Semivalues: weighting coefficients and allocations on unanimity games *

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Abstract

Each semivalue, as a solution concept defined on cooperative games with a finite set of players, is univocally determined by weighting coefficients that apply to players' marginal contributions. Taking into account that a semivalue induces semivalues on lower cardinalities, we prove that its weighting coefficients can be reconstructed from the last weighting coefficients of its induced semivalues. Moreover, we provide the conditions of a sequence of numbers in order to be the family of the last coefficients of any induced semivalues. As a consequence of this fact, we give two characterizations of each semivalue defined on cooperative games with a finite set of players: one, among all semivalues; another, among all solution concepts on cooperative games.

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1 Introduction

Semivalues were introduced by Weber [26] in the context of simple games (see also Einy [17]). In 1981, Dubey et al. [16] proposed the family of *semivalues* on all cooperative games, each one of which is defined by *weighting coefficients* that apply to the marginal contributions $v(S \cup \{i\}) - v(S)$ and are common to all coalitions of the same size. This new notion includes both, the Shapley value [25] and the Banzhaf value, defined by Owen [22] —an extension to all cooperative games of the original Banzhaf power index [3]—. In 1988, Weber [27] went further, dropped anonymity, and defined the family of *probabilistic values*, each one of which requires weighting coefficients p_S^i for each player i and each coalition $S \subseteq N \backslash \{i\}$ (of course, anonymity characterizes semivalues within this new family).

Several authors have been especially concerned with semivalues. Dragan [12] gives different interesting properties of semivalues as well as in [14, 15] the inverse problem is considered. Calvo and Santos [5] and Dragan [13] independently get a potential for every semivalue. Grabisch and Roubens [18] consider some properties of probabilistic interactions among the players in order to characterize solution concepts. In [19] axiomatizations of the probabilistic interaction indices are proposed and in [28] the additive efficient normalization of a semivalue is axiomatized. Our research group has also been studying *semivalues*. Carreras and Freixas [6] introduce regular semivalues. Carreras and Freixas [8] show the semivalue versatility as a very interesting tool for practical applications. Puente [24] devotes most of her PhD. thesis to semivalues and, especially, to their action on simple games. In [9], a restricted notion of semivalue as a power index, i.e., as a value for simple games, is axiomatically introduced. In the analysis of certain cooperative problems we have successfully used *binomial semivalues*, a single parametric subfamily defined by Puente [24] (see also [20] and [1]) that includes the Banzhaf value [22]. In [10] the binomial semivalues are used to study the effects of the partnership formation in cooperative games, comparing the joint effect on the involved players with the alternative alliance formation. Carreras and Puente [11] introduced *symmetric coalitional binomial semivalues*, a new family of coalitional values designed to take into account players' attitudes with regard cooperation. This new family applies to cooperative games with a coalition structure by combining the Shapley value and the binomial semivalues.

The payoff that a semivalue allocates to every player in a game is a weighted sum of his marginal contributions in the game, provided that all the coalitions of a common size have the same weighting coefficient. Following the characterization of semivalues by means of weighting coefficients given by Dubey et al. [16], every family of weighting coefficients having as many components as players in the game is associated to a unique semivalue if, and only if, these weighting coefficients satisfy a simple condition of *normalization*.

As it is well known, semivalues for cooperative games are defined on cardinalities rather than on specific player sets: this means that the n weighting coefficients define a semivalue on all set N such that $n = |N|$. Moreover, a semivalue on N induces semivalues for all cardinalities $t < n$, recurrently defined by the Pascal triangle (inverse) formula given by Dragan [12].

The present paper focuses on the weighting coefficients that univocally characterize a semivalue. First of all, we prove that these weighting coefficients can be reconstructed from the last weighting coefficients of its induced semivalues. After that, in order to obtain the weighting coefficients of the original semivalue, several conditions for the last weighting coefficients of its induced semivalues are provided.

As it is well known, the unanimity games form a basis of the vector space of the cooperative games and a semivalue is completely determined by its action on any unanimity game. The fact that the payoff that a semivalue allocates to each player in these games is closely related with the last weighting coefficient of the induced semivalues allows us to characterize each semivalue within the set of all semivalues defined on cooperative games with a finite set of players. In order to provide an individual axiomatization of each semivalue among all solution concepts on cooperative games, one nonstandard property, called "*Successively bounded allocations in unanimity games*" will be introduced and combined with the classical properties of *linearity, anonymity* and *dummy player property.*

According to the above considerations, the paper is organized as follows. Section 2 includes a minimum of preliminaries related to cooperative games and semivalues, paying special attention to the weighting coefficients that define each semivalue. In Section 3 we focus on the last weighting coefficients of the induced semivalues in order to establish conditions that allow us to reconstruct the weighting coefficients associated to the original semivalue. Finally, in Section 4, we give two characterizations of each semivalue defined on games with a finite set of players: one, among all semivalues; another, among all solution concepts on cooperative games.

