Characterisation of Unidimensional Averaged Similarities

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Abstract

A T-indistinguishability operator (or fuzzy similarity relation) E is called unidimensional when it may be obtained from one single fuzzy subset (or fuzzy criterion). In this paper, we study when a T-indistinguishability operator that has been obtained as an average of many unidimensional ones is unidimensional too. In this case, the single fuzzy subset used to generate E is explicitly obtained as the quasi-arithmetic mean of all the fuzzy criteria primarily involved in the construction of E.

Keywords: Indistinguishability operator, generator, quasi-arithmetic mean, Representation Theorem

1. Introduction

Indistinguishability operators with respect to a given t-norm T, or simply T-indistinguishability operators, are the natural fuzzification of classical equivalence relations. They are found under many names in the literature, depending on the author and on the chosen t-norm. *Similarity* is perhaps the most common name applied to such fuzzy relations (Zadeh [7]), although it is sometimes associated with the particular t-norm $T = MIN$. Other names are *Likeness*, *Fuzzy Equality* or *Fuzzy Equivalence Relation*. We will use *T-indistinguishability operator* (following Trillas and Valverde [6]), and also the term *similarity* in an informal way.

Crisp equivalence relations are generally regarded as the mathematical construct for dealing with classifications. They are defined as those relations being reflexive, symmetric and transitive. If E is such a relation on a set X, for each element $x \in X$ we may consider all the elements $y \in Y$ that are related to x, that is, all $y \in Y$ such that $E(x, y) = 1$. These these elements are *the class of x*. Here x acts as a prototype, and all the objects y in its class as its likes. As a result, the set X becomes partitioned into classes.

Often, equivalence relations are induced by attributes. For example, a given set X of plane closed polygonal lines becomes naturally partitioned into classes according to their number of sides. In addition to that, if the polygonal are real (drawn) lines, we may consider also color as an attribute, and the set becomes furtherly partitioned into, say, black and white lines. Each attribute is responsible for a partition of X and, therefore, for an equivalence relation E. The final partition or equivalence relation is the intersection of the two, meaning with this that every two elements x and $y \in X$ are E-related if they have the same number of sides and the same color, but they are not if they fail to meet one of the two criteria, or both of them. Formally, if E_s stands for *number of sides* and E_c means *color* then $E(x, y) = MIN\,(E_s(x, y), E_c(x, y))$.

Attributes, though, may be of a graded nature. We may consider the attribute *perimeter* of a polygonal, whose range is the positive real numbers, or lines may be drawn in a variety of shades of gray which can be expressed as real numbers between 0 and 1. Attributes that take values on continuous universes are generally regarded as vague, and they are represented by fuzzy sets. Instead of considering a rectangle whose perimeter equals 5 as entirely different from another one of perimeter 5.15, and therefore belonging to two different classes, we can regard them as very similar objects whenever perimeter is the only attribute considered. They could share the same class, provided that classes are fuzzy sets and belonging to a class is a matter of degree.

The definition of T-indistinguishability operator axiomatically captures the intuitive idea of fuzzy equivalence relation.

Definition 1.1. *Let X be a universe and T a t-norm. A T-indistinguishability operator E on X is a fuzzy relation $E : X \times X \to [0, 1]$ satisfying, for all $x, y, z \in X$,*

1. $E(x, x) = 1$ *(Reflexivity)*
2. $E(x, y) = E(y, x)$ *(Symmetry)*
3. $T(E(x, y), E(y, z)) \leq E(x, z)$ *(T-Transitivity)*

A *t-norm* T is an operation on the unit interval which is associative, commutative, non decreasing in both variables, and satisfies the boundary condition $T(x,1) = x$ for all $x, y \in [0,1]$. It is generally accepted that t-norms are the AND connectives of Fuzzy Logic [2].

We will assume within this paper that the t-norm T is continuous and Archimedean [3], or else $T = MIN$. Every continuous Archimedean t-norm is isomorphic to the sum of positive real numbers, bounded or unbounded, according to Ling's theorem [3]. The order reversing isomorphism $t : [0,1] \to [0,+\infty]$ is called *an additive generator of* T, and $T(a,b) = t^{[-1]}(t(a)+t(b))$ for all $a, b \in [0,1]$ where $t^{[-1]}$ is the pseudoinverse of t.

