Towards Mixed Gröbner Basis Algorithms: the Multihomogeneous and Sparse Case

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Abstract

One of the biggest open problems in computational algebra is the design of efficient algorithms for Gröbner basis computations that take into account the sparsity of the input polynomials. We can perform such computations in the case of unmixed polynomial systems, that is systems with polynomials having the same support, using the approach of Faugère, Spaenlehauer, and Svartz [ISSAC'14]. We present two algorithms for sparse Gröbner bases computations for mixed systems. The first one computes with mixed sparse systems and exploits the supports of the polynomials. Under regularity assumptions, it performs no reductions to zero. For mixed, square, and 0-dimensional multihomogeneous polynomial systems, we present a dedicated, and potentially more efficient, algorithm that exploits different algebraic properties that performs no reduction to zero. We give an explicit bound for the maximal degree appearing in the computations.

Keywords: Mixed Sparse Gröbner Basis; Gröbner Basis; Multihomogeneous Polynomial System; Solving Polynomial System; Sparse Polynomial System; Toric variety;

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1 Introduction

Gröbner bases are in the heart of many algebraic algorithms. One of the most important applications is to solve 0-dimensional polynomial systems. A common strategy is, first to compute a Gröbner basis in some order, usually degree lexicographic, deduce from it multiplication maps in the corresponding quotient ring, and finally recover the lexicographic order using FGLM [20].

Toric geometry [12] studies the geometric and algebraic properties of varieties given by the image of monomial maps and systems of *sparse* polynomial equations; that is systems with polynomials having monomials from a restrictive set. Sparse resultant [23], that generalizes the classical multivariate resultant, extends these ideas in (sparse) elimination theory. There are a lot of algorithms to compute the sparse resultant and to solve sparse systems, for example see [34, 18, 13]. For the related problem of fewnomial systems see [4]. Numerical continuation methods can also benefit from sparsity [29], as well as other symbolic algorithms [24, 27].

Recently Faugère et al. [22] introduced the first algorithm to solve *unmixed* sparse systems, that is systems of sparse polynomials that have the same monomials, using Gröbner basis that exploits sparsity. Their idea is to consider the polytopal algebra associated to the supports of the input polynomials. Roughly speaking, the polytopal algebra is like the standard polynomial algebra, where the variables are the monomials in the supports of the input polynomials. They compute a Gröbner basis of the ideal generated by the polynomials, in the polytopal algebra, by introducing a matrix F5-like algorithm [19, 15]. They homogenize the polynomials and compute a Gröbner basis degree by degree. By dehomogenizing the computed basis, they recover a Gröbner basis of the original ideal. In the 0-dimensional case, they apply a FGLM-like algorithm [20] to obtain a lexicographical Gröbner basis. If the homogenized polynomials form a regular sequence over the polytopal algebra, then the algorithm performs no reductions to zero. When the system is also 0-dimensional, they bound the complexity using the Castelnuovo-Mumford regularity. In this case, taking advantage of the sparsity led to large speedups. Hence, our goal is to extend [22] to *mixed* sparse polynomial systems, i.e. systems where the polynomials *do not* have necessarily the same monomials.

The Castelnuovo-Mumford regularity is a fundamental invariant in algebraic geometry, related to the maximal degrees appearing in the minimal resolutions and the vanishing of the local cohomology. It is related to the complexity of computing Gröbner basis [2, 9]. The extension of this regularity in the context of toric varieties is known as multigraded Castelnuovo-Mumford regularity [31, 30, 6].

The multihomogeneous systems form an important subclass of mixed sparse systems as they are ubiquitous in applications. Their properties are well understood, for example, the degree (number of solutions) of the system [37], the arithmetic Nullstellensätze [14], and the (multigraded) Castelnuovo-Mumford regularity [25, 1, 33, 5, 6]. We can solve these systems using general purpose algorithms based on resultants [18] and in some cases benefit from the existence of determinantal formulas [35, 38], or we can use homotopy methods [26, 17]. For unmixed bilinear systems, we compute a Gröbner basis [21] with no reductions to zero. Using determinantal formulas we can solve mixed bilinear systems with two supports using eigenvalues/eigenvectors [3]. In the unmixed case, [22] presents bounds for the complexity of computing a sparse Gröbner basis. Our goal is to present a potentially more efficient algorithm and bounds for square mixed multihomogeneous systems.

Our contribution We present two algorithms to solve 0-dimensional mixed sparse polynomial systems based on Gröbner basis computations. Both of them, under assumptions, compute with no reductions to zero, thus they avoid useless computations.

The first algorithm (Alg. 3.1) takes as input a mixed sparse system and computes a *sparse Gröbner basis* (Def. 3.3). This is a basis for the corresponding ideal over a polytopal algebra and has similar properties to the usual Gröbner basis. Using this basis, we compute normal forms by a modified division algorithm (Lem. 3.4). The orders for the monomials that we consider take into account the supports of the polynomials and they *are not* necessarily monomial orders (Sec. 2.4). We prove that for any of these orders and any ideal there is a finite sparse Gröbner basis (Corollaries 3.11 and 3.13) that we compute with a matrix F5-like algorithm, that we call $\mathtt{M}^2$. Moreover, we introduce a *sparse F5 criterion* to avoid useless computations. Under regularity assumptions, we avoid every reduction to zero (Lem. 3.19). When the ideal is 0-dimensional, we can use a sparse Gröbner basis to compute a Gröbner basis for unmixed systems introduced in [22] using FGLM.

Our second algorithm, $\mathtt{M}_3\mathtt{H}$, takes as input a 0-dimensional square multihomogeneous mixed system, that has no solutions at infinity. It outputs a monomial basis and the multiplication map of every affine variable. Both lie in the quotient ring of the dehomogenization of the (input) ideal. Using the multigraded Castelnuovo-Mumford regularity, we present an algorithm (Alg. 4.1) that avoids all reductions to zero (Cor. 4.11). Over $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$, if the input polynomials have multidegrees $\boldsymbol{d}_1, \ldots, \boldsymbol{d}_{(n_1+\cdots+n_r)} \in \mathbb{N}^r$, then the dimension of the biggest matrix appearing in the computations is the number of monomials of multidegree $\sum_{i=1}^{n_1+\cdots+n_r} \boldsymbol{d}_i + (1, \ldots, 1) - (n_1, \ldots, n_r)$. This bounds the maximal degree of the polynomials appearing in the computations and generalizes the classical Macaulay bound [28], which we recover for $r = 1$. Using the multiplication matrices, we can recover the usual Gröbner basis for the dehomogenized ideal via FGLM.

2 Preliminaries

Let $\mathbb{K}$ be a field of characteristic 0, $\boldsymbol{y} := (y_0, \ldots, y_m)$, and $\mathbb{K}[\boldsymbol{y}] := \mathbb{K}[y_0, \ldots, y_m]$. For $\alpha \in \mathbb{N}^{m+1}$, let $\boldsymbol{y}^\alpha := \prod_{i=0}^{m} y_i^{\alpha_i}$. Let $\bar{0} := (0 \ldots 0)$.

2.1 Semigroup Algebra

An affine semigroup S is a finitely-generated additive subsemigroup of $\mathbb{Z}^n$, for some $n \in \mathbb{N}$, such that it contains $0 \in \mathbb{Z}^n$. The semigroup algebra $\mathbb{K}[S]$ is the $\mathbb{K}$-algebra generated by $\{\boldsymbol{X}^s, s \in S\}$, where $\boldsymbol{X}^s \cdot \boldsymbol{X}^t = \boldsymbol{X}^{s+t}$. The set of monomials of $\mathbb{K}[S]$ is $\{\boldsymbol{X}^s, s \in S\}$.

Let $\{a_0, a_1, \ldots, a_m\}$ be a set of generators of $S \subset \mathbb{Z}^n$. Let $e_0 \ldots e_m$ be the canonical basis of $\mathbb{Z}^{m+1}$. Consider the homomorphism $\rho : \mathbb{Z}^{m+1} \to S$ that sends e_i to a_i, for $0 \leq i \leq m$. Then, $\mathbb{K}[S]$ is isomorphic to the quotient ring $\mathbb{K}[\boldsymbol{y}]/T$, where T is the lattice ideal $T := \langle \boldsymbol{y}^u - \boldsymbol{y}^v | u, v \in \mathbb{N}^{m+1}, \rho(u - v) = 0 \rangle$ [32, Thm 7.3]. Moreover, the ideal T is prime and $\mathbb{K}[S]$ is an integral domain [32, Thm 7.4].

An affine semigroup S is pointed if it does not contain non-zero invertible elements, that is for all $s, t \in S \setminus \{\bar{0}\}$, $s + t \neq 0$ [32, Def 7.8]. As in [22], we consider only pointed affine semigroups.

Let $M_1, \ldots, M_k \subset \mathbb{R}^n$ be polytopes containing 0. We consider two different semigroups associated to them. First, we consider the affine semigroup $(S_{M_1,\ldots,M_k}, `+`)$ generated by the elements in $\cup_{i=1}^{k} (M_i \cap \mathbb{Z}^n)$ with the addition over $\mathbb{Z}^n$. Second, we consider the affine semigroup $(S^h_{M_1,\ldots,M_k}, `+`)$, generated by the elements in $\cup_{i=1}^{k} \{(s, e_i) : s \in M_i \cap \mathbb{Z}^n\}$, with the addition over $\mathbb{Z}^{n+k}$, where $e_1, \ldots, e_k$ is the standard basis of $\mathbb{R}^k$.

2.2 Sparse degree and homogenization

Given a monomial $\boldsymbol{X}^{(s,d)} \in \mathbb{K}[S^h_{M_1,\dots,M_k}]$, we define its *degree* as $\deg(\boldsymbol{X}^{(s,d)}) := d \in \mathbb{N}^k$. With this grading, the semigroup algebra $\mathbb{K}[S^h_{M_1,\dots,M_k}]$ is multigraded by $\mathbb{N}^k$ and generated, as a $\mathbb{K}$-algebra, by the elements of degrees $e_1, \dots, e_k$, so it is multihomogeneous. For each $d \in \mathbb{N}^k$, let $\mathbb{K}[S^h_{M_1,\dots,M_k}]_d$ be the vector space of the multihomogeneous polynomials in $\mathbb{K}[S^h_{M_1,\dots,M_k}]$ of degree $d \in \mathbb{N}^k$.

We define the dehomogenization of $\boldsymbol{X}^{(s,d)}$ as the epimorphism that takes $\boldsymbol{X}^{(s,d)} \in \mathbb{K}[S^h_{M_1 \dots M_k}]$ to $\chi(\boldsymbol{X}^{(s,d)}) = \boldsymbol{X}^s \in \mathbb{K}[S_{M_1 \dots M_k}]$. For an ideal I^h, $\chi(I^h)$ means that we apply χ to the elements of I^h.

