

# Classification of $b^m$ -Nambu structures of top degree

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## Abstract

In this paper we classify  $b^m$ -Nambu structures via  $b^m$ -cohomology. The complex of  $b^m$ -forms is an extension of De Rham complex which allows to consider *singular* forms.  $b^m$ -Cohomology is well-understood thanks to Scott [12] and it can be expressed in terms of De Rham cohomology of the manifold and the critical hypersurface using a Mazzeo-Melrose-type formula. Each of the terms in  $b^m$ -Mazzeo-Melrose formula acquires a geometrical interpretation in this classification. We also give equivariant versions of this classification scheme. *To cite this article: A. Name1, A. Name2, C. R. Acad. Sci. Paris, Ser. I 340 (2005).*

## Résumé

### Classification de structures $b^m$ -Nambu de degré maximal

On classe les structures  $b^m$ -Nambu de degré maximal en utilisant la  $b^m$ -cohomologie. Le complexe des  $b^m$ -formes est une extension du complexe de De Rham et permet considérer des formes *singulières*. La  $b^m$ -cohomologie est bien comprise grâce à Scott [12] et elle peut être exprimée en termes de la cohomologie de De Rham de la variété et de l'hypersurface critique en utilisant une formule de type Mazzeo-Melrose. Chacun des termes dans la formule de  $b^m$ -Mazzeo-Melrose acquiert une interprétation géométrique dans cette classification. On donne aussi des versions équivariantes des théorèmes de classification.

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## 1. Introduction

In this article we focus our attention on  $b^m$ -Nambu structures. Nambu structures were introduced by Nambu [11] and Takhtajan [13] as a generalization of Poisson structures. Unlike the domain of Poisson Geometry, Nambu geometry is not so well-explored. In this short note we give a classification theorem for a class of Nambu structures using a generalization of De Rham cohomology called  $b^m$ -cohomology. Our result generalizes a former classification theorem by Martínez-Torres for generic Nambu structures of top degree [8].

Recently a class of Poisson structures called in the literature  $b$ -Poisson structures (see for instance, [3],[4],[6] and [2]) has been widely studied. A  $b$ -Poisson manifold is an even dimensional Poisson manifold  $(M^{2n}, \Pi)$  where the Poisson structure  $\Pi$  satisfies the following transversality condition:  $\Pi^n$  cuts the zero section of the bundle  $\Lambda^{2n}T(M^{2n})$  transversally. As a consequence the vanishing set of  $\Pi^n$  is a smooth submanifold of codimension 1 which is called *critical hypersurface*.

The transversality condition can be relaxed in a way the critical hypersurface is still a smooth submanifold. This is the case of  $b^m$ -Poisson manifolds introduced by Scott [12]. In this paper we generalize this setting to the Nambu world and classify these structures. This class of singular Nambu structures was already considered by Arnold in [1]. The classification theorem we prove here is an extension of Moser's classification theorem [10] for volume forms on a manifold. As an outcome of this classification scheme a geometrical interpretation is given to the Mazzeo-Melrose decomposition theorem (see section 2.16 in [9] for  $m=1$  and [12] for general  $m$ ) which expresses the  $b^m$ -cohomology in terms of the classical De Rham cohomology groups of the manifold and the critical hypersurface.

## 2. Constructions and classification of $b^m$ -Nambu structures

Nambu structures of  $b^m$ -type can be described using forms which are singular along a smooth hypersurface. These forms, called  $b^m$ -forms, were studied by Scott [12] in his thesis. We start introducing the language of  $b^m$ -forms: We follow [12] for these definitions and main properties. The set-up in Scott [12] allows to consider smooth hypersurfaces without a globally defining function. For the sake of simplicity in this paper we will consider  $Z$  a smooth hypersurfaces (not necessarily connected) and attach to it a defining function  $f$ .

Take a local set of coordinates  $(x, \dots, x_{n-1})$  in a neighborhood of a point  $p$  in the critical set, the  $b^m$ -tangent bundle can be defined as the bundle whose sections are locally generated by:

$$\left\{ x^m \frac{\partial}{\partial x}, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{n-1}} \right\}, \quad (1)$$

with  $x$  such that  $|x| = \lambda$ , and  $\lambda$  is the distance function to  $z$ . For globally defining functions  $f = x$ .

As done in the case  $m = 1$  in [3] we can define the dual bundle, the  $b^m$ -cotangent bundle  $b^m T^*(M)$ . Sections of powers of these bundles are called  $b^m$ -forms.

