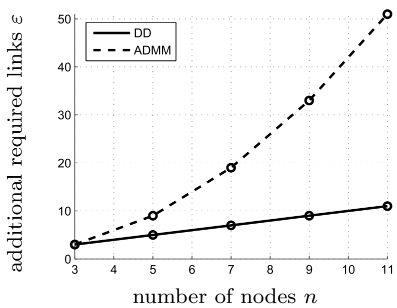

Fig. 2. Number of additional required links when implementing the DD method or the ADMM technique with respect to the population-game approaches for the optimization problem in (2), and for the collection of serial triangles in Figure 1.

It can be seen in Figure 2 that the ADMM requires more information interaction than the DD method. Besides, the difference in the amount of required extra links gets bigger as the number of decision variables increases. In addition, the population-game approaches do not require the inclusion of more communication links under the presented optimization problem. Therefore, the benefits of using the population-game approaches are more significant as the number of variables in the problem becomes larger.

4. ILLUSTRATIVE EXAMPLE AND COMMUNICATION DEPENDENCE DISCUSSION

An example is presented in order to illustrate the communication requirements and performance of the presented optimization methods, i.e., distributed population dynamics (DPD), population dynamics with mass dynamics (PD-MD), dual decomposition (DD), and alternative direction method of multipliers (ADMM). Consider the following QP problem:

$$\max_{\mathbf{x}} \; V(\mathbf{x}) = -\sum_{i=1}^{7} (\vartheta_i - x_i)^2, \quad (9a)$$

$$\text{s. t.} \; \sum_{i=1}^{7} x_i = m, \quad (9b)$$

$$x_i \geq 0, \; \forall\, i = 1,\dots,7, \quad (9c)$$

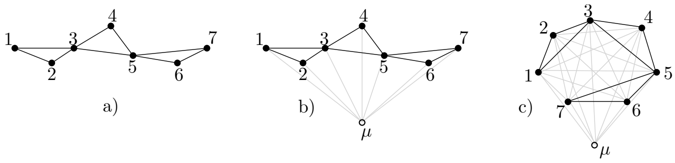

Fig. 3. Different graphs for the information requirements. a) $\mathcal{G} = (\mathcal{S}, \mathcal{E})$: DPD and PD-MD, b) $\mathcal{G}_2 = (\mathcal{S}_2, \mathcal{E}_2)$: DD, c) $\mathcal{G}_3 = (\mathcal{S}_3, \mathcal{E}_3)$: ADMM.

where $\boldsymbol{\vartheta} = [10 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7]^\top$, and $m = 37$. Suppose that (9) has limitations in the information sharing given by the graph shown in Figure 3a). Then, the optimization problem is solved with the different methods as follows.

Distributed population dynamics (DPD): In this particular case, the set of strategies is defined as $\mathcal{S} = \{1, \ldots, 7\}$. Moreover, this approach uses the function $V(\mathbf{x})$ as the potential function of a potential game. In addition, the constraint (9b) defines the set of possible strategic distributions given by the simplex with population mass m, i.e., $\Delta = \{\mathbf{x} \in \mathbb{R}^n_{\geq 0} : \sum_{i \in \mathcal{S}} x_i = 37\}$ [1]. The fitness functions are $\nabla V(\mathbf{x}) = \mathbf{F}(\mathbf{x})$, i.e., $f_i = -2(x_i - \vartheta_i)$. Notice that the required information for each proportion of agents is satisfied, then the problem is solved with the communication constraint given by the graph shown in Figure 3a) and by using the distributed replicator dynamics (1). Finally, it should be taken into account that this method requires that the initial condition belongs to the simplex, i.e., $\sum_{i \in \mathcal{S}} x_i(0) = 37$. The evolution of the decision variables for this method is shown in Figures 4a) and 4e).

Population dynamics with mass dynamics (PD-MD): Likewise, the function $V(\mathbf{x})$ is used to construct a potential game defining all the fitness functions as $\nabla V(\mathbf{x}) = \mathbf{F}(\mathbf{x})$, i.e., $f_i = -2(x_i - \vartheta_i)$. Besides, the constraint (9b) defines the social simplex with social mass m as in the DPD method. Regarding the information sharing, notice that the problem can also be solved with a social topology given by the graph shown in Figure 3a), which have three cliques and two intersection nodes. Then, the optimization problem is solved in a distributed manner by using (3) and (4). The initial conditions for this method must belong to the positive orthant. The evolution of the decision variables for this method is shown in Figures 4b), and 4f).