Remark 2.1. *For an ideal I^h, for every $f \in I^h \cap \mathbb{K}[S^h_{M_1,\dots,M_k}]_d$, and $D \geq d$, component-wise, there is $f' \in I^h \cap \mathbb{K}[S^h_{M_1,\dots,M_k}]_D$ such that $\chi(f) = \chi(f') \in \chi(I^h)$.*

When we work only with one polytope M, that is $k = 1$, we define the *affine degree* of $\boldsymbol{X}^s \in \mathbb{K}[S_M]$, $\delta^A(\boldsymbol{X}^s)$, as the smallest $d \in \mathbb{N}$ such that $\boldsymbol{X}^{(s,d)} \in \mathbb{K}[S^h_M]$. We extend this definition to the affine polynomials in $\mathbb{K}[S_M]$ as the maximal affine degree of each monomial. That is, for $f := \sum_{s \in S_M} c_s \boldsymbol{X}^s \in \mathbb{K}[S_M]$, the affine degree of f is $\delta^A(f) := \max_{s \in S_M}(\delta^A(\boldsymbol{X}^s) : c_s \neq 0)$. Let $\mathbb{K}[S_M]_{\leq d}$ be the set of all polynomials in $\mathbb{K}[S_M]$ of degree at most d. The map $\chi^{-1} : \mathbb{K}[S_M] \to \mathbb{K}[S^h_M]$ defines the homogenization of $f := \sum_{s \in S_M} c_s \boldsymbol{X}^s \in \mathbb{K}[S_M]$, where $\chi^{-1}(f) := \sum_{s \in S_M} c_s \boldsymbol{X}^{(s, \delta^A(f))} \in \mathbb{K}[S^h_M]$. Note that this map is not a homomorphism. For an ideal I, $\chi^{-1}(I)$ is the homogeneous ideal generated by applying χ^{-1} to every element of I.

Finally, given a polynomial $f \in \mathbb{K}[S^h_M]$ we define its *sparse degree* as $\delta(f) := \delta^A(\chi(f))$. Note that, the degree is always bigger or equal to the sparse degree. Even though we use the name sparse degree, it does not give a graded structure to the $\mathbb{K}$-algebra $\mathbb{K}[S^h_M]$.

2.3 Mixed systems and Regularity

Consider polytopes $M_1, \dots, M_k$ and a polynomial system $(f_1 \dots f_k)$ such that $f_i \in \mathbb{K}[S^h_{M_1,\dots,M_k}]_{e_i}$. We say the system is *regular* if $f_1, \dots, f_k$ form a regular sequence over $\mathbb{K}[S^h_{M_1,\dots,M_k}]$. Similarly, $(\chi(f_1) \dots \chi(f_k))$, that is the dehomogenization of $(f_1, \dots, f_k)$, is *regular* if $(\chi(f_1) \dots \chi(f_k))$ form a regular sequence over $\mathbb{K}[S_{M_1 \dots M_k}]$.

When all the polytopes are the same these definitions *match* the definition of regularity for unmixed systems [22]. When every polytope is a

n-simplex, these definitions are related to the standard definition of regularity [16, Chp. 17].

Like in the (standard) homogeneous case, the order of the polynomials does not affect the regularity of the system $(f_1.\dots,f_k)$. In addition, the dehomogenization preserves the regularity property.

Lemma 2.2. *Consider* $f_i \in \mathbb{K}[S^h_{M_1,\dots,M_k}]_{e_i}$ *and* σ *a permutation of* $\{1,\dots,k\}$. *If* $f_1,\dots,f_k$ *is a regular sequence over* $\mathbb{K}[S^h_{M_1,\dots,M_k}]$*, then* $(\chi(f_{\sigma_1}),\dots,\chi(f_{\sigma_k}))$ *is a regular sequence over* $\mathbb{K}[S_{M_1,\dots,M_k}]$.

Proof. If $f_1,\dots,f_k$ is a regular sequence, then any permutation of them it is regular [7, §9, Cor. 2]. Hence, we just have to prove that $\chi(f_1),\dots,\chi(f_k)$ is a regular sequence. For $w \leq k$, consider a polynomial $\bar{g}_w \in \mathbb{K}[S_{M_1,\dots,M_k}]$ such that $\bar{g}_w \cdot \chi(f_w) \in \langle \chi(f_1),\dots,\chi(f_{w-1})\rangle$. Then, there are polynomials $\bar{g}_1,\dots,\bar{g}_{w-1} \in \mathbb{K}[S_{M_1,\dots,M_k}]$ such that $\sum_{i=1}^{w} \bar{g}_i\chi(f_i) = 0$. As χ is an epimorphism, for each $\bar{g}_i$, there is $g_i \in \mathbb{K}[S^h_{M_1,\dots,M_k}]$ multihomogeneous such that $\chi(g_i) = \bar{g}_i$. Consider a vector D, such that $\forall i,j,\ D - \deg(f_i) \geq \deg(g_j)$. Then, by Rem. 2.1, there are multihomogeneous polynomials $g'_i \in \mathbb{K}[S^h_{M_1,\dots,M_k}]_{D-\deg(f_i)}$, such that $\chi(g'_i) = \bar{g}_i$. Note that, χ restricted to $\mathbb{K}[S^h_{M_1,\dots,M_k}]_D$ is injective. Hence $\chi(\sum_{i=1}^{w} g'_i f_i) = \sum_{i=1}^{w} \bar{g}_i\chi(f_i) = 0$ implies $\sum_{i=1}^{w} g'_i f_i = 0$. As $f_1,\dots,f_w$ is a regular sequence, $g'_w \in \langle f_1,\dots,f_{w-1}\rangle$ and $\bar{g}_w \in \langle \chi(f_1),\dots,\chi(f_{w-1})\rangle$. □

The proof of existence of regular systems is beyond the scope of this paper. Nevertheless, we can report that we have performed several experiments with many different sparse mixed systems, taking generic coefficients, and all them were regular.

2.4 Orders for Monomials

As in the standard case, a monomial order $<$ for $\mathbb{K}[S]$ is a well-order compatible with the multiplication on $\mathbb{K}[S]$, that is $\forall s \in S, s \neq 0 \implies \boldsymbol{X}^0 < \boldsymbol{X}^s$ and $\forall s,r,t \in S, \boldsymbol{X}^s < \boldsymbol{X}^r \implies \boldsymbol{X}^{s+t} < \boldsymbol{X}^{r+t}$. These orders exist on $\mathbb{K}[S]$ if and only if S is pointed, [22, Def 3.1].

Given any well-order $<$ for $\mathbb{K}[S_M]$, we can extend it to a well-order $<_h$, the grading of $<$, for $\mathbb{K}[S^h_M]$ as follows:

$$\boldsymbol{X}^{(s,d)} < \boldsymbol{X}^{(r,d')} \iff \begin{cases} d < d' \\ d = d' \wedge \boldsymbol{X}^s < \boldsymbol{X}^r \end{cases} \tag{1}$$

If $<$ is a monomial order, then $<_h$ is a monomial order too.

Given an ideal $I \subset \mathbb{K}[S_M]$, a common issue is to study the vector space $I \cap \mathbb{K}[S_M]_{\leq d}$, i.e. the elements of I of degree smaller or equal to d. This information allow us, for example, to compute the Hilbert Series of the affine ideal. It is also important for computational reasons. For example, to maintain the invariants in the signature-based Gröbner basis algorithms, as the F5 algorithm [19, 15].

In our setting, to compute a basis of $I \cap \mathbb{K}[S_M]_{\leq d}$, we have to work with an order for the monomials in $\mathbb{K}[S_M]$ that takes into account the sparse degree. This order, $\prec$, is such that for any $\boldsymbol{X}^s, \boldsymbol{X}^r \in \mathbb{K}[S_M]$, $\delta^A(\boldsymbol{X}^s) < \delta^A(\boldsymbol{X}^r) \implies \boldsymbol{X}^s \prec \boldsymbol{X}^r$. Unfortunately, for most of the polytopal algebras $\mathbb{K}[S_M]$, *there is no monomial order* with this property. Therefore, we are forced to work with well-orders that are not monomial orders.

Example 2.3. *Consider the semigroup generated by* $M := \{[0,0],[1,0],[0,1],[1,1]\} \subset \mathbb{N}^2$. *Consider a monomial order* $<$ *for* $\mathbb{K}[S_M]$. *Without loss of generality, assume* $\boldsymbol{X}^{[1,0]} < \boldsymbol{X}^{[0,1]}$. *Then,* $\boldsymbol{X}^{[2,0]} < \boldsymbol{X}^{[1,1]} < \boldsymbol{X}^{[0,2]}$. *But,* $\delta^A(\boldsymbol{X}^{[2,0]}) = 2$ *and* $\delta^A(\boldsymbol{X}^{[1,1]}) = 1$. *So, no monomial order on* $\mathbb{K}[S_M]$ *takes into account the sparse degree.*

Given a monomial order $<_M$ for $\mathbb{K}[S_M]$, we define the *sparse order* $\prec$ for $\mathbb{K}[S_M]$ as follows.

$$\boldsymbol{X}^s \prec \boldsymbol{X}^r \iff \begin{cases} \delta^A(\boldsymbol{X}^s) < \delta^A(\boldsymbol{X}^r) \\ \delta^A(\boldsymbol{X}^s) = \delta^A(\boldsymbol{X}^r) \wedge \boldsymbol{X}^s <_M \boldsymbol{X}^r \end{cases} \tag{2}$$

Let $\prec_h$ be the grading of the sparse order of $\mathbb{K}[S_M^h]$ (Eq. 1). We call this order the *graded sparse order*.

Remark 2.4. *By definition, these two orders are the same for monomials of the same degree. That is,*

$$\forall \boldsymbol{X}^{(s,d)}, \boldsymbol{X}^{(r,d)} \in \mathbb{K}[S_M^h],\ \boldsymbol{X}^{(s,d)} \prec_h \boldsymbol{X}^{(r,d)} \iff \boldsymbol{X}^s \prec \boldsymbol{X}^r\ .$$

Usually, this order is not compatible with the multiplication. But,

Lemma 2.5. *If* $\boldsymbol{X}^s \prec \boldsymbol{X}^t$ *and* $\delta^A(\boldsymbol{X}^r) + \delta^A(\boldsymbol{X}^t) = \delta^A(\boldsymbol{X}^t \cdot \boldsymbol{X}^r)$, *then* $\boldsymbol{X}^s \cdot \boldsymbol{X}^r \prec \boldsymbol{X}^t \cdot \boldsymbol{X}^r$.

Proof. Note that δ^A satisfies the triangular inequality, $\delta^A(\boldsymbol{X}^{s+r}) \leq \delta^A(\boldsymbol{X}^s) + \delta^A(\boldsymbol{X}^r)$. As $\boldsymbol{X}^s \prec \boldsymbol{X}^t$, $\delta^A(\boldsymbol{X}^s) \leq \delta^A(\boldsymbol{X}^t)$. By assumption, $\delta^A(\boldsymbol{X}^t) + \delta^A(\boldsymbol{X}^r) = \delta^A(\boldsymbol{X}^{t+r})$. So, $\delta^A(\boldsymbol{X}^{s+r}) \leq \delta^A(\boldsymbol{X}^s) + \delta^A(\boldsymbol{X}^r) \leq \delta^A(\boldsymbol{X}^t) + \delta^A(\boldsymbol{X}^r) \leq \delta^A(\boldsymbol{X}^{t+r})$. Hence, either $\delta^A(\boldsymbol{X}^{s+r}) < \delta^A(\boldsymbol{X}^{t+r})$ or the sparse degree is the same. In the second case, we conclude $\delta^A(\boldsymbol{X}^s) = \delta^A(\boldsymbol{X}^t)$, and so $\boldsymbol{X}^s <_M \boldsymbol{X}^t$. As $<_M$ is a monomial order, $\boldsymbol{X}^{s+r} <_M \boldsymbol{X}^{t+r}$. Hence, $\boldsymbol{X}^s \cdot \boldsymbol{X}^r \prec \boldsymbol{X}^t \cdot \boldsymbol{X}^r$. ◻

We extend this property to the homogeneous case.