In practice, this means that T-transitivity (Definition 1.1.3) is simply a version of the *triangle inequality* for metrics, since $T\left(E\left(x,y\right), E\left(y,z\right)\right) \leq E\left(x,z\right)$ can be rewritten as $t\left(E\left(x,y\right)\right) + t\left(E\left(y,z\right)\right) \geq t\left(E\left(x,z\right)\right)$ or, in a more convenient notation for the purposes of this paper,

$$t \circ E\left(x,y\right) + t \circ E\left(y,z\right) \geq t \circ E\left(x,z\right).$$

Thus, the underlying semantics of T-indistinguishability operators is enhanced to include proximity in a metric sense in addition to fuzzy equivalence.

T-indistinguishability operators may be induced by fuzzy attributes. These fuzzy attributes may be represented as fuzzy sets $h : X \to [0,1]$, and then some procedure is needed to obtain the fuzzy relation E from the fuzzy subsets h. Such procedure is provided by the Representation Theorem ([6])

For a given continuous t-norm T, we consider the *natural indistinguishability* on $[0,1]$, E_T. Since in this paper only the most used t-norms, namely Archimedean t-norms and the MIN t-norm are considered, we will provide separate definitions for each of the two, thus avoiding the general case ([6]).

Definition 1.2. *Given an Archimedean t-norm T, the natural indistinguishability E_T associated with T is the indistinguishability on the unit interval* $E_T\left(x,y\right) = t^{[-1]}\left(\left|t\left(x\right) - t\left(y\right)\right|\right)$

Definition 1.3. *The natural indistinguishability E_{MIN} associated with the t-norm $T = MIN$ is the indistinguishability on the unit interval $E_{MIN}\left(x,y\right) = MIN\left(x,y\right)$ if $x \neq y$. Otherwise, $E_{MIN}\left(x,y\right) = 1$.*

The natural indistinguishability E_T is a proper T-indistinguishability operator in the sense of Definition 1.1.

Any arbitrary fuzzy subset $h : X \to [0,1]$ induces an indistinguishability on X via $E_h(x,y) = E_T(h(x), h(y))$. The *Representation Theorem* states that every indistinguishability E can be obtained from indistinguishabilities E_h induced by single fuzzy attributes h.

Theorem 1.4. [6] *Representation Theorem. Let E be a fuzzy relation on a set X and T a t-norm. E is a T-indistinguishability operator if and only if there exists a family $H = (h_i)_{i \in I}$ of fuzzy subsets of X such that for all $x, y \in X$*

$$E(x,y) = \inf_{i \in I} E_{h_i}(x,y).$$

We say that E is *generated* by H, or that H is a *generating family* of E. T-indistinguishability operators E that are generated by a single fuzzy subset h are called *unidimensional*.

Intuitively, H is a set of attributes relevant to the classification induced by E. Each attribute $h_i : X \to [0,1]$ is responsible for a singly generated T-indistinguishability E_{h_i} which is computed by

$$E_{h_i}(x,y) = E_T(h_i(x), h_i(y)) = t^{[-1]}(|t \circ h_i(x) - t \circ h_i(y)|)$$

The metric interpretation becomes clear when we write the previous equation as

$$t \circ E_{h_i}(x,y) = |t \circ h_i(x) - t \circ h_i(y)|$$

since the right hand side is the real line distance between images of h_i via the isomorphism t.

The Representation Theorem is central to many theoretical developments in the field of fuzzy relations. Also, it provides a straight translation into the fuzzy framework of the crisp procedure described above to obtain an equivalence relation starting from a set of criteria. It first generates the equivalence relations for each attribute, and then combines all of them by taking infima.

However, from an applied perspective this way of combining information is far from satisfactory. The notion of error is essential to applied domains, and a common way to deal with errors is by averaging information. If, for example, we perceive two different objects as somehow similar under a sequence of observations, we are not likely to thing of them as entirely different just because one particular observation indicates so. We may discard the conflicting piece of information or, more likely, we may aggregate all the evidence gathered throughout the sampling process by using some averaging operator.

Quasi-arithmetic means [1] are a family of averaging operators which are widely used. Quasi-arithmetic means, or q-a means for short, may also be obtained from additive generators, in a very similar way to that of Archimedean t-norms.

