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Abstract

A fundamental problem in computational algebraic geometry is the computation of the resultant. A central question is when and how to compute it as the determinant of a matrix. whose elements are the coefficients of the input polynomials up-to sign. This problem is well understood for unmixed multihomogeneous systems, that

is for systems consisting of multihomogeneous polynomials with the same support. However, little is known for mixed systems, that is for systems consisting of polynomials with different supports.

We consider the computation of the multihomogeneous resultant of bilinear systems involving two different supports. We present a constructive approach that expresses the resultant as the exact determinant of a *Koszul resultant matrix*, that is a matrix constructed from maps in the Koszul complex. We exploit the resultant matrix to propose an algorithm to solve such systems. In the process we extend the classical eigenvalues and eigenvectors criterion to a more general setting. Our extension of the eigenvalues criterion applies to a general class of matrices, including the Sylvester-type and the Koszul-type ones.

Keywords: Resultant; Sparse Resultant; Determinantal formula; Bilinear system; Mixed Multihomogeneous system; Polynomial solving

1 Introduction

The resultant is a central object in elimination theory and computational algebraic geometry. We use it to decide when an overdetermined polynomial system has a solution and to solve well-defined (square) systems. Moreover, it is one of the few tools that take into account the sparsity of supports of the polynomials.

Usually, we compute the resultant as a quotient of determinants of two matrices [Mac02, Jou97, DD01, D'A02]. If we can compute the resultant as a determinant of only one matrix whose non-zero entries are forms evaluated at the coefficients of the input polynomials, then we have a *determinantal formula*. Among these cases, the best we can hope for is to have linear forms. In general, determinantal formulas do not exist and it is an open problem to decide when they do.

The matrices appearing in the computation of resultants have a strong structure and we can classify them according to it. For a system $(f_0, \ldots, f_n)$, a *Sylvester-type formula* is a matrix that represents a map $(g_0, \ldots, g_n) \mapsto \sum_i g_i f_i$. It extends the classical Sylvester matrix and it corresponds to the last map of the Koszul complex of $(f_0, \ldots, f_n)$. Another kind of formula is the *Koszul-type formula* that involves the other maps of the Koszul complex. We call the matrices related to this formula *Koszul resultant matri-*

ces [MT17, BMT17]. For both formulas, the elements of the matrices are linear polynomials in the coefficients of $(f_0, \ldots, f_n)$. Other important resultant matrices include *Bézout-* and *Dixon-type*; we refer to [EM99] and references therein for details. We consider *Koszul-type determinantal formulas* for mixed multihomogeneous bilinear systems with two supports.

A well-known tool to derive determinantal formulas [WZ94, DE03, EM12, EMT16, MT17, BMT17] is the Weyman complex [Wey94], a generalization of the Koszul complex. For an introduction we refer to [Wey03, Sec. 9.2] and [GKZ08, Sec. 2.5.C, Sec. 3.4.E]. We follow this approach.

For *unmixed* multihomogeneous systems, that is systems where all the polynomial share the same support, determinantal formulas are well studied, e.g., [SZ94, WZ94, KS97, CK00, DD01, Wey03, DE03]. On the other hand, when we consider polynomials with different supports, that is *mixed systems*, little is known about determinantal formulas; with the exception of scaled multihomogeneous systems [EM12], that is when the supports are scaled copies of one of them, and the bivariate tensor-product case [MT17, BMT17].

The resultant is also a tool to solve 0-dimensional square polynomial systems $(f_1, \ldots, f_n)$. There are different variants, for example by hiding a variable, or using the u-resultant; we refer to [CLO06, Chp. 3] for a general introduction. When a *Sylvester-type formula* is available, we can use the corresponding resultant matrix to obtain the matrix of the multiplication map of a polynomial f_0 in $\mathbb{K}[\boldsymbol{x}]/\langle f_1, \ldots, f_n \rangle$. Then, we can solve the system by computing the eigenvalues and eigenvectors of the latter matrix, e.g., [AS88, Emi96]. The eigenvalues correspond to the evaluation of f_0 at every zero of the system. From the eigenvectors we can recover the coordinates of the zeros. To our knowledge similar techniques involving matrices coming from Koszul-type formulas do not exist up to now.

We consider mixed bilinear polynomial systems. On the one hand, this is simplest case of mixed multihomogeneous systems where no resultant formula was known. On the other hand, bilinear, and their generalization multilinear, polynomial systems are common in applications, for example in cryptography [FLDVP08, Jou14] and game theory [EV14]. We refer to [FSEDS11], see also [Spa12], for computing the roots of unmixed multilinear systems by means of Gröbner bases, and to [EMT16] by using resultants. We refer to [BFT18] for a Gröbner bases approach to solve square mixed multihomogeneous systems.

Our contribution We introduce a new algorithm to solve square mixed multihomogeneous systems consisting of bilinear polynomials with two different supports. It relies on eigenvalues and eigenvectors computations. Following classic resultant techniques we add a polynomial, f_0, to make the system overdetermined. The polynomial f_0 must be trilinear, as this is simplest one that can separate the roots. Then, we introduce a determinantal formula for the resultant of this overdetermined system. This is the first determinantal formula for a mixed multilinear polynomial system. Using Weyman's complex, we derive a *Koszul-type* formula and compute the resultant as the determinant of a *Koszul resultant matrix*.

We present a general extension of the eigenvalue criterion that works for a general class of formulas (see Def. 4.1), which include the Koszul-type and Sylvester-type formulas as special cases. We consider a square matrix M whose determinant is a multiple of the resultant of a system $(f_0, \ldots, f_n)$. If there is a monomial $\boldsymbol{x}^\sigma$ in f_0 such that we can partition M as $\left[\begin{smallmatrix} M_{1,1} & M_{1,2} \\ M_{2,1} & M_{2,2} \end{smallmatrix}\right]$ where $M_{1,1}$ is invertible, the coefficient of the monomial $\boldsymbol{x}^\sigma$ in f_0 appears solely in the diagonal of $M_{2,2}$ and this diagonal contains only this coefficient, then the evaluations of $\frac{f_0(\boldsymbol{x})}{\boldsymbol{x}^\sigma}$ at the solutions of $(f_1, \ldots, f_n)$, that is $\{\frac{f_0(\boldsymbol{x})}{\boldsymbol{x}^\sigma}|_{\boldsymbol{x}=\alpha} : (\forall i > 0)f_i(\alpha) = 0, \boldsymbol{x}^\sigma|_{\boldsymbol{x}=\alpha} \neq 0\}$, are eigenvalues of the Schur complement of $M_{2,2}$, that is $M_{2,2} - M_{2,1} \cdot M_{1,1}^{-1} \cdot M_{1,2}$.

We extend the eigenvector criteria for these mixed bilinear systems. When M is our *Koszul resultant matrix*, we show how to recover the coordinates of the solutions from the eigenvectors of the Schur complement of $M_{2,2}$. This approach works for systems whose solutions have no multiplicities.

Algorithm 1 summarizes our strategy to solve square 0-dimensional 2-bilinear systems whose solutions have no multiplicities.

Future work. Weyman complex leads to determinantal formulas for mixed multihomogeneous systems. A possible extension is to classify all the possible determinantal formulas for mixed multihomogeneous systems of this construction, similarly to [WZ94]. The structure of the Koszul resultant matrix could lead to more efficient algorithms to perform linear algebra with these matrices, and hence to solve faster, theoretically and practically, square mixed multihomogeneous systems. Finally, our eigenvector criterion should be extensible to any Koszul resultant matrix. This approach might be adapted to recover the coordinates of the solutions with multiplicities.

Algorithm 1 `Solve2Bilinear`$((\bar{f}_1, \ldots, \bar{f}_n))$

Input: $(\bar{f}_1, \ldots, \bar{f}_k)$ is a square 2-bilinear system such that $V_{\mathcal{P}}(\bar{f}_1, \ldots, \bar{f}_k)$ is finite and has no multiplicities.

1: $A \leftarrow$ Random linear change of coordinates preserving the structure.

2: $(f_1, \ldots, f_n) \leftarrow (\bar{f}_1 \circ A, \ldots, \bar{f}_n \circ A)$.   (Thm. 4.7)

3: $f_0 \leftarrow$ Random trilinear polynomial in $S(1,1,1)$.

4: $\begin{bmatrix} M_{1,1} & M_{1,2} \\ M_{2,1} & M_{2,2} \end{bmatrix} \leftarrow \begin{cases} \text{Matrix corresponding to } \delta_1((f_0, \ldots, f_n), \boldsymbol{m}), \text{ split wrt} \\ \text{the monomial } \boldsymbol{w^\theta}. \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \text{(Def. 4.1)} \end{cases}$

5: $\left\{ \left(\frac{f_0}{\boldsymbol{w^\theta}}(\alpha), \bar{v}_\alpha \right) \right\}_\alpha \leftarrow \begin{cases} \text{Set of pairs Eigenvalue-Eigenvector of the Schur} \\ \text{complement of } M_{2,2}. \quad\quad\quad\quad\quad\quad\quad \text{(Thm. 4.2)} \end{cases}$

6: **for all** $\left(\frac{f_0}{\boldsymbol{w^\theta}}(\alpha), \bar{v}_\alpha \right) \in \left\{ \left(\frac{f_0}{\boldsymbol{w^\theta}}(\alpha), \bar{v}_\alpha \right) \right\}_\alpha$ **do**

7:     Extract the coordinates α_x, α_y from $\rho_\alpha(\widehat{\boldsymbol{\lambda}_\alpha})$ by recovering it from $\begin{bmatrix} M_{1,1}^{-1} \cdot M_{2,1} \\ I \end{bmatrix} \cdot \bar{v}$.   (Thm. 4.13)

8:     Let $\alpha_z \in \mathbb{P}^{n_z}$ be the unique solution to the linear system given by $\{f_1(\alpha_x, \alpha_y, \boldsymbol{z}) = 0, \ldots, f_n(\alpha_x, \alpha_y, \boldsymbol{z}) = 0\}$, over $\mathbb{K}[\boldsymbol{z}]$.

9:     Recover the solution of the system $(\bar{f}_1, \ldots, \bar{f}_n)$, as $A\big((\alpha_x, \alpha_y, \alpha_z)\big)$.

10: **end for**

Paper organization In Sec. 2 we introduce notation and the resultant of mixed multihomogeneous systems. In Sec. 3, we present the Weyman complex in our setting and we prove the existence of a Koszul-type formula. Then, in Sec. 4, we present algorithms for solving 2-bilinear systems; Sec. 4.1 extends the eigenvalue criterion to a general class of matrices and Sec. 4.2 studies the eigenvectors to recover the coordinates of the solutions. Finally, in Sec. 5, we compare the size of our matrix with the experimental size of the matrices in Gröbner basis computation.

2   Preliminaries

Consider $n_x, n_y, n_z \in \mathbb{N}$ and let $\mathcal{P} := \mathbb{P}^{n_x} \times \mathbb{P}^{n_y} \times \mathbb{P}^{n_z}$ be a multiprojective space over an algebraic closed field $\mathbb{K}$ of characteristic 0. Consider $\boldsymbol{x} := \{x_0, \ldots, x_{n_x}\}$, $\boldsymbol{y} := \{y_0, \ldots, y_{n_y}\}$, $\boldsymbol{z} := \{z_0, \ldots, z_{n_z}\}$ and let $S_x(d_x) := \mathbb{K}[\boldsymbol{x}]_{d_x}$, $S_y(d_y) := \mathbb{K}[\boldsymbol{y}]_{d_y}$, and $S_z(d_z) := \mathbb{K}[\boldsymbol{z}]_{d_z}$ be the spaces of homogeneous polynomials in variables $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{z}$ and degrees d_x, d_y and d_z, respectively. Let $S(d_x, d_y, d_z) := S_x(d_x) \otimes S_y(d_y) \otimes S_z(d_z)$ be the multi-

homogeneous polynomials in $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$ of degrees d_x, d_y, and d_z, respectively. We say that the polynomials in $S(d_x, d_y, d_z)$ have multidegree $\boldsymbol{d} := (d_x, d_y, d_z) \in \mathbb{N}_0^3$. To avoid the repetition of the various definitions for x, y, and z, we consider $t \in \{x, y, z\}$. The dual space of $S_t(d_t)$ is $S_t(d_t)^*$. For $\sigma_t \in \mathbb{N}_0^{n_t+1}$, we define $\boldsymbol{t}^{\sigma_t} := \prod_{i=0}^{n_t} t_i^{\sigma_{t,i}}$. Then $\mathcal{A}(d_t) := \{\sigma_t : \boldsymbol{t}^{\sigma_t} \in S_t(d_t)\}$ is the set of the exponents of all the monomials of degree d_t in $\boldsymbol{t}$ and $\mathcal{A}(\boldsymbol{d}) := \mathcal{A}(d_x) \times \mathcal{A}(d_y) \times \mathcal{A}(d_z)$ is the set of all the exponents of the monomials of multidegree $\boldsymbol{d}$. If $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z) \in \mathcal{A}(\boldsymbol{d})$, then $\boldsymbol{w}^{\boldsymbol{\sigma}} := \boldsymbol{x}^{\sigma_x} \boldsymbol{y}^{\sigma_y} \boldsymbol{z}^{\sigma_z}$. Let $n := n_x + n_y + n_z$. For multidegrees $\boldsymbol{d} = (\boldsymbol{d}_0, \ldots, \boldsymbol{d}_n) \in (\mathbb{N}_0^3)^{n+1}$, we consider square multihomogeneous polynomial system

$$\boldsymbol{f} := (f_1, \ldots, f_n) \in S(\boldsymbol{d}_1) \times \cdots \times S(\boldsymbol{d}_n) \ . \tag{1}$$

Let $V_{\mathcal{P}}(\boldsymbol{f})$ be the set of solutions of $\boldsymbol{f}$ over $\mathcal{P}$. The multihomogeneous Bézout bound (`MHB`) [VdW78] bounds the number of isolated solutions of $\boldsymbol{f}$ over $\mathcal{P}$ [Ber75, Kus76, Kho78]. The bound is attained for any *generic* square system $\boldsymbol{f}$. It is the mixed volume of the polytopes $\mathcal{A}(\boldsymbol{d}_1), \ldots, \mathcal{A}(\boldsymbol{d}_n)$ [CLO06, Chp. 7] and appears as the coefficient of the monomial $\prod_{t \in \{x,y,z\}} X_t^{n_t}$ in $\prod_{j=1}^{n} \sum_{t \in \{x,y,z\}} \boldsymbol{d}_{j,t} X_t$ [MS87].