Dual decomposition (DD): Considering the problem (2) as a minimization of the function $\tilde{V}(\mathbf{x}) = -V(\mathbf{x})$, the Lagrangian function is as follows:

$$l(\mathbf{x}, \mu) = \sum_{j=1}^{7} (\vartheta_j - x_j)^2 + \mu \left(\sum_{j=1}^{7} x_j - 37 \right),$$

then, the $\arg\min_{x_i} \; l(\mathbf{x}(k), \mu(k))$ is found for all $i = 1, \ldots, 7$ by computing $\nabla_{x_i} l(\mathbf{x}(k), \mu(k)) = 0$. In this case,

$$x_i(k+1) = \vartheta_i - \frac{\mu(k)}{2}.$$

Consequently, it is shown that each x_i, for all $i = 1, \ldots, 7$, requires information from μ. The evolution of the Lagrange multiplier is as follows:

$$\mu(k+1) = \mu(k) + \alpha \left(\sum_{j=1}^{7} x_j(k+1) - 37 \right).$$

Notice that the evolution of the Lagrange multiplier μ also requires information about the whole system, i.e., the required topology graph is the one shown in Figure 3b). From this point of view, the DPD and PD-MD approaches have an advantage in terms of information requirements. The evolution of the decision variables for this method is shown in Figures 4c), and 4g).

Alternating Direction Method of Multipliers (ADMM): First, the same assumption as in the DD method with respect to the cost function is considered, i.e., $\min_{\mathbf{x}} \tilde{V}(\mathbf{x}) = -V(\mathbf{x})$, then the augmented Lagrangian function is

$$l_\varrho(\mathbf{x}, \mu, \varrho) = \sum_{j=1}^{7} (\vartheta_j - x_j)^2 + \mu \left(\sum_{j=1}^{7} x_j - 37 \right) + \frac{\varrho}{2} \left\| \sum_{j=1}^{7} x_j - 37 \right\|_2^2,$$

and $\arg\min_{x_i} \; l_\varrho(\mathbf{x}(k), \mu(k))$ is found for all $i = 1, \ldots, 7$ by computing $\nabla_{x_i} l_\varrho(\mathbf{x}(k), \mu(k)) = 0$, i.e.,

$$x_i(k+1) = \left(2\vartheta_i - \mu(k) - \varrho \left(\sum_{j=1:j \neq i}^{7} x_j(k) - m \right) \right) \left(\frac{1}{2+\varrho} \right), \; \forall i = 1, \ldots, 7.$$

Each x_i, for all $i = 1, \ldots, 7$, requires information from μ and from all the other decision variables x_j for all $j = 1, \ldots, 7$, such that $j \neq i$. The evolution of the Lagrange multiplier is given by

$$\mu(k+1) = \mu(k) + \varrho \left(\sum_{j=1}^{7} x_j - 37 \right). \tag{10}$$

Notice that the evolution of the Lagrange multiplier μ requires information about the whole system, i.e., the required graph is the one shown in Figure 3c). From this point of view, the DPD, and the PD-MD approaches have an advantage in terms of information requirements. The evolution of the decision variables for this method is shown in Figures 4d) and 4h).

For the example in (9), the initial conditions can belong to any point in the positive orthant for the PD-MD, DD, and ADMM, i.e., $\sum_{i=1}^{7} x_i(0) \neq m$. Additionally, both the DD and the ADMM approaches require more amount of information than the population-game approaches (i.e., DPD, PD-MD). It has been shown that the problem (9) can be solved in a distributed way with the population approach proposed in Barreiro-Gomez et al. (2017), Pantoja and Quijano (2011), or Pantoja and Quijano (2012) if and only if the initial condition is feasible, i.e., $\sum_{i=1}^{7} x_i(0) = m$.