Definition 1.5. [1] *The quasi-arithmetic mean M in* [0,1] *generated by a continuous monotonic map $t : [0,1] \to [-\infty, \infty]$ is defined for all $n \in \mathbb{N}$ and $x_1, ..., x_n \in [0,1]$ by*

$$M(x_1, ...x_n) = t^{-1}\left(\frac{t(x_1) + ... + t(x_n)}{n}\right).$$

M is continuous if and only if Ran $t \neq [-\infty, \infty]$.

Proposition 1.6. [5] *The map assigning to every continuous Archimedean t-norm T with additive generator t the quasi-arithmetic mean m_t generated by t is a canonical bijection between the set of continuous Archimedean t-norms and continuous quasi-arithmetic means with $t(1) \neq \pm\infty$.*

Similarly weighted quasi-arithmetic means can be defined in the following way.

Definition 1.7. *Consider $\alpha_1, \alpha_2, ..., \alpha_n \in [0,1]$ such that $\sum_{i=1}^{n} \alpha_i = 1$. The weighted quasi-arithmetic mean $M^{\alpha_1,\alpha_2,...,\alpha_n}$ of $x_1, x_2, ..., x_n \in [0,1]$ with weights $\alpha_1, \alpha_2, ..., \alpha_n$ generated by a continuous strict monotonic map $t : [0,1] \to [-\infty, \infty]$ is*

$$M^{\alpha_1,\alpha_2,...,\alpha_n}(x_1, x_2, ..., x_n) = t^{-1}\left(\sum_{i=1}^{n} \alpha_i \cdot t(x_i)\right).$$

$M^{\alpha_1,\alpha_2,...,\alpha_n}$ is continuous if and only if Ran $t \neq [-\infty, \infty]$.

Proposition 1.8. [5] *The map assigning to every continuous Archimedean t-norm T with additive generator t the quasi-arithmetic mean $M^{\alpha_1,...,\alpha_n}$ generated by t is a canonical bijection between the set of continuous Archimedean t-norms and continuous weighted quasi-arithmetic means with weights $\alpha_1, ..., \alpha_n$ and with $t(1) \neq \pm\infty$.*

For simplicity, we will write $M(\alpha_i, x_i)$ instead of $M^{\alpha_1,...,\alpha_n}(x_1, x_2, ..., x_n)$.

2. Quasi-arithmetic means of attributes and their relationship with quasi-arithmetic means of indistinguishabilities

In this section we deal with a family of fuzzy sets $H = (h_i)_{i\in I}$ which we assume to represent a set of attributes or criteria applicable to all $x \in X$. Examples of such attributes are perimeter, gray level, weight, suitability, smoothness etc. and, since they are obtained through empirical measuring or subjective assessment, they are bound to errors and uncertainty.

Each fuzzy set $h \in H$ allows for any pair of elements $x, y \in X$ to be regarded as similar up to a degree $E_h(x, y)$ and, since h is only an approximate instantiation of some theoretical graded variable, so is $E_h(x, y)$. Standard proceedings in such situations include averaging the empirically measured features or the subjectively assessed criteria in order to obtain a more reliable fuzzy set $\bar{h}$ and, therefore, a more accurate relation $E_{\bar{h}}(x, y)$.

Let M be a quasi-arithmetic mean with weights $(\alpha_i)_{i\in I}$ and additive generator t, the same additive generator as that of a given Archimedean t-norm T.

In order to average the information via M there are two possible courses of action. We may first compute the quasi-arithmetic mean of all the fuzzy sets in the generating family H, $\bar{h} = M(\alpha_i, h_i)$ and use this single fuzzy set $\bar{h}$ to generate the indistinguishability operator $E_{\bar{h}}(x, y)$ afterwards. Or, we may start by generating a family of indistinguishability operators $(E_{hi})_{i\in I}$ and then averaging all the indistinguishabilities in the family as $\bar{E}_H = M(\alpha_i, E_{hi})$. We will show that the two procedures may throw different results, depending on how different are the orders induced by the fuzzy sets h on X.

Proposition 2.1. *$E_{\bar{h}}$ is an indistinguishability operator with respect to T.*

Proof. Obvious, since $E_{\bar{h}}$ is the T-indistinguishability generated by the fuzzy set $\bar{h}$. □

Proposition 2.2. [5][4] *$\bar{E}_H$ is an indistinguishability operator with respect to T.*

Proposition 2.3. [4] $\bar{E}_H \leq E_{\bar{h}}$

Each fuzzy set $h_i \in H$ induces a preorder $\leq_i$ on X as follows.

