

FORBIDDEN SUBGRAPHS IN THE NORM GRAPH

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ABSTRACT. We show that the norm graph with n vertices about $\frac{1}{2}n^{2-1/t}$ edges, which contains no copy of the complete bipartite graph $K_{t,(t-1)!+1}$, does not contain a copy of $K_{t+1,(t-1)!-1}$.

1. INTRODUCTION

Let H be a fixed graph. The *Turán number* of H , denoted $ex(n, H)$, is the maximum number of edges a graph with n vertices can have, which contains no copy of H . The Erdős-Stone theorem from [7] gives an asymptotic formula for the Turán number of any non-bipartite graph, and this formula depends on the chromatic number of the graph H .

When H is a complete bipartite graph, determining the Turán number is related to the “Zarankiewicz problem” (see [3], Chap. VI, Sect.2, and [9] for more details and references). In many cases even the question of determining the right order of magnitude for $ex(n, H)$ is not known.

Let $K_{t,s}$ denote the complete bipartite graph with t vertices in one class and s vertices in the other. Kővari, Sós and Turán [14] proved that for $s \geq t$

$$(1.1) \quad ex(n, K_{t,s}) \leq \frac{1}{2}(s-1)^{1/t}n^{2-1/t} + \frac{1}{2}(t-1)n.$$

The norm graph $\Gamma(t)$, which we will define the next section, has n vertices and about $\frac{1}{2}n^{2-1/t}$ edges. In [1] (based on results from [13]) it was proven that the graph $\Gamma(t)$ contains no copy of $K_{t,(t-1)!+1}$, thus proving that for $s \geq (t-1)!+1$,

$$ex(n, K_{t,s}) > cn^{2-1/t}$$

for some constant c .

In [2], it was shown that $\Gamma(4)$ contains no copy of $K_{5,5}$, which improves on the probabilistic lower bound of Erdős and Spencer [6] for $ex(n, K_{5,5})$. In this article, we will generalise

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this result and prove that $\Gamma(t)$ contains no copy of $K_{t+1,(t-1)!-1}$. For $t \geq 5$, this does not improve the probabilistic lower bound of Erdős and Spencer,

$$ex(n, K_{t,s}) \geq cn^{2-(s+t-2)/(st-1)}.$$

As far as we are aware, it is however the deterministic construction of a graph with n vertices containing no $K_{t+1,(t-1)!-1}$ with the most edges.

2. THE NORM GRAPH

Suppose that $q = p^h$, where p is a prime, and denote by \mathbb{F}_q the finite field with q elements. We will use the following properties of finite fields. For any $a, b \in \mathbb{F}_q$, $(a+b)^{p^i} = a^{p^i} + b^{p^i}$, for any $i \in \mathbb{N}$. For all $a \in \mathbb{F}_{q^i}$, $a^q = a$ if and only if $a \in \mathbb{F}_q$. Finally $N(a) = a^{1+q+\dots+q^{k-1}} \in \mathbb{F}_q$, for all $a \in \mathbb{F}_{q^k}$, since $N(a)^q = N(a)$.

Let \mathbb{F} denote an arbitrary field. We denote by $\mathbb{P}_n(\mathbb{F})$ the projective space arising from the $(n+1)$ -dimensional vector space over \mathbb{F} . Throughout \dim will refer to projective dimension. A point of $\mathbb{P}_n(\mathbb{F})$ (which is a one-dimensional subspace of the vector space) will often be written as $\langle u \rangle$, where u is a vector in the $(n+1)$ -dimensional vector space over \mathbb{F} .

