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Decomposing almost complete graphs by random trees

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Abstract

An old conjecture of Ringel states that every tree with m edges decomposes the complete graph K_{2m+1}. The best known lower bound for the order of a complete graph which admits a decomposition by every given tree with m edges is $O(m^3)$. We show that asymptotically almost surely a random tree with m edges and $p = 2m + 1$ a prime decomposes $K_{2m+1}(r)$ for every $r \geq 2$, the graph obtained from the complete graph K_{2m+1} by replacing each vertex by a coclique of order r. Based on this result we show, among other results, that a random tree with $m + 1$ edges a.a.s. decomposes the compete graph K_{6m+5} minus one edge.

Keywords: Graph decomposition, Ringel's conjecture, Polynomial method
2000 MSC: 05C70

1. Introduction

Given two graphs H and G we say that H decomposes G if G is the edge–disjoint union of isomorphic copies of H. The following is a well–known conjecture of Ringel.

Conjecture 1 (Ringel [17]). *Every tree with m edges decomposes the complete graph K_{2m+1}.*

The conjecture has been verified by a number of particular classes of trees, see the extensive survey by Gallian [11]. Recently, there have been substantial contributions in the area of graph decomposition problems which are partly motivated by Conjecture 1 leading to impressive results. Böttcher, Hladký, Piguet and Taraz [3] show that, for any $\epsilon > 0$ and any Δ, every family of trees with orders at most n maximum degree at most Δ and at most $\binom{n}{2}$ edges in total packs into the complete graph $K_{(1+\epsilon)n}$ for every sufficiently large n. This provides an approximate result to several tree packing conjectures including Conjecture 1. The result was extended by Mesuti, Rödl and Schacht [16] from trees to general minor closed classes of bounded degree graphs and by Ferber, Lee and Moussat [9] to separable classes of bounded degree graphs. Recently, Joos, Kim, Kühn and Osthus [10] give a general result on packing trees which in particular definitely proves Conjecture 1 for bounded degree trees and sufficiently large m.

In this paper we aim at results for almost all trees. By a random tree of size m we mean an unlabelled tree chosen uniformly at random among all the unlabelled trees with m edges. Drmota and the author [8] used structural results on random trees to show that asymptotically almost surely (a.a.s.) a random tree with m edges decomposes the complete bipartite graph $K_{2m,2m}$, thus providing an aproximate result to another decomposition conjecture by Graham and Häggkvist (see e.g. [12]) which asserts that the complete bipartite graph $K_{m,m}$ can be decomposed by any given tree with m edges.

Let $g(m)$ be the smallest integer n such that any tree with m edges decomposes the complete graph K_n. It was shown by Yuster [20] that $g(m) = O(m^{10})$ and the upper bound was reduced by Kezdy and Snevily [14] to $g(m) = O(m^3)$. Since $K_{2m,2m}$ decomposes the complete graph K_{8m^2+1} (see Snevily [19]), the above mentioned result on the decomposition of $K_{2m,2m}$ shows that a tree with m edges decomposes a.a.s. the complete graph K_{8m^2+1}, giving a quadratic bound on m for almost all trees. Our aim is to reduce this quadratic bound to linear getting much closer to Ringel's conjecture.

For positive integers n, r we denote by $K_n(r)$ the blow–up graph obtained from the complete graph K_n by replacing each vertex by a coclique with order r and joining every pair of vertices which do not belong to the same coclique. Our main result is the following one.

Theorem 1. *For every $r \geq 2$, asymptotically almost surely a random tree with m edges such that $p = 2m + 1$ is a prime decomposes $K_{2m+1}(r)$.*

Theorem 1 will be derived from the following deterministic result which considers trees with sufficiently many leaves.

Theorem 2. *Let $p > 10$ be a prime and $r \geq 2$ an integer. Let T be a tree with $m = (p-1)/2$ edges. If T has at least $2m/5$ leaves, then T decomposes $K_{2m+1}(r)$.*

As an application of Theorem 1 we obtain the following corollaries, which are approximate results for random trees of Ringel's conjecture. The following statement is a direct consequence of Theorem 1 with $r = 2$.

Corollary 1. *A random tree with m edges such that $p = 2m + 1$ is a prime a.a.s. decomposes $K_{2m+2} \setminus M$, where M is any perfect matching.*

The next theorems also follow from Theorem 2 with some additional work.

Theorem 3. *A random tree with m edges such that $p = 2m - 1$ is a prime a.a.s. decomposes $K_{6m-1} \setminus e$, where e is an edge of the complete graph. .*

The following extension of Theorem 3 can be seen as an approximation to a more general conjecture by Ringel which states that every tree with m edges decomposes the complete graph K_{rm+1} whenever $r \geq 2$ and m are not both odd.

Theorem 4. *For each odd number $r \geq 3$ a random tree with m edges such that $p = 2m - 1$ is a prime a.a.s. decomposes*

$$K_{2rm-(r-1)/2} \setminus K_{(r+1)/2}.$$

The paper is organised as follows. In Section 2 we introduce the notion of rainbow embeddings in connection to graph decompositions and give some results which provide a rainbow embedding of a given tree in an appropriate Cayley graph. The embedding techniques use the polynomial method of Alon [1] which introduces the condition that $p = 2m+1$ is a prime in the statement of Theorem 1. The polynomial method was already used in this problem by Kézdy [13], who proved Conjecture 1 for the class of so–called *stunted* trees. The same method was used in [15] for a closely related problem. However, these techniques are not enough to ensure that the rainbow embedded copy is isomorphic to the given tree. In order to complete the proof of Theorem 1, in Section 3 we consider the blow up of the complete graph, extend to it the rainbow embedding obtained in Section 2 and perform some local modifications to obtain a true decomposition of the graph into copies of the given tree. The strategy of the proof is outlined in the beginning of Section 3. The proofs of Theorem 1 and of the Corollaries 1, 3 and 4 are given in Section 4.