Corollary 2.6. *If $\boldsymbol{X}^{(s,d_s)} \prec \boldsymbol{X}^{(t,d_t)}$ and $\delta(\boldsymbol{X}^{(r,d_r)}) + \delta(\boldsymbol{X}^{(t,d_t)}) = \delta(\boldsymbol{X}^{(r,d_r)} \cdot \boldsymbol{X}^{(t,d_t)})$, then $\boldsymbol{X}^{(s,d_s)} \cdot \boldsymbol{X}^{(r,d_r)} \prec \boldsymbol{X}^{(t,d_t)} \cdot \boldsymbol{X}^{(r,d_r)}$.*

3 Sparse Gröbner Basis (sGB)

We want to define and compute Gröbner bases in $\mathbb{K}[S_M]$ and $\mathbb{K}[S_M^h]$ with respect to a (graded) sparse order. As these orders are not compatible with the multiplication, not all the standard definitions of Gröbner basis are equivalent. For example, the set of leading monomials of an ideal in $\mathbb{K}[S_M]$ does not necessarily form an ideal. We say that a set of generators G of an ideal $I \subset \mathbb{K}[S_M]$ is a sparse Gröbner basis with respect to an order $\prec$, if for each $f \in I$, there is a $g \in G$ such that $\texttt{LM}_\prec(g)$ divides $\texttt{LM}_\prec(f)$. Similarly for $\mathbb{K}[S_M^h]$.

This definition has a drawback: The multivariate polynomial division algorithm might not terminate. This can happen when $\texttt{LM}_\prec(f) = \boldsymbol{X}^t \cdot \texttt{LM}_\prec(g)$ and $\texttt{LM}_\prec(f) \prec \texttt{LM}_\prec(\boldsymbol{X}^t \cdot g)$. Then, the reduction step "increases" the leading monomial, so that the algorithm does not necessarily terminates. We can construct examples where we have a periodic sequence of reductions. To avoid this problem, we redefine the division relation.

Definition 3.1 (Division relation)**.** *For any $\boldsymbol{X}^{(s,d_s)}, \boldsymbol{X}^{(r,d_r)} \in \mathbb{K}[S_M^h]$, we say that $\boldsymbol{X}^{(s,d_s)}$ divides $\boldsymbol{X}^{(r,d_r)}$, and write $\boldsymbol{X}^{(s,d_s)} || \boldsymbol{X}^{(r,d_r)}$, if there is a $\boldsymbol{X}^{(t,d_t)} \in \mathbb{K}[S_M^h]$ such that $\boldsymbol{X}^{(s,d_s)} \cdot \boldsymbol{X}^{(t,d_t)} = \boldsymbol{X}^{(r,d_r)}$ and $\delta(\boldsymbol{X}^{(s,d_s)}) + \delta(\boldsymbol{X}^{(t,d_t)}) = \delta(\boldsymbol{X}^{(r,d_r)})$. Similarly, for $\boldsymbol{X}^s, \boldsymbol{X}^r \in \mathbb{K}[S_M]$, we say that $\boldsymbol{X}^s$ divides $\boldsymbol{X}^r$, and write $\boldsymbol{X}^s || \boldsymbol{X}^r$, if $\chi^{-1}(\boldsymbol{X}^s) || \chi^{-1}(\boldsymbol{X}^r)$.*

Remark 3.2. *If $LM_{\prec_h}(f) || \boldsymbol{X}^{(s,d_s)}$, then there is a $\boldsymbol{X}^{(t,d_t)} \in \mathbb{K}[S_M^h]$ such that $\boldsymbol{X}^{(s,d_s)} = \boldsymbol{X}^{(t,d_t)} \cdot LM_{\prec_h}(f) = LM_{\prec_h}(\boldsymbol{X}^{(t,d_t)} \cdot f)$, by Lem. 2.5. Similarly over $\mathbb{K}[S_M]$.*

We define the sparse Gröbner bases (sGB) as follows.

Definition 3.3 (sparse Gröbner bases)**.** *Given a (graded) sparse order* $\prec$*, see Eq.* (2)*, and an ideal* $I \subset \mathbb{K}[S_M]$*, respectively* $I \subset \mathbb{K}[S_M^h]$*, a set* $sGB(I) \subset I$ *is a sparse Gröbner basis (sGB) if it generates* I *and for any* $f \in I$ *there is some* $g \in sGB(I)$ *such that* $\texttt{LM}_\prec(g) || \texttt{LM}_\prec(f)$*.*

With this definition, each step in the division algorithm reduces the leading monomial (Rem. 3.2), and so the division algorithm always terminates, see e.g. [10, Thm. 2.3.3,Prop. 2.6.1].

Lemma 3.4. *Let* $f \in \mathbb{K}[S_M]$ *and* G *be a set of polynomials in* $\mathbb{K}[S_M]$*. Using our definition of division relation (Def. 3.1), the multivariate division algorithm [10, Thm. 2.3.3] for the division of* f *by* G*, with respect to the order* $\prec$*, terminates. Moreover, if* G *is a sGB of an ideal* I *with respect to* $\prec$ *and* $f \equiv f' \mod I$*, then the remainder division algorithm for* f *and* f' *is the same and unique for any sGB.*

Proof. By Rem. 3.2, each step in the division algorithm reduces the leading monomial. The proof follows, mutatis mutandis, from [10, Thm. 2.3.3,Prop. 2.6.1]. ∎

Our next goal is to prove that for every ideal and sparse order, there is a finite sGB. A priori, this is not clear from the Noetherian property of $\mathbb{K}$ as $\texttt{LM}_\prec(I)$ is not an ideal. Our strategy is to prove that over $\mathbb{K}[S_M^h]$ there is always a finite sparse Gröbner basis, and then extend this result to $\mathbb{K}[S_M]$. We show that this sGB is related to a standard Gröbner basis over some Noetherian ring, so it is finite.

3.1 Finiteness of sparse Gröbner Bases

Homogeneous case.

Let $<_M$ be a monomial order for $\mathbb{K}[S_M]$ and $\prec$ the sparse order related to $<_M$, Eq. (2). Consider $\prec_h$ the graded sparse order related to $\prec$ over $\mathbb{K}[S_M^h]$, Eq. (1).

Consider the lattice ideal T from Sec. 2.1. This ideal T is homogeneous and the algebra $\mathbb{K}[S_M^h]$ is isomorphic to $\mathbb{K}[\boldsymbol{y}]/T$ as a graded algebra. Let $\widetilde{\psi} : \mathbb{K}[\boldsymbol{y}]/T \to \mathbb{K}[S_M^h]$ and $\widetilde{\phi} : \mathbb{K}[S_M^h] \to \mathbb{K}[\boldsymbol{y}]/T$ be the isomorphisms related to $\mathbb{K}[S_M^h] \cong \mathbb{K}[\boldsymbol{y}]/T$, such that they are inverse of each other and $\widetilde{\psi}(\boldsymbol{X}^{(0,1)}) =$

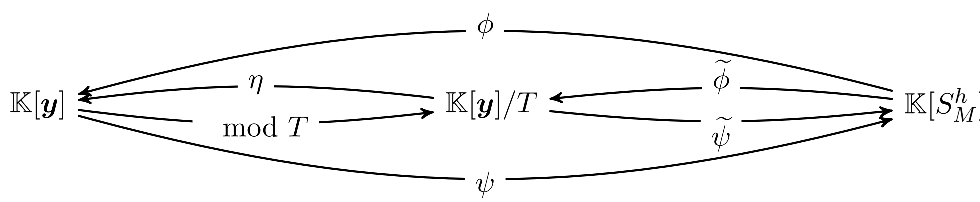
y_0. We extend $\widetilde{\psi}$ to $\psi : \mathbb{K}[\boldsymbol{y}] \to \mathbb{K}[S_M^h]$, where $\psi(\boldsymbol{y}^\alpha)$ is the image, under $\widetilde{\psi}$, of $\boldsymbol{y}^\alpha$ modulo T. The map ψ is a 0-graded epimorphism.

For $\boldsymbol{y}^\alpha \in \mathbb{K}[\boldsymbol{y}]$, let $\deg(\boldsymbol{y}^\alpha, y_0)$ be the degree of $\boldsymbol{y}^\alpha$ with respect to y_0 and $\deg(\boldsymbol{y}^\alpha)$ be the total degree. Given a (standard) monomial order $\widetilde{<}$ for $\mathbb{K}[\boldsymbol{y}]$, consider the graded monomial order $<_y$ for $\mathbb{K}[\boldsymbol{y}]$ defined as follows,

$$\boldsymbol{y}^a <_y \boldsymbol{y}^b \iff \begin{cases} \deg(\boldsymbol{y}^a) < \deg(\boldsymbol{y}^b) & \\ \deg(\boldsymbol{y}^a) = \deg(\boldsymbol{y}^b) \quad \wedge & \deg(\boldsymbol{y}^a, y_0) > \deg(\boldsymbol{y}^b, y_0) \\ \deg(\boldsymbol{y}^a) = \deg(\boldsymbol{y}^b) \quad \wedge & \deg(\boldsymbol{y}^a, y_0) = \deg(\boldsymbol{y}^b, y_0) \quad \wedge \\ & \psi(\boldsymbol{y}^a) <_M \psi(\boldsymbol{y}^b) \\ \deg(\boldsymbol{y}^a) = \deg(\boldsymbol{y}^b) \quad \wedge & \deg(\boldsymbol{y}^a, y_0) = \deg(\boldsymbol{y}^b, y_0) \quad \wedge \\ & \psi(\boldsymbol{y}^a) = \psi(\boldsymbol{y}^b) \quad \wedge \quad \boldsymbol{y}^a \,\widetilde{<}\, \boldsymbol{y}^b \end{cases} \tag{3}$$

This order is a monomial order, because it is a total order, $\boldsymbol{y}^0$ is the unique smallest monomial (it is the only one of degree 0), and it is compatible with the multiplication (every case is compatible).

For each $f \in \mathbb{K}[\boldsymbol{y}]$, we define η as the normal form (the remainder of the division algorithm) of f with respect to the ideal T and the monomial order $<_y$. Recall that $\eta = \eta \circ \eta$ and $\texttt{coker}(\eta) \cong \mathbb{K}[\boldsymbol{y}]/T$. We notice that for each poset in $\mathbb{K}[\boldsymbol{y}]/T$, η assigns the same normal form to all the elements that it contains. Therefore, we abuse notation, and we also use η to denote the map $\mathbb{K}[\boldsymbol{y}]/T \to \mathbb{K}[\boldsymbol{y}]$ that maps each poset to this unique normal form. As T is homogeneous, η is a 0-graded map. We extend $\widetilde{\phi}$ to $\phi : \mathbb{K}[S_M^h] \to \mathbb{K}[\boldsymbol{y}]$ as $\phi := \eta \circ \widetilde{\phi}$. This map is 0-graded and linear, but not a homomorphism. It holds $\psi \circ \phi = Id$ and $\phi \circ \psi = \eta$.