Definition 2.4. *$x \leq_i y$ if and only if $h_i(x) \leq h_i(y)$ for all $x, y \in X$.*

Note that the induced preorders $\leq_i$ are *total* preorders because $h_i : X \to [0,1]$ and $[0,1]$ is a totally ordered set.

Definition 2.5. *Two preorders $\leq_i$ and $\leq_j$ on X are compatible if and only if $x <_i y \Rightarrow x \leq_j y$ and $x <_j y \Rightarrow x \leq_i y$ for all $x, y \in X$*

Lemma 2.6. *For any discrete family of real numbers $\{a_i\}_{i\in I}$, $\left|\sum_{i\in I} a_i\right| = \sum_{i\in I} |a_i|$ if, and only if, $a_i \geq 0$ for all $i \in I$, or else $a_i \leq 0$ for all $i \in I$.*

Note that, in general, only $\left|\sum_{i\in I} a_i\right| \leq \sum_{i\in I} |a_i|$ holds.

Proposition 2.7. *$\bar{E}_H = E_{\bar{h}}$ if, and only if, $\leq_i$ and $\leq_j$ are compatible orders for all $i, j \in I$.*

Proof. Let t be the additive generator of both the quasi-arithmetic mean M and the Archimedean t-norm T. Let us take $a_i = \alpha_i(t \circ h_i(x)) - \alpha_i(t \circ h_i(y))$ for all $i \in I$.

Since t is monotonous, $\leq_i$ and $\leq_j$ being compatible is both a necessary and sufficient condition for all the a_i to have the same sign, which in turn is necessary and sufficient for the equality (*) to hold in the following equations, according to the previous lemma:

$$
\begin{aligned}
&E_{\bar{h}}(x,y) = E_T(\bar{h}(x),\bar{h}(y)) \\
&= t^{[-1]} \circ \left(\left|t \circ \bar{h}(x) - t \circ \bar{h}(y)\right|\right) \\
&= t^{[-1]} \circ \left(\left|t \circ t^{[-1]}\left(\sum_{i\in I}\alpha_i t \circ h_i(x)\right) - t \circ t^{[-1]}\left(\sum_{i\in I}\alpha_i t \circ h_i(y)\right)\right|\right) \\
&= t^{[-1]} \circ \left(\left|\sum_{i\in I}\alpha_i t \circ h_i(x) - \sum_{i\in I}\alpha_i t \circ h_i(y)\right|\right) \\
&= t^{[-1]} \circ \left(\left|\sum_{i\in I}\alpha_i (t \circ h_i(x) - t \circ h_i(y))\right|\right) \\
&\underset{(*)}{=} t^{[-1]} \circ \left(\sum_{i\in I}\alpha_i \left|t \circ h_i(x) - t \circ h_i(y)\right|\right) \\
&= t^{[-1]} \circ \left(\sum_{i\in I}\alpha_i t \circ t^{[-1]} \left|t \circ h_i(x) - t \circ h_i(y)\right|\right) \\
&= t^{[-1]} \circ \left(\sum_{i\in I}\alpha_i t \circ E_T\left(h_i(x), h_i(y)\right)\right) \\
&= t^{[-1]} \circ \left(\sum_{i\in I}\alpha_i t \circ E_{h_i}(x,y)\right) = M\left(\alpha_i, E_{h_i}(x,y)\right) \\
&= \overline{E}_H\left(x,y\right)
\end{aligned}
$$

□

Next, we are going to compute how different $E_{\bar{h}}(x,y)$ and $\overline{E}_H\left(x,y\right)$ are, provided that the orders induced by the fuzzy sets $h_i \in H$ are not compatible orders on X. The natural choice for measuring this difference or dissimilarity is E_T, so that we define:

$$
C_H(x,y) = E_T(E_{\bar{h}}(x,y), \overline{E}_H(x,y))
$$

for every pair $(x,y) \in X \times X$.

$C_H(x,y) = 1$ if all the fuzzy sets h induce compatible orders on $\{x,y\}$, that is, if $h_i(x) \leq h_i(y)$ or either $h_i(x) \geq h_i(y)$ for all $h \in H$. When this does not happen $C_H(x,y)$ provides a measure of how compatible these orders are.