In the sequel we consider overdetermined systems which we construct by adding an $f_0 \in S(\boldsymbol{d}_0)$ to $\boldsymbol{f}$, that is,

$$\boldsymbol{f_0} := (f_0, f_1, \ldots, f_n) \in S(\boldsymbol{d}_0) \times \cdots \times S(\boldsymbol{d}_n) \ . \tag{2}$$

Typically, we will consider $\boldsymbol{d}_0 = (1, 1, 1)$, as we would like f_0 to be as simple as possible while still depending on all the variables.

2.1 Multihomogeneous sparse resultant

The multihomogeneous sparse resultant of $\boldsymbol{f_0}$ is a polynomial in the coefficients of the polynomials in $\boldsymbol{f_0}$, which vanishes if and only if the system has a solution over $\mathcal{P}$. Following [CLO06], for fixed $\boldsymbol{d}_0 \ldots \boldsymbol{d}_n \in \mathbb{N}_0^3$, we introduce a set of variables $\boldsymbol{u}_i := \{u_{i,\boldsymbol{\sigma}}\}_{\boldsymbol{\sigma} \in \mathcal{A}(\boldsymbol{d}_i)}$, for $0 \leq i \leq n$, and $\boldsymbol{u} := \{\boldsymbol{u}_0, \ldots, \boldsymbol{u}_n\}$. Given $P \in \mathbb{K}[\boldsymbol{u}]$, we let $P(\boldsymbol{f_0})$ denote the value obtained by replacing each variable $u_{i,\boldsymbol{\sigma}}$ with the coefficient of the monomial $\boldsymbol{w}^{\boldsymbol{\sigma}}$ in the polynomial f_i of $\boldsymbol{f_0}$. In this way we obtain polynomials over the coefficients of a polynomial system. The "universal" system

$F_{d_0,\ldots,d_n} \in \mathbb{K}[\boldsymbol{u_0}][\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}] \times \cdots \times \mathbb{K}[\boldsymbol{u_n}][\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}]$ is

$$F_{d_0,\ldots,d_n} := \left(\sum_{\boldsymbol{\sigma} \in \mathcal{A}(d_0)} u_{0,\boldsymbol{\sigma}} \boldsymbol{w}^{\boldsymbol{\sigma}}, \ \ldots, \ \sum_{\boldsymbol{\sigma} \in \mathcal{A}(d_n)} u_{n,\boldsymbol{\sigma}} \boldsymbol{w}^{\boldsymbol{\sigma}} \right). \tag{3}$$

Here the variables of $\boldsymbol{u}$ parametrize the systems described by polynomials in $S(\boldsymbol{d}_0) \times \cdots \times S(\boldsymbol{d}_n)$ over $\mathbb{K}^{\#\mathcal{A}(\boldsymbol{d}_0)} \times \cdots \times \mathbb{K}^{\#\mathcal{A}(\boldsymbol{d}_n)}$.

Consider the set of all tuples of $n + 1$ multihomogeneous polynomials together with their common solutions over $\mathcal{P}$, $\{(f_0, \ldots, f_n, \alpha) \in S(\boldsymbol{d}_0) \times \cdots \times S(\boldsymbol{d}_n) \times \mathcal{P} : (\forall 0 \leq i \leq n) \, f_i(\alpha) = 0\}$. The projection of this set on $S(\boldsymbol{d}_0) \times \cdots \times S(\boldsymbol{d}_n)$ is the set of overdetermined systems with common solutions in $\mathcal{P}$, $\{(f_0, \ldots, f_n) \in S(\boldsymbol{d}_0) \times \cdots \times S(\boldsymbol{d}_n) : V_{\mathcal{P}}(f_0, \ldots, f_n) \neq \emptyset\}$. By the Projective Extension Theorem [CLO92, Chp. 8 Sec. 5], this projection is a closed set under the Zariski topology and it forms an irreducible hypersurface over the vector space $S(\boldsymbol{d}_0) \times \cdots \times S(\boldsymbol{d}_n)$ [GKZ08, Chp. 8]. More formally, there is an irreducible polynomial $\mathrm{Res}_{\mathcal{P}}(\boldsymbol{d}_0, \ldots, \boldsymbol{d}_n) \in \mathbb{Z}[\boldsymbol{u}]$ such that for all the systems $\boldsymbol{f_0} \in S(\boldsymbol{d}_0) \times \cdots \times S(\boldsymbol{d}_n)$, $V_{\mathcal{P}}(\boldsymbol{f_0}) \neq \emptyset$ if and only if $\mathrm{Res}_{\mathcal{P}}(\boldsymbol{d}_0, \ldots, \boldsymbol{d}_n)(\boldsymbol{f_0}) = 0$. This polynomial is the sparse resultant over $\mathcal{P}$ for multihomogeneous systems of multidegrees $(\boldsymbol{d}_0, \ldots, \boldsymbol{d}_n)$.

The resultant $\mathrm{Res}_{\mathcal{P}}(\boldsymbol{d}_0, \ldots, \boldsymbol{d}_n)$ is itself a multihomogeneous polynomial, homogeneous in each block of variables $\boldsymbol{u}_i$. For each i, its degree with respect to $\boldsymbol{u}_i$ is $\mathtt{MHB}(\boldsymbol{d}_0, \ldots, \boldsymbol{d}_{i-1}, \boldsymbol{d}_{i+1}, \ldots, \boldsymbol{d_n})$.

2.2   2-bilinear systems

A square *2-bilinear system* of type $(n_x, n_y, n_z \ ; \ r, s)$ is a bilinear system $\boldsymbol{f} := (f_1, \ldots, f_n)$ with two different supports, namely $f_1, \ldots, f_r \in S(1, 1, 0)$ and $f_{r+1}, \ldots, f_n \in S(1, 0, 1)$, such that $n = r + s$, $n_y \leq r$ and $n_z \leq s$. It holds $\mathtt{MHB}(\boldsymbol{f}) = \binom{r}{n_y}\binom{n-r}{n_z}$.

Example 2.1. *The following (Eq. (4)) is a square 2-bilinear system of type* $(1, 1, 1 \ ; \ 2, 1)$ *and has two solutions over* $\mathcal{P}$, *namely* $\alpha_1 := (1{:}1 \ ; \ 1{:}1 \ ; \ 1{:}1)$ *and* $\alpha_2 := (1{:}3 \ ; \ 1{:}2 \ ; \ 1{:}3)$.

$$\begin{cases} f_1 := 7x_0y_0 - 8x_0y_1 - x_1y_0 + 2x_1y_1 \\ f_2 := -5x_0y_0 + 7x_0y_1 - x_1y_0 - x_1y_1 \\ f_3 := -6x_0z_0 + 9x_0z_1 - x_1z_0 - 2x_1z_1 \end{cases} . \tag{4}$$

Consider the trilinear $f_0 \in S(1, 1, 1)$. We refer to the systems $\boldsymbol{f_0} := (f_0, f_1, \ldots, f_n)$ as overdetermined 2-bilinear systems. We can also consider f_0 in $S(1, 1, 0)$, $S(1, 0, 1)$, $S(1, 0, 0)$, $S(0, 1, 0)$ and $S(0, 0, 1)$. We work with a trilinear f_0 because in the other cases it is not always possible to separate all the solutions of $V_{\mathcal{P}}(\boldsymbol{f})$.

Example 2.2 (Cont.). *Consider the overdetermined 2-bilinear system* $\boldsymbol{f_0} := (f_0, f_1, f_2, f_3)$, *where*

$$\begin{aligned} f_0 := {}& 3\,x_0 y_0 z_0 - x_0 y_0 z_1 - 4\,x_0 y_1 z_0 + 2\,x_0 y_1 z_1 \\ & + x_1 y_0 z_0 + 2\,x_1 y_0 z_1 + 2\,x_1 y_1 z_0 - 2\,x_1 y_1 z_1. \end{aligned}$$

In the following, we use $\boldsymbol{F^{(2)}}$ to denote the "universal" system of overdetermined 2-bilinear systems (see Sec. 2.1). Similarly, we use $\mathrm{Res}_{\mathcal{P}}^{(2)}$, for the resultant of the "universal" system $\boldsymbol{F^{(2)}}$.

Lemma 2.3. *Let* $\mathrm{MHB}(\boldsymbol{f}) = \binom{r}{n_y}\binom{s}{n_z}$. *The degree of* $\mathrm{Res}_{\mathcal{P}}^{(2)}$ *is*

$$\mu := (n_x + 1)\,\mathrm{MHB}(\boldsymbol{f})\,\frac{r \cdot s - n_y \cdot n_z + r + s + 1}{(r - n_y + 1)(s - n_z + 1)} \quad . \tag{5}$$

3 Determinantal formulas for 2-bilinear systems

A complex $K_\bullet$ is a sequence of modules $\{K_v\}_{v \in \mathbb{Z}}$ together with homomorphisms $\delta_v : K_v \to K_{v-1}$, such that $(\forall v \in \mathbb{Z})\ \mathrm{Im}(\delta_v) \subseteq \mathrm{Ker}(\delta_{v-1})$, i.e., $\delta_v \circ \delta_{v-1} = 0$. We say that the complex is exact if $(\forall v \in \mathbb{Z})\ \mathrm{Im}(\delta_v) = \mathrm{Ker}(\delta_{v-1})$. A complex is bounded when there are two constants a and b such that for every $v < a$ or $b < v$, it holds $K_v = 0$. If all the K_v are finite dimensional free-modules, then we can choose a basis of them and we can represent the maps δ_v using matrices. Under certain assumptions (see [GKZ08, App. A]) given a bounded complex of finite dimensional free-modules we can define its determinant. It is the quotient of minors of the matrices of δ_v and it is not zero if and only if the complex is exact. If there are only two non-zero modules of the same dimensions in the complex (that is all the other modules are the zero module), the determinant of the complex reduces to the determinant of the (matrix of the) map between these modules.

The Weyman Complex [Wey94, WZ94, Wey03] of a multihomogeneous system $\boldsymbol{f}$ is a bounded complex that is exact if and only if the sparse resultant

of the system $\boldsymbol{f}$ does not vanish [Wey03, Thm. 9.1.2]. The determinant of the complex is a power of the resultant [Wey03, Prop. 9.1.3]. When all the multidegrees are bigger than zero, the determinant of this complex is a non-zero constant multiple of the sparse resultant [GKZ08, Thm. 3.4.11]. If the Weyman Complex only involves two non-zero modules, the resultant of the corresponding system is the determinant of the map between these modules, and it has a determinantal formula.

Let $\boldsymbol{f_0} := (f_0, f_1, \ldots, f_n)$ be an overdetermined 2-bilinear system. Consider $E := \mathbb{K}^{n+1}$ and its canonical basis $e_0, \ldots, e_n$. Given a set $I \subset \{0, \ldots, n\}$, we define $\boldsymbol{e}_I := e_{I_1} \wedge \cdots \wedge e_{I_{\#I}}$ as the exterior product of the elements $e_{I_1}, \ldots, e_{I_{\#I}}$. As the exterior product is antisymmetric, that is $e_i \wedge e_j = -e_j \wedge e_i$, when we write $e_{I_1} \wedge \cdots \wedge e_{I_{\#I}}$ we assume that $(\forall i)\, I_i < I_{i+1}$. Let $\bigwedge\limits_{a,b,c} E$ be the vector space over $\mathbb{K}$ generated by $\{\boldsymbol{e}_{K \cup I \cup J} : K \subset \{0\}, I \subset \{1, \ldots, r\}, J \subset \{r+1, \ldots, n\}, \#I = a, \#J = b, \#K = c\}$.