2. Rainbow embeddings

The general approach to show that a tree T decomposes a complete graph consists in showing that T cyclically decomposes the corresponding graph. We first recall the basic principle behind this approach in slightly different terminology from the usual one in the labeling literature [11], by introducing rainbow embeddings in Cayley graphs. Rainbow embeddings of a graph H in a Cayley graph $X = Cay(G, S)$ naturally lead to decompositions of X by H by the action of the base group G. At this point we use the polynomial method to obtain rainbow embeddings of a given tree. This is the purpose of this Section. As it happens, the goal is only partially fulfilled because the embedded graph may be not isomorphic to the tree, a problem that we will address in Section 3.

Let X be a directed graph with a coloring of the arcs. A rainbow embedding of a graph H into X is an injective homomorphism f of some orientation $\overrightarrow{H}$ of H in X such that no two directed edges of $f(\overrightarrow{H})$ have the same color.

Let $X = Cay(G, S)$ be a Cayley digraph on an abelian group G with respect to an antisymmetric subset $S \subset G$ (namely, $S \cap -S = \emptyset$.) We consider X as an arc–colored directed graph, by giving to each arc $(x, x+s)$, $x \in G, s \in S$, the color s. The underlying graph of X is the graph obtained from X by ignoring the orientation of the arcs and their colors.

Lemma 1. *Let G be an abelian group and S an antisymmetric subset of G. If a graph H admits a rainbow embedding in the Cayley directed graph $X = Cay(G, S)$ then the underlying graph of X contains n edge–disjoint copies of H. In particular, if H has $|S|$ edges then H decomposes the underlying graph of X.*

Proof. Let $f : H \to X$ be a rainbow embedding. For each $a \in G$ the translation $x \to x + a$, $x \in G$, is an automorphism of X which preserves the colors and has no fixed points. Since S is antisymmetric, each translation sends $f(\overrightarrow{H})$ to an isomorphic copy which is edge–disjoint from it. Thus the sets of translations for all $a \in G$ give rise to n edge–disjoint copies of $\overrightarrow{H}$ in X. By ignoring orientations and colors, we thus have n edge disjoint copies of H in the underlying graph of X. In particular, if H has $|S|$ edges then H decomposes the underlying graph of X. □

The proof of the main Theorem uses the Lemma 1 for a rainbow subgraph of an appropriate Cayley graph X. Instead of finding a rainbow embedding of a tree T we will find a rainbow edge–injective homomorphism of T in X in two steps, first embedding T_0, the tree with some leaves removed, and then embedding the remaining forest F of stars to complete T.

For the first step we use the the so–called Combinatorial Nullstellensatz of Alon [1] that we next recall.

Theorem 5 (Combinatorial Nullstellensatz). *Let $P \in F[x_1, \ldots, x_k]$ be a polynomial of degree d in k variables with coefficients in a field F.*

If the coefficient of the monomial $x_1^{d_1} \cdots x_k^{d_k}$, where $\sum_i d_i = d$, is nonzero, then P takes a nonzero value in every grid $A_1 \times \cdots \times A_k \subset F^k$ with $|A_i| > d_i$, for $1 \le i \le k$. □

In Lemma 2 below we use Theorem 5 in a way inspired by Kézdy [13]. A *peeling ordering* of a tree T is an ordering $x_0, \ldots, x_m$ of $V(T)$ such that for every $0 \le t \le m$ the induced subgraph $T[x_0, \ldots, x_t]$ is a subtree of T. We assume that T is an oriented tree rooted at x_0 with all its edges oriented downwards from the root x_0.

We denote by $\mathbb{Z}_p$ the finite field with p elements, p a prime. Lemma 2 shows that any tree with k edges admits a rainbow embedding in a Cayley graph $Cay(\mathbb{Z}_p, S)$ for some S with $|S| = k$ provided that k is not too large with respect to p.

Lemma 2. *Let $p > 10$ be a prime and T a tree with $k < 3(p-1)/10$ edges. There is an antisymmetric set $S \subset \mathbb{Z}_p^*$ with $|S| = k$ such that T admits a rainbow embedding in $Cay(\mathbb{Z}_p, S)$.*

Proof. Let $x_0, x_1, \ldots, x_k$ be a peeling ordering of T. Label the edges of T by variables $y_1, \ldots, y_k$ such that the edge labelled y_i joins x_i with $T[x_0, x_1, \ldots, x_{i-1}]$, $0 < i \le k$. For each i we denote by $T(i)$ the set of subscripts j such that the edges y_j lie in the unique path from x_0 to x_i in T. Consider the polynomial $P \in \mathbb{Z}_p[y_1, \ldots, y_k]$ defined as

$$P(y_1, \ldots, y_k) = \prod_{1 \le i < j \le k} (y_j^2 - y_i^2) \prod_{0 \le i < j \le k} \Big(\sum_{r \in T(i)} y_r - \sum_{s \in T(j)} y_s \Big),$$

where $T(0)=\emptyset$ and $\sum_{r\in T(0)} y_r = 0$. The polynomial P has degree

$$2\binom{k}{2}+\binom{k+1}{2}=\frac{3k(k-1)}{2}+k.$$

Suppose that $P(a_1,a_2,\cdots,a_k)\neq 0$ for some point $(a_1,\dots,a_k)\in(\mathbb{Z}_p^*)^k$. Then, since the first factor $Q=\prod_{i<j}(y_i^2-y_j^2)$ of P is nonzero at $(a_1,\dots,a_k)$, we have $a_i\neq \pm a_j$ for each pair $i\neq j$. Hence the elements $a_1,\dots,a_k$ are pairwise distinct and the set $S=\{a_1,\dots,a_k\}$ is antisymmetric.