Given $(x,y) \in X$, we split the set $H = (h_i)_{i\in I}$ of all generators into two subsets, $I = P \cup N$, where $P = \{j \in I/h_j(x) \geq h_j(y)\}$ and $N = \{k \in I/h_k(x) < h_k(y)\}$. Note that both P and N may be empty, $P \cap N = \emptyset$, $P \cup N = I$ and $H = (h_j)_{j\in P} \cup (h_k)_{k\in N}$.

We may then split the sum $t \circ \overline{E}_H(x, y) = \sum_{i \in I} \alpha_i t \circ E_{h_i}(x, y)$ accordingly,

$$t \circ \overline{E}_H(x, y) = \sum_{j \in P} \alpha_j t \circ E_{h_j}(x, y) + \sum_{k \in N} \alpha_k t \circ E_{h_k}(x, y)$$

and rename

$$A(x, y) = \sum_{j \in P} \alpha_j t \circ E_{h_j}(x, y)$$

$$B(x, y) = \sum_{k \in N} \alpha_k t \circ E_{h_k}(x, y)$$

We are now in condition to compute the error made when we replace the T-indistinguishability $\overline{E}_H$ by $E_{\bar{h}}$, which is a lot simpler since it is generated by one single fuzzy set. Also, this error provides a measure of the compatibility C_H of the orders induced by H on (x, y).

Proposition 2.8. $C_H(x, y) = t^{[-1]}(\min(2A(x, y), 2B(x, y)))$

Proof. We will show that

$$C_H(x, y) = t^{[-1]} \circ (A(x, y) + B(x, y) - |A(x, y) - B(x, y)|)$$
$$= \begin{cases} t^{[-1]}(2B) & \text{if } A \geq B \\ t^{[-1]}(2A) & \text{if } A < B \end{cases}$$

From this, the result follows immediately.

$$
\begin{aligned}
C_H(x,y) &= E_T\left(\bar{E}_H(x,y), E_{\bar{h}}(x,y)\right) \\
&= \overrightarrow{T}\left(E_{\bar{h}}(x,y)\,|\bar{E}_H(x,y)\right) \\
&= t^{[-1]} \circ \left(t \circ \bar{E}_H(x,y) - t \circ E_{\bar{h}}(x,y)\right) \\
&= t^{[-1]} \circ \left(\sum_{i \in I} \alpha_i t\left(E_{h_i}(x,y)\right) - \left|t \circ \bar{h}(x) - t \circ \bar{h}(y)\right|\right) \\
&= t^{[-1]} \circ \left(\sum_{i \in I} \alpha_i t\left(E_{h_i}(x,y)\right) - \left|\sum_{i \in I} \alpha_i t \circ h_i(x) - \sum_{i \in I} \alpha_i t \circ h_i(y)\right|\right) \\
&= t^{[-1]} \circ \left(\sum_{i \in I} \alpha_i t\left(E_{h_i}(x,y)\right) - \left|\sum_{i \in I} \alpha_i \left(t \circ h_i(x) - t \circ h_i(y)\right)\right|\right) \\
&= t^{[-1]} \circ \left(\sum_{j \in P} \alpha_j t\left(E_{h_j}(x,y)\right) + \sum_{k \in N} \alpha_k t\left(E_{h_k}(x,y)\right)\right. \\
&- \left.\left|\sum_{j \in P} \alpha_j \left(t \circ h_j(x) - t \circ h_j(y)\right) - \sum_{k \in N} \alpha_k \left(t \circ h_k(y) - t \circ h_k(x)\right)\right|\right) \\
&= t^{[-1]} \circ \left(\sum_{j \in P} \alpha_j t\left(E_{h_j}(x,y)\right) + \sum_{k \in N} \alpha_k t\left(E_{h_k}(x,y)\right)\right. \\
&- \left|\sum_{j \in P} \alpha_j t \circ t^{[-1]} \circ \left(t \circ h_j(x) - t \circ h_j(y)\right)\right. \\
&- \left.\left.\sum_{k \in N} \alpha_k t \circ t^{[-1]} \circ \left(t \circ h_k(y) - t \circ h_k(x)\right)\right|\right) \\
&= t^{[-1]} \circ \left(\sum_{j \in P} \alpha_j t\left(E_{h_j}(x,y)\right) + \sum_{k \in N} \alpha_k t\left(E_{h_k}(x,y)\right)\right. \\
&- \left.\left|\sum_{j \in P} \alpha_j t\left(E_{h_j}(x,y)\right) - \sum_{k \in N} \alpha_k t\left(E_{h_k}(y,x)\right)\right|\right) \\
&= t^{[-1]} \circ \left(A(x,y) + B(x,y) - \left|A(x,y) - B(x,y)\right|\right)
\end{aligned}
$$

□

Example 2.9.