For a *degree vector* $\boldsymbol{m} \in \mathbb{Z}^3$, the Weyman complex is $K_\bullet(\boldsymbol{f_0}, \boldsymbol{m})$. Each module of the complex is $K_v(\boldsymbol{m}) := \bigoplus_{p=0}^{n+1} K_{v,p}(\boldsymbol{m})$, where

$$
K_{v,p}(\boldsymbol{m}) := \bigoplus_{\substack{a+b+c=p \\ 0 \leq a \leq r \\ 0 \leq b \leq s \\ 0 \leq c \leq 1}} H_{\mathcal{P}}^{p-v}(\boldsymbol{m} - (p, p-b, p-a)) \otimes \bigwedge_{a,b,c} E,
$$

and $H_{\mathcal{P}}^q(\boldsymbol{m}')$ is the q-th cohomology of $\mathcal{P}$ with coefficients in the sheaf $\mathcal{O}(\boldsymbol{m}')$, and the space of global sections is $H_{\mathcal{P}}^0(\boldsymbol{m}')$ [Har77]. Note that the terms $K_{v,p}(\boldsymbol{m})$ do not depend on $\boldsymbol{f_0}$ [WZ94, Prop. 2.1]. Since $\mathcal{P}$ is a product of projective spaces, by Künneth's formula

$$
H_{\mathcal{P}}^{p-v}\left(m_x', m_y', m_z'\right) \cong \bigotimes_{t \in \{x,y,z\}} H_{\mathbb{P}^{n_t}}^{j_t}(m_t'), \tag{6}
$$

where $j_x + j_y + j_z = p - v$. By Serre's duality [Har77, Ch.III, Thm. 5.1] we have the identifications:

Proposition 3.1. *For each $t \in \{x, y, z\}$, $m_t' \in \mathbb{Z}$, it holds (1) $H_{\mathbb{P}^{n_t}}^0(m_t') \cong S_t(m_t')$ if $m_t' \geq 0$, (2) $H_{\mathbb{P}^{n_t}}^{n_t}(m_t') \cong S_t(-m_t' - 1 - n_t)^*$ if $m_t' < n_t$, where "$*$" denotes the dual space, and (3) $H_{\mathbb{P}^{n_t}}^q(m_t') \cong 0$, of all other values of q and m_t.*

As a corollary from Eq. (6), for each $t \in \{x, y, z\}$, $j_t \in \{0, n_t\}$. Moreover, we can identify dual complexes.

Proposition 3.2 ([Wey03, Thm. 5.1.4]). *Let $\boldsymbol{m}$ and $\boldsymbol{m}'$ be degree vectors such that $\boldsymbol{m} + \boldsymbol{m}' = (n_y + n_z, n_x + n_z - s, n_x + n_y - r)$. Then, $K_v(\boldsymbol{m}) \cong K_{1-v}(\boldsymbol{m}')^*$ for all $v \in \mathbb{Z}$ and $K_\bullet(\boldsymbol{f_0}, \boldsymbol{m})$ is dual to $K_\bullet(\boldsymbol{f_0}, \boldsymbol{m}')$.*

3.1 Degree vectors and determinantal formulas

If $K_1(\boldsymbol{m})$, $K_0(\boldsymbol{m})$ are the only non-zero modules in the Weyman complex $K_\bullet(\boldsymbol{f_0}, \boldsymbol{m})$, then the determinant of the complex is the determinant of the map, between them, $\delta_1(\boldsymbol{f_0}, \boldsymbol{m})$. In this case, we have a determinantal formula for the resultant. In the following, when it is clear from the context, we write δ_1 instead of $\delta_1(\boldsymbol{f_0}, \boldsymbol{m})$.

Theorem 3.3. *Let $\boldsymbol{f_0}$ be a 2-bilinear overdetermined system of type $(n_x, n_y, n_z; r, s)$, with $f_0 \in S(1,1,1)$. The degree vectors (1) $(n_y - 1, -1, n_x + n_y - r + 1)$, (2) $(n_z + 1, n_x + n_z - s + 1, -1)$, (3) $(n_z - 1, n_x + n_z - s + 1, -1)$, (4) $(n_y + 1, -1, n_x + n_y - r + 1)$ lead to determinantal Weyman complexes for $\mathrm{Res}_{\mathcal{P}}^{(2)}(\boldsymbol{f_0})$.*

Observation 3.4. *The four degree vectors of Thm. 3.3 provide a single matrix formula. Vector 1 (resp. 2) is obtained from 3 (resp. 4) by exchanging the variables $\boldsymbol{y}$ and $\boldsymbol{z}$. By Prop. 3.2, we can see that 1,2 and 3, 4 are dual pairs, yielding the same matrix transposed.*

Proof. We consider only the first degree vector $\boldsymbol{m} := (n_y - 1, -1, n_x + n_y - r + 1)$. By Obs.3.4, the other cases are similar.

First, we show that the complex has only two non-zero terms. Since $K_v(\boldsymbol{m}) := \bigoplus_{p=0}^{n+1} K_{v,p}(\boldsymbol{m})$, and in view of Eq. (6), for each $K_{v,p}(\boldsymbol{m})$, we have to consider sums $\sum_{t \in \{x,y,z\}} j_t = p - v$. By Prop. 3.1, if $j_t \notin \{0, n_t\}$, then $K_{v,p} = 0$. The remaining cases are summarized in the following table and their analysis follows.

j_x	j_y	j_z	Case
0	**0**	0	(1)
n_x	**0**	0	(1)

j_x	j_y	j_z	Case
0	**0**	n_z	(1)
n_x	**0**	n_z	(1)

j_x	j_y	j_z	Case
0	n_y	$\boldsymbol{n_z}$	(2)
n_x	n_y	$\boldsymbol{n_z}$	(2)

j_x	j_y	j_z	Case
0	$\boldsymbol{n_y}$	0	(3)
$\boldsymbol{n_x}$	$\boldsymbol{n_y}$	**0**	(4)

Case 1: $\boldsymbol{j_y} = \boldsymbol{0}$. The second term in the tensor product of $K_{v,p}$ is $H^0_{\mathbb{P}^{n_y}}(-1 - a - c) \cong S_y(-1 - a - c)$, by Prop. 3.1. As $a, c \geq 0$, $S_y(-1 - a - c) = 0$. Hence, $K_{v,p} = 0$.

Case 2: $j_z = n_z$. The third term in the tensor product of $K_{v,p}$ is $H_{\mathbb{P}^{n_z}}^{n_z}(n_x + n_y - r + 1 - b - c) \cong S_z(-(n_x + n_y + n_z) + r - 2 + b + c)^*$, by Prop. 3.1. As $n_x + n_y + n_z = r + s$, $-(n_x + n_y + n_z) + r - 2 + b + c = -s - 2 + b + c < 0$ because $b \leq s$ and $c \leq 1$. Hence, $H_{\mathbb{P}^{n_z}}^{n_z}(n_x + n_y - r + 1 - b - c) = 0$ and so $K_{v,p} = 0$.

Case 3: $j_x = 0$, $j_y = n_y$. As $j_y = n_y$, the second term in the tensor product $K_{v,p}$ is $H_{\mathbb{P}^{n_y}}^{n_y}(-1 - a - c) \cong S_y(a + c - n_y)^*$, by Prop. 3.1. This module is not zero iff $a + c \geq n_y$. Consider the first term in the tensor product, $H_{\mathbb{P}^{n_x}}^0(n_y - 1 - p) \cong S_x(n_y - 1 - p)$. If $a + c \geq n_y$, as $p = a + b + c$, then $n_y - 1 - p \leq -1 - b < 0$. Hence, either $H_{\mathbb{P}^{n_y}}^{n_y}(-1 - a - c) = 0$ or $H_{\mathbb{P}^{n_x}}^0(n_y - 1 - p) = 0$, and so $K_{v,p} = 0$.

Case 4: $j_x = 0$, $j_y = n_y$, $j_z = 0$. The first term in the tensor product $K_{v,p}$ is $H_{\mathbb{P}^{n_x}}^{n_x}(n_y - 1 - p) \cong S_x(-n_x - n_y + p) = S_x(v)$, as $p - v = j_x + j_y + j_z = n_x + n_y$. Hence, $H_{\mathbb{P}^{n_x}}^{n_x}(n_y - 1 - p) \neq 0$ iff $v \geq 0$. As $j_z = 0$ the third term in the tensor product of $K_{v,p}$ is $H_{\mathbb{P}^{n_z}}^0(n_x + n_y - r + 1 - b - c) \cong S_z(n_x + n_y - r + 1 - b - c)$. This term is not zero iff $n_x + n_y - r + 1 \geq b + c$. Moreover, as $p = a + b + c$, $v = a + b + c - n_x - n_y$. Then, if $H_{\mathbb{P}^{n_z}}^0(n_x + n_y - r + 1 - b - c) \neq 0$, then $v \leq a - r + 1$. By definition $a \leq r$, so $v \leq 1$.

We deduce that all other modules apart from $K_{1,n_x+n_y+1}(\boldsymbol{m})$ and $K_{0,n_x+n_y}(\boldsymbol{m})$ are equal to zero. Hence, by [Wey03, Prop. 9.1.3] the determinant of (a matrix expressing) δ_1 is a power [1] of $\mathrm{Res}_{\mathcal{P}}^{(2)}(\boldsymbol{f_0})$.

To conclude, it suffices to show that the exponent is equal to one. Due to the form $\delta_1 : K_{1,q+1}(\boldsymbol{m}) \to K_{0,q}(\boldsymbol{m})$, the elements in a matrix that represents δ_1 have degree $(q + 1) - q = 1$ as polynomials in $\mathbb{K}[\boldsymbol{u}]$ [Wey03, Prop. 5.2.4]. Therefore, the exponent is one iff the degree of the resultant is equal to the dimension of the matrix of

$$K_{\bullet}(\boldsymbol{f_0}, \boldsymbol{m}) : 0 \to K_{1,n_x+n_y+1}(\boldsymbol{m}) \xrightarrow{\delta_1} K_{0,n_x+n_y}(\boldsymbol{m}) \to 0 \,.$$

We analyze the possible values for (a, b, c) to compute the dimension. Following **Case 4**, if $H_{\mathbb{P}^{n_z}}^0(n_x + n_y - r + 1 - b - c) \neq 0$, then the possible values for a are $v + r - 1 \leq a \leq r$, for $v \in \{0, 1\}$. As $b = p - a - c$, and $0 \leq c \leq 1$, we enumerate all the options for (a, b, c) and write our modules as

---

[1]The exponent is known to be one for any very ample supports [GKZ08], i.e. $(\forall i, j)$ $\boldsymbol{d}_{i,j} > 0$. However, due to the zero degrees, 2-bilinear supports are ample but not very ample.

$$K_1 = K_{1,n_x+n_y+1} \cong L_{1,1} \oplus L_{1,2} \tag{7}$$
$$= \Big(S_x(1)^* \otimes S_y(r-n_y)^* \otimes S_z(0) \otimes \bigwedge_{r,s-n_z+1,0} E \Big) \oplus$$
$$\Big(S_x(1)^* \otimes S_y(r-n_y+1)^* \otimes S_z(0) \otimes \bigwedge_{r,s-n_z,1} E \Big).$$
$$K_0 = K_{0,n_x+n_y} \cong L_{0,1} \oplus L_{0,2} \oplus L_{0,3} \oplus L_{0,4} \tag{8}$$
$$= \Big(S_x(0)^* \otimes S_y(r-n_y-1)^* \otimes S_z(0) \otimes \bigwedge_{r-1,s-n_z+1,0} E \Big) \oplus$$
$$\Big(S_x(0)^* \otimes S_y(r-n_y)^* \otimes S_z(1) \otimes \bigwedge_{r,s-n_z,0} E \Big) \oplus$$
$$\Big(S_x(0)^* \otimes S_y(r-n_y)^* \otimes S_z(0) \otimes \bigwedge_{r-1,s-n_z,1} E \Big) \oplus$$
$$\Big(S_x(0)^* \otimes S_y(r-n_y+1)^* \otimes S_z(1) \otimes \bigwedge_{r,s-n_z-1,1} E \Big).$$

To compute their dimensions we notice that $\dim \left(\bigwedge_{a,b,c} E \right) = \binom{r}{a} \binom{s}{b}$, and we recall that $\dim S_t(q) = \dim S_t(q)^* = \binom{n_t+q}{q}$. The calculation leads to $\dim(K_1) = \dim(K_0) = \mu$, see Eq. (5). $\qquad\square$

The four degree vectors of Thm. 3.3 are not the only ones that lead to determinantal formulas. We are interested in them because, experimentally, there are no Sylvester-type formulas and only these degree vectors lead to Koszul-type formulas [EMT16, MT17].

3.2   Construction of the map $\delta_1(f_0, m)$

Following [Wey03, Sec. 5.5], we construct the map $\delta_1(\boldsymbol{f_0}, \boldsymbol{m}) : K_1(\boldsymbol{m}) \to K_0(\boldsymbol{m})$. By Obs. 3.4, we only consider $\boldsymbol{m} = (n_y - 1, -1, n_x + n_y - r + 1)$.