Moreover, since the second factor $R=\prod_{i<j}(\sum_{y_r\in T(i)} y_r-\sum_{y_r\in T(j)} y_r)$ is nonzero at $(a_1,\dots,a_k)$, the map $f: V(T)\to Cay(\mathbb{Z}_p,S)$ defined as

$$f(x_i)=\sum_{r\in T(i)} a_r,\ 0\leq i\leq k,$$

is injective and the edge $x_ix_{i'}$ in T joining x_i with $T[x_0,x_1,\dots,x_{i-1}]$ is sent to $f(x_i)-f(x_{i'})=a_i$ (the value of the variable y_i associated to this edge). Therefore f is a rainbow embedding of T in $Cay(\mathbb{Z}_p,S)$. Hence the Theorem will be proved if we show that P is not identically zero in $(\mathbb{Z}_p^*)^k$.

Let us show that P is nonzero at some point of $(\mathbb{Z}_p^*)^k$. To this end we consider the monomial

$$M=y_k^{3(k-1)+1}y_{k-1}^{3(k-2)+1}\cdots y_1^1,$$

in P. The monomial M has maximum degree $3k(k-1)/2+k$ and it can be obtained in the expansion of P by collecting y_k in all the factors of Q where it appears, giving $y_k^{2(k-1)}$, and also in all terms of R where it appears, which, since y_k is a leaf of T, gives y_k^k. This is the unique way to obtain $y_k^{3(k-1)+1}$ in a monomial of P. Thus the coefficient of $y_k^{3(k-1)+1}$ in P is

$$[y_k^{3(k-1)+1}]P=\pm P_{k-1},$$

where

$$P_{k-1}(y_1,\dots,y_{k-1})=\prod_{1\leq i<j\leq k-1}(y_i^2-y_j^2)\prod_{0\leq i<j\leq k-1}(\sum_{r\in T(i)} y_r-\sum_{s\in T(j)} y_s).$$

By iterating the same argument we conclude that the coefficient in P of

$$y_k^{3(k-1)+1}y_{k-1}^{3(k-2)+1}\cdots y_1^1$$

is ± 1 and, in particular, different from zero. Since $3(k-1)+1 < 9p/10 < p-1$ for $p > 10$, we conclude from Theorem 5 that P takes a nonzero value in $(\mathbb{Z}_p^*)^k$. This concludes the proof. $\square$

In the second step we try to obtain a rainbow embedding of a forest of stars. We still use Theorem 5, or rather the following consequence derived from it by Alon [2].

Theorem 6 (Alon [2]). *Let p be a prime and $k < p$. For every sequence $a_1, \ldots, a_k$ (possibly with repeated elements) and every set $\{b_1, \ldots, b_k\}$ of elements of $\mathbb{Z}_p$ there is a permutation σ of $\{1, 2, \ldots, k\}$ such that the sums $a_1 + b_{\sigma(1)}, \ldots, a_k + b_{\sigma(k)}$ are pairwise distinct.* $\square$

Let F be a forest of stars. If a component of F has more than one edge, its center is the vertex of degree largest than one and its endvertices are the vertices of degree one. If a component consists of a single edge, we distinguish one vertex as its center and the other one as its endvertex.

One consequence of Theorem 6 is that, for every antisymmetric set $S \subset \mathbb{Z}_p$ with h elements, every forest of stars with h edges admits an edge–injective rainbow map in $Cay(\mathbb{Z}_p, S)$. Moreover, the centers of the stars in the forest can be placed at prescribed vertices. The following is the precise statement, which specifies how close the image of the injective map is to the original forest.

Lemma 3. *Let p be a prime. Let F be a forest of k stars centered at $x_1, \ldots, x_k$ and $h \leq (p-1)/2$ edges. Let $S \subset \mathbb{Z}_p^*$ be an antiymmetric set with $|S| = h$.*

Every injection $f : \{x_1, \ldots, x_k\} \to \mathbb{Z}_p$ can be extended to a rainbow edge-injective homomorphism, $f_1 : F \longrightarrow Cay(\mathbb{Z}_p, S)$ such that $f_1(F)$ is a directed graph with maximum indegree one.

Proof. Let h_i be the number of edges of the star centered at x_i in F, $\sum_i h_i = h$. Let $y_1, \ldots, y_h$ denote the endvertices of F, where y_j is adjacent to x_i whenever $\sum_{r=1}^{i-1} h_r < j \leq \sum_{r=1}^{i} h_r$. Orient the edges of F from the centers of the stars to their endvertices.

Consider the sequence $(f(x_1)^{(h_1)}, \ldots, f(x_k)^{(h_k)})$, where h_i denotes the multiplicity of $f(x_i)$ in the sequence.

By Theorem 6 there is an ordering $s_1, \ldots, s_h$ of the elements of S such that for any $1 \leq i \leq k$ and any $\sum_{r=1}^{i-1} h_r < j \leq \sum_{r=1}^{i} h_r$, the sums

$$f(x_i) + s_j,$$

are pairwise distinct.