2 Preliminaries on semivalues

Let N be a finite set of *players* and 2^N be the set of its *coalitions* (subsets of N). A *cooperative game* with transferable utility on N is a function $v : 2^N \to \mathbb{R}$, which assigns a real number $v(S)$ to each coalition $S \subseteq N$ and satisfies $v(\emptyset) = 0$. A game v is *monotonic* if $v(S) \leq v(T)$ whenever $S \subset T$. A game v is *additive* (or *inessential*) if $v(S \cup T) = v(S) + v(T)$ whenever $S \cap T = \emptyset$.

A player $i \in N$ is called *dummy* in a game $v \in \mathcal{G}_N$ if all marginal contributions of the player to the coalitions to which it belongs equal its individual utility in the game, i.e., $v(S \cup \{i\}) = v(S) + v(\{i\})$ for all $S \subseteq N \backslash \{i\}$.

Endowed with the natural operations for real–valued functions, denoted by $v + v'$ and λv for $\lambda \in \mathbb{R}$, the set of all cooperative games on N is a vector space $\mathcal{G}_N$. For every nonempty coalition $T \subseteq N$, the *unanimity game* u_T is defined on N by $u_T(S) = 1$ if $T \subseteq S$ and $u_T(S) = 0$ otherwise, and it is easily checked that the set of all unanimity games is a basis for $\mathcal{G}_N$, so that $\dim(\mathcal{G}_N) = 2^n - 1$ if $n = |N|$. Each game $v \in \mathcal{G}_N$ can then be uniquely written as a linear combination of unanimity games, and its components are the *Harsanyi dividends* (Harsanyi [21]):

$$v = \sum_{T \subseteq N:\, T \neq \emptyset} \alpha_T u_T, \quad \text{where} \quad \alpha_T = \alpha_T(v) = \sum_{S \subseteq T} (-1)^{t-s} v(S) \tag{1}$$

and, as usual, $t = |T|$ and $s = |S|$. The additive games form a linear subspace of $\mathcal{G}_N$ that we denote as $\mathcal{AG}_N$ and is spanned by the set $\{u_{\{i\}} : i \in N\}$ formed by unanimity games on singletons. Finally, every permutation θ of N induces a linear automorphism of $\mathcal{G}_N$ that leaves invariant $\mathcal{AG}_N$ and is defined by $(\theta v)(S) = v(\theta^{-1} S)$ for all $S \subseteq N$ and all $v \in \mathcal{G}_N$.

By a *solution* or a *value* on $\mathcal{G}_N$ we will mean a map $f : \mathcal{G}_N \to \mathbb{R}^N$, which assigns to every game v on N a vector $f[v]$ with components $f_i[v]$ for all $i \in N$.

Following Weber's [27] axiomatic description, $\psi : \mathcal{G}_N \to \mathbb{R}^N$ is a *semivalue* iff it satisfies the following properties:

(A1) *linearity*: $\psi[v + v'] = \psi[v] + \psi[v']$ (*additivity*) and $\psi[\lambda v] = \lambda \psi[v]$ for all $v, v' \in \mathcal{G}_N$ and $\lambda \in \mathbb{R}$;

(A2) *anonymity*: $\psi_{\theta i}[\theta v] = \psi_i[v]$ for all θ permutation on N, $i \in N$, and $v \in \mathcal{G}_N$;

(A3) *dummy player property*: if $i \in N$ is a dummy in game v, then $\psi_i[v] = v(\{i\})$.

(A4) *positivity*: if v is monotonic, then $\psi[v] \geq 0$;

There is an interesting characterization of semivalues, by means of *weighting coefficients*, due to Dubey, Neyman and Weber [16].

Theorem 2.1 *(a) For every nonnegative weighting coefficients $(p_s)_{s=0}^{n-1}$ such that*

$$\sum_{s=0}^{n-1} \binom{n-1}{s} p_s = 1, \tag{2}$$

the expression

$$\psi_i[v] = \sum_{S \subseteq N \setminus \{i\}} p_s [v(S \cup \{i\}) - v(S)] \quad \text{for all } i \in N \text{ and all } v \in \mathcal{G}_N \tag{3}$$

where $s = |S|$, defines a semivalue ψ; (b) conversely, every semivalue can be obtained in this way; (c) the correspondence $(p_s)_{s=0}^{n-1} \mapsto \psi$ is bijective.

Thus, the payoff that a semivalue allocates to every player in any game is a weighted sum of his marginal contributions in the game. If p_k is interpreted as the probability that a given player i joins a coalition of size k, provided that all the coalitions of a common size have the same probability of being joined, then $\psi_i[v]$ is the expected marginal contribution of that player to a random coalition he joins.