Let $\Gamma(t)$ be the graph with vertices $(a, \alpha) \in \mathbb{F}_{q^{t-1}} \times \mathbb{F}_q$, $\alpha \neq 0$, where (a, α) is joined to (a', α') if and only if $N(a + a') = \alpha\alpha'$. The graph $\Gamma(t)$ was constructed in [13], where it was shown to contain no copy of $K_{t,t!+1}$. In [1] Alon, Rónyai and Szabó proved that $\Gamma(t)$ contains no copy of $K_{t,(t-1)!+1}$. Our aim here is to show that it also contains no $K_{t+1,(t-1)!-1}$, generalizing the same result for $t = 5$ presented in [2].

Let

$$V = \{(1, a) \otimes (1, a^q) \otimes \dots \otimes (1, a^{q^{t-2}}) \mid a \in \mathbb{F}_{q^{t-1}}\} \subset \mathbb{P}_{2^{t-1}-1}(\mathbb{F}_{q^{t-1}}).$$

The set V is the affine part of an algebraic variety that is in turn a subvariety of the Segre variety

$$\Sigma = \underbrace{\mathbb{P}_1 \times \mathbb{P}_1 \times \dots \times \mathbb{P}_1}_{t-1 \text{ times}},$$

where $\mathbb{P}_1 = \mathbb{P}_1(\mathbb{F}_q)$.

The affine point $(1, a) \otimes (1, a^q) \otimes \dots \otimes (1, a^{q^{t-2}})$ has coordinates indexed by the subsets of $T := \{0, 1, \dots, t-1\}$, where the S -coordinate is

$$\left(\prod_{i \in S} a^{q^i} \right),$$

for any non-empty subset S of T and

$$\prod_{i \in S} a^{q^i} = 1$$

when $S = \emptyset$ (see [15]).

Let $n = 2^{t-1} - 1$.

We order the coordinates of $\mathbb{P}_n(\mathbb{F}_{q^{t-1}})$ so that if the i -th coordinate corresponds to the subset S , then the $(n - i)$ -th coordinate corresponds to the subset $T \setminus S$.

Embed the $\mathbb{P}_n(\mathbb{F}_{q^{t-1}})$ containing V as a hyperplane section of $\mathbb{P}_{n+1}(\mathbb{F}_{q^{t-1}})$ defined by the equation $x_{n+1} = 0$.

Let β be the symmetric bilinear form on the $(n + 2)$ -dimensional vector space over $\mathbb{F}_{q^{t-1}}$ defined by

$$\beta(u, v) = \sum_{i=0}^n u_i v_{n-i} - u_{n+1} v_{n+1}.$$

Let \perp be defined in the usual way, so that given a subspace Π of $\mathbb{P}_{n+1}(\mathbb{F}_{q^{t-1}})$, Π^\perp is the subspace of $\mathbb{P}_{n+1}(\mathbb{F}_{q^{t-1}})$ defined by

$$\Pi^\perp = \{v \mid \beta(u, v) = 0, \text{ for all } u \in \Pi\}.$$

We wish to define the same graph $\Gamma(t)$, so that adjacency is given by the bilinear form. Let $P_\infty = (0, 0, 0, \dots, 1)$. Let Γ' be a graph with vertex set the set of points on the lines joining the points of V to P_∞ obtained using only scalars in \mathbb{F}_q , distinct from P_∞ and not contained in the hyperplane $x_{n+1} = 0$. Join two vertices $\langle u \rangle$ and $\langle u' \rangle$ in Γ' if and only if $\beta(u, u') = 0$. It is a simple matter to verify that the graph Γ' is isomorphic to the graph $\Gamma(t)$ since

$$N(a + b) = \sum_{S \subseteq T} \prod_{i \in S, j \in T \setminus S} a^{q^i} b^{q^j} = \beta(u, v) + u_{n+1} v_{n+1},$$

where

$$u = (1, a) \otimes (1, a^q) \otimes \cdots \otimes (1, a^{q^{t-2}}),$$

and

$$v = (1, b) \otimes (1, b^q) \otimes \cdots \otimes (1, b^{q^{t-2}}).$$

We shall refer to Γ' as $\Gamma(t)$ from now on.