In the proof of Thm. 3.3 we saw that the map $\delta_1(\boldsymbol{F^{(2)}}, \boldsymbol{m})$ has linear coefficients in $\mathbb{K}[\boldsymbol{u}]$. As it is a linear map between free modules, it is enough to define it over a basis of K_0 and K_1.

First we introduce some notation. Let $t \in \{x, y, z\}$. For each $\sigma_t \in \mathcal{A}(d)$, $d \in \mathbb{N}_0$, consider $\boldsymbol{\partial t}^{\sigma_t} \in S_t(d)^*$ such that $\boldsymbol{\partial t}^{\sigma_t}(\sum c_{\theta_t} \boldsymbol{t}^{\theta_t}) = c_{\sigma_t}$. The set

$\{\boldsymbol{\partial t}^{\sigma_t} : \sigma_t \in \mathcal{A}(d)\}$ forms a basis of $S_t(d)^*$. The map $\star_t : \mathbb{K}[\boldsymbol{t}] \times \mathbb{K}[\boldsymbol{t}]^* \to \mathbb{K}[\boldsymbol{t}]^*$, acts as $(\boldsymbol{t}^{\theta_t}, \boldsymbol{\partial t}^{\sigma_t}) \mapsto \boldsymbol{t}^{\theta_t} \star_t \boldsymbol{\partial t}^{\sigma_t}$, where

$$\boldsymbol{t}^{\theta_t} \star_t \boldsymbol{\partial t}^{\sigma_t} = \begin{cases} \boldsymbol{\partial t}^{\sigma_t - \theta_t} & \text{if } (\forall i,\ 0 \le i \le n_t)\ \sigma_{t,i} \ge \theta_{t,i} \\ 0 & \text{otherwise} \end{cases} . \tag{9}$$

This map is graded, that is, for each $(d, \bar{d}) \in \mathbb{Z}^2$, it maps the elements in $S_t(d) \times S_t(\bar{d})^*$ to $S_t(\bar{d} - d)^*$. We will denote the map by "$\star$" when the variable is clear from the context. We define the graded map ψ,

$$\psi : (\mathbb{K}[\boldsymbol{x}]^* \otimes \mathbb{K}[\boldsymbol{y}]^* \otimes \mathbb{K}[\boldsymbol{z}]) \times (\mathbb{K}[\boldsymbol{x}] \otimes \mathbb{K}[\boldsymbol{y}] \otimes \mathbb{K}[\boldsymbol{z}])$$
$$\to (\mathbb{K}[\boldsymbol{x}]^* \otimes \mathbb{K}[\boldsymbol{y}]^* \otimes \mathbb{K}[\boldsymbol{z}]) \tag{10}$$

$$\psi(\boldsymbol{\partial x}^{\sigma_x} \otimes \boldsymbol{\partial y}^{\sigma_y} \otimes \boldsymbol{z}^{\sigma_z}, \boldsymbol{x}^{\theta_x} \otimes \boldsymbol{y}^{\theta_y} \otimes \boldsymbol{z}^{\theta_z}) :=$$
$$(\boldsymbol{x}^{\theta_x} \star \boldsymbol{\partial x}^{\sigma_x}) \otimes (\boldsymbol{y}^{\theta_y} \star \boldsymbol{\partial y}^{\sigma_y}) \otimes (\boldsymbol{z}^{\theta_z + \sigma_z})$$

For each $(d_x, d_y, d_z, \bar{d}_x, \bar{d}_y, \bar{d}_z) \in \mathbb{Z}^6$, it maps $(S_x(d_x)^* \otimes S_y(d_y)^* \otimes S_z(d_z)) \times \left(S_x(\bar{d}_x)^* \otimes S_y(\bar{d}_y)^* \otimes S_z(\bar{d}_z)\right)$ to $S_x(d_x - \bar{d}_x)^* \otimes S_y(d_y - \bar{d}_y)^* \otimes S_z(d_z + \bar{d}_z)$.

As $\delta_1(\boldsymbol{f_0}, \boldsymbol{m}) : K_1 \to K_0$ is linear and $K_1 \cong L_{1,1} \oplus L_{1,2}$, we define the map over a basis of $L_{1,1}$ and $L_{1,2}$. For each $\boldsymbol{\ell} \in S_x(1)^* \otimes S_y(r - n_y)^* \otimes S_z(0)$ and $\boldsymbol{e}_I \in \bigwedge_{r, s-n_z+1, 0} E$, we consider $\boldsymbol{\ell} \otimes \boldsymbol{e}_I \in L_{1,1}$ and

$$\delta_1(\boldsymbol{f_0}, \boldsymbol{m})(\boldsymbol{\ell} \otimes \boldsymbol{e}_I) := \sum_{i=1}^{n_x + n_y + 1} (-1)^{i-1} \psi(\boldsymbol{\ell}, f_{I_i}) \otimes \boldsymbol{e}_{I \setminus \{I_i\}} \in L_{0,1} \oplus L_{0,2}.$$

For each $\boldsymbol{\ell} \in S_x(1)^* \otimes S_y(r - n_y + 1)^* \otimes S_z(0)$ and $\boldsymbol{e}_J \in \bigwedge_{r, s-n_z, 1} E$, we consider $\boldsymbol{\ell} \otimes \boldsymbol{e}_J \in L_{1,2}$ and

$$\delta_1(\boldsymbol{f_0}, \boldsymbol{m})(\boldsymbol{\ell} \otimes \boldsymbol{e}_J) := \sum_{i=1}^{n_x + n_y + 1} (-1)^{i-1} \psi(\boldsymbol{\ell}, f_{J_i}) \otimes \boldsymbol{e}_{J \setminus \{J_i\}} \in L_{0,2} \oplus L_{0,3} \oplus L_{0,4}.$$

The map $\delta_1(\boldsymbol{f_0}, \boldsymbol{m})$ corresponds to a Koszul-type formula, involving multiplication and dual multiplication maps. The matrix that represents this map is a Koszul resultant matrix [MT17, BMT17].

Example 3.5 (Cont.). *In this case,* $\boldsymbol{m} = (0, -1, 1)$. *We consider the following monomial basis,*

Basis of K_1 (Columns)	
(A)	$\partial x_0 \partial y_1^2 \boldsymbol{e}_{\{0,1,2\}}$
(B)	$\partial x_1 \partial y_0^2 \boldsymbol{e}_{\{0,1,2\}}$
(C)	$\partial x_1 \partial y_1^2 \boldsymbol{e}_{\{0,1,2\}}$
(D)	$\partial x_0 \partial y_0 \boldsymbol{e}_{\{1,2,3\}}$
(E)	$\partial x_0 \partial y_1 \boldsymbol{e}_{\{1,2,3\}}$
(F)	$\partial x_1 \partial y_0 \boldsymbol{e}_{\{1,2,3\}}$
(G)	$\partial x_1 \partial y_1 \boldsymbol{e}_{\{1,2,3\}}$
(H)	$\partial x_0 \partial y_0 \partial y_1 \boldsymbol{e}_{\{0,1,2\}}$
(I)	$\partial x_0 \partial y_0^2 \boldsymbol{e}_{\{0,1,2\}}$
(J)	$\partial x_1 \partial y_0 \partial y_1 \boldsymbol{e}_{\{0,1,2\}}$

Basis of K_0 (Rows)	
(I)	$\boldsymbol{e}_{\{1,3\}}$
(II)	$\boldsymbol{e}_{\{2,3\}}$
(III)	$\partial y_0 \boldsymbol{e}_{\{0,1\}}$
(IV)	$\partial y_1 \boldsymbol{e}_{\{0,1\}}$
(V)	$\partial y_0 \boldsymbol{e}_{\{0,2\}}$
(VI)	$\partial y_1 \boldsymbol{e}_{\{0,2\}}$
(VII)	$\partial y_0 z_1 \boldsymbol{e}_{\{1,2\}}$
(VIII)	$\partial y_1 z_1 \boldsymbol{e}_{\{1,2\}}$
(IX)	$\partial y_0 z_0 \boldsymbol{e}_{\{1,2\}}$
(X)	$\partial y_1 z_0 \boldsymbol{e}_{\{1,2\}}$

The following matrix represents $\delta_1(\boldsymbol{f_0}, \boldsymbol{m})$ *wrt the basis above.*

	(A)	(B)	(C)	(D)	(E)	(F)	(G)	(H)	(I)	(J)
(I)	0	0	0	5	−7	1	1	0	0	0
(II)	0	0	0	7	−8	−1	2	0	0	0
(III)	0	−1	0	0	0	0	0	−1	−5	7
(IV)	7	0	−1	0	0	0	0	−1	0	−5
(V)	0	1	0	0	0	0	0	−2	−7	8
(VI)	8	0	−2	0	0	0	0	1	0	−7
(VII)	0	2	0	9	0	−2	0	−2	−1	2
(VIII)	2	0	−2	0	9	0	−2	2	0	−1
(IX)	0	1	0	−6	0	−1	0	2	3	−4
(X)	−4	0	2	0	−6	0	−1	1	0	3

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The 2×2 splitting illustrated above will be used in the next section.

4 Solving 2-bilinear systems

Consider a 0-dimensional system $f_1, \ldots, f_n \in \mathbb{K}[\boldsymbol{x}]$. A common strategy for solving is to work over $\mathbb{K}[\boldsymbol{x}]/\langle f_1, \ldots, f_n \rangle$, which is a finite a dimensional vector space over $\mathbb{K}$. We fix a monomial basis, choose $f_0 \in \mathbb{K}[\boldsymbol{x}]$, and compute the matrix that represents the multiplication by f_0 in the quotient ring. Its eigenvalues are the evaluations of f_0 at the solutions. For a suitable basis, from the eigenvectors we can recover the coordinates of all the solutions [EM07, CLO06, Cox05]. To compute these matrices we can use the Sylvester-type formulas [AS88, Emi96, CLO06]. We extend these techniques to a general family of matrices, that includes the Koszul resultant matrix (Sec. 3.2).

4.1 Eigenvalues criteria

In this section we assume fixed multidegrees $\boldsymbol{d}_0, \ldots, \boldsymbol{d}_n$.

Definition 4.1 (property Π_θ). *Given* $\theta \in \mathcal{A}(\boldsymbol{d}_0)$ *and a matrix* $M := \begin{bmatrix} M_{1,1} & M_{1,2} \\ M_{2,1} & M_{2,2} \end{bmatrix} \in \mathbb{K}[\boldsymbol{u}]^{\mathcal{K} \times \mathcal{K}}$ *(Sec. 2.1), we say that M has the property* $\Pi_\theta(\boldsymbol{d}_0, \ldots, \boldsymbol{d}_n)$, *or simply* Π_θ, *when:*

- $Res_\mathcal{P}(\boldsymbol{d}_0, \ldots, \boldsymbol{d}_n)$ *divides* $\det(M)$,
- *the submatrix $M_{2,2}$ is square and its diagonal entries equal to $u_{0,\boldsymbol{\theta}}$, and*
- *the coefficient $u_{0,\boldsymbol{\theta}}$ does not appear anywhere in M expect from the diagonal of $M_{2,2}$.*

For a system $\boldsymbol{f_0}$, Eq. (2), let $M(\boldsymbol{f_0})$ be the specialization of M at $\boldsymbol{f_0}$ (see Sec. 2.1). If $M_{1,1}(\boldsymbol{f_0})$ is invertible, then the Schur complement of $M_{2,2}(\boldsymbol{f_0})$ is $M_{2,2}(\boldsymbol{f_0}) - M_{2,1}(\boldsymbol{f_0}) \cdot (M_{1,1}(\boldsymbol{f_0}))^{-1} \cdot M_{1,2}(\boldsymbol{f_0})$. To simplify, we write $(M_{2,2} - M_{2,1} \cdot M_{1,1}^{-1} \cdot M_{1,2})(\boldsymbol{f_0})$.

Theorem 4.2. *Consider $\theta \in \mathcal{A}(\boldsymbol{d}_0)$ and a matrix $M \in \mathbb{K}[\boldsymbol{u}]^{\mathcal{K} \times \mathcal{K}}$ such that Π_θ holds (Def. 4.1). Assume a system $\boldsymbol{f_0}$, Eq.(2), such that the specialization $M_{1,1}(\boldsymbol{f_0})$ is non-singular. Then, for all $\alpha \in V_\mathcal{P}(\boldsymbol{f})$ such that $\boldsymbol{w}^{\boldsymbol{\theta}}(\alpha) \neq 0$, $\frac{f_0}{\boldsymbol{w}^{\boldsymbol{\theta}}}(\alpha)$ is an eigenvalue of the Schur complement of $M_{2,2}(\boldsymbol{f_0})$.*

Proof. The idea of the proof is as follows: For each $\alpha \in V_{\mathcal{P}}(\boldsymbol{f})$, Eq. (1), we consider a system $\boldsymbol{g_0}$, slightly different from $\boldsymbol{f_0}$, with α as a solution. We study the matrices $M(\boldsymbol{f_0})$ and $M(\boldsymbol{g_0})$ and from the kernel of $M(\boldsymbol{g_0})$ we construct an eigenvector for the Schur complement of $M_{2,2}(\boldsymbol{f_0})$ corresponding to an eigenvalue equal to $\frac{f_0}{\boldsymbol{w^\theta}}(\alpha)$.