For each i and each $\sum_{r=1}^{i-1} h_r < j \leq \sum_{r=1}^{i} h_r$,, we obtain the desired rainbow embedding by defining,

$$f_1(x_i) = f(x_i), \quad f_1(y_j) = f(x_i) + s_j.$$

The edges of the star receive the different colors $s_1, \ldots, s_h$, so the map f_1 is rainbow. Since all sums are distinct, no two endvertices of F are sent to the same vertex by f_1 and each of them has indegree one in $f_1(F)$; by the same reason, every $f_1(x_i)$ can coincide with at most one $f_1(y_j)$ for some y_j not in the same star as x_i. Thus the image $f_1(F)$ has indegree at most one. □

3. The decomposition

In this Section we prove Theorem 2. The strategy of the proof is as follows. We decompose the given tree T into a tree T_0 and a forest F of stars centred at some vertices of T_0,

$$T = T_0 \oplus F.$$

By using Lemma 2 we find a rainbow embedding of T_0 in $Cay(\mathbb{Z}_p, S_0)$ where $S_0 \subset \mathbb{Z}_p^*$ is an antisymmetric set of cardinality $|S_0| = |E(T_0)|$. We choose $S_1 \subset \mathbb{Z}_p^* \setminus S_0$ such that $S = S_0 \cup S_1$ is an antisymmetric set, and use Lemma 3 to find an injective rainbow homomorphism from F to $Cay(\mathbb{Z}_p, S_1)$. In this second step we may fail to obtain an isomorphic image of F, so that the combination of the two steps produces a graph which is not isomorphic to T. The last step in the proof consists in extending the injective rainbow homomorphism to $Cay(\mathbb{Z}_p \times \mathbb{Z}_r, S \times \mathbb{Z}_r)$ and rearranging some arcs to obtain a decomposition of this directed graph into copies of T.

For a graph G and a positive integer r we denote by $G(r)$ the graph obtained form G by replacing each vertex with a coclique of order r and every edge xy in G by the complete bipartite graph joining the cocliques corresponding to x and y. Figure 1 illustrates the definition with $K_5(3)$. The same notation is used when G is a directed graph.

For the proof of Theorem 2 we will use the following technical Lemma.

Lemma 4. *Let $r \geq 2$ be an integer and let $M = (M_a, M_b)$ be the matrix*

$$\left(\begin{array}{cccc|cccc} 1 & 2 & \cdots & r & \sigma_1 & \sigma_2 & \cdots & \sigma_r \\ r+1 & r+2 & \cdots & 2r & \sigma_{r+1} & \sigma_{r+2} & \cdots & \sigma_{2r} \\ \vdots & & & \vdots & \vdots & & & \vdots \\ r(r-1)+1 & r(r-1)+2 & \cdots & r^2 & \sigma_{r(r-1)+1} & \sigma_{r(r-1)+2} & \cdots & \sigma_{r^2} \end{array}\right).$$

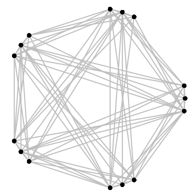


Figure 1: The blow-up $K_5(3)$ of K_5.

where $(\sigma_1, \ldots, \sigma_{r^2})$ *is a permutation of* $\{1, \ldots, r^2\}$.

There are permutations of the elements in each column of M *in such a way that the resulting matrix* M' *has no row with repeated entries.*

Proof. We proceed row by row. By the definition of M, each column has r distinct entries. Let $M_{a,i}$ be the set of entries in the i-th column of M_a and $M_{b,j}$ be the set of entries in the j-th column of M_b.

We use Hall's theorem to find a transversal of the family

$$\mathcal{M} = \{M_{a,1}, M_{a,2}, \cdots M_{a,r}, M_{b,1}, M_{b,2}, \cdots M_{b,r}\}.$$

For each pair of subsets $I, J \subset \{1, 2, \ldots, r\}$ we have,

$$|I| + |J| \leq 2\max\{|I|, |J|\} \leq r\max\{|I|, |J|\} \leq |(\cup_{i\in I} M_{a,i}) \cup (\cup_{j\in J} M_{b,j})| \quad (1)$$

which shows that Hall's condition holds and therefore $\mathcal{M}$ has a transversal. We place this transversal in the first row of the new matrix M'.

By deleting each element of the transversal from its set of $\mathcal{M}$ we get a family of $(r-1)$–sets for which the inequalities in (1) hold with r replaced by $(r-1)$ as long as $r-1 \geq 2$. Hence there is a transversal of this new family of sets which we place in the second row of M'. We can proceed with the same argument up to the $(r-1)$ row. Now, if each of the first $r-1$ rows of M' have their entries pairwise distinct, the remaining elements are also pairwise distinct and can be placed in the last row of M'. □

The next Lemma ensures the existence of a rainbow copy of a graph which is itself the edge–disjoint union of r copies of a given tree T in $Cay(\mathbb{Z}_p \times \mathbb{Z}_r, S \times \mathbb{Z}_r)$. Combined with Lemma 1 it will lead to a proof of Theorem 2.

Lemma 5. *Let* $r \geq 2$ *and let* p *be a prime. Let* T *be a tree with* $m = (p-1)/2$ *edges and at least* $2m/5$ *leaves.*

There is an antisymmetric set $S \subset \mathbb{Z}_p^*$ *with* $|S| = (p-1)/2$ *and a rainbow edge–injective homomorphism of* T *in* $X = Cay(\mathbb{Z}_p, S)$ *such that* *(i)* $H = f(T)$ *has in–degree at most two, and*

(ii) $H(r) \subset X(r) = Cay(\mathbb{Z}_p \times \mathbb{Z}_r, S \times \mathbb{Z}_r)$ *admits a decomposition into* r^2 *copies of* T

$$H(r) = T_1 \oplus T_2 \oplus \cdots \oplus T_{r^2},$$

and

(iii) $\pi(T_i) = H$ *for each* $1 \le i \le r^2$*, where* $\pi : \mathbb{Z}_p \times \mathbb{Z}_r \to \mathbb{Z}_p$ *is the canonical projection.*

Proof. Remove $\lceil 2m/5 \rceil$ leaves from T and denote by T_0 the resulting tree. Let F be the forest of stars with centers in vertices of T_0 so that

$$T = T_0 \oplus F.$$

We split the proof of the Lemma into three steps.