We recall some known properties of Σ and its subvariety

$$\mathcal{V} = \{(a, b) \otimes (a^q, b^q) \otimes \cdots \otimes (a^{q^{t-2}}, b^{q^{t-2}}) \mid (a, b) \in \mathbb{P}_1(\mathbb{F}_{q^{t-1}})\}$$

and prove a new one in Theorem 2.5.

Let $\overline{\mathbb{F}_q}$ denote the algebraic closure of \mathbb{F}_q and consider Σ as the Segre variety over $\overline{\mathbb{F}_q}$.

THEOREM 2.1. Σ is a smooth irreducible variety.

THEOREM 2.2. The dimension of Σ (as algebraic variety) is $t - 1$ and its degree is $(t - 1)!$.

Proof. The (Segre) product $X \times Y$ of two varieties X and Y of dimension d and e has dimension $d + e$, see, for example [12], page 138. The Hilbert polynomial of $X \times Y$ is the product of the Hilbert polynomials of X and Y (see [12, Chapter 18]). The Hilbert

polynomial $h(m)$ of \mathbb{P}_1 is $m + 1$, hence the Hilbert polynomial of $\Sigma = \underbrace{\mathbb{P}_1 \times \mathbb{P}_1 \times \cdots \times \mathbb{P}_1}_{t-1 \text{ times}}$ is $h_\Sigma(m) = (m + 1)^{t-1}$. Since the leading term of h_Σ is 1 and the dimension of Σ is $t - 1$, we have that the degree of Σ is $(t - 1)!$. \square

THEOREM 2.3. [15] *Any t points of \mathcal{V} are in general position.*

THEOREM 2.4. [11] *If $t + 1$ points span a $(t - 1)$ -dimensional projective space, then that space contains $q + 1$ points of \mathcal{V} .*

THEOREM 2.5. *If a subspace of codimension t contains a finite number of points of Σ then it contains at most $(t - 1)! - 2$ points of Σ .*

Proof. By Theorem 2.1, Σ is smooth, so it is regular at each of its points, i.e., if $T_P\Sigma$ is the tangent space of Σ at a point $P \in \Sigma$, then $\dim T_P\Sigma = t - 1$.

Let Π be a subspace of codimension t containing a finite number of points of Σ . Let $P \in \Pi \cap \Sigma$. Then $\dim \langle T_P\Sigma, \Pi \rangle \leq n - 1$. Therefore, there is a hyperplane H containing $\langle T_P\Sigma, \Pi \rangle$.

Suppose that H contains another tangent space $T_R\Sigma$, with $R \in \Pi \cap \Sigma$. The algebraic variety $H \cap \Sigma$ has dimension $t - 2$ (since Σ is irreducible) and it has two singular points, P and R . Since $\dim H \cap \Sigma = t - 2$ as an algebraic variety, there must be a linear subspace Π_1 of codimension $t - 2$ in H containing Π and such that $\Pi_1 \cap H \cap \Sigma$ consists of $\deg(H \cap \Sigma) \leq (t - 1)!$ points of Σ counted with their multiplicity. Since Π_1 contains P and R , which are singular points and so with multiplicity at least 2, we have that

$$|\Pi \cap \Sigma| \leq |\Pi_1 \cap \Sigma| \leq (t - 1)! - 2.$$

Suppose now that H does not contain any other tangent space $T_R\Sigma$ with $R \in \Pi \cap \Sigma$, $R \neq P$. Then take $R \in \Pi \cap \Sigma$ and consider a hyperplane $H' \neq H$ containing $\langle T_R\Sigma, \Pi \rangle$. Then the tangent spaces of P and R with respect to $H \cap H' \cap \Sigma$ are $T_P\Sigma \cap H'$ and $T_R\Sigma \cap H$, and they both have dimension $t - 2$ (as linear spaces).