Well known examples of semivalues are the *Shapley value* φ, for which $p_s = \frac{1}{n\binom{n-1}{s}}$, and the *Banzhaf value* β, for which $p_s = 2^{1-n}$. The Shapley value φ is the only *efficient* semivalue, in the sense that $\sum_{i \in N} \varphi_i[v] = v(N)$ for every $v \in \mathcal{G}_N$. It is noteworthy that these two classical values are defined for each N.

As a generalization of the Banzhaf value we find the family of *binomial semivalues* (Puente [24], Giménez [20], Amer and Giménez [1]) introduced as follows. Let $\alpha \in [0,1]$ and $p_s = \alpha^s(1-\alpha)^{n-s-1}$ for $s = 0, 1, \ldots, n-1$. Then $\{p_s\}_{s=0}^{n-1}$ are the weighting coefficients and define a semivalue that will be denoted as ψ^α and called the *α–binomial semivalue.* Using the convention that $0^0 = 1$, the definition makes sense also for $\alpha = 0$ and $\alpha = 1$, where we respectively get the *dictatorial index* $\psi^0 = \delta$ and the *marginal index* $\psi^1 = \mu$, introduced by Owen [23] and such that $\delta_i[v] = v(\{i\})$ and $\mu_i[v] = v(N) - v(N \backslash \{i\})$ for all $i \in N$ and all $v \in \mathcal{G}_N$. Of course, $\alpha = 1/2$ gives the Banzhaf value.

In fact, semivalues are defined on cardinalities rather than on specific player sets: this means that a family of weighting coefficients $(p_s)_{s=0}^{n-1}$ defines a semivalue ψ on all N such that $n = |N|$. When necessary, we shall write ψ^n for a semivalue on cardinality n and p_s^n for its weighting coefficients. This often matters since a semivalue $\psi = \psi^n$ on cardinality n gives rise to *induced semivalues* ψ^t for all cardinalities t such that $1 \leq t \leq n-1$, recurrently defined by their weighting coefficients, which are given by an expression obtained by Dubey et al. [16] and referred by Dragan ([12], [13]) as the *Pascal triangle (inverse) formula*:

$$p_s^t = p_s^{t+1} + p_{s+1}^{t+1} \quad \text{for} \quad 0 \leq s < t \leq n-1. \tag{4}$$

It is not difficult to check that the induced semivalues of the Shapley value (resp., the α–binomial semivalue) are all Shapley values (resp., α–binomial semivalues for the same α). By applying (4) repeatedly, one gets the expression of the weighting coefficients of any induced semivalue in terms of the coefficients of the original semivalue,

$$p_s^t = \sum_{j=0}^{n-t} \binom{n-t}{j} p_{s+j}^n \quad \text{for} \quad 0 \leq s < t \leq n-1 \tag{5}$$

Example 2.2 Let ψ^4 be a semivalue defined on four-player games with weighting coefficients $(p_s^4)_{s=0}^3 = (0.16, 0.14, 0.12, 0.06)$. The weighting coefficients of its induced semivalues ψ^t, for all cardinalities t such that $1 \leq t \leq 3$, are computed according to (4) and their values can be summarized on a triangular table as follows:

	ψ^4	ψ^3	ψ^2	ψ^1
p_0^n	0.16	0.30	0.56	1.00
p_1^n	0.14	0.26	0.44	
p_2^n	0.12	0.18		
p_3^n	0.06			

3 Recovering weighting coefficients

In particular, Eq. (5) allows us to determine the last weighting coefficient of each induced semivalue of ψ^n from its weighting coefficients:

$$p_{t-1}^t = \sum_{j=0}^{n-t} \binom{n-t}{j} p_{t-1+j}^n \quad \text{for} \quad 1 \leq t \leq n-1. \tag{6}$$

The latter expression is useful for cases $2 \leq t \leq n-1$. In case $t = 1$, according to condition (2), $p_0^1 = 1$. The following result provides a kind of inverse formula for the weighting coefficients of any semivalue in terms of the last weighting coefficients of its induced semivalues.

Lemma 3.1 *Let ψ^n be a semivalue defined on $\mathcal{G}_N$. Its weighting coefficients p_s^n can be recovered from the last weighting coefficients p_{t-1}^t of its induced semivalues ψ^t as follows:*

$$p_s^n = \sum_{t=s+1}^{n} (-1)^{t-s-1} \binom{n-s-1}{t-s-1} p_{t-1}^t \quad \textit{for} \quad 0 \leq s \leq n-2. \tag{7}$$

Proof. Since $p_{n-2}^{n-1} = p_{n-2}^n + p_{n-1}^n$, it follows $p_{n-2}^n = p_{n-2}^{n-1} - p_{n-1}^n$, that proves (7) for $s = n-2$.