Step 1. *Define a rainbow edge–injective homomorphism of* T *into* $X = Cay(\mathbb{Z}_p, S)$ *where* $S \subset \mathbb{Z}_p^*$ *is an antisymmetric set with* $|S| = (p-1)/2$.

Let $t \le 3m/5 < 3(p-1)/10 < (p-1)/3$ be the number of edges of T_0. By Lemma 2, there is an antisymmetric subset $S_0 \subset \mathbb{Z}_p^*$ with $|S_0| = t$ and a rainbow embedding

$$f_0 : T_0 \to Cay(\mathbb{Z}_p, S_0).$$

Let $x_0, \ldots, x_t$ be a peeling ordering of T_0. Since $t > \lceil 2m/5 \rceil$, we may assume that x_0 is not incident to a leaf in F. By exchanging elements s of S by their opposite ones $-s$ if necessary, we may assume that $f_0(T_0)$ has all its edges oriented from x_0 to the leaves of T_0. By abuse of notation we still denote by $x_0, \ldots, x_t$ the images of the vertices of T_0 by f_0. We may assume that $f_0(x_0) = 0$.

Let S be an antisymmetric subset of $\mathbb{Z}_p^*$ with $|S| = (p-1)/2$ which contains S_0, so that $|S - S_0| = |E(F)|$. Let $x_{i_1} = v_1, \cdots, , x_{i_k} = v_k$ be the centers of the stars of F. By Lemma 3 there is an edge–injective rainbow homomorphism of the forest F into $Cay(\mathbb{Z}_p, S \setminus S_0)$,

$$f_1 : F \to Cay(\mathbb{Z}_p, S \setminus S_0),$$

such that $f_1(v_i) = f_0(v_i)$, $i = 1, \ldots, k$. Moreover $\tilde{F} = f_1(F)$ is an oriented graph with maximum in–degree one.

The map $f : V(T) \to Cay(\mathbb{Z}_p, S)$ defined by f_0 on $V(T_0)$ and by f_1 on $V(F)$ is well defined, since $f_1(v_i) = f_0(v_i)$, and the two graphs $f_0(T_0), f_1(F)$ are edge–disjoint, so that

$$f(T) = f_0(T_0) \oplus f_1(F) = H$$

is a rainbow subgraph of $X = Cay(\mathbb{Z}_p, S)$.

We note that f may fail to be a rainbow embedding of T in $X = Cay(\mathbb{Z}_p, S)$ to the effect that some endvertices of T may have been sent by f_1 to some vertices of $f_0(V(T_0))$. Thus H may be not isomorphic to T and contain some cycles (see Figure 2 for an illustration.)

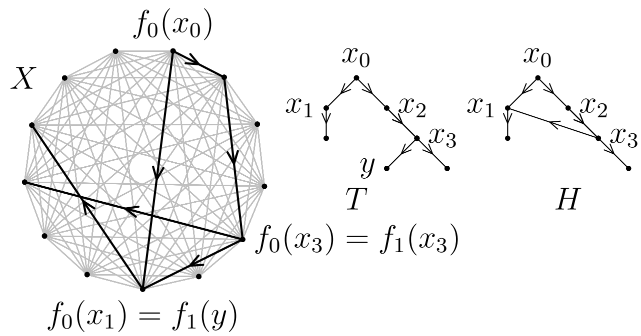


Figure 2: A rainbow map of T with conflicting arcs at $f_0(x_1) = f_1(y)$.

We observe however that, if $f_1(y) = f_0(x)$ for some endvertex $y \in F$ and some $x \in V(T_0)$, then y is not adjacent to x in T because f_1 is an edge–injective homomorphism. In other words, $H = f(T)$ has maximum in–degree at most two. This proves (i).

Step 2. *Extending the rainbow map from $X = Cay(\mathbb{Z}_p, S)$ to $X(r) = Cay(\mathbb{Z}_p \times \mathbb{Z}_r, S \times \mathbb{Z}_r)$.*

For each pair $i, j \in \mathbb{Z}_r$ we define an injective homomorphism

$$f_{ij} : H \to X(r),$$

by $f_{ij}(0) = (0, i)$ and every arc $(x, x+s) \in E(H)$ is sent to the arc $(f_{ij}(x), f_{ij}(x)+(s, j))$ of $E(X(r))$. Since H is a connected subgraph of X, the map f_{ij} is well defined. Let

$$H_{ij} = f_{ij}(H).$$

We observe that, by the definition, H_{ij} is a rainbow subgraph of $X(r)$. Figure 3 illustrates, for $p = 13$, the subgraphs $H_{0,j}$ corresponding to the example of Figure 2.

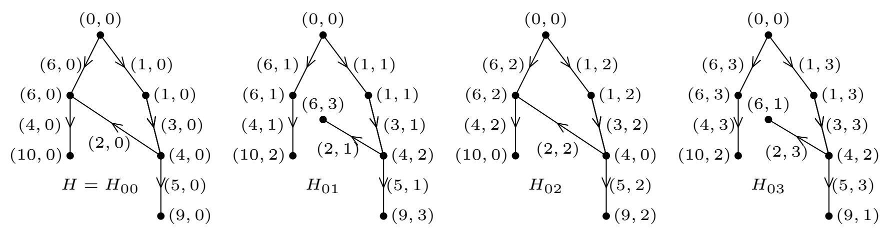


Figure 3: The rainbow subgraphs $H_{0,j} = f_{0j}(H)$ of $X(4) = Cay(\mathbb{Z}_{13} \times \mathbb{Z}_4, S \times \mathbb{Z}_4)$.