A specially simple case occurs when the additive generator $t : [0,1] \to [0,1]$ defined by $t(a) = 1 - a$ is considered. Its quasi-inverse is

$$t^{[-1]}(b) = \begin{cases} 0 \text{ if } b \geqslant 1 \\ 1 - b \text{ if } 0 \leqslant b < 1 \\ 1 \text{ if } b < 0 \end{cases}$$

The t-norm $T(a,b) = t^{[-1]}(t(a) + t(b))$ is then the Łukasiewicz t-norm, $Ł(a,b) = MAX(a+b-1, 0)$, and the associated quasi-arithmetic mean is $M(\alpha_i, a_i) = \sum_{i=1}^{n} \alpha_i a_i = \sum_{i=1}^{n} \frac{1}{n} a_i$, the standard arithmetic mean, provided that $\alpha_i = \frac{1}{n}$ for all $i = 1, ..., n$.

The next matrix displays three fuzzy sets $h_0, h_1, h_2 : X \to [0,1]$, with $X = \{x_0, x_1, x_2, x_3, x_4\}$

$$\begin{array}{c} \\ x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{ccc} h_0 & h_1 & h_2 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0.4 & 0.5 & 0 \\ 0.5 & 0.4 & 0 \\ 0.6 & 0.6 & 1 \\ 1 & 1 & 1 \end{pmatrix} \end{array}$$

The fuzzy sets h_0, h_1 are quite similar, but their induced orders are not compatible, since $h_0(x_1) < h_0(x_2)$ and $h_1(x_1) > h_1(x_2)$, while h_0, h_2 are strikingly different fuzzy sets, but they induce compatible orders.

If we note the generators' means by $\bar{h}_{01} = \frac{1}{2}h_0 + \frac{1}{2}h_1$, $\bar{h}_{02} = \frac{1}{2}h_0 + \frac{1}{2}h_2$ and the means of the indistinguishabilies by $\bar{E}_{01} = \frac{1}{2}E_{h_0} + \frac{1}{2}E_{h_1}$, $\bar{E}_{02} = \frac{1}{2}E_{h_0} + \frac{1}{2}E_{h_2}$ then it is easy to check that $\bar{E}_{01}(x_1, x_2) \neq E_{\bar{h}_{01}}(x_1, x_2)$ but $\bar{E}_{02}(x_1, x_2) = E_{\bar{h}_{02}}(x_1, x_2)$.

The example also shows that generators cannot be grouped into classes according to the orders they induce. In the present case, h_0, h_2 and h_1, h_2 can be grouped together, but not so h_0, h_1.

3. Quasi-arithmetic means of attributes and of indistinguishabilities. The case $T = MIN$.

In the previous section we have considered an Archimedean t-norm T with the same additive generator t as the quasi-arithmetic mean M. In this section, a quasi-arithmetic mean of generator t is still used for averaging purposes, but the indistinguishabilities are generated with the t-norm $T = MIN$, a non-Archimedean t-norm and therefore, one lacking an additive generator.

If $T = MIN$, the indistinguishability operator generated by a fuzzy set $h_i : X \to [0,1]$ is given by

$$E_{h_i}(x,y) = E_{MIN}(h_i(x), h_i(y))$$
$$= \begin{cases} 1 \text{ if } h_i(x) = h_i(y) \\ MIN(h_i(x), h_i(y)) \text{ otherwise.} \end{cases}$$

We want to compare the quasi-arithmetic mean of the indistinguishabilities generated by a family $H = (h_i)_{i \in I}$ of fuzzy sets,

$$\bar{E}_H = M(\alpha_i, E_{h_i})$$

with the indistinguishability generated by the fuzzy set

$$\bar{h} = M(\alpha_i, h_i)$$

the quasi-arithmetic mean of the fuzzy sets of H, that is

$$E_{\bar{h}}(x,y) = E_{MIN}\left(\bar{h}(x), \bar{h}(y)\right)$$
$$= \begin{cases} 1 \text{ if } \bar{h}(x) = \bar{h}(y) \\ MIN\left(\bar{h}(x), \bar{h}(y)\right) \text{ otherwise.} \end{cases}$$

Note that $\bar{E}_H$ is not, in general, a MIN-transitive relation, but a T-transitive one with respect to the t-norm T generated by t, the generator of the quasi-arithmetic mean.