Assume that Eq. (7) holds for values s with $k \leq s \leq n-2$; we will prove that it still holds for $s = k-1$. Eq. (6) for $t = k$,

$$p_{k-1}^k = \sum_{j=0}^{n-k} \binom{n-k}{j} p_{k-1+j}^n,$$

allows us to separate coefficient p^n_{k-1} and to apply the induction hypothesis:

$$\begin{aligned}
p^n_{k-1} &= p^k_{k-1} - \sum_{j=1}^{n-k} \binom{n-k}{j} p^n_{k-1+j} \\
&= p^k_{k-1} - \sum_{j=1}^{n-k} \binom{n-k}{j} \sum_{t=k+j}^{n} (-1)^{t-k-j} \binom{n-k-j}{t-k-j} p^t_{t-1} \\
&= p^k_{k-1} - \sum_{t=k+1}^{n} \left[\sum_{j=1}^{t-k} (-1)^{t-k-j} \binom{n-k}{j} \binom{n-k-j}{t-k-j} \right] p^t_{t-1} \\
&= p^k_{k-1} - \sum_{t=k+1}^{n} (-1)^{t-k} \binom{n-k}{t-k} \left[\sum_{j=1}^{t-k} \binom{t-k}{j} (-1)^j \right] p^t_{t-1},
\end{aligned}$$

where the last equality derives from $\binom{n-k}{j}\binom{n-k-j}{t-k-j} = \binom{n-k}{t-k}\binom{t-k}{j}$. Now, it suffices to see that $\sum_{j=1}^{t-k} \binom{t-k}{j}(-1)^j = -1$ and hence

$$p^n_{k-1} = p^k_{k-1} + \sum_{t=k+1}^{n} (-1)^{t-k} \binom{n-k}{t-k} p^t_{t-1} = \sum_{t=k}^{n} (-1)^{t-k} \binom{n-k}{t-k} p^t_{t-1}. \ \Box$$

Following Theorem 2.1, each selection of a family of weighting coefficients $(p_s)_{s=0}^{n-1}$ univocally defines a semivalue on cooperative games with n players if, and only if, these weights satisfy Eq. (2). The weighting coefficients of the semivalue lead us to the last weighting coefficients of its induced semivalues p^t_{t-1} $(1 < t \leq n-1)$ and, conversely, the previous Lemma shows that all these last weighting coefficients are able to recover the weighting coefficients of the semivalue. The following result gives some restrictions for the last weighting coefficients of the induced semivalues.

Proposition 3.2 *(a) Let ψ be a semivalue on $\mathcal{G}_N$ with weighting coefficients $(p^n_s)_{s=0}^{n-1}$. The last weighting coefficients of its induced semivalues are successively bounded according to the following expressions*

$$\begin{aligned}
&(i) \quad 0 \leq p^n_{n-1} \leq 1 \\
&(ii) \quad q_s \leq p^s_{s-1} \leq Q_s \quad \textit{for } s = n-1, n-2, \ldots, 2,
\end{aligned}$$

where

$$q_s = \sum_{t=s+1}^{n} (-1)^{t-s-1} \binom{n-s}{t-s} p^t_{t-1} \tag{8}$$

and

$$Q_s = \binom{n-1}{s-1}^{-1} + (s-1) \sum_{t=s+1}^{n} \frac{(-1)^{t-s+1}}{t-1} \binom{n-s}{t-s} p^t_{t-1}. \tag{9}$$

(b) In the set of all semivalues on $\mathcal{G}_N$, it is always possible to find semivalues satisfying the equalities in (i) or in (ii).

Proof. (a) The nonnegativity condition of all weighting coefficients $(p_s^n)_{s=0}^{n-1}$ leads us to $p_{n-1}^n \geq 0$ and, at the same time, from Eq. (2), $p_{n-1}^n \leq 1$ holds and Part (i) in the statement has been proved. Now we prove the inequalities in (ii). Lemma 3.1 shows that every weighting coefficient p_s^n can be written in terms of the last weighting coefficients of its induced semivalues. Applying it to p_{s-1}^n,

$$p_{s-1}^n = \sum_{t=s}^{n} (-1)^{t-s} \binom{n-s}{t-s} p_{t-1}^t \geq 0 \quad \text{for } s = n-1, \ldots, 2, \tag{10}$$

and hence

$$p_{s-1}^s \geq \sum_{t=s+1}^{n} (-1)^{t-s-1} \binom{n-s}{t-s} p_{t-1}^t \quad \text{for } s = n-1, \ldots, 2. \tag{11}$$