If $\dim H \cap H' \cap \Sigma = t - 3$ as an algebraic variety, then P and R are two singular points of $H \cap H' \cap \Sigma$ and we can find, as before, a linear subspace Π_1 of codimension $t - 3$ in $H \cap H'$ such that it contains Π and intersects $H \cap H' \cap \Sigma$ in $\deg(H \cap H' \cap \Sigma) \leq (t - 1)!$ points, counted with their multiplicity. Since P and R have multiplicity at least 2, we have

$$|\Pi \cap \Sigma| \leq |\Pi_1 \cap \Sigma| \leq (t - 1)! - 2.$$

If $\dim H \cap H' \cap \Sigma = t - 2$ as an algebraic variety, then $H \cap \Sigma$ is reducible. Hence, we have

$$H \cap \Sigma = \mathcal{V}_1 \cup \mathcal{V}_2 \cup \cdots \cup \mathcal{V}_r,$$

where \mathcal{V}_i is an irreducible variety of dimension $t - 2$, for all $i = 1, \dots, r$. So we have

$$H \cap H' \cap \Sigma = \mathcal{V}_1 \cup \mathcal{V}_2 \cup \cdots \cup \mathcal{V}_s \cup \mathcal{W}_{s+1} \cup \mathcal{W}_{s+2} \cup \cdots \cup \mathcal{W}_r,$$

where \mathcal{W}_i is a hyperplane section of \mathcal{V}_i , for all $i = s+1, \dots, r$. We observe that also $H' \cap \Sigma$ has to be reducible and, since the decomposition in irreducible components is unique, we have

$$H' \cap \Sigma = \mathcal{V}_1 \cup \mathcal{V}_2 \cup \dots \cup \mathcal{V}_s \cup \mathcal{V}'_{s+1} \cup \mathcal{V}'_{s+2} \cup \dots \cup \mathcal{V}'_r,$$

where \mathcal{V}_i and \mathcal{V}'_j are irreducible varieties of dimension $t-2$.

We have, by hypothesis, that $T_P \Sigma \subset H$ and $P \in \Pi$. So either $P \in \mathcal{V}_i$ and it is singular for \mathcal{V}_i , for some $i \in \{1, 2, \dots, r\}$, or it is not singular for \mathcal{V}_ℓ , for any $\ell \in \{1, 2, \dots, r\}$.

Suppose we are in the first case. We know that $P \in \Pi \subset H'$. If $\mathcal{V}_i \subseteq H'$, then P is singular for an irreducible component of $H' \cap \Sigma$ and so $T_P \Sigma \subset H'$, contradicting our hypothesis, so \mathcal{V}_i is not contained in H' and $H' \cap \mathcal{V}_i = \mathcal{W}_i$. We have that $\dim T_P \Sigma \cap H' = t-2$ (as linear subspace) and $\dim \mathcal{W}_i = t-3$ (as algebraic variety), so P is singular for \mathcal{W}_i .

Suppose now that P is not singular for any \mathcal{V}_i , so the dimension of $T_P \mathcal{V}_i$, as a subspace, is $t-2$. If $P \notin \mathcal{V}_j$, for any $i \neq j$, then

$$T_P(H \cap \Sigma) = T_P(\mathcal{V}_i) = T_P(\Sigma),$$

a contradiction since the dimension of $T_P(\Sigma)$ is $t-1$. Hence $P \in \mathcal{V}_i \cap \mathcal{V}_j$, and so P is contained in the intersection of two components of $H' \cap \Sigma$, so it is again a singular (or multiple) point. The same is true for the point R such that $T_R \Sigma \subset H'$, so in

$$\mathcal{V}_1 \cup \mathcal{V}_2 \cup \dots \cup \mathcal{V}_s \cup \mathcal{W}_{s+1} \cup \mathcal{W}_{s+2} \cup \dots \cup \mathcal{W}_r$$

there are at least two multiple points and when we sum up all the degrees, we count at least two points twice, hence, by