Let $\alpha \in V_{\mathcal{P}}(\boldsymbol{f})$ such that $\boldsymbol{w^\theta}(\alpha) \neq 0$. Consider the polynomial $g_0 := f_0 - \frac{f_0}{\boldsymbol{w^\theta}}(\alpha) \cdot \boldsymbol{w^\theta}$ and a new system $\boldsymbol{g_0} := (g_0, f_1, \ldots, f_n)$. The coefficients of the polynomials g_0 and f_0 are the same, with exception of the coefficient of the monomial $\boldsymbol{w^\theta}$, so the specializations $u_{i,\boldsymbol{\sigma}}(\boldsymbol{f_0})$ and $u_{i,\boldsymbol{\sigma}}(\boldsymbol{g_0})$ (Sec. 2.1) differ if and only if $i = 0$ and $\boldsymbol{\sigma} = \boldsymbol{\theta}$. Hence, as Π_θ holds, $u_{0,\boldsymbol{\theta}}$ does not appear in $M_{1,1}$, $M_{2,1}$, and $M_{1,2}$, and $M_{1,1}(\boldsymbol{g_0}) = M_{1,1}(\boldsymbol{f_0})$, $M_{1,2}(\boldsymbol{g_0}) = M_{1,2}(\boldsymbol{f_0})$, and $M_{2,1}(\boldsymbol{g_0}) = M_{2,1}(\boldsymbol{f_0})$. The specialization of $u_{0,\boldsymbol{\theta}}$ is a ring homomorphism, so $u_{0,\boldsymbol{\theta}}(\boldsymbol{g_0}) = u_{0,\boldsymbol{\theta}}(\boldsymbol{f_0}) - \frac{f_0}{\boldsymbol{w^\theta}}(\alpha)$. By Π_θ, $u_{0,\boldsymbol{\theta}}$ only appears in the diagonal of $M_{2,2}$. Hence, $M_{2,2}(\boldsymbol{g_0}) = M_{2,2}(\boldsymbol{f_0}) - \frac{f_0}{\boldsymbol{w^\theta}}(\alpha) \cdot I$, where I is the identity matrix. Therefore,

$$M(\boldsymbol{g_0}) = \begin{bmatrix} M_{1,1} & M_{1,2} \\ M_{2,1} & M_{2,2} \end{bmatrix} (\boldsymbol{f_0}) - \frac{f_0}{\boldsymbol{w^\theta}}(\alpha) \cdot \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}.$$

By construction $g_0(\alpha) = 0$, $\alpha \in V_{\mathcal{P}}(\boldsymbol{f})$, thus $\alpha \in V_{\mathcal{P}}(\boldsymbol{g_0})$, and so $\mathrm{Res}_{\mathcal{P}}(\boldsymbol{g_0})$ vanishes. By property Π_θ, $\det(M)$ is a multiple of $\mathrm{Res}_{\mathcal{P}}(\boldsymbol{d_0}, \ldots, \boldsymbol{d_n})$, hence $M(\boldsymbol{g_0})$ is singular. Let $v \in \ker(M(\boldsymbol{g_0}))$, then

$$M(\boldsymbol{g_0}) \cdot v = 0 \iff \begin{bmatrix} M_{1,1} & M_{1,2} \\ M_{2,1} & M_{2,2} \end{bmatrix} (\boldsymbol{f_0}) \cdot v = \frac{f_0}{\boldsymbol{w^\theta}}(\alpha) \cdot \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \cdot v .$$

Multiplying this equality by the non-singular matrix related to the Schur complement of $M_{2,2}(\boldsymbol{f_0})$, $\begin{bmatrix} I & 0 \\ -M_{2,1} \cdot M_{1,1}^{-1} & I \end{bmatrix} (\boldsymbol{f_0})$, we obtain

$$\begin{bmatrix} M_{1,1} & M_{1,2} \\ 0 & (M_{2,2} - M_{2,1} \cdot M_{1,1}^{-1} \cdot M_{1,2}) \end{bmatrix} (\boldsymbol{f_0}) \cdot v = \frac{f_0}{\boldsymbol{w^\theta}}(\alpha) \cdot \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \cdot v .$$

Consider the lower part of the matrices in the previous identity,

$$\begin{bmatrix} 0 & M_{2,2} - M_{2,1} \cdot M_{1,1}^{-1} \cdot M_{1,2} \end{bmatrix} (\boldsymbol{f_0}) \cdot v = \frac{f_0}{\boldsymbol{w^\theta}}(\alpha) \cdot \begin{bmatrix} 0 & I \end{bmatrix} \cdot v$$

and let $\bar{v} := \begin{bmatrix} 0 & I \end{bmatrix} \cdot v$ be a truncation of the vector v. Then,

$$(M_{2,2} - M_{2,1} \cdot M_{1,1}^{-1} \cdot M_{1,2})(\boldsymbol{f_0}) \cdot \bar{v} = \frac{f_0}{\boldsymbol{w^\theta}}(\alpha) \cdot \bar{v} .$$

This equality proves that $\frac{f_0}{\boldsymbol{w^\theta}}(\alpha)$ is an eigenvalue of the Schur complement of $M_{2,2}(\boldsymbol{f_0})$ with eigenvector $\bar{v}$. $\qquad\square$

Let $\boldsymbol{f} \in S(\boldsymbol{d}_1) \times \cdots \times S(\boldsymbol{d}_n)$, Eq. (1), be a square system. Consider $f_0 \in S(\boldsymbol{d}_0)$ and $\theta \in \mathcal{A}(\boldsymbol{d}_0)$. We say that the rational function $\frac{f_0}{\boldsymbol{w}^{\boldsymbol{\theta}}}$ separates the zeros of the system, if for all $\alpha \in V_{\mathcal{P}}(\boldsymbol{f})$, $\boldsymbol{w}^{\boldsymbol{\theta}}(\alpha) \neq 0$ and for all $\alpha, \alpha' \in V_{\mathcal{P}}(f_1, \ldots, f_n)$, $\frac{f_0}{\boldsymbol{w}^{\boldsymbol{\theta}}}(\alpha) = \frac{f_0}{\boldsymbol{w}^{\boldsymbol{\theta}}}(\alpha') \iff \alpha = \alpha'$.

Corollary 4.3. *Under the assumptions of Thm. 4.2, if the row dimension of $M_{2,2}$ is $\mathtt{MHB}(\boldsymbol{d}_1, \ldots, \boldsymbol{d}_n)$, $\frac{f_0}{\boldsymbol{w}^{\boldsymbol{\theta}}}$ separates the zeros of $(f_1, \ldots, f_n)$ and there are $\mathtt{MHB}(\boldsymbol{d}_1, \ldots, \boldsymbol{d}_n)$ different solutions for this subsystem (over $\mathcal{P}$), then the Schur complement of $M_{2,2}(\boldsymbol{f_0})$ is diagonizable with eigenvalues $\frac{f_0}{\boldsymbol{w}^{\boldsymbol{\theta}}}(\alpha)$, for $\alpha \in V_{\mathcal{P}}(f_1, \ldots, f_n)$.*

Proof. As a consequence of Thm. 4.2, for each $\alpha \in V_{\mathcal{P}}(\boldsymbol{f})$ we have an eigenvalue $\frac{f_0}{\boldsymbol{w}^{\boldsymbol{\theta}}}(\alpha)$ for the Schur complement of $M_{2,2}(\boldsymbol{f_0})$. As $\frac{f_0}{\boldsymbol{w}^{\boldsymbol{\theta}}}$ separates these zeros, all the eigenvalues are different. Hence, we have as many different eigenvalues as the dimension of the matrix, so the matrix is diagonalizable. $\square$

Note that, as the $\mathtt{MHB}$ bounds the number of isolated solutions counting multiplicities, we can not use Thm. 4.3 when we have a square system $\boldsymbol{f}$ such that its solutions over $\mathcal{P}$ have multiplicities.

Lemma 4.4. *Under the assumptions of Thm. 4.2, assume that $\mathrm{Res}_{\mathcal{P}}(\boldsymbol{f_0}) \neq 0$ and $\det(M) = q \cdot \mathrm{Res}_{\mathcal{P}}(\boldsymbol{d}_0, \ldots, \boldsymbol{d}_n)$, where q is a non-zero constant in $\mathbb{K}$. If λ is an eigenvalue of the Schur complement of $M_{2,2}(\boldsymbol{f_0})$, then there is $\alpha \in V_{\mathcal{P}}(\boldsymbol{f})$ such that $\lambda = \frac{f_0}{\boldsymbol{w}^{\boldsymbol{\theta}}}(\alpha)$.*

Proof. Consider the system $\boldsymbol{g_0} := ((f_0 - \lambda \cdot \boldsymbol{w}^{\boldsymbol{\theta}}), f_1, \ldots, f_n)$. As the matrix of the Schur complement in the proof of 4.2 is invertible, we extend $\bar{v}$ to $v = \begin{bmatrix} M_{1,1}^{-1} \cdot M_{2,1} \\ I \end{bmatrix}(\boldsymbol{f_0}) \bar{v}$, and reverse the argument in this proof to show that $M(\boldsymbol{g_0})$ is singular. As the determinant of M is a non-zero constant multiple of the resultant, we deduce that $Res_{\mathcal{P}}(\boldsymbol{g_0})$ is zero. Let $\alpha \in V_{\mathcal{P}}(\boldsymbol{g_0})$, then $\alpha \subset V_{\mathcal{P}}(\boldsymbol{f})$ and $(f_0 - \lambda \cdot \boldsymbol{w}^{\boldsymbol{\theta}})(\alpha) = 0$, equivalently, $f_0(\alpha) = \lambda \cdot \boldsymbol{w}^{\boldsymbol{\theta}}(\alpha)$. As we assumed that $\mathrm{Res}_{\mathcal{P}}(\boldsymbol{f_0}) \neq 0$, then $f_0(\alpha) \neq 0$ and so $\frac{f_0}{\boldsymbol{w}^{\boldsymbol{\theta}}}(\alpha) = \lambda$. $\square$

Proposition 4.5. *Under the assumptions of Thm. 4.2, assume $\det(M) = q \cdot \mathrm{Res}_{\mathcal{P}}(\boldsymbol{d}_0, \ldots, \boldsymbol{d}_n)$, where q is a non-zero constant in $\mathbb{K}$, and that the (row) dimension of $M_{2,2}$ is $\mathtt{MHB}(\boldsymbol{d}_1, \ldots, \boldsymbol{d}_n)$. Then for any system $\boldsymbol{f_0} := (f_0, \ldots, f_n)$, $V_{\mathcal{P}}(\boldsymbol{w}^{\boldsymbol{\theta}}, f_1, \ldots, f_n) = \emptyset$ if and only if $M_{1,1}(\boldsymbol{f_0})$ is non-singular.*

Proof. Consider the determinant of M. As it is a multiple of the resultant (Sec. 2.1) and the resultant is a multihomogeneous polynomial of degree $\mathtt{MHB}(\boldsymbol{d}_1, \ldots, \boldsymbol{d}_n)$ with respect to $\boldsymbol{u_0}$, we can write $\det(M) = P(\boldsymbol{u}) \cdot$

$u_{0,\boldsymbol{\theta}}^{\text{MHB}(\boldsymbol{d}_1,\ldots,\boldsymbol{d}_n)} + Q(\boldsymbol{u})$, where $P(\boldsymbol{u}) \in \mathbb{K}[\boldsymbol{u}]$ does not involve the variables in $\boldsymbol{u}_0$ and $Q(\boldsymbol{u}) \in \mathbb{K}[\boldsymbol{u}]$ is a polynomial such that none of its monomials are multiple of $u_{0,\boldsymbol{\theta}}^{\text{MHB}(\boldsymbol{d}_1,\ldots,\boldsymbol{d}_n)}$. As Π_θ holds, $u_{0,\theta}$ only appears in the diagonal of $M_{2,2}$. Consider the expansion by minors of $\det(M)$. If the (row) dimension of $M_{2,2}$ is $\text{MHB}(\boldsymbol{d}_1,\ldots,\boldsymbol{d}_n)$, then $P(\boldsymbol{u}) = \pm\det(M_{1,1})$. The polynomial $P(\boldsymbol{u})$ is a constant multiple of the cofactor of $u_{0,\boldsymbol{\theta}}^{\text{MHB}(\boldsymbol{d}_1,\ldots,\boldsymbol{d}_n)}$ in the resultant $\text{Res}_{\mathcal{P}}(\boldsymbol{d}_0,\ldots,\boldsymbol{d}_n)$.