A **Laurent Series** of a closed  $b^m$ -form  $\omega$  is a decomposition of  $\omega$  in a tubular neighborhood  $U$  of the critical set  $Z$  of the form

$$\omega = \frac{dx}{x^m} \wedge \left( \sum_{i=0}^{m-1} \pi^*(\alpha_i) x^i \right) + \beta \quad (2)$$

with  $\pi : U \rightarrow Z$  the projection of the tubular neighborhood onto  $Z$ ,  $\alpha_i$  a closed smooth De Rham form on  $Z$  and  $\beta$  a De Rham form on  $M$ .

In [12] it is proved that in a neighborhood of  $Z$ , every closed  $b^m$ -form  $\omega$  can be written in a Laurent form of type (2) once a defining function has been fixed.

The complex of  $b^m$ -forms endowed with a natural extension of De Rham differential defines  $b^m$ -Cohomology. The follow theorem tells us that  $b^m$ -cohomology can be read off from de Rham cohomology thus generalizing the classical Mazzeo-Melrose decomposition theorem in Section 2.16 in [9]:

**Theorem 2.1** ( *$b^m$ -Mazzeo-Melrose, [12]*) *The  $b^m$ -cohomology groups can be determined from De Rham cohomology groups as follows:*

$${}^{b^m}H^p(M) \cong H^p(M) \oplus (H^{p-1}(Z))^m. \quad (3)$$

We now introduce  $b^m$ -Nambu structures of top degree,

**Definition 2.2** *A  $b^m$ -Nambu structure of top degree on a pair  $(M^n, Z)$  with  $Z$  a smooth hypersurface is given by a smooth  $n$ -multivector field  $\Lambda$  such that there exists a local system of coordinates for which*

$$\Lambda = x_1^m \frac{\partial}{\partial x_1} \wedge \dots \wedge \frac{\partial}{\partial x_n} \quad (4)$$

and  $Z$  is defined by  $x_1 = 0$  in a neighborhood of  $Z$ .

Dualizing the local expression of the Nambu structure we obtain the form

$$\Theta = \frac{1}{x_1^m} dx_1 \wedge \dots \wedge dx_n \quad (5)$$

(which is not a smooth de Rham form), but it is a  $b^m$ -form of degree  $n$  defined on a  $b^m$ -manifold. As it is done in [3], we can check that this dual form is non-degenerate. So we may define a  $b^m$ -Nambu form as follows.

Mimicking the same condition as for  $b^m$ -symplectic forms we can talk about non-degenerate  $b^m$ -forms of top degree. This means that seen as a section of  $\Lambda^n({}^bT^*M)$  the form does not vanish.

**Notation:** We will denote by  $\Lambda$  the Nambu multivectorfield and by  $\Theta$  its dual.

**Definition 2.3** *A  $b^m$ -Nambu form is a non-degenerate  $b^m$ -form of top degree.*

We first include a collection of motivating examples, and then prove an equivariant classification theorem.

### 2.1. Examples

- (i)  **$b^m$ -symplectic surfaces:** Any  $b^m$ -symplectic surface is a  $b^m$ -Nambu manifold with Nambu structure of top degree.
- (ii)  **$b^m$ -symplectic manifolds as  $b^m$ -Nambu manifolds:** Let  $(M^{2n}, \omega)$  be a  $b^m$ -symplectic manifold, then  $(M^{2n}, \underbrace{\omega \wedge \dots \wedge \omega}_n)$  is automatically  $b^m$ -Nambu.
- (iii) **Orientable manifolds:** Let  $(M^n, \Omega)$  be any orientable manifold (with  $\Omega$  a volume form) and let  $f$  be a defining function for  $Z$ , then  $(1/f^m)\Omega$  defines a  $b^m$ -Nambu structure of top degree having  $Z$  as critical set.

Any Nambu structure can be written in this way if the hypersurface can be globally described as the vanishing set of a smooth function.

- (iv) **Spheres:** In [8], it was given special importance to the example  $(S^n, \sqcup_i S_i^{(n-1)})$  because of the Schoenflies theorem <sup>2</sup>, which imposes the associated graph to be a tree. The nice feature of this ex-

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2. The nature of this theorem is purely topological in dimension equal or greater than four, and so is its construction.

ample is that  $O(n)$  acts on the  $b^m$ -manifold  $(S^n, S^{(n-1)})$ , and it makes sense to consider its classification under these symmetries. This also works for other homogeneous spaces of type  $(G_1/G_2, G_2/G_3)$  with  $G_2$  and  $G_3$  with codimension 1 in  $G_1$  and  $G_2$  respectively.