Analogously, for $s = n-1, \ldots, 2$, from Eq. (2),

$$\sum_{k=s-1}^{n-1} \binom{n-1}{k} p_k^n = \sum_{k=s-1}^{n-1} \binom{n-1}{k} \sum_{t=k+1}^{n} (-1)^{t-k-1} \binom{n-k-1}{t-k-1} p_{t-1}^t \leq 1, \tag{12}$$

and consequently,

$$\sum_{t=s}^{n} \left[\sum_{k=s-1}^{t-1} (-1)^{t-k-1} \binom{n-1}{k} \binom{n-k-1}{t-k-1} \right] p_{t-1}^t \leq 1,$$

$$\binom{n-1}{s-1} p_{s-1}^s \leq 1 + \sum_{t=s+1}^{n} \left[\sum_{k=s-1}^{t-1} (-1)^{t-k} \binom{n-1}{k} \binom{n-k-1}{t-k-1} \right] p_{t-1}^t,$$

$$p_{s-1}^s \leq \binom{n-1}{s-1}^{-1} + \sum_{t=s+1}^{n} \left[\sum_{k=s-1}^{t-1} (-1)^{t-k} \binom{n-1}{k} \binom{n-k-1}{t-k-1} \binom{n-1}{s-1}^{-1} \right] p_{t-1}^t.$$

After some calculations in the inner sum, we can write

$$p_{s-1}^s \leq \binom{n-1}{s-1}^{-1} + \sum_{t=s+1}^{n} \frac{(n-s)!(s-1)!}{(n-t)!(t-1)!} \left[\sum_{k=s-1}^{t-1} (-1)^{t-k} \binom{t-1}{k} \right] p_{t-1}^t.$$

Finally, it is not difficult to see that

$$\sum_{k=s-1}^{t-1} (-1)^{t-k} \binom{t-1}{k} = (-1)^{t-s+1} \frac{s-1}{t-1} \binom{t-1}{s-1}$$

and, in this way, the previous inequality becomes

$$p_{s-1}^s \leq \binom{n-1}{s-1}^{-1} + (s-1) \sum_{t=s+1}^{n} \frac{(-1)^{t-s+1}}{t-1} \binom{n-s}{t-s} p_{t-1}^t \quad \text{for } s = n-1, n-2, \ldots, 2.$$

(b) According to Eq. (2), $p^n_{n-1} = 0$ is achieved by all semivalue satisfying $\sum_{s=0}^{n-2} \binom{n-1}{s} p^n_s = 1$, whereas $p^n_{n-1} = 1$ is only reached by the marginal index μ.

In the extreme case $p^s_{s-1} = q_s$ in (ii), Eqs. (11) and (10) become equalities and then $p^n_{s-1} = 0$. The other extreme case, $p^s_{s-1} = Q_s$, leads us to equality in Eq. (12), so that, in addition to (2):

$$\sum_{k=s-1}^{n-1} \binom{n-1}{k} p^n_k = 1 \quad \Rightarrow \quad p^n_0 = \cdots = p^n_{s-2} = 0 \quad \text{for } s = n-1, \ldots, 2. \ \square$$

The previous Proposition proves that the last weighting coefficients of each induced semivalue are bounded and their bounds depend on the last weighting coefficients of the induced semivalues defined on games with more players. From now on, we ask what amounts may be admissible for p^t_{t-1} $(1 < t \leq n)$ so that when we go back and find the coefficients p^n_s $(0 \leq s \leq n-1)$, they effectively define a semivalue on n-person games. In other words, is it possible to find a formula similar to expression (2) for the last weighting coefficients of any induced semivalues? A partial answer can be found in the next theorem, where some conditions are given so that a sequence of real numbers becomes admissible as a family of the last weighting coefficients.

Theorem 3.3 *A family of numbers $(k_n, k_{n-1}, ..., k_2)$ can be sequentially chosen in order to be the last weighting coefficients of the induced semivalues of a semivalue ψ defined on $\mathcal{G}_N$, starting by p^n_{n-1} and decreasing until p^2_1, if, and only if, it is verified:*

$$\begin{aligned} &(i) \quad 0 \leq k_n \leq 1 \\ &(ii) \quad q_s \leq k_s \leq Q_s \quad \textit{for } s = n-1, n-2, \ldots, 2, \end{aligned}$$

where

$$q_s = \sum_{t=s+1}^{n} (-1)^{t-s-1} \binom{n-s}{t-s} k_t \tag{13}$$

and

$$Q_s = \binom{n-1}{s-1}^{-1} + (s-1) \sum_{t=s+1}^{n} \frac{(-1)^{t-s+1}}{t-1} \binom{n-s}{t-s} k_t. \tag{14}$$

Proof. ($\Rightarrow$) It easily follows from part (a) in Proposition 3.2, identifying $k_n = p^n_{n-1}$, ..., $k_2 = p^2_1$.