Theorem 3.5. *Let $I^h \subset \mathbb{K}[S_M^h]$ be a homogeneous ideal and consider the homogeneous ideal $J^h := \langle \phi(I^h) + T \rangle \subset \mathbb{K}[\boldsymbol{y}]$. If the Gröbner base of J^h with respect to $<_y$ is $GB_{<_y}(J^h)$, then $\psi(GB_{<_y}(J^h))$ is a sparse Gröbner base of I^h with respect to $\prec_h$.*

To prove the theorem we need the following lemmas.

Lemma 3.6. *For all $\boldsymbol{y}^\alpha \in \mathbb{K}[\boldsymbol{y}]$, $\deg(\eta(\boldsymbol{y}^\alpha), y_0) = \deg(\boldsymbol{y}^\alpha) - \delta(\psi(\boldsymbol{y}^\alpha))$.*

Proof. Let $\boldsymbol{X}^{(s,d)} := \psi(\boldsymbol{y}^\alpha)$ and $\bar{d} = \delta(\boldsymbol{X}^{(s,d)})$. Note that $d = \deg(\boldsymbol{y}^\alpha)$, because ψ is 0-graded. We can write $\psi(\boldsymbol{y}^\alpha) = \chi^{-1}(\boldsymbol{X}^s) \cdot \boldsymbol{X}^{(0,d-\bar{d})}$. Recall that $\phi \circ \psi = \eta$. Applying ϕ to the previous equality we get, $\eta(\boldsymbol{y}^\alpha) = \eta(\bar{\phi}(\chi^{-1}(\boldsymbol{X}^s)) \cdot \bar{\phi}(\boldsymbol{X}^{(0,d-\bar{d})})) = \eta(\bar{\phi}(\chi^{-1}(\boldsymbol{X}^s)) \cdot y_0^{d-\bar{d}})$. Note that the order $>_y$ acts as the degree reverse lexicographical with respect to y_0, hence $\eta(\bar{\phi}(\chi^{-1}(\boldsymbol{X}^s)) \cdot y_0^{d-\bar{d}}) = \phi(\chi^{-1}(\boldsymbol{X}^s)) \cdot y_0^{d-\bar{d}}$. If y_0 divides $\phi(\chi^{-1}(\boldsymbol{X}^s))$, then there is a monomial $\boldsymbol{y}^\beta$ such that $y_0 \cdot \boldsymbol{y}^\beta = \phi(\chi^{-1}(\boldsymbol{X}^s))$, and so, $\psi(y_0 \cdot \boldsymbol{y}^\beta) = \psi(\phi(\chi^{-1}(\boldsymbol{X}^s)))$. As $\psi \circ \phi = Id$ and ψ is a 0-graded epimorphism, then $\boldsymbol{X}^{(0,1)} \cdot \psi(\boldsymbol{y}^\beta) = \chi^{-1}(\boldsymbol{X}^s)$, but this is not possible by definition of homogenization (Sec. 2.2). Hence, $\deg(\phi(\chi^{-1}(\boldsymbol{X}^s)), y_0) = 0$ and $\deg(\eta(\boldsymbol{y}^\alpha), y_0) = 0 + d - \bar{d}$. □

Corollary 3.7. *For all $\boldsymbol{X}^{(s,d)} \in \mathbb{K}[S_M^h]$, it holds*

$$\delta(\boldsymbol{X}^{(s,d)}) = d - \deg(\phi(\boldsymbol{X}^{(s,d)}), y_0).$$

As ψ and ϕ are 0-graded maps, by Lem. 3.6 and Cor. 3.7, they preserve the order.

Corollary 3.8. $\eta(y^\alpha) <_y \eta(y^\beta) \implies \psi(y^\alpha) \prec_h \psi(y^\beta)$.

Lemma 3.9. $\boldsymbol{y}^\alpha | \phi(\boldsymbol{X}^{(s,d)}) \implies \psi(\boldsymbol{y}^\alpha) || \boldsymbol{X}^{(s,d)}$.

Proof. Let $\boldsymbol{y}^\beta$ such that $\boldsymbol{y}^\alpha \cdot \boldsymbol{y}^\beta = \phi(\boldsymbol{X}^{(s,d)})$, so $\psi(\boldsymbol{y}^\alpha) \cdot \psi(\boldsymbol{y}^\beta) = \boldsymbol{X}^{(s,d)}$. As η is a normal form, $\eta(\phi(\boldsymbol{X}^{(s,d)})) = \phi(\boldsymbol{X}^{(s,d)})$ and then, $\eta(\boldsymbol{y}^\alpha) = \boldsymbol{y}^\alpha$ and $\eta(\boldsymbol{y}^\beta) = \boldsymbol{y}^\beta$. Hence, by Cor. 3.7, $\delta(\psi(\boldsymbol{y}^\alpha \cdot \boldsymbol{y}^\beta)) = \deg(\boldsymbol{y}^\alpha \cdot \boldsymbol{y}^\beta) - \deg(\eta(\boldsymbol{y}^\alpha \cdot \boldsymbol{y}^\beta), y_0) = \deg(\boldsymbol{y}^\alpha) - \deg(\eta(\boldsymbol{y}^\alpha), y_0) + \deg(\boldsymbol{y}^\beta) - \deg(\eta(\boldsymbol{y}^\beta), y_0) = \delta(\psi(\boldsymbol{y}^\alpha)) + \delta(\psi(\boldsymbol{y}^\beta))$, by Lem. 3.6. □

Corollary 3.10. *For all $f \in \mathbb{K}[S_M^h]$, for all $g \in \mathbb{K}[\boldsymbol{y}]$, it holds*

$$\mathtt{LM}_{<_y}(\eta(g)) | \mathtt{LM}_{<_y}(\phi(f)) \implies \mathtt{LM}_{\prec_h}(\psi(g)) || \mathtt{LM}_{\prec_h}(f).$$

Proof. By Cor. 3.8, $\psi(\mathtt{LM}_{<_y}(\eta(g))) = \mathtt{LM}_{\prec_h}(\psi(g))$ and $\psi(\mathtt{LM}_{<_y}(\phi(f))) = \mathtt{LM}_{<_y}(\psi(\phi(f))) = \mathtt{LM}_{\prec_h}(f)$. The proof follows from Lem. 3.9. □

Proof of Thm. 3.5. Consider $f \in I^h$, then $\phi(f) \in J^h$. Hence, there are $g_1, \ldots, g_k \in GB_{<_y}(J^h)$ and $p_1, \ldots, p_k \in \mathbb{K}[\boldsymbol{y}]$ such that $\phi(f) = \sum_{i=1}^k p_i \cdot g_i$. As $\psi \circ \phi = Id$ and ψ is an epimorphism such that $\psi(T) = 0$, then $\psi(\phi(f)) =$

$f = \sum_{i=1}^{k} \psi(p_i) \cdot \psi(g_i)$ and $\psi(g_i), \dots, \psi(g_k) \in I^h$. Hence, $\psi(GB_{<_y}(J^h))$ generates I^h.

The set $GB_{<_y}(J^h)$ is a Gröbner basis, then there is a $g \in GB_{<_y}(J^h)$ such that $\mathtt{LM}_{<_y}(g)|\mathtt{LM}_{<_y}(\phi(f))$. As $\phi(f) = \eta(\phi(f))$, $\eta(\mathtt{LM}_{<_y}(\phi(f))) = \mathtt{LM}_{<_y}(\phi(f))$ and $\eta(\mathtt{LM}_{<_y}(g)) = \mathtt{LM}_{<_y}(g)$. As η is a normal form wrt $<_y$, $\eta(\mathtt{LM}_{<_y}(g)) = \mathtt{LM}_{<_y}(\eta(g))$. By Cor. 3.10, $\mathtt{LM}_{\prec_h}(\psi(g))||\mathtt{LM}_{\prec_h}(f)$. Hence, $\psi(GB_{<_y}(J^h))$ is a sGB for I^h with respect to $\prec_h$. □

Corollary 3.11. *Given an ideal $I^h \subset \mathbb{K}[S_M^h]$ and a graded sparse order $\prec_h$, its sGB with respect to this order is finite.*

Proof. In Thm. 3.5 we construct $sGB_{\prec_h}(I^h)$ from a (standard) Gröbner basis of an ideal of $\mathbb{K}[\boldsymbol{y}]$, finite as $\mathbb{K}[\boldsymbol{y}]$ is Noetherian. □

Non-homogeneous case. Let $\prec$ be a sparse order for $\mathbb{K}[S_M]$.

Lemma 3.12. *Let $I^h \subset \mathbb{K}[S_M^h]$ be a homogeneous ideal. Let $\prec_h$ be the graded sparse order for $\mathbb{K}[S_M^h]$ related to $\prec$. Then, $\chi(sGB_{\prec_h}(I^h))$ is a sparse Gröbner Basis for $\chi(I^h)$ with respect to $\prec$.*

Proof. The set $\chi(sGB_{\prec_h}(I^h))$ generates $\chi(I^h)$. Note that for homogeneous polynomials, $\mathtt{LM}_{\prec_h}$ commutes with the dehomogenization, that is for any homogeneous polynomial $g \in \mathbb{K}[S_M^h]$, $\mathtt{LM}_{\prec}(\chi(g)) = \chi(\mathtt{LM}_{\prec_h}(g))$. Consider $\bar{f} \in \chi(I^h)$, then there is an $f \in I^h$ such that $f = \chi(\bar{f})$. In addition, there is $g \in sGB_{\prec_h}(I^h)$ such that $\mathtt{LM}_{\prec_h}(g)||\mathtt{LM}_{\prec_h}(f)$. Let $\boldsymbol{X}^{(s,d)} \in \mathbb{K}[S_M^h]$ such that $\mathtt{LM}_{\prec_h}(g) \cdot \boldsymbol{X}^{(s,d)} = \mathtt{LM}_{\prec_h}(f)$ and $\delta(\mathtt{LM}_{\prec_h}(g)) + \delta(\boldsymbol{X}^{(s,d)}) = \delta(\mathtt{LM}_{\prec_h}(f))$. The sparse degree δ is independent of the homogeneous degree, so $\delta(\chi(\mathtt{LM}_{\prec_h}(g))) + \delta(\boldsymbol{X}^s) = \delta(\chi(\mathtt{LM}_{\prec_h}(f)))$. Hence, $\delta(\mathtt{LM}_{\prec}(\chi(g))) + \delta(\boldsymbol{X}^s) = \delta(\mathtt{LM}_{\prec}(\bar{f}))$ and $\mathtt{LM}_{\prec}(\chi(g)) \cdot \boldsymbol{X}^s = \mathtt{LM}_{\prec}(\bar{f})$, so $\mathtt{LM}_{\prec}(\chi(g))||\mathtt{LM}_{\prec}(\bar{f})$ and $\chi(sGB_{\prec_h}(I^h))$ is a sGB of $\chi(I^h)$ wrt $\prec$. □

Corollary 3.13. *The sGB of $I \subset \mathbb{K}[S_M]$ with respect to $\prec$ is finite.*

Proof. For $\chi^{-1}(I)$, the homogenization of I, $\chi(\chi^{-1}(I)) = I$. So by Lem. 3.12 $\chi(sGB_{\prec}(\chi^{-1}(I)))$ is a sGB of I and is finite by Cor. 3.11. □

3.2 Computing sparse Gröbner Bases

Homogeneous case. To compute a sGB of a homogeneous ideal $I^h := \langle f_1, \dots, f_k \rangle$ with respect to $\prec_h$, we introduce the D-sparse Gröbner bases

[28, Sec. III.B]. A D-sparse Gröbner basis of I^h is a finite set of polynomials $\mathcal{J}^h \subset I^h$ such that for each $f \in I^h$ with $\deg(f) \leq D$, it holds $f \in \langle \mathcal{J}^h \rangle$ and there is a $g \in \mathcal{J}^h$ such that $\texttt{LM}_{\prec_h}(g) || \texttt{LM}_{\prec_h}(f)$. For big enough D, for example equal to the maximal degree in the polynomials in $sGB_{\prec_h}(I^h)$, a D-sparse Gröbner basis is a sparse Gröbner basis. The *witness degree* of I^h is the minimal D such that a D-sparse Gröbner basis is a sGB. We compute D-sparse Gröbner bases by using linear algebra.