## 2.2. $b^m$ -Nambu structures of top degree and orientability

We start proving:

**Theorem 2.4** *A compact  $n$ -dimensional manifold  $M$  admitting a  $b^{2k}$ -Nambu structure is orientable.*

**Proof:** Consider a collar of charts for the  $b^{2k}$ -Nambu structure such that in local coordinates the Nambu structure can be written as  $x_1^{2k} \frac{\partial}{\partial x_1} \wedge \dots \wedge \frac{\partial}{\partial x_n}$  with compatible orientations in a neighborhood of each connected component of  $Z$ .

Consider a 2:1 orientable covering  $(\tilde{M}, \tilde{Z})$  of the manifold and denote by  $\rho : \mathbb{Z}/2\mathbb{Z} \times \tilde{M} \rightarrow \tilde{M}$  the deck transformation. For each point  $p \in \tilde{Z}$  take a neighborhood  $U_p$  which does not contain other points identified by  $\rho$  thus  $U_p \cong \pi(U_p) =: V_p$ , and  $\Theta = \frac{1}{x_1^{2k}} dx_1 \wedge \dots \wedge dx_n$ . This form defines an orientation on  $V_p \setminus \pi(Z)$ . Take a symmetric covering of such neighborhoods to define a collar of  $Z$  with compatible orientations, and compatible with the covering. The compatible orientations and the symmetric coverings descend to  $(M, Z)$ , thus defining an orientation in  $(M, Z)$ . Thus, we have an orientation in  $V \setminus Z$ . By perturbing  $\Theta$  in  $V$  we obtain a volume form on  $V$ ,  $\tilde{\omega}$ , and thus an orientation in  $V$ . These can be glued to define an orientation via the volume form  $\tilde{\Theta}$  on the whole  $M$  proving that  $M$  is oriented.

## 2.3. Classification of $b^m$ -Nambu structures of top degree and $b^m$ -cohomology

We present the definitions contained in [8] of modular period attached to the connected component of an orientable Nambu structure using the language of  $b^m$ -forms.

Let  $\Theta$  be the dual to the multivectorfield  $\Lambda$  defining a Nambu structure. From the general decomposition of  $b^m$ -forms as it was set in Equation 2 we may write:

$$\Theta = \Theta_0 \wedge \frac{df}{f^m}$$

with  $\Theta_0 \in \Omega^{n-1}(M)$ .

This decomposition is valid in a neighborhood of  $Z$  whenever the defining function is well-defined. For non-orientable manifolds a similar decomposition can be proved by replacing the defining function  $f$  by an adapted distance (see [7]).

With this language in mind, the the **modular**  $(n-1)$ -**vector field** in [8] of  $\Theta$  along  $Z$  is the dual of the form  $\Theta_0$  in the decomposition above which is indeed the **modular**  $(n-1)$ -**form** along  $Z$  in [8].

Recall from [8] in our language:

**Definition 2.5** *The **modular period**  $T_\Lambda^Z$  of the component  $Z$  of the zero locus of  $\Lambda$  is*

$$T_\Lambda^Z := \int_Z \Theta_0 > 0.$$

In fact, this positive number determines the Nambu structure in a neighborhood of  $Z$  up to isotopy as it was proved in [8].

The following theorem gives a classification of  $b^m$ -Nambu structures.

**Theorem 2.6** *Let  $\Theta_0$  and  $\Theta_1$  be two  $b^m$ -Nambu forms of degree  $n$  on a compact orientable manifold  $M^n$ . If  $[\Theta_0] = [\Theta_1]$  in  $b^m$ -cohomology then there exists a diffeomorphism  $\phi$  such that  $\phi^*\Theta_1 = \Theta_0$ .*

**Proof:** We will apply the techniques of [10] with the only difference that we work with  $b^m$ -volume forms instead of volume forms.

Since  $\Theta_0$  and  $\Theta_1$  are non-degenerate  $b^m$ -forms both of them are a multiple of a volume form and thus the linear path  $\Theta_t = (1-t)\Theta_0 + t\Theta_1$  is a path of non-degenerate  $b^m$ -forms.

Because  $\Theta_0$  and  $\Theta_1$  determine the same cohomology class:

$$\Theta_1 - \Theta_0 = d\beta$$

with  $d$  the  $b^m$ -De Rham differential and  $\beta$  a  $b^m$ -form of degree  $n-1$ .