Every H_{ij} can be decomposed into

$$H_{ij} = T_{ij} \oplus F_{ij},$$

where, since T_0 is acyclic, T_{ij} is isomorphic to T_0. As in the Step 1, H_{ij} may be non isomorphic to the original tree T, but only due to the fact that some end vertex of F_{ij} may have been identified with some vertex of T_{ij}. However, the in–degree of every vertex in H_{ij} is again at most two as this was the case in H. If there is a vertex with indegree two in H_{ij} we call its incoming arcs to be *conflicting*.

We note that the H_{ij}'s are edge-disjoint (they hold pairwise distinct labels for j fixed and these labels emerge from distinct vertices for each i). Let

$$H_i = \oplus_{0 \le j < r} H_{ij}.$$

By the definition of f_{ij}, we observe that each H_i is a rainbow subgraph of $X(r)$ with $r(p-1)/2$ edges, so that all colors of the generating set $S \times \mathbb{Z}_r$ of $X(r)$ appear in H_i precisely once. Hence, if there are no conflicting arcs (all vertices of indegree one) in H_{ij} then H_{ij} is a rainbow copy of T.

We observe that

$$\oplus_{0 \le i < r} H_i = \oplus_{0 \le i,j < r} H_{ij} = H(r), \tag{2}$$

since, for every edge in H, there are r^2 edges between the corresponding cocliques in $\oplus_{0 \le i,j < r} H_{ij}$.

Step 3. *The final step consists of modifying each H_{ij} into H'_{ij}, which will be isomorphic to the original tree T, in such a way that,*

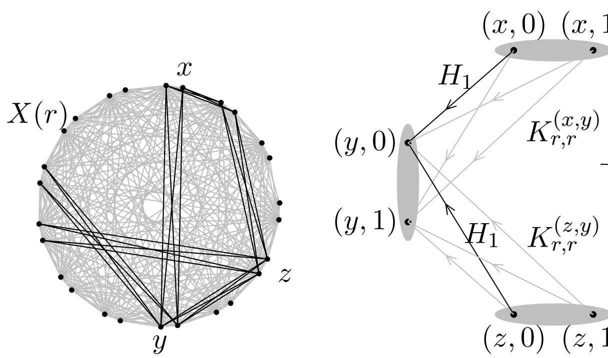
$$H(r) = \oplus_{0\leq i,j<r} H'_{ij}.$$

In this step we will perform some local modifications to the H_{ij} in order to eliminate its conflicting arcs, that is, to obtain H'_{ij} with all vertices of indegree one and isomorphic to T.

Each arc (x, y) in H is split in $H(r)$ into a (oriented) complete bipartite graph $K_{r,r}$ that we denote by $K_{r,r}^{(x,y)}$. The H'_{ij} will be constructed by rearranging the arcs in $K_{r,r}^{(x,y)}$ whenever y has indegree two in H. This rearrangement of arcs will be performed locally not affecting the remaining arcs of H_{ij}.

Suppose that $y = f_1(u)$, where $y \in V(T_0)$ and $u \in V(F)$, so that y is incident with a conflicting arc of H.

Let x be the vertex of T_0 adjacent to y in T_0 and let $z \neq x$ be the vertex of F adjacent to y in H (which creates an undesired cycle as illustrated in Figure 2.)



	$(x, 0)$	$(x, 1)$	$(z, 0)$	$(z, 1)$
$(y, 0)$	1	2	1	4
$(y, 1)$	3	4	3	2

Figure 4: Conflicting arcs at y are $((x, 0), (y, 0))$ and $((z, 0), (y, 0))$, both belonging to H_1 (in solid lines in the figure), and $((x, 0), (y, 1))$, $((z, 0), (y, 1))$, both belonging to H_3. There are no conflicts in H_2 or H_4 in this example, according to the matrix (M_{xy}, M_{zy}) on the righthand side of the figure.

Each edge in $K_{r,r}^{(x,y)}$ belongs to one of r^2 trees T_{ij} isomorphic to T_0 in the decomposition (2) of $H(r)$ and likewise, each edge in $K_{r,r}^{(z,y)}$ belongs to one of the F_{ij}. For simplicity we label these copies with the numbers $(i + 1) + rj \in \{1, 2, \ldots, r^2\}$ and denote H_{ij} by H_s with $s = (i + 1) + rj$. We thus have copies $H_1, \ldots, H_{r^2}$.

To each directed complete bipartite graph $K_{r,r}^{(x,y)}$ in $X(r)$ we associate an $(r \times r)$ matrix M_{xy} where the entry (i,j) in M_{xy} is s if the arc $((x,i),(y,j)$ belongs to H_s. Without loss of generality we may assume that the matrix (M_{xy}, M_{zy}) is

$$\left(\begin{array}{cccc|cccc} 1 & 2 & \cdots & r & \sigma_1 & \sigma_2 & \cdots & \sigma_r \\ r+1 & r+2 & \cdots & 2r & \sigma_{r+1} & \sigma_{r+2} & \cdots & \sigma_{2r} \\ \vdots & & & \vdots & \vdots & & & \vdots \\ r(r-1)+1 & r(r-1)+2 & \cdots & r^2 & \sigma_{r(r-1)+1} & \sigma_{r(r-1)+2} & \cdots & \sigma_{r^2} \end{array}\right),$$

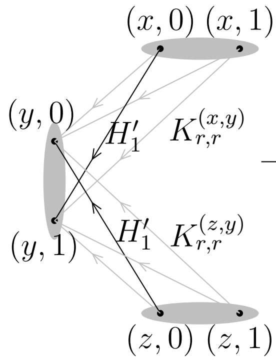
for some permutation $\sigma = (\sigma_1, \ldots, \sigma_{r^2})$ of $\{1, \ldots, r^2\}$. The righthand side of Figure 4 displays an example of such a matrix.