($\Leftarrow$) All number k_n with $0 \leq k_n \leq 1$ is admissible as the weighting coefficient p^n_{n-1} of some semivalue. Suppose sequentially given the numbers $(k_n, k_{n-1}, ..., k_t)$ admissible as $(p^n_{n-1}, p^{n-1}_{n-2}, ..., p^t_{t-1})$ for $n-2 \geq t \geq 2$, then from part (b) in Proposition 3.2, and by continuity, all the values of k_{t-1} with

$q_{t-1} \le k_{t-1} \le Q_{t-1}$ are admissible as the last weighting coefficient p_{t-2}^{t-1} of some semivalue. $\square$

Introduced by Carreras and Freixas [6], the so-called *regular* semivalues constitute a particular type of semivalues whose weighting coefficients are positive. In this case, all marginal contributions of each player matter when the allocation by such a semivalue is considered. The Shapley value and every α–binomial semivalue for $\alpha \in (0,1)$ are regular semivalues. Interesting properties on regular semivalues can be found in [6], specially their nice behavior with regard to strict monotonicity, which characterizes them. Even, a system of axioms characterizing each regular semivalue is provided in [2].

With the same notations as in Theorem 3.3, a result specially adapted to regular semivalues can be formulated.

Corollary 3.4 *A family of numbers $(k_n, k_{n-1}, ..., k_2)$ can be sequentially chosen in order to be the last weighting coefficients of the induced semivalues of a regular semivalue ψ, starting by p_{n-1}^n and decreasing until p_1^2, if, and only if, it is verified:*

$$\begin{array}{ll} (i) & 0 < k_n < 1 \\ (ii) & q_s < k_s < Q_s \quad \textit{for } s = n-1, n-2, \ldots, 2. \quad \square \end{array}$$

Remark 3.5 In a first approximation, it seems that Eqs. (13) and (14) for the bounds q_s and Q_s are rather complicated. Next, we offer formulae for the first values of q_s and Q_s.

Bounds for p_{n-2}^{n-1} $(n > 2)$: $\quad q_{n-1} = k_n; \quad Q_{n-1} = \dfrac{1}{n-1} + \dfrac{n-2}{n-1} k_n.$

Bounds for p_{n-3}^{n-2} $(n > 3)$: $\quad q_{n-2} = 2k_{n-1} - k_n;$

$$Q_{n-2} = \frac{2}{(n-1)(n-2)} + (n-3)\left[\frac{2k_{n-1}}{n-2} - \frac{k_n}{n-1}\right].$$

Bounds for p_{n-4}^{n-3} $(n > 4)$: $\quad q_{n-3} = 3k_{n-2} - 3k_{n-1} + k_n;$

$$Q_{n-3} = \frac{3!}{(n-1)(n-2)(n-3)} + (n-4)\left[\frac{3k_{n-2}}{n-3} - \frac{3k_{n-1}}{n-2} + \frac{k_n}{n-1}\right].$$

Example 3.6 Let $\mathcal{G}_N$ be the vector space of five-player games.
(a) We will prove that the sequence of numbers (0.24, 0.36, 0.49, 0.65) is admissible as the last weighting coefficients $(p_4^5, p_3^4, p_2^3, p_1^2)$ of the induced semivalues of some semivalue ψ^5 on $\mathcal{G}_N$.

It is clear that $0 < 0.24 < 1$ and, by using the formulae given in the previous Remark for $n = 5$, we have

$$\begin{gathered} q_4 = 0.24 < 0.36 < 0.43 = Q_4; \quad q_3 = 0.48 < 0.49 < 0.52\widehat{6} = Q_3 \quad \text{and} \\ q_2 = 0.63 < 0.65 < 0.685 = Q_2. \end{gathered} \tag{15}$$

From these last weighting coefficients

$$(p_4^5, p_3^4, p_2^3, p_1^2, p_0^1) = (0.24,\ 0.36,\ 0.49,\ 0.65,\ 1.00),$$

the remaining weighting coefficients of every induced semivalue ψ^t, $1 \le t \le 5$, can be reconstructed and, in particular, of the semivalue ψ^5. To do that, it is enough to consider a triangular table as in Example 2.2 but now starting with the known coefficients (in bold) and then proceeding from left to right and from bottom to top:

	ψ^5	ψ^4	ψ^3	ψ^2	ψ^1
p_0^n	0.14	0.16	0.19	0.35	**1.00**
p_1^n	0.02	0.03	0.16	**0.65**	
p_2^n	0.01	0.13	**0.49**		
p_3^n	0.12	**0.36**			
p_4^n	**0.24**				

The strict inequalities in (15) guarantee the condition of regular semivalue, as it has been obtained.

(b) On the contrary, the sequence of numbers (0.13, 0.24, 0.42, 0.65) is not admissible as a family of the last weighting coefficients $(p_4^5, p_3^4, p_2^3, p_1^2)$ of any semivalue. Notice that $0 < 0.13 < 1$ holds and

$$q_4 = 0.13 < 0.24 < 0.3475 = Q_4 \quad \text{and} \quad q_3 = 0.35 < 0.42 < 0.421\widehat{6} = Q_3,$$

whereas $q_2 = 0.67$ and $Q_2 = 0.6725$, so that $0.65 \notin [q_2, Q_2]$.