Furthermore, consider the case with more constraints on the proportion of agents, e.g., an optimization problem of the following form:

$$\max_{\mathbf{x}} \; V(\mathbf{x}), \text{ s. t. } \mathbf{Hx} = \mathbf{h}, \; \mathbf{Gx} \leq \mathbf{g}, \; x_i \geq 0, \; \forall \, i \in \mathcal{S},$$

[1] This simplex set corresponds to the feasible set for the optimization problem (9).

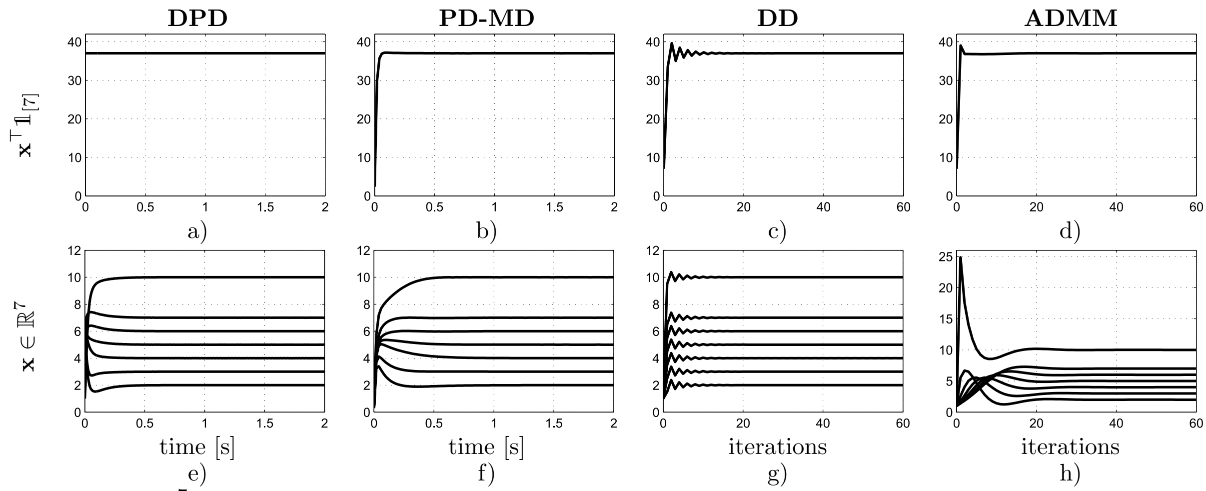

Fig. 4. Evolution of $\sum_{i=1}^{7} x_i$, and evolution of variables $\mathbf{x} \in \mathbb{R}^7$ for the problem (9) with different optimization methods. Figures a) and e) correspond to DPD. Figures b) and f) correspond to PD-MD. Figures c) and g) correspond to DD. Figures d) and h) correspond to ADMM.

where $V : \mathbb{R}^n_{\geq 0} \mapsto \mathbb{R}$ is concave, and $\mathbf{H}$, $\mathbf{G}$, $\mathbf{h}$, and $\mathbf{g}$ are matrices and vectors of suitable dimensions defining coupled constraints. This optimization problem may be solved with the PD-MD method, and by using a graph of information dependence that is the same as the graph required by the DD method. Besides, the mentioned graph is less restrictive than the one required by the ADMM approach. The fact ADMM requires more information depends on the form of the augmented Lagrangian function in (7). Therefore, even though the fact that the communication graph required by the DD method is the same as the one required by the PD-MD, the latter mentioned method has advantages for solving resource allocation problems as it has been presented in Section 4.

5. CONCLUSIONS

The communication requirements of two population-game approaches, i.e., the distributed population dynamics, and the population dynamics with mass dynamics, have been compared with the communication requirements of two classical optimization methods, i.e., dual decomposition and alternative direction of multipliers. The advantages of the population-game approaches have been highlighted. These latter methods require less information with respect to the former mentioned methods to solve the optimization problem corresponding to a resource allocation. Furthermore, comparing the population-game approaches, it has been shown that there is an advantage of the population dynamics with mass dynamics (PD-MD) approach over the distributed population dynamics (DPD) regarding the initial condition, i.e., the DPD initial condition should belong to the feasible region, whereas the initial condition for the PD-MD has to belong to the positive orthant.

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