The results differ substantially from those in the Archimedean case.

In the following propositions x, y will represent a pair of elements of X such that $x \neq y$, and $T = MIN$.

Lemma 3.1. *If $h_i(x) \neq h_i(y)$ for all $h_i \in H$ then $\bar{E}_H(x,y) \leqslant E_{\bar{h}}(x,y)$*

Proof. Given a pair x, y of elements such that $x \neq y$ we have that

$$\begin{aligned}
&E_{\bar{h}}(x,y) = E_{MIN}\left(\bar{h}(x),\bar{h}(y)\right) \\
&= \left\{ \begin{array}{c} 1 \text{ if } \bar{h}(x) = \bar{h}(y) \\ MIN\left(\bar{h}(x),\bar{h}(y)\right) \text{ otherwise.} \end{array} \right\} \geqslant MIN\left(\bar{h}(x),\bar{h}(y)\right) \\
&= MIN\left(t^{[-1]} \circ \left(\sum_{i\in I} \alpha_i t \circ h_i(x)\right), t^{[-1]} \circ \left(\sum_{i\in I} \alpha_i t \circ h_i(y)\right)\right) \\
&= t^{[-1]} \circ MAX\left(\sum_{i\in I} \alpha_i t \circ h_i(x), \sum_{i\in I} \alpha_i t \circ h_i(y)\right) \\
&\geqslant t^{[-1]} \circ \left(\sum_{i\in I} \alpha_i MAX\left(t \circ h_i(x), t \circ h_i(y)\right)\right)
\end{aligned}$$

$$\begin{aligned}
&= t^{[-1]} \circ \left(\sum_{i\in I} \alpha_i t \circ MIN\left(h_i(x), h_i(y)\right)\right) \\
&\underset{(*)}{=} t^{[-1]} \circ \left(\sum_{i\in I} \alpha_i t \circ E_{MIN}\left(h_i(x), h_i(y)\right)\right) \\
&= t^{[-1]} \circ \left(\sum_{i\in I} \alpha_i t \circ E_{h_i}(x,y)\right) \\
&= M\left(\alpha_i, E_{h_i}(x,y)\right) = \bar{E}_H(x,y)
\end{aligned}$$

(*) follows from the hypothesis $h(x) \neq h(y)$. The rest of equalities and inequalites are straightforward since t is a decreasing function.

□

Proposition 3.2. *If* $h_i(x) < h_i(y)$ *for all* $h_i \in H$ *then* $\bar{E}_H(x,y) = E_{\bar{h}}(x,y)$

Proof. If we assume that $h_i(x) < h_i(y)$ for all $h_i \in H$ then we may rewrite the proof of Lemma 3.1 , with the only difference that the two inequalities become equalities.

Indeed, $\bar{h}(x) < \bar{h}(x)$ follows from $h(x) < h(y)$ for all $h \in H$, and therefore $E_{MIN}\left(\bar{h}(x),\bar{h}(y)\right) \neq 1$, and $E_{MIN}\left(\bar{h}(x),\bar{h}(y)\right) = MIN\left(\bar{h}(x),\bar{h}(y)\right)$.

Also,

$$MAX\left(\sum_{i\in I}\alpha_i t\circ h_i(x),\sum_{i\in I}\alpha_i t\circ h_i(y)\right)=$$

$$=\sum_{i\in I}\alpha_i t\circ h_i(x)=\left(\sum_{i\in I}\alpha_i MAX\left(t\circ h_i(x),t\circ h_i(y)\right)\right)$$

which accounts for the second inequality turning into an equality.