Definition 3.14. *A Macaulay matrix $\mathcal{M}$ is a matrix whose columns are indexed by monomials in $\mathbb{K}[S_M^h]$ and the rows by polynomials in $\mathbb{K}[S_M^h]$. The set of monomials that index the columns contain all the monomial in the supports of the polynomials of the rows. For a monomial m in a polynomial f, the entry in the matrix indexed by (m, f) is the coefficient of the monomial m in f. We define $\texttt{Columns}(\mathcal{M})$ as the sequence of the monomials of $\mathcal{M}$ in the order that they index the columns. We define $\texttt{Rows}(\mathcal{M})$ as the set of* non-zero *polynomials that index the rows of $\mathcal{M}$.*

If we apply a row operation to a Macaulay matrix, we obtain a new Macaulay matrix, where we replace one of the polynomials (that is one of the rows) by linear combinations of some of them. We say that we have a *reduction to zero*, if after we perform a row operation, the resulting row is zero. As observed by Lazard [28], if we sort the columns in decreasing order by $\prec_h$, we can compute a Gröbner basis using Gaussian elimination. The proof of the following lemma follows from [28].

Lemma 3.15. *Consider the ideal $I^h := \langle f_1, \ldots, f_k \rangle \subset \mathbb{K}[S_M^h]$. Let $\mathcal{M}_D$ be the Macaulay matrix whose columns are all the monomials in $\mathbb{K}[S_M^h]_D$ sorted in decreasing order by $\prec_h$, and the rows are all the products of the form $\boldsymbol{X}^{(s, D-\deg(f_i))} \cdot f_i \in \mathbb{K}[S_M^h]_D$. Let $\widetilde{\mathcal{M}_D}$ be the matrix obtained by applying Gaussian elimination to $\mathcal{M}_D$ to obtain a reduced row echelon form. Then, the polynomials in $\bigcup_{i=1}^D \texttt{Rows}(\widetilde{\mathcal{M}_i})$ form a D-sparse Gröbner basis. Moreover, if we only consider the set of polynomials whose leading monomial can not be divided by the leading monomial of a polynomial obtained in smaller degree, that is*

$$\bigcup_{i=1}^{D} \{ f \in \texttt{Rows}(\widetilde{\mathcal{M}_i}) : (\nexists\, g \in \bigcup_{j=1}^{i-1} \texttt{Rows}(\widetilde{\mathcal{M}_j}))\ \texttt{LM}_{\prec_h}(g) || \texttt{LM}_{\prec_h}(f) \},$$

then this subset is a D-sparse Gröbner basis too.

Non-homogeneous case.

Given an ideal $I := \langle \bar{f}_1 \dots \bar{f}_r \rangle \subset \mathbb{K}[S_M]$, we homogenize the polynomials and use Lem. 3.15 to compute a sparse Gröbner basis with respect to $\prec_h$. By Lem. 3.12, if we dehomogenize the computed basis, we obtain a sparse Gröbner basis with respect to $\prec$ of I. Instead of homogenizing all polynomials $\bar{f}_i$ simultaneously, we consider an iterative approach, which, under regularity assumptions, involves only full-rank matrices, and hence avoids all reductions to zero. The following lemma allows us to compute a sparse Gröbner basis in the homogeneous case, from the non-homogeneous one.

Lemma 3.16. *If G is a sGB of I with respect to $\prec$, then $G^h := \chi^{-1}(G)$ is a sGB of $\langle \chi^{-1}(I) \rangle$ with respect to $\prec_h$.*

Proof. First note that the homogenization commutes with the leading monomial, that is $\forall \bar{g} \in \mathbb{K}[S_M]$, $\mathtt{LM}_{\prec_h}(\chi^{-1}(\bar{g})) = \chi^{-1}(\mathtt{LM}_{\prec}(\bar{g}))$. Let $f \in \langle \chi^{-1}(I) \rangle$. We can write f as $\boldsymbol{X}^{(0, \deg(f) - \delta(f))} \cdot \chi^{-1}(\chi(f))$. Consider $\bar{g} \in G$ such that $\mathtt{LM}_{\prec}(\bar{g}) || \mathtt{LM}_{\prec}(\chi(f))$. By definition (Def. 3.1), $\chi^{-1}(\mathtt{LM}_{\prec}(\bar{g})) || \chi^{-1}(\mathtt{LM}_{\prec}(\chi(f)))$, and by commutativity, it holds that $\mathtt{LM}_{\prec_h}(\chi^{-1}(\bar{g})) || \mathtt{LM}_{\prec_h}(\chi^{-1}(\chi(f))$. The sparse degree and the leading monomials with respect to $\prec_h$ are invariants under the multiplication by $\boldsymbol{X}^{(0,1)}$. Hence, $\mathtt{LM}_{\prec_h}(\chi^{-1}(\bar{g})) || \mathtt{LM}_{\prec_h}(f)$. To conclude, we have to prove that G^h is a basis of $\langle \chi^{-1}(I) \rangle$. As for each $f \in \chi^{-1}(I)$ there is a $\bar{g} \in G$ such that $\mathtt{LM}_{\prec_h}(\chi^{-1}(\bar{g})) || \mathtt{LM}_{\prec_h}(f)$. Thus, the remainder of the division algorithm (Lem. 3.4) is zero, and so we obtain a representation of f in the basis $\chi^{-1}(G)$. ☐

Corollary 3.17. *Let $I \subset \mathbb{K}[S_M]$ be an (non-homogeneous) ideal and consider the (non-homogeneous) polynomial $\bar{f} \in \mathbb{K}[S_M]$. Let G be a (non-homogeneous) sGB of I wrt $\prec$ and $G^h_{\bar{f}}$ be a (homogeneous) sGB of $\langle \chi^{-1}(G) + \chi^{-1}(\bar{f}) \rangle$ wrt $\prec_h$. Then, $\chi(G^h_{\bar{f}})$ is a (non-homogeneous) sGB of $\langle I + \bar{f} \rangle$ wrt $\prec$.*

Cor. 3.17 supports an iterative algorithm to compute a sGB of I. For each $i \leq n$, let $I_i := \langle \bar{f}_1, \dots, \bar{f}_i \rangle$ and $G_i := sGB_{\prec}(I_i)$. Consider $I_i^h := \langle \chi^{-1}(G_{i-1}) + \chi^{-1}(\bar{f}_i) \rangle$. By Cor. 3.17, we can consider G_i as $\chi(sGB_{\prec_h}(I_i^h))$. To compute $sGB_{\prec_h}(I_i^h)$ we use Def. 3.14.

Many rows of the Macaulay matrices reduces to zero during the Gaussian elimination procedure. We can adapt the F5 criterion [19, 15] to identify these rows and avoid them.

Lemma 3.18. *Let G be a sGB of the homogeneous ideal I^h wrt $\prec_h$. Let $\mathcal{N} \subset \mathbb{K}[S_M^h]_D$ be the set of monomials of degree D such that for each of them there is a polynomial in G whose leading term divides it, that is $\mathcal{N} = \left\{ \boldsymbol{X}^{(s,D)} \in \mathbb{K}[S_M^h]_D : \exists\, g \in G \text{ s.t. } \mathtt{LM}_{\prec_h}(g) || \boldsymbol{X}^{(s,D)} \right\}$. To each $\boldsymbol{X}^{(s,D)} \in \mathcal{N}$ associate only one polynomial $g \in G$, such that $\mathtt{LM}_{\prec_h}(g) || \boldsymbol{X}^{(s,D)}$. Let $\mathcal{R}$ be the set formed by the polynomials $\frac{\boldsymbol{X}^{(s,D)}}{\mathtt{LM}_{\prec_h}(g)} \cdot g$ where g is the polynomial associated to $\boldsymbol{X}^{(s,D)} \in \mathcal{N}$.*

Consider the Macaulay matrix $\mathcal{M}'_D$ with columns indexed by the monomials in $\mathbb{K}[S_M^h]_D$ in decreasing order w.r.t. $\prec_h$ and rows indexed by $\mathcal{R}$. Let $\widetilde{\mathcal{M}'_d}$ be the Macaulay matrix obtained after applying Gaussian elimination to $\mathcal{M}'_d$ to obtain a reduced row echelon form. Then, $\mathtt{Rows}(\widetilde{M'_D}) = \mathtt{Rows}(\widetilde{M_D})$, where $\widetilde{M}_d$ is the Macaulay matrix of Lem. 3.15 with respect to G^h. Moreover, the matrix M'_D is full-rank and in row echelon form.

Proof. By construction, we are skipping the polynomials whose leading monomials already appear in M'_D. Hence, each row has a different leading monomial and so, the matrix M'_D is full-rank. If we add to M'_D a new homogeneous polynomial of degree D belonging to the ideal I^h, then it must be linear dependent with the polynomials in $\mathtt{Rows}(M'_D)$. If not, after reducing the polynomial by the previous rows, we discovered a new polynomial in the ideal I^h with a leading monomial which is not divisible by G^h. But this is not possible because G^h is a sparse Gröbner basis. ☐

Lemma 3.19 (Sparse F5 criterion)**.** *Let G^h be a sparse Gröbner basis of the homogeneous ideal I^h wrt $\prec_h$ and let $\mathcal{M}'_D$ be the Macaulay matrix of Lem. 3.18 of degree D. Let $d \in \mathbb{N}$ and consider the set $\mathfrak{b} = \{\boldsymbol{X}^{(s,D-d)} \in \mathbb{K}[S_M^h]_{D-d} : \nexists\, g \in G^h \text{ s.t. } \mathtt{LM}_{\prec_h}(g) || \boldsymbol{X}^{(s,D-d)}\}$. Let $f \in \mathbb{K}[S_M^h]_d$; consider the Macaulay matrix $\mathcal{M}^*_D$ obtained after appending to $\mathcal{M}'_D$ rows indexed by $\{\boldsymbol{X}^{(s,D-d)} \cdot f : \boldsymbol{X}^{(s,D-d)} \in \mathfrak{b}\}$.*

Let $\widetilde{\mathcal{M}^*_D}$ be the matrix obtained after applying Gaussian elimination to $\mathcal{M}^*_D$. Then, $\mathtt{Rows}(\widetilde{\mathcal{M}^*_D}) = \mathtt{Rows}(\widetilde{\mathcal{M}_D})$, where $\widetilde{\mathcal{M}_D}$ is the Macaulay matrix of Lem. 3.15 for the ideal $\langle G^h, f\rangle$. Moreover, if f is not a zero-divisor in $\mathbb{K}[S_M^h]/I^h$, then $\mathcal{M}^*_D$ is full-rank.