By construction, $Q(\boldsymbol{u})$ is a homogeneous polynomial with respect to the variables $\boldsymbol{u}_0$ of degree $\text{MHB}(\boldsymbol{d}_1,\ldots,\boldsymbol{d}_n)$. As $u_{0,\boldsymbol{\theta}}^{\text{MHB}(\boldsymbol{d}_1,\ldots,\boldsymbol{d}_n)}$ does not divide any monomial in $Q(\boldsymbol{u})$, each monomial involves a variables of $\boldsymbol{u}_0$ different to $u_{0,\theta}$. Hence, for any system $\boldsymbol{f_0}$, we have $Q(\boldsymbol{w}^{\boldsymbol{\theta}}, f_1, \ldots, f_n) = 0$. By construction, the polynomial $P(\boldsymbol{u})$ does not involve any of the variables of $\boldsymbol{u}_0$. Therefore $\det(M_{1,1})(\boldsymbol{f_0}) = \det(M_{1,1})(\boldsymbol{w}^{\boldsymbol{\theta}}, f_1, \ldots, f_n)$. Therefore, for any system $\boldsymbol{f_0}$, $q \cdot \text{Res}_{\mathcal{P}}(\boldsymbol{d}_0,\ldots,\boldsymbol{d}_n)(\boldsymbol{w}^{\boldsymbol{\theta}}, f_1, \ldots, f_n) = \det(M)(\boldsymbol{w}^{\boldsymbol{\theta}}, f_1 \ldots f_n) = \pm\det(M_{1,1})(\boldsymbol{w}^{\boldsymbol{\theta}}, f_1 \ldots f_n) = \pm\det(M_{1,1})(\boldsymbol{f_0})$. The determinant of M is a non-zero constant multiple of the resultant, hence $\det(M_{1,1})(\boldsymbol{f_0}) \neq 0$ if and only if the system $(\boldsymbol{w}^{\boldsymbol{\theta}}, f_1, \ldots, f_n)$ has no solutions over $\mathcal{P}$, i.e., $V_{\mathcal{P}}(\boldsymbol{w}^{\boldsymbol{\theta}}, f_1, \ldots, f_n) = \emptyset$. $\qquad\square$

If the square system $\boldsymbol{f} = (f_1, \ldots, f_n)$ has no solutions at infinity in $\mathcal{P}$, that is all the coordinates of the solutions are not zero, then the evaluation of the solutions of $\boldsymbol{f}$ at any monomial in $S(\boldsymbol{d}_0)$ is not zero. Hence, for any $\boldsymbol{w}^{\boldsymbol{\theta}} \in S(\boldsymbol{d}_0)$, $V_{\mathcal{P}}(\boldsymbol{w}^{\boldsymbol{\theta}}, f_1, \ldots, f_n) = \emptyset$. By Prop. 4.5, $M_{1,1}(f_0, f_1, \ldots, f_n)$ is invertible. To avoid solutions at infinity, in the 0-dimensional multihomogeneous case, we perform a generic linear change of coordinates that preserves the multihomogeneous structure. We state the following corollary without proof.

Corollary 4.6. *Consider a square multihomogeneous system $\boldsymbol{f} \in S(\boldsymbol{d}_1) \times \cdots \times S(\boldsymbol{d}_n)$ with finite $V_{\mathcal{P}}(\boldsymbol{f})$. Choose $\theta \in \mathcal{A}(\boldsymbol{d}_0)$ and let M be a resultant matrix for $\text{Res}_{\mathcal{P}}(d_0, \ldots, d_n)$, such that Π_θ holds. Consider any $f_0 \in S(\boldsymbol{d}_0)$. Then, for a generic linear change of coordinates A, preserving the multihomogeneous structure, the matrix $M_{1,1}(f_0, f_1 \circ A, \ldots, f_n \circ A)$ is invertible.*

We can use Thm. 4.2 to solve the 2-bilinear systems.

Theorem 4.7. *Assume a 2-bilinear system $f_1, \ldots, f_n$ of type $(n_x, n_y, n_z; r, t)$, such that $V_{\mathcal{P}}(f_1, \ldots, f_n)$ is finite. Choose $\theta \in \mathcal{A}(\boldsymbol{d}_0)$ and consider the M be the matrix of $\delta_1(\boldsymbol{F}^{(2)}, \boldsymbol{m})$ (Sec. 3.2) for the "universal" system*

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$\boldsymbol{F^{(2)}}$ *rearranged with respect to the monomial $\boldsymbol{w^\theta}$. Choose $f_0 \in S(1,1,1)$. Then, after applying a generic linear change of coordinates A, preserving the multihomogeneous structure, the eigenvalues of the Schur complement of $M_{2,2}(f_0, f_1 \circ A, \ldots, f_n \circ A)$ are the evaluations of $\frac{f_0}{\boldsymbol{w^\theta}}$ over $V_{\mathcal{P}}(f_1 \circ A, \ldots, f_n \circ A)$.*

Proof. We only need to check if the Koszul resultant matrix has the property Π_θ. The entries of our matrix are the variables of $\boldsymbol{u}$ up to sign. Note that if $u_{i,\sigma} \in \boldsymbol{u}$ appears in an entry, then it does not appear in the other entries in the same row, or column. Hence, we can rearrange the matrix in such a way that the coefficient $u_{0,\theta}$ only appears in the diagonal of $M_{2,2}$. As the determinant of the system is a constant multiple of the resultant, the dimension of $M_{2,2}$ the degree of $\boldsymbol{u_0}$ in the determinant, which equals the MHB. □

Example 4.8 (Cont.)**.** *In the previous example (Ex. 3.5), we choose $\boldsymbol{\theta} = ((1,0),(1,0),(1,0)) \in \mathcal{A}(1,1,1)$ and partition the matrix as $\left[\begin{smallmatrix} M_{1,1} & M_{1,2} \\ M_{2,1} & M_{2,2} \end{smallmatrix}\right]$. If we consider the Schur complement, we get $\left[\begin{smallmatrix} 5 & -2 \\ 4 & -1 \end{smallmatrix}\right]$. The characteristic polynomial of this matrix is $X^2 - 4X + 3$, whose roots are $\frac{f_0}{\boldsymbol{w^\theta}}(\alpha_1) = 3$ and $\frac{f_0}{\boldsymbol{w^\theta}}(\alpha_2) = 1$.*

4.2 Eigenvectors for 2-bilinear systems

We fix $\theta \in \mathcal{A}(\boldsymbol{d_0})$. We consider the degree vector $\boldsymbol{m} = (n_y - 1, -1, n_x + n_y - r + 1)$ and the determinantal formula M for the map $\delta_1(\boldsymbol{F^{(2)}}, \boldsymbol{m})$ (Sec. 3.2). We study the right eigenvectors of the Schur complement of $M_{2,2}$ to recover the coordinates of all the solutions of a 2-bilinear system $\boldsymbol{f}$ of type $(n_x, n_y, n_z; r, s)$ (Sec. 2.2). We assume that the number of different solutions is $\#V_{\mathcal{P}}(\boldsymbol{f}) = \mathtt{MHB}(\boldsymbol{f})$.

We augment $\boldsymbol{f}$ to $\boldsymbol{f_0}$ by adding a trilinear polynomial f_0, which we specify in the sequel. We study the right eigenvalues of the Schur complement of $M_{2,2}(\boldsymbol{f_0})$. We reduce the analysis of the kernel of $\delta_1(\boldsymbol{f_0}, \boldsymbol{m})$ to the analysis of a map in a strand of the Koszul complex of a system with common solutions.

Let $\alpha = (\alpha_x, \alpha_y, \alpha_z) \in \mathcal{P}$, and without loss of generality assume that $\alpha_{t,0} \neq 0$, for $t \in \{x, y, z\}$. First, we study the kernel of $\delta_1(\boldsymbol{f_0}, \boldsymbol{m})$, when the overdetermined system $\boldsymbol{f_0}$ *has* common solutions. We relate this kernel to the eigenvectors, as we did in the proof of thm. 4.2. For each variable $t \in \{x, y, z\}$, consider the dual form

$$1_\alpha^t(d_t) := \sum_{\theta_t \in \mathcal{A}(d_t)} \frac{\boldsymbol{t^{\theta_t}}}{t_0^{d_t}}(\alpha_t) \, \boldsymbol{\partial t^\theta} \in S_t(d_t)^*$$

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for $d_t \geq 0$. If $d_t < 0$, then we take $1_\alpha^t(d_t) := 0$.

Observation 4.9. *For each variable $t \in \{x, y, z\}$, given a polynomial $g_t \in S_t(\bar{d}_t)$, such that $\bar{d}_t \leq d_t$, then operator $\star_t$, Eq. (9), acts over g_t and $1_\alpha^t(d_t)$ as the evaluation of $\frac{g_t}{t_0^{\bar{d}_t}}$ at α, that is*

$$g_t \star_t 1_\alpha^t(d_t) = \frac{g_t}{t_0^{\bar{d}_t}}(\alpha_t) \cdot 1_\alpha^t(d_t - \bar{d}_t).$$

To simplify notation, given $f \in S(d_x, d_y, d_z)$ and $(\alpha_x, \alpha_y, \alpha_z) \in \mathcal{P}$, we denote by $f(\alpha_x, \alpha_y) \in S_z(d_z)$ the partial evaluation of $\frac{f}{x_0^{d_x} y_0^{d_y}}$ at $\boldsymbol{x} = \alpha_x$ and $\boldsymbol{y} = \alpha_y$. This evaluation is well-defined because the numerator and denominator share the same degrees w.r.t. $\boldsymbol{x}$ and $\boldsymbol{y}$.

Lemma 4.10. *Consider $\boldsymbol{d} = (d_x, d_y, d_z)$, $\bar{\boldsymbol{d}} = (\bar{d}_x, \bar{d}_y, \bar{d}_z)$. Let $f \in S(\bar{\boldsymbol{d}})$ and $g_z \in S_z(d_z)$. If $d_x \geq \bar{d}_x$ and $d_y \geq \bar{d}_y$, then the map ψ (Eq. (10)) acts over $1_\alpha^x(d_x) \otimes 1_\alpha^y(d_y) \otimes g_z$ and f, as the multiplication of g_z and $f(\alpha_x, \alpha_y)$, that is*

$$\psi(1_\alpha^x(d_x) \otimes 1_\alpha^y(d_y) \otimes g_z, f) = 1_\alpha^x(d_x - \bar{d}_x) \otimes 1_\alpha^y(d_y - \bar{d}_y) \otimes \left(g_z \cdot f(\alpha_x, \alpha_y)\right).$$

Let $\omega^{(1)} := \{I : \boldsymbol{e}_I \in \bigwedge_{r, s-n_z+1, 0} E\}$ and $\omega^{(2)} := \{J : \boldsymbol{e}_J \in \bigwedge_{r, s-n_z, 1} E\}$. Let $\rho_\alpha : \mathbb{K}^{\#\omega^{(1)}} \times \mathbb{K}^{\#\omega^{(2)}} \to L_{1,1} \oplus L_{1,2}$, Eq. (7),

$$\rho_\alpha(\boldsymbol{\lambda^{(1)}}, \boldsymbol{\lambda^{(2)}}) := \sum_{I \in \omega^{(1)}} \lambda_I^{(1)} \cdot \left(1_\alpha^x(1) \otimes 1_\alpha^y(r - n_y) \otimes 1 \otimes \boldsymbol{e}_I\right)$$

$$+ \sum_{J \in \omega^{(2)}} \lambda_J^{(2)} \cdot \left(1_\alpha^x(1) \otimes 1_\alpha^y(r - n_y + 1) \otimes 1 \otimes \boldsymbol{e}_J\right)$$

As $\#\omega^{(1)} + \#\omega^{(2)} = \binom{s+1}{s-n_z+1}$, we write $\rho_\alpha : \mathbb{K}^{\binom{s+1}{s-n_z+1}} \to K_1$.

Lemma 4.11. *The linear map $\delta_1(\boldsymbol{f_0}, \boldsymbol{m}) \circ \rho_\alpha : \mathbb{K}^{\binom{s+1}{s-n_z+1}} \to K_0$ is equivalent to the $(s - n_z + 1)$-th map of the Koszul complex of the following system, consisting of $s + 1$ linear polynomials in $\boldsymbol{z}$,*

$$\boldsymbol{f_z} := \left(f_0(\alpha_x, \alpha_y), f_{r+1}(\alpha_x, \alpha_y), \dots, f_n(\alpha_x, \alpha_y)\right), \tag{11}$$

restricted to its 0-graded part, i.e. the strand of the Koszul complex such that its $(s - n_z + 1)$-th module is isomorphic to $\mathbb{K}^{\binom{s+1}{s-n_z+1}}$.

If $\boldsymbol{f_0}$ has a solution $(\alpha_x, \alpha_y, \alpha_z) \in V_{\mathcal{P}}(\boldsymbol{f_0})$, then, α_z is a solution of the linear system $\boldsymbol{f_z}$, that is $\alpha_z \in V_{\mathcal{P}}(\boldsymbol{f_z})$. As $\boldsymbol{f_z}$ is an overdetermined system, the Koszul complex $\boldsymbol{f_z}$ is not exact [Lan02, Thm. XXI.4.6].