If all the rows of (M_{xy}, M_{zy}) have pairwise distinct entries, then every vertex in $y \times \mathbb{Z}_r$ has indegree one in each H_s. If this is the case for every conflicting arc then each H_s is a rainbow isomorphic copy of T and our task in this Step 3 is finished.

Suppose on the contrary that there are rows with repeated entries in (M_{xy}, M_{zy}). By Lemma 4, there is a matrix $M' = (M'_{xy}, M'_{zy})$ obtained from M by permuting the entries within columns which have no repeated entries in the same row. We use M' as a new incidence matrix of arcs to copies, which amounts to redistribute the edges in $K_{r,r}^{(x,y)}$ and $K_{r,r}^{(z,y)}$ among the copies of H. Since M'_{xy} still has all entries pairwise distinct, each copy of H has exactly one edge of $K_{r,r}^{(x,y)}$ assigned to it, and the same is true for $K_{r,r}^{(z,y)}$. Since rows of (M'_{xy}, M'_{yz}) have no repeated entries, each vertex in $y \times \mathbb{Z}_r$ has indegree one in the resulting copies of H. Figure 3 illustrates this application of Lemma 4 in our example.

Our local rearrangement is completed by performing, for each vertex u adjacent from y in T_0, the same permutations in the matrix M_{uy} as the ones made in M_{xy} to obtain M'_{xy} . The purpose of this additional rearrangement is to make sure that the modified T'_{ij} is isomorphic to T_0 (otherwise a copy may land at (y, i') from (x, j) and continue from a vertex (y, i), $i \neq i'$, to a vertex in (u, j'), see Figure 3 for an illustration in our example.)

We can make the local arrangements described above by following the original peeling order of T_0. We proceed to modify the distribution of the arcs as we encounter vertices incident with conflicting arcs in that order. In this way we travel through directed arcs from the root of each T_{ij}, so that rearrangements of arcs do not affect modifications made previously until all conflicting arcs have been processed. This completes Step 3.



	$(x,0)$	$(x,1)$	$(z,0)$	$(z,1)$
$(y,0)$	3	2	1	4
$(y,1)$	1	4	3	2

Figure 5: Distribution of arcs after rearrangment: the conflict in H_1 has been eliminated.

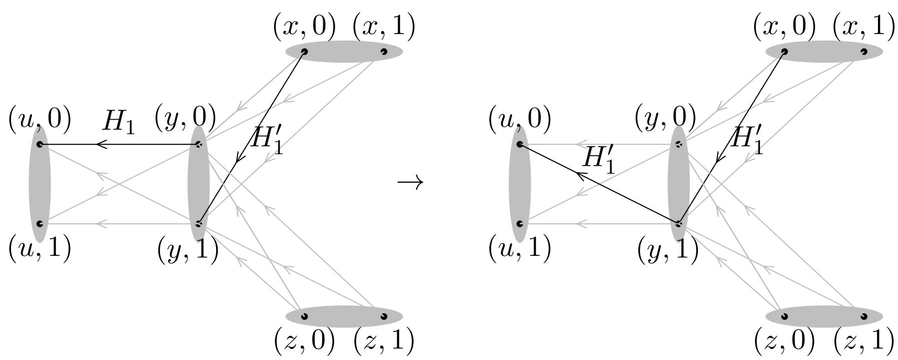


Figure 6: Completing the local rearrangament at $K_{r,r}^{(y,u)}$.

At this point we obtain an edge decomposition of $H(r)$ into the r^2 oriented graphs $H'_{i,j}$, each one isomorphic to our given tree T. Since each H'_{ij} is obtained from H_{ij} only by rearrangements within $K_{r,r}^{(x,y)}$ for some edges $xy \in T$, we still have $\pi(H'_{ij}) = H$. This completes the proof. $\square$

4. Proofs of the results

In this section we include the proofs of the statements from the Introduction. We start with the proof of Theorem 2

Proof of Theorem 2 By Lemma 5 there is a subgraph $H(r)$ of $X(r) = Cay(\mathbb{Z}_p \times \mathbb{Z}_r, S \times \mathbb{Z}_r)$ which is itself an edge–disjoint union of r^2 copies of T. Since $H = \pi(H(r)$, where $\pi : \mathbb{Z}_p \times \mathbb{Z}_r \to \mathbb{Z}_p$ is the canonical projection, is a rainbow subgraph of $X = Cay(\mathbb{Z}_p, S)$ and $H(r)$ has mr^2 edges, the set of translates

$$\{H(r) + (x, 0) : \ x \in \mathbb{Z}_p\}$$

is a decomposition of $X(r)$. Therefore T decomposes the underlying graph of $X(r)$, which is isomorphic to $K_{2m+1}(r)$. □

Theorem 2 leads directly to a proof of Theorem 1 by using known results on random trees.