Now consider the Moser equation:

$$\iota_{X_t}\Theta_t = -\beta. \tag{6}$$

Observe that since  $\beta$  is a  $b^m$ -form and  $\Theta_t$  is non-degenerate. The vector field  $X_t$  is a  $b^m$ -vector field. Let  $\phi_t$  be the  $t$ -dependent flow integrating  $X_t$ .

The  $\phi_t$  gives the desired diffeomorphism  $\phi_t : M \rightarrow M$ , leaving  $Z$  invariant (since  $X_t$  is tangent to  $Z$ ) and  $\phi_t^*\Theta_t = \Theta_0$ .

In particular we recover the classification of  $b$ -Nambu structures of top degree in [8]:

**Theorem 2.7 (Classification of  $b$ -Nambu structures of top degree, [8])** *A generic  $b$ -Nambu structure  $\Theta$  is determined, up to orientation preserving diffeomorphism, by the following three invariants: the diffeomorphism type of the oriented pair  $(M, Z)$ , the modular periods and the regularized Liouville volume.*

By Theorem 2.1,

$${}^bH^n(M) \cong H^n(M) \oplus H^{n-1}(Z).$$

The first term on the right hand side is the Liouville volume image by the De Rham theorem, as it was done in [4] for  $b$ -symplectic forms. The second term collects the periods of the modular vector field. So if the three invariants coincide then they determine the same  $b$ -cohomology class.

In other words, the statement in [8] is equivalent to the following theorem in the language of  $b$ -cohomology.

**Theorem 2.8** *Let  $\Theta_1$  and  $\Theta_2$  be two  $b$ -Nambu forms on an orientable manifold  $M$ . If  $[\Theta_1] = [\Theta_2]$  in  $b$ -cohomology then there exists a diffeomorphism  $\phi$  such that  $\phi^*\Theta_1 = \Theta_2$ .*

This global Moser theorem for  $b^m$ -Nambu structures admits an equivariant version,

**Theorem 2.9** *Let  $\Theta_0$  and  $\Theta_1$  be two  $b^m$ -Nambu forms of degree  $n$  on a compact orientable manifold  $M^n$  and let  $\rho : G \times M \rightarrow M$  be a compact Lie group action preserving both  $b^m$ -forms. If  $[\Theta_0] = [\Theta_1]$  in  $b^m$ -cohomology then there exists an equivariant diffeomorphism  $\phi$  such that  $\phi^*\Theta_1 = \Theta_0$ .*

**Proof:** As in the former proof, write

$$\Theta_1 - \Theta_0 = d\beta$$

with  $d$  the  $b^m$ -De Rham differential and  $\beta$  a  $b^m$ -form of degree  $n-1$ . Observe that the path  $\Theta_t = (1-t)\Theta_0 + t\Theta_1$  is a path of invariant  $b^m$ -forms.

Now consider Moser's equation:

$$\iota_{X_t}\Theta_t = -\beta. \tag{7}$$

Since  $\Theta_t$  is invariant we can find an invariant  $\tilde{\beta}$ . For instance take  $\tilde{\beta} = \int_G \rho_g^*(\beta) d\mu$  with  $\mu$  a de Haar measure on  $G$  and  $\rho_g$  the induced diffeomorphism  $\rho_g(x) := \rho(g, x)$ .

Now replace  $\beta$  by  $\tilde{\beta}$  to obtain,

$$\iota_{X_t^G}\Theta_t = -\tilde{\beta} \tag{8}$$

with  $X_t^G = \int_G \rho_{g*} X_t d\mu$ . The vector field  $X_t^G$  is an invariant  $b$ -vector field. Its flow  $\phi_t^G$  preserves the action and  $\phi_t^{G*}\Theta_t = \Theta_0$ .

Playing the equivariant  $b^m$ -Moser trick using the 2:1 cover of a non-orientable manifold and taking as  $G$  the group of deck transformations we obtain,

**Corollary 2.10** *Let  $\Theta_0$  and  $\Theta_1$  be two  $b^m$ -Nambu forms of degree  $n$  on a manifold  $M^n$  (not necessarily oriented). If  $[\Theta_0] = [\Theta_1]$  in  $b^m$ -cohomology then there exists a diffeomorphism  $\phi$  such that  $\phi^*\Theta_1 = \Theta_0$ .*

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