Lemma 4.12. *Let $\boldsymbol{f_0}$ be an overdetermined 2-bilinear system. If $\alpha \in V_{\mathcal{P}}(\boldsymbol{f_0})$, then there is a non-zero $\widehat{\boldsymbol{\lambda}}_{\boldsymbol{\alpha}} \in \mathbb{K}^{\binom{s+1}{s-n_z+1}}$ such that $\delta_1(\boldsymbol{f_0}, \boldsymbol{m}) \circ \rho_\alpha(\widehat{\boldsymbol{\lambda}}_{\boldsymbol{\alpha}}) = 0$.*

Proof. Following Lem. 4.11, if we compose $\delta_1(\boldsymbol{f}, \boldsymbol{m})$ and ρ_α, then we obtain a map which is similar to the 0-graded part of the $(s - n_z + 1)$-th map of the Koszul complex of the $s + 1$ linear polynomials in $\boldsymbol{z}$, $\boldsymbol{f_z}$, Eq. (11). As the linear system $\boldsymbol{f_z}$ has a solution α_z, at most n_z of its polynomials are linearly independent. Hence, the Koszul complex of $\boldsymbol{f_z}$ is isomorphic to a Koszul complex $K(\widetilde{f}_1, \ldots, \widetilde{f}_{n_z}, 0, \ldots, 0)$ of a system of $s+1$ linear polynomials, where $(s+1-n_z)$ of them are equal to zero [Lan02, Lem. XXI.4.2]. The $(s+1-n_z)$-th map of $K(\widetilde{f}_1, \ldots, \widetilde{f}_{n_z}, 0, \ldots, 0)$ maps $e_{n_z+1} \wedge \ldots \wedge e_{s+1-n_z}$ to zero. Hence, its 0-graded part has a non-trivial kernel, and so there is a non-zero $\widehat{\boldsymbol{\lambda}}_{\boldsymbol{\alpha}} \in \mathbb{K}^{\binom{s}{s-n_z+1}}$ such that $\delta_1(\boldsymbol{f}, \boldsymbol{m}) \circ \rho_\alpha(\widehat{\boldsymbol{\lambda}}_{\boldsymbol{\alpha}}) = 0$. □

Theorem 4.13. *Let $\boldsymbol{f} = (f_1, \ldots, f_n)$ be a square 2-bilinear system of type $(n_x, n_y, n_z; r, s)$, such that it has $\binom{r}{n_y} \cdot \binom{s}{n_z}$ different solutions over $\mathcal{P}$. Consider $\theta \in \mathcal{A}(1, 1, 1)$ such that*

$$\mathrm{Res}_{\mathcal{P}}^{(2)}(\boldsymbol{w}^{\boldsymbol{\theta}}, f_1, \ldots, f_n) \neq 0$$

and $f_0 \in S(1, 1, 1)$ such that $\frac{f_0}{\boldsymbol{w}^{\boldsymbol{\theta}}}$ separates the elements in $V_{\mathcal{P}}(\boldsymbol{f})$. Let $\boldsymbol{m} := (n_y - 1, -1, n_x + n_y - r + 1)$ and $M \in \mathbb{K}[\boldsymbol{u}]^{\mathcal{K} \times \mathcal{K}}$ related to $\delta_1(\boldsymbol{F^{(2)}}, \boldsymbol{m})$ for the overdetermined 2-bilinear "universal" system (Thm. 3.3). Then, the Schur complement of $M_{2,2}(\boldsymbol{f_0})$ is diagonalizable, each eigenvalue is $\frac{f_0}{\boldsymbol{w}^{\boldsymbol{\theta}}}(\alpha)$, for $\alpha \in V_{\mathcal{P}}(f_1, \ldots, f_n)$, and we can extend the eigenvector $\bar{v}_\alpha$ related to α to $v_\alpha := \left[\begin{smallmatrix} M_{1,1}^{-1} \cdot M_{2,1} \\ I \end{smallmatrix}\right](\boldsymbol{f_0}) \cdot \bar{v}_\alpha$ such that v_α is the element $\rho_\alpha(\widehat{\boldsymbol{\lambda}}_{\boldsymbol{\alpha}})$, for some $\widehat{\boldsymbol{\lambda}}_{\boldsymbol{\alpha}} \in \mathbb{K}^{\binom{s+1}{s-n_z+1}}$.

Proof. By Cor. 4.3, the Schur complex of $M_{2,2}(\boldsymbol{f_0})$ is diagonalizable and every eigenvalues is different. For each $\alpha \in V_{\mathcal{P}}(\boldsymbol{f})$, consider the eigenvalue $\frac{f_0}{\boldsymbol{w}^{\boldsymbol{\theta}}}(\alpha)$, related eigenvector $\bar{v}_\alpha$, and the system $\boldsymbol{g_\alpha} := (f_0 - \frac{f_0}{\boldsymbol{w}^{\boldsymbol{\theta}}}(\alpha), f_1, \ldots, f_n)$. By Lem. 4.12, there is a $\boldsymbol{\lambda_\alpha} \in \mathbb{K}$ such that $\delta_1(\boldsymbol{g_\alpha}, \boldsymbol{m}) \circ \rho(\boldsymbol{\lambda_\alpha}) = 0$. Hence, there is a w_α, representing $\rho(\boldsymbol{\lambda_\alpha}) = 0$, in the kernel of $M(\boldsymbol{g_\alpha})$. Following the proof of Thm. 4.2, each element in the kernel of the Schur complement of

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$M_{2,2}(\boldsymbol{g_\alpha})$ is related to an eigenvector of the Schur complement of $M_{2,2}(\boldsymbol{f_0})$ with corresponding eigenvalue $\frac{f_0}{\boldsymbol{w^\theta}}(\alpha)$. As for each eigenvalue we have only one eigenvector, then the dimension of this kernel is 1. Hence, the truncation of w_α, $\bar{w}_\alpha := (0|I) \cdot w_\alpha$, is a multiple of $\bar{v}_\alpha$, where 0 is the zero matrix of appropriate dimension.

As $M_{1,1}(\boldsymbol{g_\alpha})$ is invertible and $M(\boldsymbol{g_\alpha}) \cdot w_\alpha = 0$, it holds that $\left[\begin{smallmatrix} M_{1,1}^{-1} \cdot M_{2,1} \\ I \end{smallmatrix}\right](\boldsymbol{g_\alpha})\bar{w}_\alpha = w_\alpha$. As $\left[\begin{smallmatrix} M_{1,1}^{-1} \cdot M_{2,1} \\ I \end{smallmatrix}\right](\boldsymbol{g_\alpha})$ does not involve $u_{0,\theta}$, then $\left[\begin{smallmatrix} M_{1,1}^{-1} \cdot M_{2,1} \\ I \end{smallmatrix}\right](\boldsymbol{g_\alpha}) = \left[\begin{smallmatrix} M_{1,1}^{-1} \cdot M_{2,1} \\ I \end{smallmatrix}\right](\boldsymbol{f_0})$. Therefore, we conclude that, as $\bar{v}_\alpha$ is a multiple of $\bar{w}_\alpha$, then $v_\alpha = \left[\begin{smallmatrix} M_{1,1}^{-1} \cdot M_{2,1} \\ I \end{smallmatrix}\right](\boldsymbol{f_0}) \cdot \bar{v}_\alpha$ is a multiple of w_α. $\qquad\square$

In the following example we use Thm. 4.13 to recover α_2.

Example 4.14 (Cont.). *The eigenvalue of $\frac{f_0}{\boldsymbol{w^\theta}}(\alpha_2) = 1$ is $\bar{v}_{\alpha_2} := (1, 2)^\top$. By extending $\bar{v}_{\alpha_2}$, we get*

$$v_{\alpha_2} := \left[\begin{smallmatrix} M_{1,1}^{-1} \cdot M_{2,1} \\ I \end{smallmatrix}\right](\boldsymbol{f_0}) \cdot \left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right) = (4, 3, 12, 1, 2, 3, 6, 6, 1, 2)^\top$$

which represents $\rho_{\alpha_2}(1, 1) =$

$$\left(\boldsymbol{\partial x}^{(1,0)} + 3\,\boldsymbol{\partial x}^{(0,1)}\right) \otimes \left(\boldsymbol{\partial y}^{(2,0)} + 2\,\boldsymbol{\partial y}^{(1,1)} + 4\,\boldsymbol{\partial y}^{(0,2)}\right) \otimes 1 \otimes \boldsymbol{e}_{\{0,1,2\}}$$

$$+ \left(\boldsymbol{\partial x}^{(1,0)} + 3\,\boldsymbol{\partial x}^{(0,1)}\right) \otimes \left(\boldsymbol{\partial y}^{(1,0)} + 2\,\boldsymbol{\partial y}^{(0,1)}\right) \otimes 1 \otimes \boldsymbol{e}_{\{1,2,3\}}$$

Hence, $1_{\alpha_2}^x(1) = \left(1\,\boldsymbol{\partial x}^{(1,0)} + 3\,\boldsymbol{\partial x}^{(0,1)}\right)$, and so $\alpha_{2,x} = (1 : 3) \in \mathbb{P}^1$. Also, $1_{\alpha_2}^y(1) = \left(1\,\boldsymbol{\partial y}^{(1,0)} + 2\,\boldsymbol{\partial y}^{(0,1)}\right)$, and then $\alpha_{2,y} = (1 : 2) \in \mathbb{P}^1$. We note that $1_{\alpha_2}^y(2) = \left(1 \cdot 1 \cdot \boldsymbol{\partial y}^{(2,0)} + 1 \cdot 2 \cdot \boldsymbol{\partial y}^{(1,1)} + 2 \cdot 2 \cdot \boldsymbol{\partial y}^{(0,2)}\right)$.

We can recover $\alpha_{2,z}$ as the solution of $\boldsymbol{f}(\alpha_{2,x}, \alpha_{2,y}, \boldsymbol{z}) = 0$,

$$\begin{cases} f_1(\alpha_{2,x}, \alpha_{2,y}, \boldsymbol{z}) = 0 \\ f_2(\alpha_{2,x}, \alpha_{2,y}, \boldsymbol{z}) = 0 \\ f_3(\alpha_{2,x}, \alpha_{2,y}, \boldsymbol{z}) = -9\,z_0 + 3\,z_1 \end{cases}$$

Hence, $\alpha_{2,z} = (1 : 3) \in \mathbb{P}^1$ and so $\alpha_2 = (1{:}3 \; ; \; 1{:}2 \; ; \; 1{:}3) \in \mathcal{P}$.

5 Size of matrices and FGb

As there are no tight bounds for the complexity of Gröbner basis algorithms for solving 2-bilinear systems, we compare against our algorithms experimentally in Table 1. We consider the state-of-the-art Gröbner basis implementation, FGb [Fau10]. For each set of parameters, we consider a random square 2-bilinear system and we dehomogenize the system to compute its Gröbner basis. We compared the ratio between the size of the maximal matrix appearing in the Gröbner basis computation and the size of our Koszul resultant matrix, for all the cases $n \leq 15$. For reasons of space we only present some indicative examples for $n = 12$. The rest of the cases can be found in `http://www-polsys.lip6.fr/~bender/2bilinear/`. The results are promising and motivate the study of the structure Koszul resultant matrix to develop algorithms for faster linear algebra with such matrices.

Table 1: Matrix sizes and ratios of Koszul matrix and FGb.

n_x	n_y	n_z	r	s	Size δ_1	Size FGb	Ratio
2	6	4	7	5	630×630	1769×1158	$5.1 \sim$
10	1	1	10	2	352×352	709×422	$2.4 \sim$
5	5	2	9	3	6804×6804	8941×8390	$1.6 \sim$
4	4	4	6	6	4125×4125	5436×4262	$1.3 \sim$
5	5	2	6	6	2106×2106	2007×1164	$1/1.9 \sim$
6	3	3	6	6	7000×7000	4708×3801	$1/2.7 \sim$
6	4	2	5	7	2450×2450	1773×1125	$1/3 \sim$

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Appendix

Proof of Lem. 4.10. Consider $f = \sum_\sigma c_\sigma \boldsymbol{x}^{\sigma_x} \boldsymbol{y}^{\sigma_y} \boldsymbol{z}^{\sigma_z}$. As ψ is a bilinear map and the tensor product is multilinear, it is enough to prove this lemma only for the monomials $\boldsymbol{x}^{\sigma_x} \boldsymbol{y}^{\sigma_y} \boldsymbol{z}^{\sigma_z} \in S(\bar{\boldsymbol{d}})$.
, $\psi(1_\alpha^x(d_x) \otimes 1_\alpha^y(d_y) \otimes g, f) = \sum_\sigma c_\sigma \psi(1_\alpha^x(d_x) \otimes 1_\alpha^y(d_y) \otimes g,$
$\boldsymbol{x}^{\sigma_x} \boldsymbol{y}^{\sigma_y} \boldsymbol{z}^{\sigma_z})$. For that reason, we study the monomial case,

$$\psi\big(1_\alpha^x(d_x) \otimes 1_\alpha^y(d_y) \otimes g_z, \boldsymbol{x}^{\sigma_x} \otimes \boldsymbol{y}^{\sigma_y} \otimes \boldsymbol{z}^{\sigma_z}\big) =$$
$$\Big(\boldsymbol{x}^{\sigma_x} \star_x 1_\alpha^x(d_x)\Big) \otimes \Big(\boldsymbol{y}^{\sigma_y} \star_y 1_\alpha^y(d_y)\Big) \otimes \Big(g_z \cdot \boldsymbol{z}^{\sigma_z}\Big) =$$
$$\Big(\frac{\boldsymbol{x}^{\sigma_x}}{x_0^{\bar{d}_x}}(\alpha_x) 1_\alpha^x(d_x - \bar{d}_x)\Big) \otimes \Big(\frac{\boldsymbol{y}^{\sigma_y}}{y_0^{\bar{d}_y}}(\alpha_y) 1_\alpha^y(d_y - \bar{d}_y)\Big) \otimes \Big(g_z \cdot \boldsymbol{z}^{\sigma_z}\Big) =$$
$$\Big(1_\alpha^x(d_x - \bar{d}_x)\Big) \otimes \Big(1_\alpha^y(d_y - \bar{d}_y)\Big) \otimes \Big(g_z \cdot \frac{\boldsymbol{x}^{\sigma_x}}{x_0^{\bar{d}_x}}(\alpha_x) \frac{\boldsymbol{y}^{\sigma_y}}{y_0^{\bar{d}_y}}(\alpha_y) \cdot \boldsymbol{z}^{\sigma_z}\Big)$$