Proof. Let $\boldsymbol{X}^{(s,D-d)}$ be a monomial such that there is a $g \in G^h$ such that $\mathtt{LM}_{\prec_h}(g) || \boldsymbol{X}^{(s,D-d)}$. Consider $p := \frac{\boldsymbol{X}^{(s,D-d)}}{\mathtt{LM}_{\prec_h}(g)} \cdot g$. By Rem. 3.2, $\mathtt{LM}_{\prec_h}(p) = \boldsymbol{X}^{(s,D-d)}$. Consider $p_{red} := LT_{\prec_h}(h) + q$, where q is the remainder of the division of $p - LT_{\prec_h}(p)$ by G^h. It holds $p_{red} \in I^h$. Also all the monomials in the support of q are not divisible by the leading monomials of G^h (Lem. 3.4). Then, using the rows of $\mathcal{M}_D^*$ we can form the polynomial $f \cdot q$. If we add the row corresponding to $f \cdot \boldsymbol{X}^{(s,D-d)}$, we can reduce this polynomial to zero as $f \cdot \boldsymbol{X}^{(s,D-d)} + f \cdot q = f \cdot p_{red} \in I^h$. If f is not a zero-divisor in $\mathbb{K}[S_M^h]/I^h$, then $g \cdot f \in I^h$, implies $g \in I^h$ and so $LM_{\prec_h}(g) \in G^h$. Hence, we skip every row reducing to zero involving f. ◻

Lemma 3.20. *If $\bar{f}_1, \ldots, \bar{f}_k \in \mathbb{K}[S_M]$ is a regular sequence, then for each $i \leq k$, $\chi^{-1}(\bar{f}_i)$ is not a zero-divisor of $\mathbb{K}[S_M^h]/\chi^{-1}(\langle \bar{f}_1, \ldots, \bar{f}_{i-1}\rangle)$.*

Proof. If $\chi^{-1}(\bar{f}_i)$ is a zero-divisor of $\mathbb{K}[S_M^h]/\chi^{-1}(\langle \bar{f}_1, \ldots, \bar{f}_{i-1}\rangle)$, there is a $g \in \mathbb{K}[S_M^h]$ such that $g \notin \chi^{-1}(\langle \bar{f}_1, \ldots, \bar{f}_{i-1}\rangle)$ and $g \cdot \chi^{-1}(\bar{f}_i) \in \chi^{-1}(\langle \bar{f}_1, \ldots, \bar{f}_{i-1}\rangle)$. By definition of the homogenization of an ideal, $\chi(g) \notin \langle \bar{f}_1, \ldots, \bar{f}_{i-1}\rangle$ but, as χ is a homomorphism, $\chi(g) \cdot \bar{f}_i \in \langle \bar{f}_1, \ldots, \bar{f}_{i-1}\rangle$. So, $\bar{f}_1, \ldots, \bar{f}_i$ is not a regular sequence. ◻

Hence, given the witness degrees of each I_i^h, we have the algorithm Alg. 3.1 to compute iteratively a sparse Gröbner basis.

As in the standard case, we can define the reduced sGB and adapt [10, Prop. 2.7.6] to prove their finiteness and uniqueness.

4 Multihomogeneous systems

We consider an algorithm for solving 0-dimensional square multihomogeneous systems with no solutions at infinity.

Notation. Let $n_1, \ldots n_r \in \mathbb{N}$, $N := \sum_i n_i$, and $\boldsymbol{n} := (n_1 \ldots n_r) \in \mathbb{N}^r$. For $1 \leq i \leq r$, let $\boldsymbol{x_i}$ be the set of variables $\{x_{i,0}, \ldots, x_{i,n_i}\}$. Let $\mathbb{K}[\boldsymbol{x}] := \bigotimes_{i=1}^r \mathbb{K}[\boldsymbol{x}_i]$ be the multihomogeneous $\mathbb{K}$-algebra multigraded by $\mathbb{Z}^r$, such that for all $\boldsymbol{d} := (d_1, \ldots, d_r) \in \mathbb{Z}^r$, we have $\mathbb{K}[\boldsymbol{x}]_{\boldsymbol{d}} := \bigotimes_{i=1}^r \mathbb{K}[\boldsymbol{x}_i]_{d_i}$. Given a $\mathbb{K}[\boldsymbol{x}]$-module M, we consider $[\mathrm{M}]_{\boldsymbol{d}}$ as the graded part of M of multidegree $\boldsymbol{d}$. Given two multidegrees $\boldsymbol{d}$ and $\bar{\boldsymbol{d}}$, we say that $\boldsymbol{d} \geq \bar{\boldsymbol{d}}$ if the inequality holds componentwise. We consider the multiprojective space $\mathcal{P} := \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$.

Let $\bar{\mathbf{1}} = (1, \ldots, 1) \in \mathbb{Z}^r$ be the multidegree corresponding to multilinear polynomials in $\mathbb{K}[\boldsymbol{x}]$. Let $B = \cap_{i=1}^r \langle x_{i,0}, \ldots, x_{i,n_i}\rangle$ be the ideal generated by all the polynomials in $\mathbb{K}[\boldsymbol{x}]_{\bar{\mathbf{1}}}$.

Consider multihomogeneous polynomials $f_1, \ldots, f_k \in \mathbb{K}[\boldsymbol{x}]$ and denote their multidegrees by $\deg(f_1), \ldots, \deg(f_k) \in \mathbb{N}^r$. Let $V_{\mathcal{P}}(f_1, \ldots, f_k)$ be the

Algorithm 3.1 $\mathtt{M}^2$: Mixed sparse Matrix-F5 with respect to $\prec$

Input: $\bar{f}_1, \ldots, \bar{f}_k \in \mathbb{K}[S_M]$ and $d_1^{wit}, \ldots, d_k^{wit}$ such that d_i^{wit} is the witness degree of I_i^h.

```
for i = 1 to k do
  G_i ← ∅
  for d = 1 to d_i^wit do
    M_d^i ← Macaulay matrix with columns indexed by the monomials
            in K[S_M^h]_d in decreasing order by ≺_h
    for X^(s,d) ∈ K[S_M^h]_d do
      if ∃g ∈ G_{i-1}^h : LM_{≺_h}(g) || X^(s,d) then
        Add to M_d^i the polynomial X^(s,d) / LM_{≺_h}(g) · g
      end if
    end for
    for X^(s,d-δ^A(f̄_i)) ∈ K[S_M^h]_{d-δ^A(f̄_i)} do
      if ∄g ∈ G_{i-1}^h such that LM_{≺_h}(g) || LM_{≺_h}(χ^{-1}(f̄_i)) then
        Add to M_d^i the polynomial X^(s,d-δ^A(f̄_i)) · χ^{-1}(f̄_i)
      end if
    end for
    M̃_d^i ← Gaussian elimination of M_d^i
    G_i ← G_i ∪ {h̄ ∈ χ(Rows(M̃_d^i)) : ∄ ḡ ∈ G_i ∧ LM_≺(ḡ) || LM_≺(h̄)}
  end for
  G_i^h ← χ^{-1}(G_i)
end for
return G_k
```

zero set of $f_1, \ldots, f_k$ over $\mathcal{P}$. If the dimension of $V_{\mathcal{P}}(f_1, \ldots, f_k)$ over $\mathcal{P}$ is $N-k$, then the polynomials $f_1, \ldots, f_k$ form a regular sequence at each point of $\mathbb{P}^1 \times \cdots \times \mathbb{P}^r$. That is, for each prime ideal $\mathfrak{p}$, such that $\mathfrak{p} \not\subset B$, $(f_1, \ldots, f_k)$ form a regular sequence over $\mathbb{K}[\boldsymbol{x}]_{\mathfrak{p}}$, the localization of $\mathbb{K}[\boldsymbol{x}]$ at $\mathfrak{p}$. In this case, we say that $(f_1, \ldots, f_k)$ is a *regular sequence outside* B. This kind of sequence is related to the filter regular sequence [36, Sec. 2] and the sequence of "almost" nonzero divisors [30, Sec. 3], [33, Sec. 2].

Let $\mathcal{K}_\bullet(f_1, \ldots, f_k\,; \mathbb{K}[\boldsymbol{x}])$ be the Koszul complex of $f_1, \ldots, f_k$ over $\mathbb{K}[\boldsymbol{x}]$. Let $H_i(\mathcal{K}_\bullet(f_1, \ldots, f_k\,; \mathbb{K}[\boldsymbol{x}]))$ be the i-th Koszul homology module. We also write this homology module as H_i^k.

Let $\boldsymbol{x}_h := \prod_{i=1}^r x_{i,0} \in \mathbb{K}[\boldsymbol{x}]_{\bar{\mathbf{1}}}$. We say that a multihomogeneous system $(f_1, \ldots, f_N)$ has *no solutions* at infinity if the system $(f_1, \ldots, f_N, \boldsymbol{x}_h)$ has no solutions over $\mathcal{P}$. We dehomogenize a multihomogeneous polynomial by replacing each variable $x_{i,0}$ with 1. Let $\mathbb{K}[\bar{\boldsymbol{x}}]$ be the $\mathbb{K}$-algebra obtained by the dehomogenization of $\mathbb{K}[\boldsymbol{x}]$. Given $f \in \mathbb{K}[\boldsymbol{x}]$, we consider $\bar{f} \in \mathbb{K}[\bar{\boldsymbol{x}}]$, its dehomogenization.

Remark 4.1. *There is a (multigraded) isomorphism between the multihomogeneous* $\mathbb{K}$*-algebra* $\mathbb{K}[\boldsymbol{x}]$ *and the polytopal algebra* $\mathbb{K}[S^h_{M_1,\ldots,M_r}]$*, where* M_i *are cross products of simplex polytopes.*

4.1 Multigraded regularity

Based on Maclagan and Smith [31, 30], Botbol and Chardin [6] define the multigraded Castelnuovo-Mumford regularity over $\mathbb{K}[\boldsymbol{x}]$ in terms of the vanishing of the *local cohomology* modules with respect to B. For an introduction to local cohomology, we refer to [8]. In the following we present some results from [5, Chp. 6], that we need in our setting, see also [1].

Given a module M, $H^j_B(\mathrm{M})$ is the j-th local cohomology module at B and $\mathtt{sp}(\mathrm{M}) := \{\boldsymbol{d} \in \mathbb{Z}^r : [\mathrm{M}]_{\boldsymbol{d}} \neq 0\}$ is the set of multidegrees where the module is not zero. In [6, 5], $\mathtt{sp}(M)$ is called the support of M.

Consider $\alpha \subset \{1, \ldots, r\}$. We define the Q_α as the convex region of $\mathbb{R}^r$ given by the vectors $(v_1, \ldots, v_r) \in \mathbb{R}^r$ so that for every $i \leq r$,

$$\begin{cases} v_i \leq -n_i - 1 & \text{, if } i \in \alpha \\ v_i \geq 0 & \text{, otherwise.} \end{cases}$$

Consider the multiset $\boldsymbol{\Sigma}^k_i := \{\sum_{j \in I} \deg(f_j) : I \subset \{1 \ldots k\}, \#I = i\}$ containing the sums of the degrees of i (different) polynomials from the set

$\{f_1, \ldots, f_k\}$. Given $v \in \mathbb{R}^r$, the displacement of Q_α by v is $Q_\alpha + v := \{w \in \mathbb{R}^r : w - v \in Q_\alpha\}$. Let $N_\alpha := \sum_{i\in\alpha} n_i$.