Proof of Theorem 1. Robinson and Schwenk [18] proved that the average number of leaves in an (unlabelled) random tree with m edges is asymptotically cm with $c \approx 0.438$. Drmota and Gittenberger [6] showed that the distribution of the number of leaves in a random tree with m edges is asymptotically normal with variance $c_2 m$ for some positive constant c_2. Thus, asymptotically almost surely a random tree with m edges has more than $2m/5$ leaves. It follows from Lemma 2 that a tree with at least $2m/5$ leaves decomposes $K_{2m+1}(r)$ for each $r \geq 2$ and $m = (p-1)/2 \geq 5$ edges, where $p > 10$ is a prime. □

Corollary 1 follows from Theorem 1 with $r = 2$, because $K_{2m+1}(2)$ is isomorphic to $K_{4m+2} \setminus M$, for M any matching of K_{4m+2}. □

Theorem 3 will follow from next deterministic result in the same way as Theorem 1 follows from Theorem 2.

Theorem 7. *Let p be a prime and let T be a tree with $m = (p+1)/2$ edges. If T has more than $2m/5$ leaves then T decomposes $K_{6m-1} \setminus e$ for every edge e of the complete graph.*

Proof. Let z be a vertex of degree one in T and let yz be the edge joining z to the tree. Let $T' = T \setminus yz$. The tree T' has $(p-1)/2$ edges and at at least $2(m-1)/5$ leaves.

By Lemma 5 there is an antisymmetric set $S \subset \mathbb{Z}_p$ with $|S| = (p-1)/2$ and a rainbow subgraph H of $X = Cay(\mathbb{Z}_p, S)$ with maximum in–degree two such that $H(r) \subset X(3) = Cay(\mathbb{Z}_p \times \mathbb{Z}_3, S \times \mathbb{Z}_3)$ can be decomposed into 9 copies of T', each of them with the property that the their image by the canonical projection $\pi : \mathbb{Z}_p \times \mathbb{Z}_3 \to \mathbb{Z}_p$ is H. Let y' the vertex in H which is the image of our vertex y in T' by π.

We add two additional vertices α, β to $X(3)$ and make them adjacent from every vertex in $X(3)$. Moreover we add to $X(3)$ an oriented triangle in each coclique. The underlying graph is $K_{6m+5} \setminus e$, where $e = \{\alpha, \beta\}$.

Suppose first that y' has in–degree one in H. In this case each of $(y,0),(y,1),(y,2) \in$ has in–degree three in $H(3)$. We assign the three arcs added to $X(3)$ from each (y,j) to one of its three incoming trees bijectively. By repeating this procedure to each translate $H(3)+(z,0)$, $z \in \mathbb{Z}_p$, in $X(3)$ we obtain a decomposition of $K_{6m+5} \setminus e$, into copies of T. This completes the proof in this case.

Suppose now that y' has in–degree two in H. In this case each of $(y,0),(y,1),(y,2) \in$ has in–degree six in $H(3)$. There are nine trees in total incident to the three vertices $(y,0),(y,1),(y,2)$ in $H(3)$, label them $T_1', \dots, T_9'$. Without loss of generality we may assume that

$$\begin{aligned} (y,0) &\text{ is incident from } T_1', T_2', T_3', T_4', T_5', T_6' \\ (y,1) &\text{ is incident from } T_1', T_2', T_3', T_7', T_8', T_9' \\ (y,0) &\text{ is incident from } T_4', T_5', T_6', T_7', T_8', T_9'. \end{aligned}$$

In this case each tree T_i' can be completed to a copy of T by adding to it one arc as depicted in Figure 7. By repeating this procedure to each translate

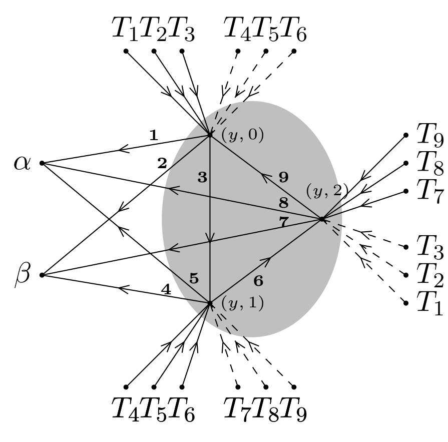


Figure 7: Completion of copies of T' to copies of T.

$H(3)+(z,0)$, $z \in \mathbb{Z}_p$, in $X(3)$ we obtain a decomposition of $K_{6m+5} \setminus e$, into copies of T. This completes the proof. □

An argument analogous to the one used in the proof of Theorem 7 can be extended to prove Theorem 4.

Proof of Theorem 4: We imitate the proof of Theorem 7. Choose avertex y of degree one in T and delete the edge xy so that the resulting tree T' has m edges and at least $2m/5$ end vertices. By Lemma 2 we obtain a decomposition of $X(r) = Cay(\mathbb{Z}_{2m+1} \times \mathbb{Z}_r, S \times \mathbb{Z}_r)$ by copies of an orientation of T'.

Consider the oriented graph $X'(r)$ obtained from $X(r)$ by adding $(r+1)/2$ new vertices $\alpha_1, \ldots, \alpha_{(r+1)/2}$ and all arcs from $X(r)$ to these vertices. Moreover we insert a regular tournament T_r in each stable set of $X(r)$. By removing the orientations, $X(r)'$ is isomorphic to $K_{r(2m+1)+\frac{r+1}{2}} \setminus K_{(r+1)/2}$ (the vertices form a stable set in Y'.)

We next add one leaf to each copy of T' by using the $(r+1)/2$ arcs to $\alpha_1, \ldots, \alpha_{(r+1)/2}$ and the $(r-1)/2$ arcs in the regular tournament through that vertex. This results in r copies of T in $X(r)'$. As in the proof of Theorem 7, this addition is straightforward if there are no conflicting arcs in the rainbow subgraph H used to obtain the decomposition of $X(r)$ through Lemma 5, and requires an argument otherwise. We omit the details of this last argument here. □

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