Then, we have

$$\psi\big(1_\alpha^x(d_x) \otimes 1_\alpha^y(d_y) \otimes g, f\big) =$$
$$\sum_\sigma c_\sigma \big(1_\alpha^x(d_x - \bar{d}_x) \otimes 1_\alpha^y(d_y - \bar{d}_y) \otimes g_z \cdot \frac{\boldsymbol{x}^{\sigma_x}}{x_0^{\bar{d}_x}}(\alpha_x) \cdot \frac{\boldsymbol{y}^{\sigma_y}}{y_0^{\bar{d}_y}}(\alpha_y) \cdot \boldsymbol{z}^{\sigma_z}\big)$$
$$1_\alpha^x(d_x - \bar{d}_x) \otimes 1_\alpha^y(d_y - \bar{d}_y) \otimes g_z \cdot \sum_\sigma c_\sigma \frac{\boldsymbol{x}^{\sigma_x}}{x_0^{\bar{d}_x}}(\alpha_x) \cdot \frac{\boldsymbol{y}^{\sigma_y}}{y_0^{\bar{d}_y}}(\alpha_y) \cdot \boldsymbol{z}^{\sigma_z}$$
$$1_\alpha^x(d_x - \bar{d}_x) \otimes 1_\alpha^y(d_y - \bar{d}_y) \otimes g_z \cdot f(\alpha_x, \alpha_y)$$

$\square$

Proof of Lem. 4.11. We split the map ρ as $\rho(\boldsymbol{\lambda^{(1)}}, \boldsymbol{\lambda^{(2)}}) := \rho_\alpha^{(1)}(\boldsymbol{\lambda^{(1)}}) + \rho_\alpha^{(2)}(\boldsymbol{\lambda^{(2)}})$, where $\rho_\alpha^{(1)} : \mathbb{K}^{\#\omega^{(1)}} \to L_{1,1}$, Eq. (7), such that,

$$\rho_\alpha^{(1)}(\boldsymbol{\lambda^{(1)}}) := \sum_{I \in \omega^{(1)}} \Big(1_\alpha^x(1) \otimes 1_\alpha^y(r - n_y) \otimes \lambda_I^{(1)} \otimes \boldsymbol{e}_I\Big),$$

and $\rho_\alpha^{(2)} : \mathbb{K}^{\#\omega^{(2)}} \to L_{1,2}$, Eq. (7), such that

$$\rho_\alpha^{(2)}(\boldsymbol{\lambda^{(2)}}) := \sum_{J \in \omega^{(2)}} \Big(1_\alpha^x(1) \otimes 1_\alpha^y(r - n_y + 1) \otimes \lambda_J^{(2)} \otimes \boldsymbol{e}_J\Big).$$

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Both maps are injective.

As $\delta_1(\boldsymbol{f_0}, \boldsymbol{m}) \circ \rho_\alpha = \delta_1(\boldsymbol{f_0}, \boldsymbol{m}) \circ \rho_\alpha^{(1)} + \delta_1(\boldsymbol{f_0}, \boldsymbol{m}) \circ \rho_\alpha^{(2)}$, we study $\delta_1(\boldsymbol{f_0}, \boldsymbol{m}) \circ \rho_\alpha^{(1)}$ and $\delta_1(\boldsymbol{f_0}, \boldsymbol{m}) \circ \rho_\alpha^{(2)}$ separately.

Following the definition of δ_1 (Sec. 3.2) we have

$$\delta_1(\boldsymbol{f_0}, \boldsymbol{m}) \circ \rho_\alpha^{(1)} =$$
$$\sum_{I \in \omega^{(1)}} \lambda_I^{(1)} \delta_1(\boldsymbol{f_0}, \boldsymbol{m}) \Big(1_\alpha^x(1) \otimes 1_\alpha^y(r - n_y) \otimes 1 \otimes \boldsymbol{e}_I\Big) =$$

$$\sum_{I \in \omega^{(1)}} \lambda_I^{(1)} \Big(\sum_{i=1}^r (-1)^{i-1} \psi(1_\alpha^x(1) \otimes 1_\alpha^y(r - n_y) \otimes 1, f_{I_i}) \otimes \boldsymbol{e}_{I \setminus \{I_i\}} +$$
$$\sum_{i=r+1}^{n_x + n_y + 1} (-1)^{i-1} \psi(1_\alpha^x(1) \otimes 1_\alpha^y(r - n_y) \otimes 1, f_{I_i}) \otimes \boldsymbol{e}_{I \setminus \{I_i\}}\Big).$$

By Lem. 4.10 we have,

$$\delta_1(\boldsymbol{f_0}, \boldsymbol{m}) \circ \rho_\alpha^{(1)} =$$
$$\sum_{I \in \omega^{(1)}} \lambda_I^{(1)} \Big(\sum_{i=1}^r (-1)^{i-1} 1_\alpha^x(0) \otimes 1_\alpha^y(r - n_y - 1) \otimes f_{I_i}(\alpha_x, \alpha_y) \otimes \boldsymbol{e}_{I \setminus \{I_i\}} +$$
$$\sum_{i=r+1}^{n_x + n_y + 1} (-1)^{i-1} 1_\alpha^x(0) \otimes 1_\alpha^y(r - n_y) \otimes f_{I_i}(\alpha_x, \alpha_y) \otimes \boldsymbol{e}_{I \setminus \{I_i\}}\Big).$$

For $i \le r$, $f_{I_i} \in S(1, 1, 0)$. Hence $f_{I_i}(\alpha_x, \alpha_y) = f_{I_i}(\alpha) = 0$.

$$\delta_1(\boldsymbol{f_0}, \boldsymbol{m}) \circ \rho_\alpha^{(1)} =$$
$$\sum_{I \in \omega^{(1)}} \lambda_I^{(1)} \sum_{i=r+1}^{n_x + n_y + 1} (-1)^{i-1} 1_\alpha^x(0) \otimes 1_\alpha^y(r - n_y) \otimes f_{I_i}(\alpha_x, \alpha_y) \otimes \boldsymbol{e}_{I \setminus \{I_i\}} =$$

$$1_\alpha^x(0) \otimes 1_\alpha^y(r - n_y) \otimes \Big(\sum_{I \in \omega^{(1)}} \sum_{i=r+1}^{n_x + n_y + 1} (-1)^{i-1} \lambda_I^{(1)} f_{I_i}(\alpha_x, \alpha_y) \otimes \boldsymbol{e}_{I \setminus \{I_i\}}\Big).$$

We conclude that the image of $\delta_1(\boldsymbol{f_0}, \boldsymbol{m}) \circ \rho_\alpha^{(1)}$ belongs to $L_{0,2}$.

Now consider $\delta_1(\boldsymbol{f_0}, \boldsymbol{m}) \circ \rho_\alpha^{(2)}$. Following a similar procedure, we deduce

$$\delta_1(\boldsymbol{f_0}, \boldsymbol{m}) \circ \rho_\alpha^{(2)} =$$

$$1_\alpha^x(0) \otimes 1_\alpha^y(r - n_y) \otimes \sum_{I \in \omega^{(2)}} \left(\lambda_I^{(2)} f_0(\alpha_x, \alpha_y) \otimes \boldsymbol{e}_{I - \{0\}} \right) +$$

$$1_\alpha^x(0) \otimes 1_\alpha^y(r - n_y - 1) \otimes \sum_{I \in \omega^{(2)}} \sum_{i=2}^{r+1} \left((-1)^{i-1} \lambda_I^{(2)} f_{I_i}(\alpha_x, \alpha_y) \otimes \boldsymbol{e}_{I - \{I_i\}} \right) +$$

$$1_\alpha^x(0) \otimes 1_\alpha^y(r - n_y + 1) \otimes \sum_{I \in \omega^{(2)}} \sum_{i=r+1}^{n_x + n_y + 1} \left((-1)^{i-1} \lambda_I^{(2)} f_{I_i}(\alpha_x, \alpha_y) \otimes \boldsymbol{e}_{I - \{I_i\}} \right)$$

For $1 \leq i \leq r + 1$, $f_{I_i} \in S(1, 1, 0)$, so $f_{I_i}(\alpha_x, \alpha_y) = f_{I_i}(\alpha) = 0$. Hence,

$$\delta_1(\boldsymbol{f_0}, \boldsymbol{m}) \circ \rho_\alpha^{(2)} =$$

$$1_\alpha^x(0) \otimes 1_\alpha^y(r - n_y) \otimes \sum_{I \in \omega^{(2)}} \left(\lambda_I^{(2)} f_0(\alpha_x, \alpha_y) \otimes \boldsymbol{e}_{I - \{0\}} \right) +$$

$$1_\alpha^x(0) \otimes 1_\alpha^y(r - n_y + 1) \otimes \sum_{I \in \omega^{(2)}} \sum_{i=r+1}^{n_x + n_y + 1} \left((-1)^{i-1} \lambda_I^{(2)} f_{I_i}(\alpha_x, \alpha_y) \otimes \boldsymbol{e}_{I - \{I_i\}} \right)$$

Therefore, the image of $\delta_1(\boldsymbol{f_0}, \boldsymbol{m}) \circ \rho_\alpha^{(2)}$ belongs to $L_{0,2} \oplus L_{0,4}$.

We can rewrite $\delta_1(\boldsymbol{f_0}, \boldsymbol{m}) \circ \rho_\alpha : \mathbb{K}^{\binom{s+1}{s - n_z + 1}} \to L_{0,2} \oplus L_{0,4}$ as

$$(\delta_1(\boldsymbol{f_0}, \boldsymbol{m}) \circ \rho_\alpha)(\boldsymbol{\lambda}) = 1_\alpha^x(0) \otimes 1_\alpha^y(r - n_y) \otimes P_1(\boldsymbol{\lambda}) +$$
$$1_\alpha^x(0) \otimes 1_\alpha^y(r - n_y + 1) \otimes (-1)^r P_2(\boldsymbol{\lambda})$$

where

$$P_1(\boldsymbol{\lambda}) := \sum_{I \subset \omega^{(2)}} \lambda_I f_0(\alpha_x, \alpha_y) \otimes \boldsymbol{e}_{I \setminus \{0\}} +$$

$$\sum_{J \subset \omega^{(1)}} \sum_{j=1}^{s - n_z + 1} (-1)^{j-1} \lambda_J f_{J_j}(\alpha_x, \alpha_y) \otimes \boldsymbol{e}_{J \setminus \{J_j\}}$$

$$P_2(\boldsymbol{\lambda}) := \sum_{I \subset \omega^{(2)}} \sum_{j=2}^{s - n_z} (-1)^{r+j-1} \lambda_I f_{I_j}(\alpha_x, \alpha_y) \otimes \boldsymbol{e}_{I \setminus \{I_j\}}$$

We observe that the intersection between the image of P_1 and $-P_2$ is trivial, because $\text{Im}(P_1) \in S_z(1) \otimes \bigwedge_{r,s-n_z,0} E$ and $\text{Im}(P_2) \in S_z(1) \otimes \bigwedge_{r,s-n_z-1,1} E$. Hence, $P_1 + P_2$ vanishes if and only if P_1 and P_2 vanish. Hence, $\delta_1 \circ \rho_\alpha$ is equivalent to the map $\boldsymbol{\lambda} \mapsto P_1(\boldsymbol{\lambda}) + P_2(\boldsymbol{\lambda})$. Note that, for all $I \in \omega^{(1)} \cup \omega^{(2)}$, $\{1, \ldots, r\} \subset I$. Therefore, if we expand this map we conclude that it is equivalent to the 0-graded part of the $(s - n_z + 1)$-th map of the Koszul complex of the linear system $\boldsymbol{f_z}$.

$$P_1(\boldsymbol{\lambda}) + P_2(\boldsymbol{\lambda}) = \sum_{\substack{J \subset \{0,r+1,\ldots,n\} \\ \#J = s-n_z+1}} \sum_{j=1}^{s-n_z+1} (-1)^{j-1} \lambda_J f_{J_j}(\alpha_x, \alpha_y) \otimes \boldsymbol{e}_{\{1\ldots r\} \cup J \setminus \{J_j\}} \quad \square$$