□

Proposition 3.3. *If $h_i(x)\leqslant h_i(y)$ for all $h_i\in H$ and $x,y\in X$, then* $\bar{E}_H(x,y)\geqslant E_{\bar{h}}(x,y)$

Proof. If $h_i(x)=h_i(y)$ for all $i\in I$, then $E_{h_i}(x,y)=1$ and $\bar{h}(x)=\bar{h}(y)$ and thus

$$\bar{E}_H(x,y)=M\left(\alpha_i,E_{h_i}(x,y)\right)=1=E_{MIN}\left(\bar{h}(x),\bar{h}(y)\right)=E_{\bar{h}}(x,y)$$

Let us suppose $h_i(x)\neq h_i(y)$ for some $i\in I$. Then we may split $I=Q\cup L$ with $Q=\{j\in I/h_j(x)=h_j(y)\}$, $L=\{k\in I/h_k(x)<h_k(y)\}$, and L is non-empty. If Q is empty, then we may apply Proposition 3.2 to conclude $\bar{E}_H(x,y)=E_{\bar{h}}(x,y)$. So let us suppose that both Q and L are non-empty, since all the other cases are already dealt with. This allows us to write:

$$\begin{aligned}
\bar{h}(x) &= t^{[-1]}\circ\left(\sum_{i\in I}\alpha_i t\circ h_i(x)\right)\\
&= t^{[-1]}\circ\left(\sum_{j\in Q}\alpha_j t\circ h_j(x)+\sum_{k\in L}\alpha_k t\circ h_k(x)\right)\\
&< t^{[-1]}\circ\left(\sum_{j\in Q}\alpha_j t\circ h_j(y)+\sum_{k\in L}\alpha_k t\circ h_k(y)\right)\\
&= t^{[-1]}\circ\left(\sum_{i\in I}\alpha_i t\circ h_i(y)\right)=\bar{h}(y)
\end{aligned}$$

We are only interested in the fact that $h_i(x)<h_i(y)$, because it excludes the possibility that $E_{MIN}\left(\bar{h}(x),\bar{h}(y)\right)=1$, and thus

$$E_{\bar{h}}(x,y)=E_{MIN}\left(\bar{h}(x),\bar{h}(y)\right)=MIN\left(\bar{h}(x),\bar{h}(y)\right)=\bar{h}(x)$$

To end the proof, we only have to show that $\bar{E}_H(x,y) \geqslant \bar{h}(x)$, which follows from:

$$\bar{E}_H(x,y) = M(\alpha_i, E_{h_i}(x,y)) = t^{[-1]} \circ \left(\sum_{i \in I} \alpha_i t \circ E_{h_i}(x,y) \right)$$

$$= t^{[-1]} \circ \left(\sum_{j \in Q} \alpha_i t \circ E_{h_j}(x,y) + \sum_{k \in L} \alpha_k t \circ E_{h_k}(x,y) \right)$$

$$= t^{[-1]} \circ \left(\sum_{k \in L} \alpha_k t \circ E_{h_k}(x,y) \right)$$

$$= t^{[-1]} \circ \left(\sum_{k \in L} \alpha_k t \circ MIN(h_k(x), h_k(y)) \right) = t^{[-1]} \circ \left(\sum_{k \in L} \alpha_k t \circ h_k(x) \right)$$

$$> t^{[-1]} \circ \left(\sum_{i \in I} \alpha_i t \circ h_i(x) \right) = \bar{h}(x)$$

□

4. Conclusions

Starting with a family of fuzzy sets, which we regard as fuzzy attributes, the Representation Theorem (Th. 1.4) provides an indistinguishability operator by taking infima of the simple indistinguishability operators associated with every single fuzzy set. When the same procedure is carried out with crisp attributes, we obtain a crisp equivalence relation in the standard way. This theorem is an important result for the theoretical study of indistinguishability operators.

However, form an applied point of view, the averaging of information is often the most advisable course of action, since noise and errors are intrinsic to empirical data. Averaging is then seen as a way of filtering, and quasi-arithmetic means are a simple way of smoothen data.

Quasi-arithmetic means can be applied in two different ways in our case. Either we average the fuzzy attributes, and then we generate the similarities with the filtered fuzzy sets, or we generate the similarities with the original fuzzy attributes and average them afterwards. The two proceedings are not equivalent.

We have given the key results that allow for comparing the two proceedings. Also, the necessary and sufficient conditions for the two of them to be equivalent, only for Archimedean t-norms.

These results suggest that, sometimes, the set of attributes used to generate an indistinguishability operator via quasi-arithmetic means, can be reduced to a much smaller set with exactly the same generating capabilities. The search for these smaller sets we regard as a key, non-trivial issue to be dealt with in the future.

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