Lemma 4.2 ([5, Lem. 6.4.7], , [1, Prop. 4.2])**.** *If* $\mu \notin \bigcup_{\substack{\alpha\in\{1,\ldots,r\}\\ N_\alpha+1=l}} Q_\alpha$*, then* $(H^l_B(\mathbb{K}[\boldsymbol{x}]))_\mu = 0$*. Equivalently,*

$$\mathtt{sp}(H^l_B(\mathbb{K}[\boldsymbol{x}])) \subset \bigcup_{\substack{\alpha\in\{1,\ldots,r\}\\ N_\alpha+1=l\\ \alpha\neq\emptyset}} Q_\alpha.$$

Proposition 4.3 ([5, Remark 6.4.10], [1, Cor. 4.3])**.** *If* $(f_1, \ldots, f_k)$ *form a regular sequence outside* B*, for every* i, j*,*

$$\mathtt{sp}(H^i_B(H^k_j)) \subset \bigcup_{\substack{\alpha\subset\{1,\ldots,k\}\\ N_\alpha+1+j-i\leq k\\ \alpha\neq\emptyset}} \bigcup_{v\in\boldsymbol{\Sigma}^k_{N_\alpha+1+j-i}} Q_\alpha + v. \tag{4}$$

Proof. As we assume that $f_1, \ldots, f_k$ form a regular sequence outside B, we have that $H^w_B(H^k_j) = 0$ for all $w > 0$. Hence, the *cohomological dimension* of H^k_j with respect to B is 0. Therefore, by [5, Rmk. 6.2.5 and Thm. 6.2.4], $\mathtt{sp}(H^0_B(H^k_j)) \subset \bigcup_{i\in\mathbb{Z}} \mathtt{sp}(H^i_B(\mathcal{K}^k_{i+j}))$. By definition $\mathcal{K}^k_{i+j} = 0$, for $i + j > k$ and $\mathcal{K}^k_{i+j} = \bigoplus_{v\in\boldsymbol{\Sigma}^k_{i+j}} \mathbb{K}[\boldsymbol{x}](-v)$, where $\mathbb{K}[\boldsymbol{x}](-v)$ is the twist (shift) of $\mathbb{K}[\boldsymbol{x}]$ by $-v$. Hence,

$$\mathtt{sp}(H^0_B(H^k_j)) \subset \bigcup_{i\in\mathbb{Z}} \mathtt{sp}(H^i_B(\mathcal{K}^k_{i+j})) = \bigcup_{\substack{i\in\mathbb{Z}\\ i+j\leq k}} \bigcup_{v\in\boldsymbol{\Sigma}^k_{i+j}} \mathtt{sp}(H^i_B(\mathbb{K}[\boldsymbol{x}](-v)))$$

By Lem. 4.2, $\mathtt{sp}(H^i_B(\mathbb{K}[\boldsymbol{x}](-v))) \subset \bigcup_{\substack{\alpha\in\{1,\ldots,r\}\\ N_\alpha+1=i\\ \alpha\neq\emptyset}} Q_\alpha + v$. The proposition follows by a change of indices. ∎

Proposition 4.4. *If* $(f_1, \ldots, f_k)$ *form a regular sequence outside* B*, then for* $i > 0$, $H^i_B(H^k_j) = 0$ *and for* $j > 0$*, it holds* $H^0_B(H^k_j) = H^k_j$*.*

The proposition follows from considering the spectral sequence of the double complex given by the Koszul complex and the Čech complex of $f_1, \ldots, f_k$ over B, when $f_1, \ldots, f_k$ is a regular sequence outside B, [1, Sec. 4].

Corollary 4.5 (Multihomogeneous Macaulay bound)**.** *Let* $f_1, \ldots, f_{N+1}$ *be regular sequence outside* B *and* $\boldsymbol{D_k} := \left(\sum_{i=1}^{k} \deg(f_i)\right) - \boldsymbol{n}$*. If* $\boldsymbol{d} \geq \boldsymbol{D_k}$*, then* $\forall i, j, k$*,* $[H_B^j(H_i^k)]_{\boldsymbol{d}} = 0$*.*

Proof. We use Prop. 4.3. Fix i and j in Eq. (4), and consider $\alpha \subset \{1, \ldots, k\}$ such that $N_\alpha + 1 + j - i \leq k$, $\#\alpha \neq \emptyset$, and $v \in \boldsymbol{\Sigma}^k_{N_\alpha+1+j-i}$. If $t \in \alpha$, then the t-th coordinate of any element in $Q_\alpha + v$ has to be $\leq -n_t - 1 + v_t$, where v_t is the t-th coordinate of v. As all the multidegrees $\deg(f_1), \ldots, \deg(f_k)$ are non-negative, $v_t \leq \sum_{i=1}^k deg(f_i)_t$. So, $-n_t - 1 + v_t < -n_t + \sum_{i=1}^k deg(f_i)_t = (\boldsymbol{D_k})_t \leq \boldsymbol{d}_t$. Hence, $\boldsymbol{d} \notin Q_\alpha + v$. By Prop. 4.3, $[H_B^0(H_i^k)]_{\boldsymbol{d}} = 0$. □

The bound $\boldsymbol{D_k}$ is not tight, e.g. see [1, Sec. 4.4].

Like with homogeneous polynomials, we define the multigraded Hilbert function, HF, of a $\mathbb{K}$-module M as the function that maps the multidegrees $\boldsymbol{d} \in \mathbb{Z}^r$ to $HF(\mathrm{M}, \boldsymbol{d}) = \dim_{\mathbb{K}}([\mathrm{M}]_{\boldsymbol{d}})$. When $\boldsymbol{d}$ is, component-wise, big enough, then $HF(\mathrm{M}, \boldsymbol{d})$ equals a polynomial $P_{\mathrm{M}} \in \mathbb{Q}[y_1, \ldots, y_r]$ evaluated at $\boldsymbol{d}$ [31, Prop. 2.8]; the Hilbert polynomial. If all the local cohomologies of M at a multidegree $\boldsymbol{d}$ vanish, that is for all i, $[H_B^i(\mathrm{M})]_{\boldsymbol{d}} = 0$, then, for this $\boldsymbol{d}$, the Hilbert function and polynomial agree, $HF(\mathrm{M}, \boldsymbol{d}) = P_{\mathrm{M}}(\boldsymbol{d})$ [31, Prop. 2.14].

Corollary 4.6. *Let* $\boldsymbol{d} \geq D_K$*, component-wise. If* $k = N$*, then the dimension of* $[\mathbb{K}[\boldsymbol{x}]/\langle f_1, \ldots, f_N\rangle]_{\boldsymbol{d}}$ *is the number of solutions, counting multiplicities, of the system* $(f_1, \ldots, f_N)$ *over* $\mathcal{P}$*. When* $k = N+1$*,* $\mathbb{K}[\boldsymbol{x}]_{\boldsymbol{d}} = [\langle f_1, \ldots, f_{N+1}\rangle]_{\boldsymbol{d}}$*.*

4.2 Computing graded parts of the ideals

Let $(f_1, \ldots, f_k)$ be multihomogeneous system over $\mathcal{P}$. Alg. 4.1 computes a set of generators of the vector space $[\langle f_1, \ldots, f_k\rangle]_{\boldsymbol{d}}$. Moreover, if $(f_1, \ldots, f_k)$ form a regular sequence outside B, and $\boldsymbol{d} \geq \boldsymbol{D_k}$, then it performs no reduction to zero.

Theorem 4.7. *Let* $(f_1, \ldots, f_k)$ *be a multihomogeneous system. Alg. 4.1 computes a matrix such that the polynomials in its rows form a set of generators of the vector space* $[\langle f_1, \ldots, f_k\rangle]_{\boldsymbol{d}}$*,* $\forall \boldsymbol{d} \in \mathbb{Z}^r$*.*

We omit the proof as it is similar to Lemmata 3.18 and 3.19.

Remark 4.8. *Following the definition of the Koszul complex,* $[H_i^k]_{\boldsymbol{d}} = 0$ *implies that, given any syzygy* $\sum_i g_i \cdot f_i = 0$ *such that* $deg(g_i f_i) = \boldsymbol{d}$*, then* $\forall j$*,* $g_j \in [\langle f_1, \ldots, f_{j-1}, f_{j+1}, \ldots, f_k\rangle]_{\boldsymbol{d} - \deg(f_j)}$*.*

Algorithm 4.1 $\mathtt{M_3H}(\{f_1,\ldots,f_k\},\boldsymbol{d},<)$

Input: $f_1,\ldots,f_k \in \mathbb{K}[\boldsymbol{x}]$, degree $\boldsymbol{d}$ and $<$ a monomial order

$\mathfrak{L} \leftarrow \emptyset$.

if $k = 1$ **then**

  $\mathcal{M}_{\boldsymbol{d}}^k \leftarrow$ Macaulay matrix with columns indexed by the monomials in $\mathbb{K}[\boldsymbol{x}]_{\boldsymbol{d}}$ in decreasing order wrt $<$

else

  $\mathcal{M}_{\boldsymbol{d}}^k \leftarrow \mathtt{M_3H}(\{f_1,\ldots,f_{k-1}\},\boldsymbol{d},<)$

  $\mathfrak{L} \leftarrow$ Leading monomials of the Gaussian elimination of $\mathtt{M_3H}(\{f_1,\ldots,f_{k-1}\},\boldsymbol{d}-\deg(f_k),<)$

end if

for $\boldsymbol{x}^\beta \in \mathbb{K}[\boldsymbol{x}]_{\boldsymbol{d}-\deg(f_k)}$ **do**

  if $\boldsymbol{x}^\beta \notin \mathfrak{L}$ **then**

    Add to $\mathcal{M}_{\boldsymbol{d}}^k$ the polynomial $\boldsymbol{x}^\beta \cdot f_k$

  end if

end for

return $\mathcal{M}_{\boldsymbol{d}}^k$

Lemma 4.9. *If $[H_1^k]_{\boldsymbol{d}} = 0$, then every polynomial $\boldsymbol{x}^\beta \cdot f_k$ in $\mathcal{M}_{\boldsymbol{d}}^k$ is linear independent to the (polynomials corresponding to) other rows.*

Proof. If there is a polynomial of the form $\boldsymbol{x}^\beta \cdot f_k$ in $\mathcal{M}_{\boldsymbol{d}}^k$ that is linearly dependent with the other rows of the matrix, then there is a syzygy of the system $(f_1,\ldots,f_k)$ involving f_k. That is, there are multihomogeneous polynomials $g_1,\ldots,g_k$ so that $\sum_i g_i\, f_i = 0$, for every $\boldsymbol{x}^\sigma$ in the support of g_i it holds $\boldsymbol{x}^\sigma \cdot f_i \in \mathtt{Rows}(\mathcal{M}_{\boldsymbol{d}}^k)$, and $\boldsymbol{x}^\beta$ belongs to the support of g_k. As H_1^k vanishes at degree $\boldsymbol{d}$, by Rem. 4.8, $g_k \in [\langle f_1,\ldots,f_k\rangle]_{\boldsymbol{d}-\deg(f_k)}$. But, by construction, $LM(g_k)\cdot f_k$ does not belong to $\mathtt{Rows}(\mathcal{M}_{\boldsymbol{d}}^k)$. Hence, this syzygy can not be formed with the rows of $\mathcal{M}_{\boldsymbol{d}}^k$. ☐