$$\sum_{i=1}^s \deg \mathcal{V}_i + \sum_{j=s+1}^r \deg \mathcal{W}_j \leq (t-1)!,$$

we get that the number of points in

$$\Pi \cap (\mathcal{V}_1 \cup \mathcal{V}_2 \cup \dots \cup \mathcal{V}_s \cup \mathcal{W}_{s+1} \cup \mathcal{W}_{s+2} \cup \dots \cup \mathcal{W}_r),$$

is at most $(t-1)! - 2$. □

Remark One could wonder whether one could try with one more hyperplane H'' such that $T_Q \Sigma \subset H''$, $T_Q \Sigma \not\subset H$, $T_Q \Sigma \not\subset H'$ and $Q \in \Pi$. However, it can happen that $H \cap H' \cap H'' = H \cap H'$, so $\dim T_Q \Sigma \cap H \cap H' \cap H'' = t-2$ (as a linear space) and $\dim H \cap H' \cap H'' \cap \Sigma = t-2$, so Q would not be a singular point of

$$H \cap H' \cap H'' \cap \Sigma = H \cap H' \cap \Sigma.$$

The locus of hyperplanes containing a tangent space to a variety X of \mathbb{P}^n is a variety X^* of the dual space $(\mathbb{P}^n)^*$ (see, e.g., [12, Chapter 15]). Let Σ^* be the dual variety of Σ . From [16], we know that Σ^* is a hypersurface, hence, if d is the degree of Σ^* , then the number of points of Σ^* on a general line of $(\mathbb{P}^n)^*$ is d . Suppose that the line of $(\mathbb{P}^n)^*$ defined by $H \cap H'$ is general, hence if $|\Pi \cap \Sigma| > d$, then we could find a point $Q \in \Pi \cap \Sigma$ such that $T_Q \Sigma \subset H''$ and H'' is a hyperplane not containing $H \cap H'$. If $d > (t-1)! - 2$ then we

would not be able to get a better bound than the bound in Theorem 2.5. The degree of Σ^* is found in [10], where it is given by N_{t-1} , where N_r is defined by the generating function

$$\sum_{r \geq 0} N_r \frac{z^r}{r!} = \frac{e^{-2z}}{(1-z)^2}.$$

Hence $d = \deg \Sigma^*$, is the evaluation of

$$\left(\frac{e^{-2z}}{(1-z)^2} \right)^{(t-1)}$$

at $z = 0$, where we denote by $f^{(n)}$ the n -th derivative of the function f .

Let $F = fg$, where f and g are two functions, then

$$F^{(n)} = \sum_{i=0}^n \binom{n}{i} f^{(i)} g^{(n-i)}.$$

Let

$$f = e^{-2z} \text{ and } g = (1-z)^{-2}.$$

It is easy to see that

$$f^{(i)} = (-2)^i f \text{ and } g^{(i)} = (i+1)!(1-z)^{-(i+2)}.$$

Since $f(0) = 1$, we have that $F^{(n)}$, evaluated at $z = 0$, is

$$\sum_{i=0}^n \binom{n}{i} (-2)^i (n+1-i)!.$$

When $n = t - 1$ and we have

$$d = N_{t-1} = \sum_{i=0}^{t-1} \binom{t-1}{i} (-2)^i (t-i)!.$$

Now

$$\sum_{i=0}^{t-1} \binom{t-1}{i} (-2)^i (t-i)! = (t-1)! \sum_{i=0}^{t-1} \frac{(-2)^i}{i!} (t-i).$$

Note that

$$\sum_{i=0}^{t-1} \frac{(-2)^i}{i!} (t-i) = 1$$

for $t = 5$ and

$$\sum_{i=0}^t \frac{(-2)^i}{i!} (t+1-i) - \sum_{i=0}^{t-1} \frac{(-2)^i}{i!} (t-i) = \sum_{i=0}^t \frac{(-2)^i}{i!}.$$