Nevertheless, a procedure as in part (a) can be carried out and a triangular table can be constructed. In this case, the obtained numbers are all positive except in the first column where we would get $(0.09, -0.02, 0.07, 0.11, 0.13)$. These amounts are not the weighting coefficients of any semivalue on five-player games.

4 Unanimity games and semivalues

Since the unanimity games form a basis of $\mathcal{G}_N$, through linearity, a semivalue is completely determined by its action on any unanimity game. In this section we raise the following question: suppose to define a linear value in a finite set of players by allocating k_s to players belonging to S in the unanimity game u_S, where $s = |S|$. Then, under which conditions do the assignments of k_s define a semivalue? This problem has been studied recently by Bernardi and Lucchetti [4]. In their work they established a sufficient condition to build semivalues via the unanimity games by means of completely monotonic sequences. Our goal is to obtain sequentially bounded intervals for all admissible players' allocations in the unanimity games.

The next Lemma relates the last weighting coefficients p_{s-1}^s of the induced semivalues ψ^s, $1 \le s \le n$, studied in Section 3, to the allocations on the unanimity games u_S in the original vector space $\mathcal{G}_N$.

Lemma 4.1 *Given a semivalue ψ defined on $\mathcal{G}_N$ with weighting coefficients $(p_s^n)_{s=0}^{n-1}$, then*

$$\psi_i[u_S] = p_{s-1}^s \quad \forall i \in S, \forall S \subseteq N \text{ with } 1 \leq s \leq n.$$

Proof. It easily follows by applying (3) and (6) to the unanimity games u_S. □

Taking into account this result, Theorem 3.3 can be rewritten in terms of players' allocations in the respective unanimity games as follows:

Proposition 4.2 *(a) For each selection of real numbers $(k_n, \dots, k_2)$ there is a unique semivalue $\psi : \mathcal{G}_N \to \mathbb{R}^N$ satisfying*

(i) $\psi_i[u_N] = k_n$ $\forall i \in N$ with $k_n \in [0, 1]$ and

(ii) $\psi_i[u_S] = k_s$ $\forall S \subset N$, $2 \leq s \leq n-1$, $\forall i \in S$ with $k_s \in [q_s, Q_s]$, where q_s and Q_S are given in Theorem 3.3.

(b) ψ is the only semivalue whose weighting coefficients $(p_s^n)_{s=0}^{n-1}$ are given by

$$p_s^n = \sum_{t=s+1}^{n} (-1)^{t-s-1} \binom{n-s-1}{t-s-1} k_t, \tag{16}$$

whereas in all cases $1 = k_1 = \psi_i[u_{\{i\}}]$ $\forall i \in N$.

Remark 4.3 An interpretation for the bounds q_s and Q_s obtained sequentially in the above Proposition derives from the following property: players' allocations given by a semivalue ψ with weighting coefficients $(p_s^n)_{s=0}^{n-1}$ in the unanimity games u_S depend on the allocations in the unanimity games u_T with $|T| > |S|$. Starting with u_N we get

$$\psi_i[u_N] = p_{n-1}^n [u_N(N) - u_N(N \setminus \{i\})] = p_{n-1}^n.$$

The admissible values for $\psi_i[u_N]$, that is, k_n, equal p_{n-1}^n, so that condition (i) in part (a) of Proposition 4.2 holds.

Following with the unanimity games $u_{N \setminus \{j\}}$, $j \in N$, and for any $i \in N \setminus \{j\}$,

$$\begin{aligned} \psi_i[u_{N\setminus\{j\}}] = {} & p_{n-1}^n [u_{N\setminus\{j\}}(N) - u_{N\setminus\{j\}}(N \setminus \{i\})] + \\ & p_{n-2}^n [u_{N\setminus\{j\}}(N \setminus \{j\}) - u_{N\setminus\{j\}}(N \setminus \{j, i\})] = p_{n-1}^n + p_{n-2}^n \end{aligned}$$

Choosing p_{n-1}^n as k_n, the admissible values for $\psi_i[u_{N\setminus\{j\}}]$, i.e., k_{n-1}, depend on p_{n-2}^n. An extreme case for $k_n + p_{n-2}^n = k_{n-1}$ arises for $p_{n-2}^n = 0$ and then $k_{n-1} = k_n$. The first lower bound q_{n-1} in (ii) is achieved. The other extreme case for $p_0^n = \cdots = p_{n-3}^n = 0$ leads us to $p_{n-2}^n = (1 - k_n)/(n-1)$ and then

$$k_{n-1} = \frac{1}{n-1} + \frac{n-2}{n-1} k_n.$$

The first upper bound Q_{n-1} in (ii) is reached. Decreasing the cardinal of S, we can repeat the same procedure for the remaining unanimity games u_S and the respective bounds in (ii) are also reached.