Lemma 4.10. *If $[H_1^s]_d = 0$, for all $s \leq k$, then M_d^k is full-rank.*

Proof. We proceed by induction on k. The case $k = 1$ is trivial, as $\langle f_1\rangle$ is a principal ideal. If M_d^k is not full-rank, we have a syzygy involving f_k, because M_d^{k-1} is full-rank by inductive hypothesis. Hence, there are multihomogeneous polynomials $g_1,\ldots,g_k$ such that $\sum_i g_i\, f_i = 0$ and we can form each g_i with the rows of M_d^k. As H_1^k vanishes at degree d, then $g_k \in [\langle f_1,\ldots,f_k\rangle]_{d-\deg(f_k)}$. Hence, the $LM(g_k)\cdots f_k$ does not belong to the $Rows(M_d^k)$, so we can not have this syzygy. ☐

Corollary 4.11. *If $(f_1, \dots, f_k)$ is a regular sequence outside B, then for $\boldsymbol{d} \geq \boldsymbol{D_k}$, all the matrices appearing in Alg. 4.1 are full-rank.*

Proof. We proceed by induction on k. When $k = 1$, the ideal is principal and so the theorem holds. In step k, note that $\boldsymbol{d} \geq \boldsymbol{D_k}$ implies $\boldsymbol{d} \geq \boldsymbol{d} - deg(f_k) \geq \boldsymbol{D_k} - \deg(f_k) = \boldsymbol{D_{k-1}}$. Hence, we have no reduction to zero in the recursive calls. As $\boldsymbol{d} \geq \boldsymbol{D_k}$, by Prop. 4.4, $H^0_B(H^k_i) = H^k_i$, and by Cor. 4.5, $[H^k_i]_{\boldsymbol{d}} = 0$. Hence, by Lem. 4.9, $\mathcal{M}^k_{\boldsymbol{d}}$ has not reduction to zero involving $\boldsymbol{x}^\beta \cdot f_k$. As, by induction, $\mathtt{M_3H}(\{f_1, \dots, f_{k-1}\}, \boldsymbol{d}, <)$ is full-rank, $\mathcal{M}^k_{\boldsymbol{d}}$ is full-rank. ☐

4.3 Solving zero-dimensional systems

Our solving strategy is to dehomogenize the system and to compute the multiplication maps for the affine variables. Then we can apply FGLM to compute a Gröbner basis or compute the eigenvalues/eigenvectors of the multiplication maps.

Let $(f_1, \dots, f_N)$ be a 0-dimensional system over $\mathcal{P}$ with no solutions at infinity. If we do not know if the system has no solutions at infinity, we can ensure it by performing a generic linear change of coordinates preserving the multihomogeneous structure, e.g. see [11, Pg. 121]. We use Alg. 4.1 to construct a monomial basis and the multiplication maps over $\mathbb{K}[\bar{\boldsymbol{x}}]/\langle \bar{f_1}, \dots, \bar{f_N} \rangle$. Following Alg. 4.1, let $\mathfrak{L}$ be the set of leading monomials of the polynomials in $[\langle f_1, \dots, f_N \rangle]_{\boldsymbol{D_N}}$, with respect to $<$. Let $\mathfrak{b}$ be a list of monomials in $\mathbb{K}[\boldsymbol{x}]_{\boldsymbol{D_k}}$ not in $\mathfrak{L}$, sorted by $<$. Consider $\boldsymbol{D_{N+1}} := \boldsymbol{D_N} + \bar{\boldsymbol{1}}$.

Definition 4.12. *For a multilinear polynomial $f_0 \in \mathbb{K}[\boldsymbol{x}]_{\bar{\boldsymbol{1}}}$, let $\breve{\mathcal{M}}^{f_0}$ be the Macaulay matrix that we obtain after we permute the columns of $\mathtt{M_3H}(\{f_1, \dots, f_N, f_0\}, \boldsymbol{D_{N+1}}, <)$ so that the columns indexed by the monomials $\{\boldsymbol{x}_h \cdot \boldsymbol{x}^\beta : \boldsymbol{x}^\beta \in \mathfrak{b}\}$ are the last ones. Let $\breve{\mathcal{M}}^{f_0}$ be $\left[\begin{smallmatrix} M^{f_0}_{1,1} & M^{f_0}_{1,2} \\ M^{f_0}_{2,1} & M^{f_0}_{2,2} \end{smallmatrix}\right]$, where the monomials indexing the columns of $\left[\begin{smallmatrix} M^{f_0}_{1,2} \\ M^{f_0}_{2,2} \end{smallmatrix}\right]$ are the monomials in $\{\boldsymbol{x}_h \cdot \boldsymbol{x}^\beta : \boldsymbol{x}^\beta \in \mathfrak{b}\}$, and the polynomials in the rows of $\left[\begin{smallmatrix} M^{f_0}_{2,1} & M^{f_0}_{2,2} \end{smallmatrix}\right]$ are of the form $\{\boldsymbol{x}^\beta \cdot f_0 : \boldsymbol{x}^\beta \in \mathfrak{b}\}$.*

Observe that, the matrix $\left[\begin{smallmatrix} M^{f_0}_{1,1} & M^{f_0}_{1,2} \end{smallmatrix}\right]$ is a permutation of $\mathtt{M_3H}(\{f_1, \dots, f_N\}, \boldsymbol{D_{N+1}}, <)$, and the polynomials in its rows do not involve f_0, so we can forget the superscripts.

Remark 4.13. *By Cor. 4.6, if* $(f_1,\dots,f_N)$ *is* 0*-dimensional, and* f_0 *does not vanish on* $V_{\mathcal{P}}(f_1,\dots,f_N)$*, then* $\widetilde{\mathcal{M}}^{f_0}$ *is invertible.*

Theorem 4.14. *Let* $\bar{\mathfrak{b}}$ *be the dehomogenization of the monomials in* $\mathfrak{b}$*. If the system* $f_1,\dots,f_n$ *has no solutions at infinity, then* $\bar{\mathfrak{b}}$ *forms a monomial basis for* $\mathbb{K}[\bar{\boldsymbol{x}}]/\langle \bar{f}_1,\dots,\bar{f}_N\rangle$*.*

Proof. The set $\bar{\mathfrak{b}}$ is a monomial basis if its elements are linear independent on $\mathbb{K}[\bar{\boldsymbol{x}}]/\langle \bar{f}_1,\dots,\bar{f}_N\rangle$ and generate this quotient ring. By Cor. 4.6, the dimension of the quotient ring, as a vector space, is the same as the number of elements in $\bar{\mathfrak{b}}$, so we only need to prove the linear independence of the elements in $\bar{\mathfrak{b}}$. Assume that there is a linear combination $\bar{p} := \sum_i c_i \bar{\mathfrak{b}}_i$ congruent to 0 in $\mathbb{K}[\bar{\boldsymbol{x}}]/\langle \bar{f}_1,\dots,\bar{f}_N\rangle$. Then, similarly to Rem. 2.1, there is a $\omega \in \mathbb{N}$, such that $(\boldsymbol{x}_h)^\omega \cdot p \in \langle f_1,\dots,f_N\rangle$, where $p := \sum_i c_i \mathfrak{b}_i$. By Rem. 4.13, as the system has no solutions at infinity, $\widetilde{\mathcal{M}}^{\boldsymbol{x}_h}$ is invertible. The rows of $\widetilde{\mathcal{M}}^{\boldsymbol{x}_h}$ contain the set $\{\boldsymbol{x}_h \cdot \mathfrak{b}_i\}_i$, so we can form $\boldsymbol{x}_h \cdot p$ by taking a linear combination of them. As the matrix is full-rank, this row is independent from the polynomials in $[\langle f_1,\dots,f_N\rangle]_{\boldsymbol{D_{N+1}}}$ (Thm. 4.7), and then $\boldsymbol{x}_h \cdot p \notin [\langle f_1,\dots,f_N\rangle]_{\boldsymbol{D_{N+1}}}$. Hence, $\omega > 1$. The multidegree of $(\boldsymbol{x}_h)^\omega \cdot p$ is $\boldsymbol{D_N} + \omega \cdot \bar{\mathbf{1}}$. As $\omega > 1$, $\boldsymbol{D_N} + \omega \cdot \bar{\mathbf{1}} \geq \boldsymbol{D_{N+1}}$. By Cor. 4.5, $[H_1(\mathcal{K}_\bullet(f_1,\dots,f_N,\boldsymbol{x}_h\,;\mathbb{K}[\boldsymbol{x}]))]_{\boldsymbol{D_N}+\omega\,\cdot\,\bar{\mathbf{1}}} = 0$. Then, by Rem. 4.8, $(\boldsymbol{x}_h)^{\omega-1} \cdot p \in \langle f_1,\dots,f_N\rangle$. But, assuming minimality of ω, $(\boldsymbol{x}_h)^{\omega-1} \cdot p \notin \langle f_1,\dots,f_N\rangle$. So, $\bar{p}$ does not exist. □

Remark 4.15. *If the system* $(f_1,\dots,f_N,\boldsymbol{x}_h)$ *has no solutions over* $\mathcal{P}$*, by Rem. 4.13, the matrix* $M^{\boldsymbol{x}_h}$ *is invertible. As* $M^{\boldsymbol{x}_h}_{2,1}$ *is zero, and* $M^{\boldsymbol{x}_h}_{2,2}$ *is the identity, the matrix* $M^{\boldsymbol{x}_h}$ *is invertible.*

Definition 4.16. *When* $(f_1 \dots f_N)$ *has no solutions at infinity, we define* $(M^{f_0}_{2,2})^c := M^{f_0}_{2,2} - M^{f_0}_{2,1} \cdot M^{-1}_{1,1} \cdot M_{1,2}$*, the* Schur complement *of* $M^{f_0}_{2,2}$*.*

Theorem 4.17. *If the system* $(f_1,\dots,f_N)$ *has no solutions at infinity, then the matrix* $(M^{f_0}_{2,2})^c$ *is the multiplication map of* $\bar{f}_0$ *over* $\mathbb{K}[\bar{\boldsymbol{x}}]/\langle \bar{f}_1,\dots,\bar{f}_N\rangle$*, with respect to the basis* $\bar{\mathfrak{b}}$*.*

Proof. By Thm. 4.14, $\bar{\mathfrak{b}}$ is a monomial basis of $\mathbb{K}[\bar{\boldsymbol{x}}]/\langle \bar{f}_1,\dots,\bar{f}_N\rangle$. Hence, for every i, $\mathfrak{b}_i \cdot f_0 \equiv \boldsymbol{x}_h \sum_j (M^{f_0}_{2,2})^c_{i,j} \mathfrak{b}_j \mod \langle f_1,\dots,f_N\rangle$. If we dehomogenize, $\bar{\mathfrak{b}}_i \cdot \bar{f}_0 \equiv \sum_j (M^{f_0}_{2,2})^c_{i,j} \bar{\mathfrak{b}}_j \mod \langle \bar{f}_1,\dots,\bar{f}_N\rangle$. □

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