Since $\sum_{i=0}^5 \frac{(-2)^i}{i!} = \frac{1}{15}$ and

$$\frac{(-2)^{n-1}}{(n-1)!} - \frac{(-2)^n}{n!} = \frac{2^{n-1}(n-2)}{n!} > 0$$

when $n \geq 3$ is odd,

$$\sum_{i=0}^t \frac{(-2)^i}{i!} > 0$$

for all $t \geq 4$ and so

$$\sum_{i=0}^{t-1} \frac{(-2)^i}{i!} (t-i)$$

is an increasing function. Thus, for $t \geq 5$,

$$\sum_{i=0}^{t-1} \frac{(-2)^i}{i!} (t-i) \geq 1,$$

and so

$$(t-1)! \sum_{i=0}^{t-1} \frac{(-2)^i}{i!} (t-i) \geq (t-1)!$$

and hence $d = N_{t-1} > (t-1)! - 2$.

THEOREM 2.6. . For $q \geq (t-1)! + 1$ the graph $\Gamma(t)$ contains no $K_{t+1, (t-1)!-1}$.

Proof. Let $X = \{x_1, x_2, \dots, x_{t+1}\}$ be $t+1$ distinct vertices of $\Gamma(t)$. The set of common neighbours of the elements of X is $\Pi^\perp \cap \Gamma(t)$, where Π is the subspace spanned by X . If any two elements of X project from P_∞ onto the same point of V , then $P_\infty \in \Pi$ and hence $\Pi^\perp \subset P_\infty^\perp$. Since P_∞^\perp is the hyperplane $x_{n+1} = 0$, $\Pi^\perp \cap \Gamma(t) = \emptyset$, and the elements of X have no common neighbour.

Therefore, we assume now that all the points in X project from P_∞ onto distinct points of V . Then, by Theorem 2.3, $\dim \Pi \geq t-1$.

If $\dim \Pi = t-1$, then by Theorem 2.3, the projection of Π onto V contains at least q points of V . Therefore, there are at least q points Y of Π on the lines joining P_∞ to the points of V . We wish to prove that the points of Y are vertices of the graph $\Gamma(t)$. To do this, we have to show that the points of Y , which are of the form $\langle(v, \lambda)\rangle$, where $v \in V$ and $\lambda \in \overline{\mathbb{F}_q}$, are of the form $\langle(u, \mu)\rangle$, where $v \in V$ and $\lambda \in \mathbb{F}_q$. Assuming that the vertices in X have at least two common neighbours, we can suppose that there is a common neighbour of the elements of X of the form $\langle(u, \mu)\rangle$, where $u \in V$, $u \neq -v$ and $\mu \in \mathbb{F}_q$, is a common neighbour of the elements of X . Then $\langle(u, \mu)\rangle$ is in Π^\perp and since $Y \subset \Pi$,

$$N(u+v) = \lambda\mu.$$

Since $N(u + v) \in \mathbb{F}_q$ and $\mu \in \mathbb{F}_q$, we have that $\lambda \in \mathbb{F}_q$ and so the points of Y are vertices of the graph $\Gamma(t)$. Therefore, the vertices of X have at least q common neighbours. Since Γ contains no $K_{t, (t-1)!+1}$, if $q \geq (t-1)! + 1$, then this case cannot occur.

If $\dim \Pi = t$ then $\dim \Pi^\perp = n - t$. Let Y be the points of Π^\perp which project from P_∞ onto V . Arguing as in the previous paragraph, the points Y are vertices of the graph $\Gamma(t)$. Since the vertices of X have at most $(t-1)! + 1$ common neighbours, there are a finite number of points in Y and so a finite number of points in the projection of Π^\perp onto V . By Theorem 2.5, this projection contains at most $(t-1)! - 2$ points of V , so there are at most $(t-1)! - 2$ points in Y . Therefore, the vertices in X have at most $(t-1)! - 2$ common neighbours. \square

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