Remark 4.4 The previous Proposition allows us to characterize a single semivalue within the set of all semivalues on cooperative games with a given set of players N. Notice that Eq. (16) can be written as

$$p_s^n = \sum_{t=s+1}^{n} (-1)^{t-s-1} \binom{n-s-1}{t-s-1} \psi_i[u_T] \quad \text{with } |T| = t \text{ and } i \in T, \tag{17}$$

taking k_t as $\psi_i[u_T]$.

As we will see, the previous result combined with the classical properties (A1), (A2) and (A3) will help us to characterize each individual semivalue among all the solution concepts defined on the set of cooperative games $\mathcal{G}_N$. To achieve this, a nonstandard property should be considered:

(A5) *Successively bounded allocations in unanimity games.* Let X be an allocation rule on $\mathcal{G}_N$. Given a family of real numbers $(k_n, k_{n-1}, \ldots, k_2)$ with $k_n \in [0,1]$ and $k_s \in [q_s, Q_s]$ for $s = n-1, \ldots, 2$, where each q_s and Q_s is defined in Eqs. (13) and (14), then $X_i[u_S] = k_s$ $\forall i \in S$ and $\forall S \subseteq N$ with $1 \leq s \leq n$.

In fact, Property (A5) is a family of properties given through each "valid" selection of numbers $(k_n, k_{n-1}, \ldots, k_2)$, where "valid" should be interpreted as the only allocations that the semivalues can offer to the players in the unanimity games, according to the relation between these allocations and the last weighting coefficients of its induced semivalues previously proved in Lemma 4.1. The recursive method used to obtain "valid" selections of numbers $(k_n, k_{n-1}, \ldots, k_2)$ has been already showed in Remark 4.3. In turn, each of these allocation families characterizes and provides the semivalue.

Theorem 4.5 *(a) For every selection of real numbers $(k_n, k_{n-1}, \ldots, k_2)$ with $k_n \in [0,1]$ and successively bounded according to Eqs. (13) and (14), there is a unique allocation rule $X : \mathcal{G}_N \to \mathbb{R}^N$ that satisfies properties (A1), (A2), (A3) and (A5) for these given numbers.*
(b) This allocation rule is the semivalue ψ with weighting coefficients $(p_s^n)_{s=0}^{n-1}$ given by

$$p_s^n = \sum_{t=s+1}^{n} (-1)^{t-s-1} \binom{n-s-1}{t-s-1} k_t, \quad \textit{where } k_1 = 1 \textit{ is imposed.} \tag{18}$$

Proof. *(Existence)* We prove that the semivalue ψ given in (b) satisfies the four properties. It is clear that properties (A1), (A2) and (A3) hold for all semivalue.

Property (A5) follows by applying Proposition 4.2 and the fact that each player $i \in N$ is a dummy in the unanimity game $u_{\{i\}}$ and consequently $\psi_i[u_{\{i\}}] = 1$ for all semivalue defined on $\mathcal{G}_N$.

(Uniqueness) Let X be an allocation rule on $\mathcal{G}_N$ that satisfies the stated properties. We will show that X is uniquely determined in all $v \in \mathcal{G}_N$, so that it must coincide with ψ.

Through linearity, we only have to prove that X is uniquely determined in each unanimity game u_S. By the dummy player property, $X_i[u_S] = 0$ if $i \notin S$. This leaves us with the players of S. Moreover, all players in S are symmetric in u_S and, according to Property (A2), their allocations are coincident. Finally, if $S = \{i\}$, according to Property (A3), $X_i[u_S]$ is well determined, and, according to Property (A5), for the remaining coalitions S with $1 < s \leq n$. $\square$

Example 4.6 Let $\mathcal{G}_N$ be the vector space of five-player games. We would like to know if there exists one solution X on $\mathcal{G}_N$ that satisfies linearity, anonymity, dummy player property and that assigns the following allocations to the players in the respective unanimity games:

$$X_i[u_N] = 0.24\ \forall i \in N; \qquad X_i[u_S] = 0.36\ S \subset N,\ s = 4, \forall i \in S;$$
$$X_i[u_S] = 0.49\ S \subset N,\ s = 3, \forall i \in S; \quad X_i[u_S] = 0.65\ S \subset N,\ s = 2, \forall i \in S.$$

These players' allocations on unanimity games, $X_i[u_S]$, $s = 5, 4, 3, 2$, for all $i \in S$, are respectively k_5, k_4, k_3, k_2 in Theorem 4.5 and, as we have seen in Example 3.6, they are adequately bounded. Then, the weighting coefficients of the semivalue ψ are:

$$(p_0^5,\ p_1^5,\ p_2^5,\ p_3^5,\ p_4^5) = (0.14,\ 0.02,\ 0.01,\ 0.12,\ 0.24).$$

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